Monte Carlo Analysis of the Existence of Self-Fulfilling Equilibria in the Krusell-Smith Algorithm

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1 Introduction

This paper aims at conducting a simplified replication of Marco Cozzi's paper "The Krusell-Smith algorithm: Are Self-Fulfilling Equilibria Likely?". The Krusell and Smith (KS) algorithm is the most widely used algorithm to compute macroeconomic equilibrium in heterogenous-agent models with incomplete markets and aggregate uncertainty. The aim is to assess the concern about the likelihood of multiple self-fulfilling equilibria (SFE) arising from this algorithm. This possibility is indeed relevant because the equilibrium Aggregate Law of Motion (ALM) is unknown and needs to be derived using a guess-and-verify iterative method. Yet, the agents' optimal decision rules have to be calculated at each step of this problem, and they in turn depend on the ALM the agents are assumed to perceive. Hence, on the face of it, this circular aspect of the process can result in a complementarity between the guess concerning agents' perception of the evolution of future prices and what they end up choosing accordingly. In their seminal paper, KS acknowledge this concern, and dismiss it a bit too hastily saying "there's nothing in the theoretical link between these perceptions and the aggregate savings behaviour that is suggestive of self-fulfilling equilibria". They do test, nonetheless, in the working paper version of their paper whether more sophisticated perceptions significantly change the equilibrium properties, and they conclude that their approximate equilibrium fixed point remains "virtually identical" throughout the experiments. A more convincing computational approach to address this skepticism might be the one adopted by Cozzi in his paper, who relies on a Monte Carlo analysis of the economy.

Here's where the possibility of SFE comes from: if postulating an ALM for capital that is below (above) the equilibrium one induces households to save less (more) resources in the aggregate, the process would be indeed subject to a complementarity between the guessed ALM and the consequent saving decisions. Unfortunately, the uniqueness of the equilibrium cannot be proved analytically, so Cozzi attempts to address the issue using a numerically intensive exercise. In his paper, he even uses the more elaborate version of KS model, featuring infinitely-lived agents and preference heterogeneity. In my replication code however, I will abstract away from preference heterogeneity. Although the model without preference heterogeneity is less vulnerable to SFE because of the low wealth concentration it generates, it's still relevant to check for SFE in the benchmark model since, (i) Cozzi does not do it, (ii) it's simpler to work with, (iii) and the onset of a complementarity between the initial guess on which agents base their savings decisions and the equilibrium ALM remains a possibility in theory since the algorithm is essentially the same.

After implementing carefully designed perturbation experiments, of which I perform a rudimentary replication, Cozzi does not find any evidence suggesting the existence of SFE such that the ALM parameters converge to different values depending on the starting guess. As he puts it: "most experiments tend to cluster around two different values for the ALM, and converge to ALM parameters that differ from the equilibrium one, but the discrepancy is always quantitatively negligible". Cozzi admits that it's hard to disentangle this gap from numerical error stemming from the discretization of the state space, the sampling variability due to the simulations and the convergence criteria, Cozzi posits that his finding makes numerical error the most plausible culprit behind the differences between the equilibrium ALM and the sequence of converged ALM found from the perturbations. What is crucial is that even if you consider the alternative ALM as different equilibria, the differences are always insignificant quantitatively.

He interprets his results as follows: although the wealth distribution in KS economy features a fat right tail, the share of households who increase their savings when they perceive an ALM that, say, overpredicts the future aggregate capital (and hence predicts lower interest rates) is far below 0.5%. Agents need to have piled up between 100 and 300 times the average income for them to start increasing their savings. Therefore, in the aggregate, this reaction is dominated by the reduction in savings of the overarching majority, and multiple SFEs are not likely to arise. This self-correcting mechanism present in the algorithm prevents the overprediction from materializing.

In Section 2, I present the KS benchmark model and algorithm. In Section 3, I explain Cozzi's numerical approach to check for SFE and discuss some results, including my replication. Section 4 concludes.

