

Cavity-Cavity interaction in the Squeezed Vacuum

PACS numbers:

I. GENERAL MASTER EQUATION OF CAVITY-CAVITY INTERACTION

In this section, we will derive the master equation for two single-mode leaky cavities placed inside the waveguide with the squeezed vacuum injected from both ends. The schematic setup is shown in Fig. 1. Then we will study how the modes inside the cavity will evolve under the influence of the squeezed vacuum. The free Hamiltonian of cavity and waveguide modes is:

$$H_0 = \sum_i \hbar \omega_i (a_i^\dagger a_i + \frac{1}{2}) + \hbar \sum_{k,s} \omega_k (a_{k,s}^\dagger a_{k,s} + \frac{1}{2}) \quad (1)$$

where a_k stands for the modes in the waveguide and a_i is the field operator of the single mode inside i th the cavity. The waveguide is saturated with the squeezed vacuum with the center frequency ω_0 . The interaction Hamiltonian between the cavity mode and waveguide modes is:

$$V = -i\hbar \sum_{\mathbf{k}s} [D a_{\mathbf{k}s} - D^\dagger a_{\mathbf{k}s}^\dagger] \quad (2)$$

where

$$D = \sum_i [g_{i,k,s}^* a_i^\dagger + g_{i,k,s} a_i] \quad (3)$$

Here we define $g_{i,k,s} = |g_{i,k,s}| e^{-ik_z r_i}$ where r_i is just a phenomenological parameter describing the location of cavity. The reduced master equation of atoms in the reservoir is[1]

$$\begin{aligned} \frac{d\rho^S}{dt} &= -\frac{1}{\hbar^2} \int_0^t d\tau \text{Tr}_F \{ [V(t), [V(t-\tau), \rho^S(t-\tau) \rho^F]] \} \\ &= -\frac{1}{\hbar^2} \int_0^t d\tau \text{Tr}_F \{ V(t) V(t-\tau) \rho^S(t-\tau) \rho^F + \rho^S(t-\tau) \rho^F V(t-\tau) V(t) \\ &\quad - V(t) \rho^S(t-\tau) \rho^F V(t-\tau) - V(t-\tau) \rho^S(t-\tau) \rho^F V(t) \}. \end{aligned} \quad (4)$$

Here we just show how to deal with the first term in Eq.(4), the remaining terms can be calculated in the same way. For the first term, we have

$$\begin{aligned} & -\frac{1}{\hbar^2} \int_0^t d\tau \text{Tr}_F \{ V(t) V(t-\tau) \rho^S(t-\tau) \rho^F \} \\ &= \int_0^t d\tau \sum_{\mathbf{k}s, \mathbf{k}'s'} \{ D(t) D(t-\tau) \text{Tr}_F [\rho^F a_{\mathbf{k}s}(t) a_{\mathbf{k}'s'}^\dagger(t-\tau)] - D(t) D^\dagger(t-\tau) \text{Tr}_F [\rho^F a_{\mathbf{k}s}(t) a_{\mathbf{k}'s'}^\dagger(t-\tau)] \\ &\quad - D^\dagger(t) D(t-\tau) \text{Tr}_F [\rho^F a_{\mathbf{k}s}^\dagger(t) a_{\mathbf{k}'s'}(t-\tau)] + D^\dagger(t) D^\dagger(t-\tau) \text{Tr}_F [\rho^F a_{\mathbf{k}s}^\dagger(t) a_{\mathbf{k}'s'}(t-\tau)] \} \rho^S(t-\tau). \end{aligned} \quad (5)$$

