The steady state population inversion of Ξ-type atoms in the squeezed vacuum reservoir

PACS numbers:

I. INTRODUCTION

In thermal equilibrium, a population inversion can never exist for a system. Traditionally, the achievements of population inversion requires pushing the system into a non-equilibrated state. The concept of population inversion is of fundamental importance in laser physics because the production of population inversion is a key step in the workings of a standard laser.

II. MASTER EQUATION OF THE THREE LEVEL ATOM IN THE SQUEEZED VACUUM

In this section, we consider a scenario where a three-level atom is located inside the waveguide with the squeezed vacuum injected from both ends, as shown in Fig. 1(a). The atomic electronic structure is shown in Fig. 1(b) where the atomic states are labeled $|a\rangle$, $|b\rangle$, $|c\rangle$ from the excited state to the ground state. We assume that $\omega_{ac} = 2\omega_0$ where ω_0 is the center frequency of the broad band squeed vacuum. ω_{ab} and ω_{bc} are not equal but they are still within the bandwith of the squeezed vacuum.

Doing the above calculation for all terms in Eq.(A3), we have the general equation for cavity-cavity interaction in the squeezed vacuum as follows:

$$\begin{aligned}\frac{d\rho^S}{dt} =& -i\sum_{i\neq j}\Lambda_{ij}[S_i^+S_j^-,\rho^S]e^{i(\omega_i-\omega_j)t} - \frac{1}{2}\sum_{i,j}\gamma_{ij}(1+N)(\rho^S S_i^+S_j^- + S_i^+S_j^-\rho^S - 2S_j^-\rho^S S_i^+)e^{i(\omega_i-\omega_j)t} \\ & - \frac{1}{2}\sum_{i,j}\gamma_{ij}N(\rho^S S_i^-S_j^+ + S_i^-S_j^+\rho^S - 2S_j^+\rho^S S_i^-)e^{-i(\omega_i-\omega_j)t} \\ & - \frac{1}{2}\sum_{\alpha=\pm}\sum_{i,j}\gamma'_{ij}Me^{2\alpha ik_{0z}R}e^{i\alpha(\omega_i+\omega_j-2\omega_0)t}(\rho^S S_i^\alpha S_j^\alpha + S_i^\alpha S_j^\alpha\rho^S - 2S_j^\alpha\rho^S S_i^\alpha)\end{aligned} \tag{1}$$

where the coefficients are

$$\begin{aligned}\gamma_{ij} &= \sqrt{\gamma_i\gamma_j}\cos(k_{0z}r_{ij}) \\ \Lambda_{ij} &= \frac{\sqrt{\gamma_i\gamma_j}}{2}\sin(k_{0z}r_{ij}) \\ \gamma'_{ij} &= \sqrt{\gamma_i\gamma_j}\cos[k_{0z}(r_i+r_j)]\end{aligned} \tag{2}$$

where γ_i is the decay rate for transition i in ordinary vacuum. For a single three level atom, we have $r_i = r_j$, for simplicity we set $R = r_i = 0$ and $\gamma_1 = \gamma_2 = \gamma$. After applying the rotating wave approximation(RWA), the master equation Eq.(1) becomes (see Appendix A)

$$\begin{aligned}\frac{d\rho^S}{dt} =& -\frac{1}{2}\sum_i\gamma(1+N)(\rho^S S_i^+S_i^- + S_i^+S_i^-\rho^S - 2S_i^-\rho^S S_i^+) \\ & - \frac{1}{2}\sum_i\gamma N(\rho^S S_i^-S_i^+ + S_i^-S_i^+\rho^S - 2S_i^+\rho^S S_i^-) \\ & - \frac{1}{2}\sum_{\alpha=\pm}\sum_{i\neq j}\gamma M(\rho^S S_i^\alpha S_j^\alpha + S_i^\alpha S_j^\alpha\rho^S - 2S_j^\alpha\rho^S S_i^\alpha)\end{aligned} \tag{3}$$

where $N = \sinh(r)^2$ and $M = \sinh(r)\cosh(r)$The steady state of Eq.(3) can be derived by re-writing Eq.(3) as:

