Cavity-Cavity interaction in the Squeezed Vacuum

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1 General master equation of cavity-cavity interaction

In this section, we consider a scenario where two single-mode leaky cavities are placed inside the waveguide with the squeezed vacuum injected from both ends. The schematic setup is shown in Fig. 1. Now we will study how the modes inside the cavity will evolve under the influence of the squeezed vacuum. The free Hamiltonian of cavity and waveguide modes is:

$$H_0 = \sum_i \hbar\omega(a^\dagger a_i + \frac{1}{2}) + \hbar \sum_{k,s} \omega_k (a^\dagger_{k,s} a_{k,s} + \frac{1}{2}) \tag{1}$$

where a_k stands for the mods in the waveguide and a is the field operator of the single mode inside the cavity. The waveguide is saturated with the squeezed vacuum with the center frequency ω_0. The interaction Hamiltonian between the cavity mode and waveguide modes is:

$$V = -i\hbar \sum_{\boldsymbol{ks}} [D a_{\boldsymbol{ks}} - D^+ a^\dagger_{\boldsymbol{ks}}] \tag{2}$$

where

$$D = \sum_i [g^*_{i,k,s} a^\dagger_i + g_{i,k,s} a_i] \tag{3}$$

Here we define $g_{i,k,s} = |g_{i,k,s}| e^{-ik_z r_i}$ where r_i is just a phenomenological parameter describing the location of cavity. The reduced master equation of atoms in the reservoir is:

$$\begin{aligned}
\frac{d\rho^S}{dt} =& -\frac{1}{\hbar^2}\int_0^t d\tau Tr_F\{[V(t),[V(t-\tau),\rho^S(t-\tau)\rho^F\} \\
=& -\frac{1}{\hbar^2}\int_0^t d\tau Tr_F\{V(t)V(t-\tau)\rho^S(t-\tau)\rho^F + \rho^S(t-\tau)\rho^F V(t-\tau)V(t) \\
& - V(t)\rho^S(t-\tau)\rho^F V(t-\tau) - V(t-\tau)\rho^S(t-\tau)\rho^F V(t)\}.
\end{aligned} \tag{4}$$

Here we just show how to deal with the first term in Eq.(4), the remaining terms can be calculated in the same way. For the first term, we have

$$
\begin{aligned}
&-\frac{1}{\hbar^2}\int_0^t d\tau Tr_F\{V(t)V(t-\tau)\rho^S(t-\tau)\rho^F\}\\
=&\int_0^t d\tau \sum_{\boldsymbol{k}s,\boldsymbol{k'}s'}\{D(t)D(t-\tau)Tr_F[\rho^F a_{ks}(t)a_{k's'}(t-\tau)]-D(t)D^{+}(t-\tau)Tr_F[\rho^F a_{ks}(t)a^{\dagger}_{k's'}(t-\tau)]\\
&-D^{+}(t)D(t-\tau)Tr_F[\rho^F a^{\dagger}_{ks}(t)a_{k's'}(t-\tau)]+D^{+}(t)D^{+}(t-\tau)Tr_F[\rho^F a^{\dagger}_{ks}(t)a^{\dagger}_{k's'}(t-\tau)]\}\rho^S(t-\tau)\}.
\end{aligned} \tag{5}
$$

