暨大資工系　線性代數　期末考　111.6.1

Theorem 1 If *S* is an orthonormal basis for an *n*-dimensional inner product space, and if $(u)_S = (u_1, u_2, \ldots, u_n)$ & $(v)_S = (v_1, v_2, \ldots, v_n)$ then:

(a) $\|u\| = \sqrt{u_1^2 + u_2^2 + \cdots + u_n^2}$

(b) $d(u,v) = \sqrt{(u_1 - v_1)^2 + (u_2 - v_2)^2 + \cdots + (u_n - v_n)^2}$

(c) $\langle u,v \rangle = u_1v_1 + u_2v_2 + \cdots + u_nv_n$

Theorem 2 If S = { $v_1, v_2, \ldots, v_n$ } is an orthonormal basis for an inner product space ∇, and u is any vector in ∇, then $u = \langle u, v_1\rangle v_1 + \langle u, v_2\rangle v_2 + \ldots + \langle u, v_n\rangle v_n$

Theorem 3. Let ϖ be a finite-dimensional subspace of an inner product space ∇.

(a) If { $v_1, v_2, \ldots, v_r$ } is an orthonormal basis for ϖ, and u is any vector in ∇, then $proj_{\varpi}u = \langle u,v_1 \rangle v_1 + \langle u,v_2 \rangle v_2 + \cdots + \langle u,v_r \rangle v_r$

(b) If { $v_1, v_2, \ldots, v_n$ } is an orthogonal basis for ϖ, and u is any vector in ∇,

then $$proj_{\varpi}u = \frac{\langle u,v_1 \rangle}{\|v_1\|^2}v_1 + \frac{\langle u,v_2 \rangle}{\|v_2\|^2}v_2 + \cdots + \frac{\langle u,v_n \rangle}{\|v_n\|^2}v_n$$

***Theorem* 4: A least squares solution of Ax=b must satisfy the equality $A^TAx=A^Tb$ (i.e. $x=(A^TA)^{-1}A^Tb$) [it is called the *normal system* associated with Ax=b].**

If A is an $m \times n$ matrix with linearly independent column vectors, then for every $n \times 1$ matrix b, the linear system Ax=b has a unique least squares solution. This solution is given by $x=(A^TA)^{-1}A^Tb$. Moreover, if ϖ is the column space of A, then the orthogonal projection of b on ϖ is $proj_{\varpi}b = Ax = A(A^TA)^{-1}A^Tb$

Theorem 5.

If $T:\mathbb{R}^n \to \mathbb{R}^n$ is a linear transformation, and $e_1, e_2, \cdots, e_n$ are the standard basic vectors for $\mathbb{R}^n$, then the standard matrix for T is $[T] = [T(e_1)|T(e_2)|\cdots|T(e_n)]$

1. (15%) Let A be a 2×2 matrix, and call a line through the origin of R^2 invariant under A if Ax lies on the line when x does.
   (a) (10%) Find equations for all lines in R^2, if any, that are invariant under the matrix A= $\begin{bmatrix} 4 & -1 \\ 2 & 1 \end{bmatrix}$.

   $Ax=\lambda x$

   (b) (5%) Find A^{1000}.

$[T]_B = \begin{bmatrix}0&0&2&0\\1&0&1&0\\0&1&-2&0\\0&0&0&1\end{bmatrix}$

$[T]_B \underset{4\times 1}{[X]_B} = \underset{4\times 1}{[\ \ ]}$

2. (15%) Let $T: M_{22} \to M_{22}$ be defined by

$$T\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \begin{bmatrix} 2c & a+c \\ b-2c & d \end{bmatrix}$$

$T(e_1)=\begin{bmatrix}0&1\\0&1\end{bmatrix}$

$T(e_2)=\begin{bmatrix}0&0\\1&0\end{bmatrix}$

$T(e_3)=\begin{bmatrix}2&1\\-2&0\end{bmatrix}$

$T(e_4)=\begin{bmatrix}0&0\\0&1\end{bmatrix}$

(a) (5%) Find the representative matrix $[T]_B$ of the linear operator relative to the standard basis B = $\left\{ \underset{e_1}{\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}}, \underset{e_2}{\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}}, \underset{e_3}{\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}}, \underset{e_4}{\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}} \right\}$.