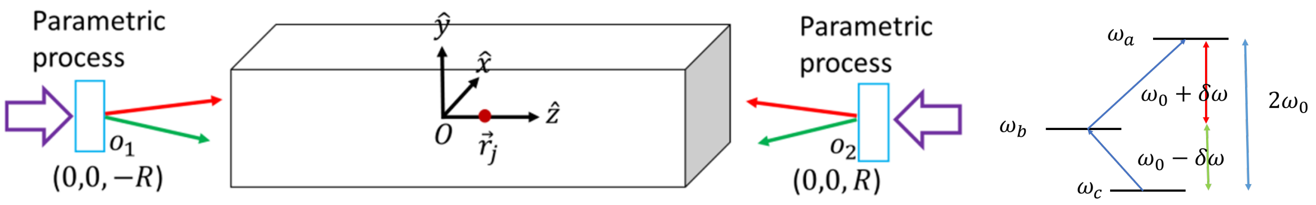

Fig. 1: (a) Schematic setup: A three-level atom is located inside the waveguide with the broadband squeezed vacuum incident from both ends. (b)The energy structure of the three level atom. Transition $|a\rangle \to |c\rangle$ is forbidden and $\omega_{ac} = 2\omega_0$ where ω_0 is the center frequency of the squeezed vacuum. ω_{ab} and ω_{bc} differ by a small amount $2\delta\omega_0$ and they are within the bandwidth of the squeezed vacuum reservoir.

$$\dot{\rho}_{aa}/\gamma = -ch^2\rho_{aa} + sh^2\rho_{bb} - \frac{1}{2}chsh(\rho_{ac} + \rho_{ca}) \tag{4a}$$

$$\dot{\rho}_{bb}/\gamma = (ch^2\rho_{aa} - sh^2\rho_{bb}) + (sh^2\rho_{cc} - ch^2\rho_{bb}) + chsh(\rho_{ac} + \rho_{ca}) \tag{4b}$$

$$\dot{\rho}_{cc}/\gamma = ch^2\rho_{bb} - sh^2\rho_{cc} - \frac{1}{2}chsh(\rho_{ac} + \rho_{ca}) \tag{4c}$$

$$(\dot{\rho}_{ac} + \dot{\rho}_{ca})/\gamma = -\frac{1}{2}(ch^2 + sh^2)(\rho_{ac} + \rho_{ca}) - shch(\rho_{aa} - 2\rho_{bb} + \rho_{cc}) \tag{4d}$$

$$\dot{\rho}_{ab}/\gamma = -(1 + \frac{3}{2}sh^2)\rho_{ab} - \frac{1}{2}chsh\rho_{cb} \tag{4e}$$

$$\dot{\rho}_{cb}/\gamma = -\frac{1}{2}chsh\rho_{ab} - (\frac{1}{2} + \frac{3}{2}sh^2)\rho_{cb} \tag{4f}$$

where $sh = \sinh(r), ch = \cosh(r)$. Eq.(4e)(4f) yield $\rho_{ab} = \rho_{cb} = 0$ for the steady state, and Eq.(4a)-(4d) yield $\rho_{aa} = \frac{sh^2}{sh^2+ch^2}$, $\rho_{cc} = \frac{ch^2}{sh^2+ch^2}$, $\rho_{ac} = -\frac{shch}{sh^2+ch^2}$. Thus, the steady state is actually a superposition state of $|a\rangle$ and $|c\rangle$: $\frac{sh}{\sqrt{sh^2+ch^2}}|a\rangle - \frac{ch}{\sqrt{sh^2+ch^2}}|c\rangle$. This phenomenon is similar to coherent trapping, but here we achieve the trapping for Ξ structure with the squeezed vacuum reservoir, which cannot be realized with coherent pump due to spontaneous emission.

In general we can study the steady state population distribution when the dipole moments of transition $|a\rangle \to |b\rangle$ and $|b\rangle \to |c\rangle$ are different, i.e., $\mu_{ab} \neq \mu_{bc}$. Although the analytical solution is quite lengthy, we can easily get the numerical solution for different μ_{ab} and μ_{bc}. In Fig. 2(a), population inversion is achieved with $\mu_{ab} < \mu_{bc}$. This can be interpreted with the help of Fig. 2(c). Figure 2(c) shows that the direct transition between $|a\rangle$, $|b\rangle$, and $|c\rangle$ are allowed just like the thermal reservoir case. However, in the squeezed vacuum, there is additional paths for population flow: electrons in any of these three states can evolve into the other two through an intermediate "state" ρ_{ac}. Although ρ_{ac} is actually not a state, and $\rho_{ac} < 0$ in our convention, it can be used to elucidate our idea. When $\mu_{ab} \ll \mu_{bc}$, the transition $|a\rangle \to |b\rangle$ can be removed. Thus state $|c\rangle$ can be excited to $|a\rangle$ through $|c\rangle \to |b\rangle \to \rho_{ac} \to |a\rangle$, but $|a\rangle$ can not decay back to $|c\rangle$. Thus, the population is trapped from $|c\rangle$ to $|a\rangle$. Although the population inversion between $|a\rangle$ and $|c\rangle$ is very sensitive to the value of M, the population inversion between $|a\rangle$ and $|b\rangle$ still holds for $M = 0.8\sqrt{N(N+1)}$, which is shown in Fig. 2(b).