Under the rotating wave approximation(RWA), we have

$$\begin{aligned} & -\frac{1}{\hbar^2} \int_0^t d\tau \text{Tr}_F \{ V(t) V(t-\tau) \rho^S(t-\tau) \rho^F \} \\ &= \sum_{ij} \sum_{\mathbf{k}s, \mathbf{k}'s'} \int_0^t d\tau \{ g_{i,k,s}^* a_i^\dagger e^{i\omega_i t} g_{j,k',s'}^* a_j^\dagger e^{i\omega_j(t-\tau)} e^{-i(\omega_{\mathbf{k}s} + \omega_{\mathbf{k}'s'})t + i\omega_{\mathbf{k}'s'}\tau} [-\sinh(r) \cosh(r) \delta_{\mathbf{k}', 2\mathbf{k}_0 - \mathbf{k}} \delta_{ss'}] \\ &\quad - g_{i,k,s}^* a_i^\dagger e^{i\omega_i t} g_{j,k',s'} a_j e^{-i\omega_j(t-\tau)} e^{-i\omega_{\mathbf{k}'s'}\tau} \cosh^2 r \delta_{\mathbf{k}\mathbf{k}'} \delta_{ss'} \\ &\quad - g_{i,k,s} a_i e^{-i\omega_i t} g_{j,k',s'}^* a_j^\dagger e^{i\omega_j(t-\tau)} e^{-i\omega_{\mathbf{k}'s'}\tau} \cosh^2 r \delta_{\mathbf{k}\mathbf{k}'} \delta_{ss'} \\ &\quad - g_{i,k,s} a_i e^{-i\omega_i t} g_{j,k',s'}^* a_j^\dagger e^{i\omega_j(t-\tau)} e^{i\omega_{\mathbf{k}'s'}\tau} \sinh^2 r \delta_{\mathbf{k}\mathbf{k}'} \delta_{ss'} \\ &\quad - g_{i,k,s}^* a_i^\dagger e^{i\omega_i t} g_{j,k',s'} a_j e^{-i\omega_j(t-\tau)} e^{i\omega_{\mathbf{k}'s'}\tau} \sinh^2 r \delta_{\mathbf{k}\mathbf{k}'} \delta_{ss'} \\ &\quad + g_{i,k,s} a_i e^{-i\omega_i t} g_{j,k',s'} a_j e^{-i\omega_j(t-\tau)} e^{i(\omega_{\mathbf{k}s} + \omega_{\mathbf{k}'s'})t - i\omega_{\mathbf{k}'s'}\tau} [-\sinh(r) \cosh(r) \delta_{\mathbf{k}', 2\mathbf{k}_0 - \mathbf{k}} \delta_{ss'}] \} \rho^S(t-\tau) \end{aligned} \quad (6)$$

Here we just calculate the first and second term to show how to get the master equation. For the second term, we have

$$\begin{aligned}
& - \sum_{k_z} \int_0^t d\tau g_{i,k,s}^* a_i^\dagger e^{i\omega_i t} g_{j,k',s'} a_j e^{-i\omega_j(t-\tau)} e^{-i\omega_{k',s'}\tau} \cosh^2 r \rho^S(t-\tau) \delta_{\mathbf{k}\mathbf{k}'} \delta_{ss'} \\
& = - \frac{L}{2\pi} e^{i(\omega_i - \omega_j)t} \int_{-\infty}^{\infty} dk_z \int_0^t d\tau e^{i\omega_j \tau} e^{-i\omega_{k_z} \tau} |g_{i,\mathbf{k},s} g_{j,\mathbf{k},s}| e^{ik_z(r_i - r_j)} \cosh^2 r a_i^\dagger a_j \rho^S(t-\tau) \\
& \approx - \frac{L}{2\pi} e^{i(\omega_i - \omega_j)t} \int_0^{\infty} dk_z \int_0^t d\tau e^{i\omega_j \tau} e^{-i[\omega_j + c^2 k_{jz}(k_z - k_{jz})/\omega_j]\tau} |g_{i,\mathbf{k},s} g_{j,\mathbf{k},s}| [e^{ik_z(r_i - r_j)} + e^{-ik_z(r_i - r_j)}] \cosh^2 r a_i^\dagger a_j \rho^S(t-\tau) \\
& \approx - \frac{L}{2\pi} e^{i(\omega_i - \omega_j)t} \int_{-k_{0z}}^{\infty} d\delta k_z \int_0^t d\tau e^{-i\tau c^2 k_{jz} \delta k_z / \omega_j} |g_{i,\mathbf{k},s} g_{j,\mathbf{k},s}| [e^{i(k_{jz} + \delta k_z)(r_i - r_j)} + e^{-i(k_{jz} + \delta k_z)(r_i - r_j)}] \cosh^2 r a_i^\dagger a_j \rho^S(t-\tau) \\
& \approx - \frac{L}{2\pi} e^{i(\omega_i - \omega_j)t} \int_{-\infty}^{\infty} d\delta k_z \int_0^t d\tau e^{-i(c^2 k_{jz} \delta k_z / \omega_j)\tau} |g_{i,\mathbf{k},s} g_{j,\mathbf{k},s}| [e^{i(k_{jz} + \delta k_z)(r_i - r_j)} + e^{-i(k_{jz} + \delta k_z)(r_i - r_j)}] \cosh^2 r a_i^\dagger a_j \rho^S(t-\tau) \\
& \approx - \frac{L}{2\pi} e^{i(\omega_i - \omega_j)t} \int_0^t d\tau |g_{i,\mathbf{k},s} g_{j,\mathbf{k},s}| 2\pi [e^{ik_{jz}(r_i - r_j)} \delta((r_i - r_j) - \frac{c^2 k_{jz}}{\omega_0} \tau) + e^{-ik_{jz}(r_i - r_j)} \delta((r_i - r_j) + \frac{c^2 k_{jz}}{\omega_0} \tau)] \cosh^2 r a_i^\dagger a_j \rho^S(t-\tau) \\
& \approx - \frac{L}{2\pi} e^{ik_{jz} r_{ij}} |g_{i,\mathbf{k},s} g_{j,\mathbf{k},s}| 2\pi \frac{\omega_j}{c^2 k_{0z}} \cosh^2 r a_i^\dagger a_j \rho^S(t) e^{i(\omega_i - \omega_j)t} \\
& \approx - [\frac{\sqrt{\gamma_i \gamma_j}}{2} \cos(k_{0z} r_{ij}) + i \frac{\sqrt{\gamma_i \gamma_j}}{2} \sin(k_{0z} r_{ij})] \cosh^2 r a_i^\dagger a_j \rho^S(t) e^{i(\omega_i - \omega_j)t} \\
& \equiv - (\frac{\sqrt{\gamma_i \gamma_j}}{2} + i \Lambda_{ij}) \cosh^2 r a_i^\dagger a_j \rho^S(t) e^{i(\omega_i - \omega_j)t}
\end{aligned} \tag{7}$$