2 Model & Algorithm

This part essentially reproduces Section 2 of KS (1998) with some minor clarifications. I'll work with KS benchmark setup. The key source of heterogeneity is the introduction of partially uninsurable idiosyncratic income shocks, and there's no preference heterogeneity (e.g., different discount factors). Apart from that, the model is identical to the standard stochastic growth model.

2.1 Environment

They consider a large population of infinitely lived agents (measure one). There's only one good per period, and the preferences of each consumer over consumption streams are described by:

$$\mathrm{E}_0 = \sum_{t=0}^{\infty} \beta^t U(c_t)$$

with

$$U(c) = \lim_{\nu \to \sigma} \frac{c^{1-\nu} - 1}{1 - \nu}$$

Production, y, of the unique good is a Cobb-Douglas function of labor l and capital k: $y = zk^{\alpha} l^{1-\alpha}$ where $\alpha \in [0, 1]$. Output can be transformed into current consumption and future capital k' following:

$$c + k' - (1 - \delta)k = y$$

where δ is the depreciation rate.

Each agent is endowed with 1 unit of time, which yields $\epsilon \tilde{l}$ units of labor input, with ϵ stochastic and equal to 0 or 1. When $\epsilon = 1$, the agent is employed and supplying $\tilde{l}$ units of labor; when $\epsilon = 0$, we think of her as unemployed. The economy is also subject to aggregate productivity shocks, denoted z. The aggregate state can either be good and $z = z_g$, or bad and $z = z_b$. The aggregate shock follows a a first-order Markov process described by the transition probabilities $\pi_{ss'}$: the probability that next period's aggregate shock is $z_{s'}$ given that it was z_s this period. The idiosyncratic and aggregate shocks are correlated, and it is assumed that the idiosyncratic shocks satisfy a law of large numbers, meaning that the sole source of aggregate uncertainty is the aggregate shock. This entails that the number of unemployed agents always equals u_g in the good state and u_b in the bad one, so when one controls for z, idiosyncratic shocks are uncorrelated. Let $\pi_{ss'\epsilon\epsilon'}$ denote the probability of transition from state (z_s, ϵ) in the current period to state (z_s', ϵ') next period. The transition probabilities must verify the following:

$$\pi_{ss'00} + \pi_{ss'01} = \pi_{ss'10} + \pi_{ss'11} = \pi_{ss'}$$

and

$$u_s \frac{\pi_{ss'00}}{\pi_{ss'}} + (1 - u_s) \frac{\pi_{ss'10}}{\pi_{ss'}} = u_{s'}$$

for all four possible combinations of (s, s').

2.2 Market arrangement

The assumption of incomplete markets is what's behind the main computational difficulty of this model. In fact, had the authors assumed the existence of complete markets, it would have been possible to determine full contingent plans for aggregate capital accumulation and to compute all state-contingent prices without knowing how the wealth distribution evolves. But KS assume incomplete markets with only one asset: capital. This asset is at once a store of value for agents and means to self-insure against income shocks. Let k denote capital holdings. To rule out Ponzi schemes and to ensure that loans are paid back, they restrict k to satisfy $k \in \mathcal{K} \equiv [0, \infty)$, and the lower bound on capital is referred to as the borrowing constraint.

Consumers collect income from supplying labor and from the services of their capital. Let $\bar{k}$ be the total amount of capital in the economy and $\bar{l}$ the total amount of labor, the implied relevant prices are $w(\bar{k}, \bar{l}, z) = (1-\alpha)z(\bar{k}/\bar{l})^{\alpha}$ and $r(\bar{k}, \bar{l}, z) = \alpha)z(\bar{k}/\bar{l})^{\alpha-1}$.