$\left[\begin{bmatrix}a&b\\c&d\end{bmatrix}\right]_B = \begin{pmatrix}a\\b\\c\\d\end{pmatrix}$

(b) (10%) Find those three bases respectively for the eigenspaces of T.

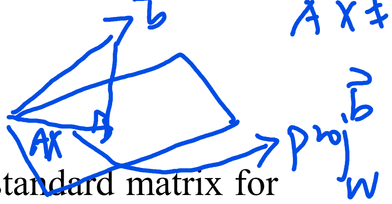
3. (35%) Let W be the plane in R^3 with equation x + z = 0.

$x+0y+z=0$

(a) (10%) Let $A = \begin{bmatrix} 1 & 0 \\ 0 & 2 \\ -1 & 0 \end{bmatrix}$, $b = \begin{bmatrix} 1 \\ -2 \\ 4 \end{bmatrix}$. Find the least squares solution of the linear system $Ax = b$.

$Ax \neq b$

$[T(e_1)]_B = \begin{pmatrix}0\\1\\0\\0\end{pmatrix}$



(b) (5%) Find an orthonormal basis for W.

(c) (5%) Utilize the above *Theorem* 4, to find the standard matrix for the orthogonal projection onto W.

(d) (5%) Use the matrix obtained in (c) to find the orthogonal projection of a point $b(1,-2,4)$ onto W.

(e) (5%) Could you solve the problem (d) directly through the inner product method by using *Theorem* 3?

(f) (5%) Find all of the points in R^3, such that all of them are orthogonally projected to the vector (1, 1, -1) through the linear transformation T (orthogonal projection).

4. (7%) Prove that if S is an orthonormal basis for an n-dimensional inner product space, and if $(u)_S = (u_1, u_2, \ldots, u_n)$ & $(v)_S = (v_1, v_2, \ldots, v_n)$ then $\langle u, v \rangle = u_1v_1 + u_2v_2 + \ldots + u_nv_n$.

$S = \{b_1\ b_2 \cdots b_n\}$

$u = \sum_{i=1}^{n} u_i b_i$

$v = \sum_{i=1}^{n} v_i b_i$

5. (8%) Let $A = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$ be an orthogonal matrix. Suppose

$A^{-1} = A^T = A$

$\begin{bmatrix} y_0 \\ y_1 \end{bmatrix} = A\begin{bmatrix} x_0 \\ x_1 \end{bmatrix}$, where $\begin{bmatrix} x_0 \\ x_1 \end{bmatrix}$ and $\begin{bmatrix} y_0 \\ y_1 \end{bmatrix}$ are the coordinate vectors relative to the old orthonormal basis $\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$ and a new basis, respectively.

$\langle u_1b_1 + u_2b_2,\ v_1b_1 + v_2b_2 \rangle$

(a) (4%) Find the new basis.

$B = \left\{ \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right), \left(\frac{1}{\sqrt{2}}, \frac{-1}{\sqrt{2}}\right) \right\}$ $\langle u_1b_1 + u_2b_2, v_1b_1 \rangle + \langle u_1b_1 + u_2b_2, v_2b_2 \rangle$

(b) (4%) For the vector (1,1), find the coordinate vector corresponding to the new basis.

