ψ-Fractional Power Series Approach for Solving Fractional Differential Equations Involving ψ-Caputo Derivative

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Abstract

1 Introduction

Fractional calculus has recently attracted considerable attention for its ability to generalize integration and differentiation to non-integer orders. This powerful mathematical tool has found successful applications across multiple disciplines, particularly for modeling systems with memory-dependent behavior and non-local interactions. Current implementations include viscoelastic material analysis [14], quantum mechanical phenomena [15, 20], electromagnetic theory [17], electrochemical systems [16], advanced signal processing [19], and biological modeling [18]. The growing adoption of these methods underscores the need to develop rigorous mathematical foundations for fractional operators and establish reliable techniques for solving fractional differential equations. Comprehensive treatments of fractional calculus theory and applications can be found in [21, 22].

Therefore, there was a critical need to determine solutions for fractional differential models to rigorously establish their theoretical properties and deepen the understanding of fractional-order physical phenomena. To address this need, fractional power series (FPS) expansions have been rigorously developed and analyzed [3–6].Specifically in [4], the authors have proposed solutions of fractional differential equations expressed in power series of the form:

$$\sum_{n=0}^{\infty} \gamma_n (t - t_0)^{n\nu} = \gamma_0 + \gamma_1 (t - t_0)^{\nu} + \gamma_2 (t - t_0)^{2\nu} + \cdots \tag{1}$$

where $k \in \mathbb{N}^*$, $\nu \in (k-1, k]$, and $t \geq t_0$. This expression represents a fractional power series (FPS) centered at t_0, where t is the independent variable and $\{\gamma_n\}_{n \in \mathbb{N}}$ are the series coefficients. In the same

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spirit, many authors have generalized the form of the last series, including the so-called generalized fractional power series [1] of the following form:

$$\sum_{n\geq 0, m\geq 0}^{\infty} \gamma_{(n,m)}(t-t_0)^{n\nu+m} = \underbrace{\gamma_{(0,0)}}_{n+m=0} + \underbrace{\left(\gamma_{(0,1)}(t-t_0) + \gamma_{(1,0)}(t-t_0)^{\nu}\right)}_{n+m=1}$$
$$+ \underbrace{\left(\gamma_{(0,2)}(t-t_0)^2 + \gamma_{(1,1)}(t-t_0)^{\nu+1} + \gamma_{(2,0)}(t-t_0)^{2\nu}\right)}_{n+m=2} + \dots \quad (2)$$

where $n, m \in \mathbb{N}$, $t \geq t_0$, and $\gamma_{(n,m)}$ are the series coefficients.

However, we note that conventional power series expansions are inadequate for many problems involving ψ-Caputo derivatives due to their intrinsic dependence on the kernel function $\psi(t)$. The ψ-Caputo operator, which differentiates with respect to a function ψ [7], offers a flexible framework whose accuracy depends on the choice of operator form and the suitability of the trial function. This was validated through various applications: [8] achieved improved data fitting using least-squares optimization with different kernels for modeling GDP and epidemics, while [9] enhanced population growth predictions with Mittag-Leffler solutions. [10] further developed the theory for ψ-Caputo thermostats, providing existence and uniqueness results and numerical methods and so on . When ψ takes specific forms, the operator recovers known derivatives such as Caputo [11], Caputo-Hadamard [12], and Caputo-Erdélyi-Kober [13], unifying various approaches in fractional calculus.

To address this limitation, we develop a generalized ψ-fractional power series (ψ-GFPS) expansion that extends analytical solutions to arbitrary ψ-Caputo derivatives. The proposed solution takes the form:

$$\sum_{n\geq 0, m\geq 0}^{\infty} \gamma_{(n,m)}(\psi(t)-\psi(t_0))^{n\nu+m} = \underbrace{\gamma_{(0,0)}}_{n+m=0} + \underbrace{\left(\gamma_{(0,1)}(\psi(t)-\psi(t_0)) + \gamma_{(1,0)}(\psi(t)-\psi(t_0))^{\nu}\right)}_{n+m=1}$$
$$+ \underbrace{\left(\gamma_{(0,2)}(\psi(t)-\psi(t_0))^2 + \gamma_{(1,1)}(\psi(t)-\psi(t_0))^{\nu+1} + \gamma_{(2,0)}(\psi(t)-\psi(t_0))^{2\nu}\right)}_{n+m=2}$$
$$+ \dots \quad (3)$$

where $n, m \in \mathbb{N}$, $t \geq t_0$, and $\gamma_{(n,m)}$ are the series coefficients and $\psi \in C^k([t_0, +\infty))$ is strictly increasing with $\psi'(t) > 0$ for all $t \geq t_0$. Notably, this framework recovers the classical Caputo-compatible expansions (Eq. 2) when $\psi(t) = t$, demonstrating its generality. The rest of work is organized as follows: Section 2 introduces the fundamental mathematical preliminaries. Section 3 presents the development of our Generalized ψ-Fractional Power Series (ψ-GFPS) methodology and its theoretical foundations. Section 4 demonstrates the application of our approach through representative examples. Finally, Section 5 summarizes the key findings and principal contributions of this research.

