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CS3 (C. Prud'homme) :

Model Order Reduction Techniques
Problem Set 1: RB for Linear Affine Elliptic Problems
Design of a Thermal Fin

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Table of contents

Chapter 1

Introduction

1.1 Fin design parameters:

We consider the problem of designing a thermal fin to effectively remove heat from a surface. The two-dimensional fin, shown in Figure 1, consists of a vertical central post and four horizontal subfins; the fin conducts heat from a prescribed uniform flux source at the root, γ_{root} , through the large-surface-area subfins to surrounding flowing air. The fin is characterized by a five-component parameter vector, or input, $\mu = (\mu_1, \mu_2, ..., \mu_5)$,where $\mu_i = k^i, i = 1, ..., 4$, and $\mu_5 = Bi$; may take on any value in a specified design set $D \in R^5$.

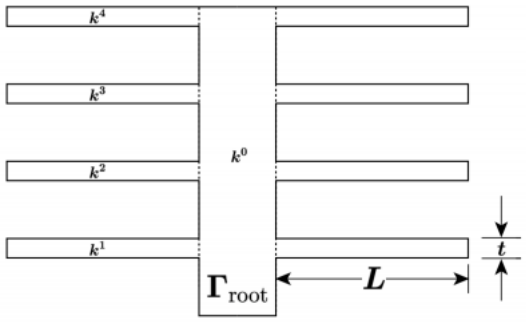

Figure 1: Thermal fin

Here k^i is the thermal conductivity of the ith subfin (normalized relative to the post conductivity $k^0$1); and Bi is the Biot number, a nondimensional heat transfer coefficient reflecting convective transport to the air at the fin surfaces (larger Bi means better heat transfer). For example, suppose we choose a thermal fin with $k^1 = 0.4, k^2 = 0.6, k^3 = 0.8, k^4 = 1.2, and Bi = 0.1$; for this particular configuration $\mu = 0.4, 0.6, 0.8, 1.2, 0.1$, which corresponds to a single point in the set of all possible configurations D (the parameter or design set).The post is of width unity and height four; the subfins are of fixed thickness t = 0.25 and length L = 2.5.

1.2 Governing equation:

We are interested in the design of this thermal fin, and we thus need to look at certain outputs or cost-functionals of the temperature as a function of μ. We choose for our output T_{root} , the average steady-state temperature of the fin root normalized by the prescribed heat flux into the fin root. The particular output chosen relates directly to the cooling efficiency of the fin lower values of T_{root} imply better thermal performance. The steadystate temperature distribution within the fin, $u(\mu)$, is governed by the elliptic partial differential equation:

$$-k^i \Delta u^i = 0 \quad in \quad \Omega^i, i = 0, ..., 4, \tag{1}$$

where Δ is the Laplacian operator, and u^i refers to the restriction of u to Ω^i. Here Ω^i is the region of the fin with conductivity k^i , i = 0, . . . , 4 : Ω^i is thus the central post, and Ω^i, i = 1, . . . , 4, corresponds to the four subfins. The entire fin domain is denoted $\Omega(\bar{\Omega} = \cup_{i=0}^{4} \bar{\Omega}^i)$.

1.3 Boundary conditions:

The boundary Ω is denoted Γ. We must also ensure continuity of temperature and heat flux at the conductivity discontinuity interfaces $\Gamma_{int}^i = \partial\Omega^0 \cap \partial\Omega^i$, i = 1, . . . , 4, where $\partial\Omega^i$ denotes the boundary of Ω^i. We have on Γ_{int}^i i = 1, . . . , 4 :

$$u^0 = u^i \tag{2}$$

$$-(\Delta u^0 . n^i) = k^i (\Delta u^i . n^i) \tag{3}$$

Here n^i is the outward normal on $\partial\Omega^i$. Finally, we introduce a Neumann flux boundary condition on the fin root:

$$-(\Delta u^0 . n^0) = 1 \quad on \quad \Gamma_{root} \tag{4}$$

which models the heat source; and a Robin boundary condition:

$$-k^i (\Delta u^i . n^i) = Bi u^i \quad on \quad \Gamma_{ext}^i i = 0, ..., 4 \tag{5}$$

which models the convective heat losses. Here iext is that part of the boundary of Ω^i exposed to the flowing fluid; note that $\cap_{i=0}^{4} \Gamma_{ext}^i = \Gamma / \Gamma_{root}$. The average temperature at the root, $\Gamma_{root}(\mu)$, can then be expressed as $l^O(u(\mu))$,where:

$$l^O(v) = \int_{\Gamma_{root}} v \tag{6}$$

(recall Γ_{root} is of length unity). Note that $l(v) = l^O(v)$ for this problem.

