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Intersection theory on the universal Picard stack

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Summary

We study the intersection theory of the universal Picard stack $\mathfrak{Pic}_{g,n}$. The universal Picard stack parametrizes a tuple $(C, p_1, \cdots, p_n, L)$ of a projective prestable algebraic curve C of genus g with distinct smooth marked points $p_1, \cdots, p_n$ and a line bundle L on C. This space is a smooth algebraic stack, locally of finite type over a base field k. There exists the universal curve $\mathfrak{C}_{g,n} \to \mathfrak{Pic}_{g,n}$ and the universal line bundle $\mathcal{L} \to \mathfrak{C}_{g,n}$ over the universal curve.

Inside the rational Chow ring $\mathsf{CH}^*(\mathfrak{Pic}_{g,n})_{\mathbb{Q}}$ of the universal Picard stack, we define tautological classes using the universal structure of the moduli problem. The tautological ring $\mathsf{R}^*(\mathfrak{Pic}_{g,n})$ is defined as the smallest subring of $\mathsf{CH}^*(\mathfrak{Pic}_{g,n})_{\mathbb{Q}}$ generated by tautological classes. The tautological ring $\mathsf{R}^*(\mathfrak{Pic}_{g,n})$ contains classical objects such as the tautological ring of the moduli space of stable curves. Moreover it has rich structure intertwining the deformation of curves with line bundles.

This thesis consists of seven papers. The first paper studies tautological relations on the moduli space of stable maps coming from the double ramification formula with target varieties. The second paper studies universal double ramification cycles and relations on the universal Picard stack. The third paper studies reduced Gromov-Witten invariants for $K3$ surfaces with imprimitive curve classes. The fourth paper studies the tautological ring of the moduli space of prestable curves. The fifth paper computes the Chow ring of the moduli space of genus zero prestable curves. The sixth paper studies tautological relations on the moduli space of stable maps from equivariant projective bundles. The seventh paper studies tautological relations on the universal Picard scheme using stable Quot schemes over the universal Picard scheme.

Zusammenfassung

Wir untersuchen die Schnitttheorie des universellen Picard-Stacks $\mathfrak{Pic}_{g,n}$. Der universelle Picard-Stack parametrisiert Tupel $(C, p_1, \cdots, p_n, L)$ bestehend aus einer projektiven prestabilen algebraischen Kurve C der Gattung g mit verschiedenen glatten markierten Punkten $p_1, \cdots, p_n$ und einem Linienbündel L über C. Dieser Raum ist ein glatter algebraischer Stack, lokal vom endlichen Typ über einem Grundkörper k. Es gibt die universelle Kurve $\mathfrak{C}_{g,n} \to \mathfrak{Pic}_{g,n}$ und das universelle Linienbündel $\mathcal{L} \to \mathfrak{C}_{g,n}$ über der universellen Kurve.

Im rationalen Chow-Ring $\mathsf{CH}^*(\mathfrak{Pic}_{g,n})_{\mathbb{Q}}$ des universellen Picard-Stacks, definieren wir tautologische Klassen unter Verwendung der universellen Struktur des Moduli-Problems. Der tautologische Ring $\mathsf{R}^*(\mathfrak{Pic}_{g,n})$ ist definiert als der kleinste Unterring von $\mathsf{CH}^*(\mathfrak{Pic}_{g,n})_{\mathbb{Q}}$, der durch tautologische Klassen erzeugt wird. Der tautologische Ring $\mathsf{R}^*(\mathfrak{Pic}_{g,n})$ enthält klassische Objekte wie den tautologischen Ring des Modulraums stabiler Kurven. Ausserdem hat er eine reichhaltige Struktur, die die Deformation von Kurven mit Linienbündeln verbindet.

Diese Arbeit besteht aus sieben Teilen. Der erste Teil untersucht tautologische Beziehungen auf dem Modulraum stabiler Abbildungen, die aus der Doppel-Verzweigungsformel mit Zielrarietäten stammen. Der zweite Teil untersucht die universelle Verzweigungszyklen und Beziehungen auf dem universellen Picard-Stack. Der dritte Teil untersucht reduzierte Gromov-Witten-Invarianten für K3 Flächen mit unprimitiven Kurvenklassen. Der vierte Teil untersucht den tautologischen Ring des Modulraums prestabiler Kurven. Der fünfte Teil berechnet den Chow-Ring des Modulraums prestabiler Kurven des Geschlechts Null. Der sechste Teil untersucht tautologische Beziehungen auf dem Modulraum stabiler Abbildungen von äquivarianten projektiven Bündeln. Der siebte Teil untersucht tautologische Beziehungen auf dem universellen Picard-Schema unter Verwendung stabiler Quot-Schemata über dem universellen Picard-Schema.

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Chapter 1. Introduction

Intersection theory on moduli spaces has been a major subject in algebraic geometry. Universal objects of the moduli problem define tautological classes inside the Chow (or cohomology) group of the moduli space. The smallest subgroup generated by tautological classes is called the *tautological ring*. The tautological ring is much easier to understand than the full Chow (or cohomology) group of the moduli spaces. For moduli space of stable curves, the study of tautological rings was initiated by Mumford's seminar paper [32].