APPENDIX A: DERIVATION OF EQ.(??)

Here we will show how to derive the master equation Eq. (1). The interaction Hamiltonian is:

$$V(t) = -i\hbar \sum_{\boldsymbol{k}s} [D(t)a_{\boldsymbol{k}s}(t) - D^+(t)a^\dagger_{\boldsymbol{k}s}(t)], \tag{A1}$$

where

$$D(t) = \sum_i [\boldsymbol{\mu}_i \cdot \boldsymbol{u}_{\boldsymbol{k},s}(r_i)S^\dagger_i(t) + \boldsymbol{\mu}^*_i \cdot \boldsymbol{u}_{\boldsymbol{k},s}(r_i)S^-_i(t)]. \tag{A2}$$

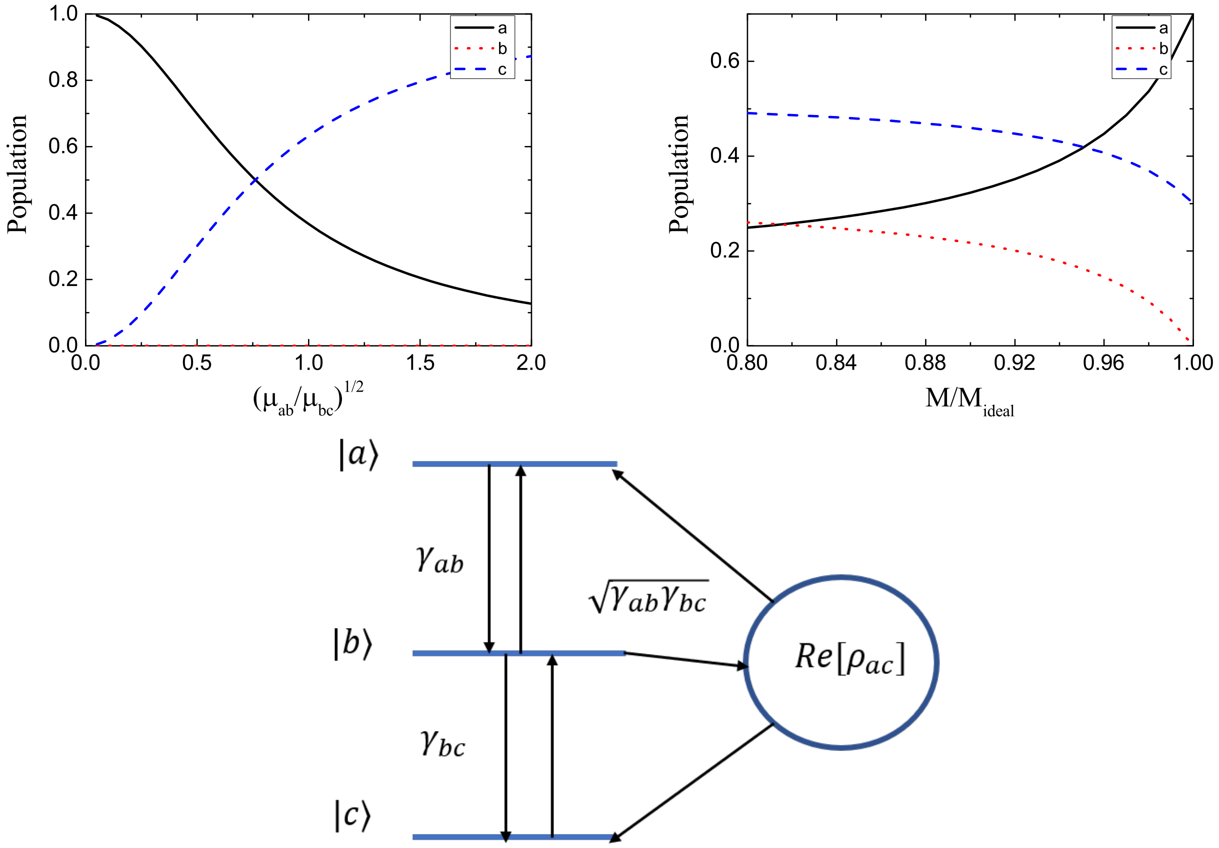

Fig. 2: (a) The steady state population distribution for different μ_{ab} andμ_{bc}. The squeezing parameter $r = 1$. (b) The steady state population distribution for non-ideal squeezed vacuum which is characterized by the ratio of M and $\sqrt{N(N+1)}$. (c) The allowed population flow in the squeezed vacuum. The squeezing parameter $r = 1$, and $\mu_{ab} = \frac{1}{4}\mu_{bc}$.

The reduced master equation of atoms in the reservoir is:

$$
\begin{aligned}
