Project

Reconstruction of 3D image using two 2d cross sections

1. Implicit method to solve the heat equation

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Major : Mathematics
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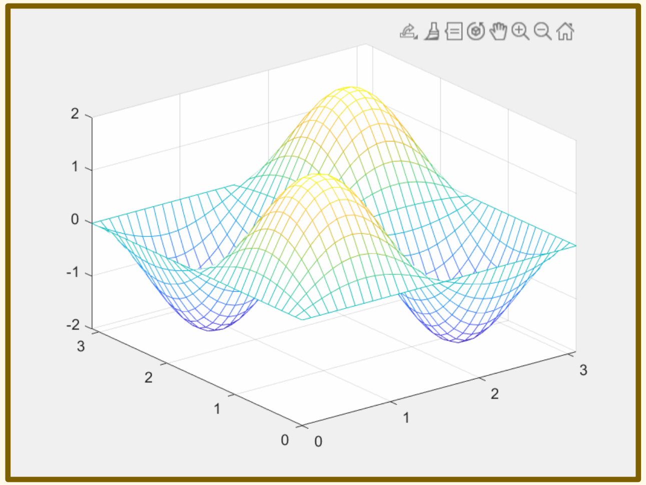
1. Heat Equation

It is a quadratic partial differential equation representing the process of heat conduction over time and is a representative parabolic partial differential equation.



It can be seen that the temperature distribution gradually becomes uniform as heat is conducted over time.

1. Heat Equation

$$\frac{\partial^2 x}{\partial x^2}(x,t) - \frac{\partial u}{\partial t}(x,t) = 0, \qquad a < x < b, 0 < t \leq T \qquad (1.1)$$

$$u(a,t) = c_0, u(b,t) = c_1, \qquad 0 < t \leq T \qquad (1.2)$$

$$u(x,0) = f(x), \qquad 0 \leq x < 1 \qquad (1.3)$$

Equation (1.2) is the boundary condition and (1.3) is the initial condition.

Actual solution $u(x,t)$ gives the temperature of the bar given to the left at point x and time t.

2. Implicit method

Discretization

The section $[0,1]$ of the space x is equally divided N and the section $[0,T]$ of the time T is equally divided M. Points belonging to each section are defined as follows.

$$x_i = ih, \qquad i = 0, 1, \cdots, N$$

$$t_j = jk, \qquad j = 0, 1, \cdots, M$$

2. Implicit method

Approximation is obtained without using algebraic methods to solve the Heat equation. To obtain an approximation solution, we use the finite difference formula below.

$$\frac{\partial^2 u}{\partial x^2}(x_i, t_j) \approx \frac{u(x_i - h, t_j) - 2u(x_i, t_j) + u(x_i + h, t_j)}{h^2}$$

$$\frac{\partial u}{\partial t}(x_i, t_j) \approx \frac{u(x_i, t_j) - u(x_i, t_j - k)}{k}$$

The derivatives for time are expressed in the form of backward differences as shown above. The error is $O(h^2, k)$.

$$\frac{\partial^2 x}{\partial x^2}(x, t) - \frac{\partial u}{\partial t}(x, t) = 0$$

$$\frac{1}{h^2}[u(x_i - h, t_j) - 2u(x_i, t_j) + u(x_i + h, t_j)] - \frac{1}{k}[u(x_i, t_j) - u(x_i, t_j - k) \approx 0$$

2. Implicit method

$$\frac{1}{h^2}\left[u(x_i - h, t_j) - 2u(x_i, t_j) + u(x_i + h, t_j)\right] - \frac{1}{k}\left[u(x_i, t_j) - u(x_i, t_j - k)\right] \approx 0$$