We study the tautological ring of the universal Picard stack. The universal Picard stack is an algebraic stack parametrizing marked prestable curves together with line bundles. The main part of this thesis consists of papers [2, 5, 4, 3, 7, 8, 6].

1 Main objects

1.1 The universal Picard stack

Let $\mathfrak{M}_{g.m}$ be the moduli stack of prestable curves of arithmetic genus g with m marked points over a base field k. It parametrizes tuples $(C, p_1, \cdots, p_m)$ where C is a connected nodal curve of genus g and $p_1, \cdots, p_m$ are distinct smooth marked points on C. The moduli space $\mathfrak{M}_{g,m}$ is a smooth algebraic stack locally of finite type of dimension $3g - 3 + m$ over k. It has the universal curve $\pi : \mathfrak{C}_{g,m} \to \mathfrak{M}_{g,m}$. Consider the relative Picard stack $\mathfrak{Pic}_{g,m}$ of π parametrizing marked nodal curves with line bundles. It is also a smooth algebraic stack locally of finite type over k [27]. The stack $\mathfrak{Pic}_{g,m}$ is called the *universal Picard stack*. It is a disjoint union

$$\mathfrak{Pic}_{g,m} = \bigsqcup_{d \in \mathbb{Z}} \mathfrak{Pic}_{g,m,d}$$

of substacks where $\mathfrak{Pic}_{g,m,d}$ is the Picard stack of curves with total degree d line bundles. The Chow ring $\mathsf{CH}^*(\mathfrak{Pic}_{g,m})$ of the universal Picard stack is well defined by [26, 7][1]. Throughout this thesis we consider Chow rings with rational coefficients.

[1]The intersection theory is not covered by [26, 7] when $(g, m) = (1, 0)$ because the stabilizer group at the general point is not a linear algebraic group. We exclude this case throughout this thesis.

1.2 Tautological ring

Inside the Chow ring $\mathsf{CH}^*(\mathfrak{Pic}_{g,m})$, we define tautological classes using the universal structure of the moduli problem. The boundary strata of $\mathfrak{Pic}_{g,m}$ can be described by combinatorial objects called prestable graphs with degrees. An element $\Gamma_\delta \in \mathsf{G}_{g,m}$ consists of a prestable graph Γ and a function $\delta : V(\Gamma) \to \mathbb{Z}$ on the set of vertices of Γ. Then there exists an algebraic stack $\mathfrak{Pic}_{\Gamma_\delta}$ parameterizing curves with degenerations imposed by the graph Γ and with line bundles which have degree $\delta(v)$ restricted to the components corresponding to the vertex $v \in V(\Gamma)$. There exists a canonical morphism

$$j_{\Gamma_\delta} : \mathfrak{Pic}_{\Gamma_\delta} \to \mathfrak{Pic}_{g,m} \tag{1}$$

which is finite. The universal Picard stack has the universal curve and m sections

$$\pi : \mathfrak{C}_{g,m} \to \mathfrak{Pic}_{g,m}\,,\ p_1, \cdots, p_m : \mathfrak{Pic}_{g,m} \to \mathfrak{C}_{g,m}$$

together with the universal line bundle

$$\mathcal{L} \to \mathfrak{C}_{g,m}\,.$$

Denote by ω_π the relative dualizing line bundle and denote by $\omega_{\log}$ the relative log dualizing line bundle on $\mathfrak{C}_{g,m}$. The *twisted κ-classes* are defined by the pushforward

$$\kappa_{a,b} = \pi_*(c_1(\omega_{\log})^{a+1} c_1(\mathcal{L})^b) \text{ for } a \geq -1, b \geq 0\,.$$

At each i-th marking, we define ψ and ξ-class by

$$\psi_i = p_i^* c_1(\omega_\pi) \text{ and } \xi_i = p_i^* c_1(\mathcal{L})\,.$$

For $\Gamma_\delta \in \mathsf{G}_{g,m}$, a *decoration* γ is given by a product of ξ, ψ and twisted κ-classes on factors $\mathfrak{Pic}_{g(v),m(v),\delta(v)}$. Each decorated stratum $[\Gamma_\delta, \gamma]$ defines a Chow class via the pushforward along (1).

Definition 1. The tautological ring $\mathsf{R}^*(\mathfrak{Pic}_{g,m}) \subset \mathsf{CH}^*(\mathfrak{Pic}_{g,m})$ is the smallest $\mathbb{Q}$-linear subspace spanned by tautological classes associated to all decorated strata $[\Gamma_\delta, \gamma]$.

Tautological classes of $\mathfrak{Pic}_{g,m}$ as operational Chow classes first appear in [4] and this definition can be extended to Chow classes. Relations among Chow classes associated to decorated strata are called *tautological relations*. We present the main question of this thesis:

Question. Can we understand the structure of $\mathsf{R}^*(\mathfrak{Pic}_{g,m})$?