\frac{d\rho^S}{dt} =& -\frac{1}{\hbar^2}\int_0^t d\tau Tr_F\{[V(t),[V(t-\tau),\rho^S(t-\tau)\rho^F\} \\
=& -\frac{1}{\hbar^2}\int_0^t d\tau Tr_F\{V(t)V(t-\tau)\rho^S(t-\tau)\rho^F + \rho^S(t-\tau)\rho^F V(t-\tau)V(t) \\
& - V(t)\rho^S(t-\tau)\rho^F V(t-\tau) - V(t-\tau)\rho^S(t-\tau)\rho^F V(t)\}.
\end{aligned} \tag{A3}
$$

Here we just show how to deal with the first term in Eq.(A3), the remaining terms can be calculated in the same way. For the first term, we have

$$
\begin{aligned}
& -\frac{1}{\hbar^2}\int_0^t d\tau Tr_F\{V(t)V(t-\tau)\rho^S(t-\tau)\rho^F\} \\
=& \int_0^t d\tau \sum_{\boldsymbol{k}s,\boldsymbol{k}'s'} \{D(t)D(t-\tau)Tr_F[\rho^F a_{ks}(t)a_{k's'}(t-\tau)] - D(t)D^+(t-\tau)Tr_F[\rho^F a_{ks}(t)a^\dagger_{k's'}(t-\tau)] \\
& - D^+(t)D(t-\tau)Tr_F[\rho^F a^\dagger_{ks}(t)a_{k's'}(t-\tau)] + D^+(t)D^+(t-\tau)Tr_F[\rho^F a^\dagger_{ks}(t)a^\dagger_{k's'}(t-\tau)]\}\rho^S(t-\tau)\}.
\end{aligned} \tag{A4}
$$