Under the rotating wave approximation(RWA), we have

$$
\begin{aligned}
&-\frac{1}{\hbar^2}\int_0^t d\tau Tr_F\{V(t)V(t-\tau)\rho^S(t-\tau)\rho^F\}\\
=&\sum_{ij}\sum_{\boldsymbol{k}s,\boldsymbol{k'}s'}\int_0^t d\tau\{g^*_{i,k,s}a^{\dagger}_i e^{i\omega_i t}g^*_{j,k',s'}a^{\dagger}_j e^{i\omega_j(t-\tau)}e^{-i(\omega_{\boldsymbol{k}s}+\omega_{\boldsymbol{k'}s'})t+i\omega_{\boldsymbol{k'}s'}\tau}[-\sinh(r)\cosh(r)\delta_{\boldsymbol{k'},2\boldsymbol{k}_0-\boldsymbol{k}}\delta_{ss'}]\\
&-g^*_{i,k,s}a^{\dagger}_i e^{i\omega_i t}g_{j,k',s'}a_j e^{-i\omega_j(t-\tau)}e^{-i\omega_{\boldsymbol{k'}s'}\tau}\cosh^2 r\delta_{\boldsymbol{kk'}}\delta_{ss'}\\
&-g_{i,k,s}a_i e^{-i\omega_i t}g^*_{j,k',s'}a^{\dagger}_j e^{i\omega_j(t-\tau)}e^{-i\omega_{\boldsymbol{k'}s'}\tau}\cosh^2 r\delta_{\boldsymbol{kk'}}\delta_{ss'}\\
&-g_{i,k,s}a_i e^{-i\omega_i t}g^*_{j,k',s'}a^{\dagger}_j e^{i\omega_j(t-\tau)}e^{i\omega_{\boldsymbol{k'}s'}\tau}\sinh^2 r\delta_{\boldsymbol{kk'}}\delta_{ss'}\\
&-g^*_{i,k,s}a^{\dagger}_i e^{i\omega_i t}g_{j,k',s'}a_j e^{-i\omega_j(t-\tau)}e^{i\omega_{\boldsymbol{k'}s'}\tau}\sinh^2 r\delta_{\boldsymbol{kk'}}\delta_{ss'}\\
&+g_{i,k,s}a_i e^{-i\omega_i t}g_{j,k',s'}a_j e^{-i\omega_j(t-\tau)}e^{i(\omega_{\boldsymbol{k}s}+\omega_{\boldsymbol{k'}s'})t-i\omega_{\boldsymbol{k'}s'}\tau}[-\sinh(r)\cosh(r)\delta_{\boldsymbol{k'},2\boldsymbol{k}_0-\boldsymbol{k}}\delta_{ss'}]\}\rho^S(t-\tau)
\end{aligned} \tag{6}
$$

Here we just calculate the first and second term to show how to get the master equation. Define a. For the second term, we have

$$
\begin{aligned}
&-\sum_{k_z}\int_0^t d\tau g^*_{i,k,s}a_i^\dagger e^{i\omega_i t}g_{j,k',s'}a_j e^{-i\omega_j(t-\tau)}e^{-i\omega_{\boldsymbol{k}'s'}\tau}\cosh^2 r\rho^S(t-\tau)\delta_{\boldsymbol{k}\boldsymbol{k}'}\delta_{ss'} \\
=&-\frac{L}{2\pi}e^{i(\omega_i-\omega_j)t}\int_{-\infty}^{\infty}dk_z\int_0^t d\tau e^{i\omega_j\tau}e^{-i\omega_{k_z}\tau}|g_{i,\boldsymbol{k},s}g_{j,\boldsymbol{k},s}|e^{ik_z(r_i-r_j)}\cosh^2 ra_i^\dagger a_j\rho^S(t-\tau) \\
\approx&-\frac{L}{2\pi}e^{i(\omega_i-\omega_j)t}\int_0^{\infty}dk_z\int_0^t d\tau e^{i\omega_j\tau}e^{-i[\omega_j+c^2k_{jz}(k_z-k_{jz})/\omega_j]\tau}|g_{i,\boldsymbol{k},s}g_{j,\boldsymbol{k},s}|[e^{ik_z(r_i-r_j)}+e^{-ik_z(r_i-r_j)}]\cosh^2 ra_i^\dagger a_j\rho^S \\
\approx&-\frac{L}{2\pi}e^{i(\omega_i-\omega_j)t}\int_{-k_{0z}}^{\infty}d\delta k_z\int_0^t d\tau e^{-i\tau c^2k_{jz}\delta k_z/\omega_j}|g_{i,\boldsymbol{k},s}g_{j,\boldsymbol{k},s}|[e^{i(k_{jz}+\delta k_z)(r_i-r_j)}+e^{-i(k_{jz}+\delta k_z)(r_i-r_j)}]\cosh^2 ra_i^\dagger a_j\rho \\
\approx&-\frac{L}{2\pi}e^{i(\omega_i-\omega_j)t}\int_{-\infty}^{\infty}d\delta k_z\int_0^t d\tau e^{-i(c^2k_{jz}\delta k_z/\omega_j)\tau}|g_{i,\boldsymbol{k},s}g_{j,\boldsymbol{k},s}|[e^{i(k_{jz}+\delta k_z)(r_i-r_j)}+e^{-i(k_{jz}+\delta k_z)(r_i-r_j)}]\cosh^2 ra_i^\dagger a_j \\
\approx&-\frac{L}{2\pi}e^{i(\omega_i-\omega_j)t}\int_0^t d\tau|g_{i,\boldsymbol{k},s}g_{j,\boldsymbol{k},s}|2\pi[e^{ik_{jz}(r_i-r_j)}\delta((r_i-r_j)-\frac{c^2k_{jz}}{\omega_0}\tau)+e^{-ik_{jz}(r_i-r_j)}\delta((r_i-r_j)+\frac{c^2k_{jz}}{\omega_0}\tau)]\cosh \\
\approx&-\frac{L}{2\pi}e^{ik_{jz}r_{ij}}|g_{i,\boldsymbol{k},s}g_{j,\boldsymbol{k},s}|2\pi\frac{\omega_j}{c^2k_{0z}}\cosh^2 ra_i^\dagger a_j\rho^S(t)e^{i(\omega_i-\omega_j)t} \\
\approx&-[\frac{\sqrt{\gamma_i\gamma_j}}{2}\cos(k_{0z}r_{ij})+i\frac{\sqrt{\gamma_i\gamma_j}}{2}sin(k_{0z}r_{ij})]\cosh^2 ra_i^\dagger a_j\rho^S(t)e^{i(\omega_i-\omega_j)t} \\
\equiv&-(\frac{\sqrt{\gamma_i\gamma_j}}{2}+i\Lambda_{ij})\cosh^2 ra_i^\dagger a_j\rho^S(t)e^{i(\omega_i-\omega_j)t}
\end{aligned}
\tag{7}
$$

