

Artificial Intelligence - Introduction to Reinforcement Learning

ENSISA 2A

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ENSISA, Université Haute-Alsace

September 12, 2023



Une école d'ingénieurs de l'Université de Haute-Alsace



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 - Time Series Generation with application on Human Motion Skeleton based datasets.

Overview

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 - Slides of Dr. Mireille Sarkiss, Telecom SudParis, Institut Polytechnique de Paris, Lectures 1-2

Sequential Decision Making

Sequential Decision Making

Suppose the following scenario

Sequential Decision Making

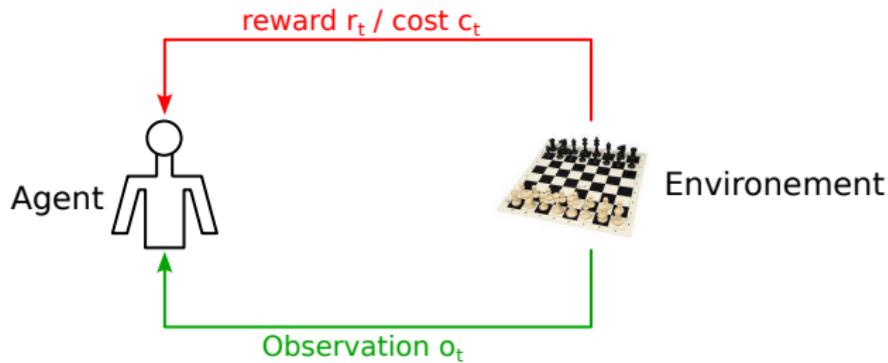
Suppose the following scenario



An agent wants to learn an environment's behavior, in this case, a game of chess

Sequential Decision Making

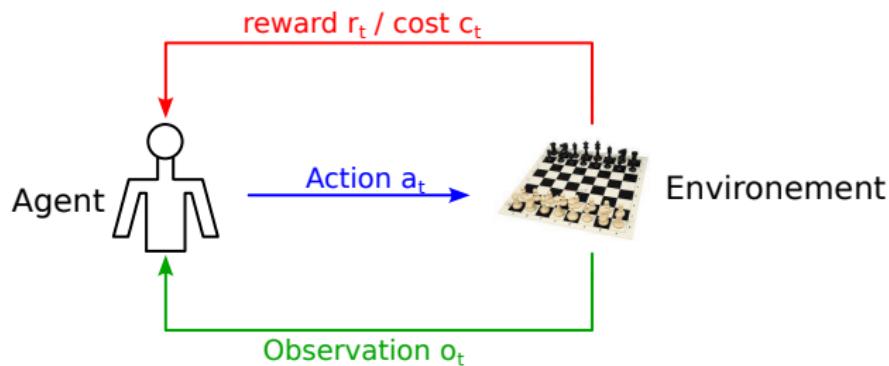
Suppose the following scenario



The environment reports a **reward r_t or cost c_t** to the agent as well as an observation o_t .

Sequential Decision Making

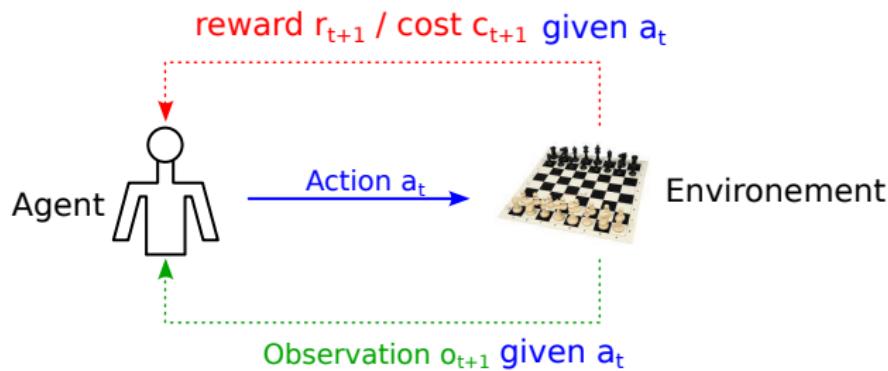
Suppose the following scenario



The agent then takes a decision given the information provided previously and applies the **action a_t** .

Sequential Decision Making

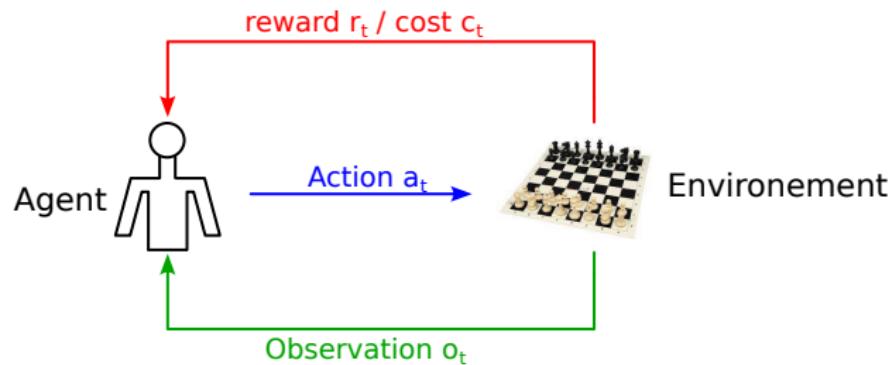
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The environment then reports, given **action a_t** , an **observation o_{t+1}** and a **reward r_{t+1} or cost c_{t+1}** .