where $r_{ij} = |r_i - r_j|$ is also a phenomenological parameter indicating the relative position between cavities. $\gamma_i = L|g_{i,k_0}|^2$ is the leaking rate for the i th cavity, and $\Lambda_{ij} = \sqrt{\gamma_i \gamma_j} \sin(k_{0z} r_{ij})/2$ is the energy shift. In the third line we expand $\omega_k = c\sqrt{(\frac{\pi}{a})^2 + (k_z)^2}$ around $k_z = k_{0z}$ since resonant modes provide dominant contributions. In the fifth line we extend the integration $\int_{-k_{0z}}^{\infty} dk_z \rightarrow \int_{-\infty}^{\infty} dk_z$ because the main contribution comes from the components around $\delta k_z = 0$. In the next line, Weisskopf-Wigner approximation is used.

Next we need to calculate the first term (squeezing term) in Eq.(6):

$$\begin{aligned}
& e^{i(\omega_i + \omega_j - 2\omega_0)t} \sum_{k_z} \int_0^t d\tau \{g_{i,2\mathbf{k}_0 - \mathbf{k}}^* a_i^\dagger g_{j,\mathbf{k}}^* a_j^\dagger e^{i(\omega_{\mathbf{k}} - \omega_j)\tau} [-\sinh(r) \cosh(r)] \rho^S(t-\tau) \\
& = - \frac{L}{2\pi} e^{i(\omega_i + \omega_j - 2\omega_0)t} \int_0^{2k_{0z}} dk_z \int_0^t d\tau e^{i(\omega_{k_z} - \omega_j)\tau} e^{i(2k_{iz} - k_z)(r_i - o_1)} e^{ik_z(r_j - o_1)} |g_{i,2\mathbf{k}_0 - \mathbf{k}} g_{j,\mathbf{k}}| \sinh(r) \cosh(r) a_i^\dagger a_j^\dagger \rho^S(t-\tau) \\
& - \frac{L}{2\pi} e^{i(\omega_i + \omega_j - 2\omega_0)t} \int_{-2k_{0z}}^0 dk_z \int_0^t d\tau e^{i(\omega_{k_z} - \omega_j)\tau} e^{i(-2k_{iz} - k_z)(r_i - o_2)} e^{ik_z(r_j - o_2)} |g_{i,2\mathbf{k}_0 - \mathbf{k}} g_{j,\mathbf{k}}| \sinh(r) \cosh(r) a_i^\dagger a_j^\dagger \rho^S(t-\tau)
\end{aligned} \tag{8}$$