KS consider a recursive equilibrium definition, with at its heart a law of motion of the aggregate state of the economy. The aggregate state is (Γ, z), where Γ is the current distribution of consumers over holdings of capital and employment status. The part of the law that is relative to z is exogenous and is described by z's transition matrix. The part that concerns Γ is labeled H, that is, $\Gamma' = H(\Gamma, z, z')$. For each agent, the state variable is his capital holdings, his idiosyncratic shock and the aggregate state: $(k, \epsilon; \Gamma, z)$. Crucially, the role of the aggregate state is allowing the agent to predict future prices. Her optimization problem can be formulated as

$$v(k, \epsilon; \Gamma, z) = \max_{c,k'}\{U(c) + \beta \mathrm{E}[v(k', \epsilon'; \Gamma', z')|z, \epsilon]\}$$

subject to

$$\begin{aligned} c + k' &= r(\bar{k}, \bar{l}, z)k + w(\bar{k}, \bar{l}, z)\tilde{l}\epsilon + (1-\delta)k \\ \Gamma' &= H(\Gamma, z, z') \\ k' &\geq 0 \end{aligned}$$

and the stochastic laws of motion for ϵ and z. Let the function f be the decision rule for capital updating implied by this problem: $k' = f(k, \epsilon; \Gamma, z)$.

We can now define what makes up a recursive equilibrium: a law of motion H, functions v and f, and pricing functions r and w such that (i) (v, f) solves the agent's problem, (ii) r and w are given by marginal productivities, and (iii) H is generated by f, meaning that it is the appropriate summing up of consumer' optimal decisions of capital given their current status in terms of employment and wealth.

2.3 KS algorithm

This part motivates the need for computational methods to solve the model and outlines KS original approach, which is necessary in order to understand why one might be worried about the existence of SFE.

The endogenous state variable, Γ, is a high-dimensional object. Yet, numerical solution of dynamic programming problems gets increasingly tricky as the state space size increases. KS idea to deal with this issue consists in assuming that consumers are boundedly rational in their perception of how Γ evolves, and increasing the sophistication of these perceptions their rationality) until the errors due to their bounded rationality become negligible.

They suppose that agents believe that future prices only depend on the first I moments of Γ, where they call these moments $m \equiv (m_1, m_2, ..., m_I)$. This is restrictive because, in order to forecast future prices, it is necessary to know how aggregate capital evolves; yet, since savings decisions do not aggregate due to idiosyncratic shocks, the future total capital stock is a nontrivial function of *all* the moments of the current distribution. So boundedly rational agents have in mind a law of motion H_I which belongs to a class $\mathcal{S}$ such that $m' = H_I(m, z, z')$, and given H_I, optimal savings are represented by f_I. Given f_I and an initial wealth and idiosyncratic shock distribution, it is possible to derive the aggregate behavior - a time-series path of of income and wealth distribution - by simulating the choices of a large number of consumers. The resulting distributions can then be used to compared the *simulated* evolution of the vector m to the perceived law of motion of m on which consumers based their choices. The approximate equilibrium is therefore a function H_I that when perceived by agents to make their decisions, (i) generates the best fit with $\mathcal{S}$ to the behavior of m in the simulated time-series and (ii) generates a fit that is almost perfect in the sens that H_I describes the behavior of m in the simulated time-series with small errors.

So here's the algorithm they use to reach their approximate equilibrium:

1. Select I.
2. Guess on a parametrized functional form H_I, and on its parameters.
3. Solve the optimization problem given H_I (using a nonlinear approximation of v)
4. Use resulting f_I to simulate the behavior of N agents (where N large) over a large number of periods, T.
5. Use the stationary region (they use the first 1000 periods as a burn-in) of the simulated time-series to estimate a set of parameters for the functional form assumed in step 2. Here we obtain a measure of goodness of fit.

6. If the estimated parameter values are very close to the initial guesses *and* the goodness of fit is good enough, we stop. If the parameters have converged but the goodness of fit is not adequate, use a bigger I or summon a different functional form for H_I.