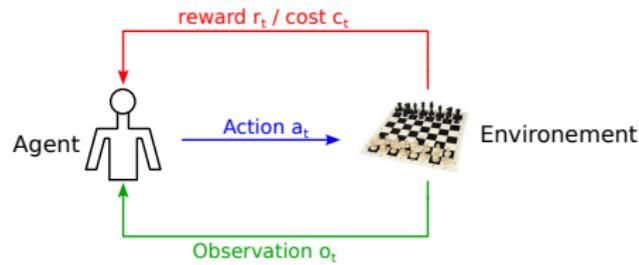
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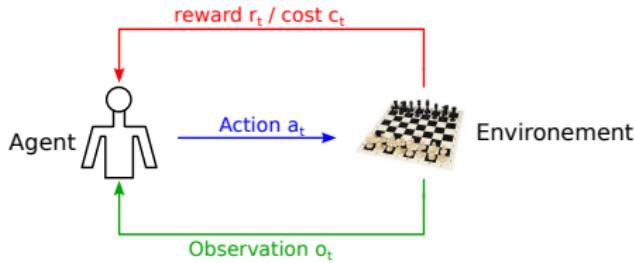


The goal of the agent is to choose the best **actions** that maximizes or minimizes the **reward or cost**.

History and State



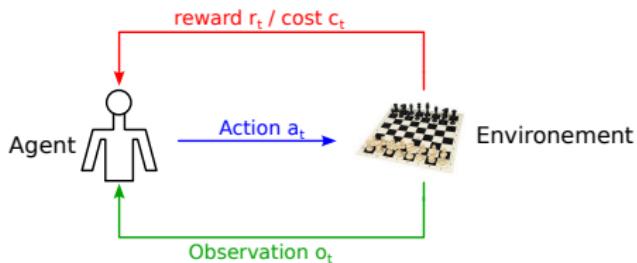
History and State



- History: sequence of past observation, actions and rewards up to the agent's decision time t

$$h_t = (o_1, r_1, a_1), (o_2, r_2, a_2), \dots, (o_t, r_t)$$

History and State

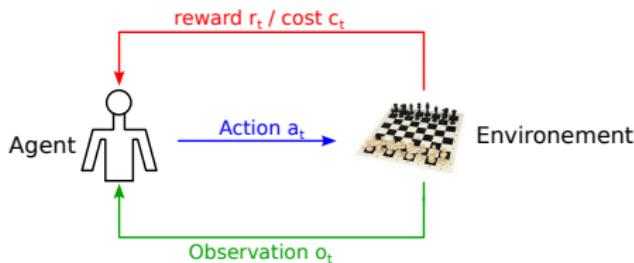


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- Agent takes decisions given the history.
- A state is produced given the history, function of history: $s_t = f(h_t)$.

Agent and Environment States

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- A process changes in a:
 - Deterministic way: given an action, one single observation and reward/cost are generated (ex: robotics)
 - Stochastic way: given an action, many possible observations and rewards/costs can be generated (ex: card game)

Components of Sequential Decision Process

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Agents the following components:

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- Value function: Given a policy that the agent follows, how good is each state and/or action over a long run.

Components of Sequential Decision Process

Agents have the following components:

- Policy π : agent's behavior (actions) function
- Value function: Given a policy that the agent follows, how good is each state and/or action over a long run.
- Model: The agent's understanding of the environment i.e. how the environment responds to a given action provided by the agent.

Policy

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- Stochastic policy: $\pi(a|s) = P(a_t = a|s_t = s)$

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- γ controls which is more important: immediate or future reward/cost.