Putting the overall factor $e^{i(\omega_i + \omega_j - 2\omega_0)t}$ aside, for $i = j$, Eq.(8) reduces to

$$\begin{aligned}
& \sum_{k_z} \int_0^t d\tau \{g_{i,2\mathbf{k}_0 - \mathbf{k}}^* a_i^\dagger g_{i,\mathbf{k}}^* a_i^\dagger e^{i(\omega_{\mathbf{k}} - \omega_i)\tau} [-\sinh(r) \cosh(r)] \rho^S(t-\tau) \\
& = - \frac{L}{2\pi} \int_0^{2k_{0z}} dk_z \int_0^t d\tau e^{i\frac{c^2 k_{iz}}{\omega_i}(k_z - k_{iz})\tau} e^{i2k_{0z}(r_i - o_1)} |g_{i,2\mathbf{k}_0 - \mathbf{k}} g_{i,\mathbf{k}}| \sinh(r) \cosh(r) a_i^\dagger a_i^\dagger \rho^S(t-\tau) \\
& - \frac{L}{2\pi} \int_{-2k_{0z}}^0 dk_z \int_0^t d\tau e^{i\frac{c^2 k_{iz}}{\omega_i}(k_z - k_{iz})\tau} e^{-i2k_{0z}(r_i - o_2)} |g_{i,2\mathbf{k}_0 - \mathbf{k}} g_{i,\mathbf{k}}| \sinh(r) \cosh(r) a_i^\dagger a_i^\dagger \rho^S(t-\tau) \\
& = - \frac{L}{2\pi} [e^{i2k_{0z}(r_i - o_1)} + e^{-i2k_{0z}(r_i - o_2)}] |g_{i,2\mathbf{k}_0 - \mathbf{k}} g_{i,\mathbf{k}}| \int_0^t d\tau 2\pi \delta(\frac{c^2 k_{iz}}{\omega_i} \tau) \sinh(r) \cosh(r) a_i^\dagger a_i^\dagger \rho^S(t-\tau) \\
& = - e^{i2k_{0z} R} \frac{\gamma_i}{2} \cos(2k_{0z} r_i) \sinh(r) \cosh(r) a_i^\dagger a_i^\dagger \rho^S(t)
\end{aligned} \tag{9}$$

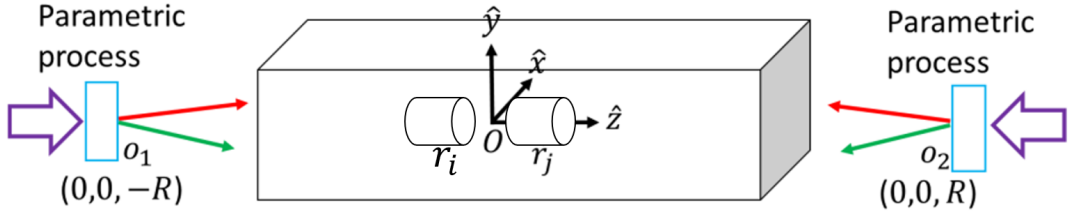


Fig. 1: (a) Schematic setup: two single-mode cavities are placed inside the waveguide with the broadband squeezed vacuum incident from both ends.

where we have used the fact that the origin of coordinate system is at equal distance from two sources (i.e., $o_2 = -o_1 = R$) in the second last line. Thus, we have $\gamma'_{ii} = \gamma_i \cos(2k_{0z}r_i)$. For $r_i \neq r_j$, Eq. (8) reduces to