3 Cozzi's numerical exercise

3.1 Why are SFE a possibility?

Cozzi suspects this algorithm of featuring SFE. He solves the model in a very similar way to KS, and he also uses a log-linear functional form for H_I which can be written thus:

$$ln\ \bar{k}' = \theta_{0,g} + \theta_{1,g}\ ln\ \bar{k}, \quad \text{if } z = z_g$$

$$ln\ \bar{k}' = \theta_{0,b} + \theta_{1,b}\ ln\ \bar{k}, \quad \text{if } z = z_b$$

Let Θ^* denote the vector of four parameters $\theta_{j,z}$ representing the ALM, with Θ^* referring to their values at the approximate equilibrium obtained with KS algorithm.

I used KS's calibration of the benchmark model, only with 10,000 agents instead of 5,000. So we have $\beta = 0.99$, $\delta = 0.025$, $\sigma = 1$, $\alpha = 0.36$. Shock values are set to $z_g = 1.01$ and $z_b = 0.99$, and unemployment rates to $u_g = 0.04$ and $u_b = 0.1$. The process for (z, ϵ) is such that the average duration of both good and bad states is eight quarters, and such that the average duration of an unemployment spell is 1.5 quarters in good states and 2.5 in bad ones. The calibration also verifies $1.25\frac{\pi_{bb00}}{\pi_{bb}} = \frac{\pi_{gb00}}{\pi_{gb}}$ and $0.75\frac{\pi_{gg00}}{\pi_{gg}} = \frac{\pi_{bg00}}{\pi_{bg}}$. And here are the ALM I obtained (remember that Cozzi works with the model with preference heterogeneity, adding unemployment benefits and using a larger N, so his ALM differ slightly):

$$ln\bar{k}' = \theta_{0,g} + \theta_{1,g}\ ln\ \bar{k} = 0.1719785 + 0.9533877 \times ln\ \bar{k}, \quad \text{for } z = z_g = 1.01$$

$$ln\bar{k}' = \theta_{0,b} + \theta_{1,b}\ ln\ \bar{k} = 0.1552321 + 0.9579019 \times ln\ \bar{k}, \quad \text{for } z = z_b = 0.99$$

To motivate his paper, he plots the differential response of savings for different household types, under two different specifications of the ALM: one is the equilibrium ALM, and the other is an alternative ALM obtained by increasing $\theta^*_{0,g}$ and $\theta^*_{0,b}$ by the same factor until the implied predicted aggregate capital is about 1% higher than its equilibrium counterpart. His graphs show clearly that asset-poor and asset-rich adjust differently to the perturbation: poor agents reduce their savings while rich ones decrease them. Hence, as Cozzi phrases it, "the onset of a complementarity between the ALM and individual savings is indeed a possibility", and this complementarity heavily depends on the shape of the wealth distribution and the magnitude of individual response (this is why, by the way, he chooses to work with the KS economy with preference heterogeneity which yields a more realistic wealth distribution, with more cross-sectional dispersion and skewness). Combining the potentiality of this complementarity, with the circular relationship in the algorithm between the guessed ALM and the simulated ALM (since the simulated ALM stems from agents' decisions based on a certain guessed ALM), we can understand the skepticism regarding SFE that I tackled in the introduction.

3.2 Cozzi's Perturbation Experiment

To assess the quantitative relevance of the SFE concern, Cozzi conducts a series of experiments where Θ^*'s parameters are perturbed randomly. Each $\theta^*_{j,z}$ is multiplied by the realization of a random variable, drawn from independent uniform distributions, whose ranges vary from experiment to the other. He increases the perturbations range progressively from $\pm 1\%$ to $\pm 25\%$, to see what happens as the quality of the initial guess deteriorates. Then the ALM is updated until convergence is reached once again, and he repeats the procedure numerous time, with various initial perturbations.

Cozzi insightfully advances that "this Monte Carlo procedure mimics the actual steps that researchers follow in their search for the equilibrium ALM". And as some researchers remark, the quality of the initial guess is often essential to the success of the algorithm. Authors tend to use the ALM derived under complete markets as an initial guess, but Cozzi consciously departs from this approach in order to put the KS algorithm to the test. He perturbs around Θ^*, playing on the fact that this solution compared to the batch of solutions obtained in the experiments, "does not satisfy any additional requirements". His experiments are intentionally designed to lead to initial guesses distant from Θ^*, to capture the possibility that a researcher starts off a poor guess.

