How Do Hyperedges Overlap in Real-World Hypergraphs? - Patterns, Measures, and Generators (ONLINE APPENDIX)

Geon Lee*
KAIST AI
Daejeon, South Korea
geonlee0325@kaist.ac.kr

Minyoung Choe*
KAIST AI
Daejeon, South Korea
minyoung.choe@kaist.ac.kr

Kijung Shin
KAIST AI & EE
Daejeon, South Korea
kijungs@kaist.ac.kr

ABSTRACT

This document provides supplementary information to the main paper "How Do Hyperedges Overlap in Real-World Hypergraphs? - Patterns, Measures, and Generators." The full results of the observations and experiments are provided.

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1 SUPPLEMENTARY EXPERIMENTS

We provide full results of the experiments presented in the main paper.

Degree distributions: Figure 1 shows the degree distributions in the real-world hypergraphs and the corresponding synthetic hypergraphs generated by different generative models. HyperCL, HyperLap, and HyperLap$^+$ accurately preserve the degree distributions of the most of the considered real-world hypergraphs, while HyperPA and HyperFF fail in many datasets.

Sampling collisions: Figure 2 shows how the average number of sampling trials for completing a hyperedge changes depending on the sizes of hyperedges in HyperCL. Note that the number of sampling trials can be larger than the hyperedge size due to collisions. That is, the same node can be sampled multiple times. The difference between the average number of sampling trials and the hyperedge size is small in most cases, while the difference increases as the hyperedge size increases.

Full results of observations: In Table 1, we provide the full results regarding the following patterns:

- **Observation 1:** Egonets in real-world hypergraphs tend to have higher density than those in randomized hypergraphs.
- **Observation 2:** Egonets in real-world hypergraphs tend to have higher overlapness than those in randomized hypergraphs.
- **Observation 3:** The number of hyperedges overlapping at each pair of nodes tends to be more skewed in real-world hypergraphs than in randomized hypergraphs.
- **Observation 4:** The number of hyperedges overlapping at each triple of nodes tends to be more skewed in real-world hypergraphs than in randomized hypergraphs.
- **Observation 5:** Hyperedges in real-world hypergraphs tend to contain structurally more similar nodes, compared to those in randomized hyperedges.

Here, we use hypergraphs generated by HyperCL as randomized hypergraphs.

Full results of comparisons: We provide full results of the comparison of hypergraph generators with respect to the reproduction of the five empirical patterns above.

- **Comparison 1:** The results regarding Observation 1 (i.e., the density of egonets) are provided in Table 3.
- **Comparison 2:** The results regarding Observation 2 (i.e., the overlapness of egonets) are provided in Table 4.
- **Comparison 3:** The results regarding Observation 3 (i.e., the number of hyperedges overlapping at each pair of nodes) are provided in Table 5. We show that their distributions are heavy-tailed in Table 8.
- **Comparison 4:** The results regarding Observation 4 (i.e., the number of hyperedges overlapping at each triple of nodes) are provided in Table 6. We show that their distributions are heavy-tailed in Table 8.
- **Comparison 5:** The results regarding Observation 5 (i.e., the homogeneity of hyperedges) are provided in Table 7. We show that their distributions are heavy-tailed in Table 8.

We also compare HyperPA and HyperCL with respect to all these patterns visually in Table 2.

Counterexamples regarding the Axioms 1, 2, and 3: In the main paper, we suggest the three axioms below, which any reasonable metrics of the degree of hyperedge overlaps should satisfy. We provide counter examples showing that, except for our proposed Overlapness, **none** of the baseline metrics (i.e., Intersection, Union Inverse, Jaccard Index, Overlap Coefficient, and Density) satisfy **all** the three axioms.

- **Axiom 1: number of hyperedges** The counter examples showing that some baseline metrics do not satisfy Axiom 1 are given in Table 9.
- **Axiom 2: number of distinct nodes** The counter examples showing that some baseline metrics do not satisfy Axiom 2 in are given in Table 10.

*Equal Contribution.

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- **Axiom 3: sizes of hyperedges** showing that some baseline metrics do not satisfy Axiom 3 in are given in Table 11.

Note that, except for our proposed Overlapness, all baseline metrics do not satisfy at least one axiom.

Results of examining macroscopic structural properties: We demonstrate that HyperLap+ reproduces the following four macroscopic structural patterns of real-world hypergraphs, which are suggested in [3].

- **Heavy-tailed degree distribution**: The degree distributions of real hypergraphs generally follow heavy-tailed distributions. The degree of a node v is defined as the number of hyperedges containing v.
- **Heavy-tailed hyperedge size distribution**: The distributions of hyperedge sizes are also heavy-tailed.
- **Heavy-tailed intersection size distribution**: The number of nodes that two hyperedges commonly have also fall under the class of heavy-tailed distributions.
- **Skewed singular values**: The singular values of the incidence matrices generally follow heavy-tailed distributions.

In Table 12, we provide the distributions regarding these four structural properties in hypergraphs generated by different models and show that all the distributions from HyperLap+ are very similar to that of real-world hypergraphs. We also check in Table 13 that all the distributions from HyperLap+ are best fitted to heavy-tailed distributions rather than exponential distributions.

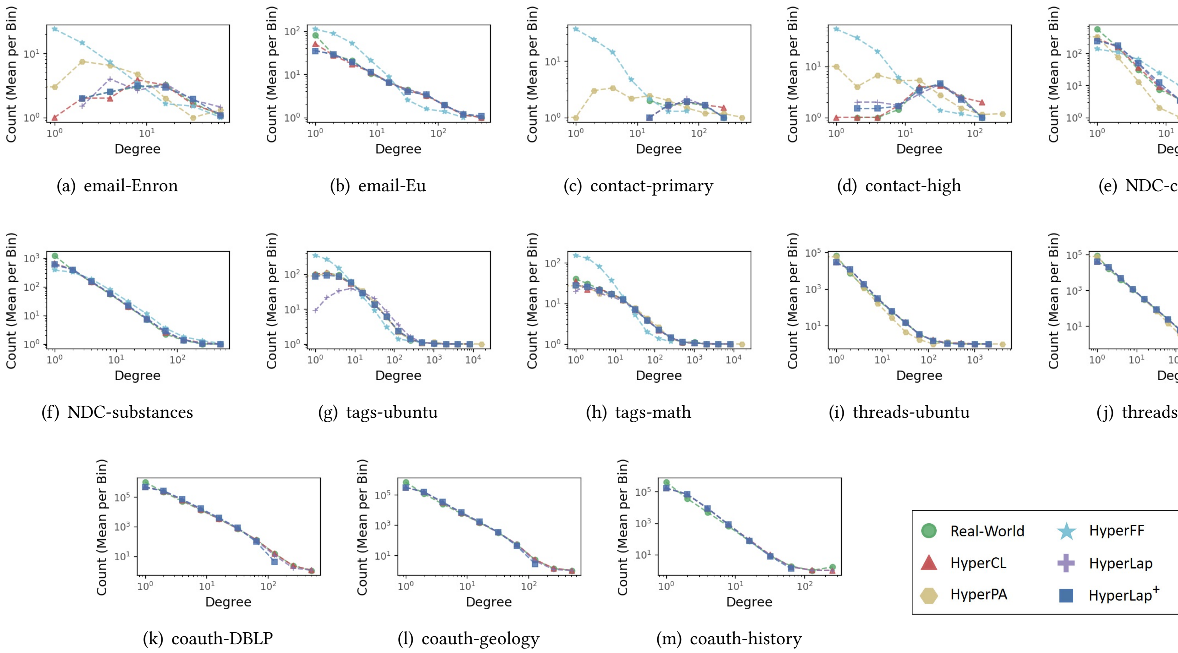
Comparison of potential null models: We compare HyperCL, which we use as a null model, with two potential alternatives: the random bipartite graph generative model [1] and the hypergraph configuration model [2]. Note that HyperCL preserves hyperedge sizes exactly and node degrees in expectation, while the configuration model preserves both exactly, and the bipartite graph generative model preserves both in expectation. As shown in Table 14, empirically, these three models generate similarly realistic hypergraphs.

REFERENCES

[1] Sinan G Aksoy, Tamara G Kolda, and Ali Pinar. 2017. Measuring and modeling bipartite graphs with community structure. *Journal of Complex Networks* 5, 4 (2017), 581–603.

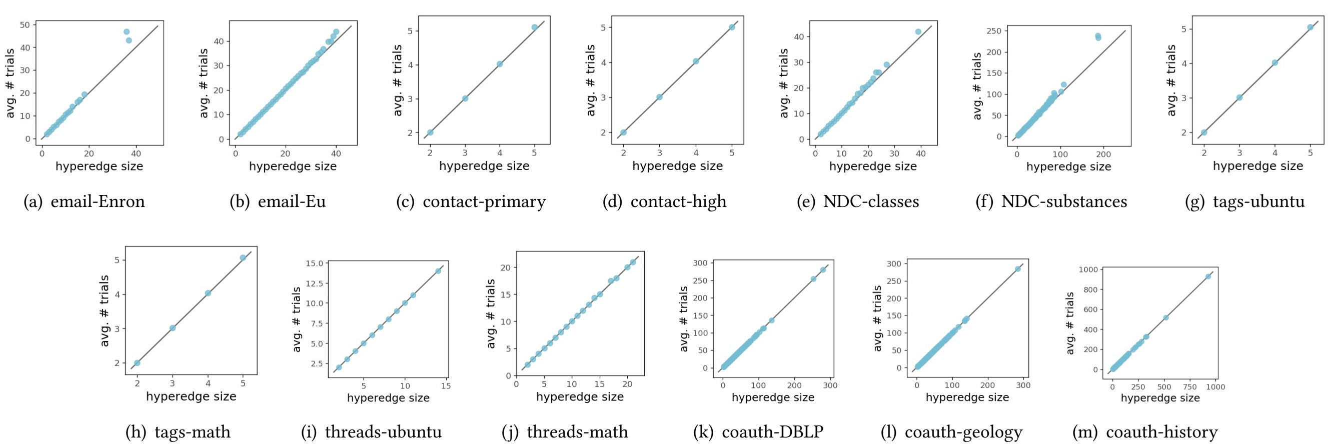
[2] Philip S Chodrow. 2020. Configuration models of random hypergraphs. *Journal of Complex Networks* 8, 3 (2020), cnaa018.

[3] Yunbum Kook, Jihoon Ko, and Kijung Shin. 2020. Evolution of Real-world Hypergraphs: Patterns and Models without Oracles. *arXiv preprint arXiv:2008.12729* (2020).



(a) email-Enron (b) email-Eu (c) contact-primary (d) contact-high (e) NDC-classes (f) NDC-substances (g) tags-ubuntu (h) tags-math (i) threads-ubuntu (j) threads-math (k) coauth-DBLP (l) coauth-geology (m) coauth-history

Figure 1: Degree distributions. The degree distributions in the real-world hypergraphs and the corresponding synthetic hypergraphs generated by different generative models. HyperCL, HyperLap, and HyperLap+ accurately preserve the degree distributions of the most of the considered real-world hypergraphs, while HyperPA and HyperFF fail in many datasets.



