$f(a, x) = f(x, a) = a$

# 離散數學期末考

2021/1/13

注意事項:

1. 禁止使用計算機、翻譯機，手機請關機。禁止攜帶計算紙。
2. 當然不可以作弊。
3. 計算與證明題需有計算過程方予計分。
4. 請於答案卷左上角填上題號以方便閱卷。
5. 使用兩張答案卷的同學記得兩張都要寫名字，並將之合併在一起交回。
6. 請努力作答。

## 一 簡答題 (58%)

1. (5-1 no.3, 5-2 no.3, 5-4 no.5, 6, 5-6 no.15) If $A = \{1, 2, 3, 4, 5\}$, $B = \{3, 4, 5\}$, and $C = \{w, x, y, z\}$, determine the following: (2% × 14)
   - 15 (a) $|A \times B|$ $= |A| \times |B| = 15$
   - $2^{15}$ (b) The number of relations from $A$ to $B$.
   - $2^{25}$ (c) The number of relations on $A$. $2^{25}$
   - $2^{23}$ (d) The number of relations on $A$ that contain (1, 2) and (3, 4).
   - $\binom{15}{4}$ (e) The number of relations from $A$ to $B$ that contain exactly four ordered pairs. $\binom{15}{4}$
   - $4^5$ (f) How many functions $f: A \to C$ are there? $\frac{4}{5} \Rightarrow 4^5$
   - 0 (g) How many functions $f: A \to C$ are one-to-one? 0
   - $4^3$ (h) How many functions $f: A \to C$ satisfy $f(1) = f(3) = x$? $\frac{4}{3} \Rightarrow 4^3$
   - $5^{25}$ (i) How many functions $f: A \times A \to A$ are there? $5^{25}$
   - $5^{10}$ (j) How many closed binary operations on $A$ are commutative? $5^4 \cdot 5$
   - $4^{15}$ (k) How many closed binary operations $f$ on $C$ satisfy $f(x, y) = z$? $4^{15}$
   - $4^4 \cdot 1$ (l) How many of the functions $f$ in part (k) have an identity? $4^4$
   - $4^4 \cdot 2$ (m) How many of the functions $f$ in part (l) are commutative?
   - $3^2 \times 2^3$ (n) How many functions $f: A \to A$ are such that $f^{-1}(\{3, 4, 5\}) = \{1, 2\}$? $3^2 \times 2^3$

2. (5-4 no.9, 7) For distinct promes $p$, $q$ let $A = \{p^m q^n \mid 0 \le m \le 31, 0 \le n \le 37\}$. If $f: A \times A \to A$ is the closed binary operation defined by $f(a, b) = \gcd(a, b)$.
   - $32 \times 38$ (a) What is $|A|$? $32 \times 38$ (2%)
   - Y (b) Is $f$ commutative? yes (2%)
   - Y (c) Is $f$ associative? yes (2%)
   - $p^{31}q^{37}$ (d) Does $f$ have an identity element? $p^{31}q^{37}$ is an identity element (2%)

3. (5-6 no.18) Let $f$, $g$, $h$ denote the following closed binary operations on $\mathscr{P}(\mathbf{Z}^+)$. For $A, B \subseteq \mathbf{Z}^+$, $f(A, B) = A \cap B$, $g(A, B) = A \cup B$, $h(A, B) = A \,\Delta\, B$. (16%)
   - (a) Are any of the functions one-to-one? h (2%)
   - (b) Are any of $f$, $g$ and $h$ onto functions? f, g, h (2%)
   - (c) Is any one of the given functions invertible? h (2%)
   - (d) Are any of the following sets infinite?

$A = \{1, 2\}$, $B = \{2, 3\}$, $f(\{1,2\}, \{2,3\}) = \{2\}$

(1) $f^{-1}(\phi)$ (2) $g^{-1}(\phi)$ (3) $g^{-1}(\{1\})$
(4) $h^{-1}(\{3\})$ (5) $f^{-1}(\{4, 7\})$

(e) Determine the number of elements in each of the finite sets in part (d).

4. (Chap 5 no.27) With $A = \{x, y, z\}$, let $f, g: A \to A$ be given by $f = \{(x, y), (y, z), (z, x)\}$, $g = \{(x, x), (y, z), (z, y)\}$. Determine each of the following: (2% × 3)

(a) $f \circ g$ $= \{(x,y), (y,x), (z,z)$

(b) $(g \circ f)^{-1}$ $gf = \{(x,z), (y,y), (z,x)\}$

(c) $g^{-1} \circ f^{-1}$ $\{(x,y), (y,x), (z,z)\}$

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7個分五堆.
3 1 1 1 1 = 5種.
2 2 1 1 1 = 10種.
分4堆.
4 1 1 1 4

## 二 計算與證明(須有計算過程或詳細證明，否則不予計分) (42%)

5. (5-2 no.22) For $n \in \mathbf{Z}^+$ define $X_n = \{1, 2, 3, \ldots, n\}$. Given $m, n \in \mathbf{Z}^+$, $f: X_m \to X_n$ is called *monotone increasing* if for all $i, j \in X_m$, $1 \le i < j \le m \Rightarrow f(i) \le f(j)$.

$)\cdot\binom{11}{7}$ (a) How many monotone increasing functions are there with domain $X_7$ and codomain $X_5$? $\{1,2,3,4,5,6,7\}\{1,2,3,4,5\}$ (5%)

$\times\binom{5}{2}$ (b) How many monotone increasing functions $f: X_7 \to X_{12}$ satisfy $f(5) = 9$? (5%)

6. (5-3 no.15) A lock has $n$ buttons labeled 1, 2, …, $n$. To open this lock we press each of the $n$ buttons exactly once. If no two or more buttons may be pressed simultaneously, then there are $n!$ ways to do this. However, if one may press two or more buttons simultaneously, then there are more than $n!$ ways to press all of the buttons. For instance, if $n = 3$ there are six ways to press the buttons one at a time. But if one may also press two or more buttons simultaneously, then we find 13 cases — namely,

(1) 1, 2, 3 (2) 1, 3, 2 (3) 2, 1, 3 (4) 2, 3, 1 (5) 3, 1, 2
(6) 3, 2, 1 (7) {1, 2}, 3 (8) 3, {1, 2} (9) {1, 3}, 2 (10) 2, {1, 3}
(11) {2, 3}, 1 (12) 1, {2, 3} (13) {1, 2, 3}.

[Here, for example, case (12) indicates that one presses button 1 first and then buttons 2, 3 (together) second.]

(a) How many ways are there to press the buttons when $n = 4$? $n = 5$? How many for $n$ general? (6%)

(b) Suppose a lock has 15 buttons. To open this lock one must press 12 different buttons (one at a time, or simultaneously in sets of two or more). In how many ways can this be done? (6%)

7. (5-5 no. 15) Let $S \subset \mathbf{Z}^+$ with $|S| = 7$. For $\phi \neq A \subseteq S$, let $s_A$ denote the sum of the elements in $A$. If $m$ is the maximum element in $S$, find the possible values of $m$ so that there will exist distinct subset $B$, $C$ of $S$ with $s_B = s_C$. (10%)

8. (Chap 5 no.15) Let $n \in Z^+$, $n$ odd. If $i_1, i_2, \ldots i_n$ is a permutation of the integers 1, 2, …, $n$, prove that $(1 - i_1)(2 - i_2)\ldots(n - i_n)$ is an even integer. (Which counting principle is at work here?) (10%)