Physics-Informed Neural Networks Assignment

CE397 and CSE393: Scientific Machine Learning

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1 1D Steady-State Heat Equation

Consider the one-dimensional steady-state heat conduction problem with a heat source on a rod of unit length. The governing equation is:

$$\frac{d^2T}{dx^2} + \frac{q(x)}{\kappa} = 0, \quad x \in [0, 1]\,,$$

with homogeneous Dirichlet boundary conditions:

$$T(0) = T(1) = 0\,.$$

Given a thermal diffusivity constant: $\kappa = 0.5$ and Heat source term: $q(x) = 15x - 2$.

a) Design a physics-informed neural network with:

- Input dimension: 1 (spatial coordinate x)
- Three hidden layers with 32 neurons each
- Hyperbolic tangent activation functions
- Output dimension: 1 (temperature T)

b) Construct the physics-informed loss function $\mathcal{L}_{Physics}$ that enforces the PDE.

c) Write the boundary loss function $\mathcal{L}_{BCs}$.

d) Train the network using:

- $N_f = 100$ equidistant collocation points
- Total loss $\mathcal{L} = \mathcal{L}_{BCs} + \mathcal{L}_{Physics}$
- Adam optimizer with learning rate 10^{-3}
- 10,000 epochs

e) Derive the analytical solution and compare it with the numerical solution. Plot both solutions and compute the relative L^2 error.

2 Loss Function Design

Investigate different formulations of the loss function:

a) Implement adaptive loss weighting:

$$\mathcal{L} = \alpha(t)\mathcal{L}_{BCs} + \beta(t)\mathcal{L}_{Physics}$$

where $\alpha(t)$ and $\beta(t)$ are dynamic weights. Compare with the original fixed-weight formulation.

3 Variational Formulation [Optional for CE397]

Implement and analyze a variational (weak) form of the problem:

a) Derive the weak form by multiplying the PDE by test functions $\phi(x)$ and integrating:

$$\int_0^1 \frac{dT}{dx}\frac{d\phi}{dx}dx = \int_0^1 \frac{q(x)}{\kappa}\phi dx$$

b) Implement this formulation using neural networks and compare with the strong form implementation

4 Data-Driven Cross-Section Identification

Consider the static bar equation where the displacement field $u(x)$ is known but the cross-sectional properties $EA(x)$ need to be identified:

$$\frac{d}{dx}(EA(x)\frac{du}{dx}) + p(x) = 0, \quad x \in [0, 1]$$

Given:

- Displacement field: $u(x) = \sin(2\pi x)$
- Distributed load: $p(x) = -2(3x^2 - 2x)\pi \cos(2\pi x) + 4(x^3 - x^2 + 1)\pi^2 \sin(2\pi x)$
- Domain: $x \in [0, 1]$
- Boundary conditions: $u(0) = u(1) = 0$

a) Design a physics-informed neural network to identify $EA(x)$ with:

- Input dimensions: 2 (spatial coordinate x and displacement u)
- Three hidden layers with 20 neurons each
- Hyperbolic tangent activation functions
- Output dimension: 1 (stiffness EA)

b) Formulate the physics-informed loss function that enforces the differential equation.

c) Train the network using:

- $N = 100$ uniformly distributed training points
- Adam optimizer with learning rate 10^{-3}
- 5,000 epochs

d) Compare the identified $EA(x)$ with the analytical solution:

$$EA(x) = x^3 - x^2 + 1$$

e) Study the influence of noise in the displacement measurements by adding Gaussian noise with a standard deviation of 0.01.