(a) email-Enron (b) email-Eu (c) contact-primary (d) contact-high (e) NDC-classes (f) NDC-substances (g) tags-ubuntu (h) tags-math (i) threads-ubuntu (j) threads-math (k) coauth-DBLP (l) coauth-geology (m) coauth-history

Figure 2: The average number of sampling trials for completing hyperedges with different sizes. The difference between the average number of sampling trials and the hyperedge size is small in most cases, while the difference increases as the hyperedge size increases.

Table 1: <u>Full results of observations.</u> The overlapping patterns of hyperedges in real-world hypergraphs (green) are distinguished from those in randomized hypergraphs (red) in thirteen datasets.

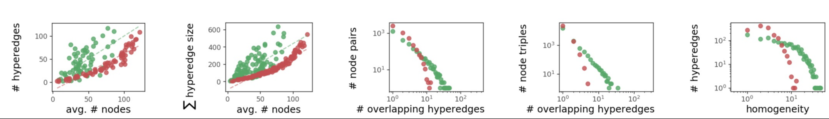
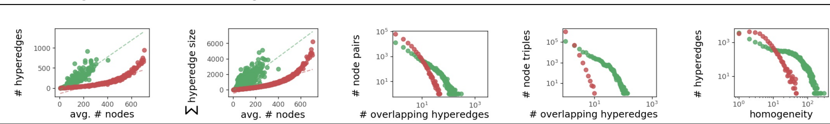
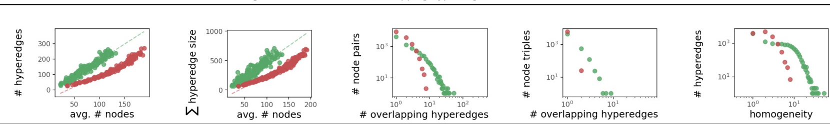
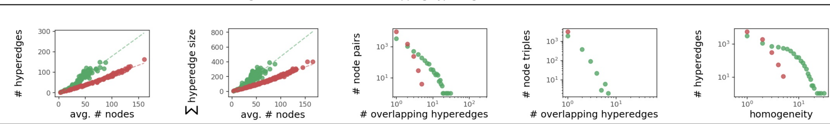
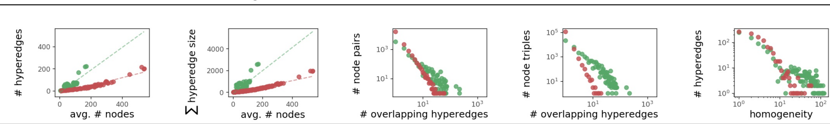
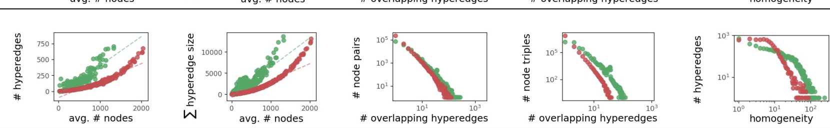
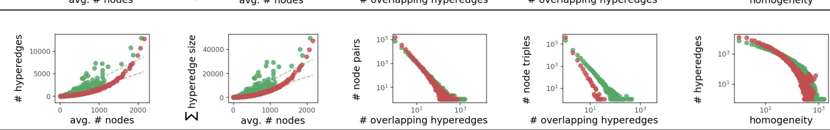
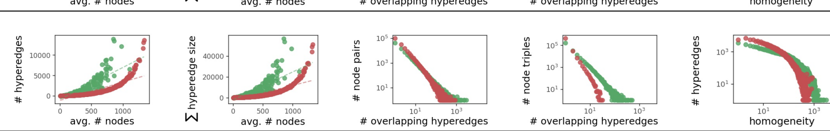
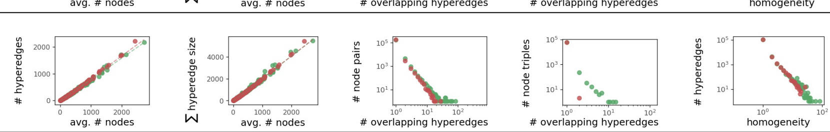
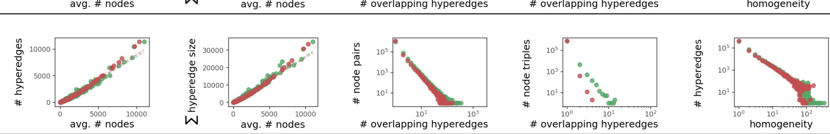
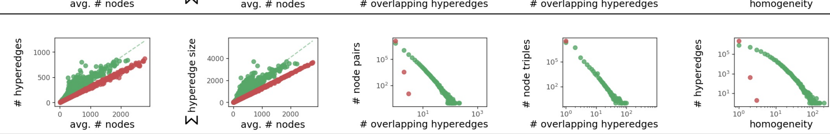
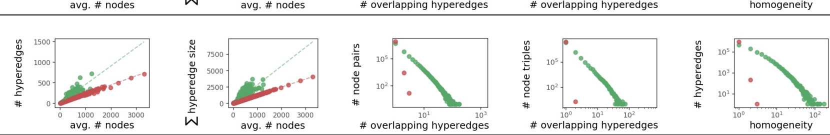
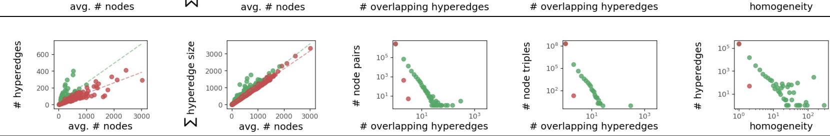
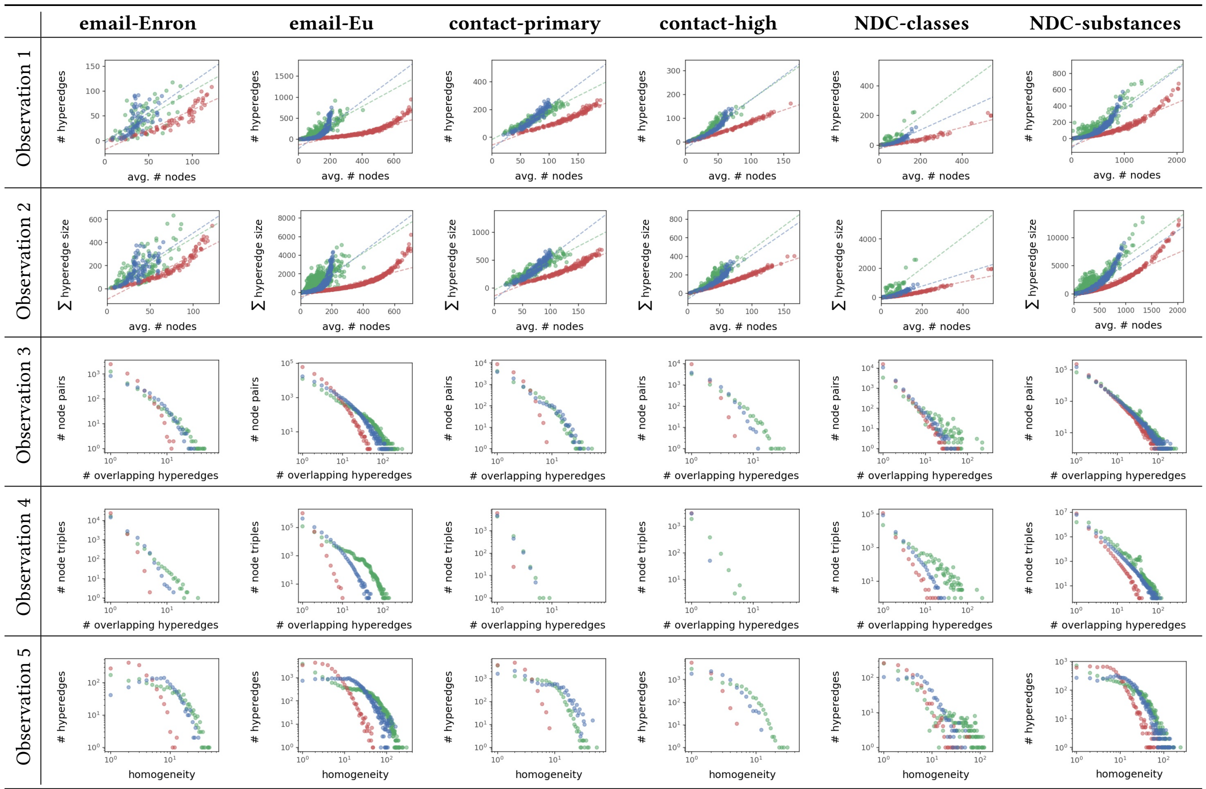
Dataset	Observation 1	Observation 2	Observation 3	Observation 4	Observation 5
email-Enron					
email-Eu					
contact-primary					
contact-high					
NDC-classes					
NDC-substances					
tags-ubuntu					
tags-math					
threads-ubuntu					
threads-math					
coauth-DBLP					
coauth-geology					
coauth-history					

Table 2: The five empirical patterns in real-world hypergraphs (green) are reproduced accurately by HyperLap+ (blue), while HyperCL (red) fails in many cases.