Chapter 2

Finite Element Approximation:

We saw in class that the reduced basis approximation is based on a truth finite element approximation of the exact (or analytic) problem statement. To begin, we have to show that the exact problem described above does indeed satisfy the affine parameter dependence and thus fits into the framework shown in class.

2.1 The variational formulation:

We start from the governing equation:

$$-k^i \Delta u^i = 0 \quad in \quad \Omega^i, i = 0, ..., 4, \tag{1}$$

We consider: $\omega = u^i$ on Ω^i with $\omega \in \mathbb{H}^1(\Omega)$,Then equation (1) implies:

$$-k^i \Delta \omega . v = 0 \quad \forall v \in \mathbb{H}^1(\Omega)$$

We integrate:

$$\int_\Omega -k^i \Delta \omega . v dA = 0 \quad \forall v \in \mathbb{H}^1(\Omega)$$

And after applying the Green formula:

$$\int_\Omega k^i \nabla \omega . \nabla v dA - \int_\Gamma k^i \nabla \omega . n^i . v dS = 0 \quad \forall v \in \mathbb{H}^1(\Omega)$$

Knowing that $\Gamma = \Gamma_{/\Gamma_{root}} \cup \Gamma_{root}$:

$$\int_\Omega k^i \nabla \omega . \nabla v dA - \int_{\Gamma_{/\Gamma_{root}}} k^i \nabla \omega . n^i . v dS - \int_{\Gamma_{root}} k^0 \nabla \omega . n^0 . v dS = 0 \quad \forall v \in \mathbb{H}^1(\Omega)$$

We use Boundary conditions (4) and (5) to obtain:

$$\int_\Omega k^i \nabla \omega . \nabla v dA + \int_{\Gamma_{/\Gamma_{root}}} Bi\omega . v dS - \int_{\Gamma_{root}} v dS = 0 \quad \forall v \in \mathbb{H}^1(\Omega)$$

We seperate Ω into to 5 subdomains:

$$\sum_{i=0}^{4} \int_{\Omega^i} k^i \nabla \omega . \nabla v dA + Bi \int_{\Gamma_{/\Gamma_{root}}} \omega . v dS = \int_{\Gamma_{root}} v dS \quad \forall v \in \mathbb{H}^1(\Omega)$$

Then we identify the above equation to the following one:

$$a(\omega, v; \mu) = l(v) \quad \forall v \in \mathbb{H}^1(\Omega) \tag{7}$$

With:

$$a(\omega, v; \mu) = \sum_{i=0}^{4} \int_{\Omega^i} k^i \nabla\omega.\nabla v dA + Bi \int_{\Gamma_{/\Gamma_{root}}} \omega.v dS$$

$$l(v) = \int_{\Gamma_{root}} v dS$$

2.2 Lax-Milgram theorem application:

Is $a(.,.;\mu)$ Bilinear and Symmetric?:

In one hand:

$$a(u, v; \mu) = \sum_{i=0}^{4} \int_{\Omega^i} k^i \nabla u.\nabla v dA + Bi \int_{\Gamma_{/\Gamma_{root}}} u.v dS = a(v, u; \mu)$$

In the other hand: $a(.,.;\mu)$ is a combination of bilinear functions, as integral and derivative are bilinear, so $a(.,.;\mu)$ is bilinear.

Is $a(.,.;\mu)$ Continuous? :

$$\mid a(u, v; \mu) \mid = \mid \sum_{i=0}^{4} \int_{\Omega^i} k^i \nabla u.\nabla v dA + Bi \int_{\Gamma_{/\Gamma_{root}}} u.v dS \mid$$

The tiangular inequality gives:

$$\mid a(u, v; \mu) \mid \leq \int_{\Omega} k^i \mid \nabla u.\nabla v \mid dA + Bi \int_{\Gamma_{/\Gamma_{root}}} \mid u.v \mid dS$$