where $r_{ij}=|r_i-r_j|$ is also a phenomenological parameter indicating the relative position between cavities. $\gamma_i=L|g_{i,k_0}|^2$ is the leaking rate for the ith cavity, and $\Lambda_{ij}=\sqrt{\gamma_i\gamma_j}\sin(k_{0z}r_{ij})/2$ is the energy shift. In the third line we expand $\omega_k=c\sqrt{(\frac{\pi}{a})^2+(k_z)^2}$ around $k_z=k_{0z}$ since resonant modes provide dominant contributions. In the fifth line we extend the integration $\int_{-k_{0z}}^{\infty}dk_z\rightarrow\int_{-\infty}^{\infty}dk_z$ because the main contribution comes from the components around $\delta k_z=0$. In the next line, Weisskopf-Wigner approximation is used.

Next we need to calculate the first term (squeezing term) in Eq.(6):

$$
\begin{aligned}
&e^{i(\omega_i+\omega_j-2\omega_0)t}\sum_{k_z}\int_0^t d\tau\{g^*_{i,2\boldsymbol{k}_0-\boldsymbol{k}}a_i^\dagger g^*_{j,\boldsymbol{k}}a_j^\dagger e^{i(\omega_{\boldsymbol{k}}-\omega_j)\tau}[-\sinh(r)\cosh(r)]\rho^S(t-\tau) \\
=&-\frac{L}{2\pi}e^{i(\omega_i+\omega_j-2\omega_0)t}\int_0^{2k_{0z}}dk_z\int_0^t d\tau e^{i(\omega_{k_z}-\omega_j)\tau}e^{i(2k_{iz}-k_z)(r_i-o_1)}e^{ik_z(r_j-o_1)}|g_{i,2\boldsymbol{k}_0-\boldsymbol{k}}g_{j,\boldsymbol{k}}|\sinh(r)\cosh(r)a_i^\dagger a_j^\dagger\rho^S \\
&-\frac{L}{2\pi}e^{i(\omega_i+\omega_j-2\omega_0)t}\int_{-2k_{0z}}^{0}dk_z\int_0^t d\tau e^{i(\omega_{k_z}-\omega_j)\tau}e^{i(-2k_{iz}-k_z)(r_i-o_2)}e^{ik_z(r_j-o_2)}|g_{i,2\boldsymbol{k}_0-\boldsymbol{k}}g_{j,\boldsymbol{k}}|\sinh(r)\cosh(r)a_i^\dagger a_j^\dagger\rho^S
\end{aligned}
\tag{8}
$$