2   Preliminaries

In this section, we present the essential notations, definitions, Properties, and results concerning the ψ-Caputo fractional derivative [7], which will be employed throughout this paper. In what follows, $\nu > 0$ denotes a real number, $y : [t_0, t_1] \to \mathbb{R}$ represents an integrable function, and $\psi \in C^k([t_0, +\infty))$ is strictly increasing with $\psi'(t) > 0$ for all $t \geq t_0$.

Definition 2.1. *For $\nu \in \mathbb{R}^+$ and $t \geq 0$, the ψ- Riemann-Liouville integral of order ν is defined as:*

$$I_{t_0+}^{\nu,\psi} y(t) := \frac{1}{\Gamma(\nu)} \int_{t_0}^{t} \psi'(\tau)(\psi(t)-\psi(\tau))^{\nu-1} y(\tau) d\tau \quad (4)$$

where the Euler gamma function $\Gamma(\cdot)$ is given by

$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t} \, dt \quad (\mathbb{R}(z) > 0).$$

Definition 2.2. *For $0 \leq k - 1 < \nu \leq k$ and $t \geq t_0$, the ψ-Riemann-Liouville derivative of order ν is defined as:*

$$\begin{aligned}
D_{t_0+}^{\nu,\psi} y(t) &:= \left(\frac{1}{\psi'(t)} \frac{d}{dt} \right)^k I_{t_0+}^{k-\nu,\psi} y(t) \\
&= \frac{1}{\Gamma(k-\nu)} \left(\frac{1}{\psi'(t)} \frac{d}{dt} \right)^k \int_{t_0}^t \psi'(\tau)(\psi(t) - \psi(\tau))^{k-\nu-1} y(\tau) d\tau
\end{aligned} \tag{5}$$

Definition 2.3. *For $0 \leq k - 1 < \nu \leq k$ and $t \geq t_0$, the ψ-Caputo derivative of order ν is*

$$^C D_{t_0+}^{\nu,\psi} y(t) = \frac{1}{\Gamma(k-\nu)} \int_{t_0}^t \psi'(\tau)(\psi(t) - \psi(\tau))^{k-\nu-1} y_\psi^{[k]}(\tau) d\tau. \tag{6}$$

where

$$y_\psi^{[k]}(t) := \left(\frac{1}{\psi'(t)} \frac{d}{dt} \right)^k y(t)$$

property 1. *If $0 \leq k - 1 < \nu \leq k$ and $\mu > 0$, then*

$$\left(^C D_{t_0+}^{\nu,\psi} (\psi(t) - \psi(t_0))^{\mu-1} \right)(\xi) = \frac{\Gamma(\mu)}{\Gamma(\mu - \nu)} (\psi(\xi) - \psi(t_0))^{\mu-\nu-1}. \tag{7}$$

Moreover, for $m = 0, 1, \ldots, k - 1$.

$$\left(^C D_{t_0+}^{\nu,\psi} (\psi(t) - \psi(t_0))^m \right)(\xi) = 0. \tag{8}$$

Definition 2.4. *If $\nu \in \mathbb{R}^+$ and $z \in \mathbb{R}$, the Mittag-Leffler function is defined as:*

$$E_\nu(z) = \sum_{n=0}^\infty \frac{z^n}{\Gamma(\nu n + 1)}.$$

Definition 2.5. *If $\nu, \mu \in \mathbb{R}^+$ and $z \in \mathbb{R}$, the generalized Mittag-Leffler function is defined as:*

$$E_{(\nu,\mu)}(z) = \sum_{n=0}^\infty \frac{z^n}{\Gamma(\nu n + \mu)}.$$

3 Generalized ψ-Fractional Power Series

In this section, we exhibit a coherent representation of ψ-fractional power series with a related convergence theorem. Unlike the well-known expansions, the exponents of the indeterminate consist of sufficient positive integers and a ψ-dependent description in terms of the fractional derivative order $\nu > 0$.