The Cauchy-Schwarz inequality for all vectors u and v of an inner product space in both $\mathbb{L}(\Omega)$ and $\mathbb{L}(\Gamma)$ gives:

$$\mid a(u, v; \mu) \mid \leq max(k^i)_{i=0,..4} \|u\|_{\mathbb{H}^1(\Omega)}.\|v\|_{\mathbb{H}^1(\Omega)} + Bi\|u\|_{\mathbb{L}(\Gamma)}.\|v\|_{\mathbb{L}(\Gamma)}$$

Note that:

$$\int_{\Omega} \mid \nabla u.\nabla v \mid dA \leq \int_{\Omega} \mid \nabla u \mid . \mid \nabla v \mid dA + \int_{\Omega} \mid u \mid . \mid v \mid dA$$

And the Trace theorem allows us to write with C ≥ 0:

$$\mid a(u, v; \mu) \mid \leq max(k^i)_{i=0,..4} \|u\|_{\mathbb{H}^1(\Omega)}.\|v\|_{\mathbb{H}^1(\Omega)} + C\|u\|_{\mathbb{L}(\Omega)}.\|v\|_{\mathbb{L}(\Omega)}$$

Thanks the continuous integration of $\mathbb{H}^1(\Omega)$ in $\mathbb{L}^2(\Omega)$:

$$\exists \gamma(\mu) \geq 0, \mid a(u, v; \mu) \mid \leq \gamma(\mu)\|u\|_{\mathbb{H}^1(\Omega)}.\|v\|_{\mathbb{H}^1(\Omega)}$$

Is $a(.,.;\mu)$ coercive? :

We assume that $a(.,.;\mu)$ is not coercive! (reductio ad absurdum). Which means:

$$\forall n, \exists u_n \quad \|u_n\|_{\mathbb{H}^1(\Omega)} > n\big(\|u_n\|_{\mathbb{L}^2(\Gamma)} + \|\nabla u_n\|_{\mathbb{L}^2(\Omega)}\big)$$

And then we consider the sequence:

$$\frac{u_n}{\|u_n\|_{\mathbb{H}^1(\Omega)}}$$

And we assume thus:

$$\|u_n\|_{\mathbb{H}^1(\Omega)} = 1$$

So the sequence is then bounded in $\mathbb{H}^1(\Omega)$ And according to Rellich theorem, $\exists u_{n^{sub}}$ a subsequence $\in \mathbb{L}^2(\Omega)$ converging to u $\in \mathbb{H}^1(\Omega)$. Moreover $\|\nabla u_{n^{sub}}\|_{\mathbb{L}^2(\Omega)}$ converge to 0. Thus, $u_{n_{sub}}$ is a Cauchy sequence $\in \mathbb{H}^1(\Omega)$. We conclude that u is constante on each of the subdomains of Ω. Given that the Trace application is continuous from $\mathbb{H}^1(\Omega)$ to $\mathbb{L}^2(\Gamma)$ the Trace of u on Γ is equal to the limit of all Traces of $u_{n^{sub}}$ on Γ.

However, $lim_n \|u_{n^{sub}}\|_{\mathbb{H}^1(\Omega)} = 0$. So u = 0 on Γ. Which is in contradiction to the fact that $\|u_n\|_{\mathbb{H}^1(\Omega)} = 1$.

Then we conclude that:

$$\exists \alpha(\mu) \geq 0, \quad \mid a(u,u;\mu) \mid \geq \alpha(\mu)\|u\|_{\mathbb{H}^1(\Omega)}$$

All conditions of lax-Milgram are fulfilled, thus there is a single solution for the equation (7) and this solution minimize the following application:

$$\mathbb{J}(\omega) = \frac{a(\omega,\omega;\mu)}{2} - l(\omega) \quad \forall \omega \in \mathbb{H}^1(\Omega) \tag{8}$$

Chapter 3

Reduced-Basis Approximation

We now consider the linear finite element space:

$$X^{\mathcal{N}} = v \in \mathbb{H}^1(\Omega) | v|_{T_h} \in \mathbf{P}^1(\mathcal{T}_h), \forall T_h \in \mathcal{T}_h,$$

and look for $u_{\mathcal{N}}(\mu) \in X^{\mathcal{N}}$ such that:

$$a(u_{\mathcal{N}}, v; \mu) = l(v) \quad \forall v \in X^{\mathcal{N}} \tag{9}$$

our output of interest is then given by:

$$T^{\mathcal{N}}_{root}(\mu) = l^o(u_{\mathcal{N}}) \tag{10}$$

Applying our usual nodal basis, we arrive at the matrix equations:

$$A^{\mathcal{N}} u_{\mathcal{N}}(\mu) = F^{\mathcal{N}} \tag{11}$$

$$T^{\mathcal{N}}_{root}(\mu) = L^T_{\mathcal{N}} u_{\mathcal{N}}(\mu) \tag{12}$$

here $\mathcal{N}$ is the dimension of the finite element space $X^{\mathcal{N}}$, which (given our natural boundary conditions) is equal to the number of nodes in $\mathcal{T}_h$.

