Accelerator Design

JUAS'22 - Lecture and Mini-Workshop - Bastian Haerer (KIT)

juas

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Today:

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- Achromat lattices for synchrotron light sources
- Dispersion suppressor
- Insertions
  → RF sections
  → mini-beta insertion
  → matching sections
- How to build an accelerator model: Step-by-step

- Adrian: Details to groups, exercises and examination

Recap: Hadron and electron storage rings

Hadron storage rings

- Heavy particles require strong B fields
- Push for highest B fields up to technical limit
- Energy limit given by maximum acceptable circumference

$$2\pi\frac{p_0}{e} = \int B\,\mathrm{d}l$$

$$\epsilon_x \propto \frac{1}{\beta\gamma}$$

$$\epsilon_x \approx \epsilon_y$$

Electron storage rings

- Synchrotron light dominated
- Push for small B fields thus large bending radius
- Energy limit given by synchrotron radiation power

$$P_\gamma = \propto \frac{\gamma^4}{\rho^2}$$

$$\epsilon_x = \frac{C_\mathrm{q}}{J_x}\gamma^2\frac{\mathcal{I}_5}{\mathcal{I}_2} \propto \gamma^2$$

$$\epsilon_y \approx 0.1 - 1\,\%\,\epsilon_x$$

Recap: e+e- colliders vs. synchrotron light sources

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Collider

- High dipole filling factor → FODO structure
- High energy → large circumference

→ Naturally small emittance

$$\mathscr{L} = \frac{N_1 N_2 n_b f}{4\pi\sigma_x^* \sigma_y^*}$$

N particles per bunch
n_b number of bunches
f revolution frequency

Jean-Luc Revol: ESRF - The European Synchrotron, JUAS 2020

Synchrotron light source

- Small footprint desired
- Low emittance beams for high brilliance

$$B(\lambda) = \frac{F(\lambda)}{(2\pi)^2 \sigma_x \sigma_{x'} \sigma_y \sigma_{y'}} \propto \frac{1}{\epsilon_x \epsilon_y}$$ with photon flux *F(λ)* [1]

$$[F] = \frac{\text{photons}}{s\ 0.1\,\%\,\text{BW A}}$$

- Achromat structures

[1] K. Wille: The Physics of Particles accelerators - an introduction

Double bend achromat lattice

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Chasman-Green-Lattice

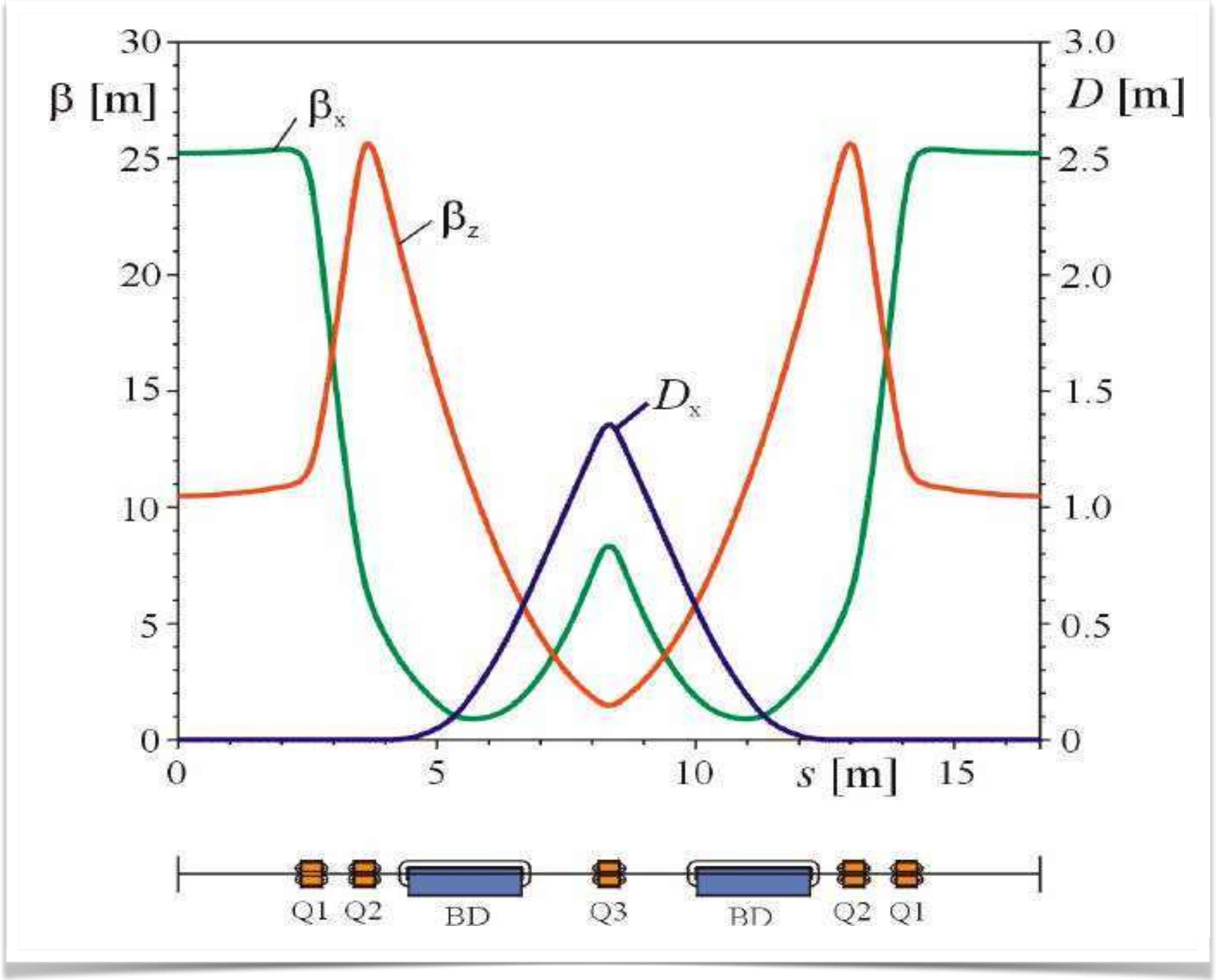
- Achromat means: Dispersion and its derivative vanish at start and end of the cell

- Dispersion is created by the first dipole. The quadrupole switches the sign of D' and the dispersion vanishes again in the second dipole.

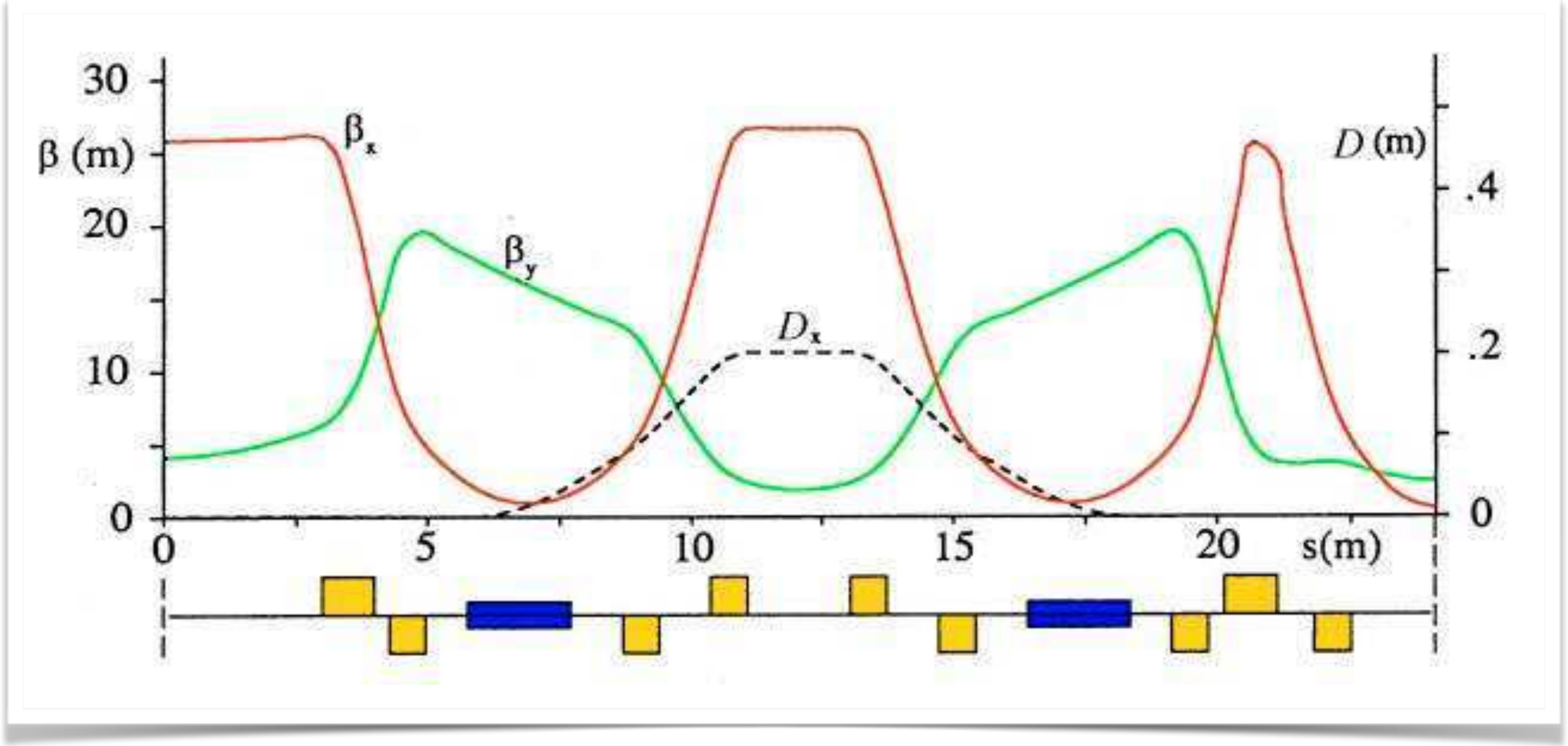
- Long drift spaces without dispersion allow

  -> installation of insertion devices
  -> small integrated dispersion thus low values
  of $\mathcal{I}_5$ and ϵ_x

- Characteristic lattice for 3rd generation synchrotron light souces



Examples of achromat lattices

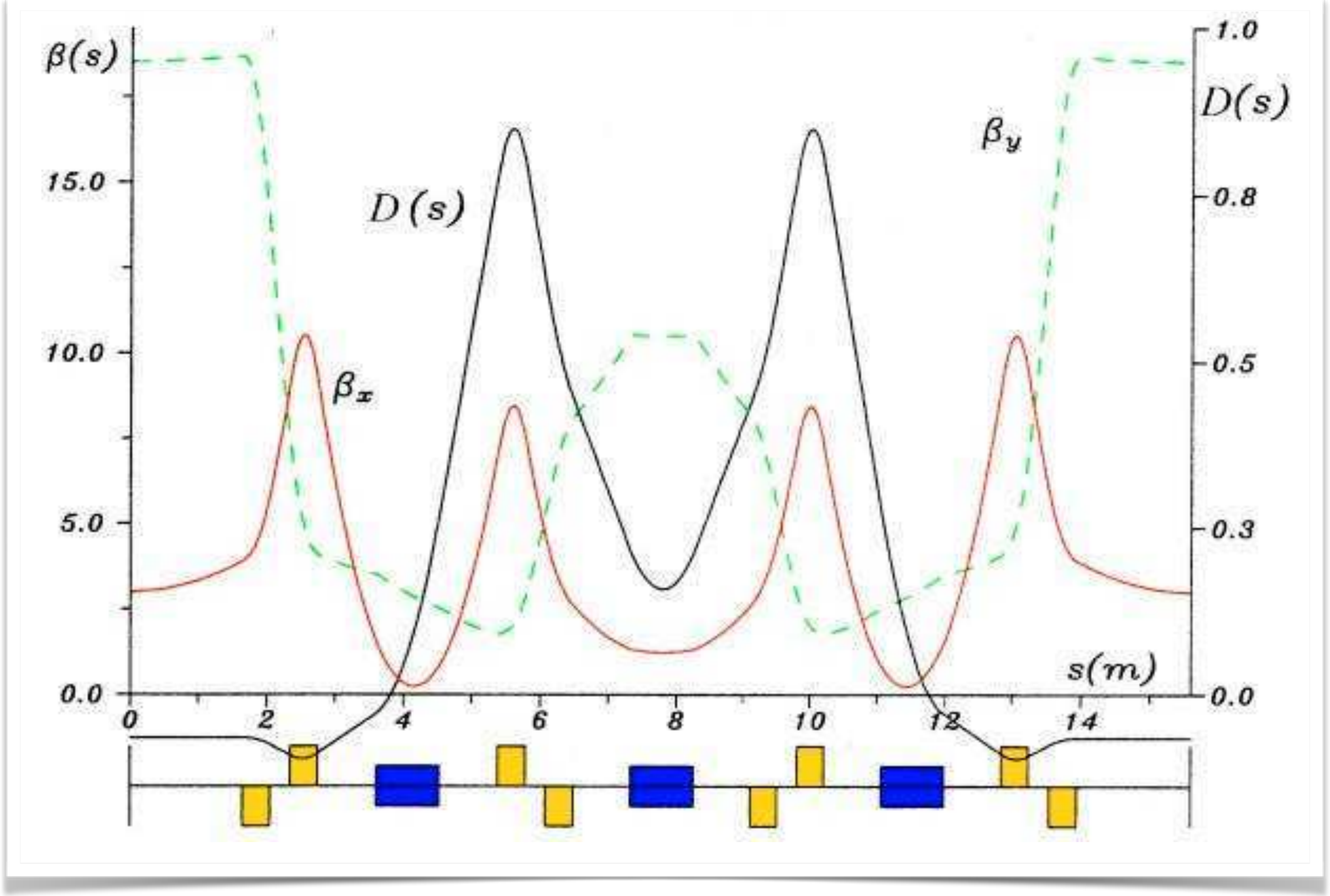


ESRF (before upgrade)
double bend achromat (DBA)

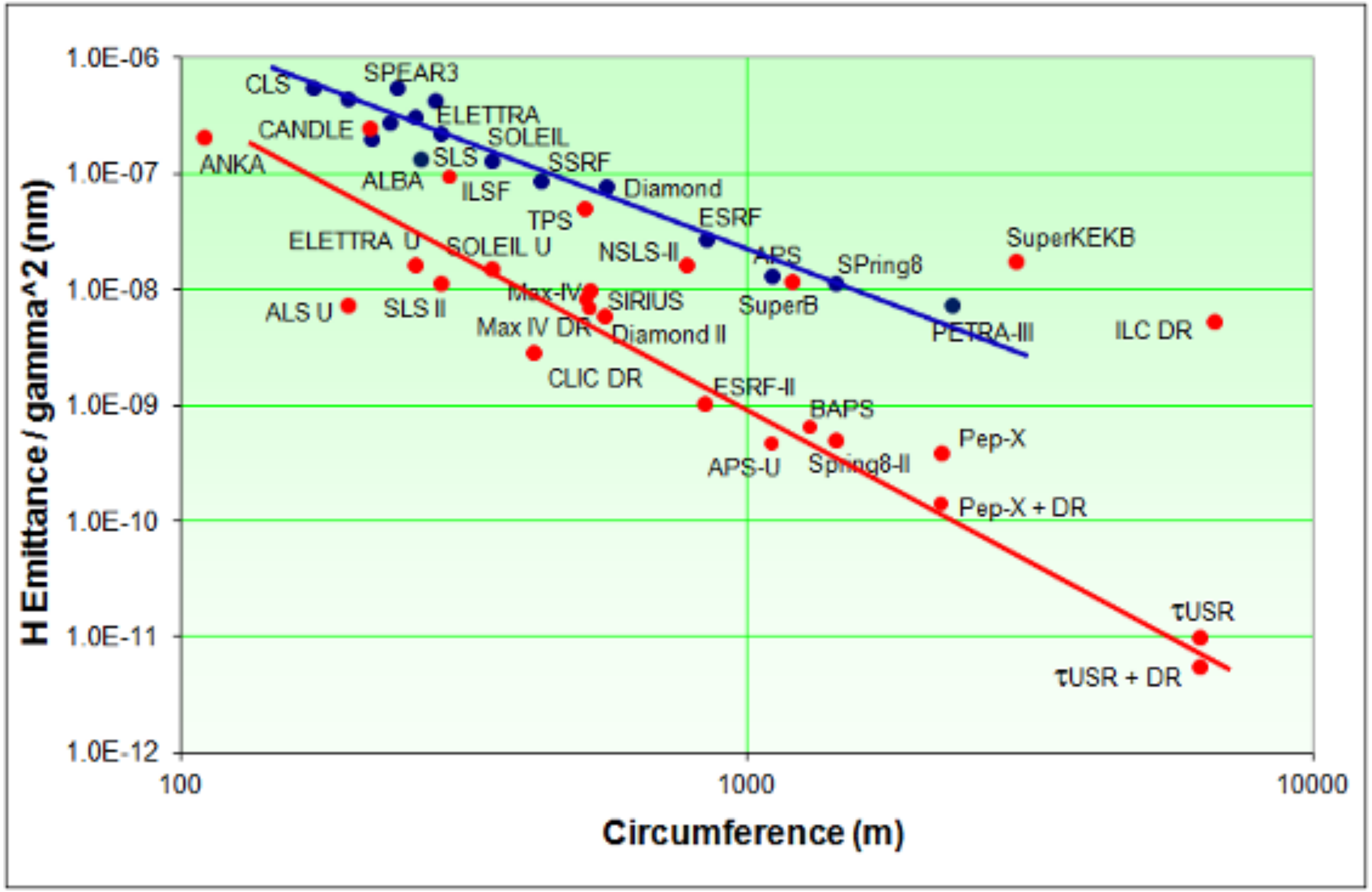
ε_x = 3.8 nm rad
C = 844 m

BESSY II
triple bend achromat (TBA)

ε_x = ~5 nm rad
C = 240 m



Emittance depending on circumference



R. Bartolini: Diamond Upgrade, Advances Optics Workshop, CERN, 2015

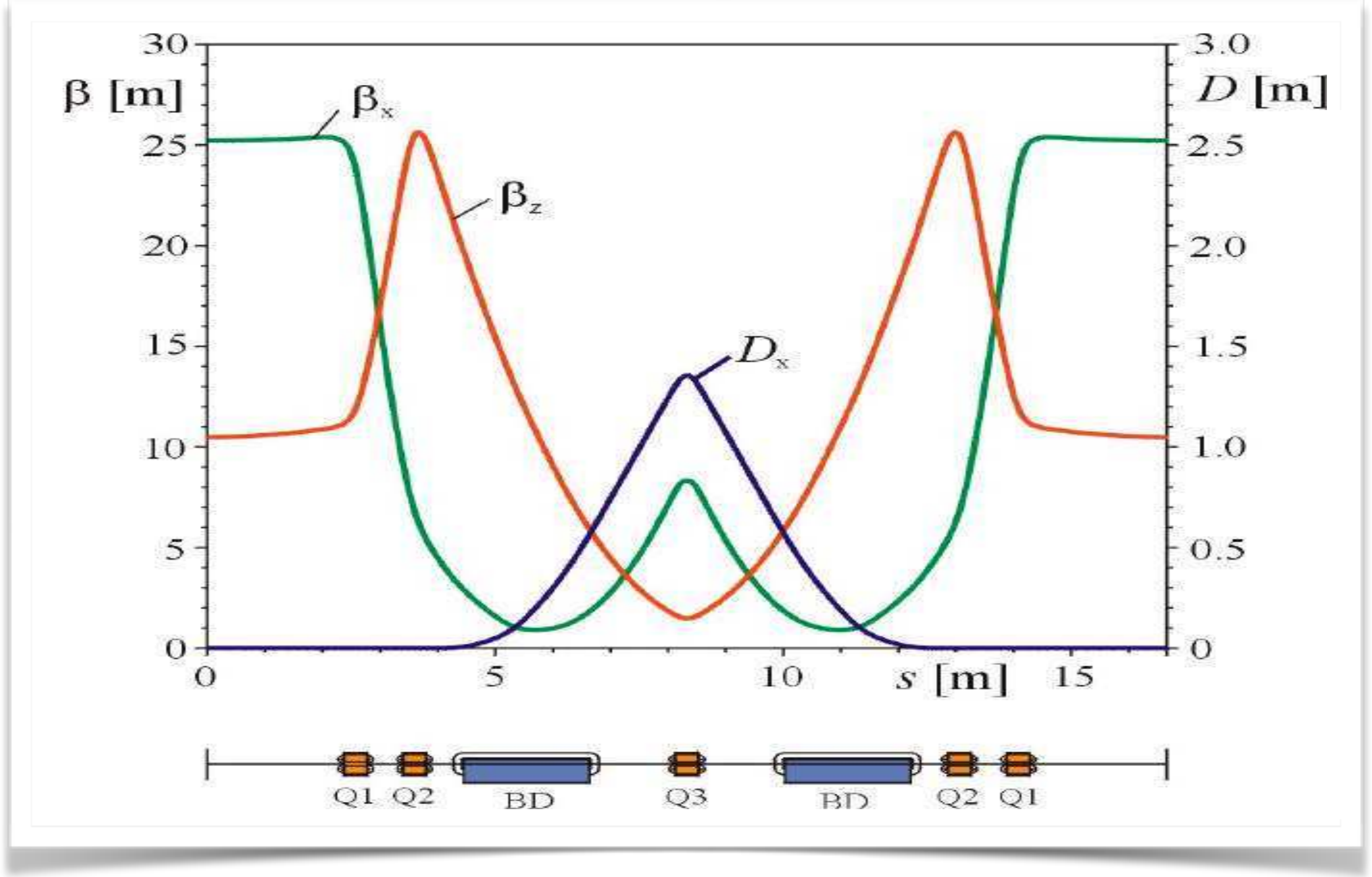
Emittance of achromat structures

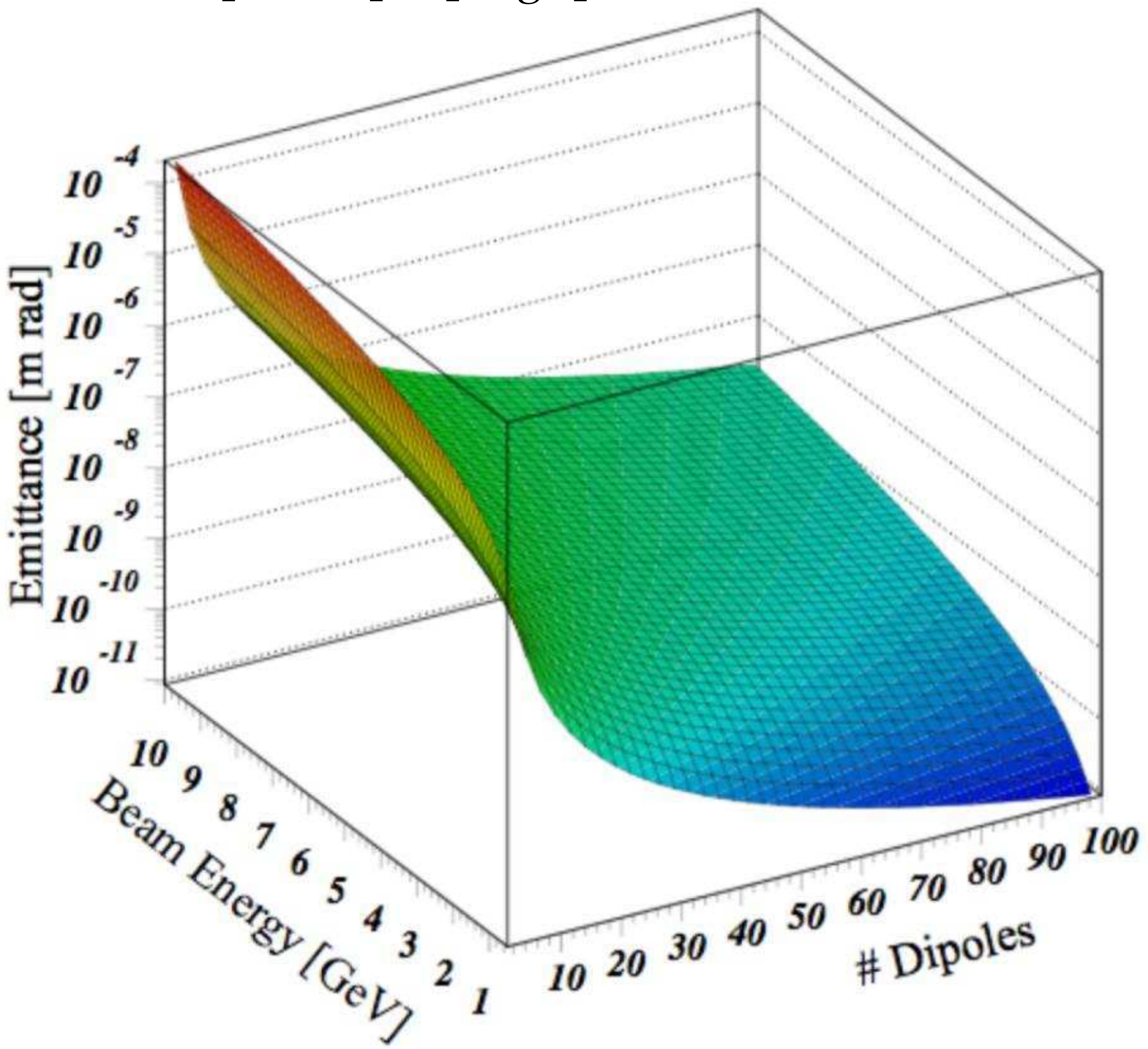
- Emittance of a DBA structure
  → beam energy
  → dipole bending angle

$$\epsilon_{\mathrm{DBA}} = \frac{C_\mathrm{q}}{4\sqrt{15}}\gamma^2\theta^3 \qquad \left(\epsilon_{\mathrm{FODO}} > C_\mathrm{q}\gamma^2\theta^3\right)$$

$$\epsilon_{\mathrm{DBA}}[\mathrm{mrad}] = 5.036\times10^{-13}E^2[GeV^2]\theta^3[deg^3]$$

- Emittance reduces for larger number of bending magnets
  → Multi bend achromat lattices





Multi bend achromat lattice

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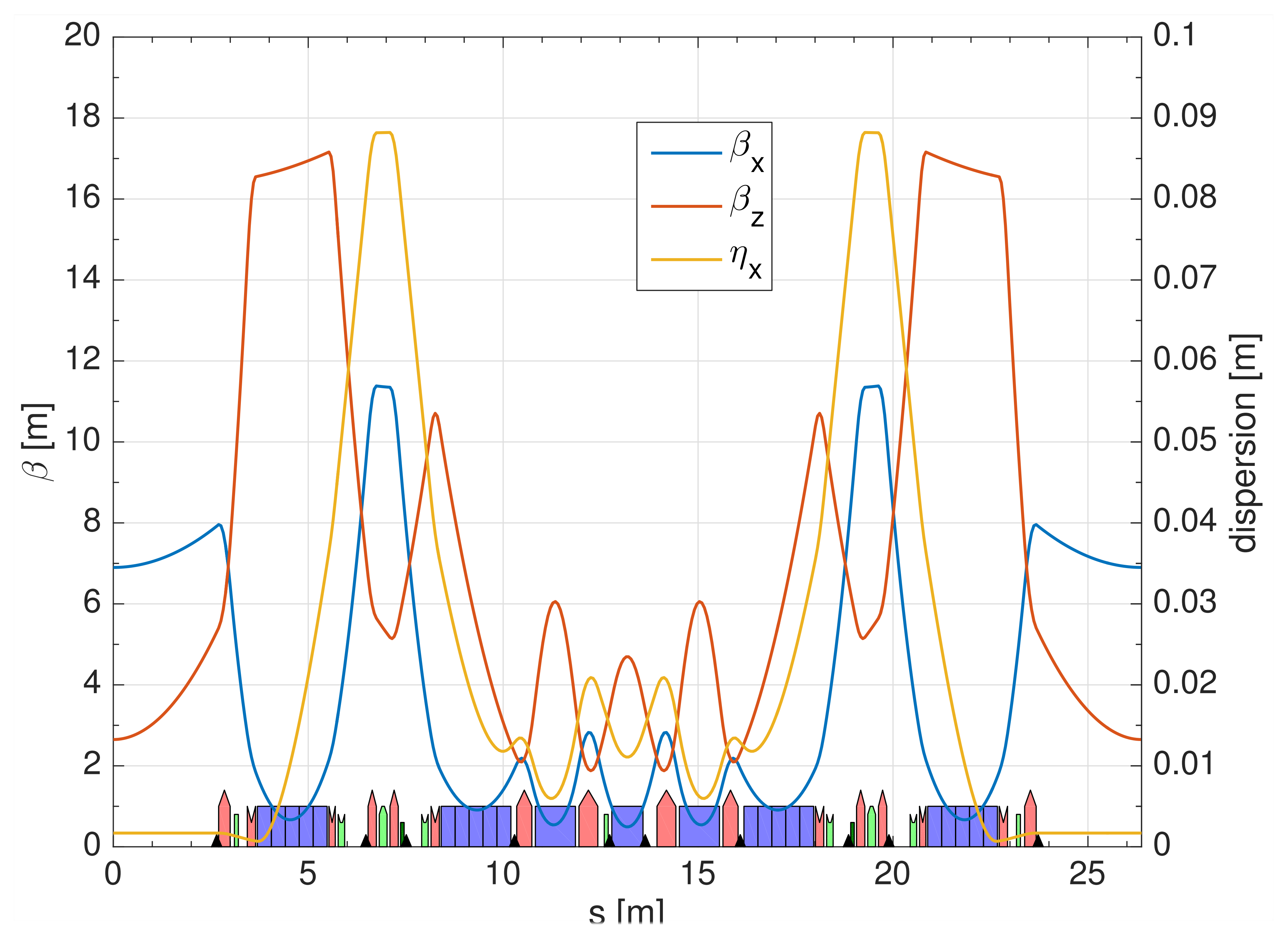
- High number of short magnets
- Special magnet technology

  -> Combined function magnets

  -> Permanent-/Hybridmagnets

  -> Modular magnets
- Full-energy injection, “top-up”, no ramping
- Highly specialised lattice with less flexibility
- **Goal: Operation 24/7 with smallest possible emittance**



7 Bend Achromat, ESRF-EBS

L. Farvacque 2015

Energy	E	6 GeV
Circumference	C	844 m
Emittance	ε_x	133 pm rad

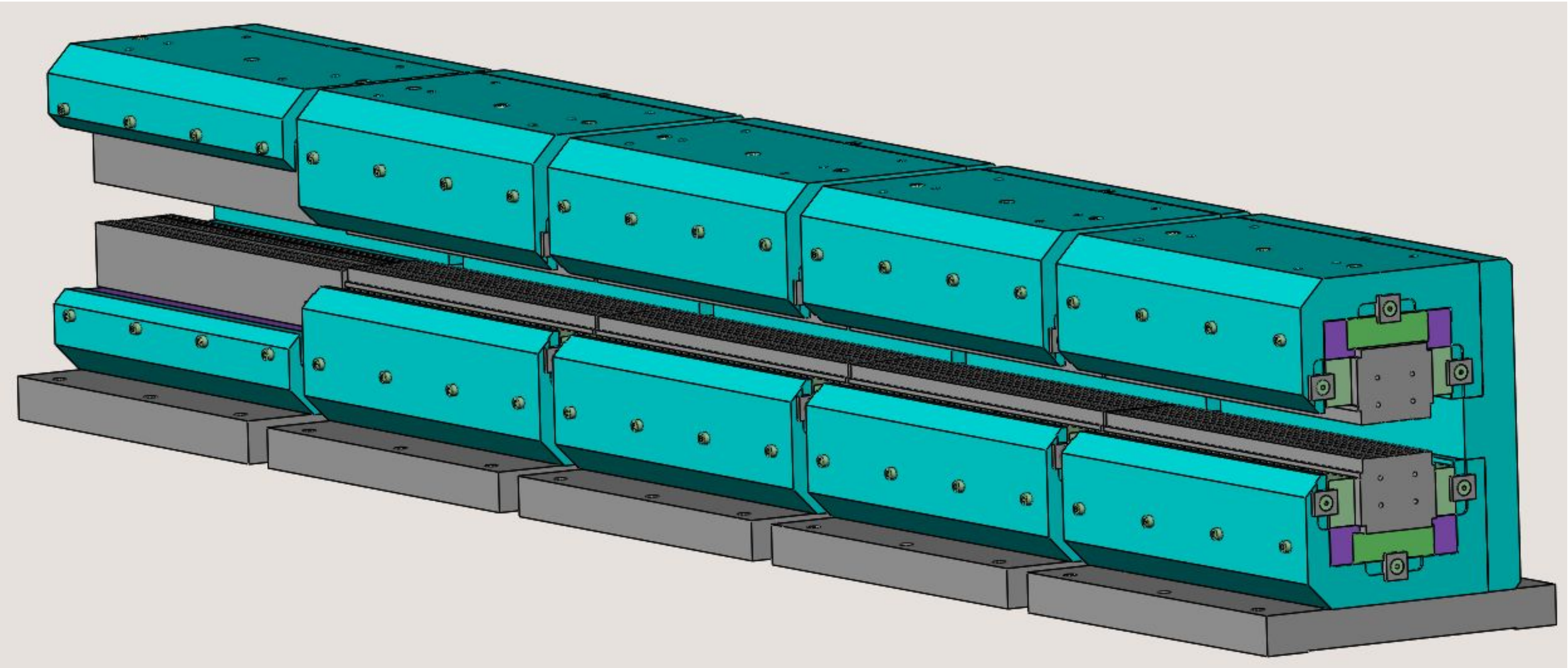
Magnet technology for MBA lattices

Modular magnets

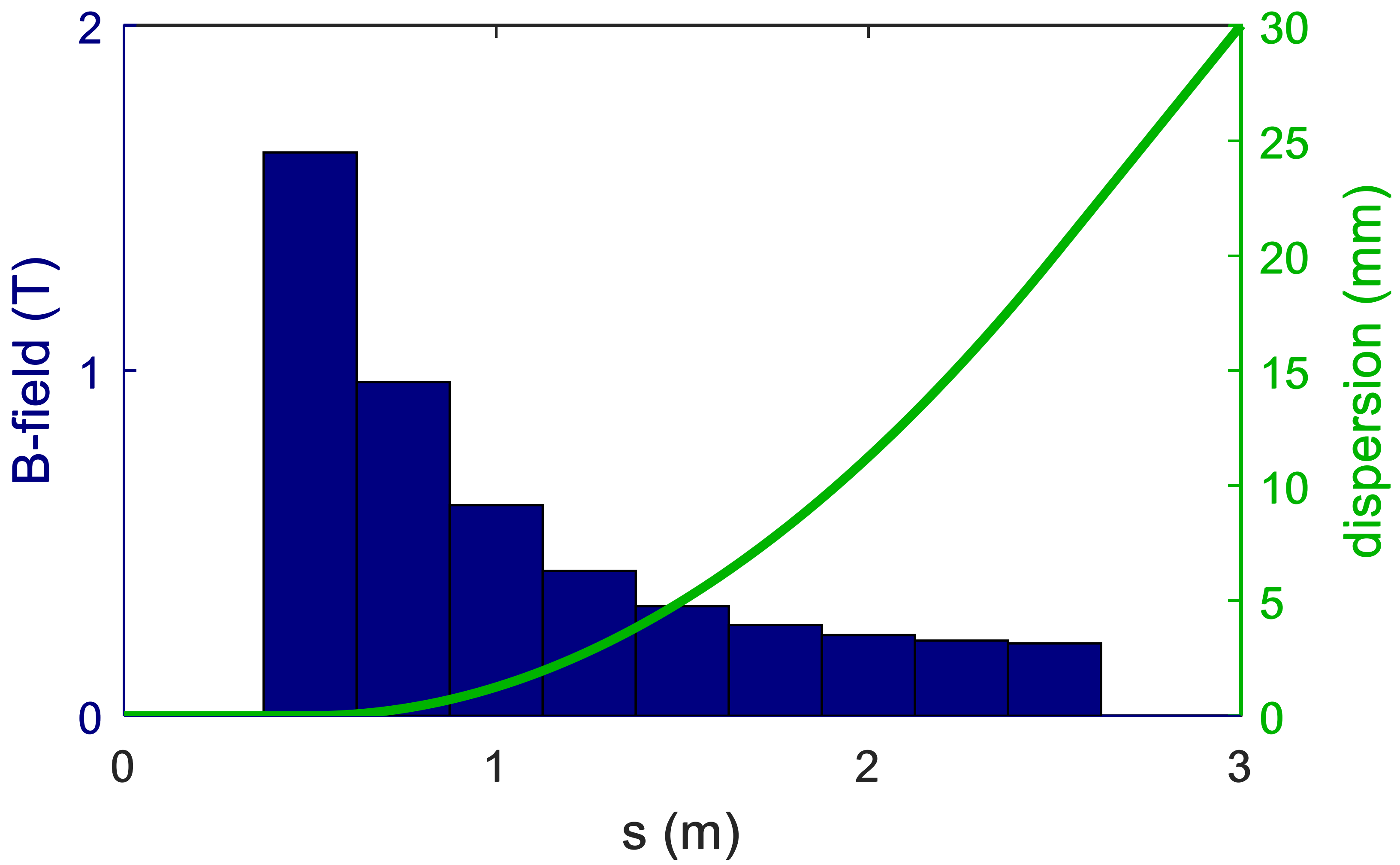
The modules feature different B field. The bending radius is reduced at locations of high dispersion and large at locations of small dispersion.

