CSIE Probability Exam I Solution

1. At NCNU, the number of the undergrads(U) is five times the number of the graduate students(G), $\mathrm{P}[U] = 5\ \mathrm{P}[G]$. In addition, the students are equally likely to have no credit card(C_0), one credit card(C_1), or two or more credit cards(C_2), $\mathrm{P}[C_0] = \mathrm{P}[C_1] = \mathrm{P}[C_2] = 1/3$. Among the students with no credit card(C_0), only 20% are graduate students, and the same is true with the students with one credit card(C_1), $\mathrm{P}[G|C_0] = \mathrm{P}[G|C_1] = 0.2$. When you meet a graduate student, the chance that she or he has two or more credit cards is 0.2, $\mathrm{P}[C_2|G] = 0.2$. Finish the following table, and find the probability when you meet a student with one or more credit cards, she or he is an undergrad $\mathrm{P}[U|C_1 \cup C_2]$. (20pts)

   U, G is a partition and $\mathrm{P}[U] = 5\mathrm{P}[G]$. $\mathrm{P}[U] = 5/6$, $\mathrm{P}[G] = 1/6$.

   $\mathrm{P}[GC_2] = \mathrm{P}[G]\ \mathrm{P}[C_2|G] = 1/30$.

   $\mathrm{P}[GC_0] = \mathrm{P}[GC_1] = \mathrm{P}[C_0]\ \mathrm{P}[G|C_0] = \mathrm{P}[C_1]\ \mathrm{P}[G|C_1] = 2/30$.

	C_0	C_1	C_2
U	8/30	8/30	9/30
G	2/30	2/30	1/30

   $$P[U|C_1 \cup C_2] = \frac{P[UC_1] + P[UC_2]}{P[C_1] + P[C_2]} = \frac{8/30 + 9/30}{10/30 + 10/30} = 0.85.$$

2. Roll a fair ten-sided die five times. The outcome is independent from roll to roll. Let $X_k =$ the k-th outcome, $k = 1, 2, ..., 5$. Find $\mathrm{P}[\max\{X_1, X_2, ..., X_5\} = 7]$. (15pts)

   $$\begin{aligned} &P[\max\{X_1, X_2, ..., X_5\} = 7] \\ &= P[\max\{X_1, X_2, ..., X_5\} \leq 7] - P[\max\{X_1, X_2, ..., X_5\} \leq 6] \\ &= P[(X_1 \leq 3), ..., (X_5 \leq 7)] - P[(X_1 \leq 6), ..., (X_5 \leq 6)] \\ &= (\frac{7}{10})^5 - (\frac{6}{10})^5 = 0.09031. \end{aligned}$$

3. You roll two fair dice until you get a double. N is the number of the rolls. Find $\mathrm{E}[N]$, and the probability that the double shows up in the first, second, or third roll, $\mathrm{P}[N \leq 3]$. (20pts)

   N is geometric with $p = 1/6$.

   $\mathrm{E}[N] = \mu_N = 1/p = 6$.

P$[N \leq 3]$ = 1 - P$[N > 3] = 1 - (1-p)^3 = 1 - (5/6)^3 = 91/216 \simeq$ 0.4212963..

4. At a base station, the number X of the messages it receives during 6:00-6:30am is a Poisson random variable with E$[X] = 3$. I is an indicator random variable such that $I = 1$ if at least one message shows up; otherwise $I = 0$. Find P$[I = 0]$, P$[I = 1]$, and E$[I]$. (20pts)

   E$[X] = \alpha = 3$.

   P$[I = 0]$ = P$[X = 0] = e^{-3} \simeq 0.04978707$.

   P$[I = 1]$ = P$[X \geq 1]$ = 1 - P$[X = 0] = 1 - e^{-3} \simeq 0.9502129$.

   I is Bernoulli, and thus E$[I]$ = P$[I = 1] = 1 - e^{-3} \simeq 0.9502129$.