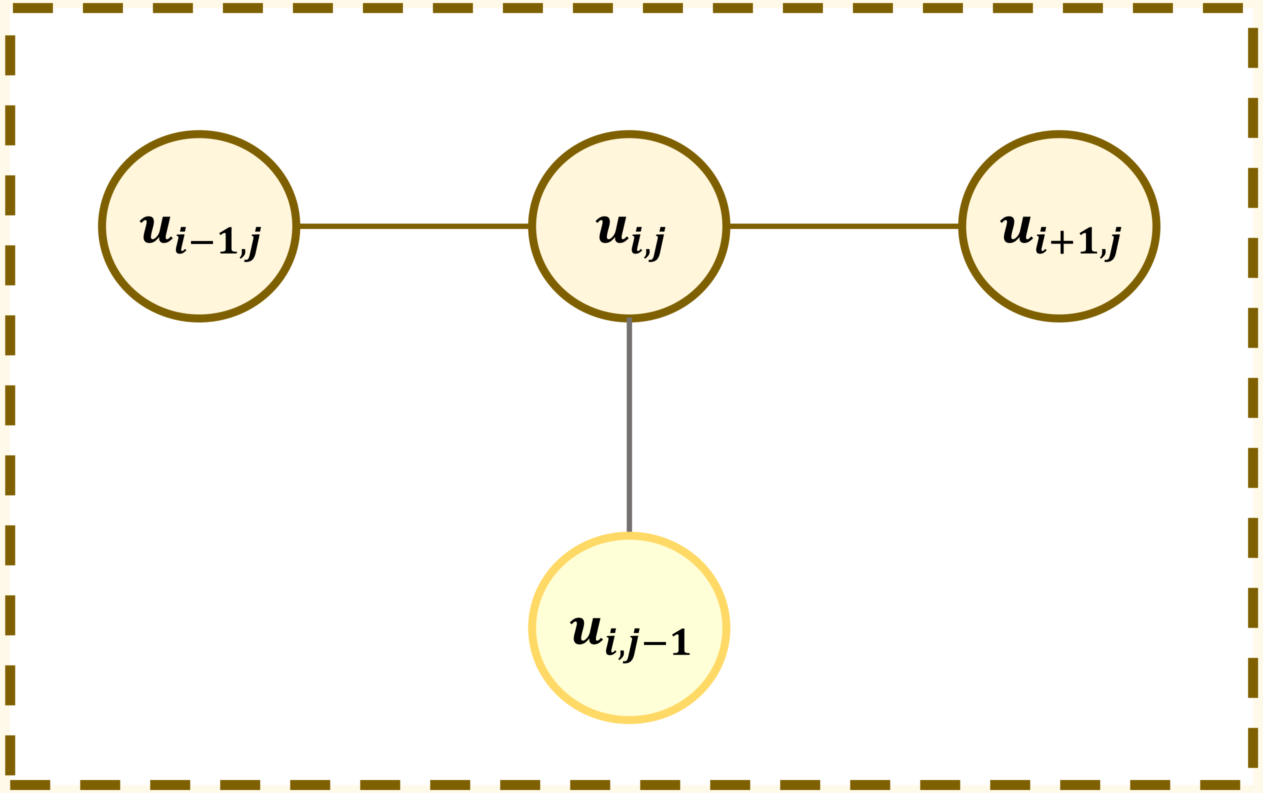
$$\alpha = \frac{k}{h^2}$$

$$-\alpha u(x_i - h, t_j) + (1 + 2\alpha)u(x_i, t_j) - \alpha u(x_i + h, t_j) \approx u(x_i, t_j - k)$$

$$u(x_i, t_j) \approx u_{i,j},$$
$$\beta = (1 + 2\alpha)$$

$$-\alpha u_{i-1,j} + \beta u_{i,j} - \alpha u_{i+1,j} = u_{i,j-1}$$

2. Implicit method



In step $t-k$, the value for point $(x_i, t_j - k)$ is known, but the values for the three points $(x_i, -h\,, t_j)$, (x_i, t_j), and $(x_i + h, t_j)$ given in the next step t are unknown.