$$\epsilon_x = C_q \gamma^2 \frac{I_5}{J_x I_2} \qquad I_5 = \oint \frac{\mathcal{H}(s)}{\rho^3(s)} ds$$

dipole magnet DL (ESRF)
permanent magnets
5 modules with B field ranging from 0.17 T to 0.67 T

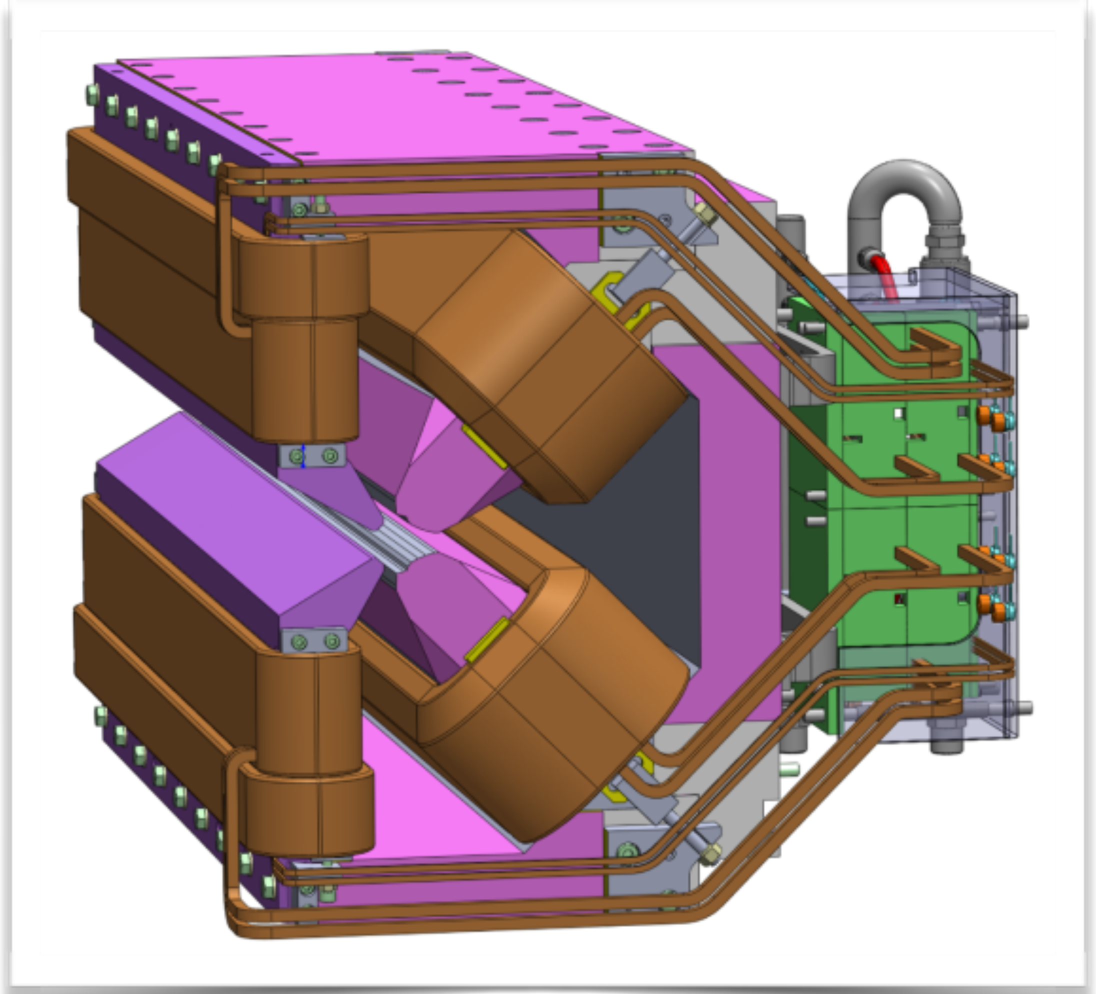


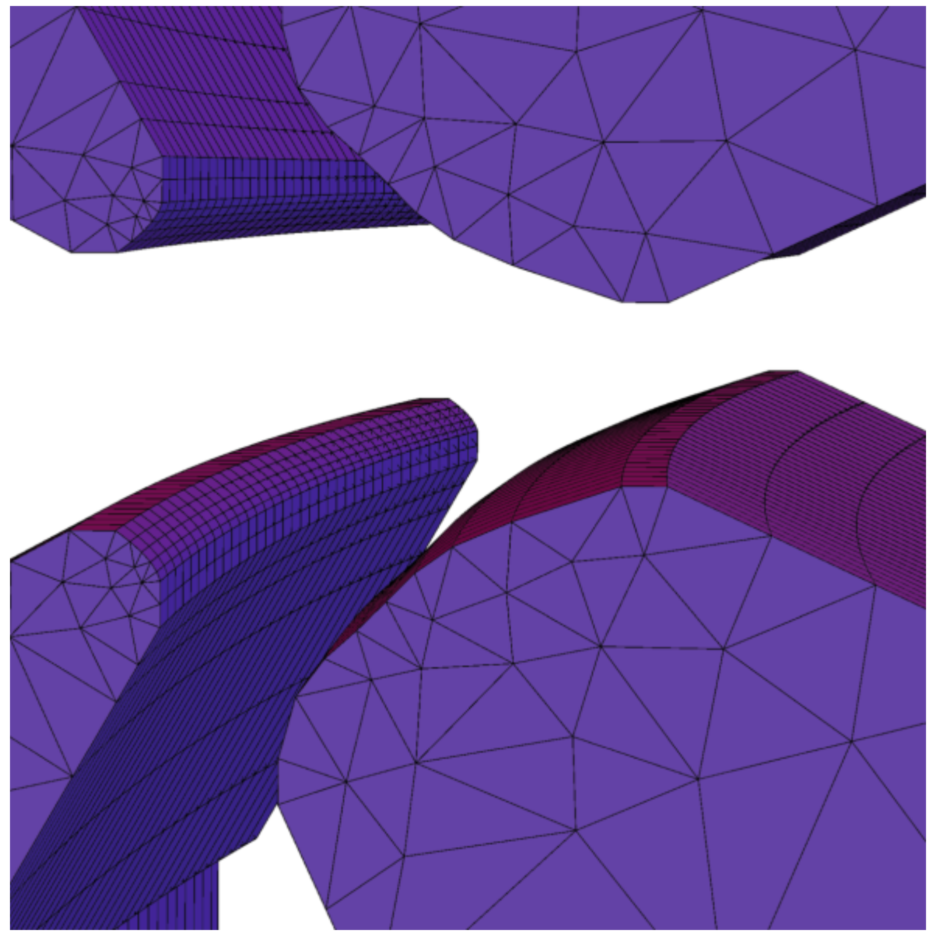
L. Farvacque, “ESRF Accelerator Upgrade Project Status”, 2015

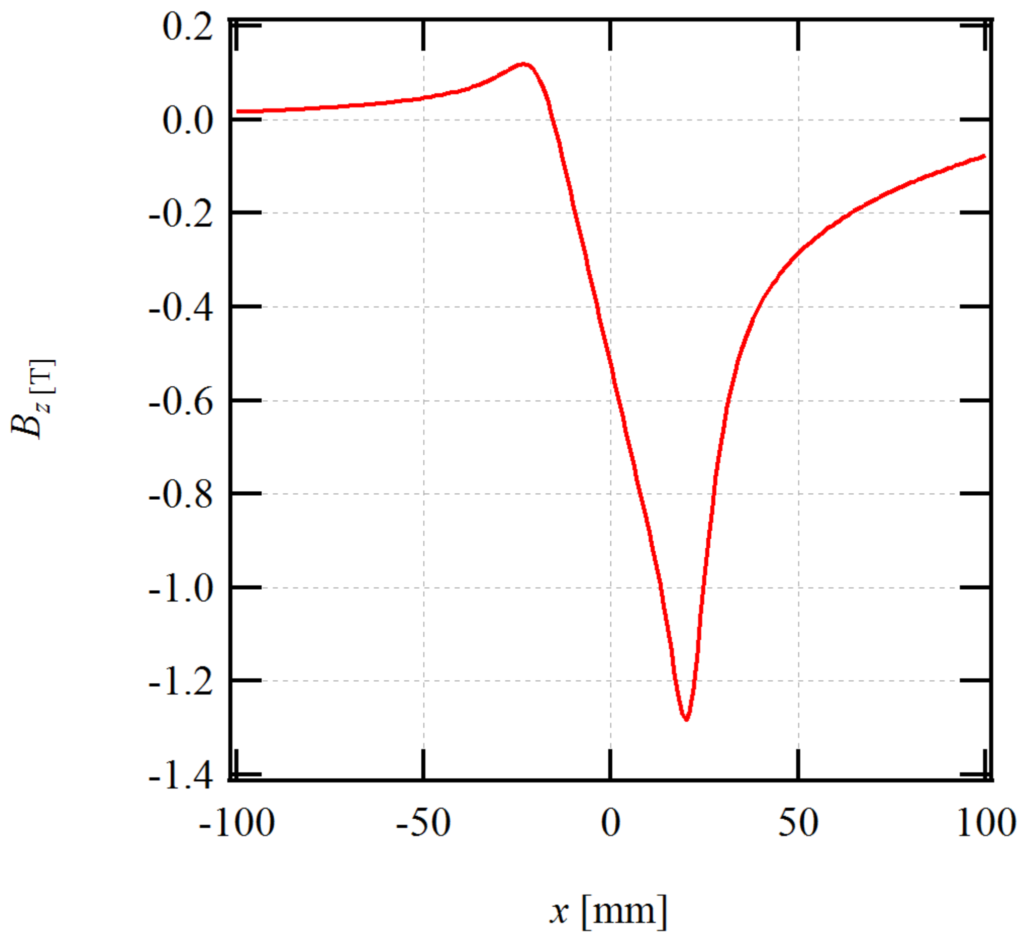


E. Karantzoulis, “From 3rd to 4th generation light sources”, 2019

Magnet technology for MBA lattices II







ESRF-EBS
dipol-quadrupol
0.5 T, 0.37 T/m

L. Farvacque, "ESRF Accelerator Upgrade Project Status", 2015

MAX-IV
single-yoke magnet design

P.F. Tavares, "Lessons learned from the MAX-IV 3 GeV Ring Commissioning", 2019

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Beam Dynamics in Electron Rings

is determined by the emission of photons and the strength of the (dipole-) fields

Key parameters

- particle energy
- dipole field strength $1/\rho$
- dispersion

… and a bit on the lattice (quadrupole structure)

and that's all.

Summarised in the so-called synchrotron radiation integrals

Summary for electron rings

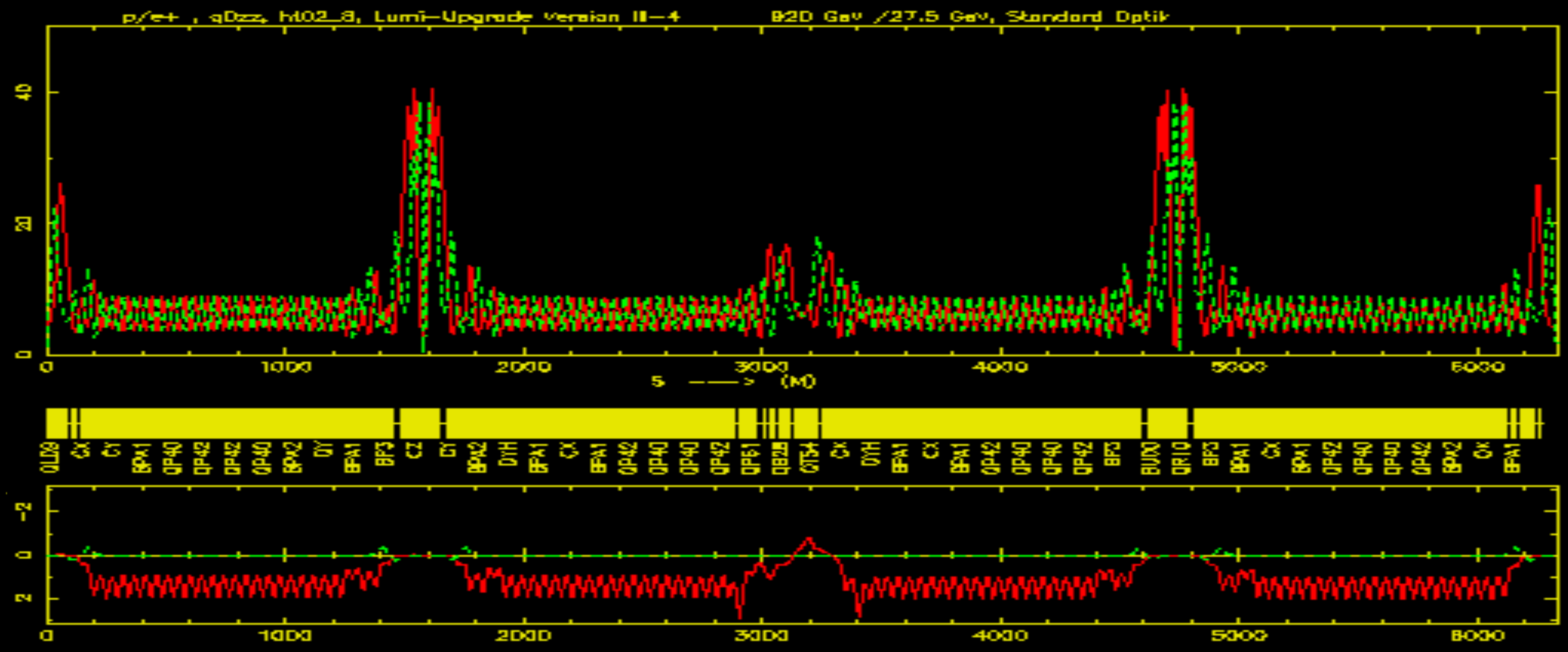
- Equilibrium of quantum excitation and radiation damping leads to equilibrium beam parameters
- Equilibrium beam emittance increases with beam energy $$\epsilon_x = \frac{C_q}{J_x}\gamma^2\frac{\mathcal{I}_5}{\mathcal{I}_2}$$
- Electron beams are flat in the sense $\epsilon_y \approx 0.1 - 1\,\%\,\epsilon_x$
- Lattice design allows to design equilibrium beam parameters

High energy storage rings

- Reduce ρ by large circumference and dipole filling factor
- FODO structure

Synchrotron light sources

- Smaller footprint and room for insertion devices require
- Achromat structures for ultra-low emittance



Arc: regular (periodic) magnet structure:

bending magnets B define the energy of the ring
main focusing & tune control, chromaticity correction,
multipoles for higher-order corrections

Straight sections: drift spaces for injection, dispersion suppressors,
low beta insertions, RF cavities, etc....
... and the high energy experiments if they cannot be avoided

Acceleration

- Lorentz force: $\vec{F}_\mathrm{L} = e(\vec{E} + \vec{v} \times \vec{B})$

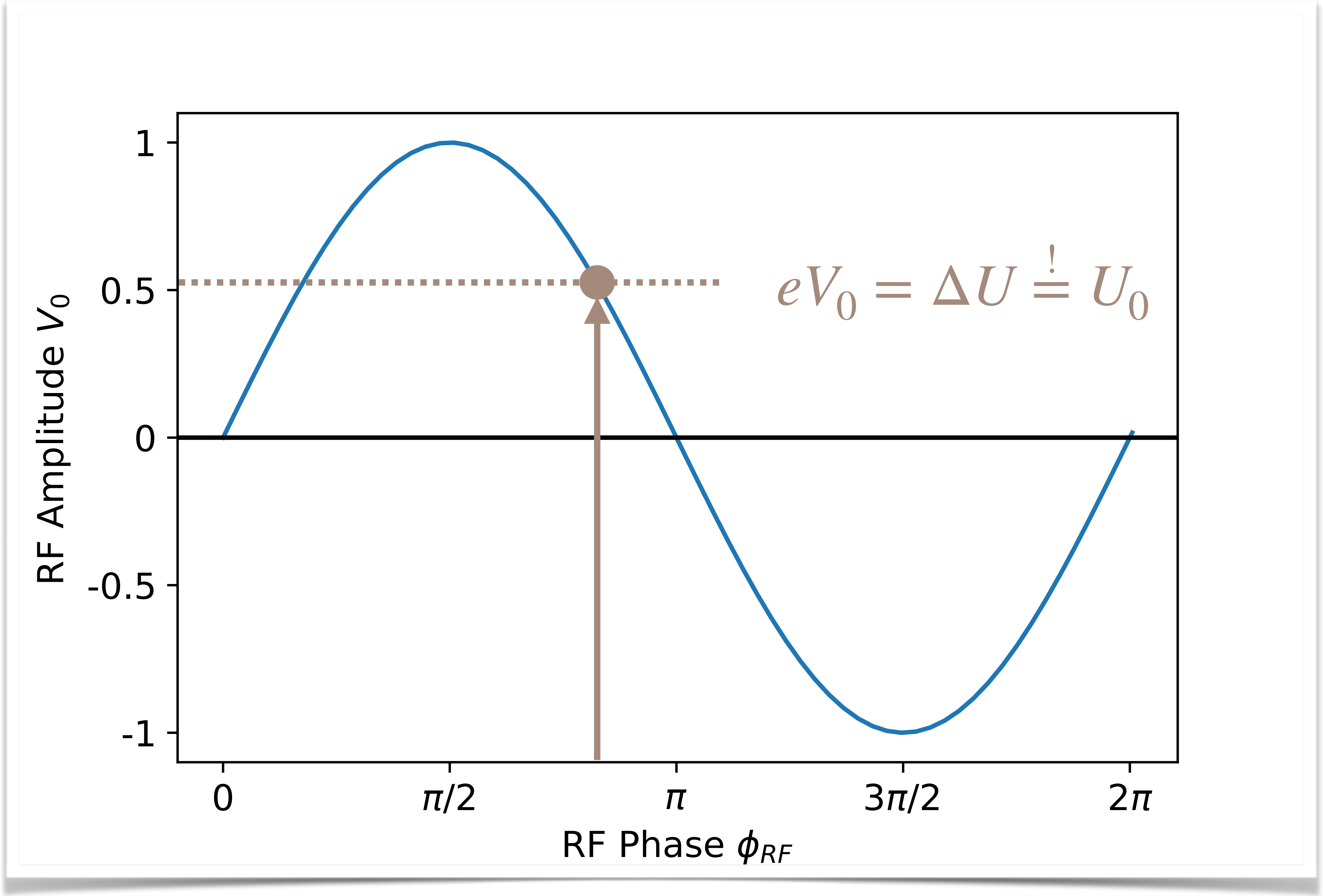
Deflection/focussing

Energy gain

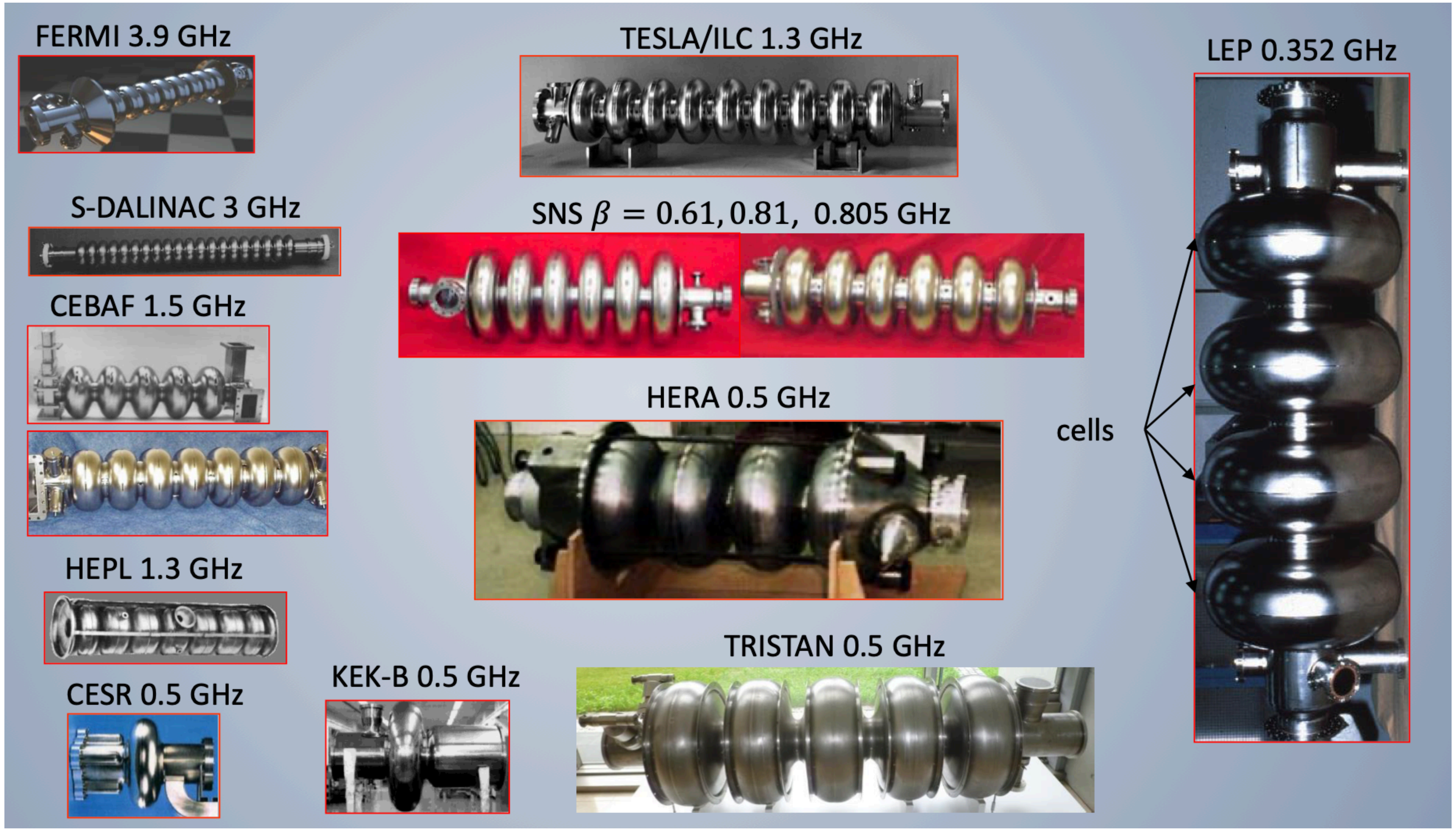
- Oscillating *E* fields in RF cavities
- Effect of cavity is given by

$$\boxed{\Delta U = eV_\mathrm{RF} \sin(\phi_\mathrm{RF} - hf_0 t)}$$

synchronous/RF phase ϕ_RF



RF cavities



high power dc type operation:

sc. cavities preferred, operational frequency range: ≈ 1 GHz

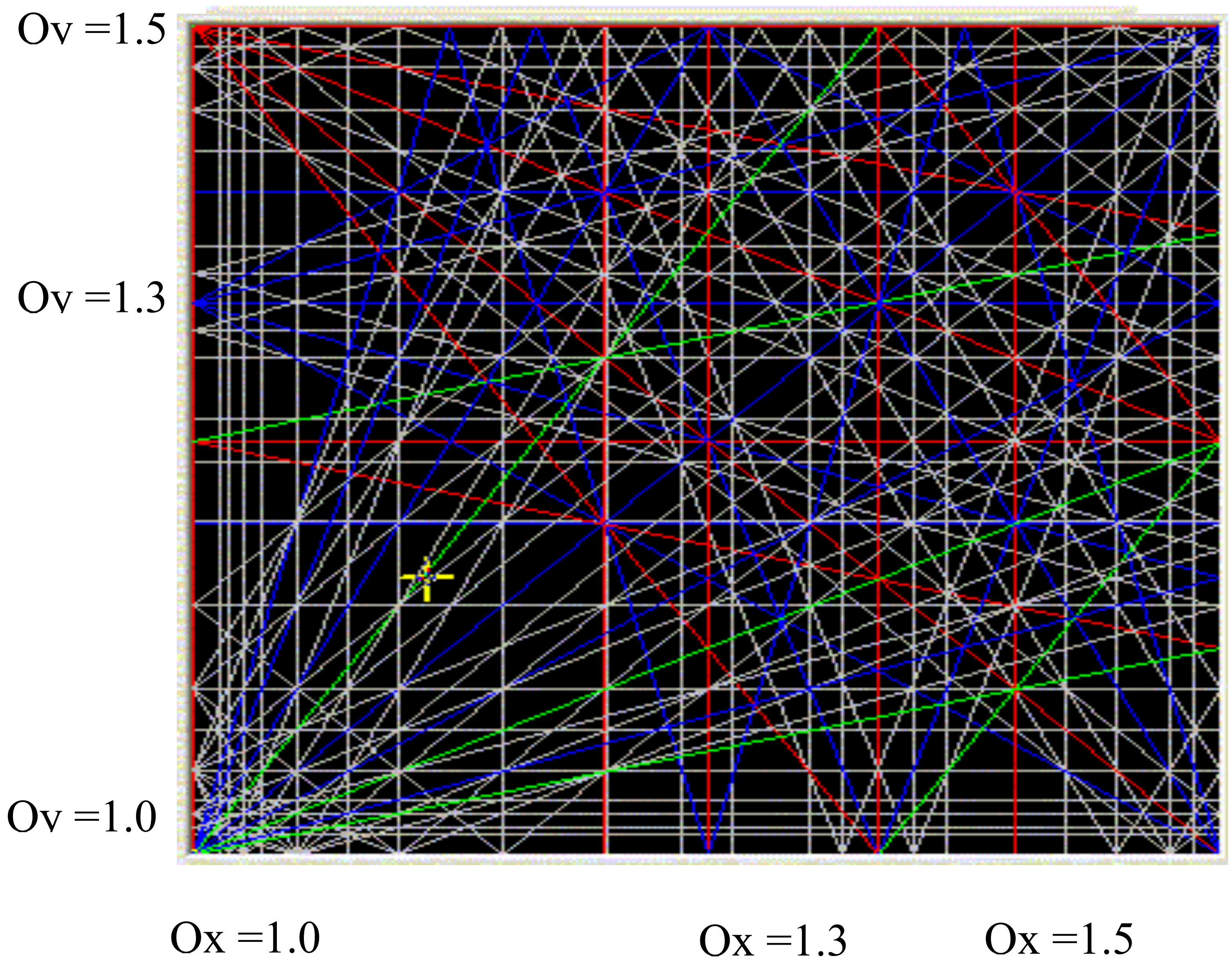
One word about RF Cavities

RF Cavities must be installed in …

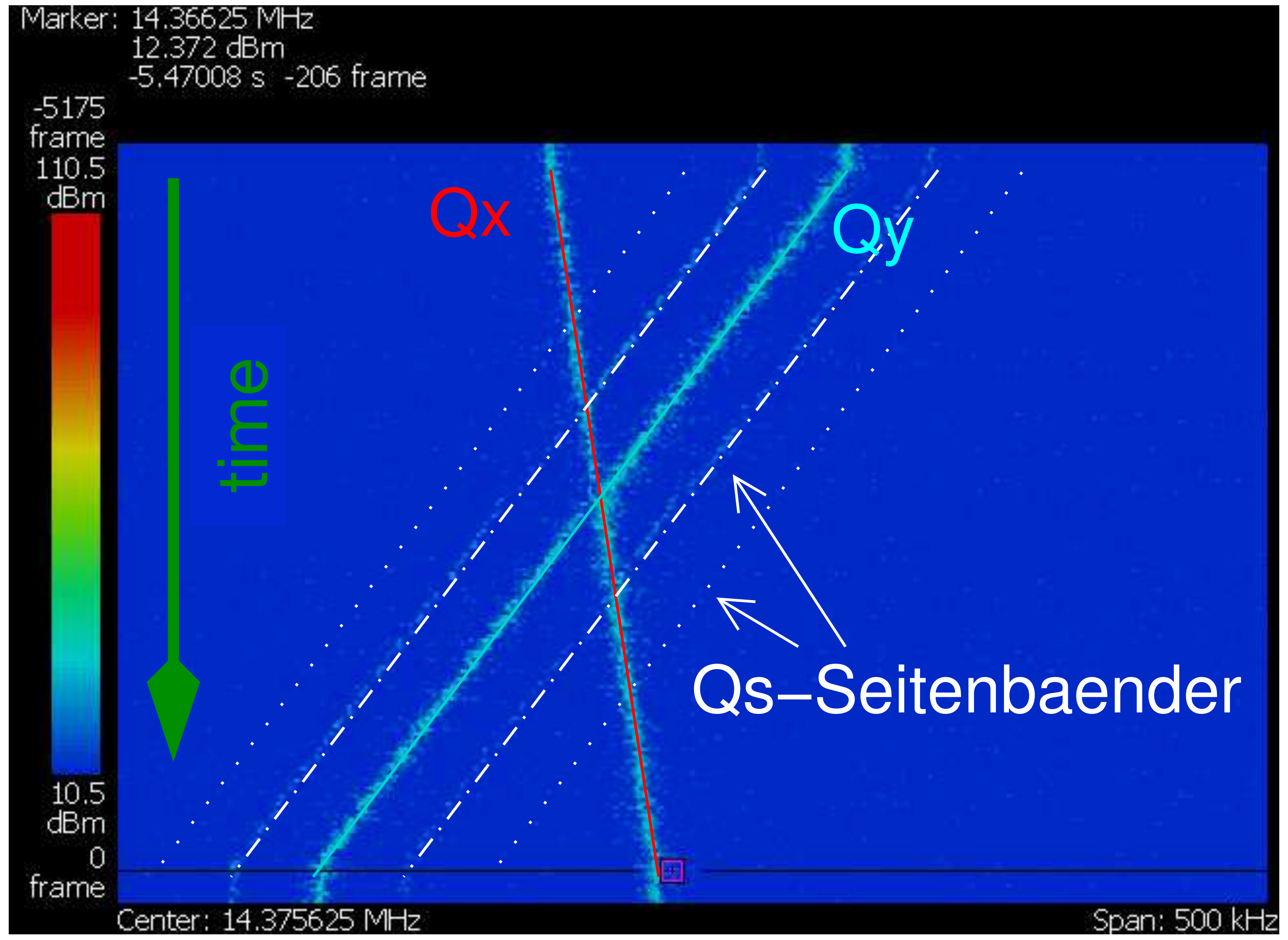
- - - - > Dispersion free sections < - - - -

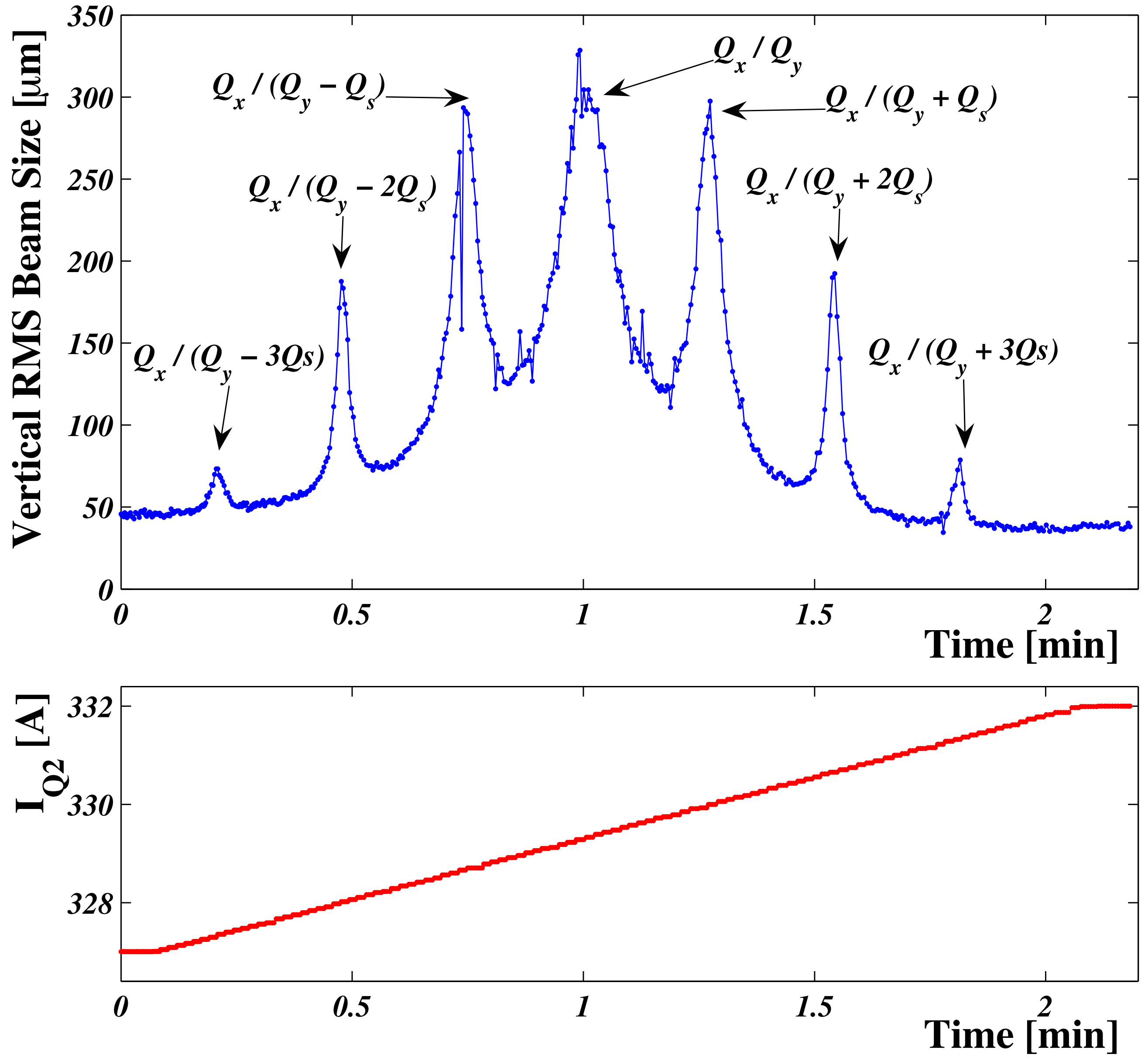
of the storage ring.