Putting the overall factor $e^{i(\omega_i+\omega_j-2\omega_0)t}$ aside, for $i=j$, Eq.(8) reduces to

$$
\begin{aligned}
&\sum_{k_z}\int_0^t d\tau\{g^*_{i,2\boldsymbol{k}_0-\boldsymbol{k}}a_i^\dagger g^*_{i,\boldsymbol{k}}a_i^\dagger e^{i(\omega_{\boldsymbol{k}}-\omega_i)\tau}[-\sinh(r)\cosh(r)]\rho^S(t-\tau)\\
&=-\frac{L}{2\pi}\int_0^{2k_{0z}}dk_z\int_0^t d\tau e^{i\frac{c^2k_{iz}}{\omega_i}(k_z-k_{iz})\tau}e^{i2k_{0z}(r_i-o_1)}|g_{i,2\boldsymbol{k}_0-\boldsymbol{k}}g_{i,\boldsymbol{k}}|\sinh(r)\cosh(r)a_i^\dagger a_i^\dagger\rho^S(t-\tau)\\
&-\frac{L}{2\pi}\int_{-2k_{0z}}^0 dk_z\int_0^t d\tau e^{i\frac{c^2k_{iz}}{\omega_i}(k_z-k_{iz})\tau}e^{-i2k_{0z}(r_i-o_2)}|g_{i,2\boldsymbol{k}_0-\boldsymbol{k}}g_{i,\boldsymbol{k}}|\sinh(r)\cosh(r)a_i^\dagger a_i^\dagger\rho^S(t-\tau)\\
&=-\frac{L}{2\pi}[e^{i2k_{0z}(r_i-o_1)}+e^{-i2k_{0z}(r_i-o_2)}]|g_{i,2\boldsymbol{k}_0-\boldsymbol{k}}g_{i,\boldsymbol{k}}|\int_0^t d\tau 2\pi\delta(\frac{c^2k_{iz}}{\omega_i}\tau)\sinh(r)\cosh(r)a_i^\dagger a_i^\dagger\rho^S(t-\tau)\\
&=-e^{i2k_{0z}R}\frac{\gamma_i}{2}\cos(2k_{0z}r_i)\sinh(r)\cosh(r)a_i^\dagger a_i^\dagger\rho^S(t)
\end{aligned}
\tag{9}
$$

where we have used the fact that the origin of coordinate system is at equal distant from two sources(i.e., $o_2=-o_1=R$) in the second last line. Thus, we have $\gamma'_{ij}=\sqrt{\gamma_i\gamma_j}\cos(2k_{0z}r_i)$. For $r_i\neq r_j$, Eq. (8) reduces to