Under the rotating wave approximation(RWA), we have

$$
\begin{aligned}
&-\frac{1}{\hbar^2}\int_0^t d\tau Tr_F\{V(t)V(t-\tau)\rho^S(t-\tau)\rho^F\}\\
=&\sum_{ij}\sum_{\boldsymbol{k}s,\boldsymbol{k'}s'}\int_0^t d\tau\{\boldsymbol{\mu}_i\cdot\boldsymbol{u}_{\boldsymbol{k}s}(r_i)S_i^+e^{i\omega_i t}\boldsymbol{\mu}_j\cdot\boldsymbol{u}_{\boldsymbol{k'}s'}(r_j)S_j^+e^{i\omega_j(t-\tau)}e^{-i(\omega_{\boldsymbol{k}s}+\omega_{\boldsymbol{k'}s'})t+i\omega_{\boldsymbol{k'}s'}\tau}[-\sinh(r)\cosh(r)\delta_{\boldsymbol{k'},2\boldsymbol{k}_0-\boldsymbol{k}}\delta_{ss'}]\\
&-\boldsymbol{\mu}_i\cdot\boldsymbol{u}_{\boldsymbol{k}s}(r_i)S_i^+e^{i\omega_i t}\boldsymbol{\mu}_j^*\cdot\boldsymbol{u}^*_{\boldsymbol{k'}s'}(r_j)S_j^-e^{-i\omega_j(t-\tau)}e^{-i\omega_{\boldsymbol{k'}s'}\tau}\cosh^2 r\delta_{\boldsymbol{k}\boldsymbol{k'}}\delta_{ss'}\\
&-\boldsymbol{\mu}_i^*\cdot\boldsymbol{u}_{\boldsymbol{k}s}(r_i)S_i^-e^{-i\omega_i t}\boldsymbol{\mu}_j\cdot\boldsymbol{u}^*_{\boldsymbol{k'}s'}(r_j)S_j^+e^{i\omega_j(t-\tau)}e^{-i\omega_{\boldsymbol{k'}s'}\tau}\cosh^2 r\delta_{\boldsymbol{k}\boldsymbol{k'}}\delta_{ss'}\\
&-\boldsymbol{\mu}_i^*\cdot\boldsymbol{u}^*_{\boldsymbol{k}s}(r_i)S_i^-e^{-i\omega_i t}\boldsymbol{\mu}_j\cdot\boldsymbol{u}_{\boldsymbol{k'}s'}(r_j)S_j^+e^{i\omega_j(t-\tau)}e^{i\omega_{\boldsymbol{k'}s'}\tau}\sinh^2 r\delta_{\boldsymbol{k}\boldsymbol{k'}}\delta_{ss'}\\
&-\boldsymbol{\mu}_i\cdot\boldsymbol{u}^*_{\boldsymbol{k}s}(r_i)S_i^+e^{i\omega_i t}\boldsymbol{\mu}_j^*\cdot\boldsymbol{u}_{\boldsymbol{k'}s'}(r_j)S_j^-e^{-i\omega_j(t-\tau)}e^{i\omega_{\boldsymbol{k'}s'}\tau}\sinh^2 r\delta_{\boldsymbol{k}\boldsymbol{k'}}\delta_{ss'}\\
&+\boldsymbol{\mu}_i^*\cdot\boldsymbol{u}^*_{\boldsymbol{k}s}(r_i)S_i^-e^{-i\omega_i t}\boldsymbol{\mu}_j^*\cdot\boldsymbol{u}^*_{\boldsymbol{k'}s'}(r_j)S_j^-e^{-i\omega_j(t-\tau)}e^{i(\omega_{\boldsymbol{k}s}+\omega_{\boldsymbol{k'}s'})t-i\omega_{\boldsymbol{k'}s'}\tau}[-\sinh(r)\cosh(r)\delta_{\boldsymbol{k'},2\boldsymbol{k}_0-\boldsymbol{k}}\delta_{ss'}]\}\rho^S(t-\tau)
\end{aligned}
\tag{A5}
$$

Here we just calculate the first and second term to show how to get the master equation Eq.(1). For the second term, we have