… in order to suppress coupling between the longitudinal motion (energy gain) and the transverse motion via dispersion.



$$m\, Q_x + n\, Q_y + l\ \ Q_s = \text{integer}$$





Dispersion function

Momentum spread of the beam adds a term on the r.h.s. of the equation of motion.
—> *inhomogeneous differential equation.*

$$x'' + x\left(\frac{1}{\rho^2} - k\right) = \frac{\Delta p}{p_0}\frac{1}{\rho}$$

—> *additional term in the solution for the particle's amplitude*

$$x(s) = x_\beta(s) + x_D(s)$$

$$x_D(s) = D(s)\frac{\Delta p}{p}$$

Normalise with respect to Δp/p:

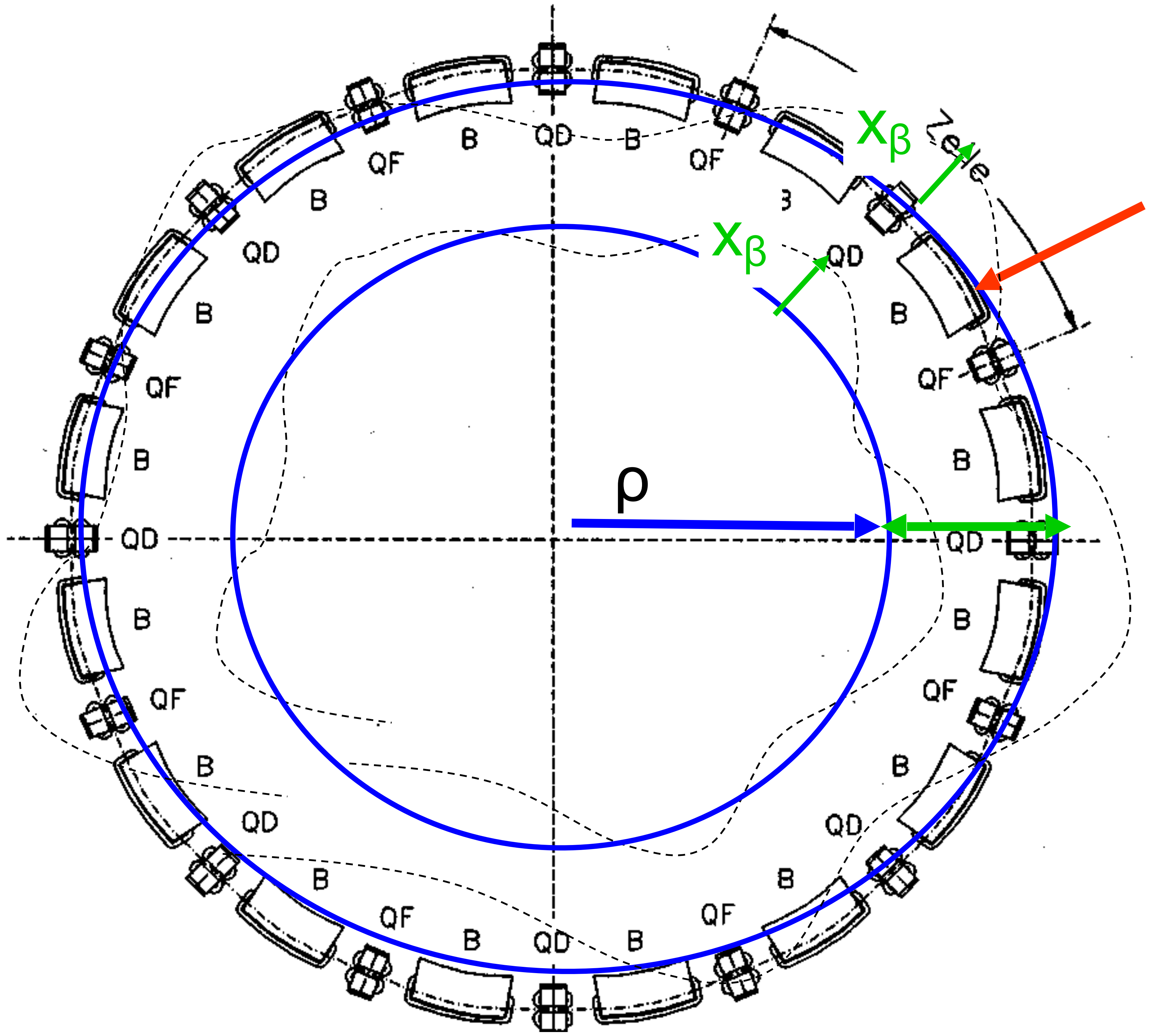
$$D(s) = \frac{x_i(s)}{\Delta p / p}$$

Dispersion function D(s) …

… is that special orbit, an ideal particle would have for Δp/p = 1

The orbit of any particle is the sum of the well known x_β and the dispersion

Dispersion orbit with homogeneous dipole field



$$x_i(s) = D(s) \cdot \frac{\Delta p}{p}$$

Example HERA

$$x_\beta = 1 \ldots 2\ mm$$

$$D(s) \approx 1 \ldots 2\, m$$

$$\Delta p / p \approx 1 \cdot 10^{-3}$$

contribution due to Dispersion ≈ beam size

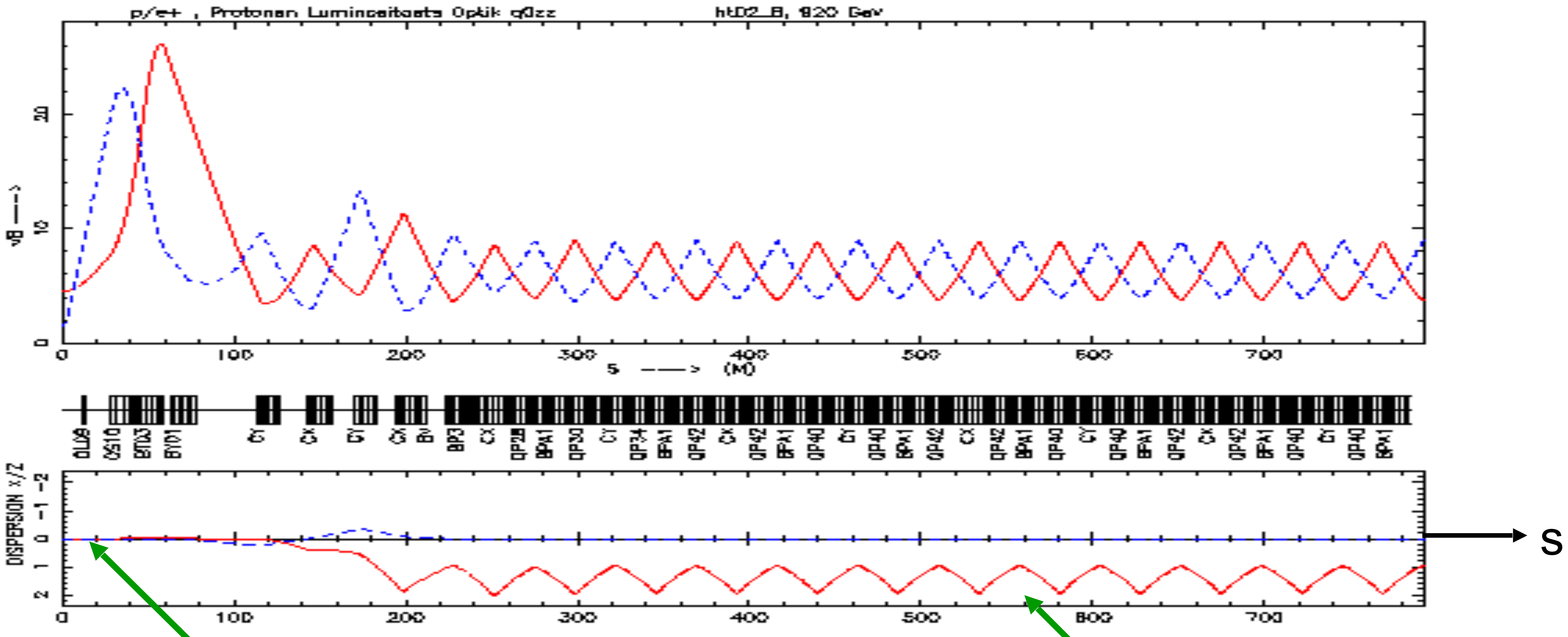
—> ***Dispersion must vanish at the collision point***

$$\sigma = \sqrt{\epsilon\beta + D^2\delta^2}$$

Dispersion suppressor - idea

D(s) is created by the dipole magnets

… and afterwards focused by the quadrupole fields



Mini Beta Section,
—> no dipoles !!!

Arc: D(s) ≈ 1 … 2 m

Think right —> left :

by clever arrangement of dipole fields & quadrupole strengths we can make D(s) vanish.

Quadrupole-based dispersion suppressor

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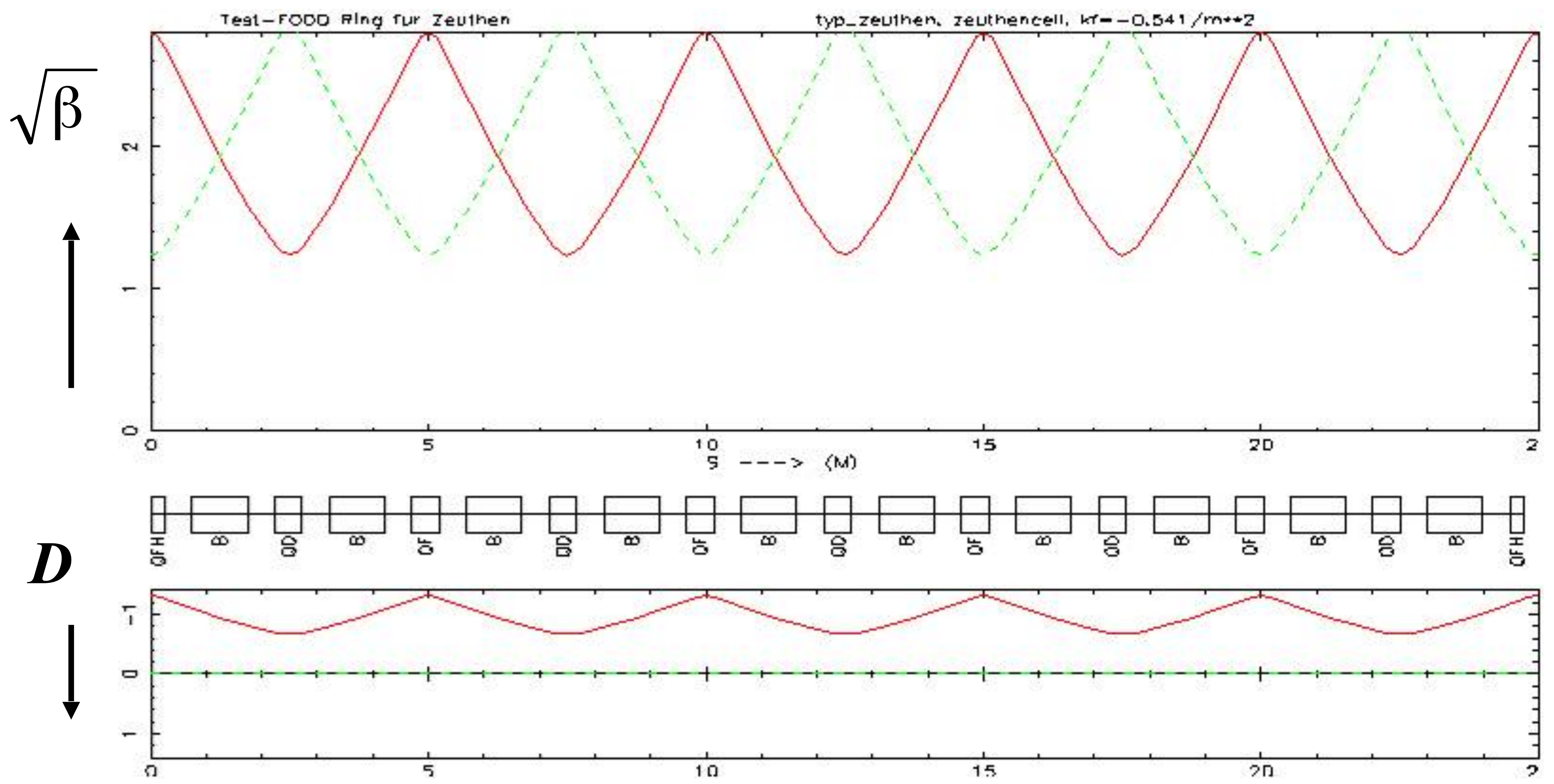
The straight forward one: ***use additional quadrupole lenses to match the optical parameters ... including the D(s), D´(s) terms***

Quadrupoles have an influence on
the optics (β function)
the phase advance
but also … the orbit.

And dispersion is "just another orbit".



FoDo cell: regular structure in the arc

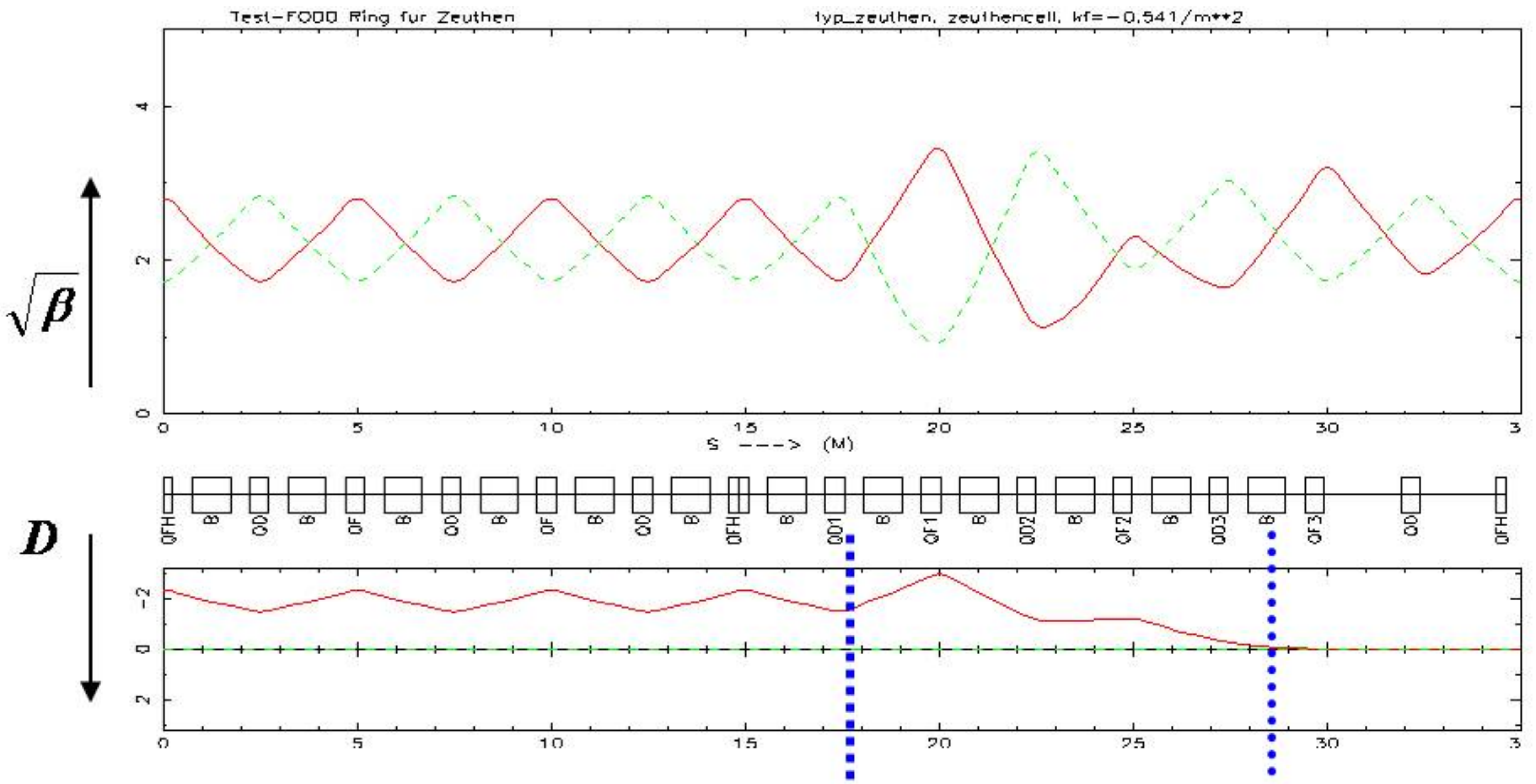
Correct the dispersion D and D' to zero by 2 quadrupole lenses,

Restore (match back) β and α to the values of the periodic solution by 4 additional quadrupoles

$$\left.\begin{array}{l} D(s), \quad D'(s) \\ \beta_x(s), \alpha_x(s) \\ \beta_y(s), \alpha_y(s) \end{array}\right\} \rightarrow$$

6 additional independent quadrupole lenses required

Quadrupole-based dispersion suppressor



periodic FoDo structure

matching section including 6 additional quadrupoles

dispersion free section, regular FoDo without dipoles

Advantage:

! easy

! flexible: it works for any phase advance per cell

! does not change the geometry of the storage ring

! can be used to match between different lattice structures (i.e. phase advances)

Disadvantage:

! additional power supplies needed (→ expensive)

! requires stronger quadrupoles

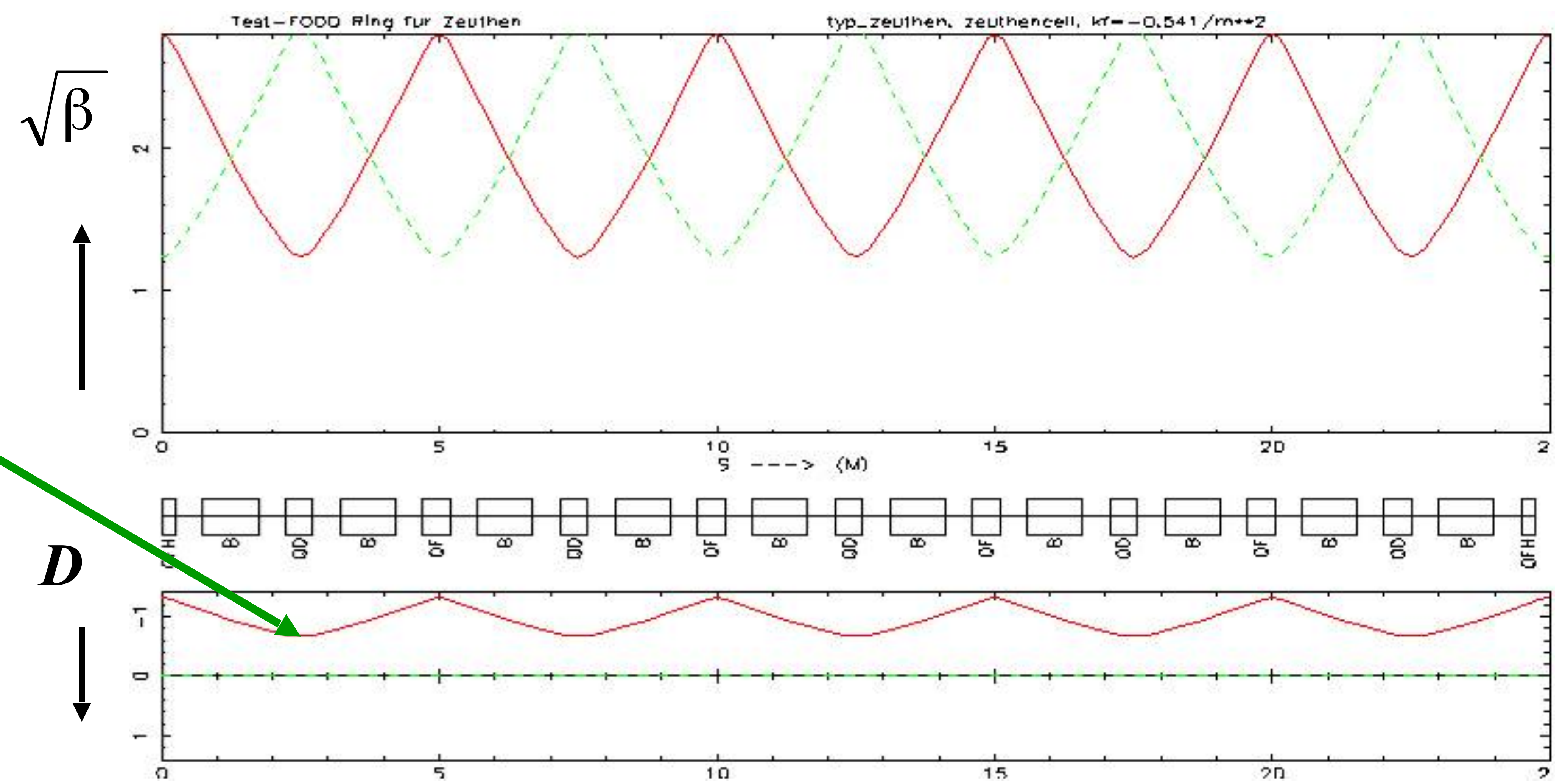
! due to higher β values: more aperture required

Dipole-based schemes

Dipole based schemes: the clever way

periodic dispersion in the arc
(FoDo in thin lens approx)

$$\hat{D} = \frac{L^2}{\rho} \frac{\left(1 + \frac{1}{2}\sin(\mu/2)\right)}{4\sin^2(\mu/2)}, \quad D' = 0$$



Think right —> left :

arrange a number of dipoles to build up — from zero — dispersion that fits to the periodic solution

$$D(s) = S(s) * \int \frac{1}{\rho(\tilde{s})} C(\tilde{s}) d\tilde{s} - C(s) * \int \frac{1}{\rho(\tilde{s})} S(\tilde{s}) d\tilde{s}$$

Arrange the dipole fields (1/ρ) in a way to create at the beginning of the regular arc cells $\hat{D}$ ***and D'=0.***

… how this is done in detail —> see appendix

Half-bend dispersion suppressor

condition for vanishing dispersion:

so if we require $D = 0$, $$2*\delta_{supr}*\sin^2\left(\frac{n\Phi_c}{2}\right) = \delta_{arc}$$

... proof ... is easy but lengthy
—> appendix

with $\boldsymbol{\delta}_{supp}$ *= dipole strength in the suppressor region*

with $\boldsymbol{\delta}_{arc}$ *= dipole strength in the arc structure*

with Φ_c *= phase advance per cell*

and we can set $\delta_{supr} = \frac{1}{2}*\delta_{arc}$ ***—> we get*** $\sin^2\left(\frac{n\Phi_c}{2}\right) = 1$

and equivalent for D'=0 ***—> we get*** $\sin(n\Phi_c) = 0$

For a given phase advance per cell we just have to add up the number of cells to get theses conditions fulfilled.

Which means … $n\Phi_c = k*\pi$, $k = 1,3,\ldots$

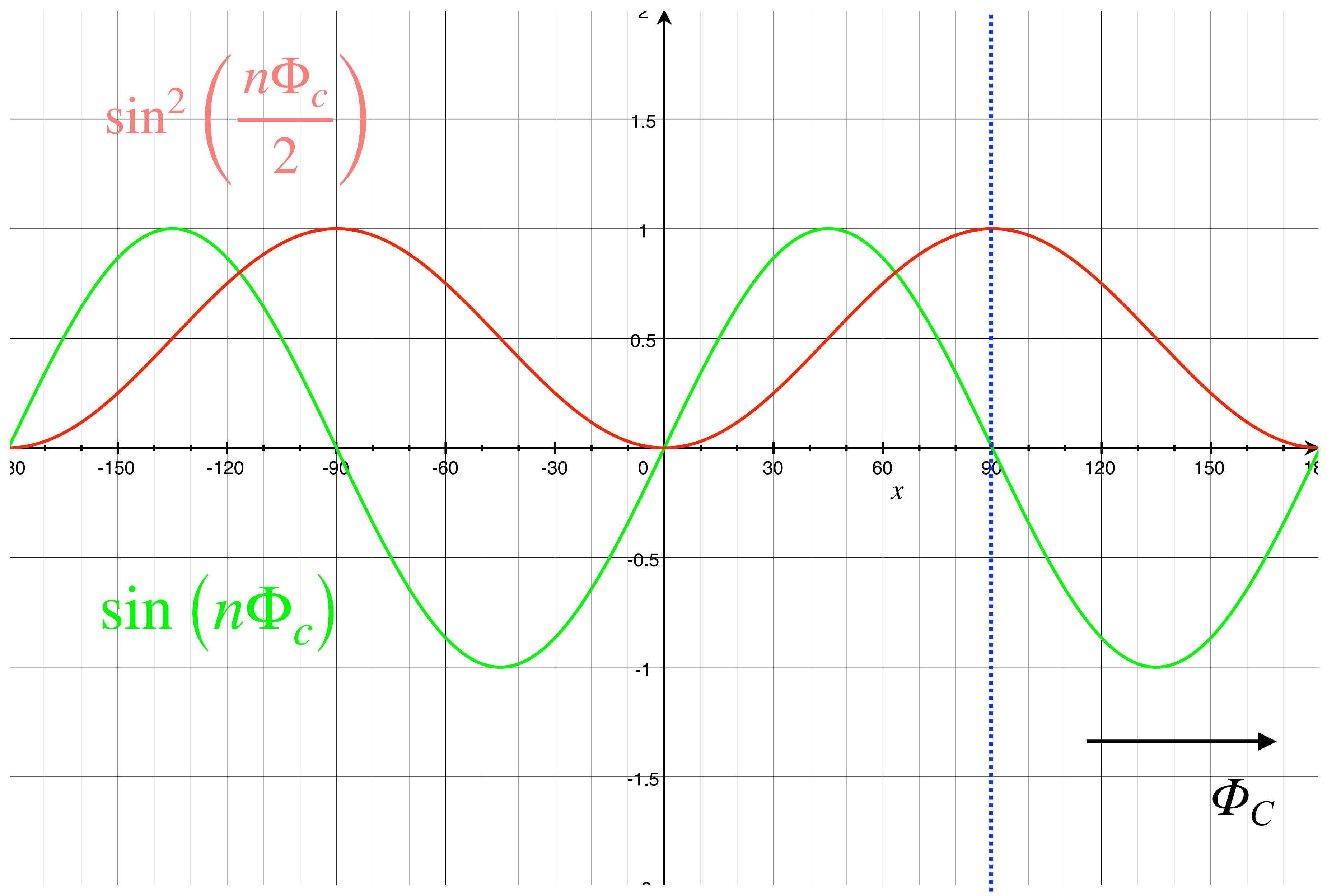
In the n suppressor cells the phase advance has to accumulate to a odd multiple of π

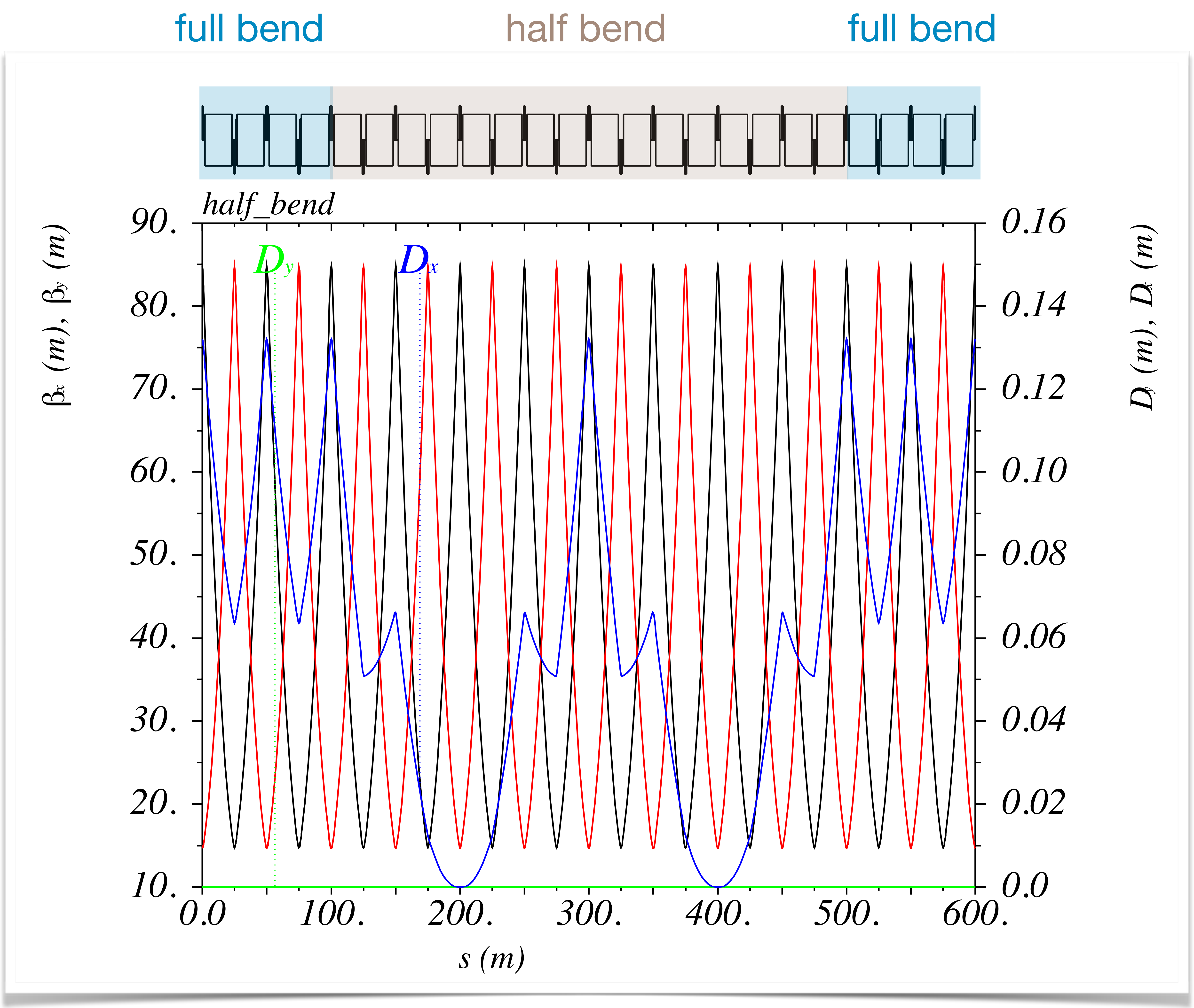
Half-bend dispersion suppressor - II

$$\sin^2\left(\frac{n\Phi_c}{2}\right) = 1 \qquad \sin\left(n\Phi_c\right) = 0$$

$$\rightarrow \quad n\Phi_c = k\pi, \quad k = 1,3,\ldots$$

$\Phi_C = 90°$: $n = 2$, $k = 1$





Half-bend dispersion suppressor - III

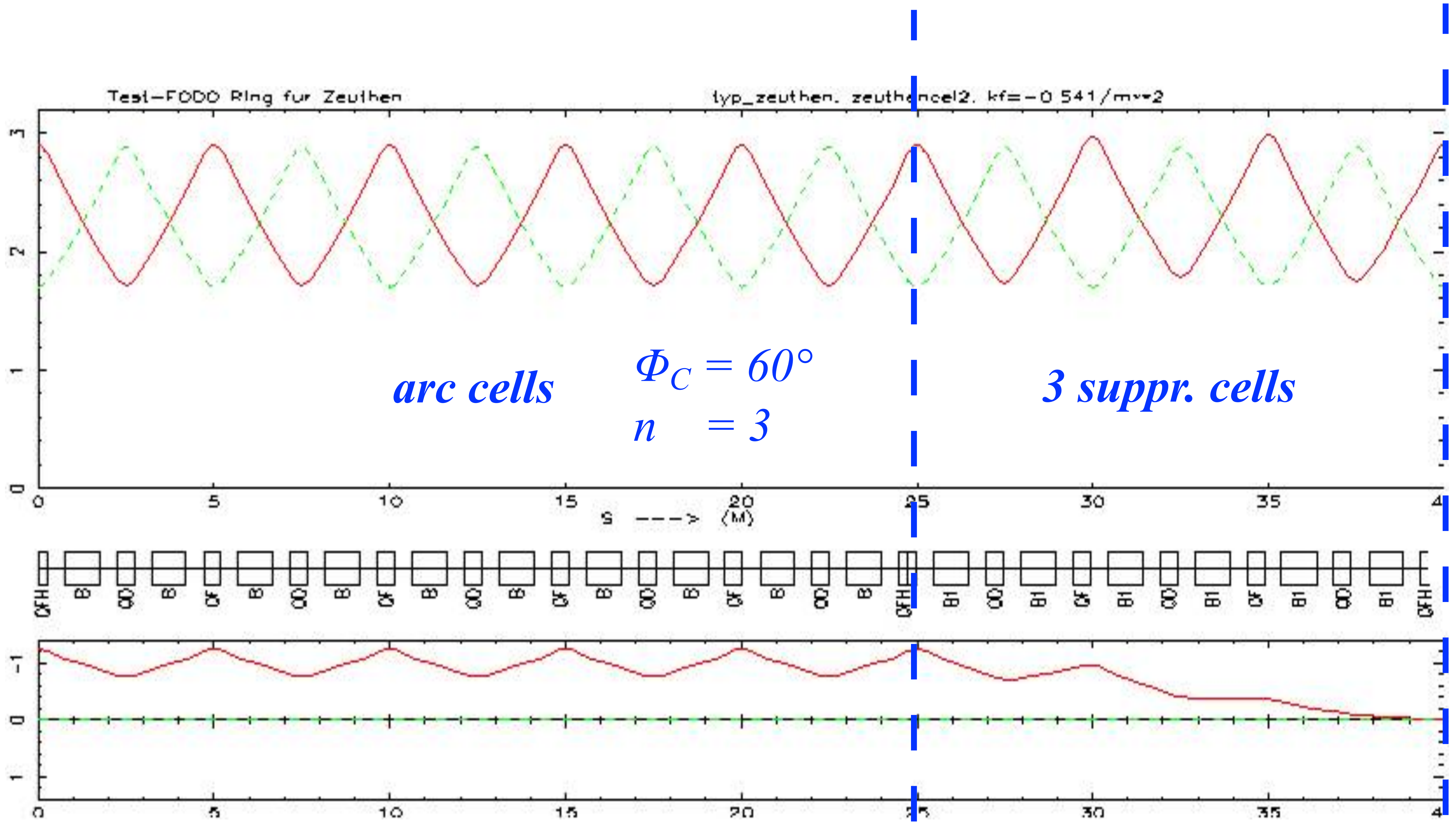
$$\sin^2\left(\frac{n\Phi_c}{2}\right) = 1 \qquad \sin\left(n\Phi_c\right) = 0$$

$$\rightarrow \quad n\Phi_c = k\pi, \quad k = 1,3,...$$

Example:

phase advance in the arc
$\Phi_C = 90°$
number of suppressor cells $n = 2$

phase advance in the arc
$\Phi_C = 60°$
number of suppressor cells $n = 3$



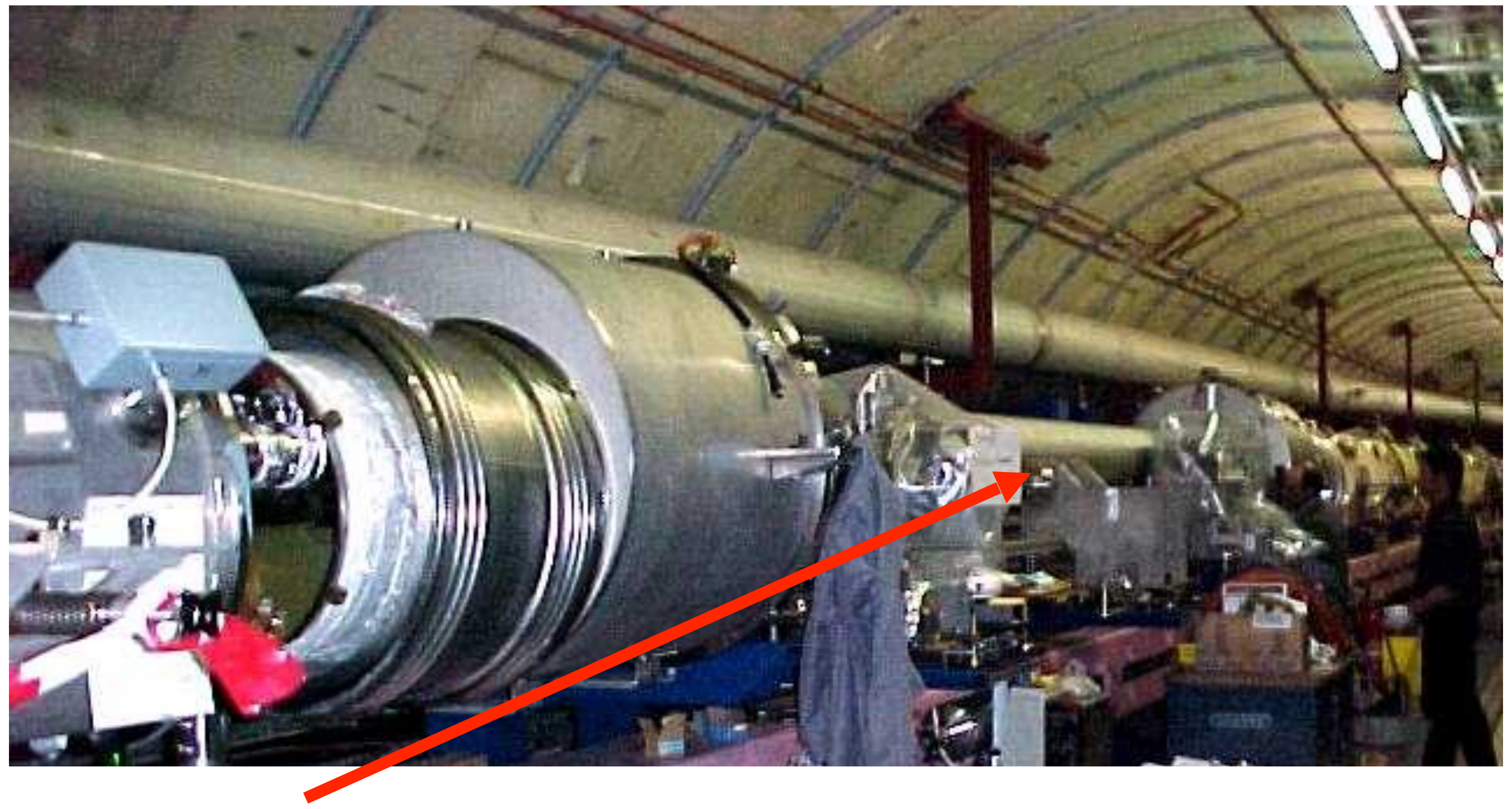
Missing-bend dispersion suppressor

conditions for the (missing) dipole fields:

$$\frac{2m+n}{2}\Phi_C = (2k+1)\frac{\pi}{2}$$

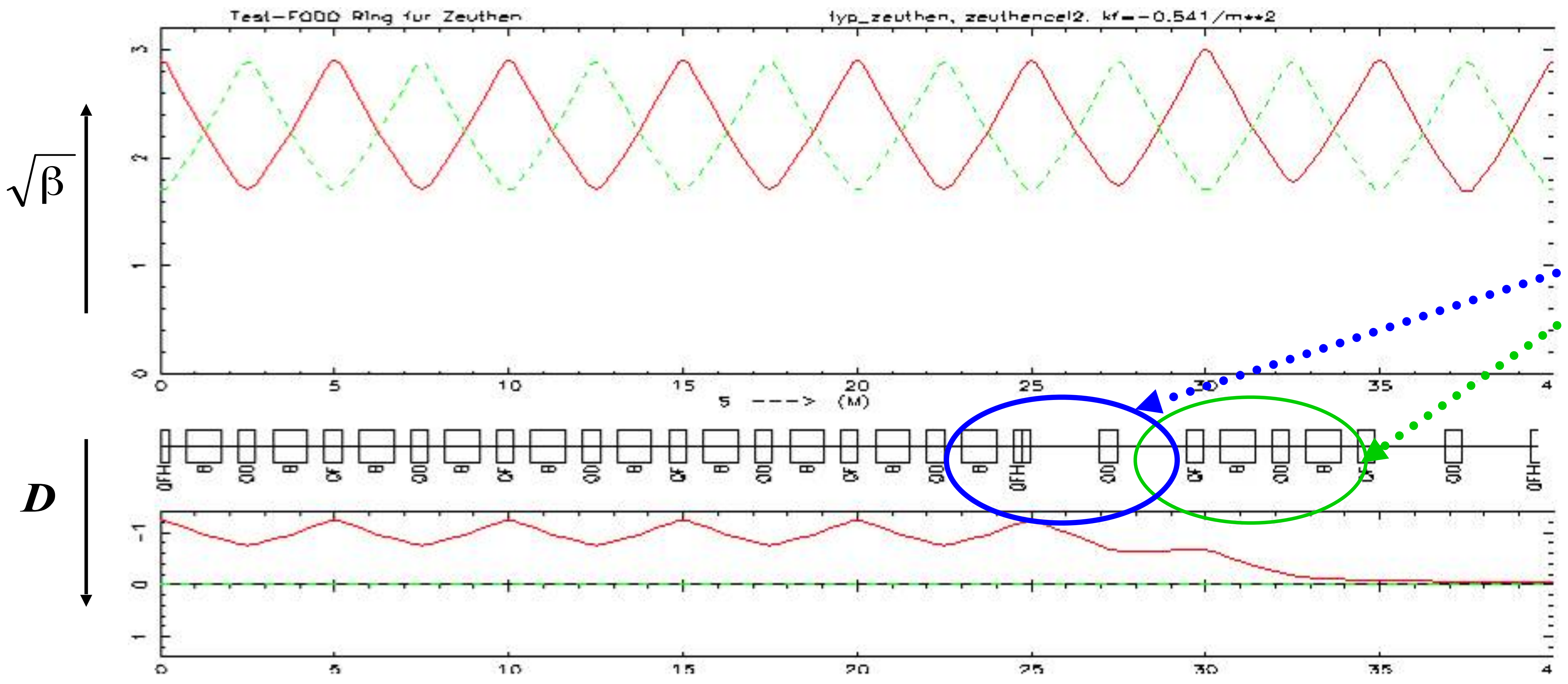
$$\sin\frac{n\Phi_C}{2} = \frac{1}{2}\ ,\ k = 0,2\ \ldots\ \textit{or}$$

$$\sin\frac{n\Phi_C}{2} = \frac{-1}{2}\ ,\ k = 1,3\ \ldots$$



Empty cell suppressor in HERA

m = number of cells without dipoles
followed by
n regular arc cells.

