Production of squeezed state of single mode cavity field by the coupling of squeezed vacuum field reservoir in nonautonomous case*

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The dissipative and decoherence properties as well as the asymptotic behavior of the single mode electromagnetic field interacting with the time-dependent squeezed vacuum field reservoir are investigated in detail by using the algebraic dynamical method. With the help of the left and right representations of the relevant $hw(4)$ algebra, the dynamical symmetry of the nonautonomous master equation of the system is found to be $su(1,1)$. The unique equilibrium steady solution is found to be the squeezed state and any initial state of the system is proved to approach the unique squeezed state asymptotically. Thus the squeezed vacuum field reservoir is found to play the role of a squeezing mold of the cavity field.

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The fundamental property of the squeezed state is that the quantum fluctuations in one quadrature component of the field can be reduced heavily. Following from the early work [1, 2] on the squeezed state, researchers have taken great effort on this specific state. The first experimental result for the generation of the squeezed state was reported by Slusher *et al.*[3] with the scheme of 4-wave mixing in an optical cavity. Currently the successful scheme for generating squeezed light can also be based on the parametric oscillator or parametric down converter [4]. Recently, due to its potential applications in the fields of quantum measurement, optical communication, and quantum information, squeezed vacuum state has been extensively studied [5, 6]. A natural problem is what effect of physical systems can be induced by the squeezed vacuum. The squeezed light field will generally be characterized as a non-stationary reservoir which contains phase dependent features, i.e. non-zero values for correlation function between pairs of photons. When the bandwidths of the squeezed light are not too small, it can be treated as a Markovian reservoir, and the master equations of the reduced systems can be obtained based on Markovian approximation [7].

Master equations, which are derived by integrating out the enormous irrelevant degrees of freedom of the environment, are of fundamental importance on the treatment of open quantum systems. The common feature of the quantum master equations is the existence of the sandwich terms of the Liouville operators where the reduced density matrix of the system is between some quantum excitation and de-excitation operators. Thus, it is very difficult to get the exact analytical solution of the master equation, only some simple cases such as a single mode of cavity field coupled a vacuum state reservoir ($T = 0$) or stationary regime properties are normally treated [7], where the master equations are normally converted into c-number equations in the coherent state representation–the Fokker-Planck equation [8, 9].

In our previous work [10, 11, 12, 13], we have proposed and developed an algebraic method to treat the sandwich terms in the Liouville operator for the nonequilibrium quantum statistical systems. This method is just a generalization of the algebraic dynamical method [14] from quantum mechanical systems to quantum statistical systems. According to the characteristics of the sandwich terms in the Liouville operator, the left and right representations [15] have been introduced and the corresponding composite algebra has been constructed. As a result, the master equation is converted into a Schrödinger-like equation and the problems can be solved exactly. Using this method, we have successfully solved the von Nuemann equation of the quantum statistical characteristic function of the two-level Janes-Cummings model [10], the master equations of sympathetic cooling of Bose-Einstein condensate in the mean field approximation [11], the nonautonomous dissipative two-level atom system [12], and the nonautonomous dissipative two identical cavity system [13]. The salient feature of our method resides in the treatment of nonautonomous systems which are related to the control of man-made quantum systems by changing the coupling constant or adjusting other parameters of the system. It would be very desirable to get the analytical solution of the corresponding master equation of such a system. The nonautonomous system is the one whose Hamiltonian is time-dependent through some parameters employed for the purpose of control of the system. However, even if the total Hamiltonian of a composite system-the system investigated plus the environment, is time-independent, the master equation of the reduced density matrix of the investigated system still becomes nonautonomous under the non-Markovian dynamics

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[16]. Therefore, the quantum master equation of the reduced density matrix, in general, should be nonautonomous.

In this letter, using the algebraic dynamical method, we shall solve the problem of single mode electromagnetic field interacting with the squeezed vacuum field reservoir and investigate what effect on the system can be induced by the squeezed vacuum field reservoir. As is well known, as the environment is ordinary thermal equilibrium fields, the systems have been investigated very well. When the environment is the squeezed vacuum field, some work has been done for the autonomous case [17, 18, 19]. For the autonomous case, the investigation of the single mode electromagnetic field in the squeezed vacuum field can be found in [18] where the master equation was converted into a Fokker-Planck equation, and in [19] where the general solution of the master equation was given using a superoperator technique. However, for the system being nonautonomous, to our knowledge, no investigation exists up to now. It is our goal to study such a nonautonomous case. By using the algebraic dynamical method, here the $su(1,1)$ dynamical symmetry of the nonautonomous master equation for the single mode electromagnetic field is explored and the system is thus proved to be integrable for the first time. The corresponding analytical solutions, both the steady solutions and the time-dependent solutions of the systems are obtained exactly. Any time-dependent solution of the systems is proven to approach the unique steady equilibrium solution and the squeezed vacuum is found to play the role of the squeezing mold of the single mode cavity field.