Definition 3.1. *The generalized ψ-Fractional Power Series (ψ-GFPS) is an infinite series of the form:*

$$\sum_{n \geq 0, m \geq 0}^{\infty} \gamma_{(n,m)}(\psi(t) - \psi(t_0))^{n\nu+m} \qquad , \qquad k - 1 < \nu \leq k, \tag{9}$$

where $k \in \mathbb{N}^*$, and $t \geq t_0$ serves as an indeterminate variable, while $\gamma(n,m)$ represents the coefficients of the series and $\psi \in C^k([t_0, +\infty))$ is strictly increasing with $\psi'(t) > 0$ for all $t \geq t_0$.

The ψ-GFPS expansion (Eq.9) constitutes a substantial extension beyond conventional fractional power series [1–5]. This formulation achieves two complementary objectives: first, it exactly recovers the classical Caputo-compatible expansions when $\psi(t) = t$ [1]; second, through $\psi(t)$ it generalizes Hadamard and Kober derivatives while preserving their fundamental operator properties. The resulting unified framework preserves all essential convergence properties while significantly broadening the scope of applicable fractional calculus problems.

Lemma 3.2. *Let* $k - 1 < \nu \leq k$, $k \in \mathbb{N}^*$. *The equality* $\left({}^C D_{t_0+}^{\nu,\psi} y \right)(t) = 0$ *is valid if, and only if,*

$$y(t) = \sum_{m=0}^{k-1} a_m (\psi(t) - \psi(t_0))^m, \tag{10}$$

where $a_m \in \mathbb{R}^*$.

Proof. Applying the ψ-Caputo fractional derivative ${}^C D_{t_0+}^{\nu,\psi}$ and using its linearity:

$$^C D_{t_0+}^{\nu,\psi} y(t) = \sum_{m=0}^{k} a_m \, {}^C D_{t_0+}^{\nu,\psi} \left(\psi(t) - \psi(t_0) \right)^m. \tag{11}$$

By (Eq.8), we know that

$$\left({}^C D_{t_0+}^{\nu,\psi} (\psi(t) - \psi(t_0))^m \right)(\xi) = 0, \tag{12}$$

for $m = 0, 1, \ldots, k - 1$. Thus, we obtain:

$$\left({}^C D_{t_0+}^{\nu,\psi} y \right)(t) = 0. \tag{13}$$

$\square$

Theorem 3.3. *Let* $y(t) = \sum_{n,m \geq 0} \gamma_{(n,m)}(\psi(t) - \psi(t_0))^{n\nu+m}$ *be a* ψ-*fractional power series with* $0 \leq k - 1 < \nu \leq k$ *for some integer* $k \geq 1$. *If the series converges with radius* $R_\psi > 0$ *and satisfies* $0 < \psi(t) - \psi(t_0) < R_\psi$, *then the following properties hold:*

$$^C D_{t_0+}^{\nu,\psi} y(t) = \sum_{m \geq k}^{\infty} \gamma_{(0,m)} \frac{m!}{\Gamma(m - \nu + 1)} (\psi(t) - \psi(t_0))^{m-\nu}$$
$$+ \sum_{n \geq 1, m \geq 0}^{\infty} \gamma_{(n,m)} \frac{\Gamma(n\nu + m + 1)}{\Gamma((n-1)\nu + m + 1)} (\psi(t) - \psi(t_0))^{(n-1)\nu+m}, \tag{14}$$

$$^C D_{t_0+}^{2\nu,\psi} y(t) = \sum_{m \geq k}^{\infty} \gamma_{(0,m)} \frac{m!}{\Gamma(m - 2\nu + 1)} (\psi(t) - \psi(t_0))^{m-2\nu}$$
$$+ \sum_{m \geq k}^{\infty} \gamma_{(1,m)} \frac{\Gamma(\nu + m + 1)}{\Gamma(m - \nu + 1)} (\psi(t) - \psi(t_0))^{m-\nu}$$
$$+ \sum_{n \geq 2, m \geq 0}^{\infty} \gamma_{(n,m)} \frac{\Gamma(n\nu + m + 1)}{\Gamma((n-2)\nu + m + 1)} (\psi(t) - \psi(t_0))^{(n-2)\nu+m}. \tag{15}$$