In general, the dimension of the finite element space, dim X = $\mathcal{N}$, will be quite large (in particular if we were to treat the more realistic three-dimensional fin problem), and thus the solution of $A^{\mathcal{N}} u_{\mathcal{N}}(\mu) = F^{\mathcal{N}}$ can be quite expensive. We thus investigate the reduced-basis methods that allow us to accurately and very rapidly predict $T^{\mathcal{N}}_{root}(\mu)$ in the limit of many evaluations that is, at many different values of μ which is precisely the limit of interest in design and optimization studies. To derive the reduced-basis approximation we shall exploit the energy principle,

$$u_{\mathcal{N}}(\mu) = argmin_{\omega \in X^{\mathcal{N}}} \mathbb{J}(\omega) \tag{13}$$

where $\mathbb{J}(\omega)$ is given by equation (8).

To begin, we introduce a sample in parameter space, $S_N = \mu_1, \mu_2, ..., \mu_N$ with $N << \mathcal{N}$. Each $\mu_i, i = 1, ..., N$, belongs in the parameter set D. For our parameter set we choose $D = [0.1, 10.0]^4 \times [0.01, 1.0]$, that is, $0.1 \leq k^i \leq 10.0, i = 1, ..., 4$ for the conductivities, and $0.01 \leq Bi \leq 1.0$ for the Biot number. We then introduce the reduced-basis space as:

$$W_N = span\big(u_{\mathcal{N}}(\mu_1), u_{\mathcal{N}}(\mu_2), ..., u_{\mathcal{N}}(\mu_N)\big) \tag{14}$$

where $u_{\mathcal{N}}(\mu_i)$ is the finite-element solution for $\mu = \mu_i$. To simplify the notation, we define $\xi_i \in X$ as $\xi_i = u_{\mathcal{N}}(\mu_i), i = 1, ..., N$; we can then write $W_N = span\big(\xi_i, i = 1, ..., N\big)$. Which means that $\forall v_N \in W_N$ we have:

$$v_N = \sum_{i=0}^{N} \beta^j \xi^j, \tag{15}$$

for some unique choice of $\beta^j \in R, j = 1, ..., N$. (We implicitly assume that the $\xi, i = 1, ..., N$, are linearly independent; it follows that W_N is an N dimensional subspace of $X_{\mathcal{N}}$.) In the reduced-basis approach we look for an approximation $u_{\mathcal{N}}(\mu)$ to $u_N(\mu)$ (which for our purposes here we presume is arbitrarily close to $u^e(\mu)$) in W_N ; in particular, we express $u_N(\mu)$ as:

$$u_N(\mu) = \sum_{i=0}^{N} u_N^j \xi^j, \tag{16}$$

We denote by $u_N(\mu) \in R^N$ the coefficient vector $(u_N^1, ..., u_N^N)$. The premise or hope is that we should be able to accurately represent the solution at some new point in parameter space,μ, as an appropriate linear combination of solutions previously computed at a small number of points in parameter space (the $\mu_i, i = 1, ..., N$). But how do we find this appropriate linear combination? And how good is it? And how do we compute our approximation efficieWe developntly? The energy principle is crucial here (though more generally the weak form would suffice). To wit, we apply the classical Rayleigh-Ritz procedure to define

$$u_N(\mu) = argmin_{\omega_N \in W_N} \mathbb{J}(\omega_N) \tag{17}$$

alternatively we can apply Galerkin projection to obtain the equivalent statement

$$a(u_N(\mu), v; \mu) = l(v) \quad \forall v \in W_N \tag{18}$$

The output can then be calculated from

$$T_{root}^N(\mu) = l^o(u_N(\mu)) \tag{19}$$

We now study this approximation in more detail.