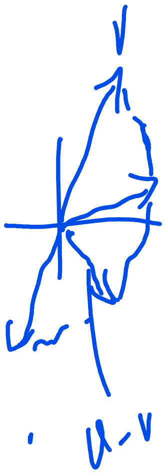
6. (10%) Let $F(-\infty,\infty)$ be the vector space of real-valued functions. Suppose that W is a subspace of $F(-\infty,\infty)$ and W is spanned by

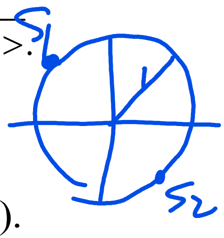
$\varphi_1(t)$ and $\varphi_2(t)$, where

$\varphi_1(t) = \sqrt{2}\cos(2\pi t)$ and $\varphi_2(t) = \sqrt{2}\sin(2\pi t)$, $0 \le t \le 1$. Let W have

the inner product $<p(t), q(t)> \triangleq \int_0^1 p(t)q(t)dt$, where $p(t), q(t) \in W$; and

the *norm*(or *length*) of $p(t)$ is defined by $\|p(t)\| = \sqrt{<p(t), p(t)>}$.





Suppose $\| (-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}) - (\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}) \|$

$$s_1(t) = \frac{-\sqrt{2}}{2}\varphi_1(t) + \frac{\sqrt{2}}{2}\varphi_2(t) \text{ and } s_2(t) = \frac{\sqrt{2}}{2}\varphi_1(t) - \frac{\sqrt{2}}{2}\varphi_2(t).$$

$= \| (-\sqrt{2}, \sqrt{2}) \| = 2$

(a) (4%) Find the coordinate vectors of $s_1(t)$ and $s_2(t)$ respectively with respect to the basis S.

(b) (6%) Find the distance between $s_1(t)$ and $s_2(t)$ (write down two methods)

$\| s_1(t) - s_2(t) \|$

$= \| -\sqrt{2}\varphi_1(t) + \sqrt{2}\varphi_2(t) \|$

7. (10%) Given the matrix

$$\boldsymbol{A} = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 2 & 2 \\ 0 & 1 & 1 \end{bmatrix} \text{ and let the column matrix } \mathbf{X} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}.$$

$\sqrt{< -\sqrt{2}\varphi_1(t) + \sqrt{2}\varphi_2(t), -\sqrt{2}\varphi_1(t) + \sqrt{2}\varphi_2(t) >}$

$= 2$

Find all possible vectors **b**, such that $AX = \mathbf{b}$ is consistent.

求 A 的 column space

$$[c_1\ c_2\ c_3]\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \quad \{ s(1\ 0\ 0) + t(2\ 2\ 1) \mid s,t \in \mathbb{R} \}$$

$$\sum_{i=1}^{3} c_i x_i$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 2 & 2 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & \frac{1}{2} \end{bmatrix}$$

令 $2t+s=k$, $l=2t$ $\quad k(1\ 0\ 0) + l(0\ 1\ \frac{1}{2})$

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1.(a)

$\lambda=2$ $\begin{bmatrix}-2 & 1\\-2 & 1\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}=0$

$-2x+y=0$

$y=2x$ #

$(4-\lambda)(1-\lambda)+2=0$

$4-5\lambda+\lambda^2+2=0$

$\lambda^2-5\lambda+6=0$

$\lambda=3,2$

$\lambda=3$ $\begin{bmatrix}-1 & 1\\-2 & 2\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}=0$

$y=x$ #

$AX=\lambda X$

$(\lambda I-A)X=0$

Note: 需寫出直線方程式

(b) $\begin{bmatrix}\lambda-4 & 1\\-2 & \lambda-1\end{bmatrix}$ $(\lambda-4)(\lambda-1)+2=0$, $\lambda^2-5\lambda+6=0$

$\lambda=3,2$

$\lambda=3$ $\begin{bmatrix}-1 & 1\\-2 & 2\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix}=0$ $-x_1+x_2=0$ $\begin{bmatrix}x_1\\x_2\end{bmatrix}=t\begin{bmatrix}1\\1\end{bmatrix}$ ← basis

$\lambda=2$ $\begin{bmatrix}-2 & 1\\-2 & 1\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix}=0$ $-2x_1+x_2=0$ $\begin{bmatrix}x_1\\x_2\end{bmatrix}=\begin{bmatrix}t\\2t\end{bmatrix}=t\begin{bmatrix}1\\2\end{bmatrix}$

