離散數學 第二次期中考

2020/12/9

注意事項：
1. 禁止使用計算機、翻譯機、禁止攜帶計算紙；**手機請關機**。
2. **計算與證明題需要計算過程方予計分。**
3. 當然不可以作弊。
4. 請於答案卷左上角填上題號以方便閱卷。
5. 使用兩張答案卷的同學記得兩張都要寫名字，並將之合併在一起交回。
6. 請努力作答。

一、簡答題 (17%)

1. (C2 no.13) Consider the open statement

$$p(x, y): y - x = y + x^2$$

where the universe for each of the variables x, y comprises all integers. Determine the truth value for each of the following statements.

(a) $p(0, 1)$ (2%)

(b) $\exists y\, p(0, y)$ (2%)

(c) $\forall x\, \exists y\, p(x, y)$. (2%)

2. (3-1 no.3) Let $A = \{1, 2, \{2\}\}$. Which of the following statements are true? (5%)

(a) $1 \in A$, (b) $\{1\} \in A$, (c) $\{\{1\}\} \subseteq A$,

(d) $\{2\} \in A$, (e) $\{2\} \subseteq A$

3. (4-5 no.8) (a) How many positive divisors are there for $n = 2^{14}3^{9}5^{8}7^{10}11^{3}13^{5}37^{10}$? (2%)

(b) For the divisors in part (a), how many are

(i) perfect squares that are divisible by $2^{2}3^{4}5^{2}11^{2}$? (2%)

(ii) perfect squares and perfect cubes? (2%)

2 4 2 0 2 0 6
4 6 4 2 1 2
6 8 6 4 4
8 3 8 6 3
10 4 8
12 16
14 6
7

二、計算與證明(須寫出完整計算過程方予計分) (83%)

4. (2-5 no.15+17) Provide a proof by contradiction for the following: "If n is an odd integer, then $n + 11$ is even". (5%)

5. (3-2 no.17b) Using the laws of set theory, simplify:

$$(A \cap B) \cup (A \cap B \cap \overline{C} \cap D) \cup (\overline{A} \cap B).$$ (10%)

6. (3-3 no.3) A binary string of length 10 is made up of 10 bits (that is, 10 symbols, each of which is a 0 or 1). How many such strings either start with three 1's or end in four 0's? (8%)

7. (3-4 no.11) Darci rolls a fair die three times. What is the probability that (a) her second and third rolls are both larger than her first roll? (b) the result of her second roll is greater than that of her first roll and result of her third roll is greater than the second? (6%)

8. (C3 no.10) Let $\mathcal{U}$ be a given universe with $A, B \subseteq \mathcal{U}$, $|A \cap B| = 3$, $|A \cup B| = 8$, and $|\mathcal{U}| = 12$.
   (a) How many subset $C \subseteq \mathcal{U}$ satisfy $A \cap B \subseteq C \subseteq A \cup B$? How many of these subsets C contain an even number of elements? (4%)
   (b) How many subset $D \subseteq \mathcal{U}$ satisfy $\overline{A \cup B} \subseteq D \subseteq \bar{A} \cup \bar{B}$? How many of these subsets D contain an even number of elements? (4%)

9. (4-1 no.14+15) Prove that for all $n \in \mathbf{Z}^+$, $n > 4 \Rightarrow n^2 < 2^n < n!$ (10%)

10. (4-2 no.3) Use the recursive definition of $\wedge$ and "Generalizes Associative Law for $\wedge$" to prove that if $p, q_1, q_2, \ldots, q_n$ are statements and $n \geq 2$, then
$$p \vee (q_1 \wedge q_2 \wedge \cdots \wedge q_n) \Leftrightarrow (p \vee q_1) \wedge (p \vee q_2) \wedge \cdots \wedge (p \vee q_n).$$ (10%)

11. (4-3 no.10) If $n \in \mathbf{Z}^+$, and n is odd, prove that $8|(n^2 - 1)$. (6%)

12. (4-4 Ex4.34) Find the greatest common divisor of 427 and 125, and express the result as a linear combination of these integers. (10%)

13. (Chap 4 no.27) F_n denote the nth Fibonacci number, prove that $F_n \leq (5/3)^n$ for all $n \in \mathbf{N}$. (10%)

註: The Fibonacci numbers may be defined recursively by

1) $F_0 = 0$, $F_1 = 1$; and

2) $F_n = F_{n-1} + F_{n-2}$, for $n \in Z^+$ with $n \geq 2$.