Proof. Starting from equation (Eq.8) and using the properties of the ψ-Caputo fractional derivative operator ${}^{C}D_{t_0+}^{\nu,\psi}$ (given in equation (7)), we can conclude that:

$$
\begin{aligned}
{}^{C}D_{t_0+}^{\nu,\psi}y(t) &= \sum_{m\geq 0}^{\infty}\gamma_{(0,m)}\left({}^{C}D_{t_0+}^{\nu,\psi}(\psi(t)-\psi(t_0))^m\right) + \sum_{n\geq 1,m\geq 0}^{\infty}\gamma_{(n,m)}\left({}^{C}D_{t_0+}^{\nu,\psi}(\psi(t)-\psi(t_0))^{n\nu+m}\right).\\
&= \sum_{m\geq k}^{\infty}\gamma_{(0,m)}\frac{m!}{\Gamma(m-\nu+1)}(\psi(t)-\psi(t_0))^{m-\nu}\\
&\quad + \sum_{n\geq 1,m\geq 0}^{\infty}\gamma_{(n,m)}\frac{\Gamma(n\nu+m+1)}{\Gamma((n-1)\nu+m+1)}(\psi(t)-\psi(t_0))^{(n-1)\nu+m}.
\end{aligned}
$$

$$(16)$$

To establish the proof, we employ the semigroup property of the ψ-Caputo fractional derivative, by which the composition of two derivatives of order ν is equivalent to a single derivative of order 2ν.

$$
\begin{aligned}
{}^{C}D_{t_0+}^{\nu,\psi}\left[{}^{C}D_{t_0+}^{\nu,\psi}y(t)\right] &= {}^{C}D_{t_0+}^{\nu,\psi}\Bigg(\sum_{m\geq k}^{\infty}\gamma_{(0,m)}\frac{\Gamma(m+1)}{\Gamma(m-\nu+1)}(\psi(t)-\psi(t_0))^{m-\nu}\\
&\quad + \sum_{n\geq 1,m\geq 0}^{\infty}\gamma_{(n,m)}\frac{\Gamma(n\nu+m+1)}{\Gamma((n-1)\nu+m+1)}(\psi(t)-\psi(t_0))^{(n-1)\nu+m}\Bigg)\\
&= {}^{C}D_{t_0+}^{\nu,\psi}\Bigg(\sum_{m\geq k}^{\infty}\gamma_{(0,m)}\frac{\Gamma(m+1)}{\Gamma(m-\nu+1)}(\psi(t)-\psi(t_0))^{m-\nu}\Bigg)\\
&\quad + {}^{C}D_{t_0+}^{\nu,\psi}\Bigg(\sum_{m\geq 0}^{\infty}\gamma_{(1,m)}\frac{\Gamma(\nu+m+1)}{m!}(\psi(t)-\psi(t_0))^{m}\\
&\quad + \sum_{n\geq 2,m\geq 0}^{\infty}\gamma_{(n,m)}\frac{\Gamma(n\nu+m+1)}{\Gamma((n-1)\nu+m+1)}(\psi(t)-\psi(t_0))^{(n-1)\nu+m}\Bigg)\\
&= \sum_{m\geq k}^{\infty}\gamma_{(0,m)}\frac{m!}{\Gamma(m-2\nu+1)}(\psi(t)-\psi(t_0))^{m-2\nu}\\
&\quad + \sum_{m\geq k}^{\infty}\gamma_{(1,m)}\frac{\Gamma(\nu+m+1)}{\Gamma(m-\nu+1)}(\psi(t)-\psi(t_0))^{m-\nu}\\
&\quad + \sum_{n\geq 2,m\geq 0}^{\infty}\gamma_{(n,m)}\frac{\Gamma(n\nu+m+1)}{\Gamma((n-2)\nu+m+1)}(\psi(t)-\psi(t_0))^{(n-2)\nu+m},
\end{aligned}
$$

$$(17)$$

where ${}^{C}D_{t_0+}^{2\nu,\psi}y(t) = {}^{C}D_{t_0+}^{\nu,\psi}\left[{}^{C}D_{t_0+}^{\nu,\psi}y(t)\right]$ [7] . $\qquad\square$

The coefficients $\gamma_{(n,m)}$ can be computed through successive evaluation of ψ-Caputo derivatives at t_0, providing a systematic method for constructing solutions as established in the following theorem.

