Dipole-Dipole/Cavity-Cavity interaction in the Squeezed Vacuum

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1 Three Level Atom

In this section, we consider a scenario where a three-level atom is located inside the waveguide with the squeezed vacuum injected from both ends, as shown in Fig. 1(a). The atomic electronic structure is shown in Fig. 1(b) where the atomic states are labeled $|a\rangle$, $|b\rangle$, $|c\rangle$ from the excited state to the ground state. We assume that $\omega_{ac} = 2\omega_0$ where ω_0 is the center frequency of the broad band squeed vacuum. ω_{ab} and ω_{bc} are not equal but they are still within the bandwith of the squeezed vacuum.

The general master equation of dipole-dipole interaction in the squeezed vacuum can be used to study the dynamics of the three-level atom[1]:

$$
\begin{aligned}
\frac{d\rho^S}{dt} =& -i\sum_{i\neq j}\Lambda_{ij}[S_i^+S_j^-, \rho^S]e^{i(\omega_i-\omega_j)t} \\
&- \frac{1}{2}\sum_{i,j}\gamma_{ij}(1+N)(\rho^S S_i^+S_j^- + S_i^+S_j^-\rho^S - 2S_j^-\rho^S S_i^+)e^{i(\omega_i-\omega_j)t} \\
&- \frac{1}{2}\sum_{i,j}\gamma_{ij}N(\rho^S S_i^-S_j^+ + S_i^-S_j^+\rho^S - 2S_j^+\rho^S S_i^-)e^{-i(\omega_i-\omega_j)t} \\
&- \frac{1}{2}\sum_{\alpha=\pm}\sum_{i,j}\gamma'_{ij}Me^{2\alpha ik_{0z}R}e^{i\alpha(\omega_i+\omega_j-2\omega_0)t}(\rho^S S_i^\alpha S_j^\alpha + S_i^\alpha S_j^\alpha\rho^S - 2S_j^\alpha\rho^S S_i^\alpha)
\end{aligned}
\tag{1}
$$

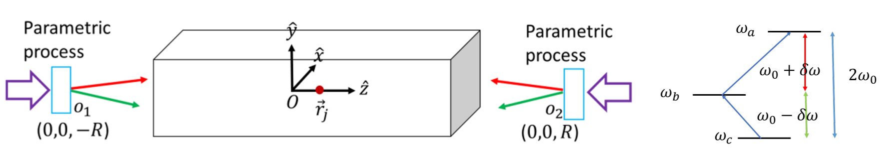

Fig. 1:

where the coefficients are

$$
\begin{aligned}
\gamma_{ij} &= \sqrt{\gamma_i\gamma_j}\cos(k_{0z}r_{ij}) \\
\Lambda_{ij} &= \frac{\sqrt{\gamma_i\gamma_j}}{2}\sin(k_{0z}r_{ij}) \\
\gamma'_{ij} &= \sqrt{\gamma_i\gamma_j}\cos[k_{0z}(r_i+r_j)]
\end{aligned} \tag{2}
$$

where γ_i is the decay rate for transition i in ordinary vacuum. For a single three level atom, we have $r_i = r_j$, for simplicity we set $R = r_i = 0$ and $\gamma_1 = \gamma_2 = \gamma$. After applying the rotating wave approximation(RWA), the master equation Eq.(1) becomes

$$
\begin{aligned}
\frac{d\rho^S}{dt} =& -\frac{1}{2}\sum_i \gamma(1+N)(\rho^S S_i^+ S_i^- + S_i^+ S_i^- \rho^S - 2S_i^- \rho^S S_i^+) \\
& -\frac{1}{2}\sum_i \gamma N(\rho^S S_i^- S_i^+ + S_i^- S_i^+ \rho^S - 2S_i^+ \rho^S S_i^-) \\
& -\frac{1}{2}\sum_{\alpha=\pm}\sum_{i\neq j}\gamma M(\rho^S S_i^\alpha S_j^\alpha + S_i^\alpha S_j^\alpha \rho^S - 2S_j^\alpha \rho^S S_i^\alpha)
\end{aligned} \tag{3}
$$

where $N = \sinh(r)^2$ and $M = \sinh(r)\cosh(r)$The steady state of Eq.(3) can be derived by re-writing Eq.(3) as:

$$\dot{\rho}_{aa}/\gamma = -ch^2\rho_{aa} + sh^2\rho_{bb} - \frac{1}{2}chsh(\rho_{ac}+\rho_{ca}) \tag{4a}$$