$$\begin{aligned}
& \sum_{k_z} \int_0^t d\tau \{g_{i,2\mathbf{k}_0-\mathbf{k}} a_i^\dagger g_{j,\mathbf{k}} a_j^\dagger e^{i(\omega_{\mathbf{k}}-\omega_j)\tau} [-\sinh(r) \cosh(r)] \rho^S(t-\tau) \\
&= -\frac{L}{2\pi} \int_0^{2k_{0z}} dk_z \int_0^t d\tau e^{i\frac{c^2 k_{jz}}{\omega_j}(k_z-k_{jz})\tau} e^{i2k_{0z}(r_c-o_1)} e^{-i(k_z-k_{0z})(r_i-r_j)} |g_{i,2\mathbf{k}_0-\mathbf{k}} g_{j,\mathbf{k}}| \sinh(r) \cosh(r) a_i^\dagger a_j^\dagger \rho^S(t-\tau) \\
&- \frac{L}{2\pi} \int_{-2k_{0z}}^0 dk_z \int_0^t d\tau e^{i\frac{c^2 k_{jz}}{\omega_j}(-k_z-k_{jz})\tau} e^{-i2k_{0z}(r_c-o_2)} e^{-i(k_z+k_{0z})(r_i-r_j)} |g_{i,2\mathbf{k}_0-\mathbf{k}} g_{j,\mathbf{k}}| \sinh(r) \cosh(r) a_i^\dagger a_j^\dagger \rho^S(t-\tau) \\
&= -\frac{L}{2\pi} e^{i2k_{0z}(r_c-o_1)} |g_{i,2\mathbf{k}_0-\mathbf{k}} g_{j,\mathbf{k}}| \int_{-\infty}^{\infty} dk_z \int_0^t d\tau e^{i\frac{c^2 k_{jz}}{\omega_j}(k_z-k_{jz})\tau} e^{-i(k_z-k_{0z})(r_i-r_j)} \sinh(r) \cosh(r) a_i^\dagger a_j^\dagger \rho^S(t-\tau) \\
&- \frac{L}{2\pi} e^{-i2k_{0z}(r_c-o_2)} |g_{i,2\mathbf{k}_0-\mathbf{k}} g_{j,\mathbf{k}}| \int_{-\infty}^{\infty} dk_z \int_0^t d\tau e^{i\frac{c^2 k_{jz}}{\omega_j}(k_z-k_{jz})\tau} e^{i(k_z-k_{0z})(r_i-r_j)} \sinh(r) \cosh(r) a_i^\dagger a_j^\dagger \rho^S(t-\tau) \\
&\approx -\frac{L}{2\pi} e^{i2k_{0z}R} |g_{i,2\mathbf{k}_0-\mathbf{k}} g_{j,\mathbf{k}}| \int_0^t d\tau 2\pi [e^{i2k_{0z}r_c} \delta(r_i-r_j - \frac{c^2 k_{0z}}{\omega_0} \tau) + e^{-i2k_{0z}r_c} \delta(r_i-r_j + \frac{c^2 k_{0z}}{\omega_0} \tau)] \sinh(r) \cosh(r) a_i^\dagger a_j^\dagger \rho^S(t-\tau) \\
&\approx -e^{i2k_{0z}R} L |g_{i,\mathbf{k}_0} g_{j,\mathbf{k}_0}| e^{i2k_{0z}r_c \text{sgn}(i-j)} a_i^\dagger a_j^\dagger \rho^S(t) \rightarrow -\frac{\sqrt{\gamma_i \gamma_j}}{2} e^{i2k_{0z}R} \cos(k_{0z}(r_i+r_j)) a_i^\dagger a_j^\dagger \rho^S(t)
\end{aligned} \tag{10}$$

where $\text{sgn}(i-j)$ is the sign function. The last arrow is because we need to sum over i, j , so the imaginary part of $e^{i2k_{0z}r_c \text{sgn}(i-j)}$ vanishes and the neat result is that $\gamma'_{ij} = e^{i2k_{0z}R} \sqrt{\gamma_i \gamma_j} \cos(k_{0z}(r_i+r_j))$. As for $a_i^\dagger \rho^S(t) a_j^\dagger$ terms, the combination of the last two terms in Eq.(4) will make the imaginary part of $e^{i2k_{0z}r_c \text{sgn}(i-j)}$ vanish. Thus, we have $\gamma'_{ij} = e^{i2k_{0z}R} \sqrt{\gamma_i \gamma_j} \cos(k_{0z}(r_i+r_j))$.