$P=\begin{bmatrix}1 & 1\\1 & 2\end{bmatrix}$ $P^{-1}=\begin{bmatrix}2 & -1\\-1 & 1\end{bmatrix}$

$D=P^{-1}AP=\begin{bmatrix}2 & -1\\-1 & 1\end{bmatrix}\begin{bmatrix}4 & -1\\2 & 1\end{bmatrix}\begin{bmatrix}1 & 1\\1 & 2\end{bmatrix}=\begin{bmatrix}6 & -3\\-2 & 2\end{bmatrix}\begin{bmatrix}1 & 1\\1 & 2\end{bmatrix}=\boxed{\begin{bmatrix}3 & 0\\0 & 2\end{bmatrix}}$

$A^{1000}=PD^{1000}P^{-1}=\begin{bmatrix}1 & 1\\1 & 2\end{bmatrix}\begin{bmatrix}3^{1000} & 0\\0 & 2^{1000}\end{bmatrix}\begin{bmatrix}2 & -1\\-1 & 1\end{bmatrix}$

$=\begin{bmatrix}3^{1000} & 2^{1000}\\3^{1000} & 2^{1001}\end{bmatrix}\begin{bmatrix}2 & -1\\-1 & 1\end{bmatrix}=\begin{bmatrix}2\times3^{1000}-2^{1000} & -3^{1000}+2^{1000}\\2\times3^{1000}-2^{1001} & -3^{1000}+2^{1001}\end{bmatrix}$ #

$A=PDP^{-1}$

$$B=\left\{\underbrace{\begin{bmatrix}1&0\\0&0\end{bmatrix}}_{B_1},\underbrace{\begin{bmatrix}0&1\\0&0\end{bmatrix}}_{B_2},\underbrace{\begin{bmatrix}0&0\\1&0\end{bmatrix}}_{B_3},\underbrace{\begin{bmatrix}0&0\\0&1\end{bmatrix}}_{B_4}\right\}$$ basis for M_{22}

2.

(a) $[T(B_1)]_B=\begin{bmatrix}0&1\\0&0\end{bmatrix}$ $[T(B_2)]_B=\begin{bmatrix}0&0\\1&0\end{bmatrix}$ $[T(B_3)]_B=\begin{bmatrix}2&1\\-2&0\end{bmatrix}$ $[T(B_4)]_B=\begin{bmatrix}0&0\\0&1\end{bmatrix}$

$[T(B_4)]_B$

$=\begin{pmatrix}0\\0\\0\\1\end{pmatrix}$

應該寫成向量 $\begin{pmatrix}0\\0\\0\\1\end{pmatrix}$

$$[T]_B=\begin{bmatrix}0&0&2&0\\1&0&1&0\\0&1&-2&0\\0&0&0&1\end{bmatrix}\ \#$$

(b) $\begin{vmatrix}\lambda&0&-2&0\\-1&\lambda&-1&0\\0&-1&\lambda+2&0\\0&0&0&\lambda-1\end{vmatrix}=0$, $(\lambda-1)^2(\lambda+1)(\lambda+2)=0$

$(\lambda-1)[\lambda(\lambda^2+2\lambda-1)-2]$

$\lambda=1,1,-1,-2.$

$\lambda=1$ $\begin{bmatrix}1&0&-2&0\\-1&1&-1&0\\0&-1&3&0\\0&0&0&0\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\\x_4\end{bmatrix}$

$x_1-2x_3=0$
$-x_1+x_2-x_3=0$
$-x_2+3x_3=0$

$\begin{bmatrix}x_1\\x_2\\x_3\\x_4\end{bmatrix}=\begin{bmatrix}2t\\3t\\t\\s\end{bmatrix}=t\begin{bmatrix}2\\3\\1\\0\end{bmatrix}+s\begin{bmatrix}0\\0\\0\\1\end{bmatrix}$

basis $\left\{\begin{bmatrix}2\\3\\1\\0\end{bmatrix}\begin{bmatrix}0\\0\\0\\1\end{bmatrix}\right\}$ #