Model

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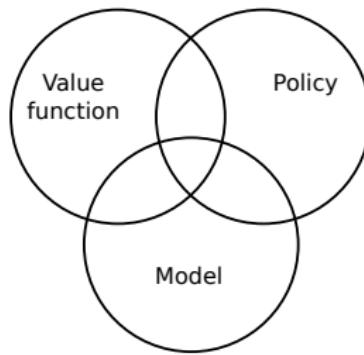
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- A reward model predicts the next immediate reward:

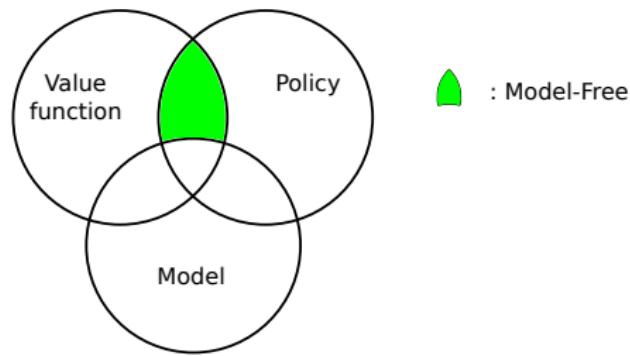
$$R_s^a = R(s, a) = \mathbb{E}[r_{t+1} | s_t = s, a_t = a]$$

Types of Agents

Types of Agents

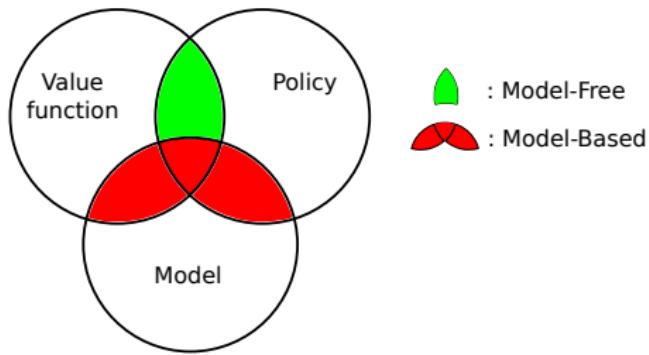


Types of Agents



A model-free agent will optimize the policy without trying to learn the environment rules and state information.

Types of Agents



A model-based agent will optimize the policy by learning the model's behavior and state information.

Sequential Decision Making - Reinforcement Learning

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 - Exploitation: select the action that maximizes the reward given past experience.
 - Exploration: discover more possibilities to find more information about the environment's behavior to avoid doing a mistake in the future.

Exploitation/Exploration Trade Off

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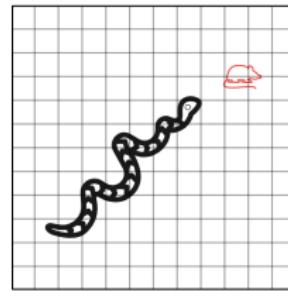
Agent Simulator

Consider an agent being a computer bot.

Exploitation/Exploration Trade Off



Agent
Simulator



Snake
Game

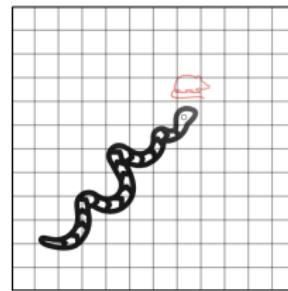
The agent, at time t , is trying to win a snake game by understanding the environment's behavior.

Exploitation/Exploration Trade Off

Hmm instead
of going up I will
try to go right this
time for future
use



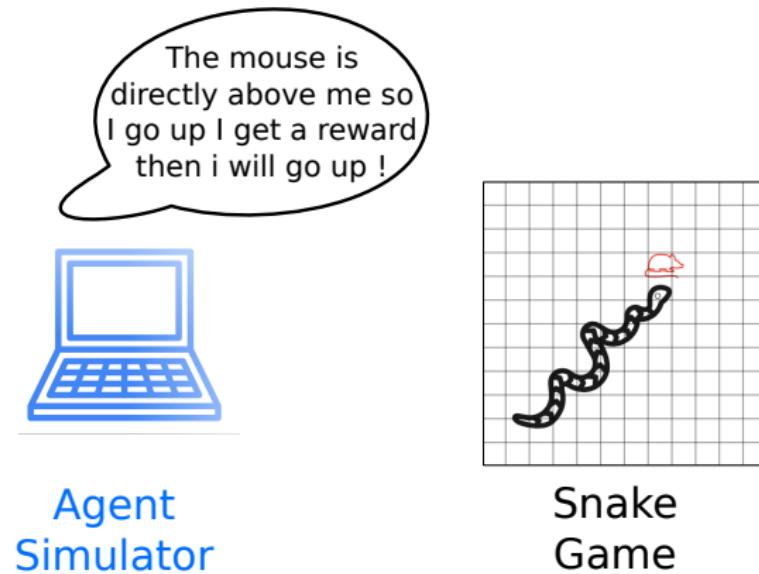
Agent
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Snake
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The agent when exploring is trying to do an action to explore.

Exploitation/Exploration Trade Off



The agent when exploiting is trying to take a decision that maximizes the reward.