The tautological ring of the universal Picard stack will be an important tool to understand the cycle theory of compactified Picard stacks [12] and the universal logarithmic Picard group constructed by Molcho-Wise [31]

2 Tautological relations from the moduli space of stable maps

2.1 Overview

The virtual geometry of the moduli space of stable maps is an important tool to understand the tautological ring of the moduli space $\overline{\mathcal{M}}_{g,m}$ of stable curves. Let X be a smooth projective variety over the base field k. The moduli space $\overline{\mathcal{M}}_{g,m}(X)$ of stable maps parametrizes maps from marked nodal curves C to X which satisfy the stability condition. The stability condition is equivalent to saying that each stabilizer group is finite. When $2g-2+m>0$, there exists a morphism

$$\mu : \overline{\mathcal{M}}_{g,m}(X) \to \overline{\mathcal{M}}_{g,m} \tag{2}$$

defined by forgetting the map and stabilizing the domain curve. The moduli space $\overline{\mathcal{M}}_{g,m}(X)$ carries a natural perfect obstruction theory [10] and hence there exists a pure degree algebraic cycle

$$[\overline{\mathcal{M}}_{g,m}(X)]^{\mathrm{vir}} \in \mathsf{CH}_{\mathrm{vdim}}(\overline{\mathcal{M}}_{g,m}(X)),$$

called the *virtual fundamental class*. *Pushforwards* of virtual fundamental classes along (2) define the *Cohomological Field Theory* (CohFT) [25]. The virtual fundamental class and the CohFT are crucial tools to study the tautological ring of $\overline{\mathcal{M}}_{g,m}$ [34].

There is an alternative way of using virtual geometry of the stable maps to study tautological rings. We explain this idea for the tautological ring of the universal Picard stack. Consider a pair (X, L) of a smooth projective variety X and a line bundle L on X. Such pair defines a morphism to the universal Picard stack

$$\nu : \overline{\mathcal{M}}_{g,m}(X) \to \mathfrak{Pic}_{g,m}, \ (f : C \to X) \mapsto (C, f^*L). \tag{3}$$

Suppose we are interested in problems on $\overline{\mathcal{M}}_{g,m}(X)$ which only depends on the pair (X, L). Then it is often the case that there is the universal answer on $\mathfrak{Pic}_{g,m}$. A shadow of this fact can be detected on $\overline{\mathcal{M}}_{g,m}(X)$ by *pulling back* the answer along ν and intersect with the virtual fundamental class. Said differently, variation of ν over all pairs (X, L) produces charts for studying intersection theory of $\mathfrak{Pic}_{g,m}$. Finding connections between the virtual geometry of moduli space of stable maps, or their variants, and the intersection theory of universal Picard stack will be a recurring theme of this thesis.

2.2 Double ramification relations with target varieties

In [2] we define the tautological ring of the moduli space of stable maps to a target variety and prove a system of tautological relations coming from Pixton's double ramification cycle formula with target varieties. Foundations of double ramification cycle formula is developed by Janda-Pandharipande-Pixton-Zvonkine [24, 23].

The main result of [2] can be presented in terms of (3). Let $A = (a_1, \cdots, a_m) \in \mathbb{Z}^m$ be a tuple of integers. For a prestable graph with degrees Γ_δ, *a weighting mod r of* Γ_δ is a function on the set of half-edges

$$w : H(\Gamma) \to \{0, 1, \cdots, r-1\}$$

satisfying a certain balancing condition depending on A (see [4]). We denote by $\mathsf{W}_{\Gamma_\delta, r}$ the finite set of all possible weightings mod r of Γ_δ. We denote by $\mathsf{P}^{c,r}_{g,A,d}$ the codimension c component of the tautological class

$$\sum_{\substack{\Gamma_\delta \in \mathsf{G}_{g,m,d} \\ w \in \mathsf{W}_{\Gamma_\delta, r}}} \frac{r^{-h^1(\Gamma_\delta)}}{|\mathrm{Aut}(\Gamma_\delta)|} j_{\Gamma_\delta *} \Bigg[\prod_{i=1}^{n} \exp\left(\frac{1}{2} a_i^2 \psi_i + a_i \xi_i\right) \prod_{v \in V(\Gamma_\delta)} \exp\left(-\frac{1}{2}\kappa_{-1,2}(v)\right) \prod_{e=(h,h') \in E(\Gamma_\delta)} \frac{1 - \exp\left(-\frac{w(h)w(h')}{2}(\psi_h + \psi_{h'})\right)}{\psi_h + \psi_{h'}} \Bigg].$$

By [24, Appendix] the expression $\mathsf{P}^{c,r}_{g,A}$ is polynomial in r for large enough r. Let $\mathsf{P}^{c}_{g,A}$ be the constant term of this polynomial.

Theorem 2. Let X be a smooth projective variety and L be a line bundle on X. For an effective curve class $\beta \in H_2(X, \mathbb{Z})$, we denote $d = \int_\beta c_1(L)$. For a tuple of integers $A = (a_1, \cdots, a_m) \in \mathbb{Z}^m$ with $\sum_{i=1}^m a_i = d$ and $c > g$, we have

$$\nu^* \mathsf{P}^c_{g,A,d} \cap [\overline{\mathcal{M}}_{g,m}(X, \beta)]^{\mathrm{vir}} = 0$$

in $\mathsf{CH}_*(\overline{\mathcal{M}}_{g,m}(X, \beta))$.