$\Rightarrow$ the basis for It's the eigenspace of $[T]_B$

真正答案應該是

$\left\{\begin{bmatrix}2&3\\1&0\end{bmatrix},\begin{bmatrix}0&0\\0&1\end{bmatrix}\right\}$ is the basis for the eigenspace of T corresponding

$\lambda=-2$ $\begin{bmatrix}-2&0&-2&0\\-1&-2&-1&0\\0&1&0&0\\0&0&0&-3\end{bmatrix}$

$-2x_1-2x_3=0$
$-x_1-2x_2-x_3=0$
$-x_2=0$
$-3x_4=0$

$\begin{bmatrix}x_1\\x_2\\x_3\\x_4\end{bmatrix}=\begin{bmatrix}-t\\0\\t\\0\end{bmatrix}=t\begin{bmatrix}-1\\0\\1\\0\end{bmatrix}$

basis $\left\{\begin{bmatrix}-1\\0\\1\\0\end{bmatrix}\right\}$ #

$\left\{\begin{bmatrix}-1&0\\1&0\end{bmatrix}\right\}$

$\lambda=-1$ $\begin{bmatrix}-1&0&-2&0\\-1&-1&-1&0\\0&1&1&0\\0&0&0&-2\end{bmatrix}$

$-x_1-2x_3=0$
$-x_1-x_2-x_3=0$
$-x_2+x_3=0$
$-2x_4=0$

$\begin{bmatrix}x_1\\x_2\\x_3\\x_4\end{bmatrix}=t\begin{bmatrix}-2\\1\\1\\0\end{bmatrix}$

basis $=\left\{\begin{bmatrix}-2\\1\\1\\0\end{bmatrix}\right\}$ #

$\left\{\begin{bmatrix}-2&1\\1&0\end{bmatrix}\right\}$

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check:

$\lambda=1$ $T\Big(\underbrace{k_1\begin{bmatrix}2&3\\1&0\end{bmatrix}+k_2\begin{bmatrix}0&0\\0&1\end{bmatrix}}_{X}\Big)=\underline{k_1\begin{bmatrix}2&3\\1&0\end{bmatrix}+k_2\begin{bmatrix}0&0\\0&1\end{bmatrix}}=X$

$\lambda=-2$ $T\Big(\underbrace{k_1\begin{bmatrix}-1&0\\1&0\end{bmatrix}}_{X}\Big)=k_1\begin{bmatrix}2&0\\-2&0\end{bmatrix}=-2X$

$\lambda=-1$ $T\Big(\underbrace{k_1\begin{bmatrix}-2&1\\1&0\end{bmatrix}}_{X}\Big)=k_1\begin{bmatrix}2&-1\\-1&0\end{bmatrix}=-X$

3. $W: x+0y+z=0$

(a) $A^TAx=A^Tb$

$$\begin{bmatrix}1&0&-1\\0&2&0\end{bmatrix}\begin{bmatrix}1&0\\0&2\\-1&0\end{bmatrix}=\begin{bmatrix}2&0\\0&4\end{bmatrix}\ (A^TA)$$

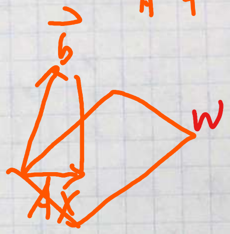
$$\begin{bmatrix}2&0\\0&4\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix}=\begin{bmatrix}1&0&-1\\0&2&0\end{bmatrix}\begin{bmatrix}-1\\-2\\4\end{bmatrix}=\begin{bmatrix}-5\\-4\end{bmatrix}$$