The Hamiltonian of the open system reads as

$$H = \hbar\omega(a^\dagger a + \frac{1}{2}) + \hbar\sum_k \omega_k(a_k^\dagger a_k + \frac{1}{2}) + \hbar\sum_k g_k(a^\dagger a_k + h.c.),$$

which describes the single mode cavity field interacting with the squeezed vacuum field reservoir. The total initial density operator is [7]

$$\rho_T = \rho(0) \otimes \prod_k S_k(\xi)\,|0_k\rangle\langle 0_k| S_k^\dagger(\xi)\,,$$

where $S_k(\xi)$ is the squeezing operator and reads as

$$S_k(\xi) = \exp(\xi^* a_{k_0+k}a_{k_0-k} - \xi a_{k_0+k}^\dagger a_{k_0-k}^\dagger)$$

with $\omega = ck_0$ and $\xi = r\exp(i\theta)$, r being the squeezing parameter and θ being the reference phase for the squeezed field. The master equation of the reduced density operator for the field can be obtained in the usual way

$$\begin{aligned}
\dot\rho &= \frac{\gamma(t)}{2}(N+1)(2a\rho a^\dagger - a^\dagger a\rho - \rho a^\dagger a) \\
&\quad+\frac{\gamma(t)}{2}N(2a^\dagger\rho a - aa^\dagger\rho - \rho aa^\dagger) \\
&\quad-\frac{\gamma(t)}{2}M(2a\rho a - aa\rho - \rho aa) \\
&\quad-\frac{\gamma(t)}{2}M^*(2a^\dagger\rho a^\dagger - a^\dagger a^\dagger\rho - \rho a^\dagger a^\dagger), \qquad (1)
\end{aligned}$$

where $\langle a_k^\dagger a_{k'}\rangle = N\delta_{kk'} = \sinh^2(r)\delta_{kk'}$ and $\langle a_k^\dagger a_{k'}^\dagger\rangle = -M\delta_{k',2k_0-k} = -\cosh(r)\sinh(r)\exp(-i\theta)\,\delta_{k',2k_0-k}$, corresponding to N is very large. When $N \to \bar n = \frac{1}{\exp(\frac{\hbar\nu_k}{k_B T})-1}$, $M \to 0$, the master equation(1) reduces to the familiar form of the master equation which describes the system coupling to the ordinary thermal equilibrium radiation field. The last two terms of the above equation exhibit the phase-sensitive nature of the investigated system. In the model considered above, we assume that one can control the squeezed field so that the decay rate γ can be changed and is time-dependent. The system thus becomes nonautonomous. Eq. (1) in the autonomous case has been applied to the reduction of laser noise through injection of the squeezed vacuum [17] and to the noise-free phase-sensitive amplification via the two-photon correlated-emission-laser [20].

To explore the algebraic structure and dynamical symmetry of Eq. (1) in the nonautonomous case, we make a squeeze transformation or a general canonical Bogoliubov transformation [21] and transform the electromagnetic field operators (a, a^+) to the squeezed ones (A, A^+) as follows

$$\begin{aligned}
A &= \cosh(r)\,a - \sinh(r)\exp(i\theta)\,a^\dagger \\
&= S^\dagger a S, \\
A^\dagger &= \cosh(r)\,a^\dagger - \sinh(r)\exp(-i\theta)\,a \\
&= S^\dagger a^\dagger S, \qquad (2)
\end{aligned}$$

where the squeeze operator $S=\exp[\frac{1}{2}(\xi^* a^2-\xi a^{\dagger 2})]$. Using this squeezed mode we can rewrite Eq. (1) as

$$\dot{\rho}=\frac{\gamma(t)}{2}(2A\rho A^{\dagger}-A^{\dagger}A\rho-\rho A^{\dagger}A). \tag{3}$$