$$
\begin{aligned}
&\sum_{k_z}\int_0^t d\tau\{g_{i,2\boldsymbol{k}_0-\boldsymbol{k}}a_i^\dagger g_{j,\boldsymbol{k}}a_j^\dagger e^{i(\omega_{\boldsymbol{k}}-\omega_j)\tau}[-\sinh(r)\cosh(r)]\rho^S(t-\tau)\\
&=-\frac{L}{2\pi}\int_0^{2k_{0z}}dk_z\int_0^t d\tau e^{i\frac{c^2k_{jz}}{\omega_j}(k_z-k_{jz})\tau}e^{i2k_{0z}(r_c-o_1)}e^{-i(k_z-k_{0z})(r_i-r_j)}|g_{i,2\boldsymbol{k}_0-\boldsymbol{k}}g_{j,\boldsymbol{k}}|\sinh(r)\cosh(r)a_i^\dagger a_j^\dagger\rho^S(t-\tau)\\
&-\frac{L}{2\pi}\int_{-2k_{0z}}^0 dk_z\int_0^t d\tau e^{i\frac{c^2k_{jz}}{\omega_j}(-k_z-k_{jz})\tau}e^{-i2k_{0z}(r_c-o_2)}e^{-i(k_z+k_{0z})(r_i-r_j)}|g_{i,2\boldsymbol{k}_0-\boldsymbol{k}}g_{j,\boldsymbol{k}}|\sinh(r)\cosh(r)a_i^\dagger a_j^\dagger\rho^S(t-\tau)\\
&=-\frac{L}{2\pi}e^{i2k_{0z}(r_c-o_1)}|g_{i,2\boldsymbol{k}_0-\boldsymbol{k}}g_{j,\boldsymbol{k}}|\int_{-\infty}^{\infty}dk_z\int_0^t d\tau e^{i\frac{c^2k_{jz}}{\omega_j}(k_z-k_{jz})\tau}e^{-i(k_z-k_{0z})(r_i-r_j)}\sinh(r)\cosh(r)a_i^\dagger a_j^\dagger\rho^S(t-\tau)\\
&-\frac{L}{2\pi}e^{-i2k_{0z}(r_c-o_2)}|g_{i,2\boldsymbol{k}_0-\boldsymbol{k}}g_{j,\boldsymbol{k}}|\int_{-\infty}^{\infty}dk_z\int_0^t d\tau e^{i\frac{c^2k_{jz}}{\omega_j}(k_z-k_{jz})\tau}e^{i(k_z-k_{0z})(r_i-r_j)}\sinh(r)\cosh(r)a_i^\dagger a_j^\dagger\rho^S(t-\tau)\\
&\approx-\frac{L}{2\pi}e^{i2k_{0z}R}|g_{i,2\boldsymbol{k}_0-\boldsymbol{k}}g_{j,\boldsymbol{k}}|\int_0^t d\tau 2\pi[e^{i2k_{0z}r_c}\delta(r_i-r_j-\frac{c^2k_{0z}}{\omega_0}\tau)+e^{-i2k_{0z}r_c}\delta(r_i-r_j+\frac{c^2k_{0z}}{\omega_0}\tau)]\sinh(r)\cosh(r)\\
&\approx-e^{i2k_{0z}R}L|g_{i,2\boldsymbol{k}_0}g_{j,\boldsymbol{k}_0}|e^{i2k_{0z}r_c sgn(i-j)}a_i^\dagger a_j^\dagger\rho^S(t)\rightarrow-\frac{\sqrt{\gamma_i\gamma_j}}{2}e^{i2k_{0z}R}\cos(k_{0z}(r_i+r_j))a_i^\dagger a_j^\dagger\rho^S(t)
\end{aligned}
\tag{10}
$$

where $sgn(i-j)$ is the sign function. The last arrow is because we need to sum over i,j, so the imaginary part of $e^{i2k_{0z}r_c sgn(i-j)}$ vanishes and the neat result is that $\gamma'_{ij}=e^{i2k_{0z}R}\sqrt{\gamma_i\gamma_j}\cos(k_{0z}(r_i+r_j))$. As for $a_i^\dagger\rho^S(t)a_j^\dagger$ terms, the combination of the last two terms in Eq.(4) will make the imaginary part of $e^{i2k_{0z}r_c sgn(i-j)}$ vanish. Thus, we have $\gamma'_{ij}=e^{i2k_{0z}R}\sqrt{\gamma_i\gamma_j}\cos(k_{0z}(r_i+r_j))$. If one needs to get γ_{ij},γ'_{ij} and Λ_{ij}in the unidirectional waveguide case, we just need to discard the second terms in the parenthesis of Eq.(7) and Eq.(10).

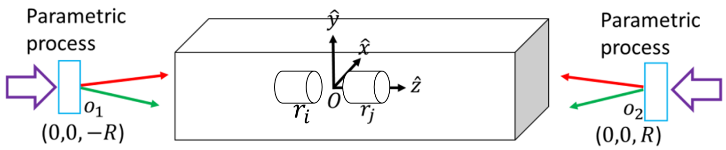

Fig. 1: (a) Schematic setup: two single-mode cavities are placed inside the waveguide with the broadband squeezed vacuum incident from both ends.