Markov Decision Process (MDP)

MDP and Markov Property

Definition: Markov State

A state s_t is Markov if and only if:

$$P(s_{t+1}|s_t) = P(s_{t+1}|s_t, s_{t-1}, \dots, s_1)$$

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a \longrightarrow b
means b depends on a

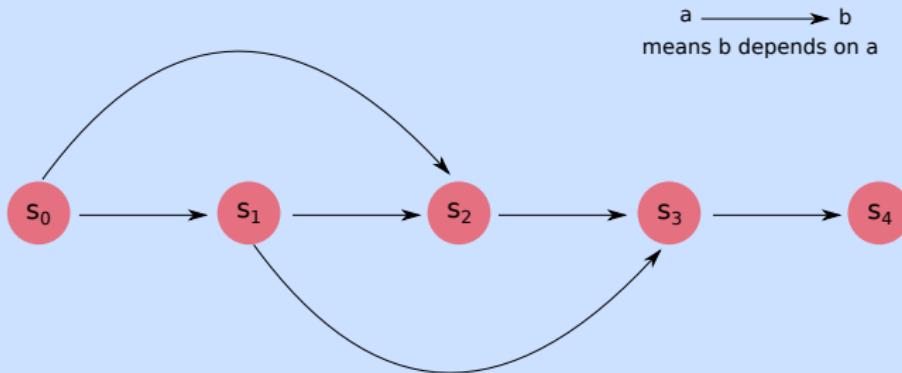


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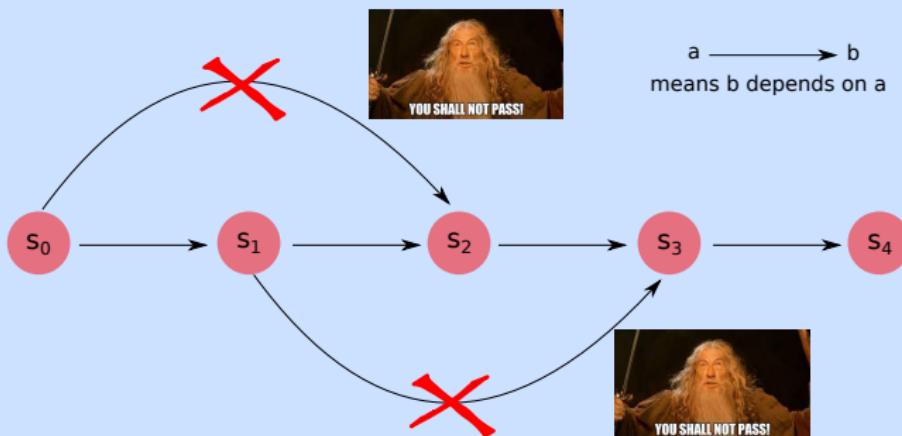


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- State transition matrix P where only the rows sum up to 1.

$$P = [P_{ss'}] \begin{bmatrix} P_{s_1 s_1} & P_{s_1 s_2} & \dots & P_{s_1 s_N} \\ P_{s_2 s_1} & P_{s_2 s_2} & \dots & P_{s_2 s_N} \\ \dots & \dots & \dots & \dots \\ P_{s_N s_1} & P_{s_N s_2} & \dots & P_{s_N s_N} \end{bmatrix}$$

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- Rows sum up to 1 but columns do not (proof next slide).

MDP and Markov Property

proof:

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MDP and Markov Property

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$$\text{Sum of row } r = \sum_{i=1}^N P_{s_r s_i} (s_{t+1} = s_i | s_t = s_r)$$

MDP and Markov Property

proof:

- Rows sum up to 1:

$$\begin{aligned}\text{Sum of row } r &= \sum_{i=1}^N P_{s_r s_i}(s_{t+1} = s_i | s_t = s_r) \\ &= \sum_{i=1}^N \frac{P_{s_r s_i}(s_{t+1} = s_i, s_t = s_r)}{P(s_t = s_r)} \text{ Bias Rule}\end{aligned}$$

MDP and Markov Property

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MDP and Markov Property

proof:

- Columns do not sum up to 1:

$$\begin{aligned}\text{Sum of column } c &= \sum_{i=1}^N P_{s_i s_c}(s_{t+1} = s_c | s_t = s_i) \\ &= \sum_{i=1}^N \frac{P_{s_i s_c}(s_{t+1} = s_c, s_t = s_i)}{P(s_t = s_i)} \text{ Bias Rule}\end{aligned}$$

Nothing can be done here as the sum includes both numerator and denominator.