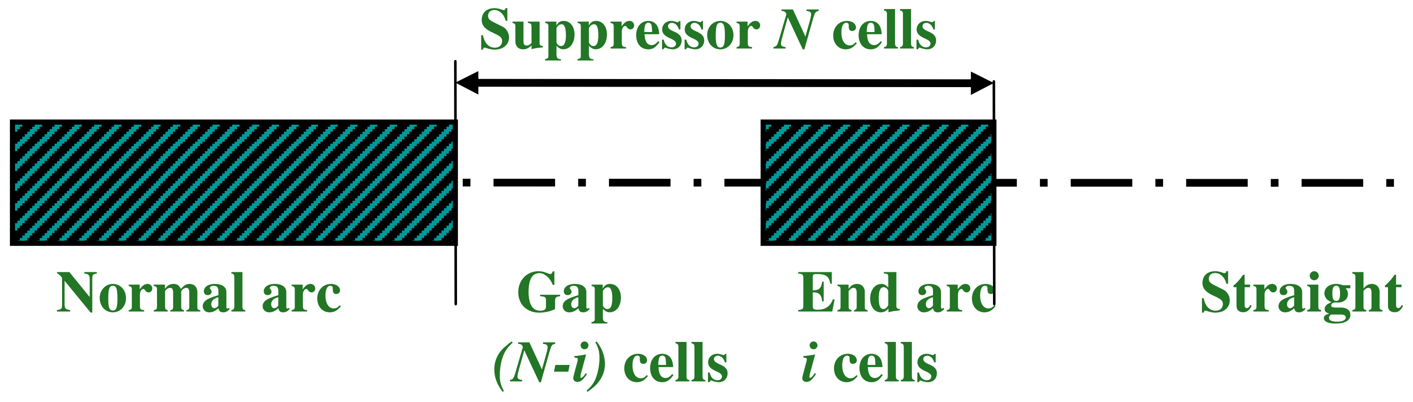


Example:

phase advance in the arc $\Phi_C = 60°$

number of suppr. cells $m = 1$
number of regular cells $n = 1$

Dispersion suppressors



- Missing-magnet suppressors for FODO arcs (Fquad. + Dipole + Dquad. + Dipole):

N	*Gap*	i	$\Delta\mu$	End arc dipole θ
2	1	1	60°	L/ρ
3	1	2	45°	$(L/\rho)/\sqrt{2}$
4	2	2	30°	$(L/\rho)/2$

- Half-field suppressors for FODO arcs ($N = i$, no gap)

N=i	*Gap*	$\Delta\mu$	End arc dipole θ
2	0	90°	$(L/\rho)/2$
3	0	60°	$(L/\rho)/2$
4	0	45°	$(L/\rho)/2$

Half-field is useful in electron machines as it reduces the synchrotron radiation into the experimental region.

JUAS17_02- P.J. Bryant - Lecture 2
Lattice Design I - Slide18

(Phil Bryant)

Comment:

- Dipole-based dispersion suppressors affect the geometry of the ring.
- If the footprint of a new accelerator is pre-defined (e. g. existing tunnel,…) or the phase advance does not match, this concept does not work 100%.

—> Dispersion suppressor has to be supported by quadrupoles.

Example of vanishing dispersion in a particle collider

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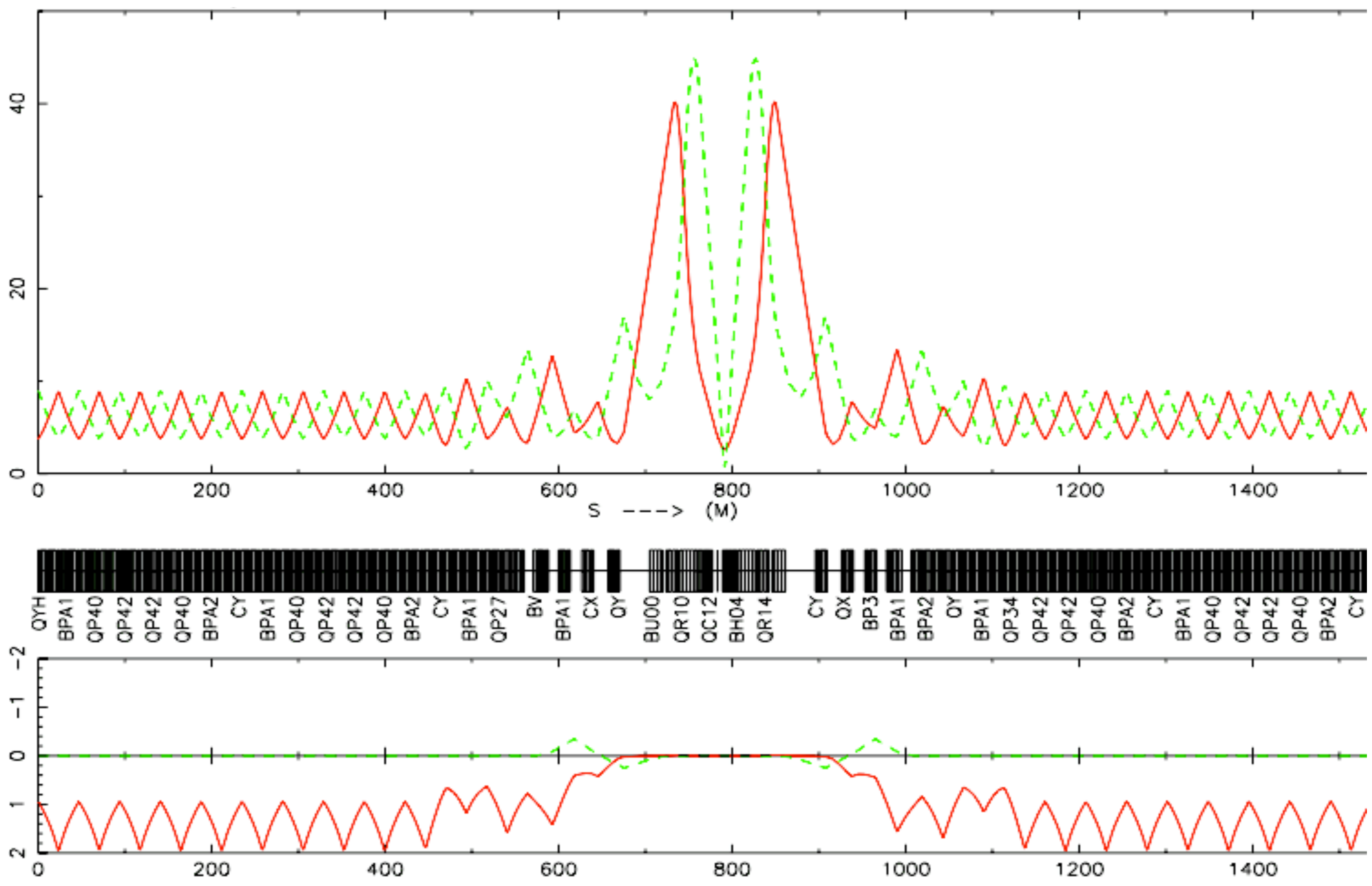

Interaction region with mini-beta insertion

Prepare for Beam collisions

… there is just a little problem

Problem: Our particles are VERY small!

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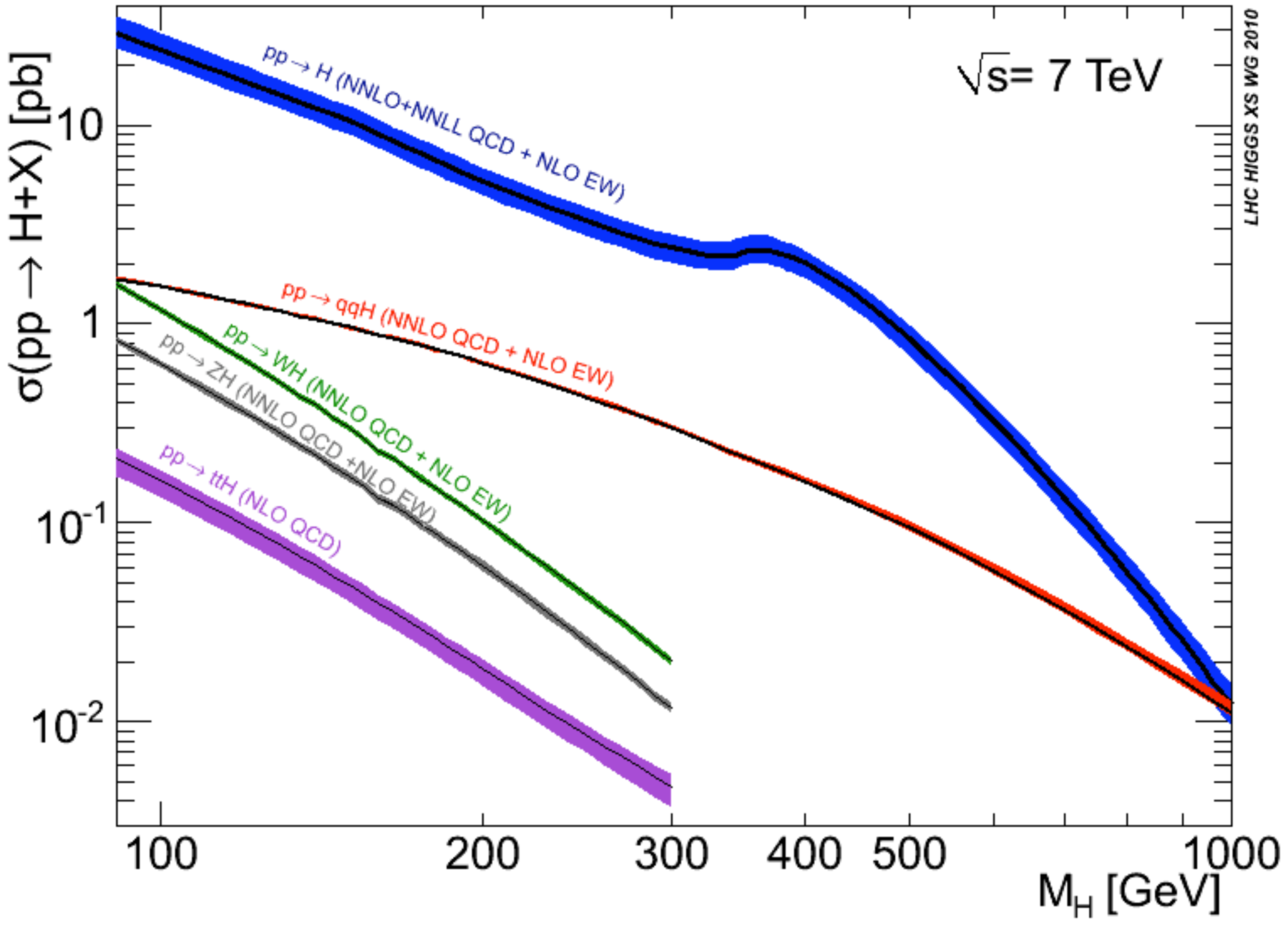
Overall cross section of the Higgs:

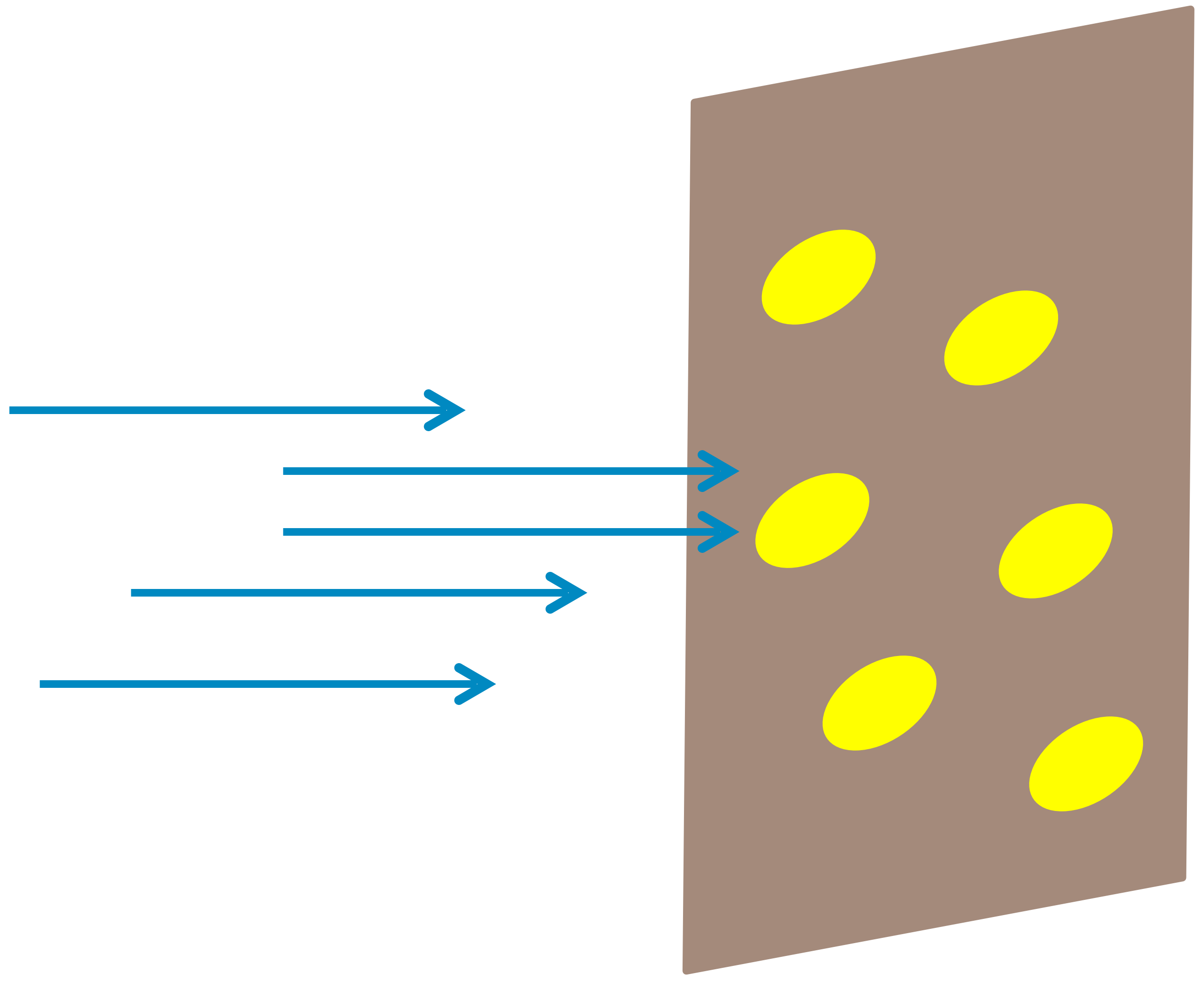


$$\Sigma_{\text{react}} \approx 1\,\text{pb}^{-1}$$

$$1\,\text{b} = 10^{-24}\,\text{cm}^2$$

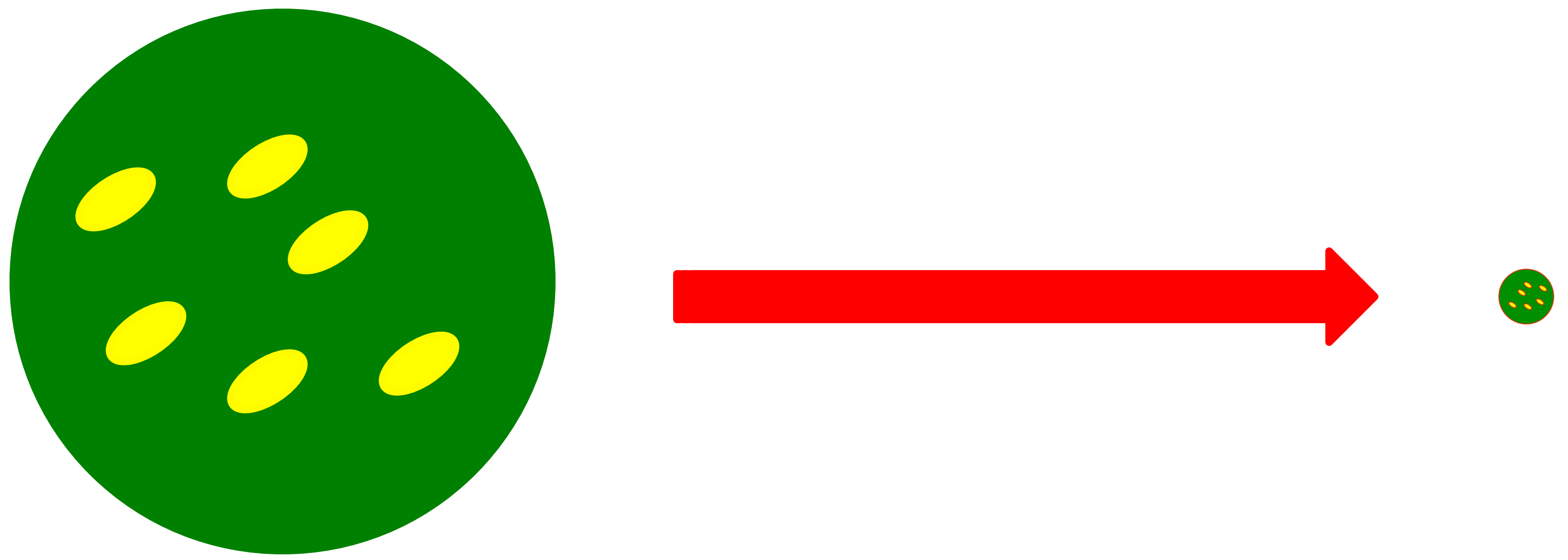
$$1\,\text{pb} = 10^{-12} \cdot 10^{-24}\,\text{cm}^2$$

$$1\,\text{pb} = \frac{1}{\text{mio}} \cdot \frac{1}{\text{mio}} \cdot \frac{1}{\text{mio}} \cdot \frac{1}{\text{mio}} \cdot \frac{1}{\text{mio}} \cdot \frac{1}{10000}\,\text{mm}^2$$



The only chance we have:

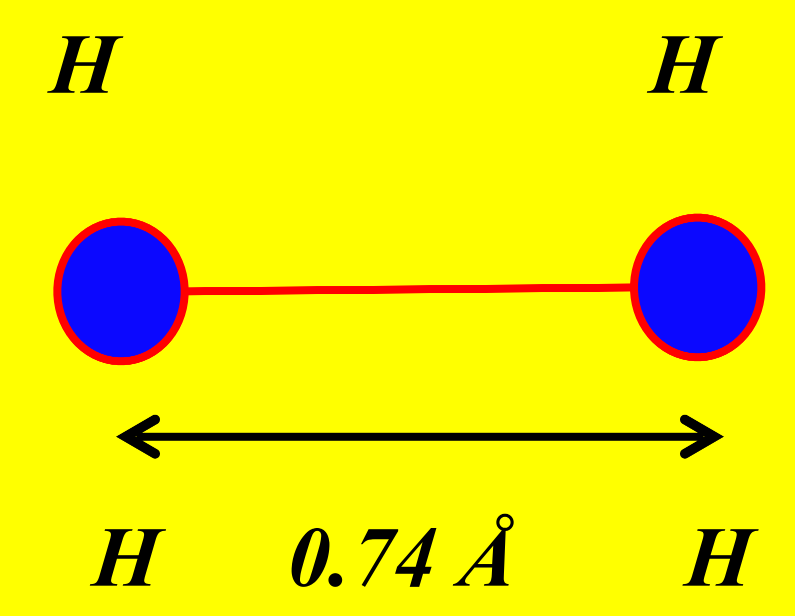
compress the transverse beam size … at the IP



LHC typical:

σ = 0.1 mm → 16 µm

Particle density in matter

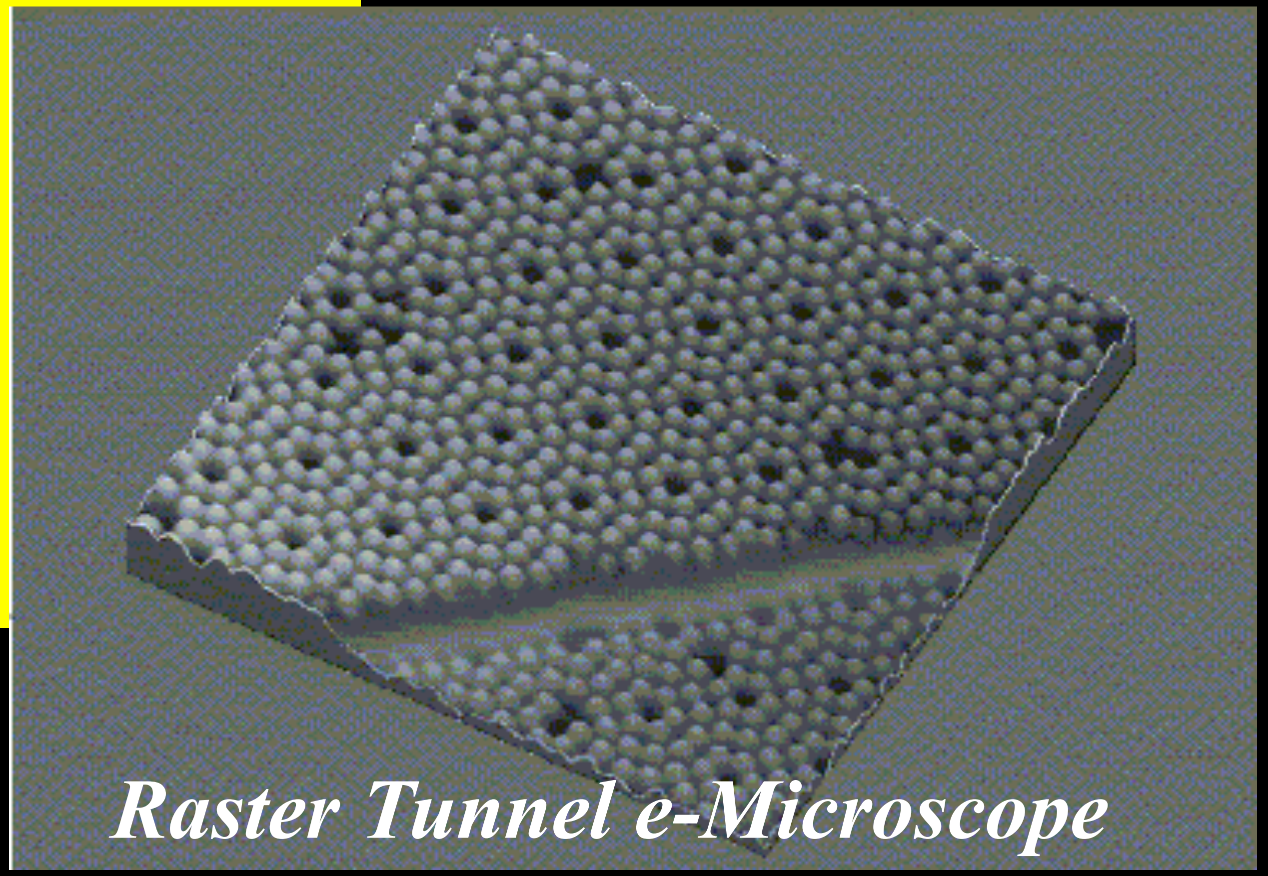


Atomic Distance in Hydrogen Molecule

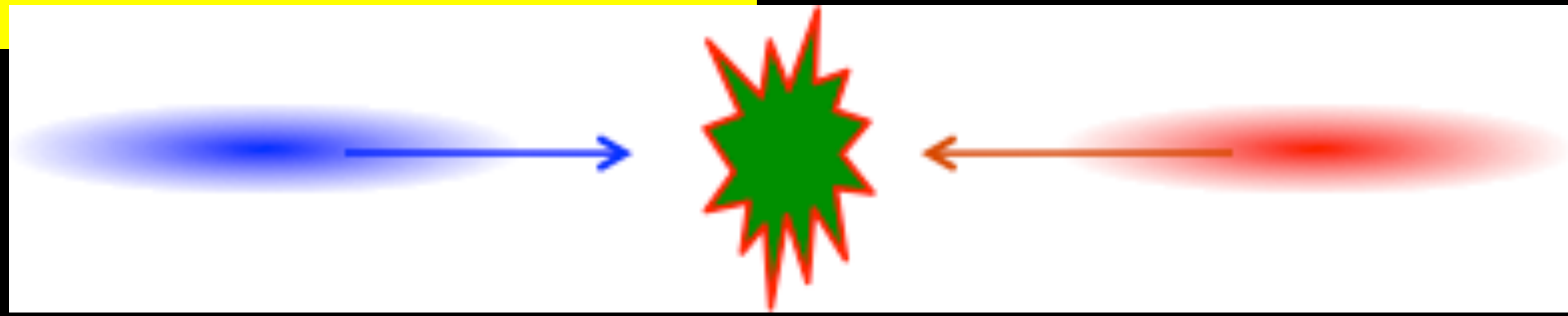
$R_B \approx 0.5$ Å

in solids / fluids $\lambda \approx 1 \dots 3$ Å

in gases $\lambda \approx 35$ Å $= 3.5$ nm



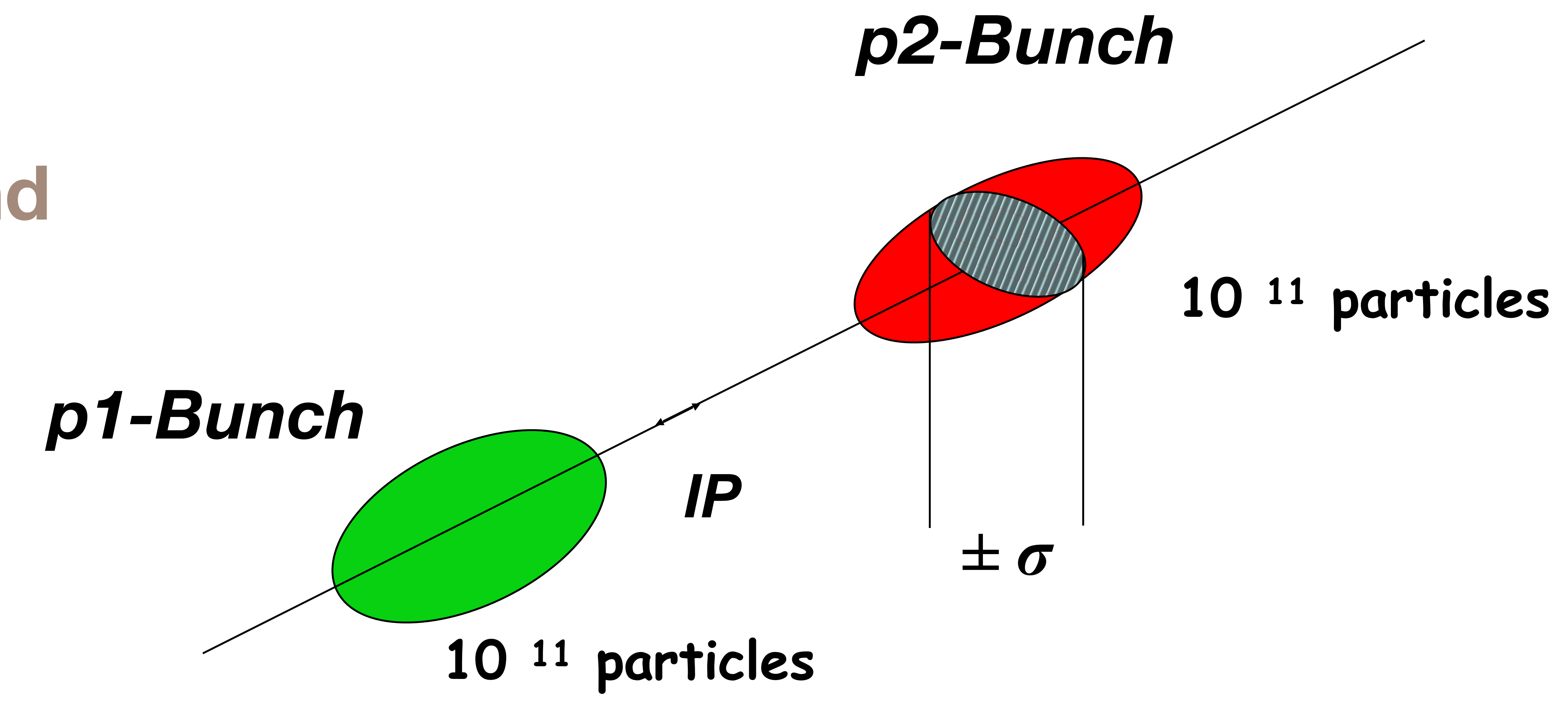
Particle Distance in Accelerators: $\lambda \approx 600$ nm (Arc) … 300nm (IP LEP)
$= 3000$ Å



Luminosity

Event Rate: “Physics“ per Second

$$R = \Sigma_{react} \cdot L$$



Example: Luminosity run at LHC

$\beta_{x,y} = 0.55\ m$

$\varepsilon_{x,y} = 5*10^{-10}\ rad\ m$

$\sigma_{x,y} = 17\ \mu m$

$f_0 = 11.245\ kHz$

$n_b = 2808$

$I_p = 584\ mA$

$$L = \frac{1}{4\pi e^2 f_0 n_b} * \frac{I_{p1} I_{p2}}{\sigma_x \sigma_y}$$

$$L = 1.0 * 10^{34}\ {}^{1}\!/_{cm^2 s}$$

Beam collisions: Mini-beta insertion

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A Mini-Beta-Insertion is basically just a long

drift space, embedded in the storage ring lattice

transformation rule for the optics parameters:

$$\begin{pmatrix}\beta\\ \alpha\\ \gamma\end{pmatrix}_{s2} = \begin{pmatrix} m_{11}^2 & -2m_{11}m_{12} & m_{12}^2 \\ -m_{11}m_{21} & m_{12}m_{21}+m_{11}m_{22} & -m_{12}m_{22} \\ m_{12}^2 & -2m_{22}m_{21} & m_{22}^2 \end{pmatrix}_{s1} = \begin{pmatrix}\beta\\ \alpha\\ \gamma\end{pmatrix}_{s1}$$

with the matrix elements given by the product-matrix of the lattice elements

$$M_{total} = \ldots M_{QD} \cdot M_{Drift} \cdot M_B \cdot M_{Drift} \cdot M_{QF} = \begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix}$$

transfer matrix for a drift:

$$M_{drift} = \begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix} = \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix}$$

transferring from 0 —> s

$$\beta(s) = \beta_0 - 2a_0 \cdot s + \gamma_0 \cdot s^2$$

$$\alpha(s) = \alpha_0 - \gamma_0 \cdot s$$

$$\gamma(s) = \gamma_0$$

What will happen to the beam parameters α, β, γ if we stop focusing for a while …?