Theorem 2 generalizes double ramification relations of Clader-Janda [13]. From the perspective of Section 2.1, it is natural to ask whether $\mathsf{P}^c_{g,A,d} = 0$ holds on $\mathfrak{Pic}_{g,m,d}$. However, Theorem 2 does not automatically produce this conclusion because we do not know whether pulling back classes along (3) and capping with the virtual fundamental class is injective. In order to understand this question, we need additional argument which we will explain in the following section.

2.3 Universal double ramification cycles and relations

In [4] (with D. Holmes, R. Pandharipande, J. Schmitt and R. Schwarz) we implement the idea of using moduli spaces of stable maps as test spaces to understand the intersection theory of $\mathfrak{Pic}_{g,m}$. For technical reasons, we use the language of operational classes for algebraic stacks. We extend the notion of double ramification cycle

[24, 23] on moduli space of stable maps to the universal Picard stack. We prove the Pixton's formula for the universal double ramification cycle, and the universal double ramification relations, using corresponding results on stable maps [23, 2].

For a tuple of integers $A = (a_1, \cdots, a_m) \in \mathbb{Z}^m$, define the Abel-Jacobi section by

$$\sigma_A : \mathfrak{M}_{g,m} \to \mathfrak{Pic}_{g,m}\,,\ (C, p_1, \cdots, p_m) \mapsto (C,\, \mathcal{O}_C(a_1 p_1 + \cdots + a_m p_m))\,. \tag{4}$$

Let $\overline{\sigma}_A$ be the schematic image of (4). The closed substack $\overline{\sigma}_A$ defines an operational class $\mathsf{DR}^{\mathsf{op}}_{g,A} \in \mathsf{CH}^g_{\mathsf{op}}(\mathfrak{Pic}_{g,m})$, which we call the *universal double ramification cycle*.

Theorem 3. Let $A = (a_1, \cdots, a_m) \in \mathbb{Z}^m$ with $\sum_{i=1}^m a_i = d$. Then we have

$$\mathsf{DR}^{\mathsf{op}}_{g,A} = \mathsf{P}^g_{g,A}$$

in $\mathsf{CH}^g_{\mathsf{op}}(\mathfrak{Pic}_{g,m,d})$. Moreover for $c > g$, we have a system of tautological relations

$$\mathsf{P}^c_{g,A} = 0 \text{ in } \mathsf{CH}^c_{\mathsf{op}}(\mathfrak{Pic}_{g,m,d})\,.$$

For the first assertion, we compare $\mathsf{DR}^{\mathsf{op}}_{g,A}$ with the cycle defined by logarithmic Abel-Jacobi section [28]. Using various invariance properties of $\mathsf{DR}^{\mathsf{op}}_{g,A}$ and $\mathsf{P}^c_{g,A}$, we reduce the case to the double ramification cycle formula with target varieties [23]. For the second assertion, we use the same invariance properties of $\mathsf{P}^c_{g,A}$ and we reduce to the double ramification relations with targets [2].

Theorem 3 has several consequences. For those applications it is necessary to lift [24, 23] to the universal Picard stack. In [4] we prove the conjecture of Farkas-Pandharipande [17] on the cycle class formula of the locus of meromorphic differentials. In [3] we study reduced Gromov-Witten invariants for K3 surfaces using the universal double ramification relations. In [21] Theorem 3 is used to compute the logarithmic double ramification cycle formula and relations.

2.4 Reduced Gromov-Witten invariants of K3 surfaces

In [3] (with T. H. Buelles) we apply Theorem 3 to study reduced Gromov-Witten invariants of K3 surfaces. Let S be a smooth projective K3 surface. When the effective curve class $\beta \in H_2(S, \mathbb{Z})$ is nontrivial, the usual virtual fundamental class of $\overline{\mathcal{M}}_{g,,m}(S, \beta)$ vanishes. There exists a *reduced* virtual fundamental class

$$[\overline{\mathcal{M}}_{g,m}(S, \beta)]^{\mathrm{red}} \in \mathsf{CH}_g(\overline{\mathcal{M}}_{g,m}(S, \beta))$$

which produces nontrivial invariants [30].