Markov Process

Markov Process

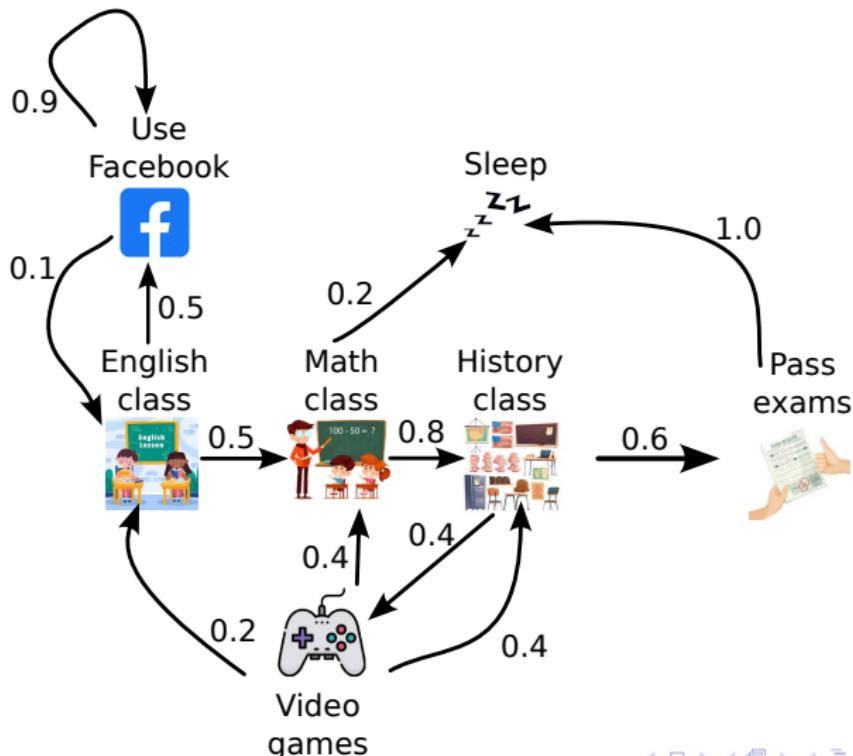
Definition: Markov Process or Markov Chain

A Markov Process or Chain is a tuple $\langle S, P \rangle$ where:

- S is a finite set of states
- P is a state transition probability matrix

Markov Process

Example: Student Markov Chain



Markov Reward Process (MRP)

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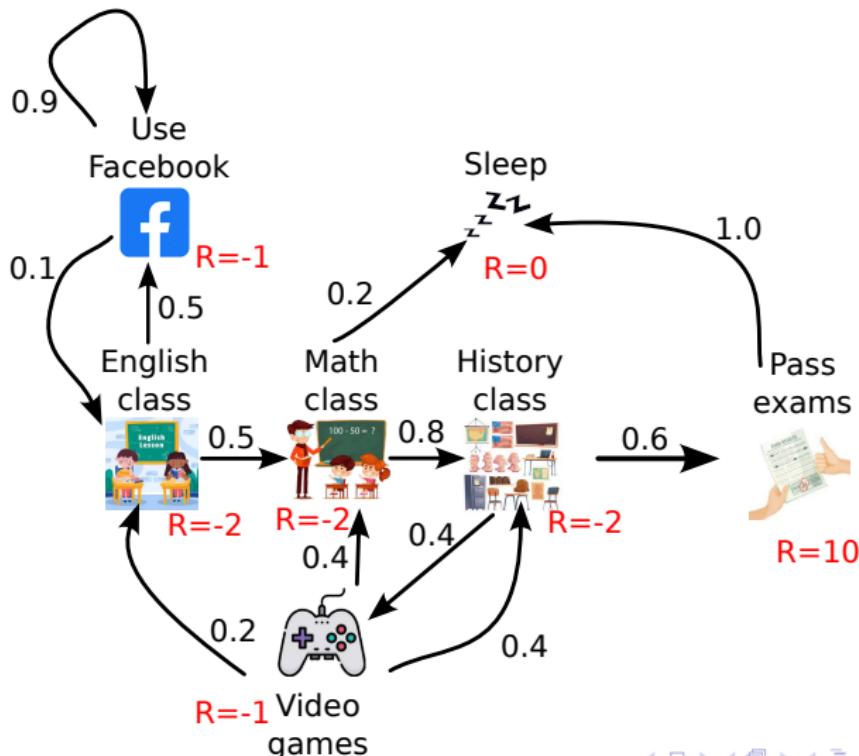
Definition: Markov Reward Process

A Markov Reward Process is a tuple $\langle S, P, R, \gamma \rangle$ where:

- S is a finite set of states
- P is a state transition probability matrix
- R is a reward function
- γ is a discount factor

Markov Reward Process (MRP)

Example: Student MRP



Properties of MRP

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Definition: Horizon

The number of time steps T in each episode, can be finite or ∞ .

Properties of MRP

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Definition: Return

The return G_t at time t of an MRP is the total discounted reward from t to T .

$$G_t = r_{t+1} + \gamma r_{t+2} + \gamma^2 r_{t+3} + \dots = \sum_{k=0}^{\infty} \gamma^k r_{t+k+1}$$

Properties of MRP

Definition: Horizon

The number of time steps T in each episode, can be finite or ∞ .