Theorem 3.4. *Suppose that y has a ψ-GFPS representation at $t=t_0$ of the form*

$$
y(t) = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\gamma_{(n,m)}(\psi(t)-\psi(t_0))^{n\nu+m}, \quad 0\leqslant k-1 < \nu \leqslant k, \quad \psi(t_0)\leqslant\psi(t) < \psi(t_0)+R_\psi.
$$

If:

- *$y(t)\in C^{\infty}[t_0, t_0+R_t)$ where $R_t = \psi^{-1}(\psi(t_0)+R_\psi) - t_0$,*
- *${}^{C}D_{t_0}^{n\nu,\psi}y(t)\in C^{\infty}(t_0, t_0+R_t)$ for all $n\in\mathbb{N}$.*

then the coefficients $\gamma_{(n,m)}$ are given by

$$\gamma_{(n,m)} = \frac{^C D_{t_0}^{m,\psi} \; ^C D_{t_0}^{n\nu,\psi} y(t_0)}{\Gamma(m+n\nu+1)}, \quad n,m \in \mathbb{N}_0. \tag{18}$$

Proof. Applying the ψ-Caputo derivative term-wise to the series representation yields:

$$^C D_{t_0}^{n\nu,\psi} y(t) = \sum_{n \geq 0, m \geq 0} \gamma_{n,m} \frac{\Gamma(n\nu + m + 1)}{\Gamma(m+1)} (\psi(t) - \psi(t_0))^m, \tag{19}$$

Evaluation at $t = t_0$ isolates the $m = 0$ term:

$$^C D_{t_0}^{n\nu,\psi} y(t_0) = \gamma_{n,0} \Gamma(n\nu + 1), \tag{20}$$

For $m \geq 1$, taking the m-th ψ-derivative and evaluating at t_0 gives:

$$^C D_{t_0}^{m,\psi} \left(^C D_{t_0}^{n\nu,\psi} y(t_0) \right) = \gamma_{(n,m)} \Gamma(n\nu + m + 1), \tag{21}$$

Solving for $\gamma_{(n,m)}$ produces the coefficient formula:

$$\gamma_{(n,m)} = \frac{^C D_{t_0}^{m,\psi} \; ^C D_{t_0}^{n\nu,\psi} y(t_0)}{\Gamma(n\nu + m + 1)}. \tag{22}$$

$\square$

Additionally, the generalized fractional power series (Eq.9) can be derived naturally as the Cauchy product of two power series, following a rearrangement, as shown below:

$$\sum_{n \geq 0, m \geq 0}^{\infty} \gamma_{(n,m)} (\psi(t) - \psi(t_0))^{n\nu + m} = \left(\sum_{n=0}^{\infty} c_n (\psi(t) - \psi(t_0))^{n\nu} \right) \left(\sum_{m=0}^{\infty} d_m (\psi(t) - \psi(t_0))^m \right). \tag{23}$$

where $\gamma_{(n,m)} = c_n d_m$.

Proposition 1. *If $\sum_{n=0}^{\infty} c_n (\psi(t) - \psi(t_0))^{n\nu}$ converges for some $\psi(t) - \psi(t_0) = c > 0$, then it converges absolutely for all t such that $\psi(t) - \psi(t_0) \in (0, c)$.*

Proof. Fix $\epsilon = 1$. By the convergence of $\sum_{n=0}^{\infty} c_n c^{n\nu}$, there exists $N \in \mathbb{N}$ such that $|c_n c^{n\nu}| < 1$ for all $n \geq N$. Thus, for $n \geq N$ and $\psi(x) - \psi(x_0) \in (0, c)$, we have:

$$|c_k (\psi(t) - \psi(t_0))^{n\nu}| < \left(\frac{\psi(t) - \psi(t_0)}{c} \right)^{n\nu}. \tag{24}$$

The series $\sum_{n=0}^{\infty} \left(\frac{\psi(t) - \psi(t_0)}{c} \right)^{n\nu}$ is a geometric series with ratio $\left(\frac{\psi(t) - \psi(t_0)}{c} \right)^{\nu} < 1$, and hence it converges. By the comparison test, the series $\sum_{n=0}^{\infty} |c_n (\psi(x) - \psi(t_0))^{n\alpha}|$ converges absolutely for $\psi(t) - \psi(t_0) \in (0, c)$. $\square$

Corollary 1. *If $\sum_{n=0}^{\infty} d_n (\psi(t) - \psi(t_0))^n$ converges for some $(\psi(t) - \psi(t_0)) = d > 0$, then it converges absolutely for $(\psi(t) - \psi(t_0)) \in (0, d)$.*

The following result extends Mertens' classical theorem by establishing necessary convergence conditions for (Eq.9).