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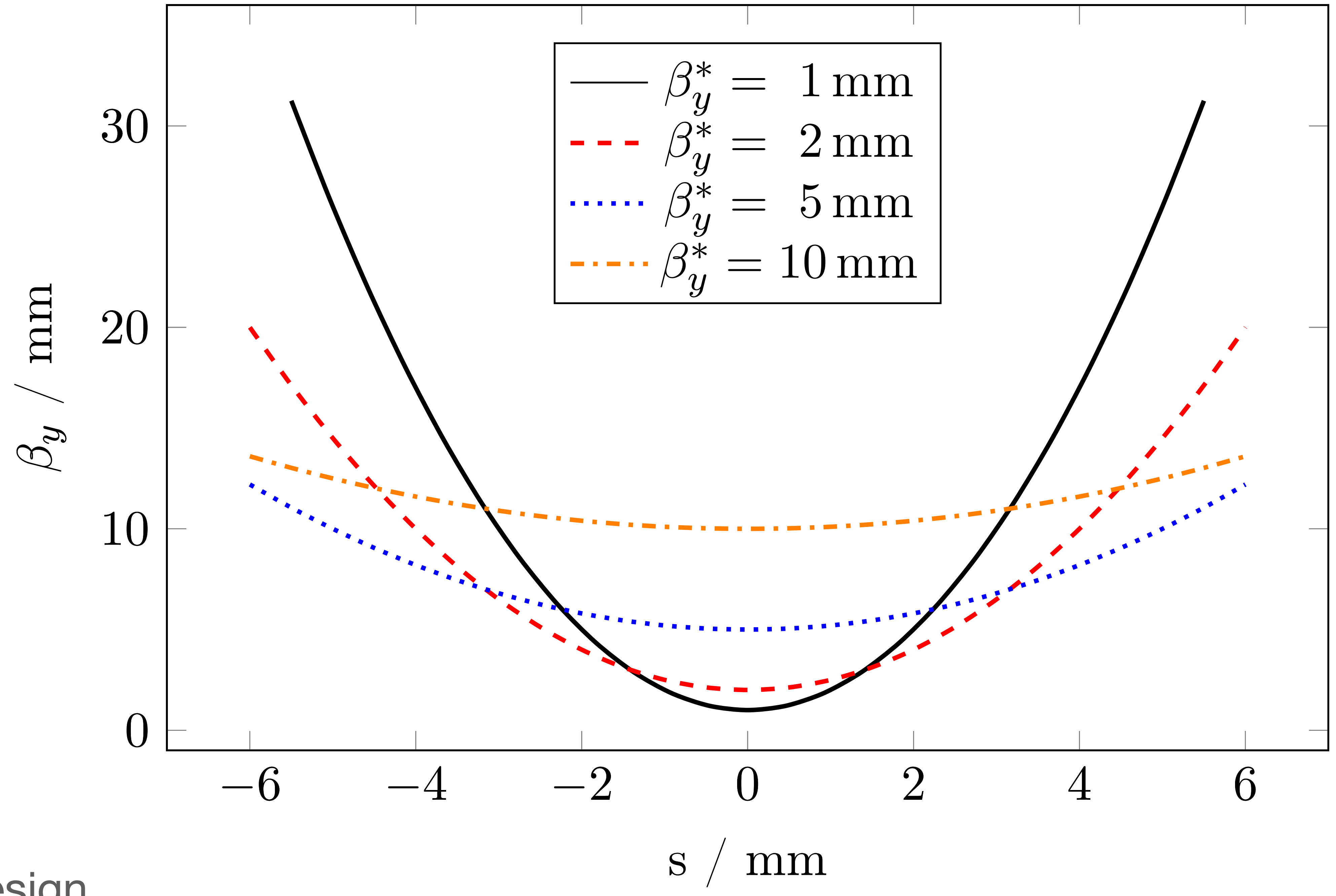
let's assume we are at a symmetry point in the center of a drift.

$$\beta(s) = \beta_0 - 2\alpha_0 s + \gamma_0 s^2 \qquad \dots \text{ as } \qquad \alpha_0 = 0, \quad \rightarrow \quad \gamma_0 = \frac{1+\alpha_0^2}{\beta_0} = \frac{1}{\beta_0}$$

so, we get for the β function in the neighbourhood of the symmetry point

$$\beta(s) = \beta_0 + \frac{s^2}{\beta_0}$$

! ! !



Nota bene:

1.) this is very bad !!!

2.) this is a direct consequence of the conservation of phase space density (… in our words: ε = const) … and there is no way out.

3.) Thank you, Mr. Liouville !!!

Mini-beta insertion: phase space

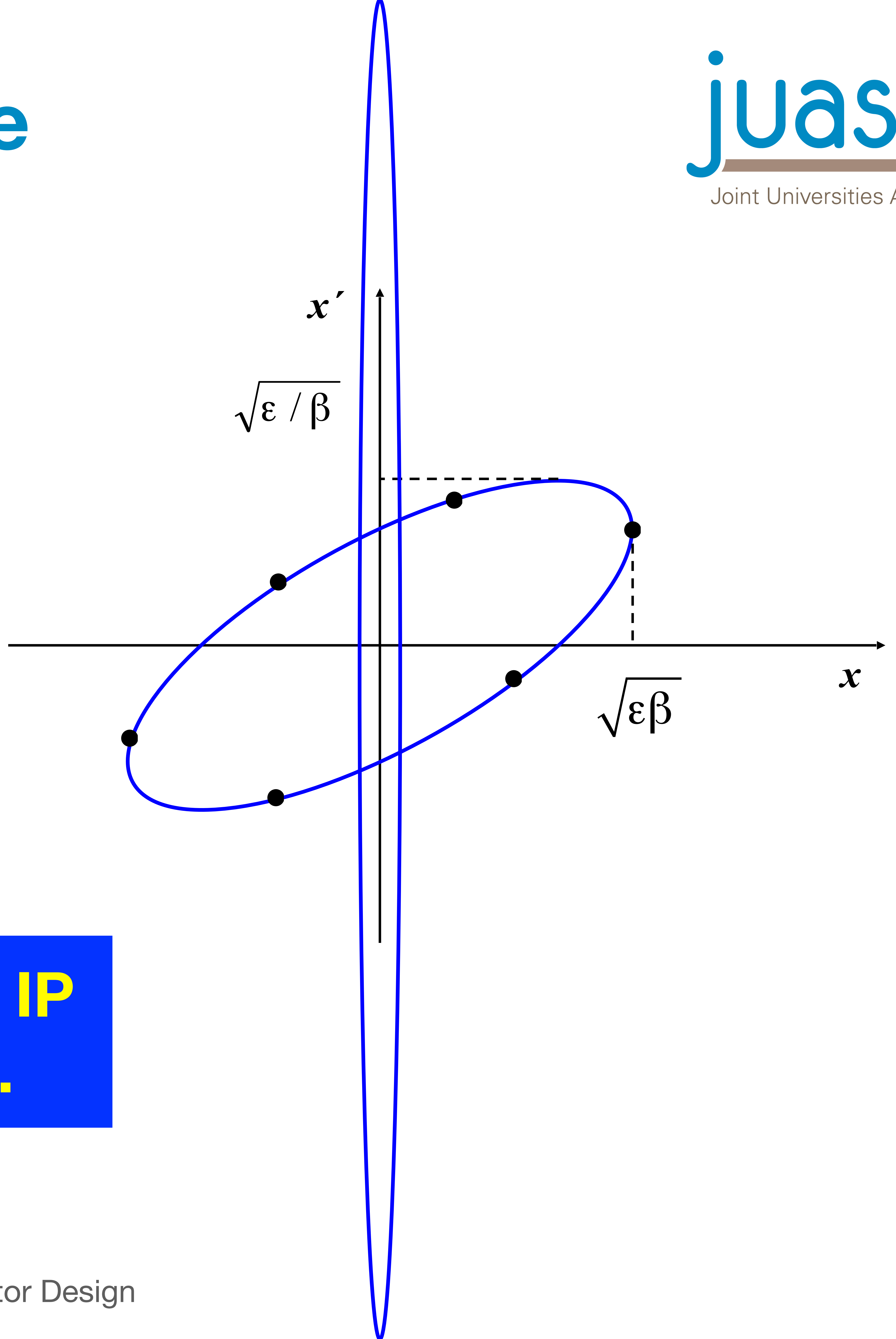
Symmetry point of a drift space: $\alpha^* = 0$

$$\epsilon = \gamma(s)x^2(s) + 2\alpha(s)x(s)x'(s) + \beta(s)x'^2(s)$$

Greetings from Liouville:

the smaller the beam size

the larger the beam divergence



The first quadrupole magnets after the IP need the largest aperture in the ring.

Betafunction in a drift

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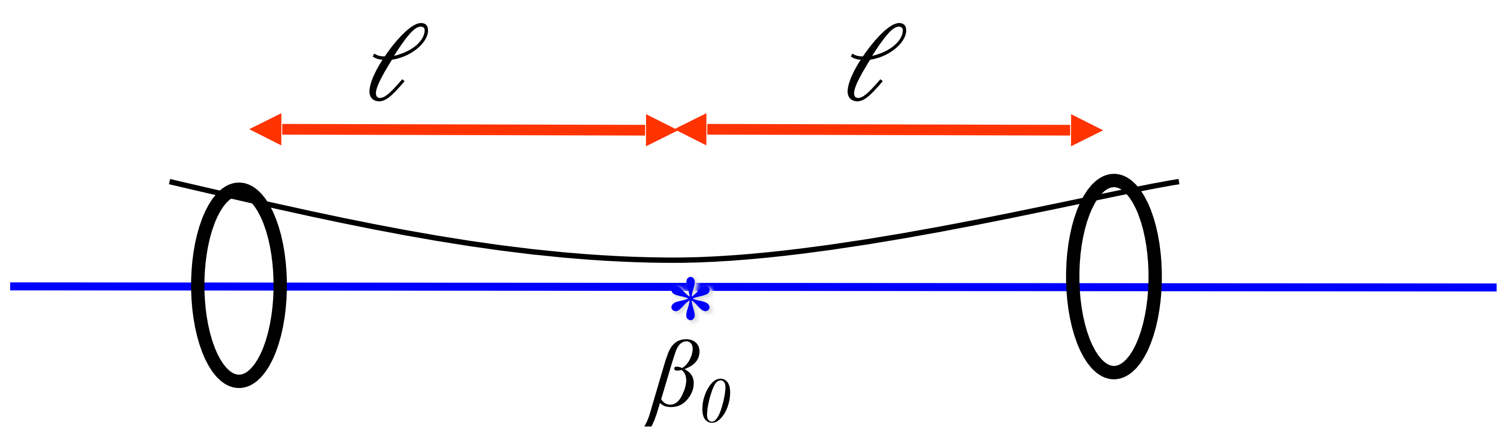
If we cannot fight against Liouville's theorem ... at least we can optimise

Optimisation of the beam dimension at position s = ℓ :

$$\beta(\ell) = \beta_0 + \frac{\ell^2}{\beta_0}$$

Find the β at the center of the drift that leads to the lowest maximum β at the end:



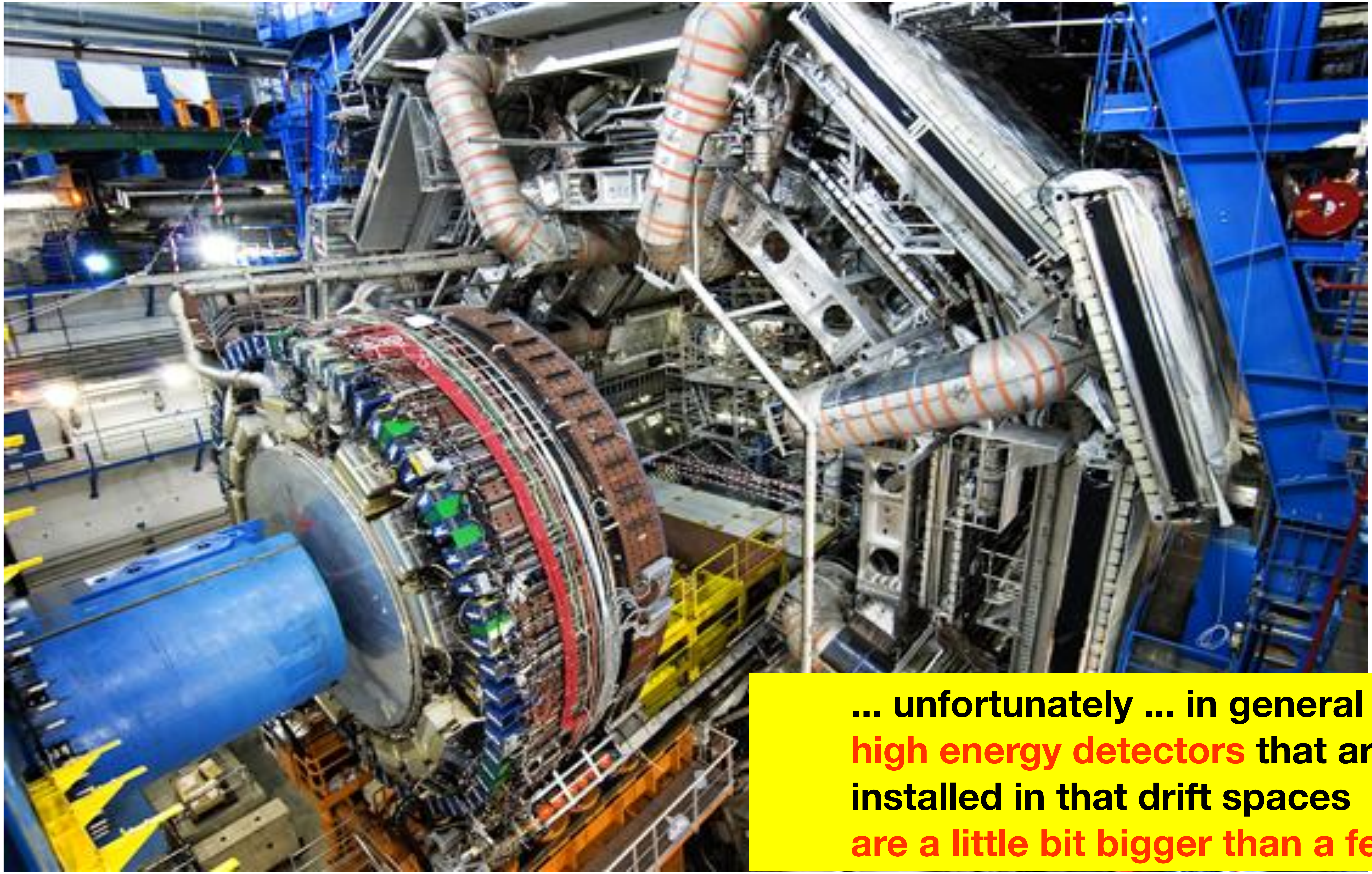
$$\frac{d\hat{\beta}}{d\beta_0} = 1 - \frac{\ell^2}{\beta_0^2} = 0$$

$$\rightarrow \beta_0 = \ell$$

$$\rightarrow \hat{\beta} = 2\beta_0$$

If we choose $\beta_0 = \ell$ we get the smallest β at the end of the drift and the maximum β is just twice the distance ℓ

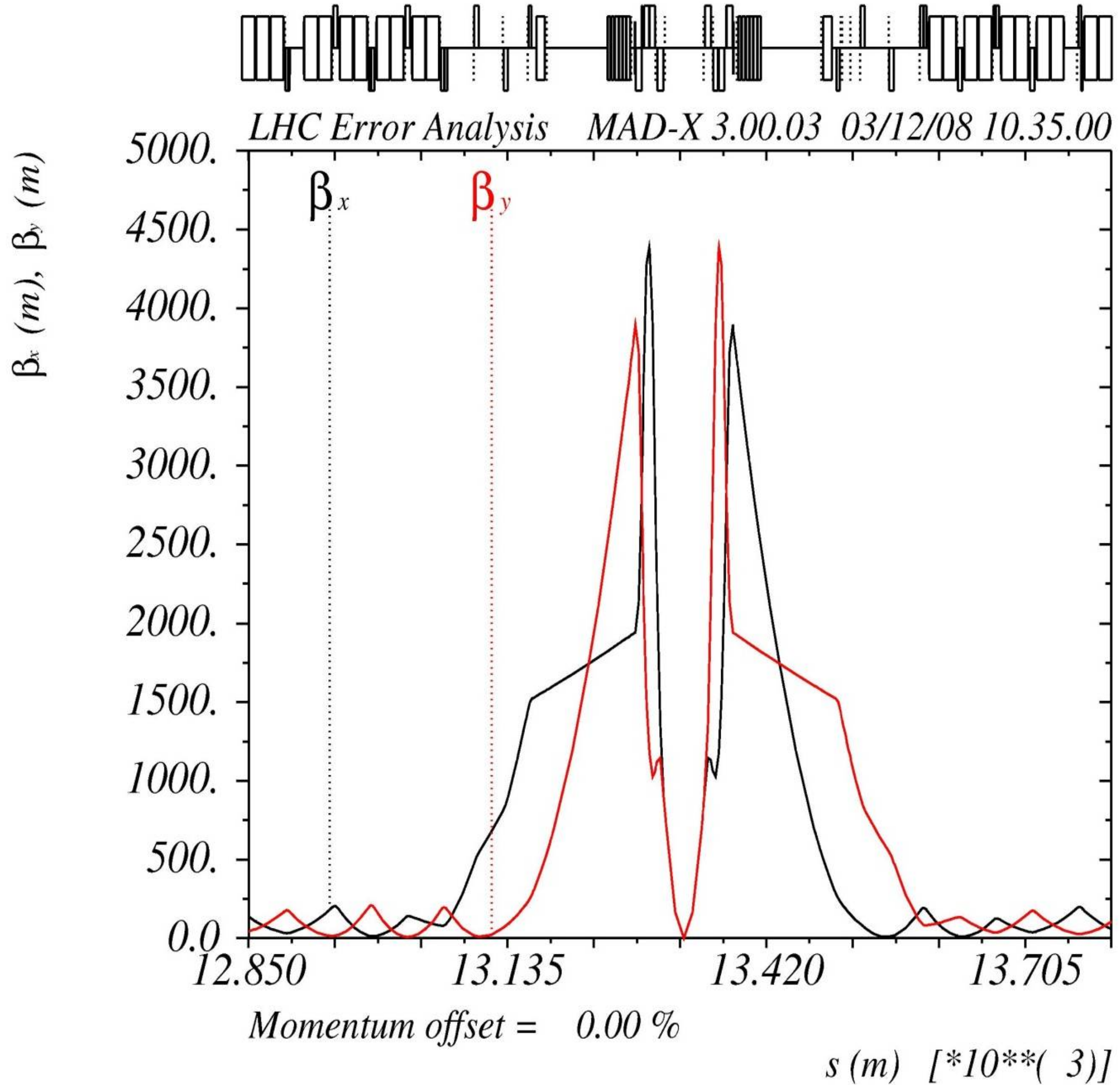
In any case: keep ℓ as SMALL as possible !!!

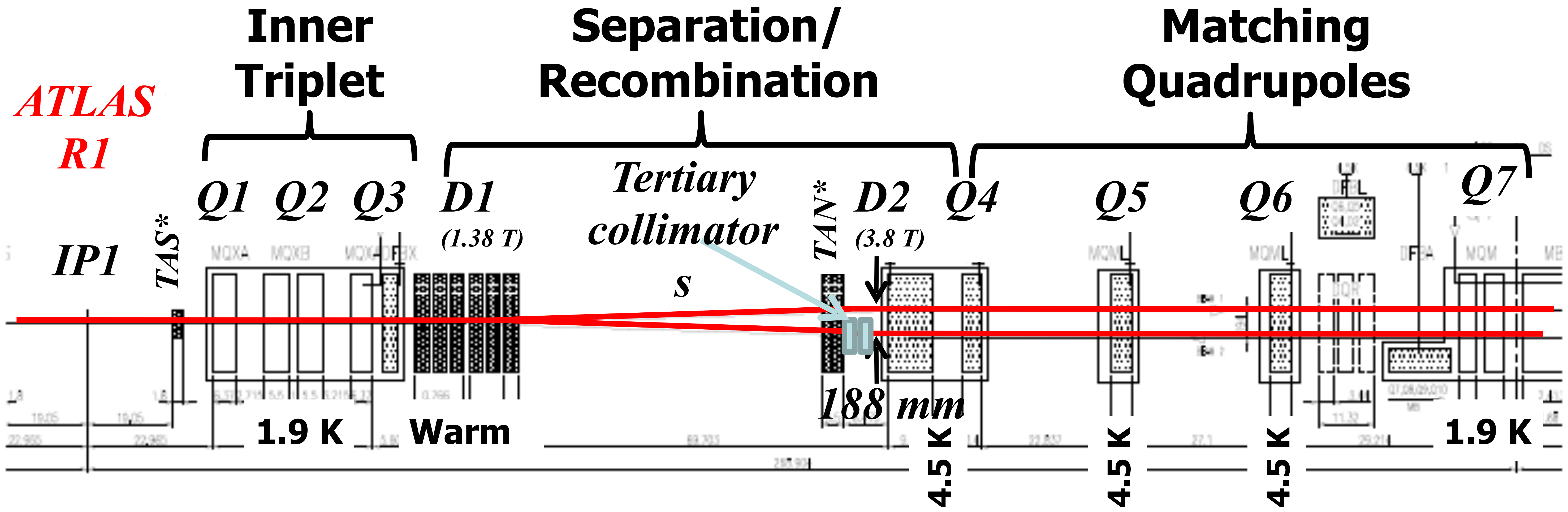


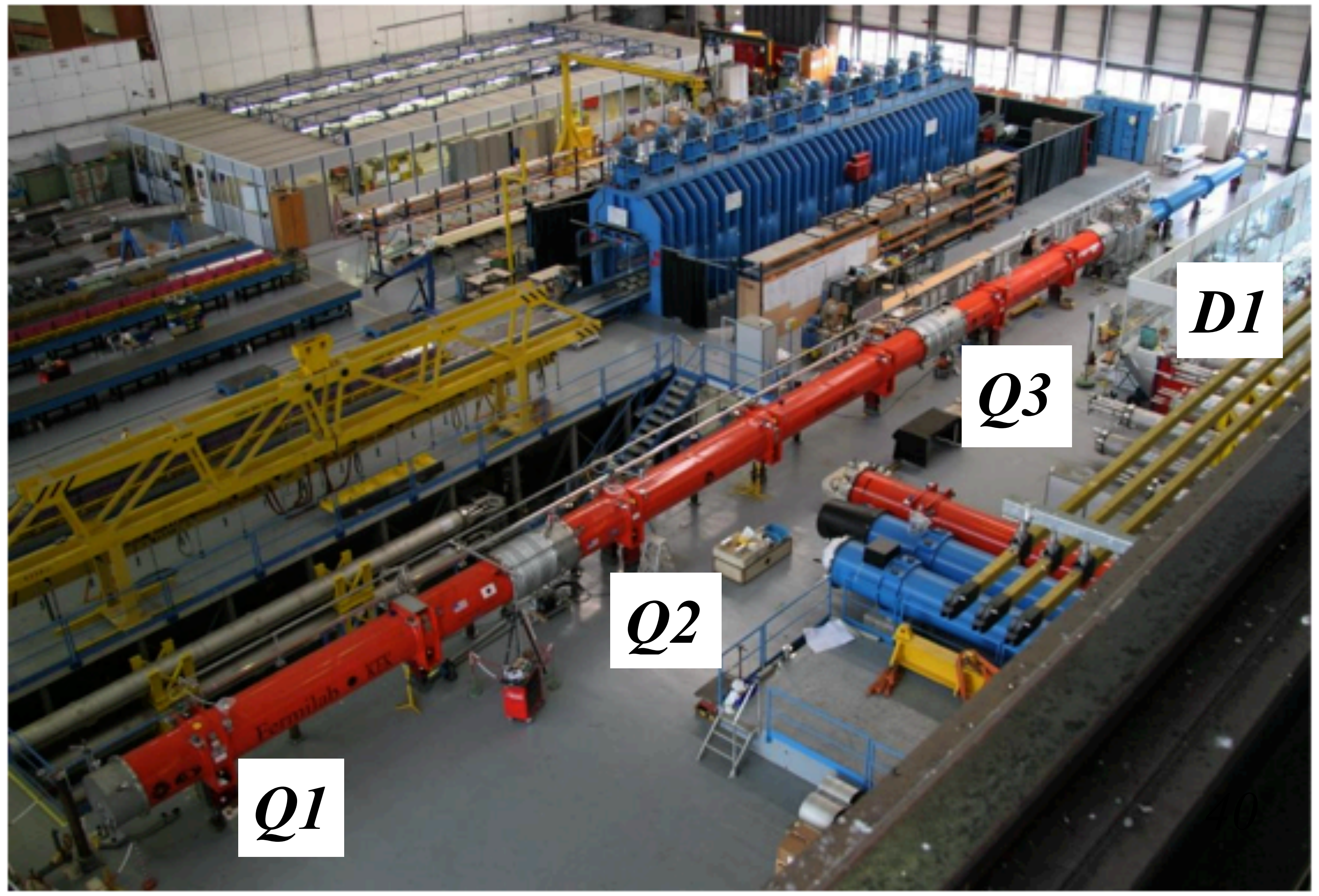
... unfortunately ... in general high energy detectors that are installed in that drift spaces are a little bit bigger than a few centimeters ...

The LHC mini-beta insertions

mini-beta optics







Mini-beta insertions: phase advance

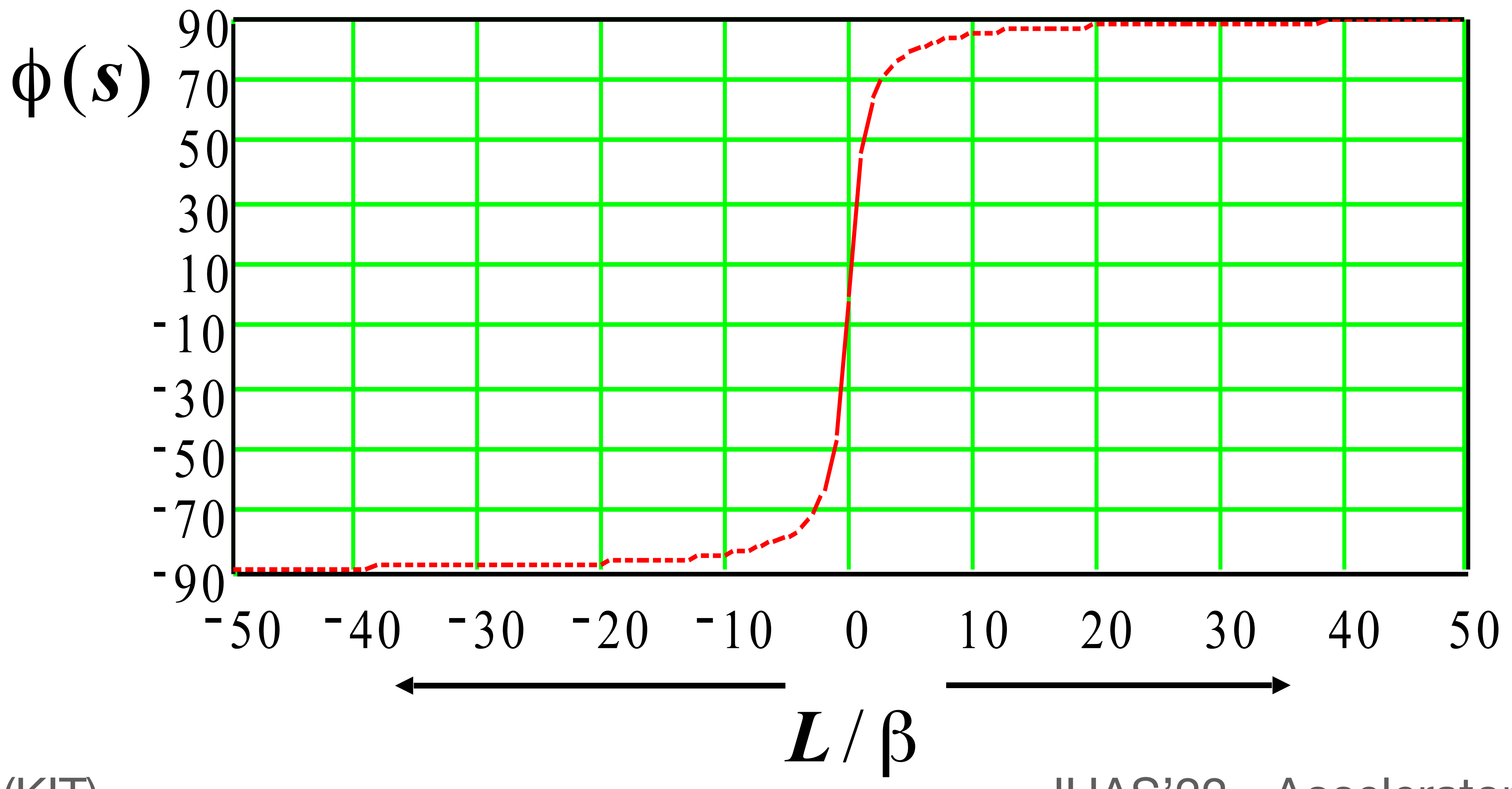
By definition the phase advance is given by:

$$\mu(s) = \int \frac{1}{\beta(s)} \, ds$$

Now in a mini-beta insertion:

$$\beta(s) = \beta_0 \left(1 + \frac{s^2}{\beta_0^2}\right)$$

$$\rightarrow \mu(s) = \frac{1}{\beta_0} \int_0^L \frac{1}{1 + s^2/\beta_0^2} \, ds = \arctan \frac{L}{\beta_0}$$



Consider the drift spaces on both sides of the IP:

the phase advance of a mini β insertion is approximately π,

in other words: the tune will increase by half an integer.

Mini-beta insertion: Guidelines

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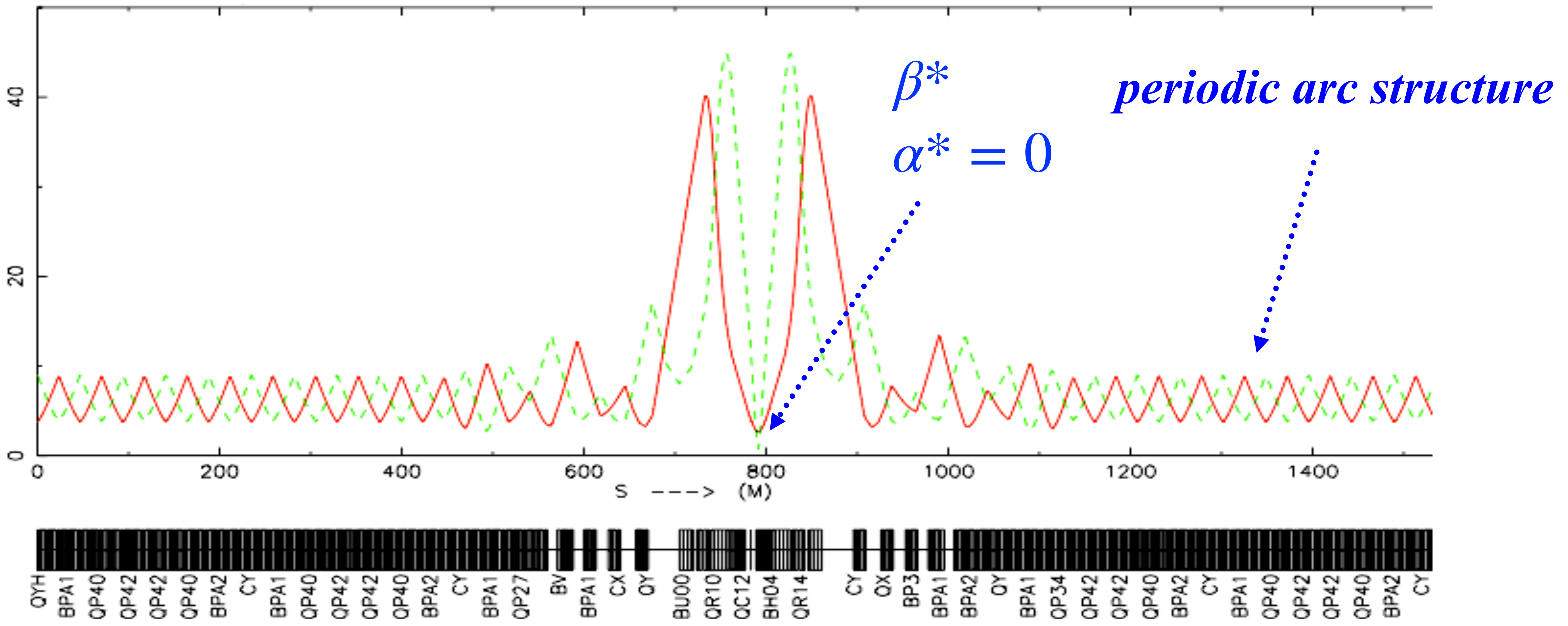
* calculate the periodic solution in the arc

* introduce the drift space needed for the insertion device (detector ...)

* put a quadrupole doublet (triplet ?) as close as possible

* introduce additional quadrupole lenses to match the beam parameters to the values at the beginning of the arc structure



Parameters to be optimised & matched to the periodic solution:

$$\alpha_x,\ \beta_x \qquad D_x,\ D_x'$$
$$\alpha_y,\ \beta_y \qquad Q_x,\ Q_y$$

8 individually powered quadrupole magnets are needed to match the insertion (... at least)

One word about mini-beta insertions:

Mini-beta insertions must be installed in

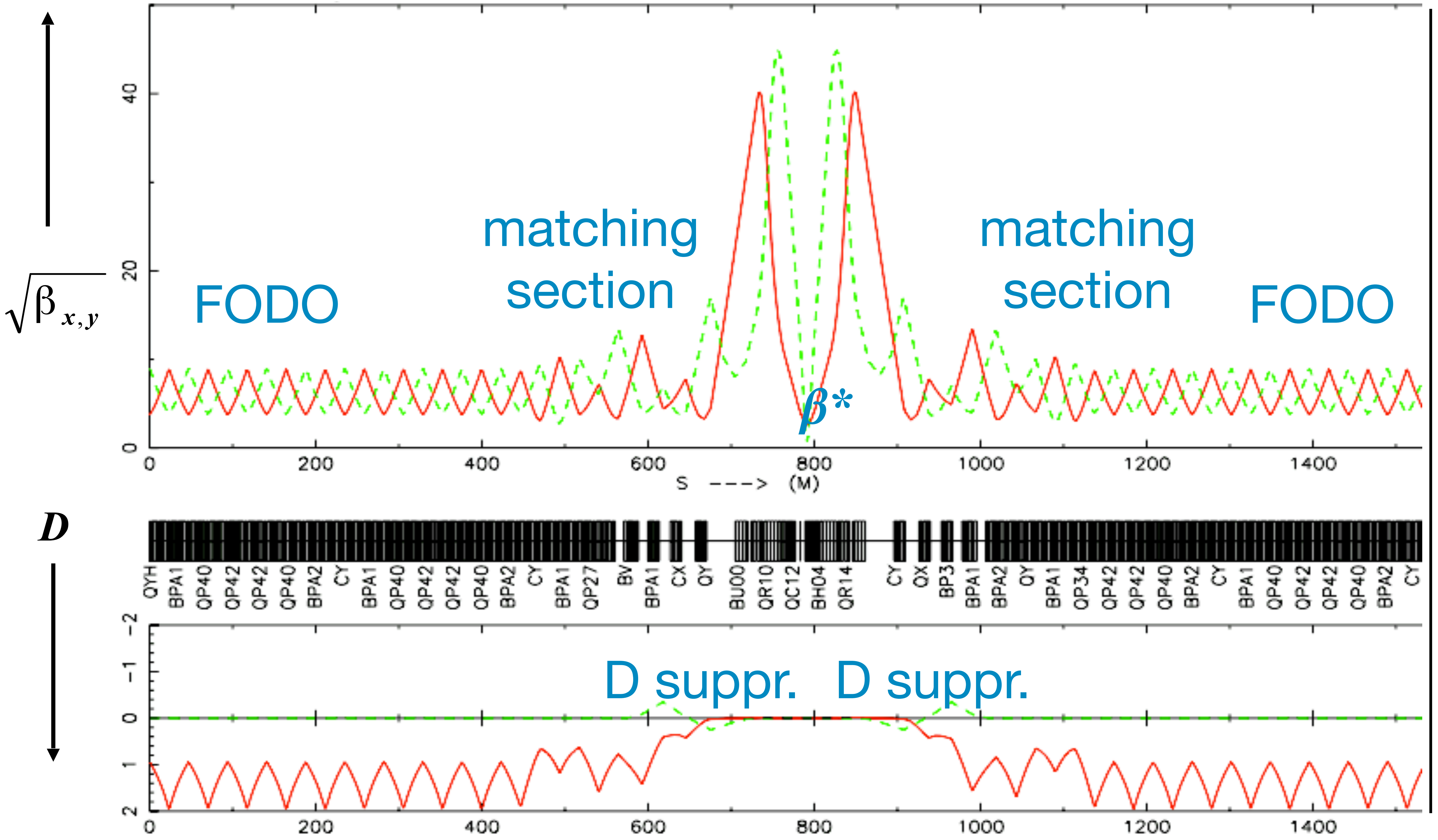
… straight sections (no dipoles that drive dispersion)

… that are dispersion free

… that are connected to the arc lattice by dispersion suppressors

if not, the dispersion dilutes the particle density and increases the effective transverse beam size.

