# CSIE Probability Exam III Solution

1. $X$ and $Y$ are Gaussian random variables with the joint PDF

$$f_{X,Y}(x,y) = k\; e^{-(\frac{x^2}{32}-\frac{xy}{12}+\frac{y^2}{9})}.$$

Find $k$. (20pts)

Observe that $\rho_{X,Y} > 0$ and $\mu_x = \mu_y = 0$.

$$(I)\quad \frac{1}{32} = \frac{1}{2(1-\rho_{X,Y}^2)\sigma_X^2}$$

$$(II)\quad \frac{1}{12} = \frac{2\rho_{X,Y}}{2(1-\rho_{X,Y}^2)\sigma_X\sigma_Y}$$

$$(III)\quad \frac{1}{9} = \frac{1}{2(1-\rho_{X,Y}^2)\sigma_Y^2}.$$

$\frac{(II)^2}{(I)(III)} = 2 = 4\rho_{X,Y}^2,\ \rho_{X,Y}^2 = 1/2,\ \rho_{X,Y} = 1/\sqrt{2}.$

$\frac{1}{32} = \frac{1}{2(1-\rho_{X,Y}^2)\sigma_X^2} = \frac{1}{\sigma_X^2},\quad \sigma_X^2 = 32,\quad \sigma_X = 4\sqrt{2}.$

$\frac{1}{9} = \frac{1}{2(1-\rho_{X,Y}^2)\sigma_Y^2} = \frac{1}{\sigma_Y^2},\quad \sigma_Y^2 = 9,\quad \sigma_Y = 3.$

Therefore, $k = \frac{1}{2\pi\sigma_X\sigma_Y\sqrt{1-\rho_{X,Y}^2}} = \frac{1}{2\pi(4\sqrt{2})(3)(1/\sqrt{2})} = \frac{1}{24\pi}.$

2. $Y = X_1 + X_2 + ... + X_{48}$, where $X_k$'s are iid random variables with the PDF $f_X(x) = 1/2,\ 0 \le x \le 2$, and thus $\mu_X = 1,\ \sigma_X^2 = 1/3$. Find $P[Y < 40]$. (15pts)

$X_k$'s are iid, $E[Y] = \mu_Y = E[X_1] +...+ E[X_{48}] = 48\mu_X = 48.$

$Var[Y] = \sigma_Y^2 = Var[X_1] +...+ Var[X_{48}] = 48\sigma_X^2 = 16.$

$Y$ is close to a Gaussian random variable Gaussian(48,4).

$P[Y < 40] \simeq \Phi(\frac{40-\mu_Y}{\sigma_Y}) = \Phi(-2) \simeq 0.02275013.$

3. $X$ and $Y$ are continuous random variables with the joint PDF

$$f_{X,Y}(x,y) = \begin{cases} 2 & 0 \le x \le 1,\ 0 \le y \le 1,\ (x+y) \le 1 \\ 0 & \text{otherwise} \end{cases}$$

Verify if $X$ and $Y$ are independent. (15pts)

$S_{X,Y}$ is not rectangular. Hence, $X$ and $Y$ are NOT independent.

4. $X$ and $Y$ are discrete random variables with the joint PMF

$$P_{X,Y}(x,y) = \begin{cases} \frac{1}{5} & (x,y) = (0,0)\ (0,1)\ (0,2) \\ \frac{2}{5} & (x,y) = (1,0) \\ 0 & \text{otherwise} \end{cases}$$

(a) Find $\rho_{X,Y}$. (20pts)

$$P_X(x) = \begin{cases} 3/5 & x = 0 \\ 2/5 & x = 1 \\ 0 & \text{otherwise} \end{cases}.$$

$\mathrm{E}[X] = 0(3/5) + 1(2/5) = 2/5$.

$\mathrm{E}[X^2] = 0(3/5) + 1(2/5) = 2/5$.

$\mathrm{Var}[X] = 2/5 - (2/5)^2 = 6/25$.

$$P_Y(y) = \begin{cases} 3/5 & y = 0 \\ 1/5 & y = 1, 2 \\ 0 & \text{otherwise} \end{cases}.$$

$\mathrm{E}[Y] = 0(3/5) + 1(1/5) + 2(1/5) = 3/5$.

$\mathrm{E}[Y^2] = 0(3/5) + 1(1/5) + 4(1/5) = 1$.

$\mathrm{Var}[Y] = 1 - (3/5)^2 = 16/25$.

$\mathrm{E}[XY] = 0$, since $xy = 0,\ \forall(x,y)$.

$\mathrm{Cov}[X,Y] = 0 - (2/5)(3/5) = -6/25$.

$\rho_{X,Y} = \frac{-6/25}{\sqrt{(6/25)(16/25)}} = -\frac{\sqrt{6}}{4} \simeq -0.6123724.$

(b) Verify if $X$ and $Y$ are independent. (10pts)

$\rho_{X,Y} \neq 0$, and hence $X$ and $Y$ are NOT independent

5. $X$ is a continuous random variables with the PDF

$$f_X(x) = \begin{cases} 1 & 0 \le x \le 1 \\ 0 & \text{otherwise} \end{cases}$$

$Y = X^{1/5}$. Find the $\mathrm{E}[Y]$. (20pts)

First find the CDF $F_Y(y)$. For $y \le 0,\ F_Y(y) = P[X^{1/5} \le y] = 0$.

For $0 \le y \le 1,\ F_Y(y) = P[X^{1/5} \le y] = P[X \le y^5] = \int_0^{y^5} 1\, dx = y^5$.

For $y \ge 1,\ F_Y(y) = P[X^{1/5} \le y] = 1$.

Thus, $f_Y(y) = \frac{d}{dy} F_Y(y) = \begin{cases} 5y^4 & 0 \le y \le 1 \\ 0 & \text{otherwise} \end{cases}$.

$\mathrm{E}[Y] = \int_0^1 y(5y^4)dy = \int_0^1 5y^5 dy = \frac{5y^6}{6}\Big|_{y=0}^{1} = \frac{5}{6}$.