$2x_1=-5$ $\quad$ $x_1=-\frac{5}{2}\ \ x_2=-1$ #

$4x_2=-4$

$AX\neq b$

but: $A^TAx=A^Tb$ 有解



$X=(A^TA)^{-1}A^Tb$

$AX=A(A^TA)^{-1}A^Tb$

(b) $x=-s \quad y=t \quad z=s$

$$\begin{bmatrix}-s\\t\\s\end{bmatrix}=t\begin{bmatrix}0\\1\\0\end{bmatrix}+s\begin{bmatrix}-1\\0\\1\end{bmatrix}$$

orthonormal basis $(0,1,0)$, $(\frac{-1}{\sqrt{2}},0,\frac{1}{\sqrt{2}})$

(c) $A(A^TA)^{-1}A^T$

$A^TA=\begin{bmatrix}2&0\\0&4\end{bmatrix}$ $\quad (A^TA)^{-1}=\frac{1}{8}\begin{bmatrix}4&0\\0&2\end{bmatrix}$

$$\frac{1}{8}\begin{bmatrix}1&0\\0&2\\-1&0\end{bmatrix}\begin{bmatrix}4&0\\0&2\end{bmatrix}\Rightarrow\begin{bmatrix}4&0\\0&4\\-4&0\end{bmatrix}\begin{bmatrix}1&0&-1\\0&2&0\end{bmatrix}=\begin{bmatrix}4&0&-4\\0&8&0\\-4&0&4\end{bmatrix}\frac{1}{8}$$

$$\Rightarrow\begin{bmatrix}\frac{1}{2}&0&-\frac{1}{2}\\0&1&0\\-\frac{1}{2}&0&\frac{1}{2}\end{bmatrix}$$ #

(d) $$\begin{bmatrix}\frac{1}{2}&0&-\frac{1}{2}\\0&1&0\\-\frac{1}{2}&0&\frac{1}{2}\end{bmatrix}\begin{bmatrix}1\\-2\\4\end{bmatrix}=\begin{bmatrix}-\frac{3}{2}\\-2\\\frac{3}{2}\end{bmatrix}$$ #

(e)

$(1,-2,4)\cdot(0\,1\,0)\times(0\,1\,0)=(0,-2,0)$

$(1,-2,4)\cdot(-\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}})\times(-\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}})=\frac{3}{\sqrt{2}}(-\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}})$

$=(-\frac{3}{2}, 0, \frac{3}{2})$

$(0,-2,0)+(-\frac{3}{2}, 0, \frac{3}{2})=(-\frac{3}{2}, -2, \frac{3}{2})$ #

(f) $\begin{bmatrix} \frac{1}{2} & 0 & -\frac{1}{2} \\ 0 & 1 & 0 \\ -\frac{1}{2} & 0 & \frac{1}{2} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}=\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$

$\frac{1}{2}x_1-\frac{1}{2}x_3=1$ $\quad$ $x_2=1$

$x_2=1$ $\Rightarrow$ $x_1=t$ $\quad\Rightarrow x_1=t \quad x_2=1 \quad x_3=t-2$

$-\frac{1}{2}x_1+\frac{1}{2}x_3=-1$ $\quad$ $x_3=t-2$

$(t, 1, t-2)\ t\in\mathbb{R}$ #

$\parallel$

$(0\,1\,0)+t(1\,0\,-2)$

4、

$S = \{e_1, e_2, \dots e_n\}$

$u = u_1e_1 + u_2e_2 + \dots u_ne_n$

$v = v_1e_1 + v_2e_2 + \dots v_ne_n$

$\langle u, v\rangle = \langle (u_1e_1 + u_2e_2 \cdots u_ne_n), (v_1e_1 + v_2e_2 + \dots v_ne_n)\rangle$

$= \langle u_1e_1, v_1e_1\rangle + \dots \langle u_ne_1, v_ne_1\rangle$

$+ \langle u_1e_2, v_1e_2\rangle + \dots \langle u_ne_2, v_ne_2\rangle$

$\vdots$

$= u_1v_1\langle e_1 e_1\rangle + \dots u_nv_n\langle e_n e_n\rangle$

$\because$ orthonormal basis $|e_n| = 1 = \langle e_n e_n\rangle$

$\therefore = u_1v_1 + u_2v_2 + \dots u_nv_n$ #

5.