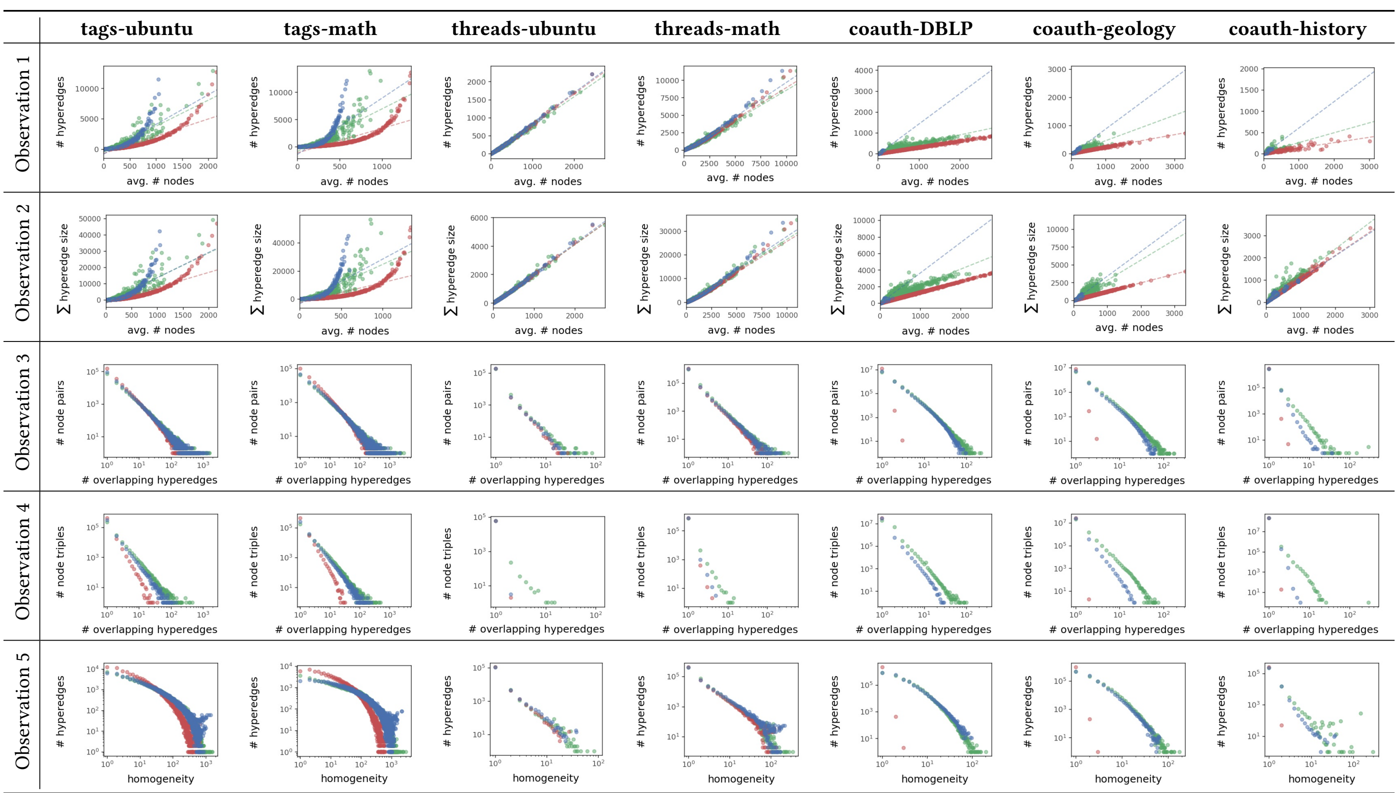


Table 3: Comparison 1. Comparison of hyperedge generators with respect to egonet density distributions. The slopes of the regression lines imply the average egonet density. The distribution generated by HyperLap and HyperLap+ look similar to the distributions generated by real-world hypergraphs in thirteen datasets.

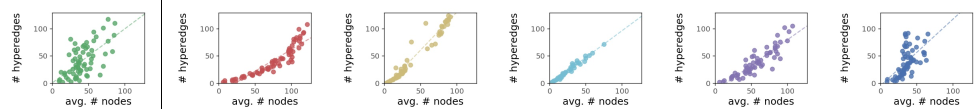
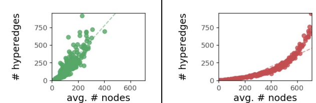
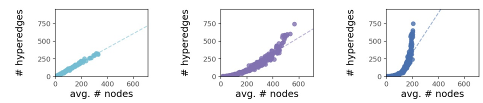
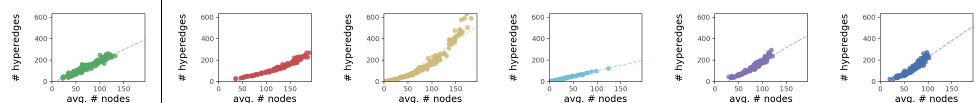
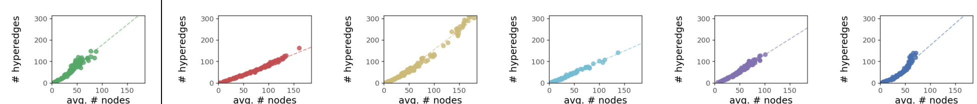
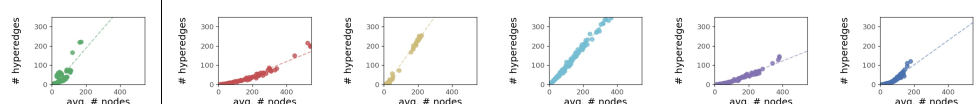
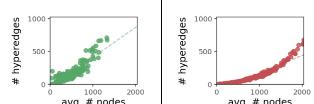
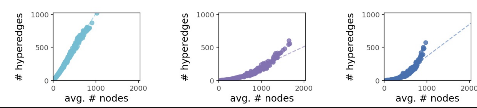
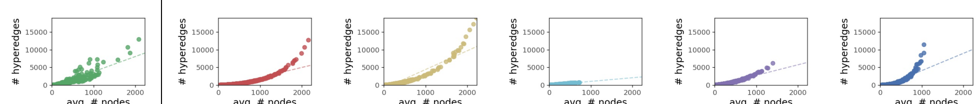
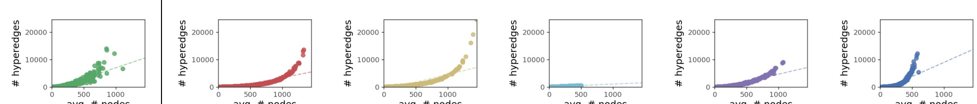
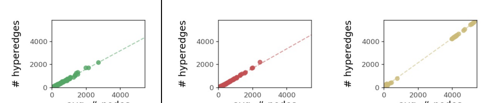
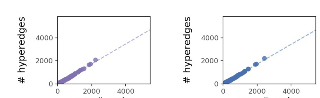
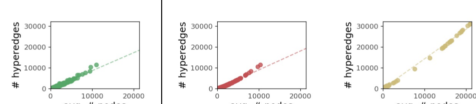
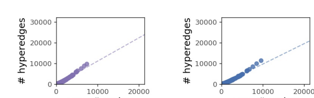
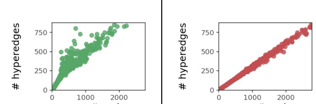
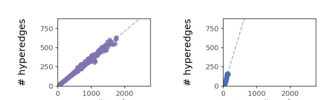
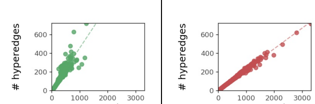
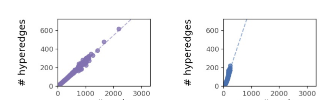
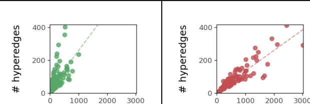
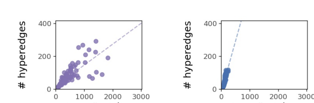
Dataset	Real	HyperCL	HyperPA	HyperFF	HyperLap	HyperLap+
email-Enron						
email-Eu			Out of time or out of memory			
contact-primary						
contact-high						
NDC-classes						
NDC-substances			Out of time or out of memory			
tags-ubuntu						
tags-math						
threads-ubuntu				Out of time or out of memory		
threads-math				Out of time or out of memory		
coauth-DBLP			Out of time or out of memory	Out of time or out of memory		
coauth-geology			Out of time or out of memory	Out of time or out of memory		
coauth-history			Out of time or out of memory	Out of time or out of memory		

Table 4: Observation 2. Comparison of hyperedge generators with respect to egonet overlapness distributions. The slopes of the regression lines imply the average egonet overlapness. The distributions generated by HyperLap and HyperLap$^+$ look similar to the distribution generated by real-world hypergraphs in thirteen datasets.

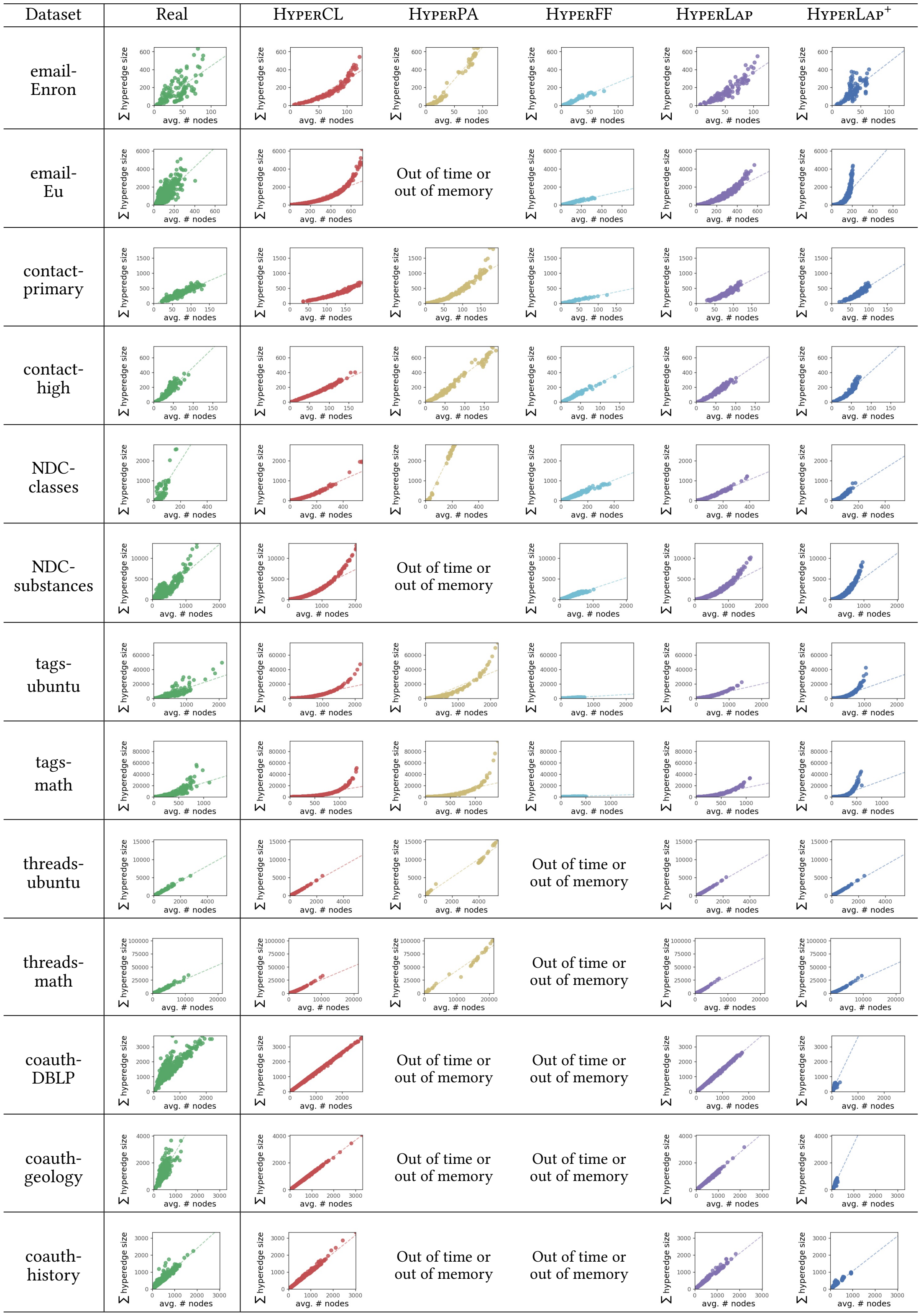


Table 5: Comparison 3. Comparison of hypergraph generators with respect to node-pair degree distributions, i.e., the number of hyperedges overlapping at each pair of nodes. The distribution is skewed with a heavy tail in real-world hypergraphs. This pattern is preserved well by HyperLap and HyperLap⁺.