Reduced Gromov-Witten invariants of K3 surfaces can be computed on an elliptic fibered K3 surface with a section. Consider an elliptic K3 surface $S \to \mathbb{P}^1$ with a section. Denote $B \in H_2(S, \mathbb{Z})$ by the curve class of the section and $F \in H_2(S, \mathbb{Z})$

be the curve class of a fiber. For integers $\ell, h, k_i \geq 0$ and cohomology classes $\gamma_i \in H^*(S,\mathbb{Q})$ we define

$$\left\langle \tau_{a_1}(\gamma_1)\dots\tau_{a_m}(\gamma_m)\right\rangle^S_{g,\ell B+hF} := \int_{[\overline{\mathcal{M}}_{g,m}(S,\ell B+hF)]^{\mathrm{red}}} \prod_{i=1}^m \psi_i^{a_i} \cup \mathrm{ev}_i^*(\gamma_i)$$

and a descendent potential

$$\mathsf{F}_{g,\ell}\big(\tau_{a_1}(\gamma_1)\dots\tau_{a_n}(\gamma_n)\big) := \sum_{h\geq 0} \left\langle \tau_{a_1}(\gamma_1)\dots\tau_{a_m}(\gamma_m)\right\rangle^S_{g,\ell B+hF} q^{h-\ell}\,.$$

There exists a notion of *modified degree* of cohomology classes coming from the monodromy of polarized K3 surfaces [33]. We say a cohomology class $\gamma \in H^*(S,\mathbb{Q})$ is *homogeneous* if it is homogeneous for the modified degree.

We consider the space $\mathsf{QMod}(2)$ of quasimodular forms for the level two congruence subgroup $\Gamma_0(2) \subset \mathrm{SL}_2(\mathbb{Z})$. Denote the modular discriminant by

$$\Delta(q) = q\prod_{m=1}^{\infty}(1-q^m)^{24}\,.$$

Theorem 4. Let S be a smooth projective K3 surface. For fixed genus g and homogeneous cohomology classes $\gamma_1,\cdots,\gamma_m \in H^*(S,\mathbb{Q})$, the series $\mathsf{F}_{g,2}$ is the Fourier expansion of a weakly holomorphic quasimodular form

$$\mathsf{F}_{g,2}\big(\tau_{a_1}(\gamma_1)\cdots\tau_{a_n}(\gamma_n)\big) \in \frac{1}{\Delta(q)^2}\mathsf{QMod}(2)\,.$$

Moreover the generating series satisfies the holomorphic anomaly equation.

The exact form of the holomorphic anomaly equation for imprimitive curve classes appears in [3]. Theorem 4 proves a conjecture of Maulik-Pandharipande-Thomas [30] for divisibility two cases. Moreover this gives partial results on the multiple cover formula conjectured by Oberdieck-Pandhariapande [33].

3 Chow ring of the moduli space of prestable curves

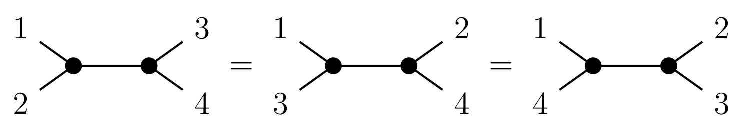
In [7, 8] (with J. Schmitt) we study the tautological ring of the moduli stack $\mathfrak{M}_{g,m}$ of prestable curves of genus g with m markings. Since there is a natural morphism from $\mathfrak{Pic}_{g,m}$ to $\mathfrak{M}_{g,m}$, tautological relations on $\mathfrak{M}_{g,m}$ produce tautological relations on the universal Picard stack. On the other hand, the moduli space $\overline{\mathcal{M}}_{g,m}$ is an open substack of $\mathfrak{M}_{g,m}$ and hence the tautological ring $\mathsf{R}^*(\mathfrak{M}_{g,m})$ can be viewed as a generalization of $\mathsf{R}^*(\overline{\mathcal{M}}_{g,m})$.

3.1 Foundation of the Chow ring of the moduli space of prestable curves

In [7] we define the tautological ring for the moduli stack of prestable curves and studied calculus among tautological classes. The integral Chow theory of algebraic stacks of finite type over k and stratified by quotients stacks was developed by Kresch [26]. It has expected functorial properties such as pushforward along projective morphisms and pullback along locally complete intersection morphisms. On the other hand the morphism $\pi : \mathfrak{C}_{g,m} \to \mathfrak{M}_{g,m}$ from the universal curve may not be projective. To define tautological classes on $\mathfrak{M}_{g,m}$ we should relax two conditions: being finite type and projective pushforward. The first part was done in [7] and the second part was done in Skowera's PhD thesis [39]. After developing technical foundation on the Chow group, we define the tautological ring $\mathsf{R}^*(\mathfrak{M}_{g,m})$ inside the Chow ring $\mathsf{CH}^*(\mathfrak{M}_{g,m})$. These techniques can be applied to the universal Picard stack and define the tautological ring $\mathsf{R}^*(\mathfrak{Pic}_{g,m})$ as a subspace of the Chow ring.

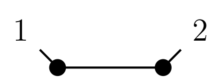
3.2 Complete description of the Chow group of the moduli space of genus zero prestable curves

In [8] we compute the Chow group of $\mathfrak{M}_{g,m}$ when the genus is zero. We first introduce two sources of tautological relations. The Witten–Dijkgraaf–Verlinde–Verlinde (WDVV) relation is a degree one relation



(5)

in $\mathsf{CH}^1(\mathfrak{M}_{0,4})$. There exists a degree one relation

$\psi_1 + \psi_2 =$ 

(6)

in $\mathsf{CH}^1(\mathfrak{M}_{0,2})$. Both relations are straight forward to check using the localization sequence.