Thus, the general equation for cavity-cavity interaction in the squeezed vacuum is:

$$\begin{aligned}\dot{\rho} = \sum_{ij} &\gamma \cosh^2 r(-\rho a_i^\dagger a_j - a_i^\dagger a_j \rho + 2a_i \rho a_j^\dagger) \\ &+ \gamma \sinh^2 r(-\rho a_i a_j^\dagger - a_i a_j^\dagger \rho + 2a_i^\dagger \rho a_j) \\ &+ \gamma \cosh r \sinh r(e^{i(\theta_i+\theta_j)}\rho a_i a_j + e^{i(\theta_i+\theta_j)} a_i a_j \rho - e^{i(\theta_i+\theta_j)} 2a_i \rho a_j + h.c.)\end{aligned} \tag{11}$$

First, we study two non-resonant cavities coupled to the squeezed vacuum reservoir. The eigen frequencies of these two cavities are $\omega_1 = \omega_0 - \delta\omega$ and $\omega_2 = \omega_0 + \delta\omega$, following exacly the same steps as the derivation in Appendix A with S replaced by a_k, we get:

$$\begin{aligned}\dot{\rho} = \sum_i &\gamma(1+N)(-\rho a_i^\dagger a_i - a_i^\dagger a_i \rho + 2a_i \rho a_i^\dagger) \\ &+ \gamma N(-\rho a_i a_i^\dagger - a_i a_i^\dagger \rho + 2a_i^\dagger \rho a_i) \\ &+ \sum_{i\neq j} \gamma M(e^{i(\theta_i+\theta_j)}\rho a_i a_j + e^{i(\theta_i+\theta_j)} a_i a_j \rho - 2e^{i(\theta_i+\theta_j)} a_i \rho a_j + h.c.)\end{aligned} \tag{12}$$

where θ_i is a phase factor which depends on the relative position of cavities and the squeezing source. The above equation can be re-arranged as:

$$\begin{aligned}\dot{\rho} = \sum_{i\neq j} &\frac{\gamma}{2}[-\rho(\cosh(r)a_i^\dagger - e^{i\theta}\sinh(r)a_j)(\cosh(r)a_i - e^{-i\theta}\sinh(r)a_j^\dagger) \\ &- (\cosh(r)a_i^\dagger - e^{i\theta}\sinh(r)a_j)(\cosh(r)a_i - e^{-i\theta}\sinh(r)a_j^\dagger)\rho \\ &+ 2(\cosh(r)a_i - e^{-i\theta}\sinh(r)a_j^\dagger)\rho(\cosh(r)a_i^\dagger - e^{i\theta}\sinh(r)a_j)]\end{aligned} \tag{13}$$

we use the following Bogoliubov transformation[2]:

$$\begin{aligned}S &= exp(\eta^\star a_i a_j - \eta a_i^\dagger a_j^\dagger) \\ A_i &= S^+ a_i S = \cosh(r)a_i - e^{-i\theta}\sinh(r)a_j^\dagger \\ A_i^+ &= S^+ a_i^+ S = \cosh(r)a_i^+ - e^{i\theta}\sinh(r)a_j\end{aligned} \tag{14}$$

so the master equation Eq.(13) becomes:

$$\dot{\rho} = \sum_i \gamma[-\rho A_i^\dagger A_i - A_i^\dagger A_i \rho + 2A_i \rho A_i^\dagger] \tag{15}$$

Next we redefine the density matrix: $\rho_s = S\rho S^\dagger$. Thus Eq.(15) becomes:

$$\begin{aligned}\dot{\rho}_s &= \sum_i \gamma[-\rho_s a_i^\dagger a_i - a_i^\dagger a_i \rho_s + 2a_i\rho_s a_i^\dagger] \\ &\equiv \sum_i \gamma[-a_i^{l\dagger} a_i^l \rho_s - a_i^{r\dagger} a_i^r \rho_s + 2a_i^r a_i^{l\dagger}\rho_s] \equiv L\rho_s\end{aligned} \tag{16}$$

Here we define superoperator $\{a_i^l, a_i^{l\dagger}\}(\{a_i^r, a_r^{l\dagger}\}$) only acting to the left(right) on density operator ρ [3, 4]. These operators have the following commutation relations:

$$[a_i^r, a_j^{r\dagger}] = \delta_{ij},\ [a_i^l, a_j^{l\dagger}] = -\delta_{ij},\ [a_i^l, a_j^{r\dagger}] = [a_i^l, a_j^r] = [a_i^{l\dagger}, a_j^r] = [a_i^{l\dagger}, a_j^{r\dagger}] = 0 \tag{17}$$