Definition: Return

The return G_t at time t of an MRP is the total discounted reward from t to T .

$$G_t = r_{t+1} + \gamma r_{t+2} + \gamma^2 r_{t+3} + \dots = \sum_{k=0}^{\infty} \gamma^k r_{t+k+1}$$

Definition: State Value Function

As defined before, slide 10. Its the expected return starting from state s :

$$V(s) = \mathbb{E}[G_t | s_t = s]$$

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The last step simply uses the rule that says $\mathbb{E}[\mathbb{E}[X|Y]|Z] = \mathbb{E}[X|Z]$ and that

$$\mathbb{E}[V(s_{t+1}) | s_t = s] = \mathbb{E}[\mathbb{E}[G_{t+1} | s_{t+1}] | s_t = s] = \mathbb{E}[G_{t+1} | s_t = s]$$

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- Better solution: Dynamic programming.

Defining the MDP

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Its a tuple $\langle S, A, P, R, \gamma \rangle$

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Model-Based Reinforcement Learning Methods

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Definition: Principle of Optimality

A policy $\pi(a|s)$ achieves the optimal value from state s , $V_\pi(s) = V^*(s)$ if and only if:

For any state s' reachable from s , π achieves the optimal value from state s' ,

$$V_\pi(s') = V^*(s')$$

Policy Iteration

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 - Step 1: Evaluate policy π using

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 - Step 2: Improve the policy by acting greedily w.r.t π using one look ahead in the Bellman equation.
$$\pi' = greedy(V_\pi)$$

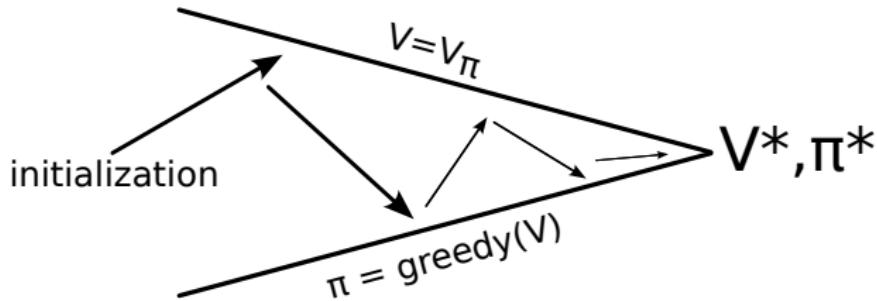
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$$V(s) = \max_{a \in A} (R_s^a + \gamma \sum_{s' \in S} P_{ss'}^a(s'))$$

- The last step consisting on choosing the policy of the optimal value function:

$$\pi(s) = \arg \max_{a \in A} (R_s^a + \gamma \sum_{s' \in S} P_{ss'}^a(s'))$$

Policy VS Value Iteration

Table: Comparing Policy and Value Iteration algorithms to solve MDP model-based problems.

Properties	Policy Iteration	Value Iteration
Initialization	Random policy	Random value function
Complexity	Very complex	Simple
Convergence	Guaranteed	Guaranteed
Number of iterations till convergence	Small number	High number
Speed	Fast	Slow

Model-Free Reinforcement Learning Methods

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Model-Free RL methods: Outline

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Monte-Carlo Simulation

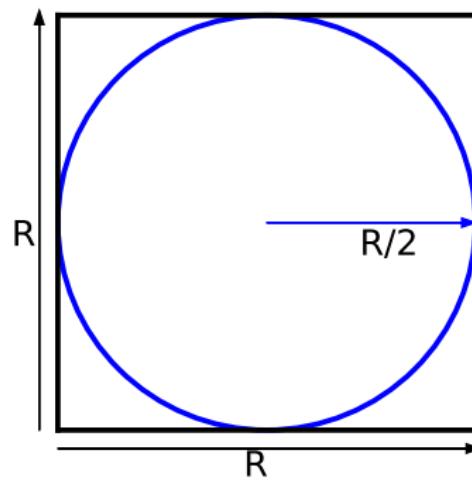
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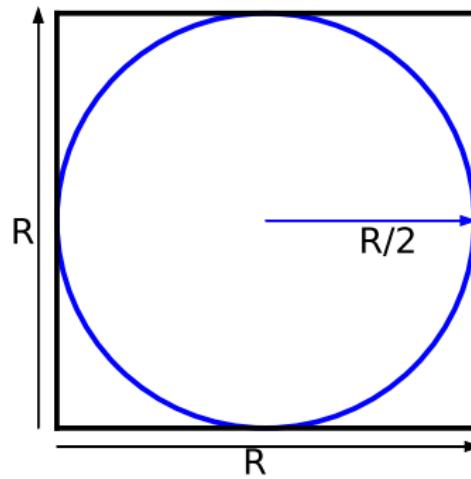
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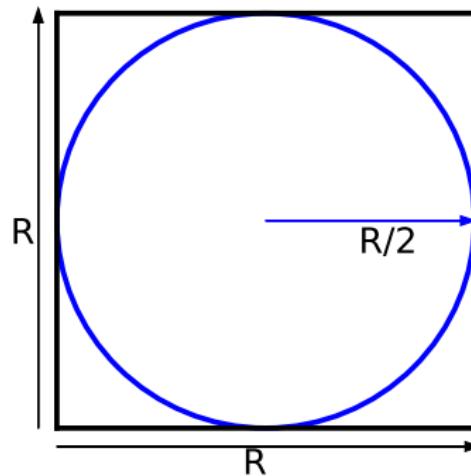


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- The area of the square is: $A_{sq} = R^2$
- The area of the circle is: $A_{cr} = \frac{\pi R^2}{4}$

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To estimate the value of π , we can randomly choose points in the $2D$ grid, but guaranteed to be inside the square.