Theorem 5. For $m \geq 0$ we have the equality

$$\mathsf{CH}^*(\mathfrak{M}_{0,m}) = \mathsf{R}^*(\mathfrak{M}_{0,m})\,.$$

Moreover the system of all linear relations in $\mathsf{CH}^*(\mathfrak{M}_{0,m})$ between the decorated strata classes $[\Gamma, \alpha]$ of normal form is generated by (5) on $\mathfrak{M}_{0,4}$ and the relation (6) on $\mathfrak{M}_{0,2}$.

For the first part, we introduce is the notion of *Chow-Künneth generating property* (CKgP) for algebraic stacks and we show that various quotient stacks appearing as strata of $\mathfrak{M}_{0,m}$. We stratify the moduli space $\mathfrak{M}_{0,n}$ by the number of nodes and use excision sequence to prove that the Chow group of $\mathfrak{M}_{0,m}$ is generated by tautological classes. For the second part, the main input is the extended localization sequence [11, 26] and the explicit computation of higher Chow groups for various strata of $\mathfrak{M}_{0,m}$. We compute the connecting homomorphism in the extended localization sequence and we show that tautological relations are generated by the WDVV relations and additional relations on $\mathfrak{M}_{0,2}$.

4 Virtual geometry and tautological relations on the universal Picard stack

It is a remarkable fact that the tautological ring of the moduli space of stable curves has organized structures. This system of tautological relations is called Pixton's relations [38]. In [37] Pixton's relations are proven by Givental-Teleman's classification of semi-simple CohFTs [18, 40]. On the other hand, one can ask whether tautological relations on the universal Picard stack also have any reasonable structure. There are several evidences why the Pixton's relations should be lifted to the universal Picard stack. Although this hope has not been fully materialized, we give partial answers on this direction.

4.1 Tautological relations from equivariant projective bundle

In [5] (with H. Lho) we study tautological relations on the moduli space of stable maps $\overline{\mathcal{M}}_{g,m}(X)$ following [22]. Let L be a line bundle on X. We consider the fiberwise $\mathbb{G}_m$ action on the projective bundle $\mathbb{P}(L \oplus \mathcal{O}_X)$. The equivariant pushforward of the virtual cycle along

$$\rho : \overline{\mathcal{M}}_{g,m}(\mathbb{P}(L \oplus \mathcal{O}_X)) \to \overline{\mathcal{M}}_{g,m}(X) \tag{7}$$

lies in the tautological ring of $\overline{\mathcal{M}}_{g,m}(X)$ by virtual localization formula [20]. The localization contribution of the pushforward can be organized by Givental's formalism on localizations [19, 18, 14]. In [5] we make a further simplification and assume $X = \mathbb{P}^1$. We obtain a generalization of Pixton's relations depending on hypergeometric series

$$A(z) = \sum_{i\geq 0} \frac{(6i)!}{(3i)!(2i)!}\left(\frac{-z}{576}\right)^i, \qquad B(z) = \sum_{i\geq 0} \frac{(6i)!}{(3i)!(2i)!}\frac{1+6i}{4}\left(\frac{-z}{576}\right)^i,$$

$$C(z) = \sum_{i\geq 0} \frac{(6i)!}{(3i)!(2i)!}\frac{1+6i}{1-6i}\left(\frac{-z}{576}\right)^i, \qquad D(z) = \sum_{i\geq 0} \frac{(6i)!}{(3i)!(2i)!}\frac{1+6i}{-4}\left(\frac{-z}{576}\right)^i,$$

where the first hypergeometric series already appears in [16].

The tautological relations on $\overline{\mathcal{M}}_{g,m}(X)$ only depend on $[L] : X \to B\mathbb{G}_m$, and not on the particular geometry of X and L. Following the idea of Section 2.1, we speculate that the relations should come from the universal Picard stack. The relevant universal object is the tautological line bundle $L \to B\mathbb{G}_m$ and consider certain stable map spaces to $\mathbb{P}(L \oplus \mathcal{O}_{B\mathbb{G}_m})$. Our use of moduli spaces of stable maps to projective bundles $\mathbb{P}(L \oplus \mathcal{O}_X)$ is a shadow of this idea.

4.2 Tautological relations from moduli spaces of stable Picard quotients

In [6] (with H. Lho) we obtain tautological relations on the relative Picard scheme by considering a certain relative Quot scheme over the relative Picard scheme. We consider the moduli space $\mathsf{Pic}^{\underline{0}}_{g,m}$ of relative Picard scheme of multi-degree zero line bundles for the universal curve $\mathcal{C}_{g,m} \to \overline{\mathcal{M}}_{g,m}$. It is a separated Deligne-Mumford stack over $\overline{\mathcal{M}}_{g,m}$. When $m \geq 1$, the relative Picard scheme $\mathsf{Pic}^{\underline{0}}_{g,m}$ is a commutative smooth group scheme of finite type over $\overline{\mathcal{M}}_{g,m}$. Therefore the Chow ring has further decomposition into weight spaces [9, 15, 1]

$$\mathsf{CH}^*(\mathsf{Pic}^{\underline{0}}_{g,m}) = \bigoplus_{j=0}^{2g} \mathsf{CH}^*_{(j)}(\mathsf{Pic}^{\underline{0}}_{g,m}) \,.$$

This decomposition refines tautological relations on $\mathsf{Pic}^{\underline{0}}_{g,m}$ into each weight piece.