Thus, the steady state of Eq.(16) can be solved by solving $L\rho = 0$, which requires the diagnolization of superoperator L. Applying the similarity transformation $U = e^{-a_1^r a_1^{l\dagger} - a_2^r a_2^{l\dagger}}$ to Eq.(16) , since we have $U^{-1}(a_i^{r\dagger}, a_i^l, a_i^r, a_i^{l\dagger})U = (a_i^{r\dagger} + a_i^{l\dagger}, a_i^r + a_i^l, a_i^r, a_i^{l\dagger})$, the right hand side of Eq.(16) becomes:

$$RHS = \sum_i \gamma U^{-1}[-a_i^{l\dagger}a_i^l - a_i^{r\dagger}a_i^r + 2a_i^r a_i^{l\dagger}]UU^{-1}\rho_s = \sum_i \gamma[-a_i^{l\dagger}a_i^l - a_i^{r\dagger}a_i^r]U^{-1}\rho_s \tag{18}$$

The only solution to $L\rho = 0$ is $U^{-1}\rho_s = |0,0\rangle\langle 0,0|$, which yields $\rho = S^\dagger \rho_S S = S^\dagger e^{-K_{-1}-K_{-2}}|0,0\rangle\langle 0,0|S = S^\dagger|0,0\rangle\langle 0,0|S$ which is the two mode squeezed vacuum.

Then we study the case where two cavities are identical, i.e., $\omega_1 = \omega_2 = \omega_0$. Then the master equation becomes:

$$\begin{aligned}\dot{\rho} = &\sum_{ij} \gamma \cosh^2 r(-\rho a_i^\dagger a_j - a_i^\dagger a_j \rho + 2a_i \rho a_j^\dagger) \\ &+ \gamma \sinh^2 r(-\rho a_i a_j^\dagger - a_i a_j^\dagger \rho + 2a_i^\dagger \rho a_j) \\ &+ \gamma \cosh r \sinh r(e^{i(\theta_i+\theta_j)}\rho a_i a_j + e^{i(\theta_i+\theta_j)} a_i a_j \rho - e^{i(\theta_i+\theta_j)} 2a_i \rho a_j + h.c.)\end{aligned} \tag{19}$$

This equation can be rearranged when $\theta_1 = \theta_2 = \frac{\theta}{2}$:

$$\begin{aligned}\dot{\rho} = &\sum_{ij} \gamma[-\rho(\cosh r a_i^\dagger - e^{i\theta}\sinh r a_i)(\cosh r a_j - e^{-i\theta}\sinh r a_j^\dagger) \\ &- (\cosh r a_i^\dagger - e^{i\theta}\sinh r a_i)(\cosh r a_j - e^{-i\theta}\sinh r a_j^\dagger)\rho \\ &+ 2(\cosh r a_j - e^{-i\theta}\sinh r a_j^\dagger)\rho(\cosh r a_i^\dagger - e^{i\theta}\sinh r a_i)]\end{aligned} \tag{20}$$

We introduce the Bogoliubov transformation:

$$\begin{aligned}S_i &= exp(\frac{1}{2}\eta^\star a_i^2 - \frac{1}{2}\eta a_i^{\dagger 2}) \\ A_i &= S_i^+ a_i S_i = \cosh(r)a_i - e^{-i\theta}\sinh(r)a_i^\dagger \\ A_i^+ &= S_i^+ a_i^+ S_i = \cosh(r)a_i^+ - e^{i\theta}\sinh(r)a_i\end{aligned} \tag{21}$$

so master equation Eq.(20) becomes

$$\dot{\rho} = \sum_{ij} \gamma[-\rho A_i^\dagger A_j - A_i^\dagger A_j \rho + 2A_j \rho A_i^\dagger] \tag{22}$$

Next we define $\rho_s = S_1 S_2 \rho \dot{S_1^+} S_2^+$ so the master equation is reduced to:

$$\dot{\rho_s} = \sum_{ij} \gamma[-\rho_s a_i^\dagger a_j - a_i^\dagger a_j \rho_s + 2a_j \rho_s a_i^\dagger] \tag{23}$$