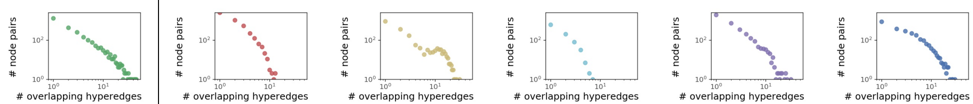
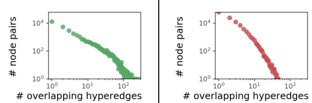
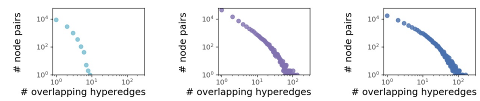
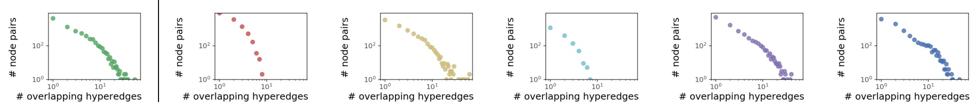
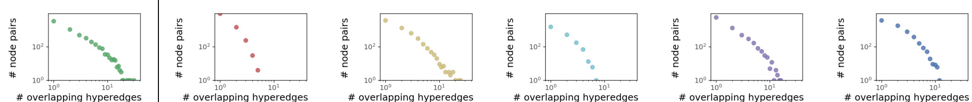
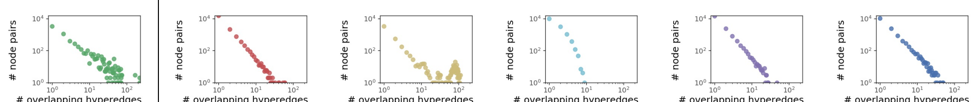
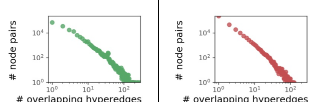
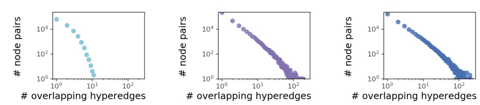
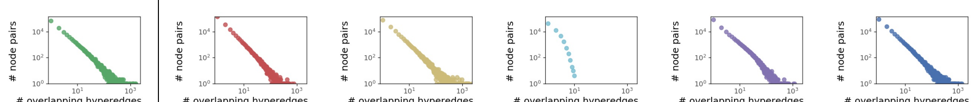
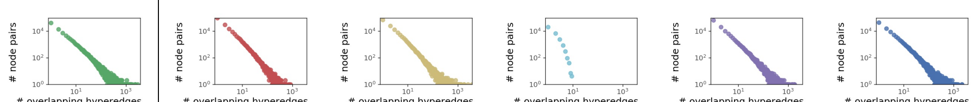
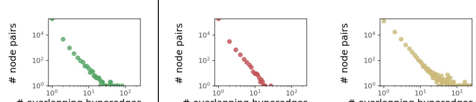
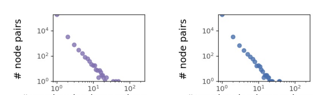
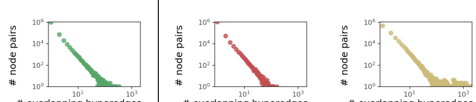
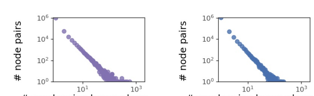
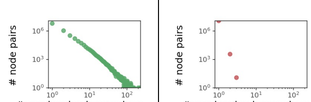
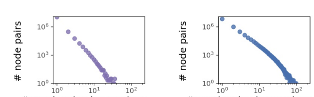
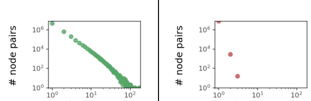
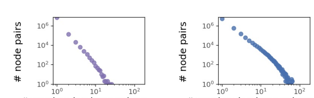
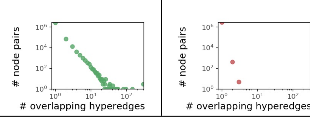
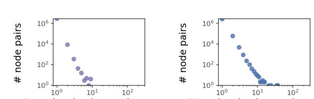
Dataset	Real	HyperCL	HyperPA	HyperFF	HyperLap	HyperLap⁺
email-Enron						
email-Eu			Out of time or out of memory			
contact-primary						
contact-high						
NDC-classes						
NDC-substances			Out of time or out of memory			
tags-ubuntu						
tags-math						
threads-ubuntu				Out of time or out of memory		
threads-math				Out of time or out of memory		
coauth-DBLP			Out of time or out of memory	Out of time or out of memory		
coauth-geology			Out of time or out of memory	Out of time or out of memory		
coauth-history			Out of time or out of memory	Out of time or out of memory		

Table 6: Comparison 4. Comparison of hypergraph generators with respect to node-triple degree distributions, i.e., the number of hyperedges overlapping at each triple of nodes. The distribution is skewed with a heavy tail in real-world hypergraphs. This pattern is preserved well by HyperLap and HyperLap+.

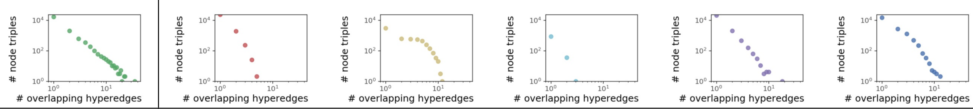
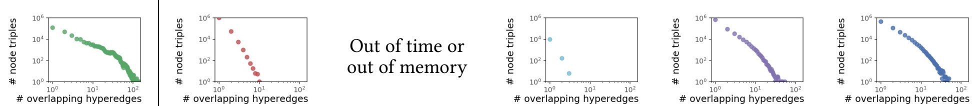
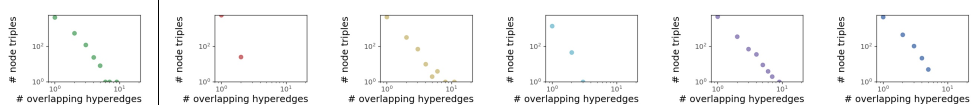
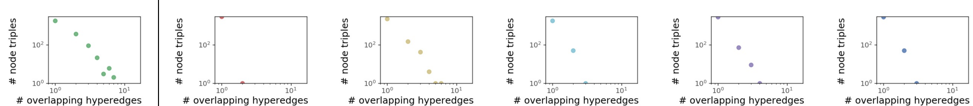
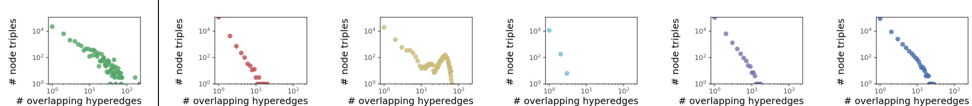
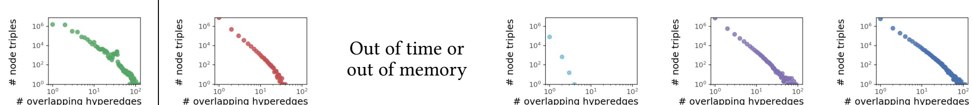
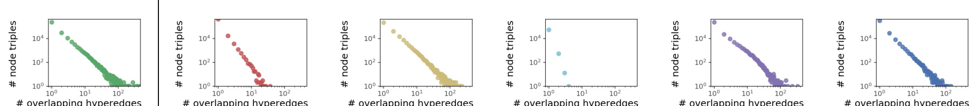
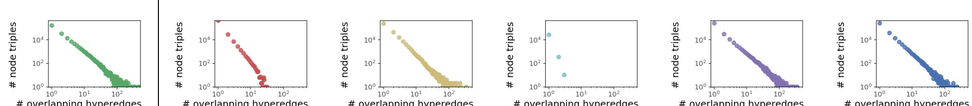
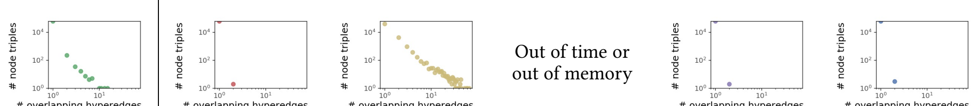
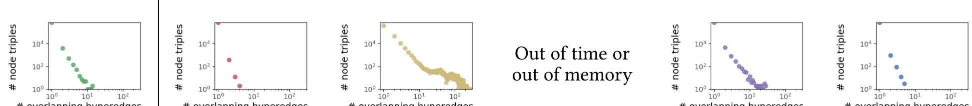
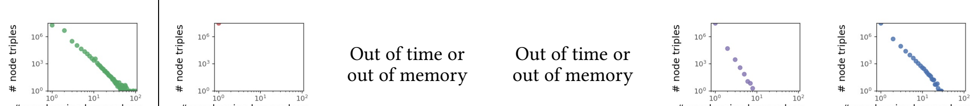
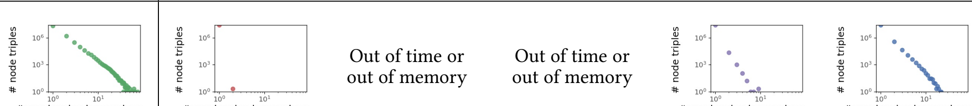
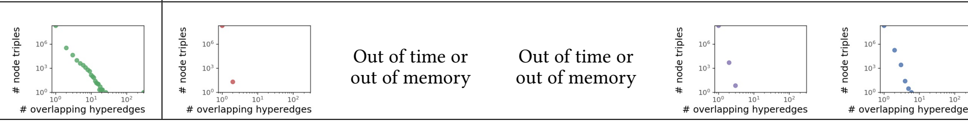
Dataset	Real	HyperCL	HyperPA	HyperFF	HyperLap	HyperLap+
email-Enron						
email-Eu			Out of time or out of memory			
contact-primary						
contact-high						
NDC-classes						
NDC-substances			Out of time or out of memory			
tags-ubuntu						
tags-math						
threads-ubuntu				Out of time or out of memory		
threads-math				Out of time or out of memory		
coauth-DBLP			Out of time or out of memory	Out of time or out of memory		
coauth-geology			Out of time or out of memory	Out of time or out of memory		
coauth-history			Out of time or out of memory	Out of time or out of memory		

Table 7: Comparison 5. Comparison of hyperedge generators with respect to hyperedge homogeneity distributions. The distribution of real-world hypergraphs shows that hyperedges in them tend to contain structurally more similar nodes, which leads to be skewed with a heavy tail. This pattern also can be shown in the distributions generated by HyperLap and HyperLap+.