3.1 Best solution approximation in the reduced-basis space:

Given the energy norm defined as follow:

$$|||.||| = a(., \mu)^{\frac{1}{2}}$$

We write $\forall w_N \in W_N$:

$$|||u(\mu) - w_N||| = a(u(\mu) - w_N, \mu)^{\frac{1}{2}}$$

$\Longleftrightarrow$:

$$|||u(\mu) - w_N|||^2 = a(u(\mu) - w_N, \mu)$$

$\Longleftrightarrow$:

$$|||u(\mu) - w_N|||^2 = a(u(\mu), \mu) - 2a(u(\mu), w_N, \mu) + a(w_N, \mu)$$

$\Longleftrightarrow$:

$$|||u(\mu) - w_N|||^2 = a(u(\mu), \mu) + 2(-a(u(\mu), w_N, \mu) + \frac{a(w_N, \mu)}{2})$$

Given that $a(u_N(\mu), w_N, \mu) = l(w_N)$ according to (18):

$$|||u(\mu) - w_N|||^2 = a(u(\mu), \mu) + 2(-l(w_N) + \frac{a(w_N, \mu)}{2})$$

According to (17),

$$\frac{a(u_N(\mu), \mu)}{2} - l(u_N(\mu)) \leq \frac{a(w_N, \mu)}{2} - l(w_N) \quad \forall w_N \in W_N$$

Then we replace and we obtain the inequality:

$$|||u(\mu) - w_N|||^2 \geq a(u(\mu), \mu) + 2(-l(u_N(\mu)) + \frac{a(u_N(\mu), \mu)}{2}$$

We replace $a(u_N(\mu), u_N, \mu) = l(u_N(\mu))$ according to (18):

$$|||u(\mu) - w_N|||^2 \geq a(u(\mu) - u_N(\mu), \mu)$$

Which gives us the result:

$$|||u(\mu) - w_N||| \geq |||u(\mu) - u_N(\mu)||| \tag{20}$$

This inequality indicates that out of all the possible choices of w_N in the space W_N, the reduced basis method defined above will choose the best one (in the energy norm). Equivalently, we can say that even if we knew the solution $u(\mu)$, we would not be able to find a better approximation to $u(\mu)$ in W_N in the energy norm than $u_N(\mu)$.

3.2 Best output approximation in the reduced-basis space:

We know that:

$$|||u(\mu) - u_N(\mu)|||^2 = a(u(\mu) - u_N(\mu), \mu)$$

$\iff$

$$|||u(\mu) - u_N(\mu)|||^2 = a(u(\mu), \mu) - 2a(u(\mu), u_N(\mu), \mu) + a(u_N(\mu), \mu)$$

We replace the terms using equation (9) and (18) and we obtain:

$$|||u(\mu) - u_N(\mu)|||^2 = l(u(\mu)) - 2l(u_N(\mu)) + l(u_N(\mu))$$

$\iff$

$$|||u(\mu) - u_N(\mu)|||^2 = l(u(\mu)) - l(u_N(\mu))$$

According to the definition given in (10) and (19) we have the result:

$$|||u(\mu) - u_N(\mu)|||^2 = T_{root}(\mu) - T^N_{root}(\mu) \tag{21}$$

Combining this result to (20), this equality indicates that out of all the possible choices of w_N in the space W_N, the reduced basis method defined above will choose the best one (in the energy norm) and the related output $T^N_{root}(\mu)$ is a good approximation to $T_{root}(\mu)$.

3.3 Large scale and reduced-order model:

We know according to (16) that:

$$u_N(\mu) = \sum_{i=0}^{N} u_N^j \xi^j,$$

Also, the exact solution (finite element solution) can be expressed in the $\mathbb{H}^1 basis$ as:

$$u_{\mathcal{N}}(\mu) = \sum_{j=0}^{\mathcal{N}} u_{\mathcal{N}}^i \phi^i,$$

Both approximations are equal at μ_i thus:

$$u_{\mathcal{N}}(\mu_j) = u_N(\mu_j) = \xi^j = \sum_{k=0}^{\mathcal{N}} u_{\mathcal{N}}^i \phi^i = Z_N^T \phi^i,$$