Doing the above calculation for all terms in Eq.(4), we have the general equation for cavity-cavity interaction in the squeezed vacuum as follows:

$$\begin{aligned}
\dot{\rho} &= \sum_{ij} \gamma_{ij} \cosh^2 r (-\rho a_i^\dagger a_j - a_i^\dagger a_j \rho + 2a_i \rho a_j^\dagger) e^{i(\omega_i-\omega_j)t} \\
&+ \sum_{ij} \gamma_{ij} \sinh^2 r (-\rho a_i a_j^\dagger - a_i a_j^\dagger \rho + 2a_i^\dagger \rho a_j) e^{-i(\omega_i-\omega_j)t} \\
&+ \sum_{ij} \gamma_{ij} \cosh r \sinh r [(e^{i\theta} \rho a_i a_j + e^{i\theta} a_i a_j \rho - e^{i\theta} 2a_i \rho a_j) e^{-i(\omega_i+\omega_j)t} + H.c.]
\end{aligned} \tag{11}$$

II. STEADY STATE OF NON-RESONANT CAVITIES

First, we study two non-resonant cavities coupled to the squeezed vacuum reservoir. The eigen frequencies of these two cavities are $\omega_1 = \omega_0 - \delta\omega$ and $\omega_2 = \omega_0 + \delta\omega$. Under the rotating wave approximation(RWA), Eq.(11) becomes:

$$\begin{aligned}\dot{\rho} = & \sum_i \gamma(1+N)(-\rho a_i^\dagger a_i - a_i^\dagger a_i \rho + 2a_i \rho a_i^\dagger) \\ & + \sum_i \gamma N(-\rho a_i a_i^\dagger - a_i a_i^\dagger \rho + 2a_i^\dagger \rho a_i) \\ & + \sum_{i \neq j} \gamma M(e^{i\theta} \rho a_i a_j + e^{i\theta} a_i a_j \rho - 2e^{i\theta} a_i \rho a_j + h.c.)\end{aligned}\quad (12)$$

where we have assumed $\gamma_{ij} = \gamma$ for simplicity. The above equation can be re-arranged as:

$$\begin{aligned}\dot{\rho} = & \sum_{i \neq j} \frac{\gamma}{2} [-\rho(\cosh(r)a_i^\dagger - e^{i\theta} \sinh(r)a_j)(\cosh(r)a_i - e^{-i\theta} \sinh(r)a_j^\dagger) \\ & - (\cosh(r)a_i^\dagger - e^{i\theta} \sinh(r)a_j)(\cosh(r)a_i - e^{-i\theta} \sinh(r)a_j^\dagger)\rho \\ & + 2(\cosh(r)a_i - e^{-i\theta} \sinh(r)a_j^\dagger)\rho(\cosh(r)a_i^\dagger - e^{i\theta} \sinh(r)a_j)]\end{aligned}\quad (13)$$

we use the following Bogoliubov transformation[3]:

$$\begin{aligned}S &= \exp(\eta^* a_i a_j - \eta a_i^\dagger a_j^\dagger) \\ A_i &= S^+ a_i S = \cosh(r)a_i - e^{-i\theta} \sinh(r)a_j^\dagger \\ A_i^+ &= S^+ a_i^+ S = \cosh(r)a_i^+ - e^{i\theta} \sinh(r)a_j\end{aligned}\quad (14)$$

so the master equation Eq.(13) becomes:

$$\dot{\rho} = \sum_i \gamma [-\rho A_i^\dagger A_i - A_i^\dagger A_i \rho + 2A_i \rho A_i^\dagger]\quad (15)$$

Next we redefine the density matrix: $\rho_s = S\rho S^\dagger$. Thus Eq.(15) becomes:

$$\begin{aligned}\dot{\rho}_s &= \sum_i \gamma [-\rho_s a_i^\dagger a_i - a_i^\dagger a_i \rho_s + 2a_i \rho_s a_i^\dagger] \\ &\equiv \sum_i \gamma [-a_i^{l\dagger} a_i^l \rho_s - a_i^{r\dagger} a_i^r \rho_s + 2a_i^r a_i^{l\dagger} \rho_s] \equiv L\rho_s\end{aligned}\quad (16)$$

Here we define superoperator $\{a_i^l, a_i^{l\dagger}\}(\{a_i^r, a_i^{r\dagger}\})$ only acting to the left(right) on density operator ρ [4, 5]. These operators have the following commutation relations:

$$[a_i^r, a_j^{r\dagger}] = \delta_{ij}, [a_i^l, a_j^{l\dagger}] = -\delta_{ij}, [a_i^l, a_j^{r\dagger}] = [a_i^l, a_j^r] = [a_i^{l\dagger}, a_j^r] = [a_i^{l\dagger}, a_j^{r\dagger}] = 0\quad (17)$$