For $r, d, n \in \mathbb{Z}_{\geq 0}$ and a tuple of integers $B = (b_1, \cdots, b_n) \in \mathbb{Z}^n$, we consider the moduli space of *stable Picard quotients* $\mathsf{P}_{g,m}(r, B, d)$ which generalizes Marian-Oprea-Pandharipande [29]. The moduli space of stable Picard quotients parametrizes tuples (C, L, q) where C is a prestable curve of genus g with m markings, L is a multi-degree zero line bundle on C and q is a short exact sequence of coherent sheaves

$$0 \to S \to L^{b_1} \oplus \cdots \oplus L^{b_n} \xrightarrow{q} Q \to 0$$

satisfying the stability condition parallel to usual stable quotients. The natural forgetful morphism

$$\mu : \mathsf{P}_{g,m}(r, B, d) \to \mathsf{Pic}^{\underline{0}}_{g,m} \tag{8}$$

is proper and $\mathsf{P}_{g,m}(r, B, d)$ carries a natural perfect obstruction theory. It has a natural evaluation map

$$\mathrm{ev}_i : \mathsf{P}_{g,m}(r, B, d) \to \mathrm{Gr}(r, B)$$

to the quotient stack $\mathrm{Gr}(r, B) = [\mathrm{Gr}(r, n)/\mathbb{G}_m]$. Here the $\mathbb{G}_m$-action is defined using the tuple B.

The equivariant pushforward of the virtual fundamental class along (8) produces a system of tautological relations on $\mathsf{Pic}^{\underline{0}}_{g,m}$. We consider the case when $r=1$ and $B=(0,1)$. We further restrict the moduli space over $\mathsf{Pic}^{\underline{0}}(\mathcal{M}_g)$. Consider an element $\widetilde{\gamma}$ in $\mathbb{Q}[\{\kappa_{a,b}\}_{a\geq -1,b\geq 0}][[u,t,x]]$

$$\begin{aligned}\widetilde{\gamma} :=& \sum_{w=2}^{\infty}\frac{1}{w(w-1)}\kappa_{-1,w}u^w t^{w-1} + \sum_{w=1}^{\infty}\frac{1}{2w}\kappa_{0,w}u^w t^w \\ &+ \sum_{w=0}^{\infty}\sum_{s=1}^{\infty}\binom{2s-2+w}{w}\frac{B_{2s}}{2s(2s-1)}\kappa_{2s-1,w}u^w t^{2s+w-1} \\ &+ \sum_{d=1}^{\infty}\sum_{r=-1}^{\infty}\sum_{w=0}^{\infty}\tilde{C}_{r,d,w}\kappa_{r,w}u^w t^{r+w}\frac{x^d}{d!}\,, \end{aligned} \tag{9}$$

where $\widetilde{C}_{r,w,d}$ are defined by the coefficients of

$$\sum_{d=1}^{\infty}\sum_{r}\sum_{w}\tilde{C}_{r,w,d}t^r u^w\frac{x^d}{d!} := \log\left(1+\sum_{d=1}^{\infty}\prod_{i=1}^{d}\frac{1}{1-it-ut}\frac{(-1)^d}{t^d}\frac{x^d}{d!}\right). \tag{10}$$

The formula (9) arises from the virtual localization formula for $\mathsf{P}_{g,m}(2,B,d)$. When we set $u=0$, the right hand side of (10) appears in the proof of Faber-Zagier relations on $\mathcal{M}_g$ [36].

Theorem 6. *For $g-2d-1<r$ and $g\equiv r+1+w \mod 2$, we have*

$$[\exp(-\widetilde{\gamma})]_{u^w t^r x^d} = 0 \text{ in } \mathsf{R}^r(\mathsf{Pic}^{\underline{0}}(\mathcal{M}_g))\,.$$

Moreover when the parity condition does not hold the expression is trivially zero.

In [6] we obtain a further set of tautological relations by inserting tautological classes on $\mathsf{P}_{g,m}(2,B,d)$. The strategy of Theorem 6 can be used to understand tautological relations on the enter moduli space $\mathsf{Pic}^{\underline{0}}_{g,m}$. The virtual fundamental classes of $\mathsf{P}_{g,m}(r,B,d)$ satisfy the splitting axiom and the unit axiom of the Picard CohFT [35]. Eventually the study of moduli spaces of stable Picard quotients in the point of view of Picard CohFT can be used to generalize Pixton's formula to the relative Picard scheme $\mathsf{Pic}^{\underline{0}}_{g,m}$ over $\overline{\mathcal{M}}_{g,m}$.