and

$$u_{\mathcal{N}}(\mu) = Z_N u_N(\mu)$$

With Z_N is an $\mathcal{N} \times N$ matrix, the j_{th} column of which is $u_N(\mu_j)$ (the nodal values of $u_{\mathcal{N}}(\mu_j)$. As convention for what is comingbelow we consider $\big(a(\phi_i, \phi_j)\big)_{i,j}$ a matrix of composed of $a(\phi_i, \phi_j) elements$ According to (11):

$$A_{\mathcal{N}}(\mu) = \big(a(\phi_i, \phi_j)\big)_{i,j} = \big(l(\phi_j)\big)_j = F_{\mathcal{N}}$$

which implies:

$$A_{\mathcal{N}}(\mu) u_{\mathcal{N}}(\mu) = \big(a(\phi_i, \phi_j)\big)_{i,j} Z_N u_N(\mu) = \big(l(\phi_j)\big)_j = F_{\mathcal{N}}(\mu)$$

$\Longrightarrow$:

$$A_{\mathcal{N}}(\mu) u_{\mathcal{N}}(\mu) = \big(a(Z_N^T \phi_i, \phi_j)\big)_{i,j} u_N(\mu) = \big(l(\phi_j)\big)_j = F_{\mathcal{N}}(\mu)$$

We multiply by Z_N^T:

$$Z_N^T A_{\mathcal{N}}(\mu) Z_N u_N(\mu) = \big(a(Z_N^T \phi_j, \xi_j)\big)_{i,j} u_N(\mu) = Z_N^T \big(l(\phi_j)\big)_j = Z_N^T F_{\mathcal{N}}(\mu)$$

Then the result:

$$Z_N^T A_{\mathcal{N}}(\mu) Z_N u_N(\mu) = \big(a(\xi_i, \xi_j)\big)_{i,j} u_N(\mu) = \big(l(Z_N^T \phi_j)\big)_j = Z_N^T F_{\mathcal{N}}$$

Given that $\big(a(\xi_i, \xi_j)\big)_{i,j} = A_N(\mu)$ and $\big(l(Z_N^T \phi_j)\big)_j = \big(l(\xi_i)\big)_i = F_N(\mu)$ we conclude the result:

$$Z_N^T A_{\mathcal{N}}(\mu) Z_N u_N(\mu) = Z_N^T F_{\mathcal{N}}(\mu)$$

Thus $u_N(\mu)$ satisfies:

$$A_N(\mu) u_N(\mu) = F_N(\mu) \tag{22}$$

With:

$$A_N(\mu) = Z_N^T A_{\mathcal{N}}(\mu) Z_N$$

And:

$$F_N(\mu) = Z_N^T F_{\mathcal{N}}(\mu)$$

With the same elements and starting from (12) we have:

$$T_{root}^{\mathcal{N}}(\mu) = L_{\mathcal{N}}^T u_{\mathcal{N}}(\mu)$$

Thus:

$$T_{root}^{\mathcal{N}}(\mu) = L_{\mathcal{N}}^T Z_N u_N(\mu)$$

And then we conclude that $u_N(\mu)$ satisfies:

$$T_{root}^{N}(\mu) = L_N^T u_N(\mu) \tag{23}$$

With:

$$L_N^T(\mu) = L_{\mathcal{N}}^T Z_N$$

3.4 Parametric coercivity:

Starting from $a(\omega, v, \mu)\ \forall v \in X$ and $\mu \in D$:

$$a(\omega, v; \mu) = \sum_{i=0}^{4} \int_{\Omega^i} k^i \nabla\omega.\nabla v dA + Bi \int_{\Gamma_{/\Gamma_{root}}} \omega.v dS$$

We can write:

$$a(\omega, v; \mu) = \sum_{q=0}^{5} \Theta^q(\mu) a^q(\omega, v) \tag{24}$$

With: $\Theta^q(\mu)$ is the q_{th} element of the following list:

$$k^0, k^1, k^2, k^3, k^4, Bi$$

And:

$$a^q(\omega, v) = \int_{\Omega^q} \nabla\omega.\nabla v dA, q = 0, 1, ..., 4$$

$$a^5(\omega, v) = \int_{\Gamma_{/\Gamma_{root}}} \omega.v dS$$

$A^{\mathcal{N}}(\mu)$ is assembled as follows:

$$A_{\mathcal{N}}(\mu) = \big(a(\phi_i, \phi_j)\big)_{i,j}$$

we replace in (24):

$$A_{\mathcal{N}}(\mu) = \big(a(\phi_i, \phi_j)\big)_{i,j} = \sum_{q=0}^{5} \Theta^q(\mu) \big(a^q(\phi_i, \phi_j)\big)_{i,j}$$