Thus, the steady state of Eq.(16) can be solved by solving $L\rho = 0$, which requires the diagonalization of superoperator L . Applying the similarity transformation $U = e^{-a_1^r a_1^{l\dagger} - a_2^r a_2^{l\dagger}}$ to Eq.(16), since we have $U^{-1}(a_i^{r\dagger}, a_i^l, a_i^r, a_i^{l\dagger})U = (a_i^{r\dagger} + a_i^{l\dagger}, a_i^r + a_i^l, a_i^r, a_i^{l\dagger})$, the right hand side of Eq.(16) becomes:

$$RHS = \sum_i \gamma U^{-1} [-a_i^{l\dagger} a_i^l - a_i^{r\dagger} a_i^r + 2a_i^r a_i^{l\dagger}] U U^{-1} \rho_s = \sum_i \gamma [-a_i^{l\dagger} a_i^l - a_i^{r\dagger} a_i^r] U^{-1} \rho_s\quad (18)$$

The only solution to $L\rho = 0$ is $U^{-1}\rho_s = |0,0\rangle\langle 0,0|$, which yields $\rho = S^\dagger \rho_s S = S^\dagger e^{-K_1 - K_2} |0,0\rangle\langle 0,0| S = S^\dagger |0,0\rangle\langle 0,0| S$ which is the two mode squeezed vacuum.

III. STEADY STATE OF RESONANT CAVITIES

Next we study the case where two cavities are identical, i.e., $\omega_1 = \omega_2 = \omega_0$. Then the master equation becomes:

$$\begin{aligned}\dot{\rho} = & \sum_{ij} \gamma \cosh^2 r (-\rho a_i^\dagger a_j - a_i^\dagger a_j \rho + 2a_j \rho a_i^\dagger) \\ & + \sum_{ij} \gamma \sinh^2 r (-\rho a_i a_j^\dagger - a_i a_j^\dagger \rho + 2a_j^\dagger \rho a_i) \\ & + \sum_{ij} \gamma \cosh r \sinh r (e^{i\theta} \rho a_i a_j + e^{i\theta} a_i a_j \rho - e^{i\theta} 2a_i \rho a_j + h.c.)\end{aligned}\quad (19)$$

This equation can be rearranged as follows:

$$\begin{aligned}\dot{\rho} = & \sum_{ij} \gamma [-\rho (\cosh r a_i^\dagger - e^{i\theta} \sinh r a_i) (\cosh r a_j - e^{-i\theta} \sinh r a_j^\dagger) \\ & - (\cosh r a_i^\dagger - e^{i\theta} \sinh r a_i) (\cosh r a_j - e^{-i\theta} \sinh r a_j^\dagger) \rho \\ & + 2(\cosh r a_j - e^{-i\theta} \sinh r a_j^\dagger) \rho (\cosh r a_i^\dagger - e^{i\theta} \sinh r a_i)]\end{aligned}\quad (20)$$

We introduce the Bogoliubov transformation:

$$\begin{aligned}S_i &= \exp\left(\frac{1}{2}\eta^* a_i^2 - \frac{1}{2}\eta a_i^{\dagger 2}\right) \\ A_i &= S_i^\dagger a_i S_i = \cosh(r) a_i - e^{-i\theta} \sinh(r) a_i^\dagger \\ A_i^\dagger &= S_i^\dagger a_i^\dagger S_i = \cosh(r) a_i^\dagger - e^{i\theta} \sinh(r) a_i\end{aligned}\quad (21)$$

so master equation Eq.(20) becomes

$$\dot{\rho} = \sum_{ij} \gamma [-\rho A_i^\dagger A_j - A_i^\dagger A_j \rho + 2A_j \rho A_i^\dagger]\quad (22)$$

Next we define $\rho_s = S_1 S_2 \rho S_1^\dagger S_2^\dagger$ so the master equation is reduced to:

$$\dot{\rho}_s = \sum_{ij} \gamma [-\rho_s a_i^\dagger a_j - a_i^\dagger a_j \rho_s + 2a_j \rho_s a_i^\dagger]\quad (23)$$