Layout of the HERA mini-beta insertion



One word about limitations:

It looks like we can get infinite luminosities by

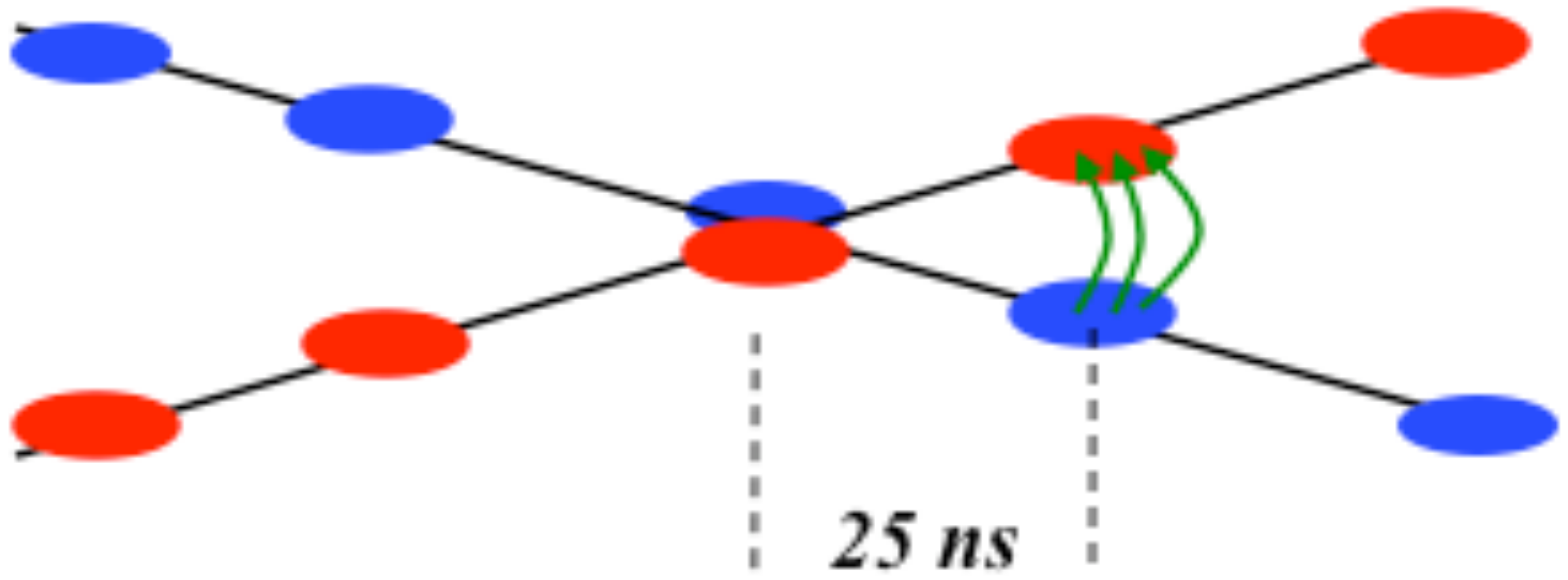
… creating smallest β* at the IP

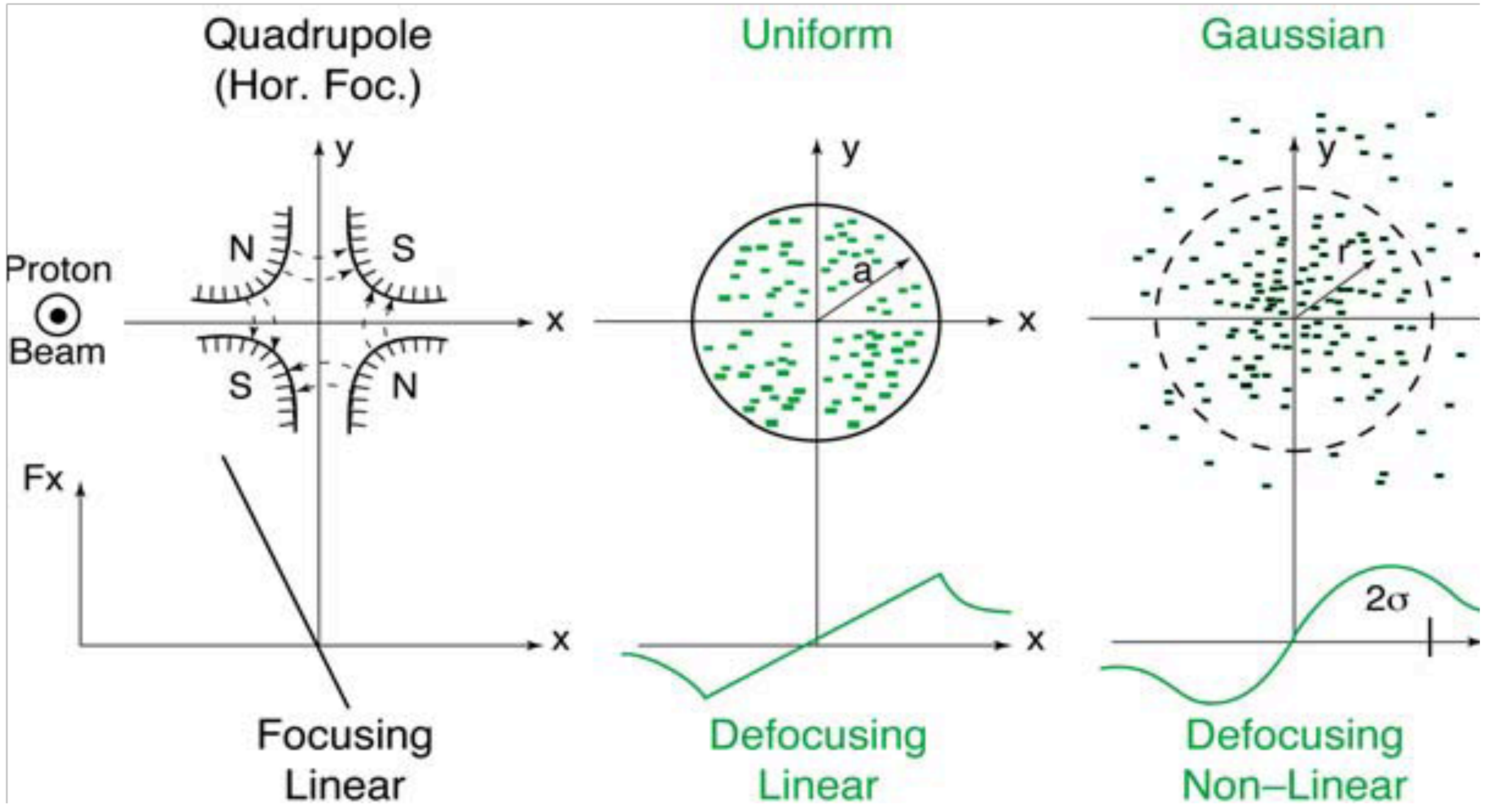
… and accumulating infinite bunch intensities.

However, that is not how life is.

Beam-Beam Effect

the colliding bunches influence each other (space charge)
=> change the focusing properties of the ring !!
for LHC a strong non-linear defocussing effect





court. K. Schindl

most simple case:
linear beam-beam tune-shift
-> puts a limit to N_p

$$\Delta Q_y = \frac{r_e}{2\pi\gamma_e} \cdot \frac{\beta_y^*}{\sigma_y} \cdot \frac{N_e}{(\sigma_x + \sigma_y)}$$

Particles are pushed onto resonances and are lost.

Beam-beam parameter

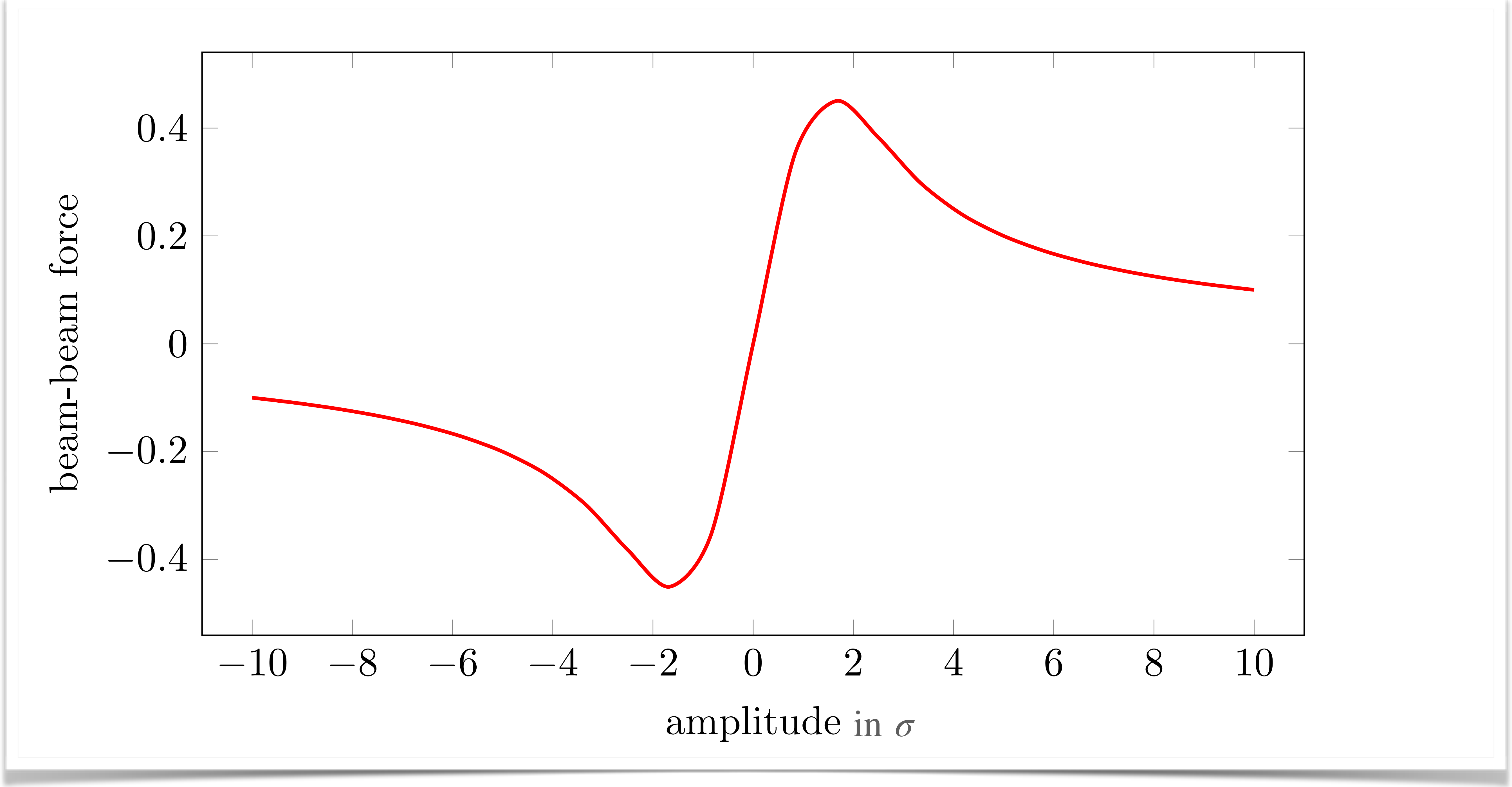
- For small amplitudes the tune shift is equal to the linear beam-beam parameter:

$$\xi_u = \frac{N r_e \beta_u^*}{2\pi\gamma\sigma_u^*(\sigma_x + \sigma_u)}$$

- It is often used to quantify the strength of the beam-beam interaction.

- However, it does not reflect its non-linear nature.

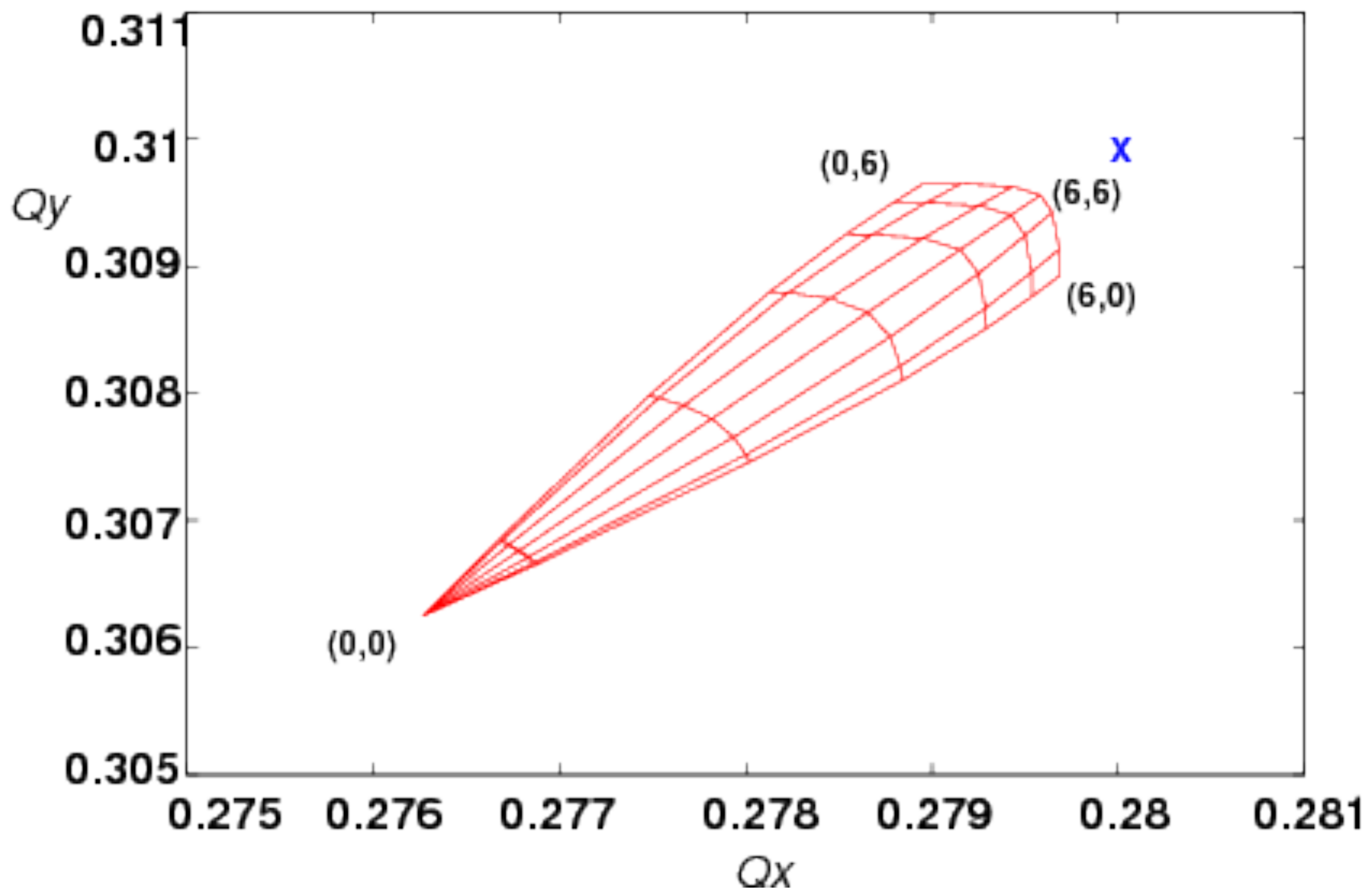
Important: $\boxed{\xi_u \propto \frac{N}{\epsilon_u}}$



Beam-beam force for round beams in arbitrary units

Beam-beam: Tune footprint and Luminosity

- Beam-beam parameter: $\xi_u \propto \dfrac{N}{\epsilon_u}$



- What are the implications for the luminosity?

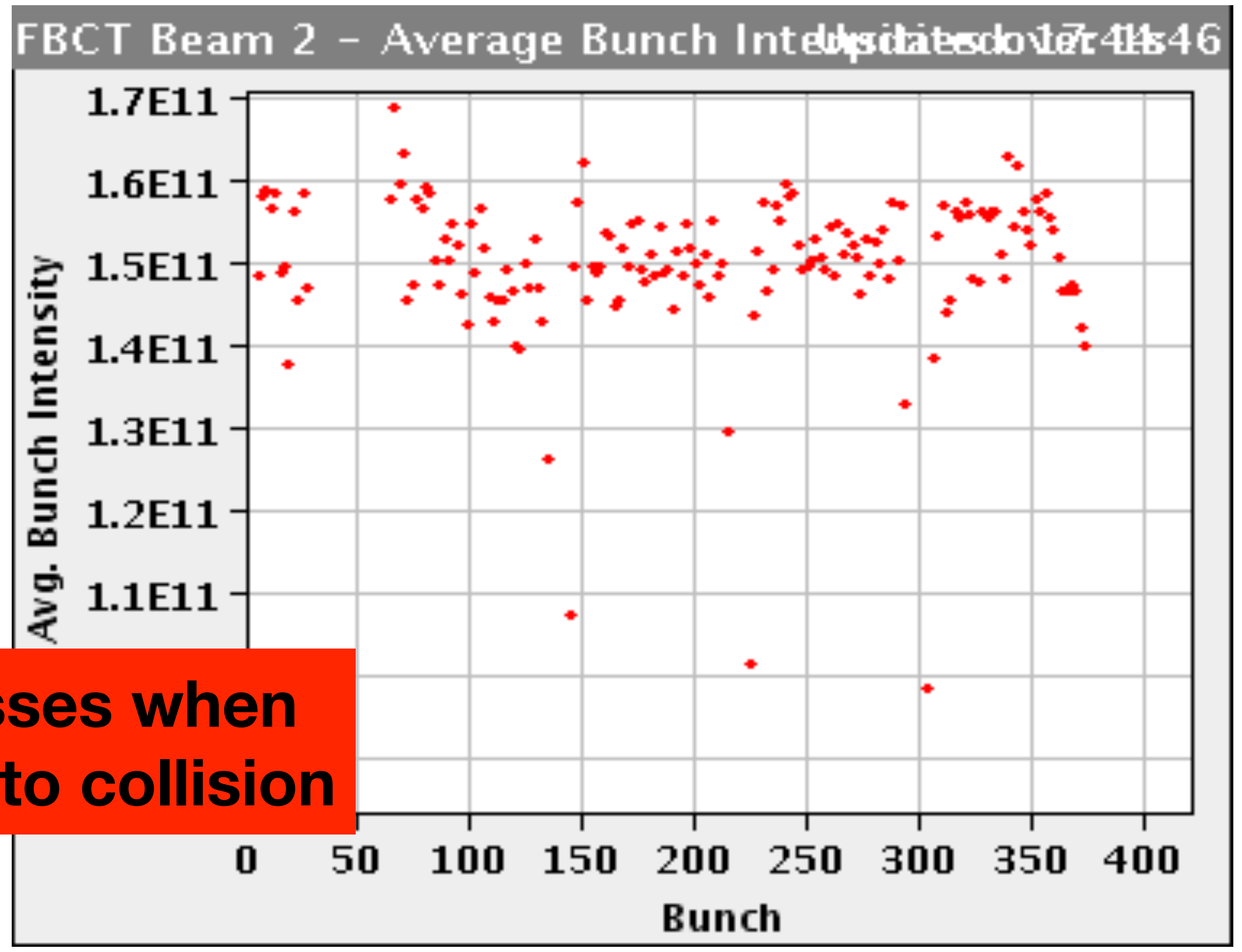
$$\mathscr{L} = \frac{1}{4\pi e^2 f_0 n_b} \frac{I_1 \cdot I_2}{\sigma_x^* \cdot \sigma_y^*}$$

with $I = n_b\ N_p\ e\ f_0$

$$\Rightarrow \mathscr{L} = \frac{N_1 N_2 n_b f}{4\pi \sigma_x^* \sigma_y^*}$$

$\Rightarrow$ Number of particles per bunch is limited!!!

$$\mathcal{O}(N) = 10^{11}$$



observed particle losses when beams are brought into collision

General Remark:

Whenever we combine two different lattice structures we need a

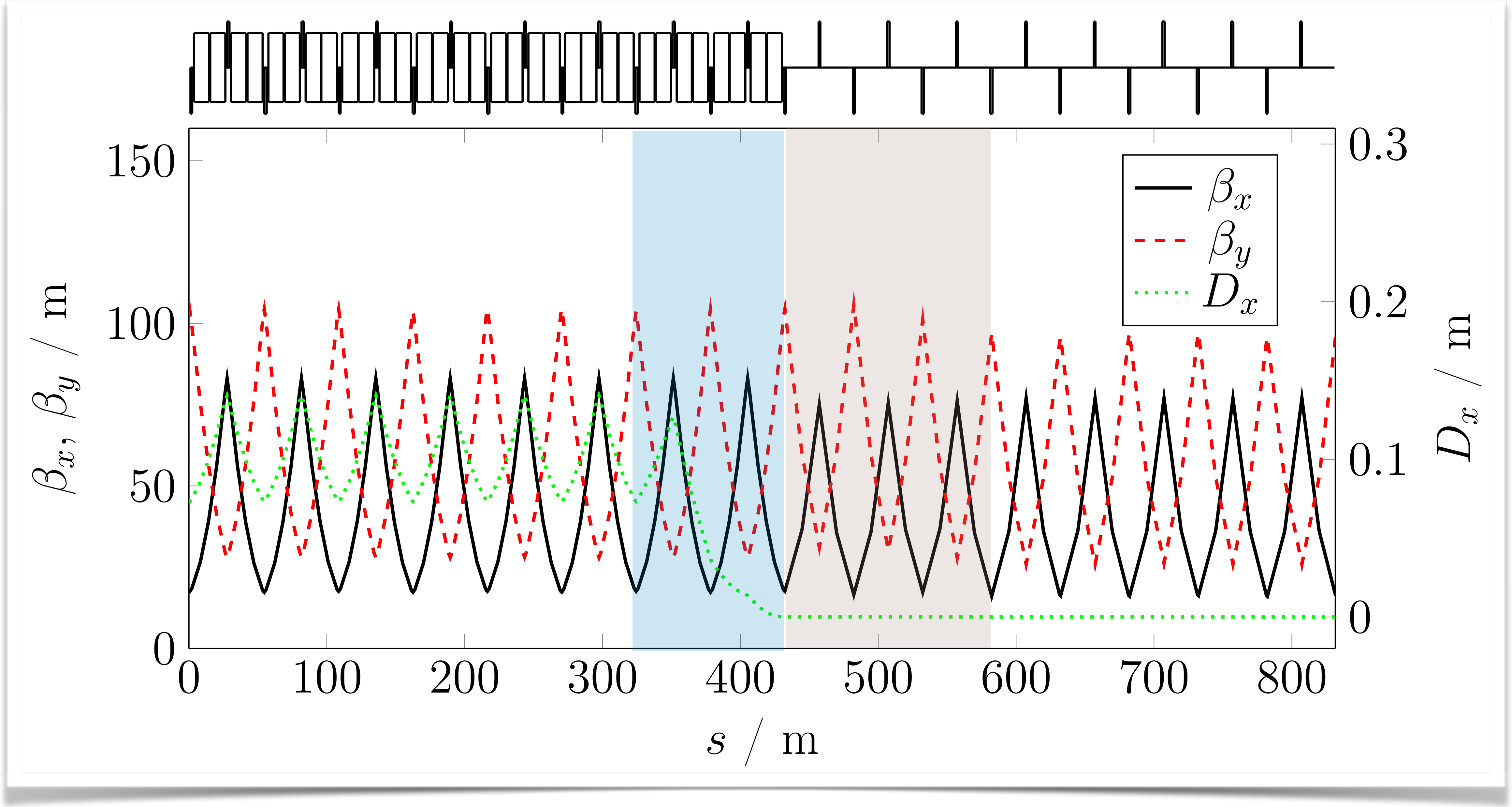
"matching section"

in between to adapt the optics functions between the two lattices.

Example: Change of phase advance per cell

54 m arc cell ⇒ 50 m straight cell

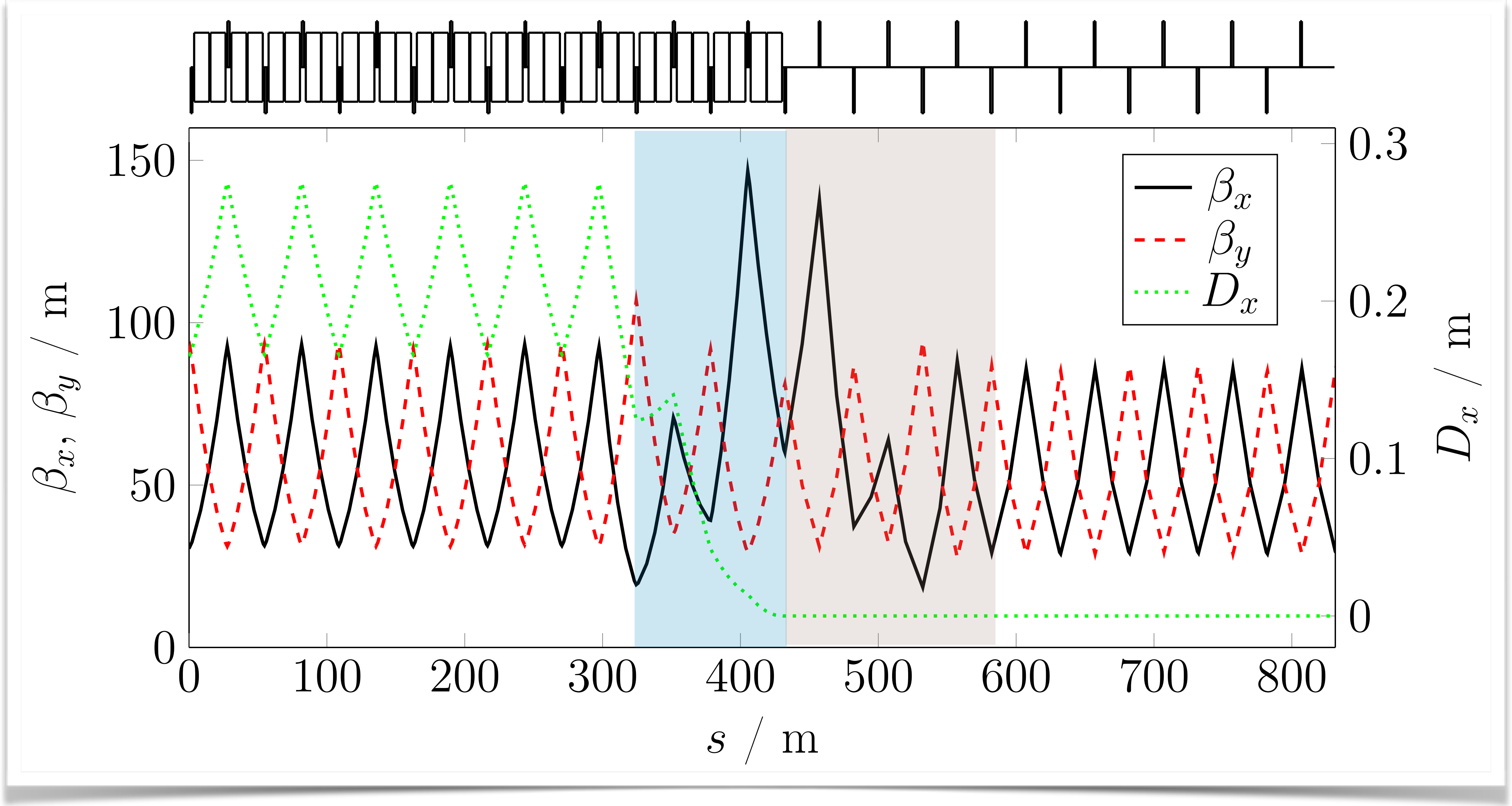
- $\mu = 90°$



half-bend DS
(4 quadrupoles)

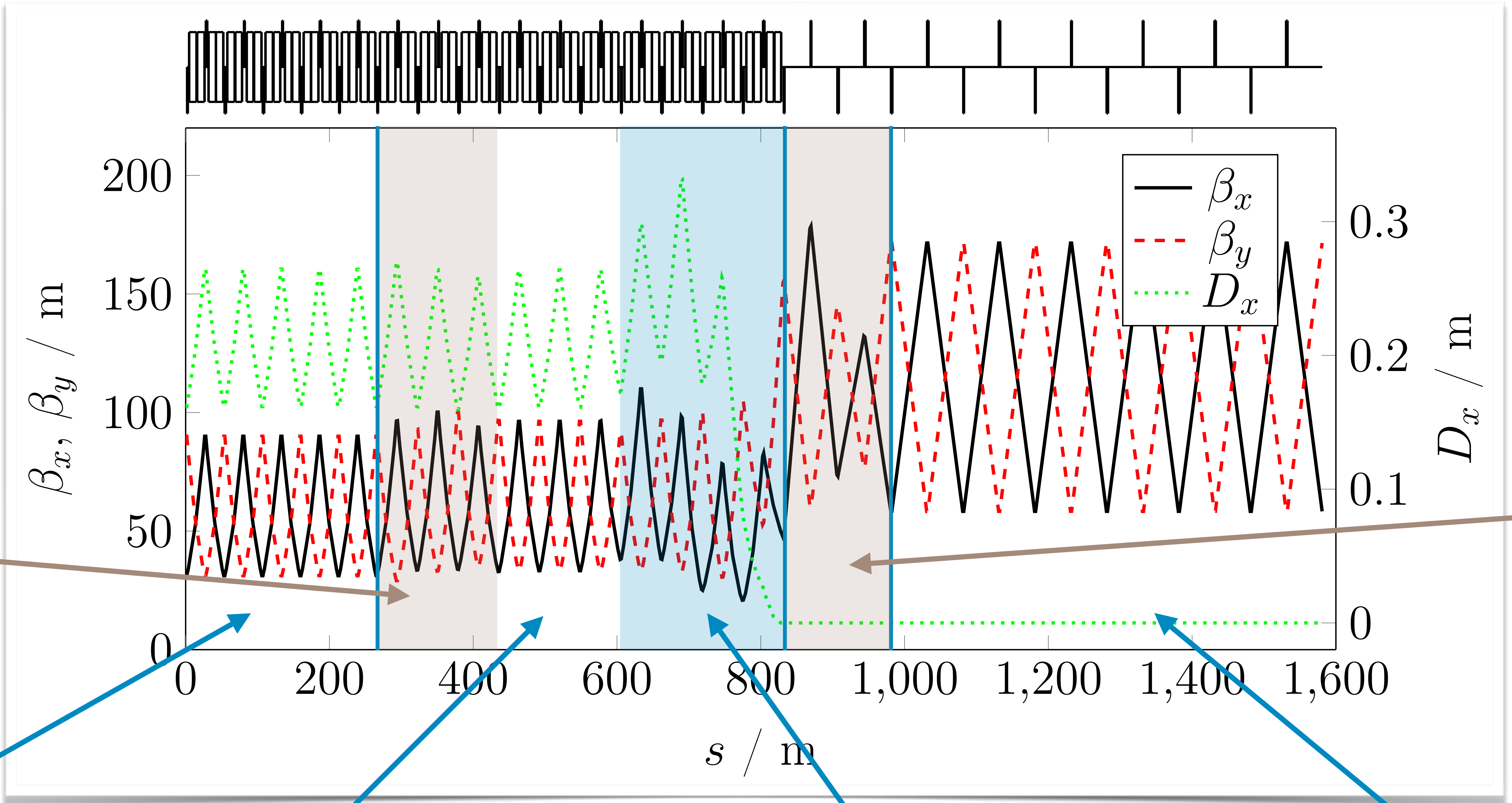
matching section
(5 quadrupoles)

- switch to $\mu = 60°$



half-bend DS needs to be supported by quadrupoles

Example: FCC-ee top-up booster synchrotron



Matching section
6 quadrupoles

Matching section
4 quadrupoles
L_{cell} = 75 m

16 km arc
L_{cell} = 54 m
rho = 13.15 km

FCC-hh dispersion suppressor section
L_{cell} = 56.6 m
rho = 15.06 km

Dispersion suppressor
8 quadrupoles
L_{cell} = 56.6 m
rho = 15.06 km

Straight section for RF installation
L_{cell} = 100 m
(doesn't contribute to ϵ_x)

Lattice design for large rings

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(Phil Bryant)

Large rings, such as the LHC, often have a **basic FODO cell** in the arcs.

The overall ring has an n-fold symmetry containing the **n-arcs and n straight regions** in which the physics experiments are mounted.

Between the arc and the straight region there is the so-called **dispersion suppressor** that brings the dispersion function to zero in the straight region in a controlled way. There are several schemes for dispersion suppressors.

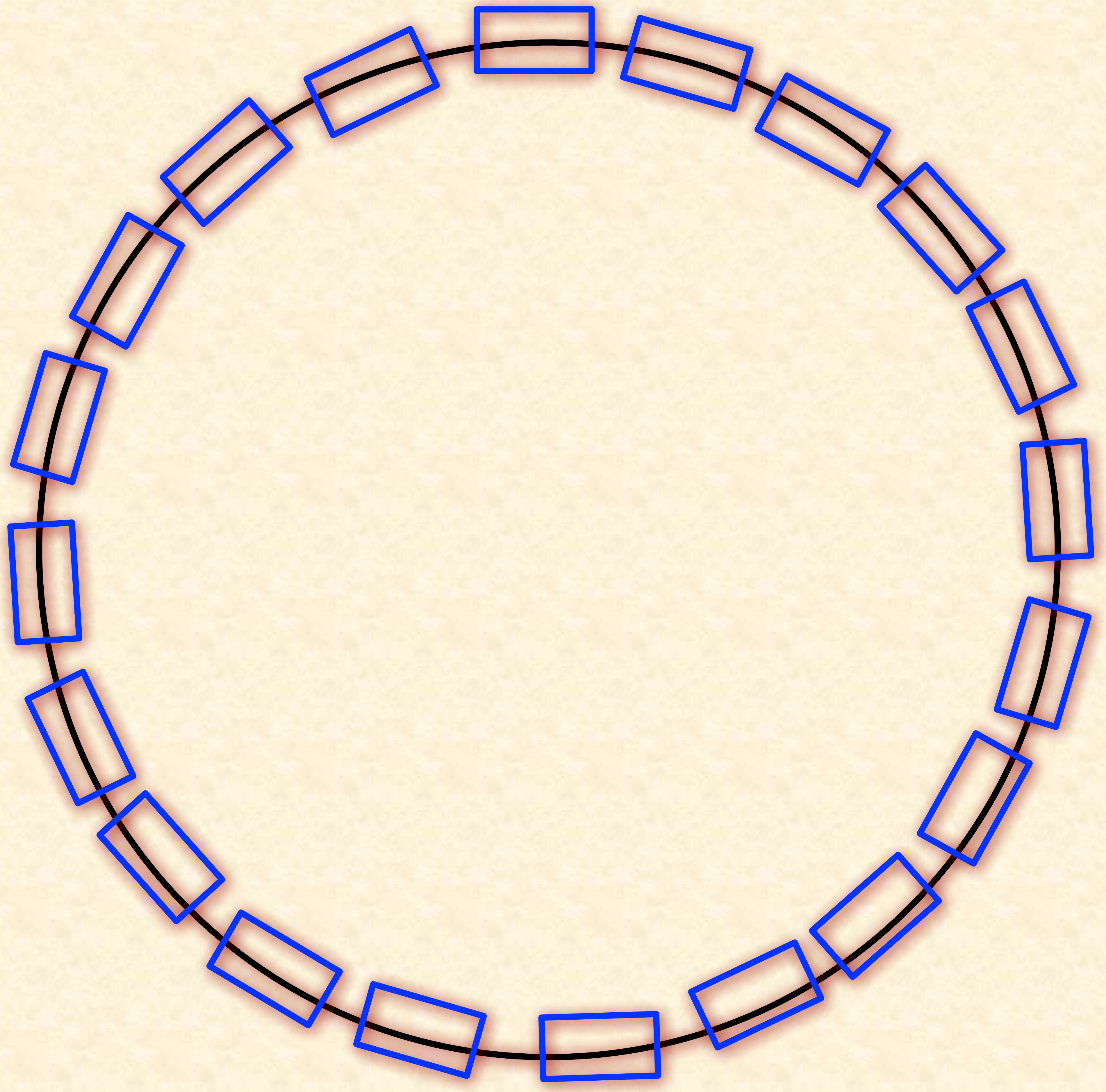
The straight regions contain the **injection and extraction and the RF cavities**, which, in an electron machine like LEP, can occupy hundreds of metres.
A dispersion-free straight region is also needed for the **low-β insertion**.