Substituting Eqs. (2) to Eq. (3) and defining $\rho_s=S\rho S^{\dagger}$ we can obtain

$$\dot{\rho}_s=\frac{\gamma(t)}{2}(2a\rho_s a^{\dagger}-a^{\dagger}a\rho_s-\rho_s a^{\dagger}a). \tag{4}$$

$\{a,a^{\dagger},a^{\dagger}a,1\}$ constitutes the algebra $hw\,(4)$. Based on the left and right representations of certain algebra [11], we define the left and right representations of this algebra as

$$\begin{aligned} hw\,(4)_r &= \left\{a^r,a^{r\dagger},n^r=a^{r\dagger}a^r,1\right\}, \\ hw\,(4)_l &= \left\{a^l,a^{l\dagger},n^l=a^{l\dagger}a^l,1\right\}, \end{aligned} \tag{5}$$

where $hw\,(4)_r$ acts to the right on the ket state $|n\rangle$ and $hw\,(4)_l$ acts to the left on the bra state $\langle n|$. They have the commutation rules

$$\begin{aligned} [a^r,a^{r\dagger}] &= 1,[a^r,n^r]=a^r,[a^{r\dagger},n^r]=-a^{r\dagger}; \\ [a^l,a^{l\dagger}] &= -1,[a^l,n^l]=-a^l,[a^{l\dagger},n^l]=a^{l\dagger}. \end{aligned}$$

It is evident that $hw\,(4)_r$ ($hw\,(4)_l$) is isomorphic (anti-isomorphic, respectively) to $hw\,(4)$. Since they act on different spaces, the operators commute with each other

$$[hw\,(4)_r\,,hw\,(4)_l]=0.$$

Having these useful algebras at hand, we can constitute the composite algebra C,

$$C=\{K_+=a^{r\dagger}a^l,K_-=a^r a^{l\dagger},K_0=\frac{n^r+n^l}{2}\}.$$

We see that C is an $su(1,1)$ algebra from the commutation rules

$$[K_0,K_{\pm}]=\pm K_{\pm},[K_-,K_+]=2K_0$$

which can be derived from Eq. (5). The operators have the following actions on the base of von Neumann space as

$$\begin{aligned} K_+|n\rangle\langle m| &= \sqrt{(n+1)\,(m+1)}|n+1\rangle\langle m+1|, \\ K_-|n\rangle\langle m| &= \sqrt{nm}|n-1\rangle\langle m-1|, \\ K_0|n\rangle\langle m| &= \frac{n+m+1}{2}|n\rangle\langle m|. \end{aligned} \tag{6}$$

Then the master equation (1) can be converted into a Schrödinger-like equation

$$\begin{aligned} \dot{\rho}_s &= \Gamma\,(t)\,\rho_s, \\ \Gamma\,(t) &= \gamma(t)(K_- - K_0+\frac{1}{2}). \end{aligned} \tag{7}$$

So the quantum master equation (1) possesses the $su\,(1,1)$ dynamical symmetry. It is thus integrable and can be solved analytically according to the algebraic dynamics [14].

To study the property of steady states, we should solve the eigen equation of the master equation for the steady case $\gamma(t)\rightarrow const.$

$$\Gamma\rho_s=\beta\rho_s. \tag{8}$$

Introducing a similarity transformation

$$U=e^{-K_-}$$

we can diagonalize the operator Γ as

$$\bar{\Gamma} = U^{-1}\Gamma U = \gamma(-K_0 + \frac{1}{2}). \tag{9}$$

From Eq. (9), we get the solution of Eq. (8) as follows

$$\begin{aligned} \beta_{nm} &= \frac{\gamma}{2}(n+m); \\ \rho_s &= e^{-K_-}|n\rangle\langle m|. \end{aligned}$$

Then unique steady solution of the master equation (1), $\rho_{s0} = e^{-K_-}|0\rangle\langle 0| = |0\rangle\langle 0|$ which is the zero-mode solution ($\beta_{00} = 0$) of the eigen equation (8), corresponds to the squeezed state of the field:

$$\rho_0 = S^\dagger \rho_{s0} S = S^\dagger |0\rangle\langle 0| S. \tag{10}$$

Next, we study the time-dependent solution of the nonautonomous master equation (3). To this end, we go back to Eq. (7) and introduce the time-dependent gauge transformation

$$U_g(t) = e^{\alpha(t)K_-}.$$

Under the gauge transformation condition

$$\dot{\alpha}(t) = \gamma(t)[\alpha(t) + 1], \tag{11}$$

the rate operator $\Gamma(t)$ can be diagonalized as

$$\bar{\Gamma}(t) = U_g^{-1}(t)\,\Gamma U_g(t) - U_g^{-1}(t)\,\dot{U}_g(t) = \gamma(t)(-K_0 + \frac{1}{2}).$$