Here's the algorithm he uses to check for SFE:

1. For a given calibration, solve the model and store $\Theta^* = [\theta^*_{0,g}, \theta^*_{1,g}; \theta^*_{0,b}, \theta^*_{1,b}]$.

2. Choose a grid $X = \{x_1, x_2, ..., x_n\}$ for the perturbation factor x.

3. Set $x = x_1$.

4. Perturb each of the four parameters by drawing four random variates from a uniform distribution with support $[-x\%, +x\%]$.

5. Check that the new ALM initial guess satisfies some requirement of non-explosive dynamics. If it doesn't, discard the perturbation.

6. Solve the economy with the new initial guess Θ^g, simulate the time-series path, and update the ALM parameters with a weighted average between the current guess and the parameters of the state-dependent OLS regressions on the simulated data.
7. Iterate until the four parameters in Θ converge, and store them.
8. Repeat the procedure n times (in Cozzi's experiments, $n = 500$).
9. Move to the next x, and redo steps from 1 to 8.

3.3 Some results: SFE Are Not Likely

The replication I conducted does not pretend to be authoritative, in the sense that it is merely for the sake of the exercise. Indeed, as Cozzi indicates in a footnote, a decent Monte Carlo analysis (for instance, with $N = 500$) would take two weeks to run on top-notch computers for each perturbation experiment, that is, for each value of x. On my computer, with reasonable convergence criteria, solving the KS economy once takes several hours. Endowed with little time and a modest computer, and having had to rerun time-consuming codes several times before settling on a satisfactory version of the code, I could only replicate the experiment for $N = 7$ and $x = 5$. Hence, although my replication may have the slight novelty of relying on the benchmark model instead of the full-fledged model that Cozzi uses, it can hardly pretend to add anything substantial to the question, mainly due to the constraints of the time and computers available.

It is also noteworthy that, in essence, the model without discount factor heterogeneity is less prone to SFE, because it yields a low wealth concentration, and hence a negligible mass of agents whose response goes in the same direction as the forecast change in k' (e.g., increasing their savings in the case of an ALM that overpredicts capital evolution). Nevertheless, it is still relevant to check for SFE for this model since Cozzi doesn't do it,and since the onset of a complementarity between the initial guess on which agents base their savings decisions and the equilibrium ALM remains a possibility since the algorithm is essentially the same.

Cozzi's quantitative finding is that SFE are not likely. Observing the set of statistics of the distributions of the converged ALM, he notes that, across experiments, the range of the converged parameters is always tiny: each parameter differs fro its counterpart in Θ^* by 10^{-5} at worse. This applies both for experiments where he perturbs all parameters evenly with $X = [1, 2, 3, 4]$ and for others where he essentially keeps all parameters at their equilibrium ALM values except one which he perturbs wildly (he alternately perturbs constants by $\pm 10\%$ and slopes by $\pm 25\%$). Plotting kernel density estimates of the distributions of the converged parameters, he finds the densities to be bimodal: each converged parameter tends to cluster around two distinct values. He provides suggestive evidence pointing to the fact that this outcome is driven by numerical error (using tighter convergence criteria reduced on average the interquartile range by a factor of 5), but he maintains that even if these were in fact two distinct SFE, the discrepancy is always quantitatively minimal.

Now turning to my replication, my code which reproduces an instance of a perturbation experiment, builds on the elaborate code provided by *QuantEcon* to solve the KS benchmark economy. I studied the code to grasp the role of each function, and incorporated it within a function, *perturb*, that runs the experiment. Again due to the runtime constraints, I ran my code with a faster updating factor at step 6 (0.5 instead of 0.3) than before, a lower tolerance level for value function iteration (10^{-5}) and for convergence of ALM parameters (10^{-3}). And to do them justice, I will compare them to equilibrium ALM obtained with these same slacker convergence criteria. The benchmark equilibrium ALM is then:

$$ln\ \bar{k}' = 0.1721241 + 0.9533481 \times ln\ \bar{k} \quad \text{for } z = z_g = 1.01$$

$$ln\ \bar{k}' = 0.1556267 + 0.9577952 \times ln\ \bar{k} \quad \text{for } z = z_b = 0.99$$