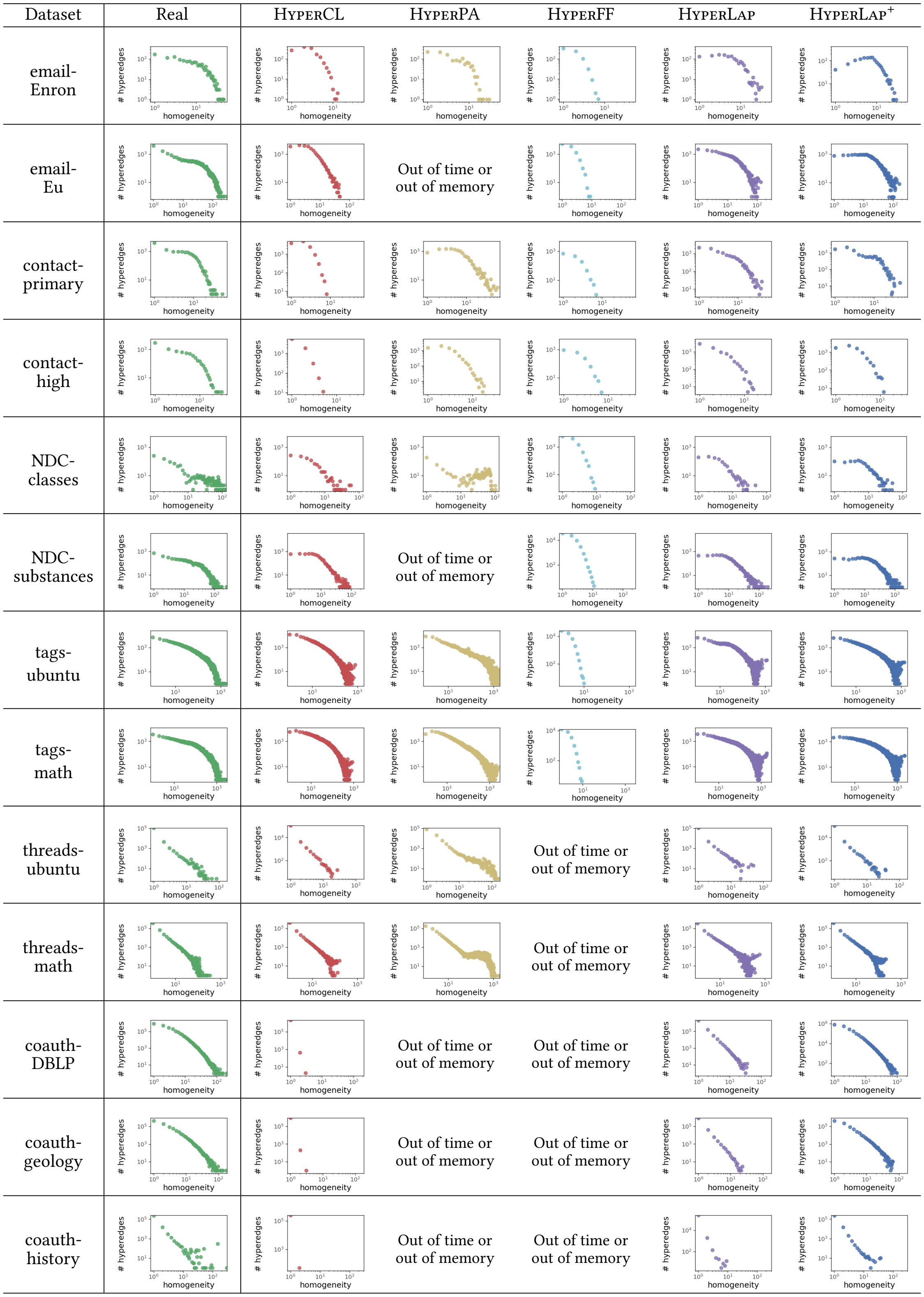


Table 8: The heavy-tailed distributions. The distributions of the number of hyperedges overlapping at each pair or triple of nodes and homogeneity of hyperedges are all heavy-tailed. The loglikehood ratios when fitting the distributions to each of three heavy-tailed distributions (i.e., the power-law distribution, the truncated power-law distribution, and the log normal distribution) against the exponential distribution support this claim. We report the maximum log-likelihood ratio, and a positive value means that the distribution is closer to a heavy-tailed distribution than to the exponential distribution.

Dataset	Pair of Nodes						Triple of Nodes						Homogeneity					
	Real	H-CL	H-PA	H-FF	H-LAP	H-LAP$^+$	Real	H-CL	H-PA	H-FF	H-LAP	H-LAP$^+$	Real	H-CL	H-PA	H-FF	H-LAP	H-LAP$^+$
email-Enron	**4.22**	**0.11**	**0.13**	**0.23**	**1.03**	-0.29	**3.88**	**0.37**	**1.69**	**0.50**	**0.74**	**0.38**	-0.26	**0.53**	**1.14**	**2.18**	**1.34**	-0.43
email-Eu	**1.48**	**1.95**	-	**0.41**	**1.71**	**2.75**	**0.77**	-0.18	-	**1.24**	**1.25**	**2.11**	**0.91**	**2.63**	-	**4.21**	**3.93**	**3.13**
contact-primary	**1.40**	-0.49	**2.54**	**0.54**	**0.40**	**15.74**	**0.48**	**2.60**	**0.44**	**0.50**	0.00	**1.23**	**2.30**	**17.67**	**1.16**	**4.25**	**1.63**	**2.26**
contact-high	**0.81**	**1.06**	**1.34**	**0.15**	**0.38**	-0.06	**0.80**	**0.10**	**0.10**	**0.50**	**0.52**	**0.50**	**1.95**	**11.08**	**2.09**	**3.23**	**2.95**	**4.72**
NDC-classes	**15.74**	**4.09**	**3.42**	**0.55**	**6.47**	**14.42**	**31.53**	**1.69**	**5.03**	**1.24**	**1.76**	**7.92**	**0.39**	**3.10**	**1.26**	**4.94**	**2.78**	**0.87**
NDC-substances	**43.87**	**34.99**	-	**0.54**	**34.78**	**40.13**	**116.45**	**12.55**	-	**0.51**	**19.29**	**55.38**	**1.22**	**3.46**	-	**9.22**	**2.78**	**2.90**
tags-ubuntu	**41.55**	**38.19**	**33.01**	**1.66**	**49.54**	**43.70**	**17.84**	**4.85**	**22.82**	**0.51**	**3.12**	**15.57**	**2.26**	**5.77**	-1.04	**8.37**	**11.53**	**7.00**
tags-math	**4.49**	**54.94**	**12.59**	**0.27**	**49.14**	**45.60**	**29.26**	**8.59**	**13.98**	**1.60**	**29.34**	**23.12**	**2.62**	**1.92**	**3.70**	**5.40**	**13.06**	**6.56**
threads-ubuntu	**3.97**	**2.33**	**7.08**	-	**2.68**	**1.75**	**0.80**	-1.41	**9.10**	-	-1.41	-1.72	**7.70**	**2.14**	**46.51**	-	**6.56**	**4.25**
threads-math	**14.78**	**18.44**	**27.31**	-	**22.46**	**16.66**	-0.09	**0.76**	**0.18**	-	**3.96**	**0.96**	**9.00**	**9.66**	-0.50	-	**27.35**	**12.10**
coauth-DBLP	**22.47**	**1.57**	-	-	**3.32**	**74.95**	**5.84**	0.00	-	-	**0.79**	**6.73**	**4.31**	**156.83**	-	-	**4.53**	**25.23**
coauth-geology	**53.39**	**1.93**	-	-	**1.52**	**45.08**	**13.73**	-1.41	-	-	**1.53**	**1.10**	**5.52**	**101.59**	-	-	**41.95**	**8.06**
coauth-history	**3.91**	**1.12**	-	-	**0.89**	**1.77**	**1.42**	-4.46	-	-	**1.32**	**0.57**	**1.73**	**4.82**	-	-	**9.91**	**4.31**

-: out of time (taking more than 10 hours) or out of memory

Table 9: An example showing that Intersection, Union Inverse, Jaccard Index, and Overlap Coefficient do not satisfy Axiom 1.

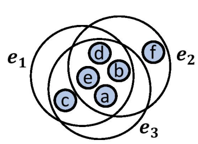
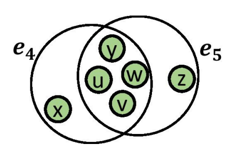
Metric	$\mathcal{E}'$		$\mathcal{E}$
			
Intersection	$\|e_1 \cap e_2 \cap e_3\| = \|\{a,b,d,e\}\| = 4$	=	$\|e_4 \cap e_5\| = \|\{u,v,w,y\}\| = 4$
Union Inverse	$\frac{1}{\|e_1 \cup e_2 \cup e_3\|} = \frac{1}{\|\{a,b,c,d,e,f\}\|} = \frac{1}{6}$	=	$\frac{1}{\|e_4 \cup e_5\|} = \frac{1}{\|\{u,v,w,x,y,z\}\|} = \frac{1}{6}$
Jaccard Index	$\frac{\|e_1 \cap e_2 \cap e_3\|}{\|e_1 \cup e_2 \cup e_3\|} = \frac{\|\{a,b,d,e\}\|}{\|\{a,b,c,d,e,f\}\|} = \frac{4}{6}$	=	$\frac{\|e_4 \cap e_5\|}{\|e_4 \cup e_5\|} = \frac{\|\{u,v,w,y\}\|}{\|\{u,v,w,x,y,z\}\|} = \frac{4}{6}$
Overlap Coefficient	$\frac{\|e_1 \cap e_2 \cap e_3\|}{min(\|e_1\|,\|e_2\|,\|e_3\|)} = \frac{\|\{a,b,d,e\}\|}{min(5,5,5)} = \frac{4}{5}$	=	$\frac{\|e_4 \cap e_5\|}{min(\|e_4\|,\|e_5\|)} = \frac{\|\{u,v,w,y\}\|}{min(5,5)} = \frac{4}{5}$
Density	$\frac{\|\mathcal{E}'\|}{\|\bigcup_{e \in \mathcal{E}'} e\|} = \frac{\|\{e_1,e_2,e_3\}\|}{\|\{a,b,c,d,e,f\}\|} = \frac{3}{6}$	>	$\frac{\|\mathcal{E}\|}{\|\bigcup_{e \in \mathcal{E}} e\|} = \frac{\|\{e_4,e_5\}\|}{\|\{u,v,w,x,y,z\}\|} = \frac{2}{6}$
Overlapness	$\frac{\sum_{e \in \mathcal{E}'} \|e\|}{\|\bigcup_{e \in \mathcal{E}'} e\|} = \frac{\|e_1\|+\|e_2\|+\|e_3\|}{\|\{a,b,c,d,e,f\}\|} = \frac{15}{6}$	>	$\frac{\sum_{e \in \mathcal{E}} \|e\|}{\|\bigcup_{e \in \mathcal{E}} e\|} = \frac{\|e_4\|+\|e_5\|}{\|\{u,v,w,x,y,z\}\|} = \frac{10}{6}$

Table 10: An example showing that Intersection, Jaccard Index, and Overlap Coefficient do not satisfy Axiom 2.