The logical path to Accelerator Design

1.) determine particle type & energy

2.) beam rigidity —> calculate integrated dipole field

magnet technology
dipole length & number
size of the ring
arrangement of the dipoles in the ring



The logical path to Accelerator Design

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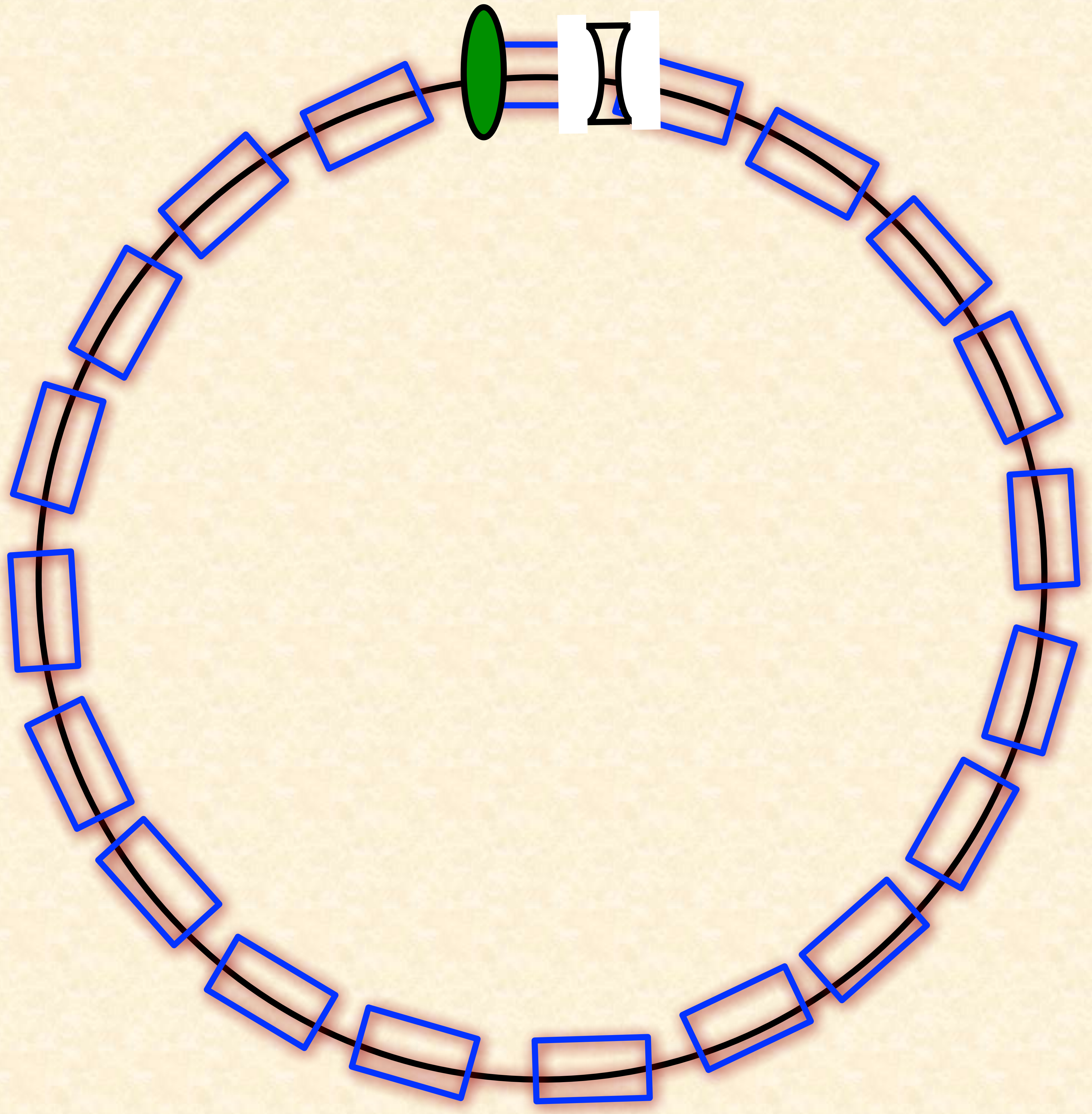
3.) determine the focusing structure of the basic cell
— FODO, DBA — etc. etc.

calculate the optics parameters of the basic cell
- beam dimension
- vacuum chamber
- magnet aperture & design
- tune



The logical path to Accelerator Design

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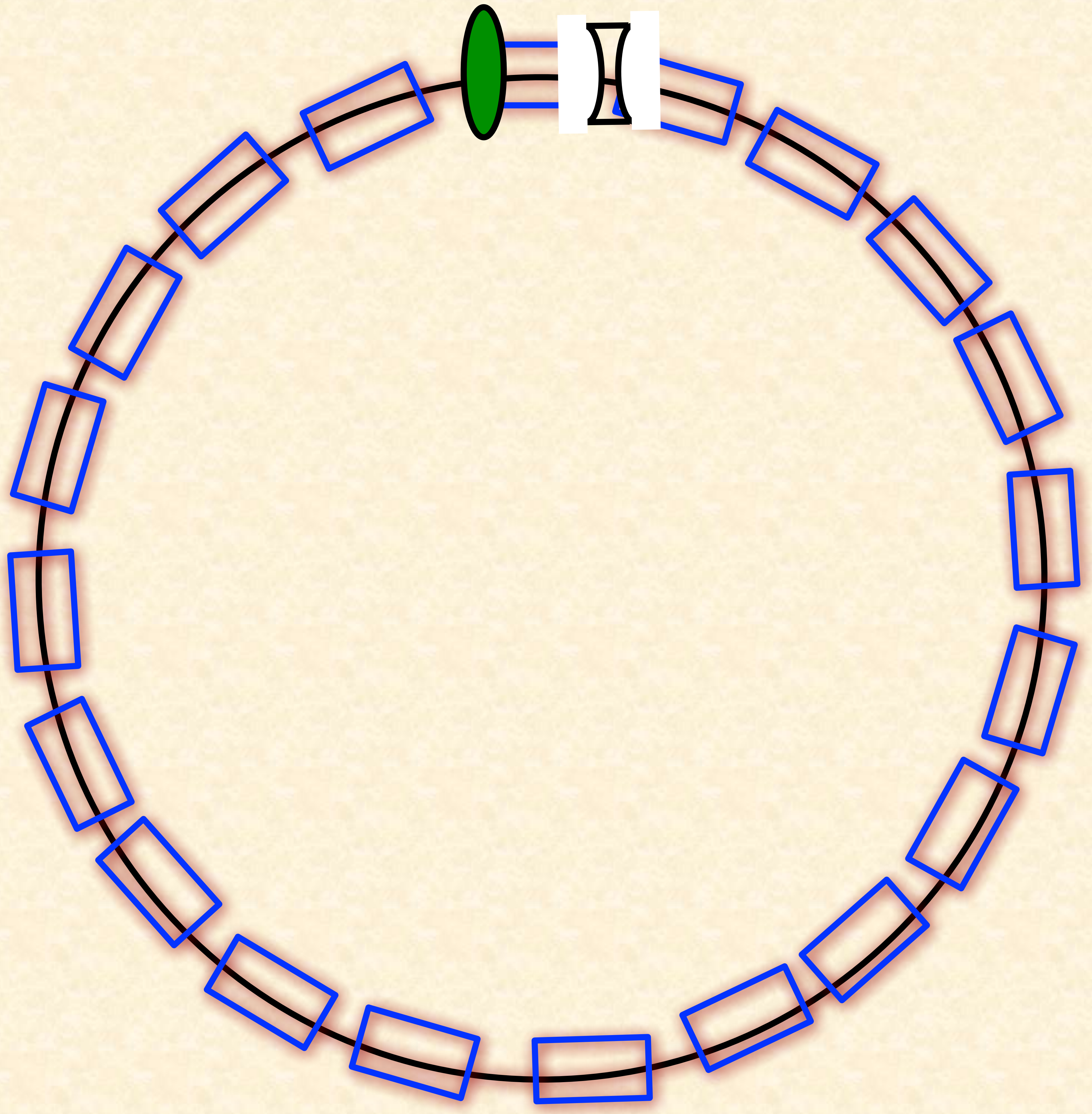
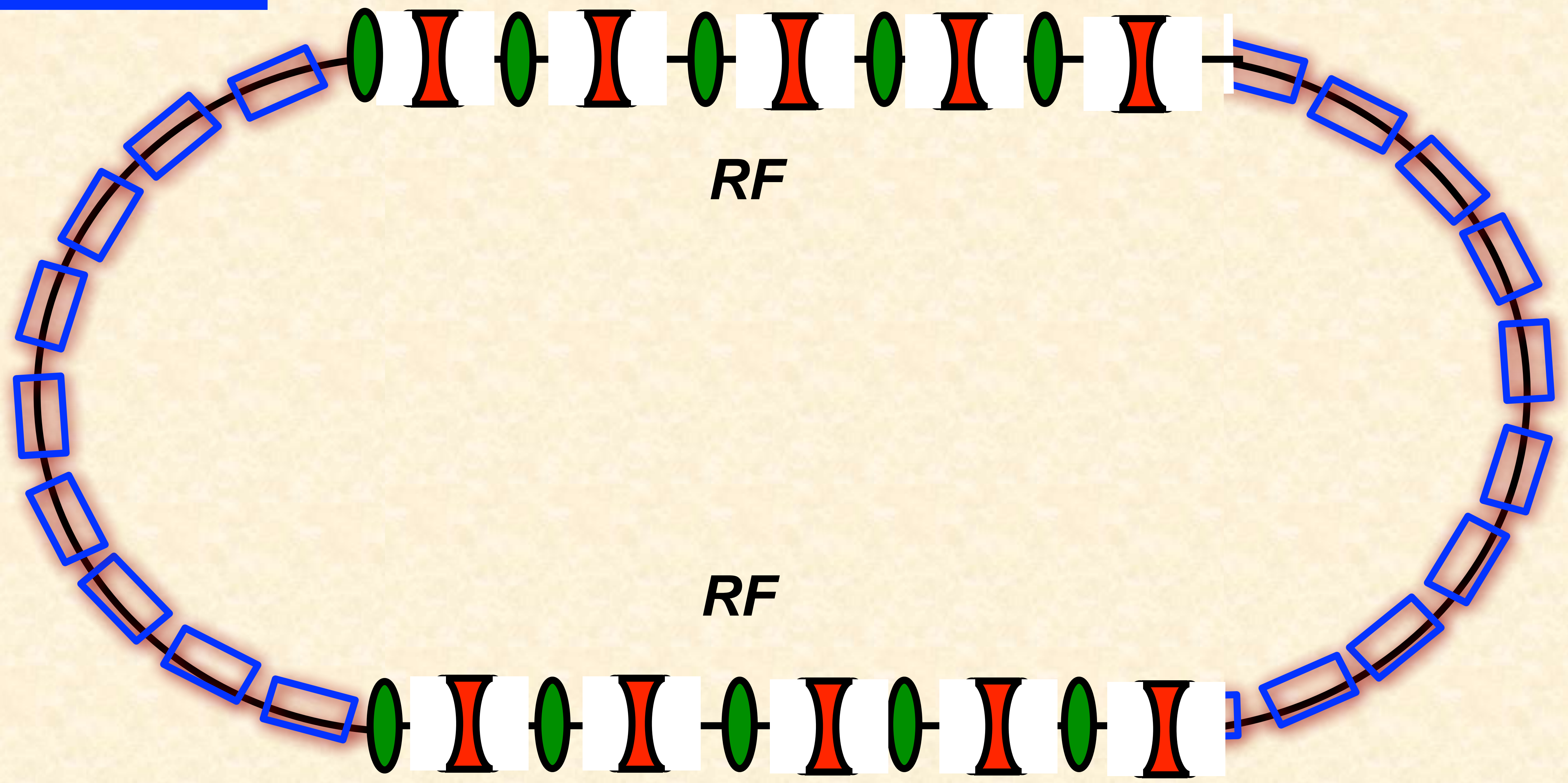
4.) Determine the radiation losses
Energy Loss per turn
Power loss Frequency
—> electrons radiate !!
—> protons do not !!

5.) Determine the parameters for the RF system
Frequency, overall Voltage,
space needed in the lattice
for the cavities



The logical path to Accelerator Design

6.) Open the lattice structure to install
- straight sections for the RF system
- optimise the phase advance per cell
- connect the straight sections to the arc lattice with dispersion suppressors
- choose which type fits best
- add eventually a matching section



The logical path to Accelerator Design

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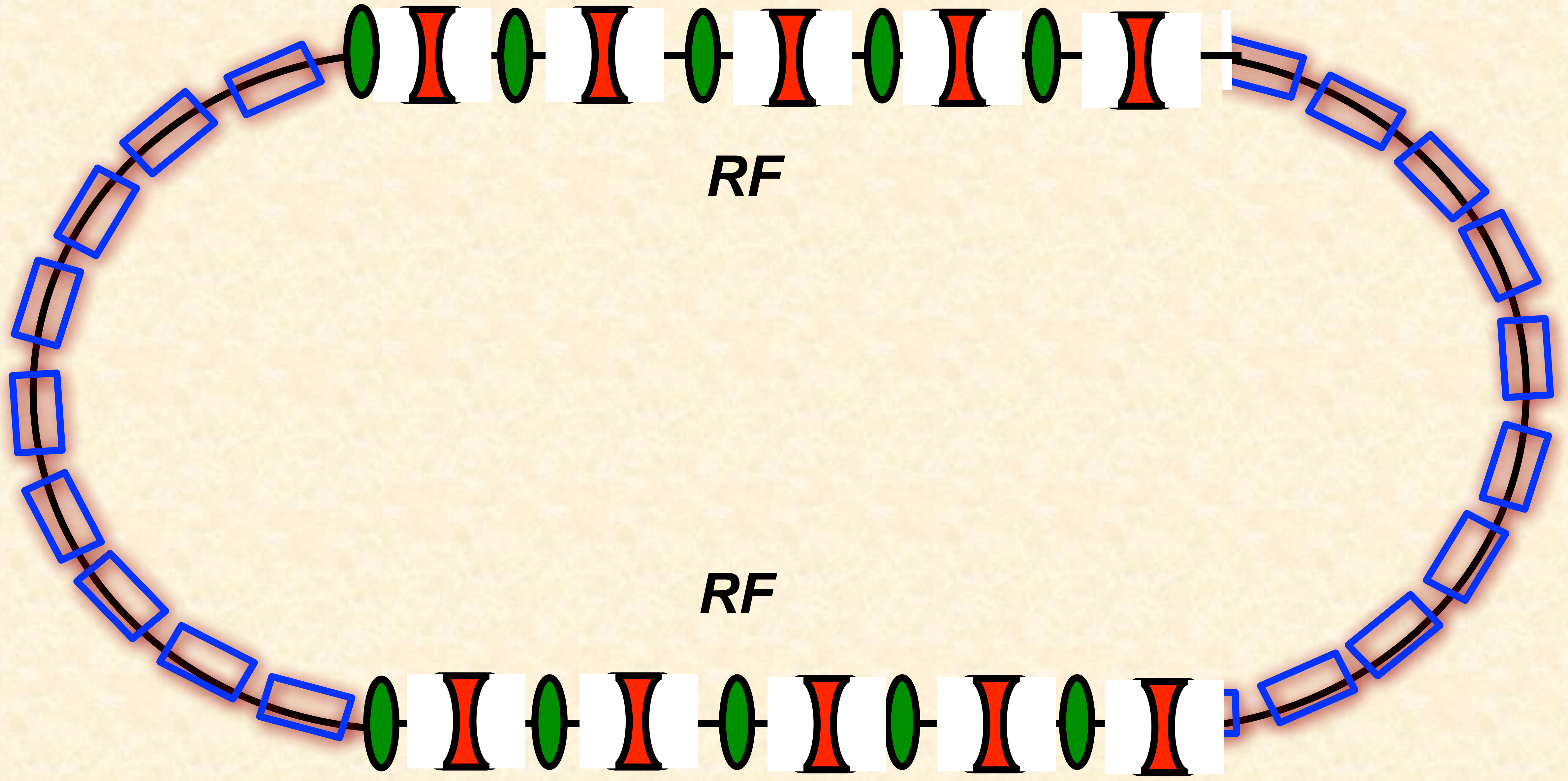
6.) Open the lattice structure to install
straight sections for the RF system
optimise the phase advance per cell
connect the straight sections to the arc
lattice with dispersion suppressors
choose which type fits best
add eventually a matching section



The logical path to Accelerator Design

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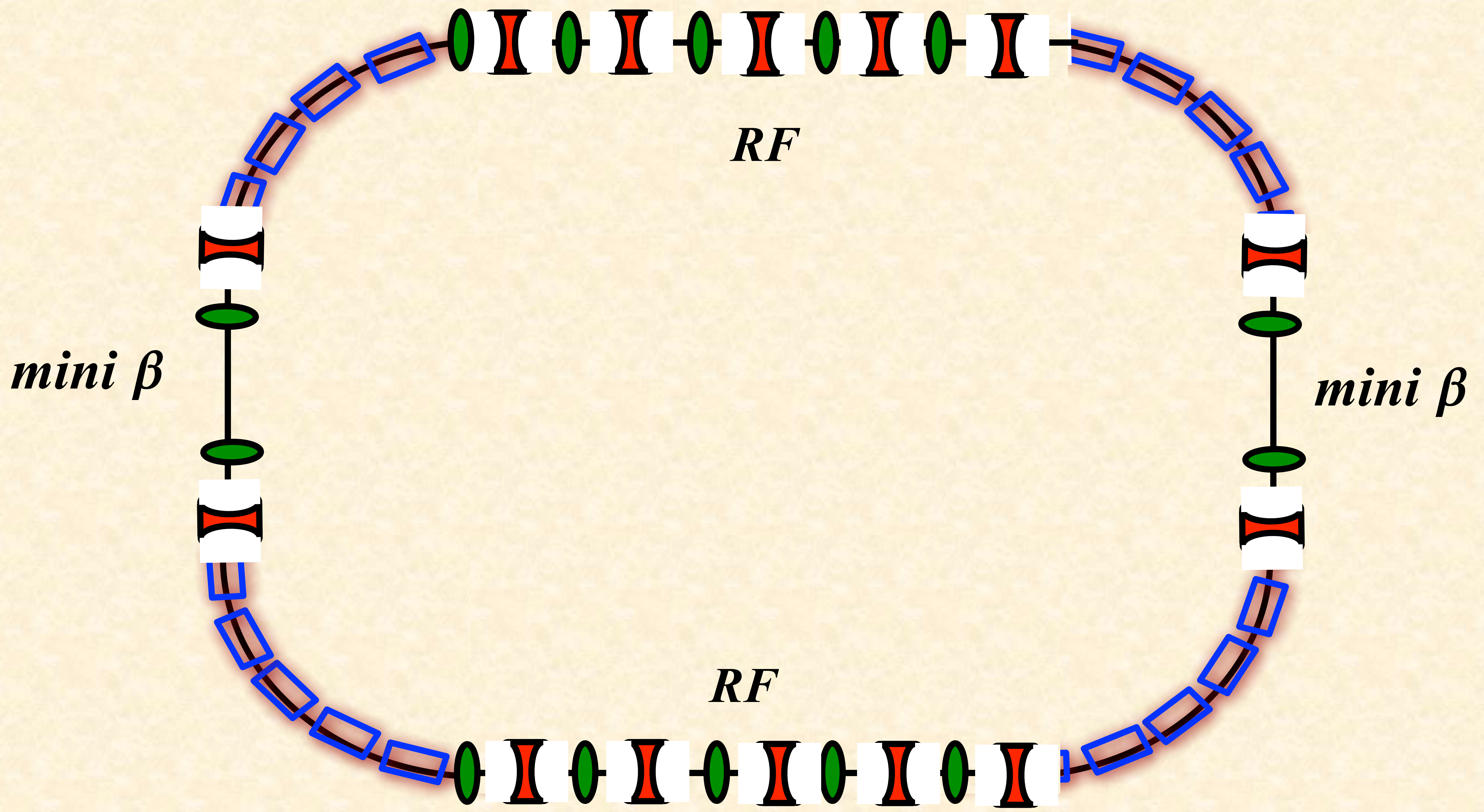
7.) Open the lattice structure to install
- a dispersion free straight section for the mini beta insertion
- define independent quadrupoles (four if D_x=0)
- connect the straight sections to the arc lattice with mini-beta quadrupoles and matching quadrupoles
- match to the desired β^*



… and then you just turn the key and run the machine.

Appendix

Appendix: Periodic solution of the Dispersion function:

$$x'' + x\left(\frac{1}{\rho^2} - k\right) = \frac{\Delta p}{p} \cdot \frac{1}{\rho}$$

general solution:

$$x(s) = x_h(s) + x_i(s)$$

$$\begin{cases} x_h''(s) + K(s) \cdot x_h(s) = 0 \\ x_i''(s) + K(s) \cdot x_i(s) = \frac{1}{\rho} \cdot \frac{\Delta p}{p} \end{cases}$$

Normalise with respect to Δp/p:

$$D(s) = \frac{x_i(s)}{\Delta p / p}$$

Dispersion function D(s)

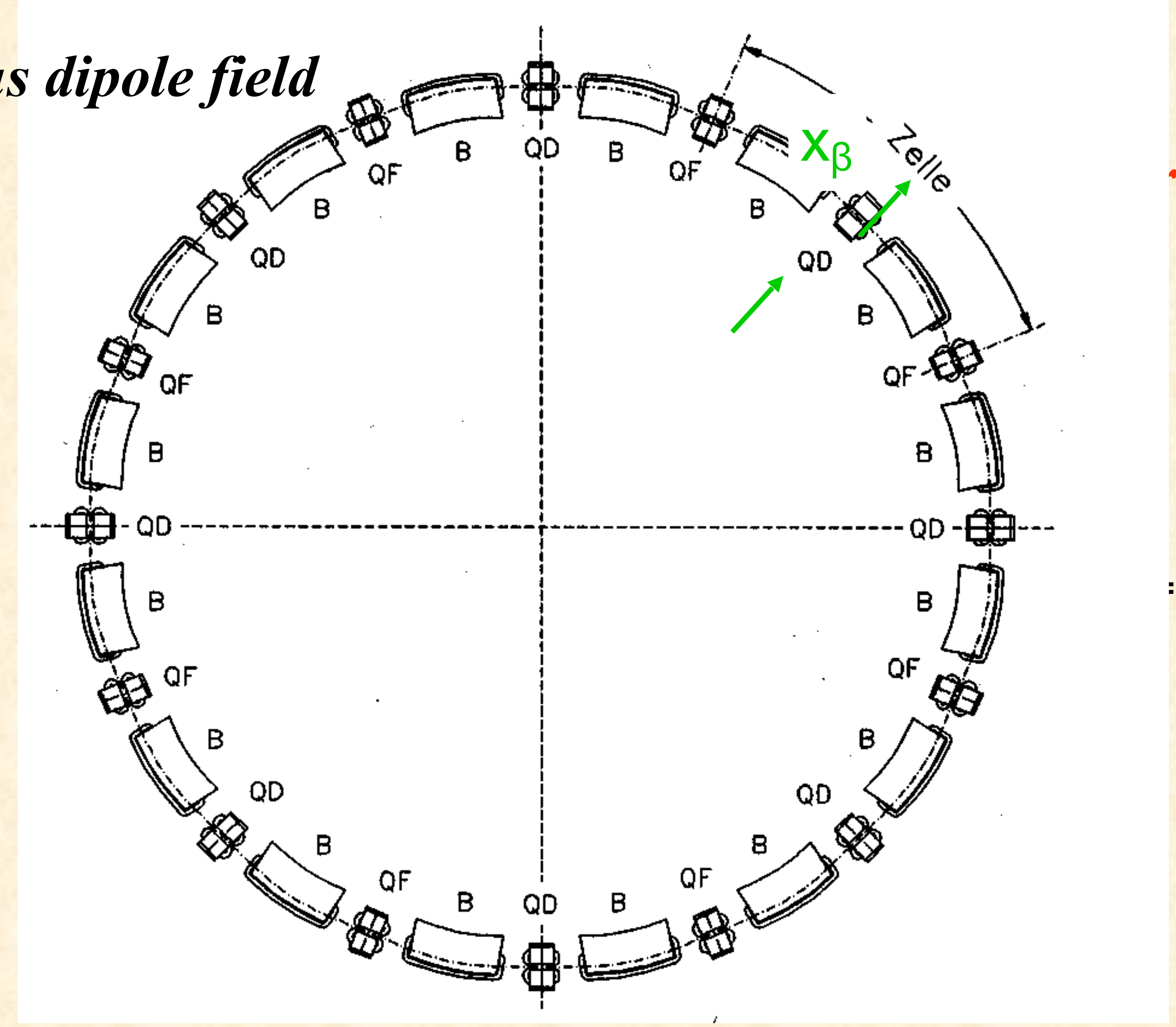
****** is that special orbit, an ideal particle would have for Δp/p = 1***

****** the orbit of any particle is the sum of the well known x_β and the dispersion***

****** as D(s) is just another orbit it will be subject to the focusing properties of the lattice***

Dispersion

Example: homogeneous dipole field



bit for Δp/p > 0

$$= D(s)\cdot\frac{\Delta p}{p}$$

Matrix formalism:

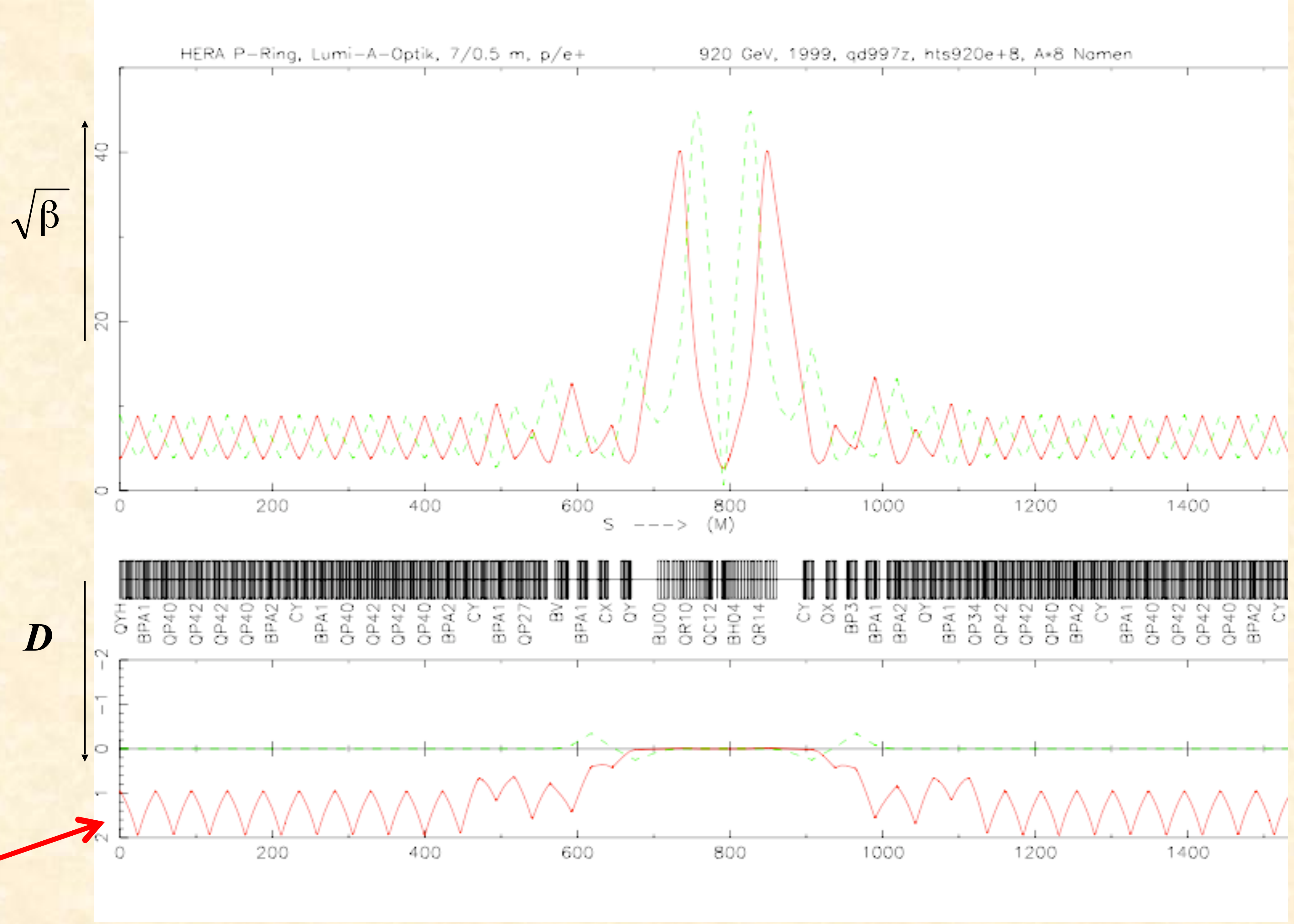
$$x(s) = x_\beta(s) + D(s)\cdot\frac{\Delta p}{p}$$

$$x(s) = C(s)\cdot x_0 + S(s)\cdot x_0' + D(s)\cdot\frac{\Delta p}{p}$$

$$\begin{pmatrix} x \\ x' \end{pmatrix}_s = \begin{pmatrix} C & S \\ C' & S' \end{pmatrix}\begin{pmatrix} x \\ x' \end{pmatrix}_0 + \frac{\Delta p}{p}\begin{pmatrix} D \\ D' \end{pmatrix}_0$$

or expressed as 3x3 matrix

$$\begin{pmatrix} x \\ x' \\ \Delta p/p \end{pmatrix}_s = \begin{pmatrix} C & S & D \\ C' & S' & D' \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} x \\ x' \\ \Delta p/p \end{pmatrix}_0$$



Example

$$x_\beta = 1 \dots 2\, mm$$
$$D(s) \approx 1 \dots 2\, m$$
$$\Delta p/p \approx 1 \cdot 10^{-3}$$

Amplitude of Orbit oscillation
contribution due to Dispersion ≈ beam size
—> Dispersion must vanish at the collision point

!

Calculate D, D´: ... takes a couple of sunny Sunday evenings !

$$D(s) = S(s)\int_{s0}^{s1} \frac{1}{\rho} C(\tilde{s})\,d\tilde{s} - C(s)\int_{s0}^{s1} \frac{1}{\rho} S(\tilde{s})\,d\tilde{s}$$

Dispersion:

Example: ***Drift***

$$M_{Drift} = \begin{pmatrix} 1 & l \\ 0 & 1 \end{pmatrix}$$

$$M_{Drift} = \begin{pmatrix} 1 & l & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$D(s) = S(s)\underbrace{\int_{s0}^{s1} \frac{1}{\rho} C(\tilde{s})\,d\tilde{s}}_{=0} - C(s)\underbrace{\int_{s0}^{s1} \frac{1}{\rho} S(\tilde{s})\,d\tilde{s}}_{=0}$$

Example: ***Dispersion in a Sector Dipole Magnet***

Remember: ***Matrix of a magnetic element***

$$M_{foc} = \begin{pmatrix} \cos(\sqrt{|K|}l) & \frac{1}{\sqrt{|K|}}\sin(\sqrt{|K|}l) \\ -\sqrt{|K|}\sin(\sqrt{|K|}l) & \cos(\sqrt{|K|}l) \end{pmatrix}$$

in general: $K = k - \frac{1}{\rho^2}$

… but in a dipole, as $k = 0$ …

$$M_{foc} = \begin{pmatrix} C & S \\ C' & S' \end{pmatrix} = \begin{pmatrix} \cos\frac{l}{\rho} & \rho\sin\frac{l}{\rho} \\ -\frac{1}{\rho}\sin\frac{l}{\rho} & \cos\frac{l}{\rho} \end{pmatrix}$$

calculate the „D" elements for the marix a Sector Dipole Magnet

$$D(s) = S(s)\int_{s0}^{s1}\frac{1}{\rho}C(\tilde{s})d\tilde{s} - C(s)\int_{s0}^{s1}\frac{1}{\rho}S(\tilde{s})d\tilde{s}$$

$$D(s) = (\rho\sin\frac{l}{\rho}) * \frac{1}{\rho} * (\rho\sin\frac{l}{\rho}) - \cos\frac{l}{\rho} * \frac{1}{\rho} * \rho\cdot(-\cos\frac{l}{\rho} + 1) * \rho$$

$$D(s) = \rho\sin^2\frac{l}{\rho} + \rho\cos\frac{l}{\rho} * (\cos\frac{l}{\rho} - 1)$$

$$\boldsymbol{D(s) = \rho\cdot(1-\cos\frac{l}{\rho})} \quad , \quad \boldsymbol{D'(s) = \sin\frac{l}{\rho}}$$

Dispersion elements in a sector dipole magnet

$$M_{dipole} = \begin{pmatrix} \cos\frac{l}{\rho} & \rho\sin\frac{l}{\rho} & D \\ -\frac{1}{\rho}\sin\frac{l}{\rho} & \cos\frac{l}{\rho} & D' \\ 0 & 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} x \\ x' \\ \Delta p/p \end{pmatrix}_{s2} = \begin{pmatrix} \cos\frac{l}{\rho} & \rho\sin\frac{l}{\rho} & \rho * (1-\cos\frac{l}{\rho}) \\ -\frac{1}{\rho}\sin\frac{l}{\rho} & \cos\frac{l}{\rho} & \sin\frac{l}{\rho} \\ 0 & 0 & 1 \end{pmatrix} * \begin{pmatrix} x \\ x' \\ \Delta p/p \end{pmatrix}_{s1}$$

Nota bene: ***even an ideal particle with x = x´= 0 will start to oscillate if it passes a dipole magnet and has a momentum error Δp/p***
A dispersion trajectory will obey the same focusing forces (i.e. will be transferred by the same matrices) as a normal betatron oscillation

Periodic Dispersion: η, η'

- The equation:

$$D(s) = S(s) \int_0^s \frac{1}{\rho(t)} C(t)\,\mathrm{d}t - C(s) \int_0^s \frac{1}{\rho(t)} S(t)\,\mathrm{d}t$$

  allows to compute **the dispersion inside a magnet**, which does not depend on the dispersion that might have been generated by the upstreams magnets.

- At the exit of a magnet of length L_m the dispersion reaches the value $D(L_\mathrm{m})$

- The dispersion (also indicated as η, with its derivative η') propagates from there, through the rest of the machine, just like any other particle:

$$\begin{pmatrix} \eta \\ \eta' \\ 1 \end{pmatrix}_s = \begin{pmatrix} C & S & D \\ C' & S' & D' \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \eta \\ \eta' \\ 1 \end{pmatrix}_0$$

Periodic Dispersion: η, η'

In a periodic lattice, also the dispersion must be periodic.