Theorem 3.5. *Let ψ be an increasing function, and consider the two series:*

$$A = \sum_{n=0}^{\infty} c_n (\psi(t) - \psi(t_0))^{n\nu}, \quad B = \sum_{m=0}^{\infty} d_m (\psi(t) - \psi(t_0))^m,$$

where:

1. *A converges absolutely to a for $\psi(t) - \psi(t_0) < R_1$,*

2. *B converges to b for $\psi(t) - \psi(t_0) < R_2$.*

Then, the Cauchy product C of A and B, defined as:

$$C = \sum_{k=0}^{\infty} \gamma_k (\psi(t) - \psi(t_0))^k, \quad where \quad \gamma_k = \sum_{i=0}^{k} c_i d_{k-i},$$

converges to ab for $\psi(t) - \psi(t_0) < R$, where $R = \min\{R_1, R_2\}$.

Proof. Consider the partial sums:

$$A_n = \sum_{k=0}^{n} c_k (\psi(t) - \psi(t_0))^{k\nu},$$

$$B_n = \sum_{k=0}^{n} d_k (\psi(t) - \psi(t_0))^k,$$

$$C_n = \sum_{k=0}^{n} \gamma_k (\psi(t) - \psi(t_0))^k,$$

where $\gamma_k = \sum_{i=0}^{k} c_i d_{k-i}$.

We can express C_n as:

$$C_n = \sum_{k=0}^{n} \left(\sum_{i=0}^{k} c_i d_{k-i} \right) (\psi(t) - \psi(t_0))^k$$

$$= \sum_{i=0}^{n} c_i (\psi(t) - \psi(t_0))^{i\nu} \sum_{j=0}^{n-i} d_j (\psi(t) - \psi(t_0))^j$$

$$= \sum_{i=0}^{n} c_i (\psi(t) - \psi(t_0))^{i\nu} B_{n-i}.$$

Let $\tilde{B}_{n-i} = B_{n-i} - b$, where $b = \lim_{n \to \infty} B_n$. Then:

$$C_n = b A_n + \sum_{i=0}^{n} c_i (\psi(t) - \psi(t_0))^{i\nu} \tilde{B}_{n-i}.$$

Since $A_n \to a$ and $\tilde{B}_n \to 0$, we focus on the remainder term. For any $\epsilon > 0$:

1. There exists $M > 0$ such that $|\tilde{B}_k| \leq M$ for all $k \in \mathbb{N}$

2. There exists $n_0 \in \mathbb{N}$ such that $|\tilde{B}_k| < \frac{\epsilon}{2a}$ for all $k \geq n_0$

We bound the remainder:

$$\left| \sum_{i=0}^{n} c_i (\psi(t) - \psi(t_0))^{i\nu} \tilde{B}_{n-i} \right| \leq \sum_{i=0}^{n-n_0-1} |c_i (\psi(t) - \psi(t_0))^{i\nu}| M$$

$$+ \sum_{i=n-n_0}^{n} |c_i (\psi(t) - \psi(t_0))^{i\nu}| \frac{\epsilon}{2a}.$$

By absolute convergence of A, there exists n_1 such that for $n > n_0 + n_1$:

$$\sum_{i=n-n_0}^{n} |c_i(\psi(t) - \psi(t_0))^{i\nu}| < \frac{\epsilon}{2M}.$$

Thus, for $n > n_0 + n_1$:

$$|C_n - ab| < \epsilon.$$

Therefore, $C_n \to ab$ as $n \to \infty$, proving the convergence of the Cauchy product C to ab for $\psi(t) - \psi(t_0) < R$, where $R = \min\{R_1, R_2\}$. $\qquad\square$

4 Application

To illustrate the effectiveness of our proposed method, we apply the generalized ψ-fractional power series expansion to solve several fractional differential equations.

Example 4.1. *Let $k - 1 < \nu \leq k, k \in \mathbb{N}^*$ and $\rho \neq 0$. Consider the initial value problem defined by the following differential equation*

$$^C D_{t_0+}^{\nu,\psi} y(t) + \rho y(t) = 0, \tag{25}$$

$$y_\psi^{[j]}(t_0) = y_\psi^j, \quad j = 0, 1, \ldots, k - 1. \tag{26}$$

Let us consider the ψ-Caputo generalized fractional power series expansion, represented by the infinite series:

$$y(t) = \sum_{n \geq 0, m \geq 0}^{\infty} \gamma_{(n,m)}(\psi(t) - \psi(t_0))^{n\nu+m} \tag{27}$$