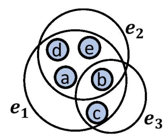
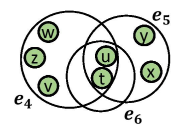
Metric	$\mathcal{E}'$		$\mathcal{E}$
			
Intersection	$\lvert e_1 \cap e_2 \cap e_3\rvert = \lvert\{b\}\rvert = 1$	$<$	$\lvert e_4 \cap e_5 \cap e_6\rvert = \lvert\{t,u\}\rvert = 2$
Union Inverse	$\frac{1}{\lvert e_1 \cup e_2 \cup e_3\rvert} = \frac{1}{\lvert\{a,b,c,d,e\}\rvert} = \frac{1}{5}$	$>$	$\frac{1}{\lvert e_4 \cup e_5 \cup e_6\rvert} = \frac{1}{\lvert\{t,u,v,w,x,y,z\}\rvert} = \frac{1}{7}$
Jaccard Index	$\frac{\lvert e_1 \cap e_2 \cap e_3\rvert}{\lvert e_1 \cup e_2 \cup e_3\rvert} = \frac{\lvert\{b\}\rvert}{\lvert\{a,b,c,d,e\}\rvert} = \frac{1}{5}$	$<$	$\frac{\lvert e_4 \cap e_5 \cap e_6\rvert}{\lvert e_4 \cup e_5 \cup e_6\rvert} = \frac{\lvert\{t,u\}\rvert}{\lvert\{t,u,v,w,x,y,z\}\rvert} = \frac{2}{7}$
Overlap Coefficient	$\frac{\lvert e_1 \cap e_2 \cap e_3\rvert}{min(\lvert e_1\rvert,\lvert e_2\rvert,\lvert e_3\rvert)} = \frac{\lvert\{b\}\rvert}{min(5,4,2)} = \frac{1}{2}$	$<$	$\frac{\lvert e_4 \cap e_5 \cap e_6\rvert}{min(\lvert e_4\rvert,\lvert e_5\rvert,\lvert e_6\rvert)} = \frac{\lvert\{t,u\}\rvert}{min(5,4,2)} = \frac{2}{2}$
Density	$\frac{\lvert\mathcal{E}'\rvert}{\lvert\bigcup_{e\in\mathcal{E}'} e\rvert} = \frac{\lvert\{e_1,e_2,e_3\}\rvert}{\lvert\{a,b,c,d,e\}\rvert} = \frac{3}{5}$	$>$	$\frac{\lvert\mathcal{E}\rvert}{\lvert\bigcup_{e\in\mathcal{E}} e\rvert} = \frac{\lvert\{e_4,e_5,e_6\}\rvert}{\lvert\{t,u,v,w,x,y,z\}\rvert} = \frac{3}{7}$
Overlapness	$\frac{\sum_{e\in\mathcal{E}'}\lvert e\rvert}{\lvert\bigcup_{e\in\mathcal{E}'} e\rvert} = \frac{\lvert e_1\rvert+\lvert e_2\rvert+\lvert e_3\rvert}{\lvert\{a,b,c,d,e\}\rvert} = \frac{11}{5}$	$>$	$\frac{\sum_{e\in\mathcal{E}}\lvert e\rvert}{\lvert\bigcup_{e\in\mathcal{E}} e\rvert} = \frac{\lvert e_4\rvert+\lvert e_5\rvert+\lvert e_6\rvert}{\lvert\{t,u,v,w,x,y,z\}\rvert} = \frac{11}{7}$

Table 11: An example showing that all the considered measures except for Overlapness do not satisfy Axiom 3.

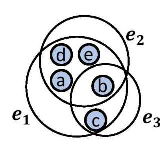
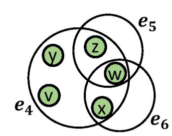
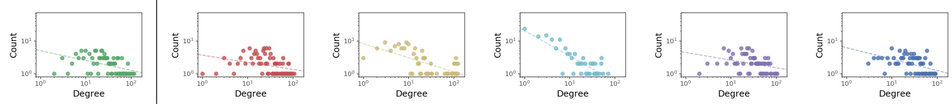
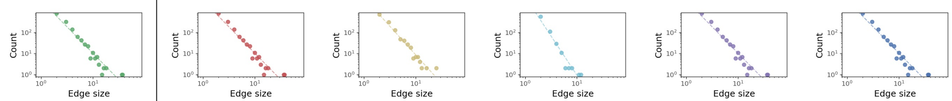
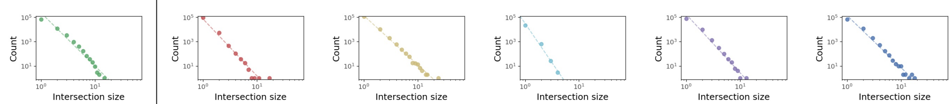
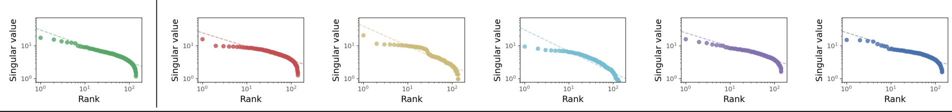
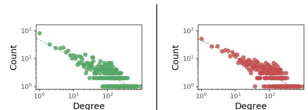
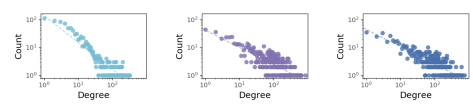
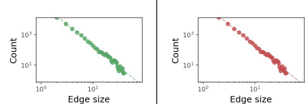
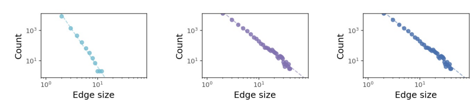
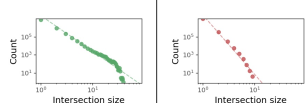
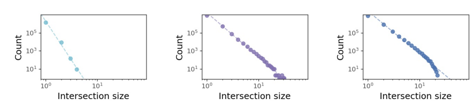
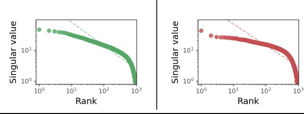
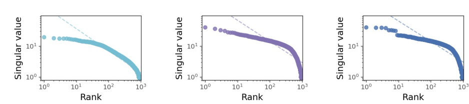
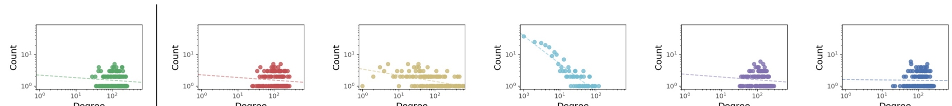
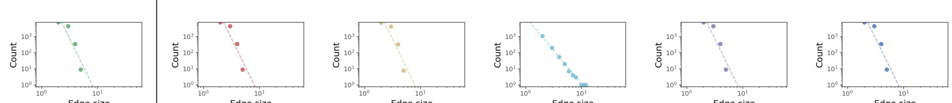
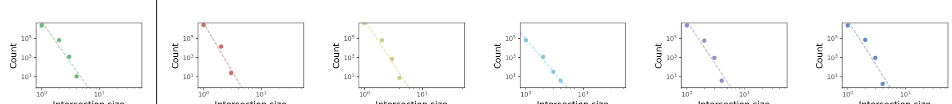
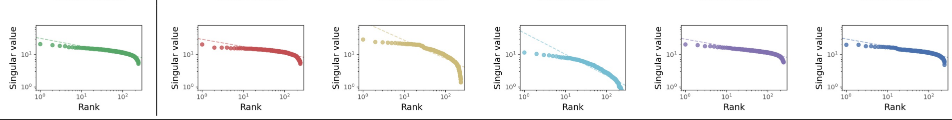
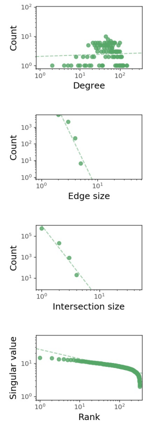
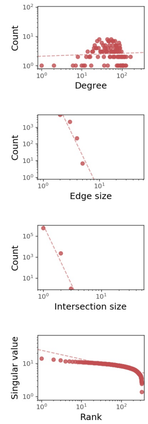
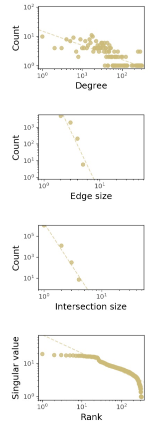
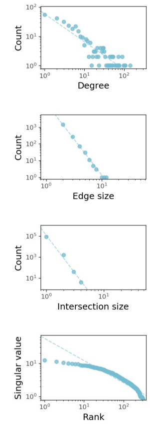
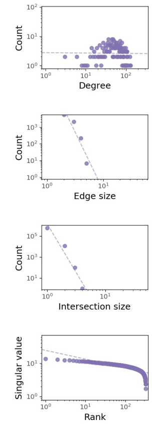
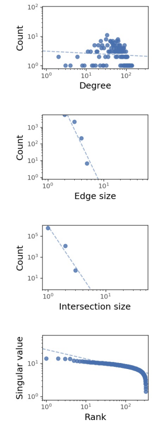
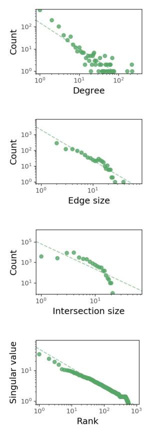
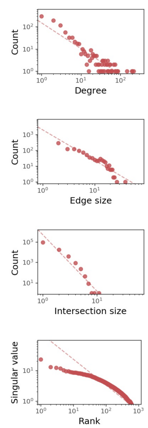
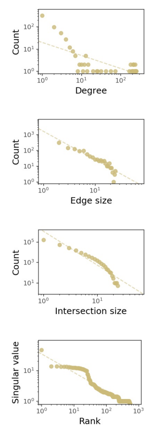
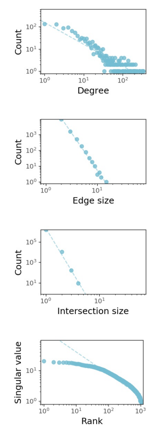
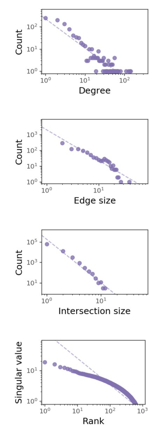
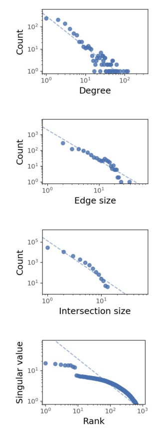
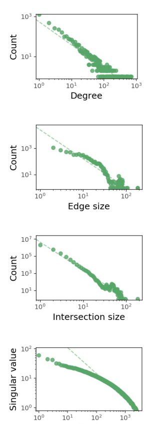
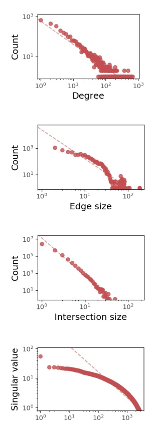
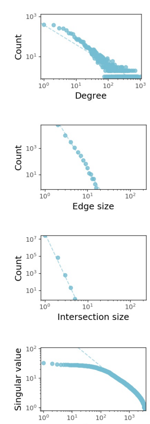
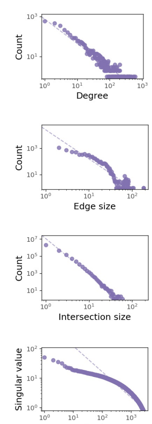
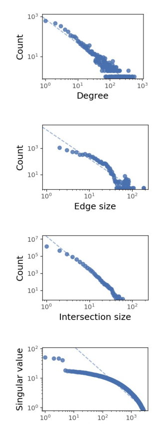
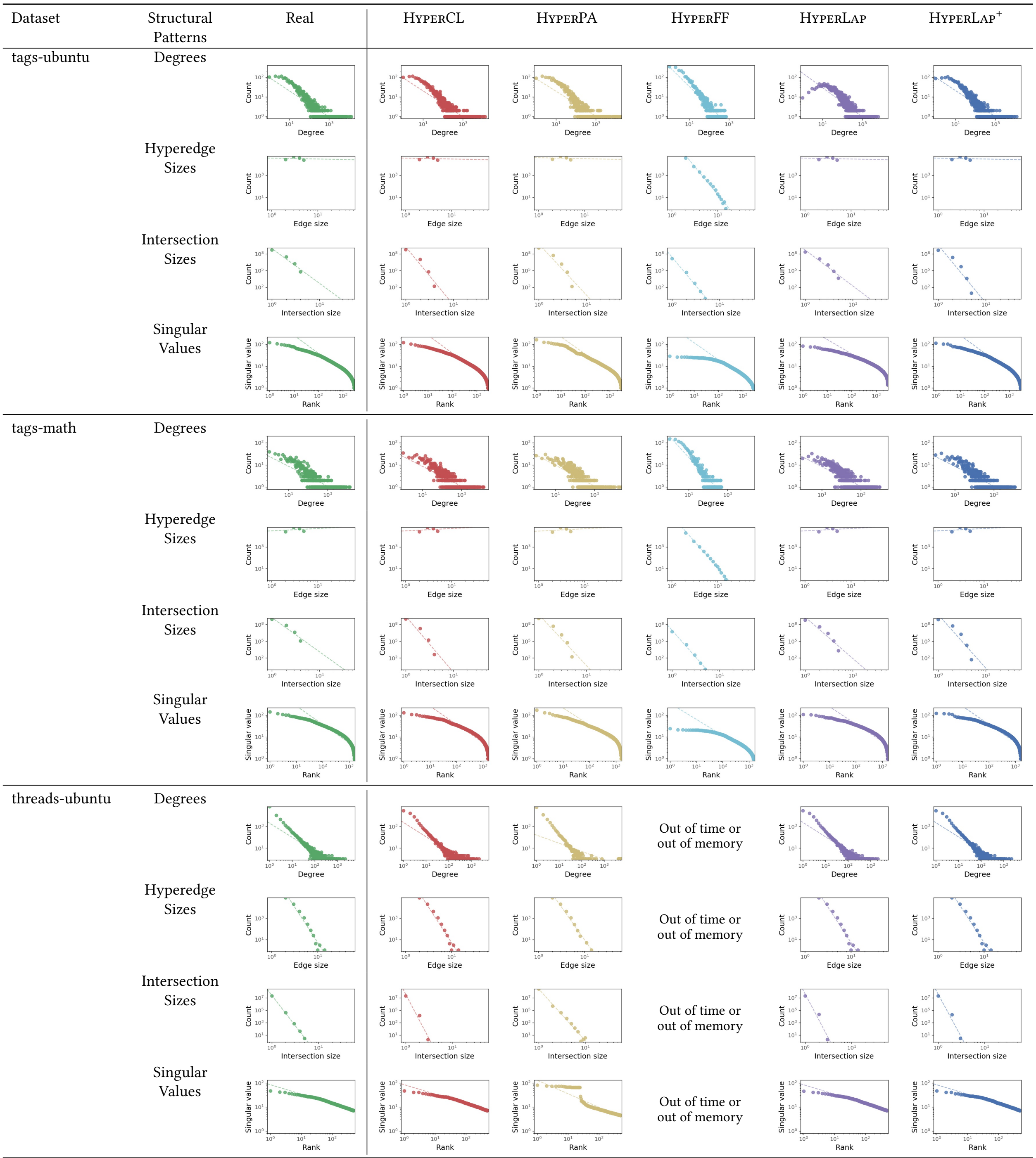
Metric	$\mathcal{E}'$		$\mathcal{E}$
			