To diagnolize this Lindblad equation, we introduce the transformation:

$$\begin{matrix} L_1 \\ L_2 \end{matrix} = \begin{matrix} \frac{1}{\sqrt{2}}(a_1 - a_2) \\ \frac{1}{\sqrt{2}}(a_1 + a_2) \end{matrix}$$

where $[L_i, L_j^\dagger] = \delta_{ij}$, and the master equation becomes:

$$\begin{aligned} \dot{\rho_s} &= \gamma[-2\rho_s L_2^\dagger L_2 - 2L_2^\dagger L_2 \rho_s + 4L_2 \rho_s L_2^\dagger] \\ &= \gamma[-2L_2^{r\dagger} L_2^r \rho_s - 2L_2^{l\dagger} L_2^l \rho_s + 4L_2^l L_2^{r\dagger} \rho_s] \\ &= L\rho \end{aligned} \tag{24}$$

Operator $L_2^\dagger$ has the following properties:

$$L_2^\dagger |0\rangle = \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle) \equiv |1_{L_2}\rangle$$

$$L_2^\dagger \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle) = \sqrt{2}[\frac{1}{2}(|02\rangle + \sqrt{2}|11\rangle + |20\rangle)] = \sqrt{2}|2_{L_2}\rangle$$

$$L_2^\dagger \frac{1}{2}(|02\rangle + \sqrt{2}|11\rangle + |20\rangle) = \sqrt{3}[\frac{1}{2\sqrt{2}}(|03\rangle + \sqrt{3}|12\rangle + \sqrt{3}|21\rangle + |30\rangle)] = \sqrt{3}|3_{L_2}\rangle$$

...

while the operator $L_1^\dagger$ has the following properties:

$$L_1^\dagger |0\rangle = \frac{1}{\sqrt{2}}(-|01\rangle + |10\rangle) \equiv |1_{L_1}\rangle$$

$$L_1^\dagger \frac{1}{\sqrt{2}}(-|01\rangle + |10\rangle) = \sqrt{2}[\frac{1}{2}(|02\rangle - \sqrt{2}|11\rangle + |20\rangle)] = \sqrt{2}|2_{L_1}\rangle$$

$$L_1^\dagger \frac{1}{2}(|02\rangle - \sqrt{2}|11\rangle + |20\rangle) = \sqrt{3}[\frac{1}{2\sqrt{2}}(-|03\rangle + \sqrt{3}|12\rangle - \sqrt{3}|21\rangle + |30\rangle)] = \sqrt{3}|3_{L_1}\rangle$$

...

Then we use the similarity transformation: $U = e^{-L_2^r L_2^{l\dagger}}$, which yields $U^{-1}(L_2^{r\dagger}, L_2^l, L_2^{l\dagger}, L_2^r)U = (L_2^{r\dagger} + L_2^{l\dagger}, L_2^l + L_2^r, L_2^{l\dagger}, L_2^r)$. Thus, the master equation Eq.(25) becomes:

$$RHS = \gamma U^{-1}[-L_2^{l\dagger} L_2^l - L_2^{r\dagger} L_2^r + 2L_2^r L_2^{l\dagger}]UU^{-1}\rho_s = \gamma[-L_2^{l\dagger} L_2^l - L_2^{r\dagger} L_2^r]U^{-1}\rho_s \tag{25}$$

The solutions to the steady state are $\rho_s = e^{-L_2^r L_2^{l\dagger}} |0_{L_2} m_{L_1}\rangle\langle 0_{L_2} n_{L_1}| = |m_{L_1}\rangle\langle n_{L_1}|$ which yields $\rho = S_1^+ S_2^+ \frac{1}{\sqrt{m!}}(\frac{a_1^\dagger - a_2^\dagger}{\sqrt{2}})^m |0\rangle\langle 0| \frac{1}{\sqrt{n!}}(\frac{a_1 - a_2}{\sqrt{2}})^n S_1 S_2$. This solution degenerates to the single mode squzeed vacuum in two modes when $m = n = 0$. Generally, an initial state $\rho(0) = \sum_{mnpq} C_{mnpq} |mn\rangle\langle pq| = \sum_{mnpq} C'_{mnpq} |m_{L_1} p_{L_2}\rangle\langle n_{L_1} q_{L_2}|$ will evolve into $\sum_{mn} G_{mn} |m_{L_1}\rangle\langle n_{L_1}|$ where $G_{mn} = \sum_{mnp} C'_{mnpp}$.

A Appendix A: Derivation of master equation

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