2. Implicit method

In order to obtain an approximation of the internal points($j = 1, 2, \cdots, M$), the following linear equations must be addressed at each step.

$$\begin{pmatrix} \beta & -\alpha & & & & \\ -\alpha & \beta & -\alpha & & 0 & \\ & -\alpha & \beta & -\alpha & & \\ & & \ddots & \ddots & \ddots & \vdots \\ & 0 & & -\alpha & \beta & -\alpha \\ & & & & -\alpha & \beta \end{pmatrix} \begin{pmatrix} u_{1,j} \\ u_{2,j} \\ u_{3,j} \\ \vdots \\ u_{N-2,j} \\ u_{N-1,j} \end{pmatrix} = \begin{pmatrix} u_{1,j-1} + \alpha u_{0,j} \\ u_{2,j-1} \\ u_{3,j-1} \\ \vdots \\ u_{N-2,j-1} \\ u_{N-1,j-1} \end{pmatrix}$$

$u_{0,j} = c_0, u_{N,j} = c_1$ for all j by boundary condition and $u_{i,0} = f(x_i), i = 1, 2, \cdots, N-1$ by initial condition.

2. Implicit method

I solved an example for practice.

Heat equation is

$$\frac{\partial^2 x}{\partial x^2}(x,t) - \frac{\partial u}{\partial t}(x,t) = 0, \qquad 0 < x < 1, t > 0,$$

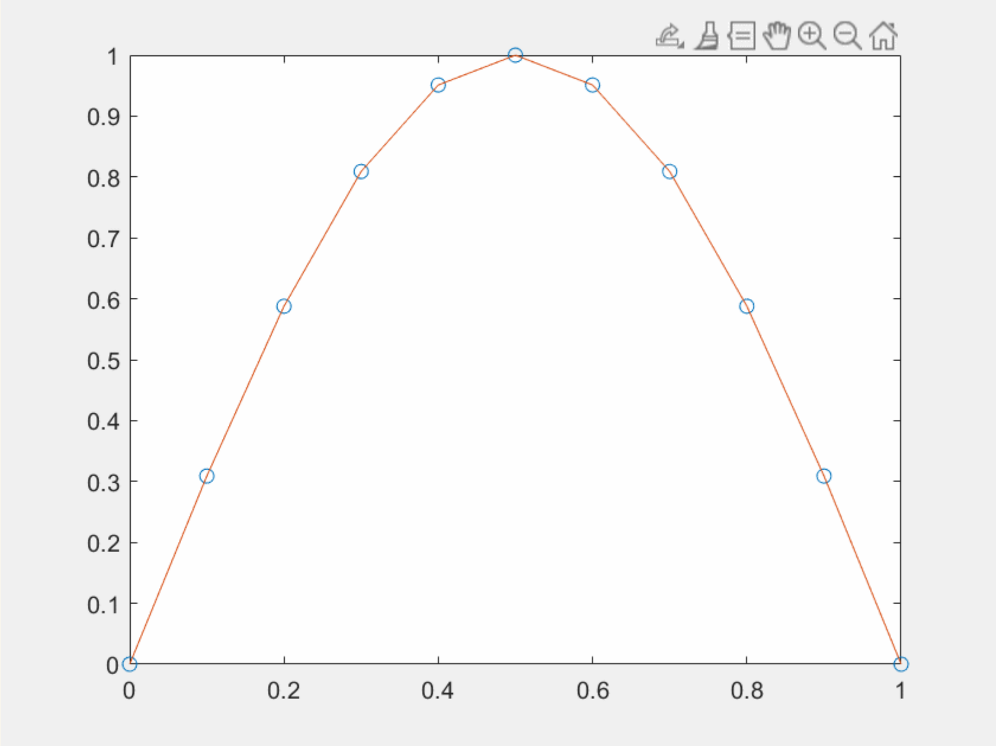
And boundary value and initial value are

$$u(0,t) = 0, u(1,t) = 0, u(x,0) = \sin(\pi x).$$

Let's try to find an approximate solution using an implicit method.

2. Implicit method

Explanation



When $h = 0.1, k = 0.005$
O : Approximation, - : Real
code : heat_equation_implicit

Thank you!