Then the transformed equation reads as

$$\begin{aligned} \frac{d\bar{\rho}_s(t)}{dt} &= \bar{\Gamma}(t)\bar{\rho}_s(t), \\ \bar{\rho}_s(t) &= U_g^{-1}(t)\,\rho_s(t). \end{aligned}$$

So the solution of Eqs. (7) can be obtained readily as

$$\begin{aligned} \rho_s(t) &= U_g(t)\, e^{\int_0^t \bar{\Gamma}(\tau)d\tau} \rho_s(0), \\ &= \sum_{m,n} C_{m,n} e^{-\frac{n+m}{2}\int_0^t \gamma(\tau)d\tau} e^{\alpha(t)K_-}|n\rangle\langle m|, \end{aligned} \tag{12}$$

where we have used the gauge condition $U_g(0) = 1$ and the initial state $\rho_s(0) = \sum_{m,n} C_{m,n}|n\rangle\langle m| (\sum_m C_{m,m} = 1)$. After the inverse squeezing transformation we obtain the solution of Eq. (1)

$$\rho(t) = \sum_{m,n} C_{m,n} e^{-\frac{n+m}{2}\int_0^t \gamma(\tau)d\tau} S^\dagger \{e^{\alpha(t)K_-}|n\rangle\langle m|\} S = \sum_{m,n} C_{m,n}\rho_{nm}(t). \tag{13}$$

In the following we shall prove that the time-dependent solution (13) will approach the steady solution (10) asymptotically. To study the asymptotical behavior of $\alpha(t)$, we define $y(t) = \alpha(t)\exp[-\int_0^t \gamma(\tau)\, d\tau]$, the time differential of $y(t)$ is given by $b(t)\exp[-\int_0^t \gamma(\tau)\, d\tau]$ where $b(t) = \dot{\alpha}(t) - \gamma(t)\alpha(t) = \gamma(t)$. Because $b(t) \to \gamma(\infty) = \gamma$ is bounded for large t, the differential $\dot{y}(t)$ tends to zero. Hence $y(t)$ is towards to a constant. This implies that $\alpha(t)$ diverges asymptotically. Therefore one has the asymptotical properties of Eq. (11)

$$\begin{aligned} y(t)|_{t\to\infty} &= const, \\ \alpha(t)|_{t\to\infty} &= \infty. \end{aligned} \tag{14}$$

Using Eq. (6) we can rewrite any component of Eq. (13) as

$$\rho_{nm}(t) = S^\dagger \{ y(t)^{\frac{n+m}{2}} \sum_{q=|\frac{n-m}{2}|}^{\frac{n+m}{2}} \alpha(t)^{-q} \frac{\sqrt{n(n-1)\cdots(q+\frac{n-m}{2}+1)m(m-1)\cdots(q-\frac{n-m}{2}+1)}}{(\frac{n+m}{2}-q)!} |q+\frac{n-m}{2}\rangle\langle q-\frac{n-m}{2}| \} S.$$

Substituting the asymptotical conditions Eqs. (14) into the above equation, we can see that only the term corresponding to $n = m$ and $q = 0$ will survive, yielding the unique equilibrium steady state. Therefore, any time-dependent solution (13) of the nonautonomous master equation (1) tends to the unique steady solution (10) asymptotically,

$$\rho_{nm}(t) \to \delta_{nm}\rho_0 = S^{\dagger}|0\rangle\langle 0|S\delta_{nm}, \rho(t) \to \rho_0 = S^{\dagger}|0\rangle\langle 0|S.$$

Since any initial state approaches the steady solution which is the squeezed state of the single mode field, one comes to the important and interesting conclusion that the squeezed vacuum field reservoir always brings any initial state of the single mode field to its squeezed state asymptotically, and the squeezed vacuum field reservoir plays the role of a squeezing mold which transforms any kind of monomode cavity field state into its squeezed state: the squeezed vacuum imprints its squeezing information to the cavity field exclusively. This property may be useful in practice to generate a squeezed state of a cavity field.

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