$$\dot{\rho}_{bb}/\gamma = (ch^2\rho_{aa} - sh^2\rho_{bb}) + (sh^2\rho_{cc} - ch^2\rho_{bb}) + chsh(\rho_{ac}+\rho_{ca}) \tag{4b}$$

$$\dot{\rho}_{cc}/\gamma = ch^2\rho_{bb} - sh^2\rho_{cc} - \frac{1}{2}chsh(\rho_{ac}+\rho_{ca}) \tag{4c}$$

$$(\dot{\rho}_{ac}+\dot{\rho}_{ca})/\gamma = -\frac{1}{2}(ch^2+sh^2)(\rho_{ac}+\rho_{ca}) - shch(\rho_{aa} - 2\rho_{bb} + \rho_{cc}) \tag{4d}$$

$$\dot{\rho}_{ab}/\gamma = -(1+\frac{3}{2}sh^2)\rho_{ab} - \frac{1}{2}chsh\rho_{cb} \tag{4e}$$

$$\dot{\rho}_{cb}/\gamma = -\frac{1}{2}chsh\rho_{ab} - (\frac{1}{2}+\frac{3}{2}sh^2)\rho_{cb} \tag{4f}$$

where $sh = \sinh(r), ch = \cosh(r)$. Eq.(4e)(4f) yield $\rho_{ab} = \rho_{cb} = 0$ for the steady state, and Eq.(4a)-(4d) yield $\rho_{aa} = \frac{sh^2}{sh^2+ch^2}$, $\rho_{cc} = \frac{ch^2}{sh^2+ch^2}$, $\rho_{ac} = -\frac{shch}{sh^2+ch^2}$. Thus, the steady state is actually a superposition state of $|a\rangle$ and $|c\rangle$: $\frac{sh}{\sqrt{sh^2+ch^2}}|a\rangle - \frac{ch}{\sqrt{sh^2+ch^2}}|c\rangle$. This phenomenon is similar to coherent trapping, but here we achieve the trapping for Ξ structure with the squeezed vacuum reservoir, which cannot be realized with coherent pump due to spontaneous emission.

2 Cavity-Cavity interaction

In this section, we consider a similar scenario but now the atoms are replaced with single mode cavity. The total Hamiltonian is:

$$H = \sum_i \hbar\omega(a^\dagger a_i + \frac{1}{2}) + \hbar \sum_{i,k} \omega_k (a_i^\dagger a_i + \frac{1}{2}) + \hbar \sum_{i,k} g_k (a_i^\dagger a_k + H.c.) \tag{5}$$

where a_k stands for the mods in the waveguide and a is the field operator of the single mode inside the cavity. The waveguide is saturated with the squeezed vacuum with the center frequency ω_0.

First, we study two non-resonant cavities coupled to the squeezed vacuum reservoir. The eigen frequencies of these two cavities are $\omega_1 = \omega_0 - \delta\omega$ and $\omega_2 = \omega_0 + \delta\omega$, following exacly the same steps as the derivation of Eq.(1), we get:

$$\begin{aligned}\dot{\rho} = &\sum_i \gamma(1+N)(-\rho a_i^\dagger a_i - a_i^\dagger a_i \rho + 2a_i \rho a_i^\dagger) \\ &+ \gamma N(-\rho a_i a_i^\dagger - a_i a_i^\dagger \rho + 2a_i^\dagger \rho a_i) \\ &+ \sum_{i\neq j} \gamma M(e^{i\theta}\rho a_i a_j + e^{i\theta} a_i a_j \rho - 2e^{i\theta} a_i \rho a_j + h.c.)\end{aligned} \tag{6}$$

which can be re-arranged as:

$$\begin{aligned}\dot{\rho} = &\sum_{i\neq j} \frac{\gamma}{2}[-\rho(\cosh r a_i^\dagger - e^{i\theta}\sinh r a_j)(\cosh r a_i - e^{-i\theta}\sinh r a_j^\dagger) \\ &- (\cosh r a_i^\dagger - e^{i\theta}\sinh r a_j)(\cosh r a_i - e^{-i\theta}\sinh r a_j^\dagger)\rho \\ &+ 2(\cosh r a_i - e^{-i\theta}\sinh r a_j^\dagger)\rho(\cosh r a_i^\dagger - e^{i\theta}\sinh r a_j)]\end{aligned} \tag{7}$$