(a) $T(\begin{bmatrix}1\\0\end{bmatrix}) = \frac{1}{\sqrt{2}}\begin{bmatrix}1 & 1\\1 & -1\end{bmatrix}\begin{bmatrix}1\\0\end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\1\end{bmatrix}$

$T(\begin{bmatrix}0\\1\end{bmatrix}) = \frac{1}{\sqrt{2}}\begin{bmatrix}1 & 1\\1 & -1\end{bmatrix}\begin{bmatrix}0\\1\end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\-1\end{bmatrix}$

$[T]_{B'} = \begin{bmatrix}\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}}\end{bmatrix}$

basis $= \left\{\begin{bmatrix}\frac{1}{\sqrt{2}}\\ \frac{1}{\sqrt{2}}\end{bmatrix}, \begin{bmatrix}\frac{1}{\sqrt{2}}\\ -\frac{1}{\sqrt{2}}\end{bmatrix}\right\}$

(b) $1 = a\frac{1}{\sqrt{2}} + b\frac{1}{\sqrt{2}}$ $\qquad \frac{2a}{\sqrt{2}} = 2$ $\qquad a = \sqrt{2}$

$1 = a\frac{1}{\sqrt{2}} - b\frac{1}{\sqrt{2}}$ $\qquad\qquad b = 0$

$\langle (1,1)\rangle_{B'} = (\sqrt{2}, 0)$ #

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(a) $\langle s_1(t)\rangle_{s} = \left(\frac{-\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$, $\langle s_2(t)\rangle_{s} = \left(\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right)$

(b) ①

$$f = s_1 - s_2 = -\sqrt{2}\,\varphi_1(t) + \sqrt{2}\,\varphi(t)$$
$$= 2\sin(2\pi t) - 2\cos(2\pi t)$$
$$= 2(\sin(2\pi t) - \cos(2\pi t))$$

$|f| = \sqrt{\langle f, f\rangle}$ $\theta = 2\pi t$

$$(2\sin\theta - 2\cos\theta)^2 = 4\sin^2\theta - 8\sin\theta\cos\theta + 4\cos^2\theta$$
$$= 4 - 8\sin\theta\cos\theta$$
$$= 4 - 4\sin 2\theta = 4 - 4\sin(4\pi t)$$

$$\int_0^1 4 - 4\sin(4\pi t)\,dt = 4 - 4\int_0^1 \sin(4\pi t)\,dt$$
$$= 4 - 4\left(-\frac{1}{4\pi}\cos(4\pi t)\right)\Big]_0^1$$
$$= 4$$

$|f| = \sqrt{4} = 2$ #

② $[f]_s = (-\sqrt{2}, \sqrt{2})$

$|f| = \sqrt{(-\sqrt{2})^2 + (\sqrt{2})^2} = 2$ #

7. $\begin{bmatrix} 1 & 2 & 1 \\ 0 & 2 & 2 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$

$$\begin{bmatrix} 1 & 2 & 1 & b_1 \\ 0 & 2 & 2 & b_2 \\ 0 & 1 & 1 & b_3 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 1 & b_1 \\ 0 & 0 & 0 & b_2 - 2b_3 \\ 0 & 1 & 1 & b_3 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 1 & 0 & b_1 - b_3 \\ 0 & 1 & 1 & b_3 \\ 0 & 0 & 0 & b_2 - 2b_3 \end{bmatrix}$$

$b_2 - 2b_3 = 0$ $\quad$ $b_2 = 2t \quad b_3 = t \quad b_1 = s$

$b = \begin{bmatrix} s \\ 2t \\ t \end{bmatrix} \quad s, t \in R$ #

$b = s\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + t\begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}$

$s, t \in \mathbb{R}$