That is, for $\begin{pmatrix} \eta \\ \eta' \\ 1 \end{pmatrix}$ we need to have:

$$\begin{pmatrix} \eta \\ \eta' \\ 1 \end{pmatrix} = \begin{pmatrix} C & S & D \\ C' & S' & D' \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \eta \\ \eta\prime \\ 1 \end{pmatrix}$$

Let's rewrite this in 2×2 form:

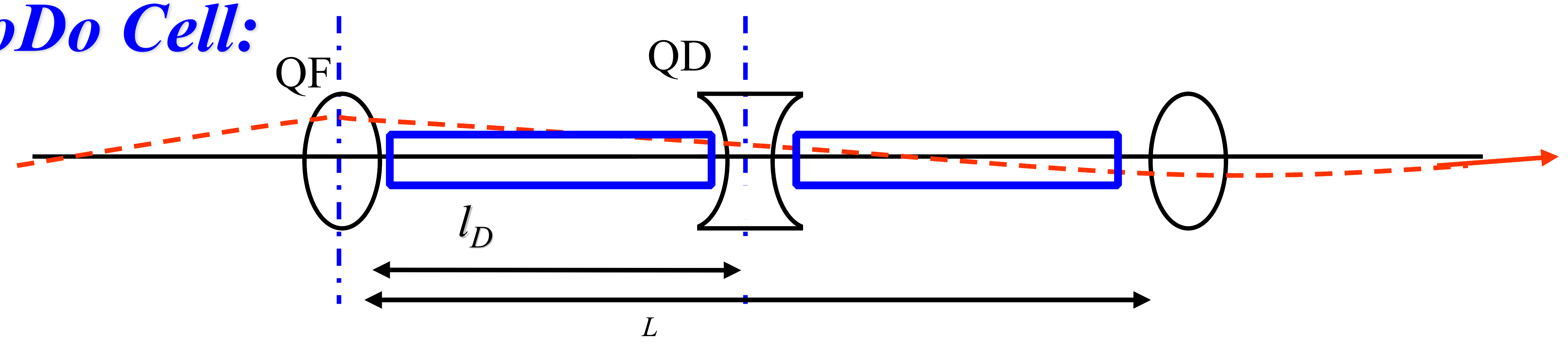
$$\begin{pmatrix} \eta \\ \eta' \end{pmatrix} = \begin{pmatrix} C & S \\ C' & S' \end{pmatrix} \begin{pmatrix} \eta \\ \eta' \end{pmatrix} + \begin{pmatrix} D \\ D' \end{pmatrix}$$

$$\begin{pmatrix} 1-C & -S \\ -C' & 1-S' \end{pmatrix} \begin{pmatrix} \eta \\ \eta' \end{pmatrix} = \begin{pmatrix} D \\ D' \end{pmatrix}$$

The solution is:

$$\begin{pmatrix} \eta \\ \eta' \end{pmatrix} = \frac{1}{(1-C)(1-S') - C'S} \begin{pmatrix} 1-S' & S \\ C' & 1-C \end{pmatrix} \begin{pmatrix} D \\ D' \end{pmatrix}$$

Appendix: Dispersion in a FoDo Cell:



!! we have now introduced dipole magnets in the FoDo:

—> we still neglect the weak focusing contribution $1/\rho^2$

—> but take into account $1/\rho$ for the dispersion effect

assume: length of the dipole = l_D

Calculate the matrix of the FoDo half cell in thin lens approximation:

in analogy to the derivations of $\hat{\beta}, \check{\beta}$

**** thin lens approximation:*** $f = \dfrac{1}{k\ell_Q} >> \ell_Q$

**** length of quad negligible*** $\ell_Q \approx 0, \rightarrow \ell_D = \dfrac{1}{2}L$

**** start at half quadrupole*** $\dfrac{1}{\tilde{f}} = \dfrac{1}{2f}$

Matrix of the half cell

$$M_{HalfCell} = M_{\frac{QD}{2}} * M_B * M_{\frac{QF}{2}}$$

$$M_{Half\ Cell} = \begin{pmatrix} 1 & 0 \\ \frac{1}{\tilde{f}} & 1 \end{pmatrix} * \begin{pmatrix} 1 & \ell \\ 0 & 1 \end{pmatrix} * \begin{pmatrix} 1 & 0 \\ \frac{-1}{\tilde{f}} & 1 \end{pmatrix}$$

$$M_{Half\ Cell} = \begin{pmatrix} C & S \\ C' & S' \end{pmatrix} = \begin{pmatrix} 1 - \frac{\ell}{\tilde{f}} & \ell \\ \frac{-\ell}{\tilde{f}^2} & 1 + \frac{\ell}{\tilde{f}} \end{pmatrix}$$

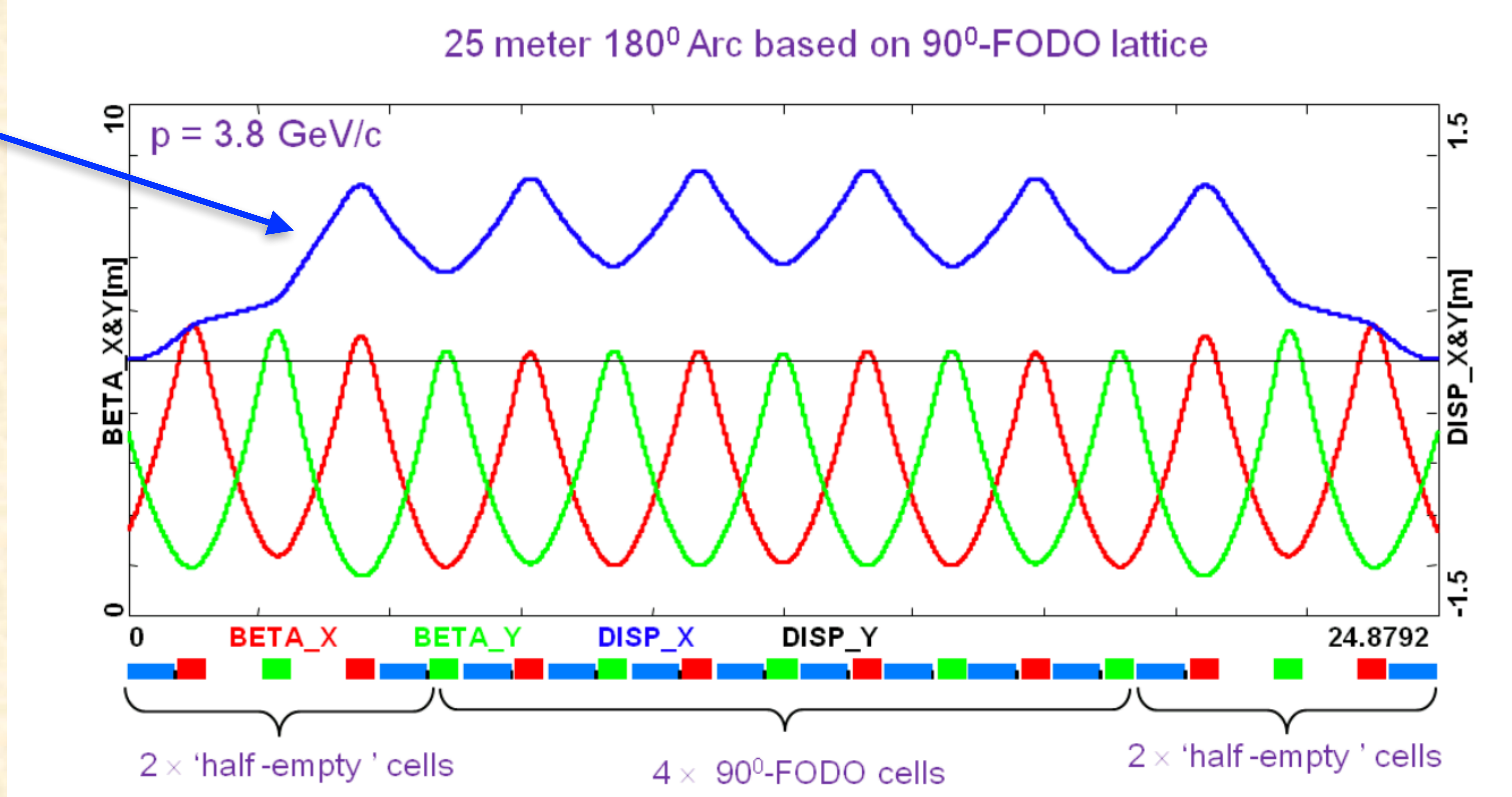
calculate the dispersion terms D, D´ from the matrix elements

$$\boldsymbol{D(s) = S(s) * \int \frac{1}{\rho(\tilde{s})} C(\tilde{s}) d\tilde{s} - C(s) * \int \frac{1}{\rho(\tilde{s})} S(\tilde{s}) d\tilde{s}}$$

$$D(\ell) = \underbrace{\ell}_{S(s)} * \frac{1}{\rho} * \int_0^\ell \underbrace{\left(1 - \frac{s}{\tilde{f}}\right)}_{C(s)} ds - \underbrace{\left(1 - \frac{\ell}{\tilde{f}}\right)}_{C(s)} * \frac{1}{\rho} * \int_0^\ell \underbrace{s \quad ds}_{S(s)}$$

$$D(\ell) = \frac{\ell}{\rho}\left(\ell - \frac{\ell^2}{2\tilde{f}}\right) - \left(1 - \frac{\ell}{\tilde{f}}\right) * \frac{1}{\rho} * \frac{\ell^2}{2} = \frac{\ell^2}{\rho} - \frac{\ell^3}{2\tilde{f}\rho} - \frac{\ell^2}{2\rho} + \frac{\ell^3}{2\tilde{f}\rho}$$

$$D(\ell) = \frac{\ell^2}{2\rho}$$



in full analogy on derives for D´:

$$D'(s) = \frac{\ell}{\rho}\left(1 + \frac{\ell}{2\tilde{f}}\right)$$

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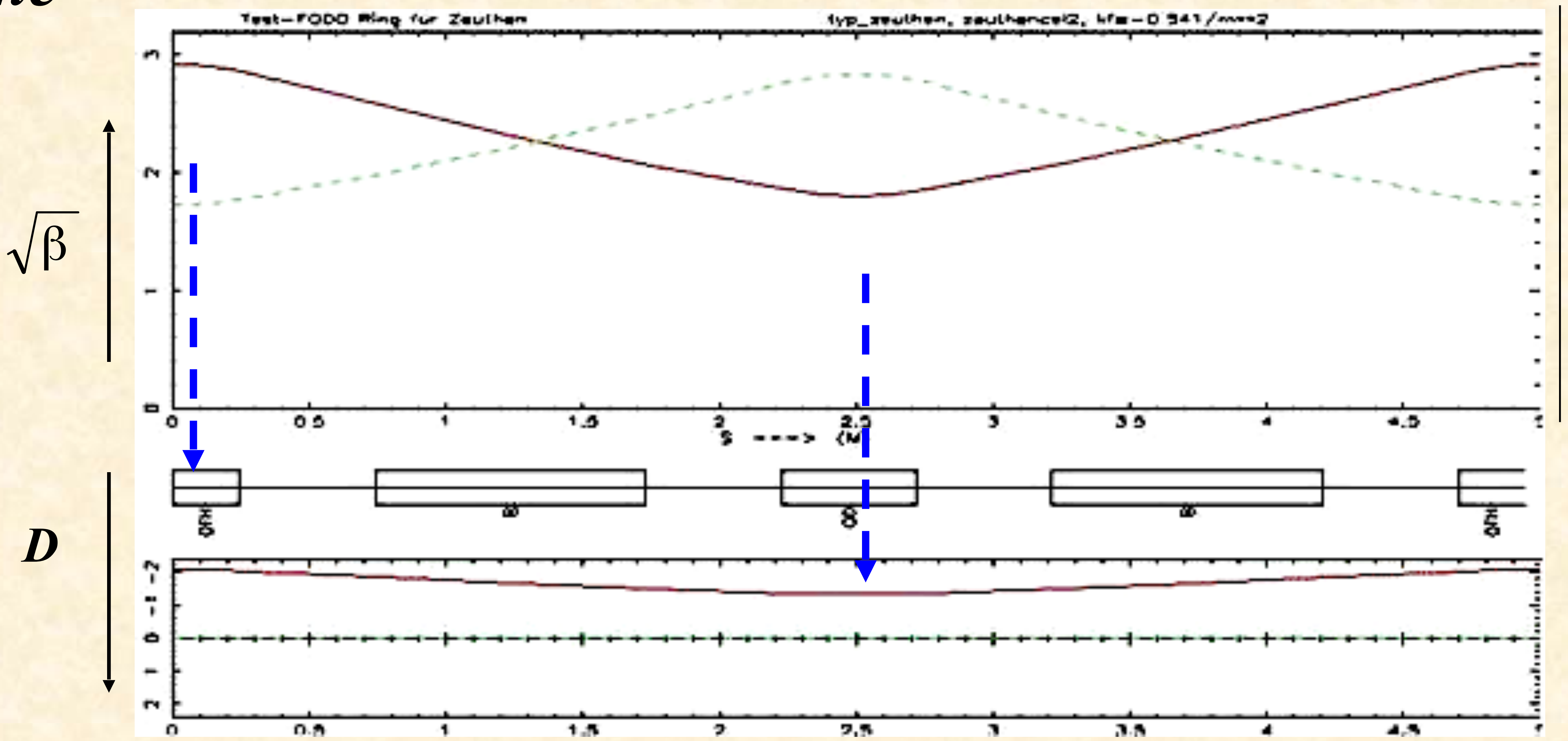
and we get the complete matrix including the dispersion terms D, D´

$$M_{halfCell} = \begin{pmatrix} C & S & D \\ C' & S' & D' \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1-\dfrac{\ell}{\tilde{f}} & \ell & \dfrac{\ell^2}{2\rho} \\ \dfrac{-\ell}{\tilde{f}^2} & 1+\dfrac{\ell}{\tilde{f}} & \dfrac{\ell}{\rho}\left(1+\dfrac{\ell}{2\tilde{f}}\right) \\ 0 & 0 & 1 \end{pmatrix}$$

boundary conditions for the transfer from the center of the foc. to the center of the defoc. quadrupole

$$\begin{pmatrix} \check{D} \\ 0 \\ 1 \end{pmatrix} = M_{1/2} * \begin{pmatrix} \hat{D} \\ 0 \\ 1 \end{pmatrix}$$



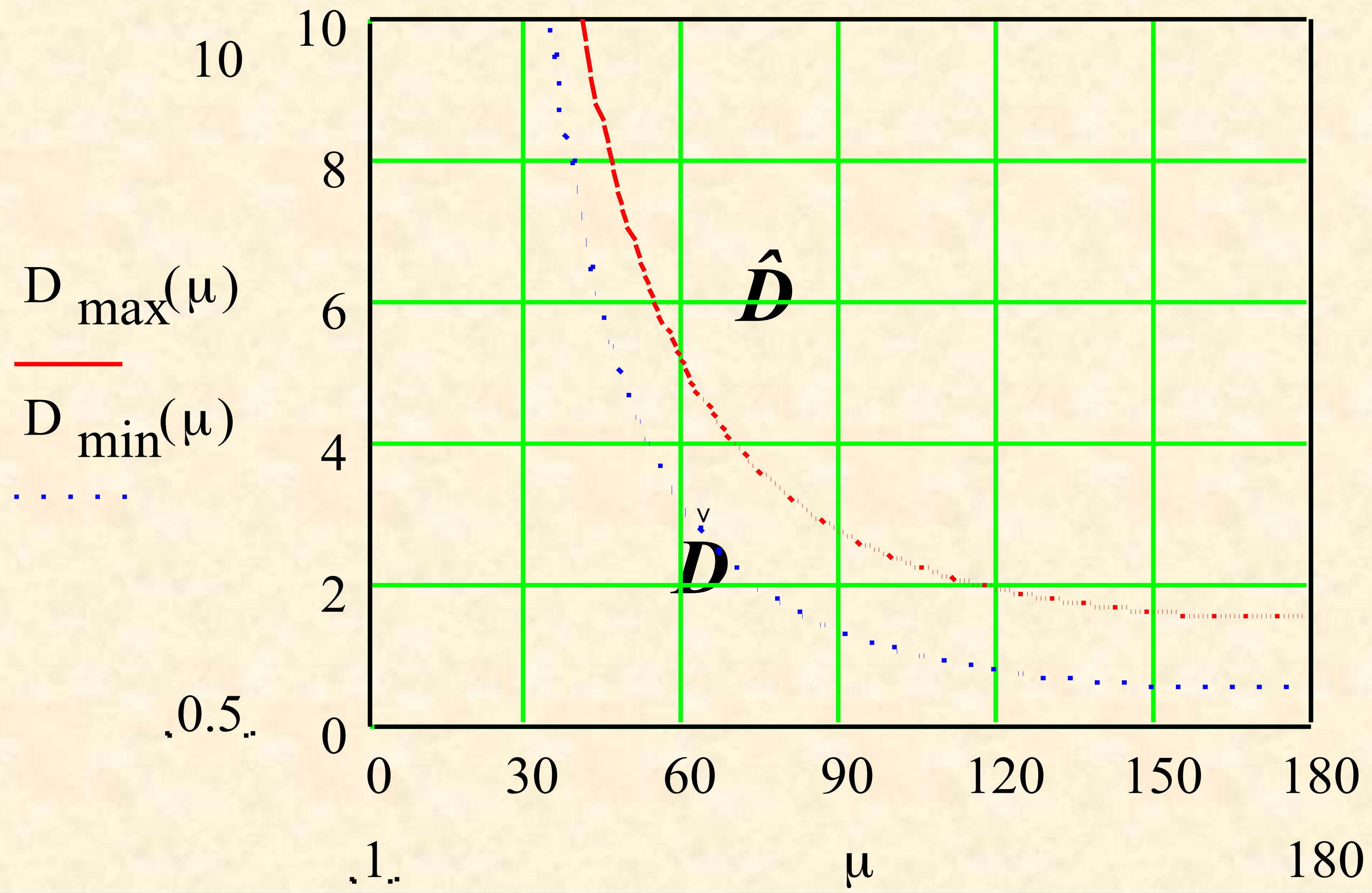
Dispersion in a FoDo Cell

$$\rightarrow \check{D} = \hat{D}\left(1-\frac{\ell}{\tilde{f}}\right)+\frac{\ell^2}{2\rho}$$

$$\rightarrow 0 = -\frac{\ell}{\tilde{f}^2} * \hat{D} + \frac{\ell}{\rho}\left(1+\frac{\ell}{2\tilde{f}}\right)$$

$$D_{max} = \frac{l^2}{r}\cdot\frac{1+\frac{1}{2}\sin\frac{\psi_{cell}}{2}}{\sin^2\frac{\psi_{cell}}{2}} \qquad D_{min} = \frac{l^2}{r}\cdot\frac{1-\frac{1}{2}\sin\frac{\psi_{cell}}{2}}{\sin^2\frac{\psi_{cell}}{2}}$$

where ψ_{cell} denotes the phase
advance of the full cell
and $l/f = \sin(\psi/2)$



Nota bene:

! small dispersion needs strong focusing
→ large phase advance

!! ↔ there is an optimum phase for small β

!!! ...do you remember the stability criterion?
½ trace = cos ψ ↔ ψ < 180°

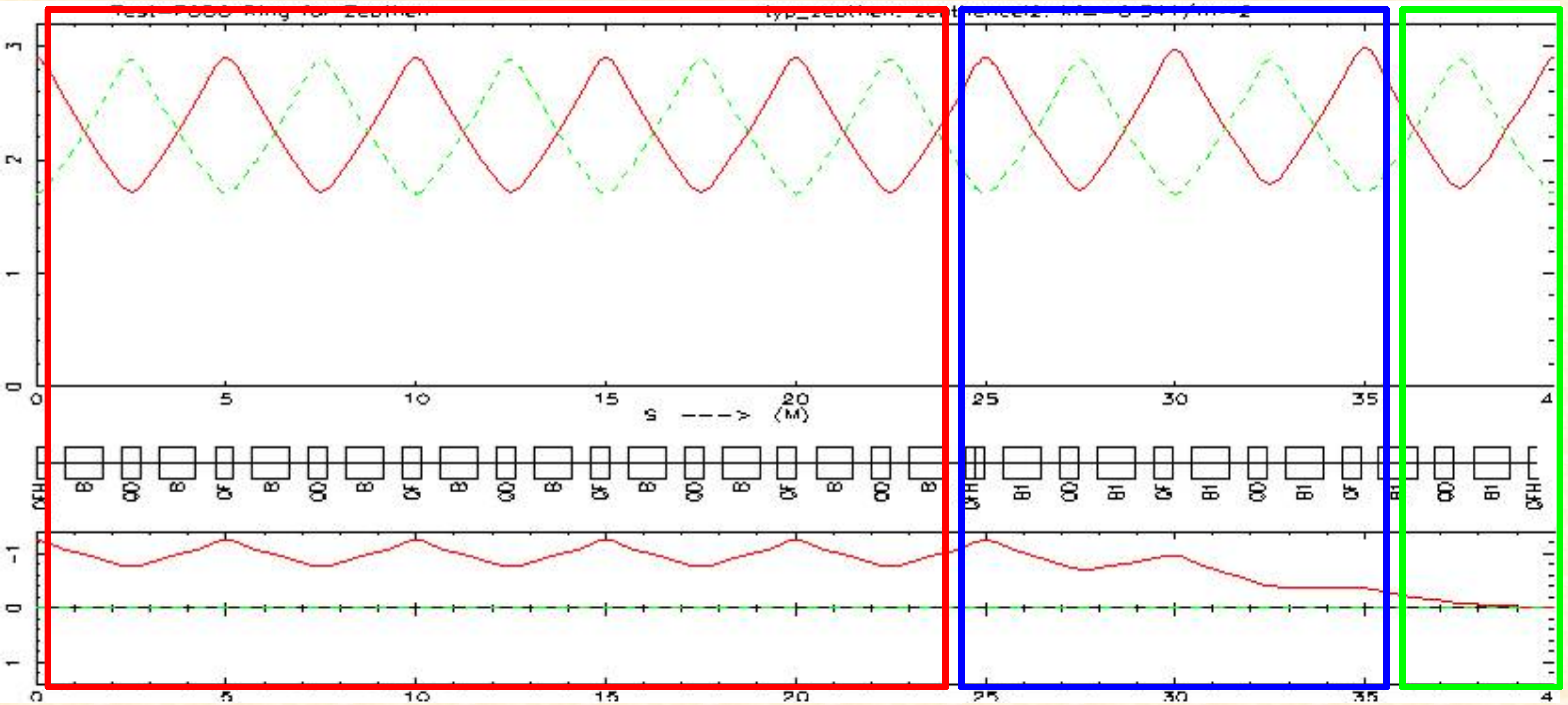
!!!! … life is not easy

Appendix: Dispersion Suppressors

... the calculation of the half bend scheme in full detail (for purists only)

1.) the lattice is split into 3 parts: *(Gallia divisa est in partes tres)*

* periodic solution of the arc	periodic β, periodic dispersion D
* section of the dispersion suppressor	periodic β, dispersion vanishes
* FoDo cells without dispersion	periodic β, D = D´ = 0



2.) calculate the dispersion D in the periodic part of the lattice

transfer matrix of a periodic cell:

$$M_{0\to S} = \begin{pmatrix} \sqrt{\frac{\beta_S}{\beta_0}}(\cos\phi + \alpha_0 \sin\phi) & \sqrt{\beta_S\beta_0}\,\sin\phi \\ \frac{(\alpha_0-\alpha_S)\cos\phi - (1+\alpha_0\alpha_S)\sin\phi}{\sqrt{\beta_S\beta_0}} & \sqrt{\frac{\beta_S}{\beta_0}}(\cos\phi - \alpha_S\sin\phi) \end{pmatrix}$$

for the transformation from one symmetriy point to the next (i.e. one cell) we have:
Φ_C = phase advance of the cell, $\alpha = 0$ at a symmetry point. The index "*c*" refers to the periodic solution of one cell.

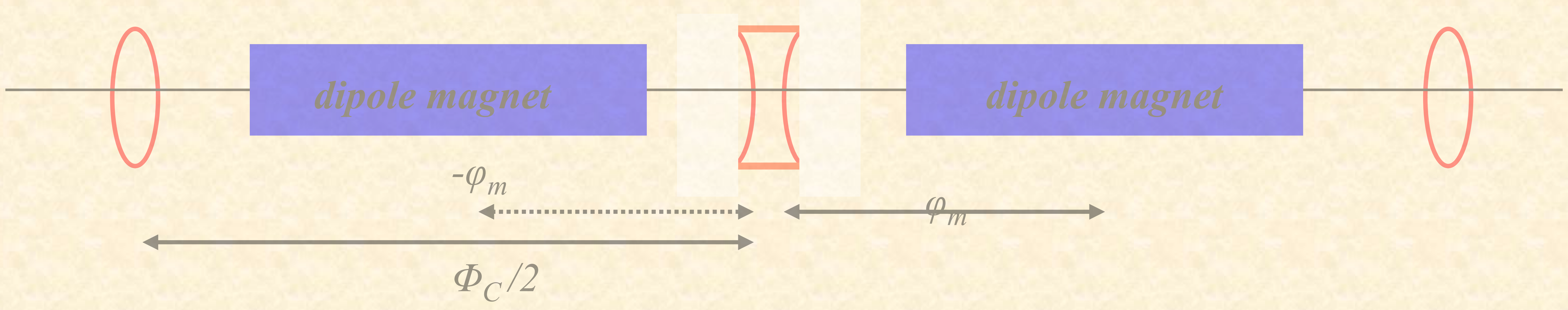
$$M_{Cell} = \begin{pmatrix} C & S & D \\ C' & S' & D' \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} \cos\Phi_C & \beta_C \sin\Phi_C & D(l) \\ \frac{-1}{\beta_C}\sin\Phi_C & \cos\Phi_C & D'(l) \\ 0 & 0 & 1 \end{pmatrix}$$

The matrix elements D and D‘ are given by the C and S elements in the usual way:

$$D(l) = S(l) * \int_0^l \frac{1}{\rho(\tilde{s})} C(\tilde{s})\,d\tilde{s} - C(l) * \int_0^l \frac{1}{\rho(\tilde{s})} S(\tilde{s})\,d\tilde{s}$$

$$D'(l) = S'(l) * \int_0^l \frac{1}{\rho(\tilde{s})} C(\tilde{s})\,d\tilde{s} - C'(l) * \int_0^l \frac{1}{\rho(\tilde{s})} S(\tilde{s})\,d\tilde{s}$$

here the values C(l) and S(l) refer to the symmetry point of the cell (middle of the quadrupole) and the integral is to be taken over the dipole magnet where $\rho \neq 0$. For ρ = const the integral over C(s) and S(s) is approximated by the values in the middle of the dipole magnet.



Transformation of C(s) from the symmetry point to the center of the dipole:

$$C_m = \sqrt{\frac{\beta_m}{\beta_C}} \cos\Delta\Phi = \sqrt{\frac{\beta_m}{\beta_C}} \cos\left(\frac{\Phi_C}{2} \pm \varphi_m\right) \qquad S_m = \beta_m \beta_C \sin\left(\frac{\Phi_C}{2} \pm \varphi_m\right)$$

where β_C is the periodic β function at the beginning and end of the cell, β_m its value at the middle of the dipole and φ_m the phase advance from the quadrupole lens to the dipole center.

Now we can solve the intergal for D and D':

$$D(l) = S(l) * \int_0^l \frac{1}{\rho(\tilde{s})} C(\tilde{s})\, d\tilde{s} - C(l) * \int_0^l \frac{1}{\rho(\tilde{s})} S(\tilde{s})\, d\tilde{s}$$

$$D(l) = \beta_C \sin\Phi_C * \frac{L}{\rho} * \sqrt{\frac{\beta_m}{\beta_C}} * \cos\left(\frac{\Phi_C}{2} \pm \varphi_m\right) - \cos\Phi_C * \frac{L}{\rho} \sqrt{\beta_m \beta_C} * \sin\left(\frac{\Phi_C}{2} \pm \varphi_m\right)$$

$$D(l) = \delta\sqrt{\beta_m\beta_C}\left\{\sin\Phi_C\left[\cos(\frac{\Phi_C}{2}+\varphi_m)+\cos(\frac{\Phi_C}{2}-\varphi_m)\right]-\right.$$

$$\left.-\cos\Phi_C\left[\sin(\frac{\Phi_C}{2}+\varphi_m)+\sin(\frac{\Phi_C}{2}-\varphi_m)\right]\right\}$$

I have put $\delta = L/\rho$ for the strength of the dipole

remember the relations $\cos x + \cos y = 2\cos\frac{x+y}{2} * \cos\frac{x-y}{2}$

$$\sin x + \sin y = 2\sin\frac{x+y}{2} * \cos\frac{x-y}{2}$$

$$D(l) = \delta\sqrt{\beta_m\beta_C}\left\{\sin\Phi_C * 2\cos\frac{\Phi_C}{2} * \cos\varphi_m - \cos\Phi_C * 2\sin\frac{\Phi_C}{2} * \cos\varphi_m\right\}$$

$$D(l) = 2\delta\sqrt{\beta_m\beta_C} * \cos\varphi_m\left\{\sin\Phi_C * \cos\frac{\Phi_C}{2} * -\cos\Phi_C * \sin\frac{\Phi_C}{2}\right\}$$

remember: $\sin 2x = 2\sin x * \cos x$

$$\cos 2x = \cos^2 x - \sin^2 x$$

$$D(l) = 2\delta\sqrt{\beta_m\beta_C} * \cos\varphi_m\left\{2\sin\frac{\Phi_C}{2} * \cos^2\frac{\Phi_C}{2} - (\cos^2\frac{\Phi_C}{2} - \sin^2\frac{\Phi_C}{2}) * \sin\frac{\Phi_C}{2}\right\}$$

$$D(l) = 2\delta\sqrt{\beta_m\beta_C} * \cos\varphi_m * \sin\frac{\Phi_C}{2}\left\{2\cos^2\frac{\Phi_C}{2} - \cos^2\frac{\Phi_C}{2} + \sin^2\frac{\Phi_C}{2}\right\}$$

$$D(l) = 2\delta\sqrt{\beta_m\beta_C} * \cos\varphi_m * \sin\frac{\Phi_C}{2}$$

in full analogy one derives the expression for D‘:

$$\boldsymbol{D'(l)} = 2\delta\sqrt{\beta_{\boldsymbol{m}} / \beta_{\boldsymbol{c}}} * \cos\varphi_{\boldsymbol{m}} * \cos\frac{\Phi_c}{2}$$

As we refer the expression for D and D‘ to a periodic struture, namly a FoDo cell we require periodicity conditons:

$$\begin{pmatrix} D_C \\ D'_C \\ 1 \end{pmatrix} = M_C * \begin{pmatrix} D_C \\ D'_C \\ 1 \end{pmatrix}$$

and by symmetry: $D'_C = 0$

With these boundary conditions the Dispersion in the FoDo is determined:

$$D_C * \cos\Phi_C + \delta\sqrt{\beta_m\beta_C} * \cos\varphi_m * 2\sin\frac{\Phi_C}{2} = D_C$$

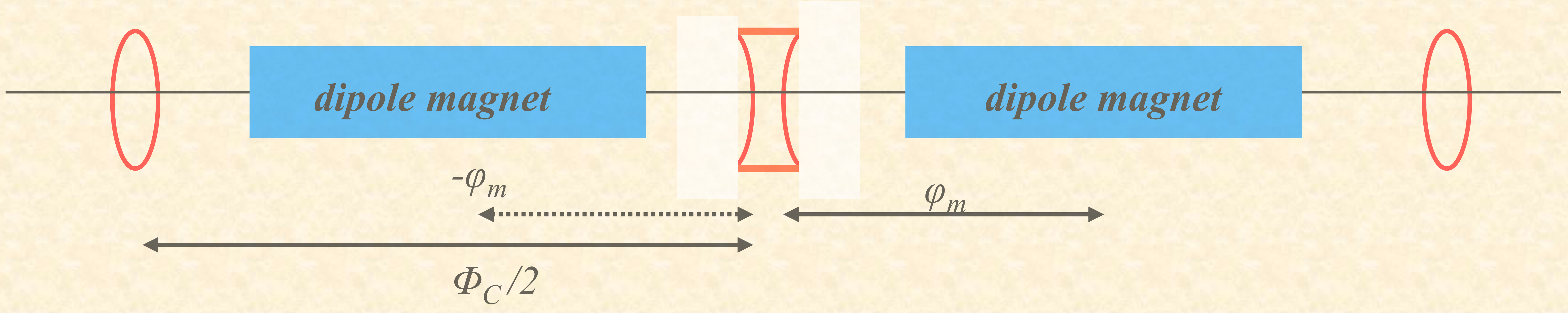
(A1)

$$D_C = \delta\sqrt{\beta_m \beta_C} * \cos\varphi_m / \sin\frac{\Phi_C}{2}$$

This is the value of the periodic dispersion in the cell evaluated at the position of the dipole magnets.

3.) Calculate the dispersion in the suppressor part:

We will now move to the second part of the dispersion suppressor: The section where ... starting from D=D‘=0 the dispesion is generated ... or turning it around where the Dispersion of the arc is reduced to zero.
The goal will be to generate the dispersion in this section in a way that the values of the periodic cell that have been calculated above are obtained.



The relation for D, generated in a cell still holds in the same way:

$$D(l) = S(l) * \int_0^l \frac{1}{\rho(\tilde{s})} C(\tilde{s})\, d\tilde{s} - C(l) * \int_0^l \frac{1}{\rho(\tilde{s})} S(\tilde{s})\, d\tilde{s}$$

as the dispersion is generated in a number of n cells the matrix for these n cells is

$$M_n = M_C^n = \begin{pmatrix} \cos n\Phi_C & \beta_C \sin n\Phi_C & D_n \\ \frac{-1}{\beta_C} \sin n\Phi_C & \cos n\Phi_C & D'_n \\ 0 & 0 & 1 \end{pmatrix}$$

$$D_n = \beta_C \sin n\Phi_C * \delta_{\sup r} * \sum_{i=1}^{n} \cos(i\Phi_C - \frac{1}{2}\Phi_C \pm \varphi_m) * \sqrt{\frac{\beta_m}{\beta_C}} -$$
$$- \cos n\Phi_C * \delta_{\sup r} * \sum_{i=1}^{n} \sqrt{\beta_m \beta_C} * \sin(i\Phi_C - \frac{1}{2}\Phi_C \pm \varphi_m)$$

$$D_n = \sqrt{\beta_m \beta_C} * \sin n\Phi_C * \delta_{\sup r} * \sum_{i=1}^{n} \cos((2i-1)\frac{\Phi_C}{2} \pm \varphi_m) - \sqrt{\beta_m \beta_C} * \delta_{\sup r} * \cos n\Phi_C \sum_{i=1}^{n} \sin((2i-1)\frac{\Phi_C}{2} \pm \varphi_m)$$

remember: $\sin x + \sin y = 2\sin\frac{x+y}{2} * \cos\frac{x-y}{2}$ $\cos x + \cos y = 2\cos\frac{x+y}{2} * \cos\frac{x-y}{2}$

$$D_n = \delta_{\sup r} * \sqrt{\beta_m \beta_C} * \sin n\Phi_C * \sum_{i=1}^{n} \cos((2i-1)\frac{\Phi_C}{2}) * 2\cos\varphi_m -$$
$$-\delta_{\sup r} * \sqrt{\beta_m \beta_C} * \cos n\Phi_C \sum_{i=1}^{n} \sin((2i-1)\frac{\Phi_C}{2}) * 2\cos\varphi_m$$

$$D_n = 2\delta_{\sup r} * \sqrt{\beta_m \beta_C} * \cos\varphi_m \left\{ \sum_{i=1}^{n} \cos((2i-1)\frac{\Phi_C}{2}) * \sin n\Phi_C - \sum_{i=1}^{n} \sin((2i-1)\frac{\Phi_C}{2}) * \cos n\Phi_C \right\}$$

$$D_n = 2\delta_{\sup r} * \sqrt{\beta_m \beta_C} * \cos\varphi_m \left\{ \sin n\Phi_C \left\{ \frac{\sin\frac{n\Phi_C}{2} * \cos\frac{n\Phi_C}{2}}{\sin\frac{\Phi_C}{2}} \right\} - \cos n\Phi_C * \left\{ \frac{\sin\frac{n\Phi_C}{2} * \sin\frac{n\Phi_C}{2}}{\sin\frac{\Phi_C}{2}} \right\} \right\}$$

$$D_n = \frac{2\delta_{\sup r} * \sqrt{\beta_m \beta_C} * \cos\varphi_m}{\sin\frac{\Phi_C}{2}} \left\{ \sin n\Phi_C * \sin\frac{n\Phi_C}{2} * \cos\frac{n\Phi_C}{2} - \cos n\Phi_C * \sin^2\frac{n\Phi_C}{2} \right\}$$

set for more convenience $x = n\Phi_C/2$

$$D_n = \frac{2\delta_{\sup r} * \sqrt{\beta_m \beta_C} * \cos\varphi_m}{\sin\frac{\Phi_C}{2}} \left\{ \sin 2x * \sin x * \cos x - \cos 2x * \sin^2 x \right\}$$

$$D_n = \frac{2\delta_{\sup r} * \sqrt{\beta_m \beta_C} * \cos\varphi_m}{\sin\frac{\Phi_C}{2}} \left\{ 2\sin x \cos x * \cos x \sin x - (\cos^2 x - \sin^2 x) \sin^2 x \right\}$$

(A2)

$$D_n = \frac{2\delta_{\sup r} * \sqrt{\beta_m \beta_C} * \cos\varphi_m}{\sin\frac{\Phi_C}{2}} * \sin^2\frac{n\Phi_C}{2}$$

and in similar calculations:

$$D'_n = \frac{2\delta_{\sup r} * \sqrt{\beta_m \beta_C} * \cos\varphi_m}{\sin\frac{\Phi_C}{2}} * \sin n\Phi_C$$

This expression gives the dispersion generated in a certain number of *n* cells as a function of the dipole kick δ in these cells.
At the end of the dispersion generating section the value obtained for D(s) and D'(s) has to be equal to the value of the periodic solution:

→equating *(A1)* and *(A2)* gives the conditions for the matching of the periodic dispersion in the arc to the values D = D'= 0 afte the suppressor.

$$D_n = \frac{2\delta_{\sup r} * \sqrt{\beta_m \beta_C} * \cos\varphi_m}{\sin\frac{\Phi_C}{2}} * \sin^2\frac{n\Phi_C}{2} = \delta_{arc}\sqrt{\beta_m \beta_C} * \frac{\cos\varphi_m}{\sin\frac{\Phi_C}{2}}$$

$$\left.\begin{array}{l} \rightarrow 2\delta_{\sup r} \sin^2\left(\dfrac{n\Phi_C}{2}\right) = \delta_{arc} \\ \rightarrow \sin(n\Phi_C) = 0 \end{array}\right\} \delta_{\sup r} = \frac{1}{2}\delta_{arc}$$

and at the same time the phase advance in the arc cell has to obey the relation:

$$n\Phi_C = k * \pi, \ k = 1, 3, \ \ldots$$