Then the seeked expression:

$$A_{\mathcal{N}}(\mu) = \sum_{q=0}^{5} \Theta^q(\mu) A^q_{\mathcal{N}} \tag{25}$$

With:

$$A^q_{\mathcal{N}} = \left(a^q(\phi_i, \phi_j)\right)_{i,j}, \quad (\phi_i)_i \, is \, the \, basis \, of \, X$$

We already proved that:

$$A_N(\mu) = Z_N^T A_{\mathcal{N}}(\mu) Z_N$$

So we can deduce that:

$$A_N(\mu) = \sum_{q=0}^{5} \Theta^q(\mu) A^q_N \tag{26}$$

With:

$$A^q_N = Z_N^T A^q_{\mathcal{N}} Z_N,$$

3.5 Bounded condition number:

The coercivity and continuity constants of the bilinear form for the continuous problem are denoted by $\alpha(\mu)$ and $\gamma(\mu)$, respectively.

First we calculate the Rayleigh Quotient $R(A_N, v)$:

$$R(A_N, v) = \frac{V_N^T A_N(\mu) V_N}{V_N^T V_N}$$

We now assume that the basis function $i, i = 1, ..., N$, are orthonormalized. And:

$$V_N = \sum_{i=1}^{N} V_N^i \xi_i$$

So:

$$R(A_N, v) = \frac{V_N^T a(\xi_i, \xi_j, \mu) V_N}{V_N^T V_N} = \frac{a(V_N, V_N, \mu)}{V_N^T V_N}$$

In one hand we know that A is symetric so:

$$\lambda_{min} \leq R(A_N, v) = \frac{a(V_N, V_N, \mu)}{V_N^T V_N} \leq \lambda_{max}$$

With λ_{min} and λ_{max} is its smallest and greatest eigen values. In the other hand according to the definition of $\alpha(\mu)$ and $\gamma(\mu)$, we have:

$$\alpha(\mu) = inf_{V \in R^N} \frac{a(V_N, V_N, \mu)}{V_N^T V_N} \leq \lambda_{min}$$

And:

$$\lambda_{max} \leq sup_{V_N \in R^N} sup_{W_N \in R^N} \frac{a(W_N, V_N, \mu)}{\|V_N\|_{R^N} \|W_N\|_{R^N}} = \gamma(\mu)$$

Knowing that A is symetrical thus normal we conclude:

$$"condition\,number" = \frac{\lambda_{max}}{\lambda_{min}} \leq \frac{\gamma(\mu)}{\alpha(\mu)}$$

Chapter 4

BONUS and Conclusion

Summary : In the first part of the problem we had proven that the fin problem has a solution garanted by lax-Milgram. The solution can be deternined by the finite element method but it's time comsuming especially because it is a design problem. So in the linear compliant case for instance, one has to change the parameters set $\mu \in D$ for each iteration at cost $O(\mathcal{N})$. Then, online forming and solving $A_N(\mu)$ at cost $O(Q\mathcal{N}^{\in}) + \mathcal{O}(\mathcal{N}^{\ni})$. This is why the reduced basis was introduced in part 2 in order to reduce the online computing independent on $\mathcal{N} >> \mathcal{N}$.

The condition number of a function measures how much the output value of the function can change for a small change in the input argument. This is used to measure how sensitive a function is to changes or errors in the input, and how much error in the output results from an error in the input. Thus the importance of the condition number.

Bonus :

I think the idea is to write $\Omega_i \, i = 1, .., 4$ as dependant to t and L. In that case, $\Omega_i = [0, t] \times [0, L] \, i = 1, .., 4$ and $\Omega_0 = [0, t_0] \times [0, L_0]$ and $\Gamma / \Gamma_{root} = [0, C_0]$ all devided by the circonference of the root. We proceed to vraiable change and we obtain an additional multiplication of $t \times L$ included in the affine decomposition equation as follows:

$$A_N(\mu) = t_0 \times L_0 \Theta^0(\mu) A_N^0 + \sum_{q=1}^{4} t \times L \Theta^q(\mu) A_N^q + circonference_{\Gamma/\Gamma_{root}} \Theta^5(\mu) A_N^5$$

all devided by the circonference of the root.