We then substitute (14) and (27) into (25) to arrive at the balance equation:

$$\sum_{m \geq k}^{\infty} \gamma_{(0,m)} \frac{m!}{\Gamma(m - \nu + 1)}(\psi(t) - \psi(t_0))^{m-\nu} + \sum_{n \geq 1, m \geq 0}^{\infty} \gamma_{(n,m)} \frac{\Gamma(n\nu + m + 1)}{\Gamma((n-1)\nu + m + 1)}(\psi(t) - \psi(t_0))^{(n-1)\nu+m}$$

$$= -\rho \sum_{n \geq 1, m \geq 0}^{\infty} \gamma_{(n-1,m)}(\psi(t) - \psi(t_0))^{(n-1)\nu+m}. \tag{28}$$

We can immediately deduce that

$$\gamma_{(0,m)} = 0 \quad m \geq k, \tag{29}$$

$$\gamma_{(n,m)} = -\rho \frac{\Gamma((n-1)\nu + m + 1)}{\Gamma(n\nu + m + 1)} \gamma_{(n-1,m)} \quad n \geq 1, m \geq 0. \tag{30}$$

From (29) and the the recurrence relation (30), it follows that $\gamma_{(n,m)} = 0$ for all $n \geq 0$ and $m \geq k$. Consequently, the only non-zero coefficients are those with $m < k$, resulting in a solution of the form:

$$y(t) = \sum_{m=0}^{k-1} \sum_{n=0}^{\infty} \gamma_{(n,m)}(\psi(t) - \psi(t_0))^{n\nu+m} = \sum_{m=0}^{k-1} (\psi(t) - \psi(t_0))^m \sum_{n=0}^{\infty} \gamma_{(n,m)}(\psi(t) - \psi(t_0))^{n\nu}$$

From (30) we have

$$\gamma_{(n,m)} = -\rho \frac{\Gamma((n-1)\nu + m + 1)}{\Gamma(n\nu + m + 1)}\gamma_{(n-1,m)} \quad n \geq 1, m \leq k-1. \tag{31}$$

Equivalently we can write

$$\gamma_{(n,m)} = (-1)^n \frac{\rho^n m!}{\Gamma(n\nu + m + 1)}\gamma_{(0,m)} \quad n \geq 1, \quad m \leq k-1. \tag{32}$$

where $\gamma_{(0,m)} = \frac{y_\psi^{[m]}(t_0)}{m!}$ and we deduce that :

$$\gamma_{(n,m)} = (-1)^n \frac{\rho^n}{\Gamma(n\nu + m + 1)} y_\psi^m \quad n \geq 1, \quad m \leq k-1. \tag{33}$$

Consequently, the solution admits the series representation:

$$y(t) = \sum_{m=0}^{k-1} y_\psi^m (\psi(t) - \psi(t_0))^m E_{\nu,m+1}\left(-\rho(\psi(t) - \psi(t_0))^\nu\right). \tag{34}$$

Let's now move on to another example, which focuses on inhomogeneous fractional differential equations to demonstrate the applicability of our approach.

Example 4.2. *Consider the ψ-Caputo fractional initial value problem:*

$$^C D_{t_0+}^{\nu,\psi}[y(t)] - \rho y(t) = \frac{(\psi(t) - \psi(t_0))^{2-\nu}}{\Gamma(3-\nu)}, \qquad y(t_0) = y_0. \tag{35}$$

where $0 < \nu \leq 1$ and $t \geq t_0$. In light of the preceding analysis and applying the initial condition, the generalized fractional power series solution to problem (35) admits the representation:

$$y(t) = \sum_{n \geq 0, m \geq 0}^{\infty} \gamma_{(n,m)}(\psi(t) - \psi(t_0))^{n\nu+m}. \tag{36}$$

By substituting equations (14) and (36) into (35) and by comparing coefficients of corresponding powers in the series expansion, we derive the recurrence relations:

$$\gamma_{(0,2)} = \frac{1}{2}, \tag{37}$$

$$\gamma_{(0,m)} = 0 \quad m \in \{1,3,4,\ldots\}, \tag{38}$$

$$\gamma_{(n,m)} = \rho \frac{\Gamma((n-1)\nu + m + 1)}{\Gamma(n\nu + m + 1)}\gamma_{(n-1,m)} \quad n \geq 1, m \geq 0. \tag{39}$$

This consequently implies that for $n \geq 1$:

$$\gamma_{(n,0)} = \frac{\rho^n}{\Gamma(n\nu + 1)} y_0, \quad \gamma_{(n,2)} = \frac{\rho^n}{\Gamma(n\nu + 3)}, \quad and \quad \gamma_{(n,m)} = 0 \quad, otherwise.$$