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This means that $\pi = 4 * P$

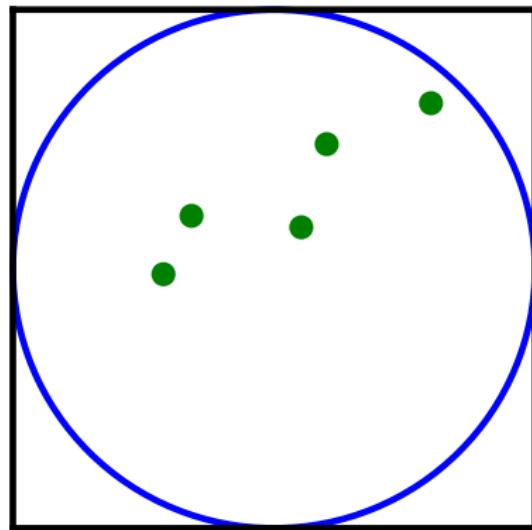
Monte-Carlo Simulation

To estimate the probability P of a point being in the circle, with enough generated points we have:

$$P = \frac{\text{number of points landing inside the circle}}{\text{total number of generated points}}$$

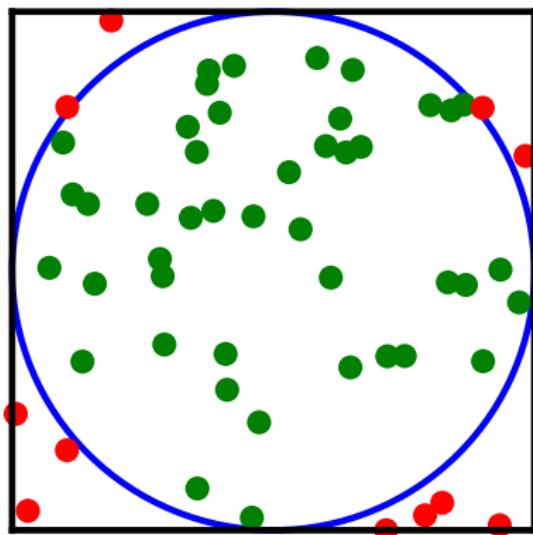
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$n = 5, \pi = 4.0$



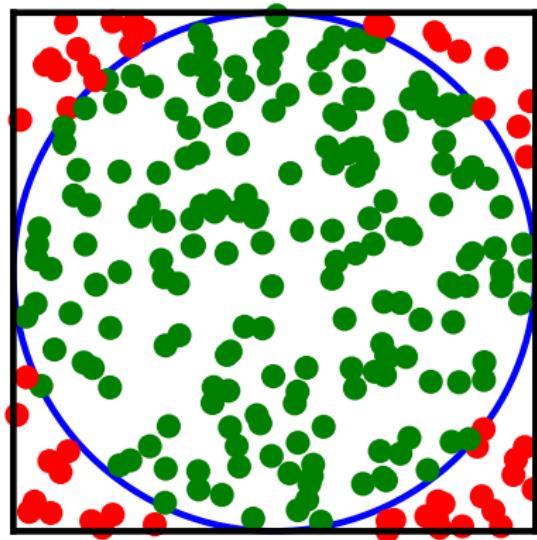
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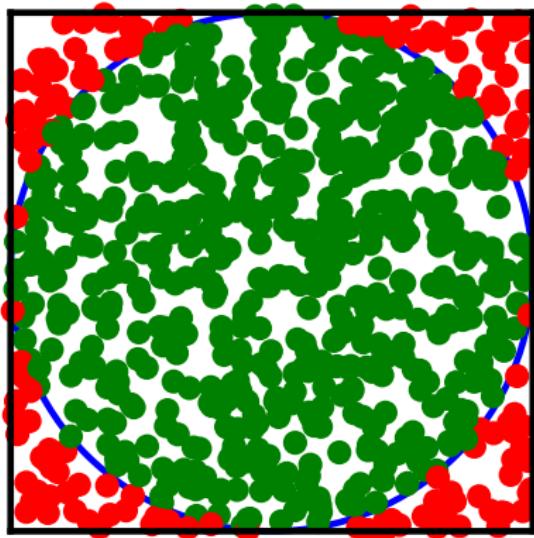
Monte-Carlo Simulation