we use the following Bogoliubov transformation[2]:

$$\begin{aligned}S &= exp(\eta^\star a_i a_j - \eta a_i^\dagger a_j^\dagger) \\ A_i &= S^+ a_i S = \cosh(r) a_i - e^{-i\theta}\sinh(r) a_j^\dagger \\ A_i^+ &= S^+ a_i^+ S = \cosh(r) a_i^+ - e^{i\theta}\sinh(r) a_j\end{aligned} \tag{8}$$

so the master equation Eq.(7) becomes:

$$\dot{\rho} = \sum_i \gamma[-\rho A_i^\dagger A_i - A_i^\dagger A_i \rho + 2A_i \rho A_i^\dagger] \tag{9}$$

Next we redefine the density matrix: $\rho_s = S\rho S^\dagger$. Thus Eq.(9) becomes:

$$\begin{aligned}\dot{\rho}_s &= \sum_i \gamma[-\rho_s a_i^\dagger a_i - a_i^\dagger a_i \rho_s + 2a_i \rho_s a_i^\dagger] \\ &\equiv \sum_i \gamma[-a_i^{l\dagger} a_i^l \rho_s - a_i^{r\dagger} a_i^r \rho_s + 2a_i^r a_i^{l\dagger}\rho_s] \equiv L\rho_s\end{aligned} \tag{10}$$

Here we define superoperator $\{a_i^l, a_i^{l\dagger}\}(\{a_i^r, a_r^{l\dagger}\}$) only acting to the left(right) on density operator ρ [3, 4]. These operators have the following commutation relations:

$$[a_i^r, a_j^{r\dagger}] = \delta_{ij},\ [a_i^l, a_j^{l\dagger}] = -\delta_{ij},\ [a_i^l, a_j^{r\dagger}] = [a_i^l, a_j^r] = [a_i^{l\dagger}, a_j^r] = [a_i^{l\dagger}, a_j^{r\dagger}] = 0 \tag{11}$$

Thus, the steady state of Eq.(10) can be solved by solving $L\rho = 0$, which requires the diagnolization of superoperator L. Applying the similarity transformation $U = e^{-a_1^r a_1^{l\dagger} - a_2^r a_2^{l\dagger}}$ to Eq.(10) , since we have $U^{-1}(a_i^{r\dagger}, a_i^l, a_i^r, a_i^{l\dagger})U = (a_i^{r\dagger} + a_i^{l\dagger}, a_i^r + a_i^l, a_i^r, a_i^{l\dagger})$, the right hand side of Eq.(10) becomes:

$$RHS = \sum_i \gamma U^{-1}[-a_i^{l\dagger}a_i^l - a_i^{r\dagger}a_i^r + 2a_i^r a_i^{l\dagger}]UU^{-1}\rho_s = \sum_i \gamma[-a_i^{l\dagger}a_i^l - a_i^{r\dagger}a_i^r]U^{-1}\rho_s \tag{12}$$

The only solution to $L\rho = 0$ is $U^{-1}\rho_s = |0,0\rangle\langle 0,0|$, which yields $\rho = S^\dagger \rho_S S = S^\dagger e^{-K_{-1}-K_{-2}}|0,0\rangle\langle 0,0|S = S^\dagger|0,0\rangle\langle 0,0|S$ which is the two mode squeezed vacuum.

Then we study the case where two cavities are identical, i.e., $\omega_1 = \omega_2 = \omega_0$. Then the master equation becomes:

$$\begin{aligned}\dot{\rho} = &\sum_{ij} \gamma \cosh^2 r(-\rho a_i^\dagger a_j - a_i^\dagger a_j \rho + 2a_i \rho a_j^\dagger)\\ &+ \gamma \sinh^2 r(-\rho a_i a_j^\dagger - a_i a_j^\dagger \rho + 2a_i^\dagger \rho a_j)\\ &+ \gamma \cosh r \sinh r(e^{i(\theta_i+\theta_j)}\rho a_i a_j + e^{i\theta} a_i a_j \rho - e^{i\theta} 2 a_i \rho a_j + h.c.)\end{aligned} \tag{13}$$