Consequently, the precise solution to equation (35) is:

$$y(t) = y_0 \sum_{n=0}^{\infty} \frac{\rho^n}{\Gamma(n\nu + 1)}(\psi(t) - \psi(t_0))^{n\nu} + \sum_{n=0}^{\infty} \frac{\rho^n}{\Gamma(n\nu + 3)}(\psi(t) - \psi(t_0)^{n\nu+2}$$
$$= y_0 E_\nu \left(\rho(\psi(t) - \psi(t_0))^\nu\right) + E_{\nu,3}\left(\rho(\psi(t) - \psi(t_0))^\nu\right). \tag{40}$$

In Particular when $t_0 = 0$, $\nu = 1$ and $\psi(t) = t$, we have the exact solution for the classical version of (Eq.35) :

$$y(t) = \left(y_0 + \frac{1}{\rho^2}\right) e^{\rho(t-t_0)} - \frac{1}{\rho^2}(\rho(t - t_0) + 1). \tag{41}$$

Notably, the solution for (Eq.35) cannot be derived through other available FPS expansions, such as those in [1–5].

Example 4.3. *Consider the ψ-Caputo fractional initial value problem:*

$$^C D_{t_0^+}^{2\nu,\psi}(y(t)) - 2 {}^C D_{t_0^+}^{\nu,\psi}[y(t)] + y(t) = \frac{(\psi(t) - \psi(t_0))^{2-2\nu}}{\Gamma(3 - 2\nu)}, \quad 0 < \nu \leq 1, t \geq t_0 \tag{42}$$

Substituting Eqs. (9), (14), and (15) into (42) and equating the coefficients of like powers of $\psi(t) - \psi(t_0)$, we get $\gamma_{(0,2)} = \frac{1}{2}, \gamma_{(0,m)} = 0$ for $m \notin \{0, 2\}, \gamma_{(1,2)} = \frac{2}{\Gamma(\nu+3)}, \gamma_{(1,m)} = 0$ for $m \notin \{0, 2\}$, and the following recursive relation

$$\gamma_{(n+2,m)} = \frac{2\Gamma((n+1)\nu + m + 1)\gamma_{(n+1,m)} - \Gamma(n\nu + m + 1)\gamma_{(n,m)}}{\Gamma((n+2)\nu + m + 1)}. \tag{43}$$

This yields that

$$\gamma_{(n,0)} = \frac{n\Gamma(\nu + 1)\gamma_{(1,0)} - (n-1)\Gamma(\nu)\gamma_{(0,0)}}{\Gamma(n\nu + 1)}, \qquad n \geq 2 \tag{44}$$

$$\gamma_{(n,2)} = \frac{n+1}{\Gamma(n\nu + 3)}, \qquad n \geq 0 \tag{45}$$

$$\gamma_{(n,m)} = 0, \qquad \text{otherwise.} \tag{46}$$

Substituting the resulting coefficients back into the Eq. (42) gives the following exact "memory" solution

$$y(t) = \gamma_{(1,0)}\Gamma(\nu + 1) \sum_{n=0}^{\infty} \frac{n}{\Gamma(n\nu + 1)}(\psi(t) - \psi(t_0))^{n\nu} - \gamma_{(0,0)}\Gamma(\nu) \sum_{n=0}^{\infty} \frac{(n-1)}{\Gamma(n\nu + 1)}(\psi(t) - \psi(t_0))^{n\nu}$$

$$+ \sum_{n=0}^{\infty} \frac{(n+1)}{\Gamma(n\nu + 3)}(\psi(t) - \psi(t_0))^{n\nu+2}. \tag{47}$$

We remark here that the existing FPS expansions [1–5] are inadequate for finding the solution to (Eq.47). Particularly when $t_0 = 0$, $\nu = 1$ and $\psi(t) = t$, we have the exact solution for the classical version of (Eq.42) :

$$y(t) = \underbrace{\gamma_{(0,0)}e^t + \left(\gamma_{(1,0)} - \gamma_{(0,0)}\right) te^t}_{\text{homogenous solution}} + \underbrace{te^t - e^t + 1}_{\text{particular solution}}. \tag{48}$$

Conclusion and comments

In this work, we introduced an adaptable framework for solving fractional differential equations of the ψ-Caputo type by employing a generalized ψ- fractional power series approach. Theoretical validations

and key theorems were provided to establish the reliability and applicability of the proposed approach. Furthermore, illustrative examples demonstrated its superiority over existing methods, highlighting its precision and efficiency in handling complex fractional models. The new methodology offers a promising tool for advancing research in fractional calculus and can be extended to various scientific and engineering applications involving fractional-order systems.

Data Availability

No data were used to support this study.

Conflicts of Interest

The author declares no conflicts of interest.

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