Here are some of my results:

Table: Coefficients with $x = 5$ and $N = 7$

"theta0g"	"theta1g"	"theta0b"	"theta1b"
0.21763600410345418	0.9411540272052454	0.1858882523396433	0.9501558403277754
0.2165714154277438	0.9407387367493254	0.1859027014684876	0.9499581361664831
0.2177611839926297	0.940263264741994	0.185793497187632	0.9496172010177895
0.21763403144124127	0.9411064690115157	0.1857802946596186	0.9491016765880576
0.2178360068544099	0.9404774974629364	0.1859244948739015	0.950345723088512
0.21767479684659063	0.9412113030010083	0.18580691276030778	0.9499595966180625
0.2178221751632633	0.9415053649323867	0.18590705621999973	0.949933633897722

Despite the small N, some meaningful observations can still be made. It is obvious that parameters are very similar across iterations, although each one starts off a different initial guess for the ALM used by agents to maximize their value function.

Table: Statistics with $x = 5$ and $N = 7$

"minimum"	"maximum"	"mean"
0.2165714154277438	0.2178360068544099	0.2175622305470475
0.940263264741994	0.9415053649323867	0.9409223804434875
0.1857802946596186	0.1859244948739015	0.18585760135851292
0.9491016765880576	0.950345723088512	0.9409223804434875

The biggest gaps between a parameter and its counterpart in $\Theta*$ do not exceed 0.03. Given the slack convergence criteria and following Cozzi's argument, these bigger gaps can be attributed to numerical error, and they do not seem worrying. More solid results can be attained with more iterations, and with increasing the dimension of X, which can be achieved by a straightforward adjustment of the code.

3.4 Economic intuition

We owe this result to a self-correcting mechanism present in the KS algorithm. Compared to the equilibrium ALM, an alternative one that overpredicts (underpredicts) future capital will reduce (increase) the equilibrium interest rate, increasing (decreasing) the savings of wealthy consumes only. Yet, the latter account for only a tiny fraction of the population and, hence, their behavior cannot outweigh the opposite change driven by poor agents, who now expect higher (lower) future wages and increase (decrease) their current consumption, inhibiting the overprediction (underprediction) from materializing. Quantitatively, the change in behavior of rich agents has a negligible influence on the determination of the change in aggregate savings, yielding stability in the algorithm as a by-product. More details regarding the decomposition of the effects affecting the intertemporal motive of savings are discussed by Cozzi in his paper. Since these effects do not go in the same direction for everyone, it is theoretically hard to tell which group will outweigh the other in the aggregate. It is hard because it depends on the threshold values for accumulated wealth such that the negative income effect starts dominating (that is, the effect rich agents experience), and on the mass of agents above them. There are several such thresholds: one for each combination of state variables. However, Cozzi shows that, quantitatively, for a plausible calibration, these thresholds are always superior to 100 times the average income. And this underlies the self-correcting mechanism: since the mass of agents holding such wealth is way below 0.5%, multiple SFE are not likely to arise.

4 Conclusion

The KS algorithm, thanks to its elegance and versatility, is often the preferred method to solve heterogeneous-agent models with aggregate shocks. Cozzi pointed out to a potential vulnerability of the model, multiple SFE, and showed through perturbation experiments that, at least quantitatively speaking, this should not be a concern for a canonical version of the model with incomplete markets, aggregate shocks and preference heterogeneity. Relying on his approach, I showed evidence suggesting that this result holds in the benchmark KS model as well (without heterogeneity in the discount factor), as one might expect due to the lower wealth concentration associated with this model. With appropriate means and time, the code I provided can be easily adapted TO conduct a more rigorous Monte Carlo analysis.

References

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