To diagonalize this Lindblad equation, we introduce the transformation:

$$\begin{aligned}L_1 &= \frac{1}{\sqrt{2}}(a_1 - a_2) \\ L_2 &= \frac{1}{\sqrt{2}}(a_1 + a_2)\end{aligned}$$

where $[L_i, L_j^\dagger] = \delta_{ij}$, and the master equation becomes:

$$\begin{aligned}\dot{\rho}_s &= \gamma [-2\rho_s L_2^\dagger L_2 - 2L_2^\dagger L_2 \rho_s + 4L_2 \rho_s L_2^\dagger] \\ &= \gamma [-2L_2^{r\dagger} L_2^r \rho_s - 2L_2^{l\dagger} L_2^l \rho_s + 4L_2^l L_2^{r\dagger} \rho_s] \\ &= L \rho\end{aligned}\quad (24)$$

Operator L_2^\dagger has the following properties:

$$L_2^\dagger |0\rangle = \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle) \equiv |1_{L_2}\rangle$$

$$L_2^\dagger \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle) = \sqrt{2} \left[\frac{1}{2}(|02\rangle + \sqrt{2}|11\rangle + |20\rangle) \right] = \sqrt{2} |2_{L_2}\rangle$$

$$L_2^\dagger \frac{1}{2}(|02\rangle + \sqrt{2}|11\rangle + |20\rangle) = \sqrt{3} \left[\frac{1}{2\sqrt{2}}(|03\rangle + \sqrt{3}|12\rangle + \sqrt{3}|21\rangle + |30\rangle) \right] = \sqrt{3} |3_{L_2}\rangle$$

...

while the operator L_1^\dagger has the following properties:

$$L_1^\dagger|0\rangle = \frac{1}{\sqrt{2}}(-|01\rangle + |10\rangle) \equiv |1_{L_1}\rangle$$

$$L_1^\dagger \frac{1}{\sqrt{2}}(-|01\rangle + |10\rangle) = \sqrt{2}[\frac{1}{2}(|02\rangle - \sqrt{2}|11\rangle + |20\rangle)] = \sqrt{2}|2_{L_1}\rangle$$

$$L_1^\dagger \frac{1}{2}(|02\rangle - \sqrt{2}|11\rangle + |20\rangle) = \sqrt{3}[\frac{1}{2\sqrt{2}}(-|03\rangle + \sqrt{3}|12\rangle - \sqrt{3}|21\rangle + |30\rangle)] = \sqrt{3}|3_{L_1}\rangle$$

...

Thus, L_1 and L_2 is just another representation of a_1 and a_2 . Then we use the similarity transformation: $U = e^{-L_2^r L_2^{l\dagger}}$, which yields $U^{-1}(L_2^{r\dagger}, L_2^l, L_2^{l\dagger}, L_2^r)U = (L_2^{r\dagger} + L_2^{l\dagger}, L_2^l + L_2^r, L_2^{l\dagger}, L_2^r)$. Thus, the master equation Eq.(25) becomes:

$$RHS = \gamma U^{-1}[-L_2^{l\dagger} L_2^l - L_2^{r\dagger} L_2^r + 2L_2^r L_2^{l\dagger}]UU^{-1}\rho_s = \gamma[-L_2^{l\dagger} L_2^l - L_2^{r\dagger} L_2^r]U^{-1}\rho_s \quad (25)$$

The solutions to the steady state are $\rho_s = e^{-L_2^r L_2^{l\dagger}}|0_{L_2} m_{L_1}\rangle\langle 0_{L_2} n_{L_1}| = |m_{L_1}\rangle\langle n_{L_1}|$ which yields $\rho = S_1^+ S_2^+ \frac{1}{\sqrt{m!}}(\frac{a_1^\dagger - a_2^\dagger}{\sqrt{2}})^m|0\rangle\langle 0|\frac{1}{\sqrt{n!}}(\frac{a_1 - a_2}{\sqrt{2}})^n S_1 S_2$. This solution degenerates to the single mode squeezed vacuum in two modes when $m = n = 0$. Generally, an initial state $\rho(0) = \sum_{mnpq} C_{mnpq}|mn\rangle\langle pq| = \sum_{mnpq} C'_{mnpq}|m_{L_1} p_{L_2}\rangle\langle n_{L_1} q_{L_2}|$ will evolve into $\sum_{mn} G_{mn}|m_{L_1}\rangle\langle n_{L_1}|$ where $G_{mn} = \sum_{mnp} C'_{mnp}$.

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