Intersection	$\lvert e_1 \cap e_2 \cap e_3\rvert = \lvert\{b\}\rvert = 1$	$=$	$\lvert e_4 \cap e_5 \cap e_6\rvert = \lvert\{w\}\rvert = 1$
Union Inverse	$\frac{1}{\lvert e_1 \cup e_2 \cup e_3\rvert} = \frac{1}{\lvert\{a,b,c,d,e\}\rvert} = \frac{1}{5}$	$=$	$\frac{1}{\lvert e_4 \cup e_5 \cup e_6\rvert} = \frac{1}{\lvert\{v,w,x,y,z\}\rvert} = \frac{1}{5}$
Jaccard Index	$\frac{\lvert e_1 \cap e_2 \cap e_3\rvert}{\lvert e_1 \cup e_2 \cup e_3\rvert} = \frac{\lvert\{b\}\rvert}{\lvert\{a,b,c,d,e\}\rvert} = \frac{1}{5}$	$=$	$\frac{\lvert e_4 \cap e_5 \cap e_6\rvert}{\lvert e_4 \cup e_5 \cup e_6\rvert} = \frac{\lvert\{w\}\rvert}{\lvert\{v,w,x,y,z\}\rvert} = \frac{1}{5}$
Overlap Coefficient	$\frac{\lvert e_1 \cap e_2 \cap e_3\rvert}{min(\lvert e_1\rvert,\lvert e_2\rvert,\lvert e_3\rvert)} = \frac{\lvert\{b\}\rvert}{min(5,4,2)} = \frac{1}{2}$	$=$	$\frac{\lvert e_4 \cap e_5 \cap e_6\rvert}{min(\lvert e_4\rvert,\lvert e_5\rvert,\lvert e_6\rvert)} = \frac{\lvert\{w\}\rvert}{min(5,2,2)} = \frac{1}{2}$
Density	$\frac{\lvert\mathcal{E}'\rvert}{\lvert\bigcup_{e\in\mathcal{E}'} e\rvert} = \frac{\lvert\{e_1,e_2,e_3\}\rvert}{\lvert\{a,b,c,d,e\}\rvert} = \frac{3}{5}$	$=$	$\frac{\lvert\mathcal{E}\rvert}{\lvert\bigcup_{e\in\mathcal{E}} e\rvert} = \frac{\lvert\{e_4,e_5,e_6\}\rvert}{\lvert\{v,w,x,y,z\}\rvert} = \frac{3}{5}$
Overlapness	$\frac{\sum_{e\in\mathcal{E}'}\lvert e\rvert}{\lvert\bigcup_{e\in\mathcal{E}'} e\rvert} = \frac{\lvert e_1\rvert+\lvert e_2\rvert+\lvert e_3\rvert}{\lvert\{a,b,c,d,e\}\rvert} = \frac{11}{5}$	$>$	$\frac{\sum_{e\in\mathcal{E}}\lvert e\rvert}{\lvert\bigcup_{e\in\mathcal{E}} e\rvert} = \frac{\lvert e_4\rvert+\lvert e_5\rvert+\lvert e_6\rvert}{\lvert\{v,w,x,y,z\}\rvert} = \frac{9}{5}$

Table 12: <u>Macroscopic Structural Patterns.</u> All four macroscopic structural properties of real-world hypergraphs [3] are reproduced accurately by HyperLap+ in all the datasets.

Dataset	Structural Patterns	Real	HyperCL	HyperPA	HyperFF	HyperLap	HyperLap+
email-Enron	Degrees						
	Hyperedge Sizes						
	Intersection Sizes						
	Singular Values						
email-Eu	Degrees			Out of time or out of memory			
	Hyperedge Sizes			Out of time or out of memory			
	Intersection Sizes			Out of time or out of memory			
	Singular Values			Out of time or out of memory			
contact-primary	Degrees						
	Hyperedge Sizes						
	Intersection Sizes						
	Singular Values						

Dataset	Structural Patterns	Real	HyperCL	HyperPA	HyperFF	HyperLap	HyperLap$^+$
contact-high	Degrees Hyperedge Sizes Intersection Sizes Singular Values						
NDC-classes	Degrees Hyperedge Sizes Intersection Sizes Singular Values						
NDC-substances	Degrees Hyperedge Sizes Intersection Sizes Singular Values			Out of time or out of memory Out of time or out of memory Out of time or out of memory Out of time or out of memory			


Dataset
Structural Patterns
Real
HyperCL
HyperPA
HyperFF
HyperLap
HyperLap+
tags-ubuntu
tags-math
threads-ubuntu
Degrees
Hyperedge Sizes
Intersection Sizes
Singular Values
Count
Degree
Edge size
Intersection size
Singular value
Rank
Out of time or out of memory

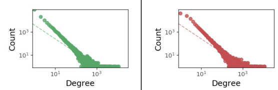
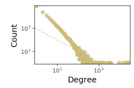
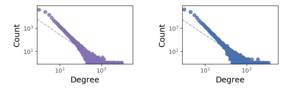
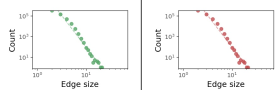
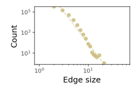
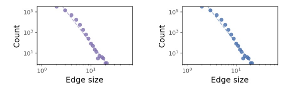
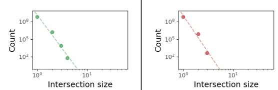
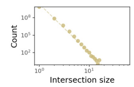
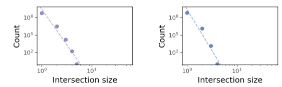
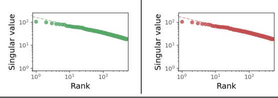
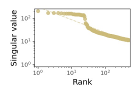
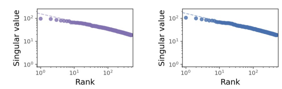
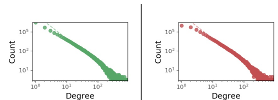
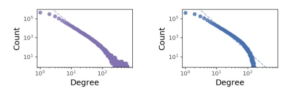
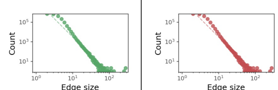
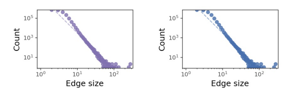
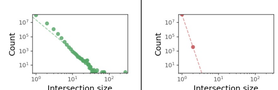
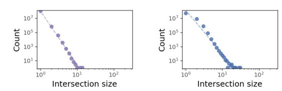
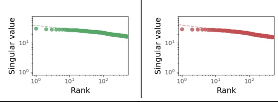
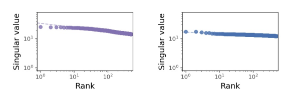
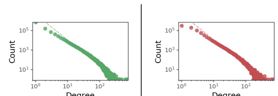
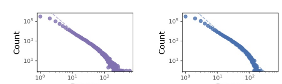
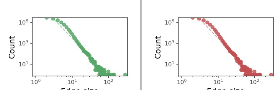
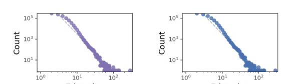
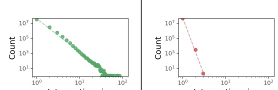
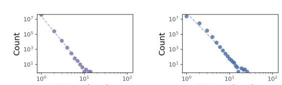
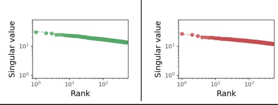
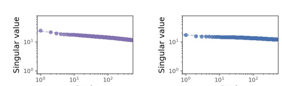
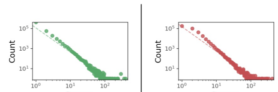
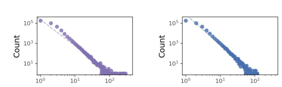
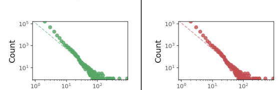
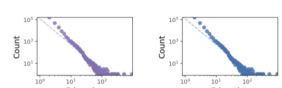
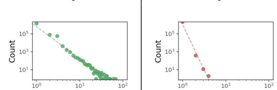
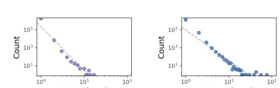
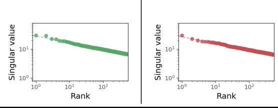
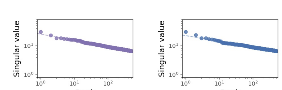
Dataset	Structural Patterns	Real	HyperCL	HyperPA	HyperFF	HyperLap	HyperLap$^+$
threads-math	Degrees				Out of time or out of memory		
	Hyperedge Sizes				Out of time or out of memory		
	Intersection Sizes				Out of time or out of memory		
	Singular Values				Out of time or out of memory		
coauth-DBLP	Degrees			Out of time or out of memory	Out of time or out of memory		
	Hyperedge Sizes			Out of time or out of memory	Out of time or out of memory		
	Intersection Sizes			Out of time or out of memory	Out of time or out of memory		
	Singular Values			Out of time or out of memory	Out of time or out of memory		
coauth-geology	Degrees			Out of time or out of memory	Out of time or out of memory		
	Hyperedge Sizes			Out of time or out of memory	Out of time or out of memory		
	Intersection Sizes			Out of time or out of memory	Out of time or out of memory		
	Singular Values			Out of time or out of memory	Out of time or out of memory		
coauth-history	Degrees			Out of time or out of memory	Out of time or out of memory		
	Hyperedge Sizes			Out of time or out of memory	Out of time or out of memory		
	Intersection Sizes			Out of time or out of memory	Out of time or out of memory		
	Singular Values			Out of time or out of memory	Out of time or out of memory		