$n = 200, \pi = 3.1$



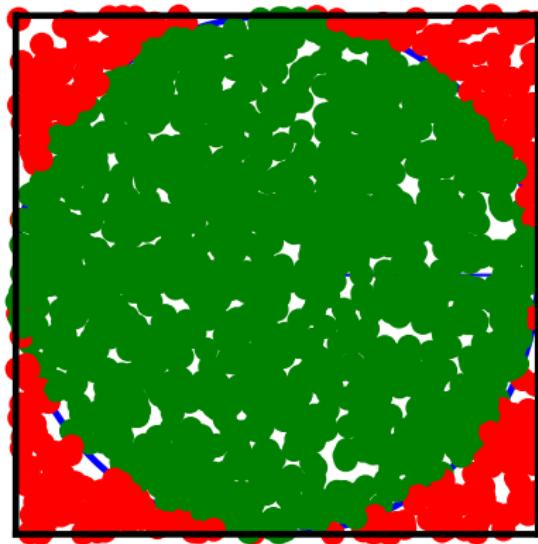
Monte-Carlo Simulation

$n = 500, \pi = 3.264$



Monte-Carlo Simulation

$n = 1000, \pi = 3.22$



Monte-Carlo Simulation

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If we keep generating points we will converge to almost 3.14

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Algorithm 1 First-Visit Monte-Carlo Prediction

Input: Policy π , int E the number of episodes

Output: Value function V_π

Initialize #visits $N(s) = 0$ and Returns $G(s) = 0 \forall s \in S$

for $e = 1$ to E **do**

 Generate using π an episode $s_0, a_0, r_1, s_1, a_1, r_2, \dots, s_{T-1}, a_{T-1}, a_T$

$g \leftarrow 0$

for $t = T - 1$ to $t = 0$ **do**

$g \leftarrow g + r_{t+1}$

if state s_t is not in the sequence s_0, s_1, \dots, s_{t-1} **then**

$G(s_t) \leftarrow G(s_t) + g$

$N(s_t) \leftarrow N(s_t) + 1$

end if

end for

end for

return $V(s) \leftarrow \frac{G(s)}{N(s)} \forall s \in S$

Every-Visit MC Policy Evaluation

Algorithm 2 Every-Visit Monte-Carlo Prediction

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 $s_0, a_0, r_1, s_1, a_1, r_2, \dots, s_{T-1}, a_{T-1}, r_T$

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 V(s) &= V(s) + \frac{1}{N(s)} (G_t - V(s))
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- New Estimate = Old Estimate + Step * (Target - Old Estimate)

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- This is done incrementally while going through the episode.

Off Policy Control: Q-Learning

Off Policy Control: Q-Learning

Definition: State-Action Value function

The state-action value function $q_{\pi}(s, a)$ of an MDP is the expected Return for when starting in a state s , and taking action a following the policy π .

$$A_{\pi}(s, a) = \mathbb{E}_{\pi}[G_t | s_t = s, a_t = a]$$
$$V_{\pi}(s) = \sum_{a \in A} \pi(a|s).Q(s, a)$$

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Bellman Expectation Equation for state-action value function:

$$A_{\pi}(s, a) = \mathbb{E}_{\pi}[r_{t+1} + \gamma A_{\pi}(s_{t+1}, a_{t+1}) | s_t = s, a_t = a] \quad (2)$$

Bellman Optimality Equation for state-action value function:

$$Q_{\pi}(s_t, a_t) = R_{s_t}^{a_t} + \gamma \max_{a \in A} Q_{\pi}(s_{t+1}, a) \quad (3)$$

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- α is the learning rate $\alpha \in [0, 1]$

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- Solution: ϵ – *greedy* approach ($\epsilon \in [0, 1]$):
 - With probability $1 - \epsilon$ choose the optimal action that maximizes the Q function
 - With probability ϵ choose a random action

The Q-Learning step-by-step Algorithm

Algorithm 3 Q-Learning Algorithm

```
Initialize  $Q(s, a) \forall s \in S, a \in A$  randomly and  $Q(\text{terminal\_state}, .) = 0$ 
for  $e = 1$  to  $E$  do
    Initialize  $s$ 
    while  $s \neq \text{terminal\_state}$  do
        Choose  $a$  from  $s$  using  $\epsilon - \text{greedy}$ 
        Take action  $a$  and observe  $s'$  and  $r$ 
         $Q(s, a) \leftarrow Q(s, a) + \alpha[r + \gamma \max_{a' \in A} Q(s', a') - Q(s, a)]$ 
    end while
end for
return  $Q$ 
```

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- RL is simply "the science of sequential decision making"
- RL consists on a smart Agent interacting with an Environment in order to learn how it works resulting in maximization of the reward
- Depending on the case scenario, a choice of algorithms should be done to solve such a problem