This equation can be rearranged when $\theta_1 = \theta_2 = \theta$:

$$\begin{aligned}\dot{\rho} = &\sum_{ij} \gamma[-\rho(\cosh r a_i^\dagger - e^{i\theta}\sinh r a_i)(\cosh r a_j - e^{-i\theta}\sinh r a_j^\dagger)\\ &- (\cosh r a_i^\dagger - e^{i\theta}\sinh r a_i)(\cosh r a_j - e^{-i\theta}\sinh r a_j^\dagger)\rho\\ &+ 2(\cosh r a_j - e^{-i\theta}\sinh r a_j^\dagger)\rho(\cosh r a_i^\dagger - e^{i\theta}\sinh r a_i)]\end{aligned} \tag{14}$$

We introduce the Bogoliubov transformation:

$$\begin{aligned}S_i &= exp(\eta^\star a_i^2 - \eta a_i^{\dagger 2})\\ A_i &= S_i^+ a_i S_i = \cosh(r)a_i - e^{-i\theta}\sinh(r)a_i^\dagger\\ A_i^+ &= S_i^+ a_i^+ S_i = \cosh(r)a_i^+ - e^{i\theta}\sinh(r)a_i\end{aligned} \tag{15}$$

so master equation Eq.(14) becomes

$$\dot{\rho} = \sum_{ij} \gamma[-\rho A_i^\dagger A_j - A_i^\dagger A_j \rho + 2A_j \rho A_i^\dagger] \tag{16}$$

Next we define $\rho_s = S_1 S_2 \rho S_1^{\dot{+}} S_2^{+}$ so the master equation is reduced to:

$$\dot{\rho_s} = \sum_{ij} \gamma[-\rho_s a_i^\dagger a_j - a_i^\dagger a_j \rho_s + 2a_j \rho_s a_i^\dagger] \tag{17}$$

To diagnolize this Lindblad equation, we introduce the transformation:

$$\begin{matrix} L_1 \\ L_2 \end{matrix} = \begin{matrix} \frac{1}{\sqrt{2}}(a_1 - a_2) \\ \frac{1}{\sqrt{2}}(a_1 + a_2) \end{matrix}$$

where $[L_i, L_j^\dagger] = \delta_{ij}$, and the master equation becomes:

$$\begin{aligned} \dot{\rho_s} &= \gamma[-2\rho_s L_2^\dagger L_2 - 2L_2^\dagger L_2 \rho_s + 4L_2 \rho_s L_2^\dagger] \\ &= \gamma[-2L_2^{r\dagger} L_2^r \rho_s - 2L_2^{l\dagger} L_2^l \rho_s + 4L_2^l L_2^{r\dagger} \rho_s] \\ &= L\rho \end{aligned} \tag{18}$$

Operator $L_2^\dagger$ has the following properties:

$$L_2^\dagger|0\rangle = \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle) \equiv |1_L\rangle$$

$$L_2^\dagger \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle) = \sqrt{2}[\frac{1}{2}(|02\rangle + \sqrt{2}|11\rangle + |20\rangle)] = \sqrt{2}|2_L\rangle$$

$$L_2^\dagger \frac{1}{2}(|02\rangle + \sqrt{2}|11\rangle + |20\rangle) = \sqrt{3}[\frac{1}{2\sqrt{2}}(|03\rangle + \sqrt{3}|12\rangle + \sqrt{3}|21\rangle + |30\rangle)] = \sqrt{3}|3_L\rangle$$

$$...$$

Then we use the similarity transformation: $e^{-L^r L^{l\dagger}}$, which yields $U^{-1}(L_2^{r\dagger}, L_2^l, L_2^{l\dagger}, L_2^r)U = (L_2^{r\dagger} + L_2^{l\dagger}, L_2^l + L_2^r, L_2^{l\dagger}, L_2^r)$. Thus, the master equation Eq.(19) becomes:

$$RHS = \gamma U^{-1}[-L_2^{l\dagger} L_2^l - L_2^{r\dagger} L_2^r + 2L_2^r L_2^{l\dagger}]UU^{-1}\rho_s = \gamma[-L_2^{l\dagger} L_2^l - L_2^{r\dagger} L_2^r]U^{-1}\rho_s \tag{19}$$

References

[1] Jieyu You, Zeyang Liao, Sheng-Wen Li, and M. Suhail Zubairy, Phys. Rev. A 97, 023810

[2] Bogolubov N N 1947 J. Phys. (USSR) 11 23

[3] Wang S J et al. 2002 Phys. Rev. A 66 033608(2002)

[4] Jun-Hong An, Shun-Jin Wang, Hong-Gang Luo, Cheng-Long Jia Chin. Phys. Lett., Vol 21, No. 1