$$
\begin{aligned}
&-\sum_{k_z}\int_0^t d\tau\boldsymbol{\mu}_i\cdot\boldsymbol{u}_{\boldsymbol{k}s}(r_i)S_i^+e^{i\omega_i t}\boldsymbol{\mu}_j^*\cdot\boldsymbol{u}^*_{\boldsymbol{k'}s'}(r_j)S_j^-e^{-i\omega_j(t-\tau)}e^{-i\omega_{\boldsymbol{k'}s'}\tau}\cosh^2 r\rho^S(t-\tau)\delta_{\boldsymbol{k}\boldsymbol{k'}}\delta_{ss'}\\
=&-\frac{L}{2\pi}e^{i(\omega_i-\omega_j)t}\int_{-\infty}^{\infty}dk_z\int_0^t d\tau e^{i\omega_j\tau}e^{-i\omega_{k_z}\tau}\frac{\omega_k\mu_i\mu_j}{\epsilon_0 LS\hbar}e^{ik_z(r_i-r_j)}\cosh^2 rS_i^+S_j^-\rho^S(t-\tau)\\
\approx&-\frac{L}{2\pi}e^{i(\omega_i-\omega_j)t}\int_0^{\infty}dk_z\int_0^t d\tau e^{i\omega_j\tau}e^{-i[\omega_j+c^2k_{jz}(k_z-k_{jz})/\omega_j]\tau}\frac{\omega_k\mu_i\mu_j}{\epsilon_0 LS\hbar}[e^{ik_z(r_i-r_j)}+e^{-ik_z(r_i-r_j)}]\cosh^2 rS_i^+S_j^-\rho^S(t-\tau)\\
\approx&-\frac{L}{2\pi}e^{i(\omega_i-\omega_j)t}\int_{-k_{0z}}^{\infty}d\delta k_z\int_0^t d\tau e^{-i\tau c^2k_{jz}\delta k_z/\omega_j}\frac{\omega_k\mu_i\mu_j}{\epsilon_0 LS\hbar}[e^{i(k_{jz}+\delta k_z)(r_i-r_j)}+e^{-i(k_{jz}+\delta k_z)(r_i-r_j)}]\cosh^2 rS_i^+S_j^-\rho^S(t-\tau)\\
\approx&-\frac{L}{2\pi}e^{i(\omega_i-\omega_j)t}\int_{-\infty}^{\infty}d\delta k_z\int_0^t d\tau e^{-i(c^2k_{jz}\delta k_z/\omega_j)\tau}\frac{\omega_k\mu_i\mu_j}{\epsilon_0 LS\hbar}[e^{i(k_{jz}+\delta k_z)(r_i-r_j)}+e^{-i(k_{jz}+\delta k_z)(r_i-r_j)}]\cosh^2 rS_i^+S_j^-\rho^S(t-\tau)\\
\approx&-\frac{L}{2\pi}e^{i(\omega_i-\omega_j)t}\int_0^t d\tau\frac{\omega_j\mu_i\mu_j}{\epsilon_0 LS\hbar}2\pi[e^{ik_{jz}(r_i-r_j)}\delta((r_i-r_j)-\frac{c^2k_{jz}}{\omega_0}\tau)+e^{-ik_{jz}(r_i-r_j)}\delta((r_i-r_j)+\frac{c^2k_{jz}}{\omega_0}\tau)]\cosh^2 rS_i^+S_j^-\rho^S(t-\tau)\\
\approx&-\frac{L}{2\pi}e^{ik_{jz}r_{ij}}\frac{\omega_j\mu_i\mu_j}{\epsilon_0 LS\hbar}2\pi\frac{\omega_j}{c^2k_{0z}}\cosh^2 rS_i^+S_j^-\rho^S(t)e^{i(\omega_i-\omega_j)t}\\
\approx&-[\frac{\sqrt{\gamma_i\gamma_j}}{2}\cos(k_{0z}r_{ij})+i\frac{\sqrt{\gamma_i\gamma_j}}{2}sin(k_{0z}r_{ij})]\cosh^2 rS_i^+S_j^-\rho^S(t)e^{i(\omega_i-\omega_j)t}\\
\equiv&-(\frac{\sqrt{\gamma_i\gamma_j}}{2}+i\Lambda_{ij})\cosh^2 rS_i^+S_j^-\rho^S(t)e^{i(\omega_i-\omega_j)t}
\end{aligned}
\tag{A6}
$$

where emitter separation $r_{ij}=|r_i-r_j|$, $\gamma_i=2\mu_i^2\omega_i^2/\hbar\epsilon_0 Sc^2k_{iz}$ which is the collective decay rate when $i=j$, and $\Lambda_{ij}=\gamma_{1d}\sin(k_{0z}r_{ij})/2$ is the collective energy shift. In the third line we expand $\omega_k=c\sqrt{(\frac{\pi}{a})^2+(k_z)^2}$ around $k_z=k_{0z}$ since resonant modes provide dominant contributions. In the fifth line we extend the integration $\int_{-k_{0z}}^{\infty}dk_z\to\int_{-\infty}^{\infty}dk_z$ because the main contribution comes from the components around $\delta k_z=0$. In the next line, Weisskopf-Wigner approximation is used. Thus, we have obtained γ_{ij} and Λ_{ij} as is shown in Eq.(2).

Next we need to calculate the first term (squeezing term) in Eq.(A5):