Table 13: The heavy-tailed distributions regarding the four macroscopic structural properties [3]. The distributions of degrees, hyperedge sizes, intersection sizes, and singular values are all heavy-tailed in hypergraphs generated by HyperLap$^+$. As in Table 8, we report the maximum log-likelihood ratios of heavy-tailed distributions against exponential distributions. We boldface positive values, which indicate that the corresponding distribution is closer to a heavy-tailed distribution than to the exponential distribution.

Dataset	Degrees						Hyperedge Sizes					
	Real	H-CL	H-PA	H-FF	H-LAP	H-LAP$^+$	Real	H-CL	H-PA	H-FF	H-LAP	H-LAP$^+$
email-Enron	-2.97	-2.21	**3.77**	**1.28**	-3.19	**0.51**	**2.46**	**2.46**	**2.55**	**1.45**	**2.46**	**2.46**
email-Eu	**0.98**	**1.21**	-	**5.47**	**0.40**	-1.86	**19.54**	**19.54**	-	-0.19	**19.54**	**19.54**
contact-primary	**1.89**	**1.23**	-2.22	**2.79**	**1.32**	**0.84**	**2.35**	**2.35**	**2.49**	**0.62**	**2.35**	**2.35**
contact-high	**1.14**	**0.13**	**4.31**	**3.16**	**1.18**	**0.61**	**12.77**	**12.77**	**12.60**	**0.31**	**12.77**	**12.77**
NDC-classes	**7.48**	**7.22**	**16.17**	**5.17**	**7.04**	**7.14**	**0.85**	**0.85**	**0.59**	**0.064**	**0.85**	**0.85**
NDC-substances	**1.95**	**1.87**	-	**8.88**	**2.85**	**1.26**	**4.80**	**4.80**	-	**0.24**	**4.80**	**4.80**
tags-ubuntu	**8.83**	**8.86**	**9.34**	**7.97**	**6.63**	**9.15**	**99.66**	**99.66**	**103.5**	**0.198**	**99.66**	**99.66**
tags-math	**5.32**	**5.57**	**3.94**	**6.74**	**5.00**	**5.50**	**104.7**	**104.7**	**102.2**	**0.94**	**104.7**	**104.7**
threads-ubuntu	**12.40**	**12.49**	**10.87**	-	**12.96**	**13.61**	**0.77**	**0.77**	**1.12**	-	**0.77**	**0.77**
threads-math	**36.91**	**31.33**	**11.02**	-	**32.92**	**31.20**	**1.02**	**1.02**	**2.07**	-	**1.02**	**1.02**
coauth-DBLP	**8.85**	**8.73**	-	-	**6.62**	**193.7**	**4.19**	**4.19**	-	-	**4.19**	**4.19**
coauth-geology	**144.5**	**124.9**	-	-	**151.1**	**143.17**	**5.77**	**5.77**	-	-	**5.77**	**5.77**
coauth-history	**8.13**	**9.47**	-	-	**10.80**	**13.98**	**2.38**	**2.38**	-	-	**2.38**	**2.38**

-: out of time (taking more than 10 hours) or out of memory. Also, the range of hyperedge sizes is small in some datasets, which make some ratio not available.

Dataset	Intersection Sizes						Singular Values					
	Real	H-CL	H-PA	H-FF	H-LAP	H-LAP$^+$	Real	H-CL	H-PA	H-FF	H-LAP	H-LAP$^+$
email-Enron	**0.68**	**0.75**	**2.54**	**0.91**	**0.94**	**2.44**	**30.58**	**32.12**	**25.84**	**17.51**	**31.55**	**32.83**
email-Eu	**33.45**	**1.10**	-	**1.54**	**6.45**	**6.12**	**65.75**	**70.35**	-	**58.63**	**72.45**	**72.75**
contact-primary	**1.69**	**2.51**	**1.42**	**1.05**	**1.01**	**0.77**	**74.84**	**73.73**	**47.64**	**24.25**	**74.98**	**76.11**
contact-high	**2.20**	**0.28**	**1.33**	**1.04**	**0.50**	**3.67**	**70.23**	**68.21**	**48.60**	**28.60**	**68.82**	**67.63**
NDC-classes	**13.76**	**1.32**	**7.59**	**1.53**	**0.20**	**0.65**	**44.35**	**33.93**	**43.97**	**62.20**	**33.17**	**34.35**
NDC-substances	**36.31**	**7.89**	-	**0.51**	**13.64**	**8.24**	**80.90**	**80.75**	-	**121.3**	**80.85**	**80.77**
tags-ubuntu	**127.5**	**6.16**	**42.08**	**0.51**	**14.24**	**1.48**	**139.3**	**137.6**	**141.9**	**105.3**	**183.7**	**140.2**
tags-math	**174.5**	**10.04**	**45.86**	**0.51**	**31.95**	**3.10**	**119.4**	**121.8**	**127.9**	**80.45**	**126.6**	**122.2**
threads-ubuntu	**0.89**	**-1.41**	**1.54**	-	**-1.41**	**0.01**	**115.7**	**116.0**	**91.06**	-	**115.9**	**115.9**
threads-math	**3.92**	**8.86**	-	-	**0.49**	**0.86**	**189.6**	**184.7**	-	-	**188.8**	**188.7**
coauth-DBLP	**2.58**	-60.3	-	-	**1.05**	**2.35**	**178.4**	**173.5**	-	-	**164.8**	**151.4**
coauth-geology	**5.13**	**0.41**	-	-	**0.64**	**2.12**	**159.5**	**151.2**	-	-	**148.3**	**150.0**
coauth-history	**3.25**	**0.76**	-	-	**0.61**	**3.11**	**112.5**	**112.1**	-	-	**112.1**	**110.4**

-: out of time (taking more than 10 hours) or out of memory. The range of intersection sizes is small in some datasets, which makes some ratios not available.

Table 14: D-statistics between the distributions regarding Observations 1-5 in real-world hypergraphs and those in hypergraphs generated by three potential null models: HyperCL, the bipartite graph generative model (Bipartite) [1], and the hypergraph configuration model (Configuration) [2]. Overall, all three models generate similarly realistic hypergraphs.

		Observation 1	Observation 2	Observation 3	Observation 4	Observation 5
email-Enron	HyperCL	0.545	0.517	0.143	0.089	0.498
	Bipartite	0.580	0.487	0.122	0.095	0.518
	Configuration	0.538	0.517	0.148	0.125	0.514
email-Eu	HyperCL	0.724	0.543	0.225	0.480	0.505
	Bipartite	0.755	0.526	0.210	0.477	0.522
	Configuration	0.732	0.545	0.225	0.482	0.506
contact-primary	HyperCL	0.896	0.867	0.196	0.137	0.430
	Bipartite	0.966	0.760	0.129	0.130	0.388
	Configuration	0.904	0.859	0.197	0.139	0.426
contact-high	HyperCL	0.948	0.874	0.277	0.210	0.423
	Bipartite	0.993	0.877	0.195	0.207	0.411
	Configuration	0.951	0.871	0.275	0.210	0.423
NDC-classes	HyperCL	0.694	0.302	0.273	0.376	0.274
	Bipartite	0.714	0.305	0.274	0.375	0.304
	Configuration	0.688	0.342	0.292	0.385	0.268
NDC-substances	HyperCL	0.451	0.321	0.272	0.521	0.377
	Bipartite	0.464	0.320	0.273	0.525	0.408
	Configuration	0.449	0.348	0.283	0.539	0.349
tags-ubuntu	HyperCL	0.522	0.432	0.091	0.148	0.245
	Bipartite	0.666	0.432	0.073	0.120	0.219
	Configuration	0.524	0.439	0.094	0.147	0.238
tags-math	HyperCL	0.496	0.460	0.095	0.209	0.337
	Bipartite	0.578	0.439	0.074	0.170	0.302
	Configuration	0.503	0.468	0.097	0.209	0.327
threads-ubuntu	HyperCL	0.159	0.299	0.011	0.004	0.020
	Bipartite	0.488	0.299	0.006	0.004	0.267
	Configuration	0.076	0.013	0.012	0.004	0.022
threads-math	HyperCL	0.137	0.232	0.041	0.006	0.060
	Bipartite	0.472	0.232	0.031	0.005	0.231
	Configuration	0.153	0.078	0.042	0.006	0.062
coauth-DBLP	HyperCL	0.228	0.302	0.224	0.215	0.715
	Bipartite	0.379	0.322	0.224	0.215	0.713
	Configuration	0.215	0.307	0.224	0.215	0.715
coauth-geology	HyperCL	0.200	0.248	0.178	0.086	0.624
	Bipartite	0.333	0.247	0.178	0.086	0.623
	Configuration	0.171	0.227	0.178	0.086	0.624
coauth-history	HyperCL	0.087	0.316	0.033	0.001	0.154
	Bipartite	0.281	0.318	0.033	0.001	0.229
	Configuration	0.045	0.065	0.033	0.001	0.155
Average	HyperCL	0.468	0.439	0.158	0.191	0.359
	Bipartite	0.590	0.428	0.140	0.185	0.395
	Configuration	0.458	0.391	0.162	0.196	0.356