$$
\begin{aligned}
&e^{i(\omega_i+\omega_j-2\omega_0)t}\sum_{k_z}\int_0^t d\tau\{\boldsymbol{\mu}_i\cdot\boldsymbol{u}_{2\boldsymbol{k}_0-\boldsymbol{k}}(r_i)S_i^+\boldsymbol{\mu}_j\cdot\boldsymbol{u}_{\boldsymbol{k}}(r_j)S_j^+e^{i(\omega_{\boldsymbol{k}}-\omega_j)\tau}[-\sinh(r)\cosh(r)]\rho^S(t-\tau)\\
=&-\frac{L}{2\pi}e^{i(\omega_i+\omega_j-2\omega_0)t}\int_0^{2k_{0z}}dk_z\int_0^t d\tau e^{i(\omega_{k_z}-\omega_j)\tau}e^{i(2k_{jz}-k_z)(r_i-o_1)}e^{ik_z(r_j-o_1)}\frac{\sqrt{\omega_{k_z}\omega_{2k_{0z}-k_z}}\mu^2}{\epsilon_0 LS\hbar}\sinh(r)\cosh(r)S_i^+S_j^+\rho^S(t-\tau)\\
&-\frac{L}{2\pi}e^{i(\omega_i+\omega_j-2\omega_0)t}\int_{-2k_{0z}}^{0}dk_z\int_0^t d\tau e^{i(\omega_{k_z}-\omega_j)\tau}e^{i(-2k_{jz}-k_z)(r_i-o_2)}e^{ik_z(r_j-o_2)}\frac{\sqrt{\omega_{k_z}\omega_{-2k_{0z}-k_z}}\mu^2}{\epsilon_0 LS\hbar}\sinh(r)\cosh(r)S_i^+S_j^+\rho^S(t-\tau)
\end{aligned}
\tag{A7}
$$

Putting the overall factor $e^{i(\omega_i+\omega_j-2\omega_0)t}$ aside, for $r_i = r_j$, Eq.(A7) reduces to

$$\begin{aligned}
&\sum_{k_z}\int_0^t d\tau\{\boldsymbol{\mu}_i\cdot\boldsymbol{u}_{2\boldsymbol{k}_0-\boldsymbol{k}}(r_i)S_i^+\boldsymbol{\mu}_j\cdot\boldsymbol{u}_{\boldsymbol{k}}(r_j)S_j^+e^{i(\omega_{\boldsymbol{k}}-\omega_j)\tau}[-\sinh(r)\cosh(r)]\rho^S(t-\tau)\\
&=-\frac{L}{2\pi}\int_0^{2k_{0z}}dk_z\int_0^t d\tau e^{i\frac{c^2k_{jz}}{\omega_j}(k_z-k_{jz})\tau}e^{i2k_{0z}(r_i-o_1)}\frac{\sqrt{\omega_{k_z}\omega_{2k_{0z}-k_z}}\mu_i\mu_j}{\epsilon_0 LS\hbar}\sinh(r)\cosh(r)S_i^+S_j^+\rho^S(t-\tau)\\
&-\frac{L}{2\pi}\int_{-2k_{0z}}^0 dk_z\int_0^t d\tau e^{i\frac{c^2k_{jz}}{\omega_j}(k_z-k_{jz})\tau}e^{-i2k_{0z}(r_i-o_2)}\frac{\sqrt{\omega_{k_z}\omega_{-2k_{0z}-k_z}}\mu_i\mu_j}{\epsilon_0 LS\hbar}\sinh(r)\cosh(r)S_i^+S_j^+\rho^S(t-\tau)\\
&=-\frac{L}{2\pi}[e^{i2k_{0z}(r_i-o_1)}+e^{-i2k_{0z}(r_i-o_2)}]\frac{\sqrt{\omega_i\omega_j}\mu_i\mu_j}{\epsilon_0 LS\hbar}\int_0^t d\tau 2\pi\delta(\frac{c^2k_{jz}}{\omega_j}\tau)\sinh(r)\cosh(r)S_i^+S_j^+\rho^S(t-\tau)\\
&=-e^{i2k_{jz}R}\frac{\omega_0^2\mu_i\mu_j}{\epsilon_0\hbar Sc^2k_{0z}}\cos(2k_{0z}r_i)\sinh(r)\cosh(r)S_i^+S_j^+\rho^S(t)\\
&=-e^{i2k_{0z}R}\frac{\sqrt{\gamma_i\gamma_j}}{2}\cos(2k_{0z}r_i)\sinh(r)\cosh(r)S_i^+S_j^+\rho^S(t)
\end{aligned}\tag{A8}$$

where we have used the fact that the origin of coordinate system is at equal distant from two sources(i.e., $o_2 = -o_1 = R$) in the second last line. Thus, we have $\gamma'_{ij} = \sqrt{\gamma_i\gamma_j}\cos(2k_{0z}r_i)$. For $r_i \neq r_j$, Eq. (A7) reduces to

$$\begin{aligned}
&\sum_{k_z}\int_0^t d\tau\{\boldsymbol{\mu}_i\cdot\boldsymbol{u}_{2\boldsymbol{k}_0-\boldsymbol{k}}(r_i)S_i^+\boldsymbol{\mu}_j\cdot\boldsymbol{u}_{\boldsymbol{k}}(r_j)S_j^+e^{i(\omega_{\boldsymbol{k}}-\omega_j)\tau}[-\sinh(r)\cosh(r)]\rho^S(t-\tau)\\
&=-\frac{L}{2\pi}\int_0^{2k_{0z}}dk_z\int_0^t d\tau e^{i\frac{c^2k_{jz}}{\omega_j}(k_z-k_{jz})\tau}e^{i2k_{0z}(r_c-o_1)}e^{-i(k_z-k_{0z})(r_i-r_j)}\frac{\sqrt{\omega_{k_z}\omega_{2k_{0z}-k_z}}\mu_i\mu_j}{\epsilon_0 LS\hbar}\sinh(r)\cosh(r)S_i^+S_j^+\rho^S(t-\tau)\\
&-\frac{L}{2\pi}\int_{-2k_{0z}}^0 dk_z\int_0^t d\tau e^{i\frac{c^2k_{jz}}{\omega_j}(-k_z-k_{jz})\tau}e^{-i2k_{0z}(r_c-o_2)}e^{-i(k_z+k_{0z})(r_i-r_j)}\frac{\sqrt{\omega_{k_z}\omega_{-2k_{0z}-k_z}}\mu_i\mu_j}{\epsilon_0 LS\hbar}\sinh(r)\cosh(r)S_i^+S_j^+\rho^S(t-\tau)\\
&=-\frac{L}{2\pi}e^{i2k_{0z}(r_c-o_1)}\frac{\sqrt{\omega_i\omega_j}\mu_i\mu_j}{\epsilon_0 LS\hbar}\int_{-\infty}^{\infty}dk_z\int_0^t d\tau e^{i\frac{c^2k_{jz}}{\omega_j}(k_z-k_{jz})\tau}e^{-i(k_z-k_{0z})(r_i-r_j)}\sinh(r)\cosh(r)S_i^+S_j^+\rho^S(t-\tau)\\
&-\frac{L}{2\pi}e^{-i2k_{0z}(r_c-o_2)}\frac{\sqrt{\omega_i\omega_j}\mu_i\mu_j}{\epsilon_0 LS\hbar}\int_{-\infty}^{\infty}dk_z\int_0^t d\tau e^{i\frac{c^2k_{jz}}{\omega_j}(k_z-k_{jz})\tau}e^{i(k_z-k_{0z})(r_i-r_j)}\sinh(r)\cosh(r)S_i^+S_j^+\rho^S(t-\tau)\\
&\approx-\frac{L}{2\pi}e^{i2k_{0z}R}\frac{\omega_0^2\mu_i\mu_j}{\epsilon_0 LS\hbar}\int_0^t d\tau 2\pi[e^{i2k_{0z}r_c}\delta(r_i-r_j-\frac{c^2k_{0z}}{\omega_0}\tau)+e^{-i2k_{0z}r_c}\delta(r_i-r_j+\frac{c^2k_{0z}}{\omega_0}\tau)]\sinh(r)\cosh(r)S_i^+S_j^+\rho^S(t-\tau)\\
&\approx-e^{i2k_{0z}R}\frac{\omega_0^2\mu_i\mu_j}{\epsilon_0\hbar Sc^2k_{0z}}e^{i2k_{0z}r_c sgn(i-j)}S_i^+S_j^+\rho^S(t)\to-\frac{\sqrt{\gamma_i\gamma_j}}{2}e^{i2k_{0z}R}\cos(k_{0z}(r_i+r_j))S_i^+S_j^+\rho^S(t)
\end{aligned}\tag{A9}$$

where $sgn(i-j)$ is the sign function. The last arrow is because we need to sum over i, j, so the imaginary part of $e^{i2k_{0z}r_c sgn(i-j)}$ vanishes and the neat result is that $\gamma'_{ij} = e^{i2k_{0z}R}\sqrt{\gamma_i\gamma_j}\cos(k_{0z}(r_i + r_j))$. As for $S_i^+\rho^S(t)S_j^+$ terms, the combination of the last two terms in Eq.(A3) will make the imaginary part of $e^{i2k_{0z}r_c sgn(i-j)}$ vanish. Thus, we have $\gamma'_{ij} = e^{i2k_{0z}R}\sqrt{\gamma_i\gamma_j}\cos(k_{0z}(r_i + r_j))$. Thus, we

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