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Chapter 2. Scientific papers

Tautological relations for stable maps to a target variety

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Abstract

We define tautological relations for the moduli space of stable maps to a target variety. Using the double ramification cycle formula for target varieties of Janda-Pandharipande-Pixton-Zvonkine [20], we construct non-trivial tautological relations parallel to Pixton's double ramification cycle relations for the moduli of curves. Examples and applications are discussed.

1 Introduction

1.1 Tautological relations on the moduli space of stable curves

Let $\overline{\mathcal{M}}_{g,n}$ be the moduli space space of stable curves of genus g with n marked points. It is a smooth Deligne-Mumford stack of dimension $3g-3+n$ and has both singular and Chow cohomology theories. Mumford [23] initiated the study of the subring of tautological classes

$$R^*(\overline{\mathcal{M}}_{g,n}) \subseteq A^*(\overline{\mathcal{M}}_{g,n})_{\mathbb{Q}}$$

of the Chow ring, which we now call the *tautological ring*—see [11, Section 1] for an introduction and basic definitions. One advantage of considering the tautological ring is that there is a canonical set of additive generators of $R^*(\overline{\mathcal{M}}_{g,n})$ by boundary strata together with basic ψ and κ classes [16]. A fundamental goal is to find all linear relations among these additive generators—such relations are called *tautological relations*. For an excellent survey of this topic, see [29].

Because few tools exist to handle the geometry of moduli spaces of stable curves in higher genus, finding new tautological relations is not easy. A breakthrough appeared in Pixton's note [30], which was inspired by the Faber-Zagier conjecture on tautological relations in $A^*(\mathcal{M}_g)$ proven in [27]. Pixton conjectured a systematic means of writing down tautological relations in terms of graph sums. His conjecture

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was proven in cohomology [28] using the theory of Witten's 3-spin class and in Chow [18] using the equivariant Gromov-Witten theory of $\mathbb{P}^1$.

1.2 Tautological relations on the moduli space of stable maps

Let X be a nonsingular projective variety over $\mathbb{C}$. For an effective curve class $\beta \in H_2(X,\mathbb{Z})$ and nonnegative integers g and n, we consider the moduli space of stable maps $\overline{\mathcal{M}}_{g,n,\beta}(X)$. A closed point of the moduli space corresponds to a morphism from a connected nodal curve C of genus g with n marked points to X,

$$f : (C, x_1, \cdots, x_n) \to X, \quad f_*[C] = \beta\,. \tag{1.1}$$

By the stability condition, the data (1.1) is required to have only finitely many automorphisms—see [14] for a foundational treatment. Unlike $\overline{\mathcal{M}}_{g,n}$, the moduli space of stable maps is typically not smooth and can have many connected components of different dimensions. Nevertheless, $\overline{\mathcal{M}}_{g,n,\beta}(X)$ has a natural perfect obstruction theory of expected dimension

$$\mathrm{vdim} = (1-g)(\dim_{\mathbb{C}} X - 3) + \int_\beta c_1(X) + n\,,$$

which allows us to define virtual fundamental classes [5] and Gromov-Witten invariants [4] for the target variety X .

The first question that we pursue here is whether a ring of tautological classes can be defined in the rational Chow theory of $\overline{\mathcal{M}}_{g,n,\beta}(X)$. In Section 2, we define tautological classes via fundamental classes and show that their span in the Chow group carries a natural product structure. The issue of the multiplication of tautological classes is delicate because we are considering cycles on a singular space $\overline{\mathcal{M}}_{g,n,\beta}(X)$. Nevertheless, this product structure can be obtained by realizing tautolgical classes as Fulton's operational Chow classes [13, Chapter 17]. In genus 0, tautological classes were studied earlier by Oprea [25] in cases where the moduli space of stable maps $\overline{\mathcal{M}}_{0,n,\beta}(X)$ is smooth. Our definition agrees with [25] in the cases he considers.

Classical examples of tautological relations on $\overline{\mathcal{M}}_{g,n,\beta}(X)$ come from tautological relations on $\overline{\mathcal{M}}_{g,n}$ by pulling-back relations via the stabilization morphism

$$st : \overline{\mathcal{M}}_{g,n,\beta}(X) \to \overline{\mathcal{M}}_{g,n}\,. \tag{1.2}$$

Formulas for the pull-back are presented in Section 2.2.

Our study is motivated by the following two basic questions concerning the structure of tautological relations on the moduli space of stable maps:

1. Can we use the geometry of X to find tautological relations on $\overline{\mathcal{M}}_{g,n,\beta}(X)$ that are not obtained via pull-back from Pixton's set of tautological relations on $\overline{\mathcal{M}}_{g,n}$?

2. Can we use the tautological relations on $\overline{\mathcal{M}}_{g,n,\beta}(X)$ to determine new relations among Gromov-Witten invariants?

For question (1), the answers should be in the form of general constructions which are valid for all target variety[1]. As an example, in [22], the authors obtained tautological relations in the Picard group of $\overline{\mathcal{M}}_{0,n,d}(\mathbb{P}^m)$ to prove a reconstruction theorem for genus 0 quantum cohomology and quantum K-theory for

$$X \subseteq \mathbb{P}^m .$$

Our main result here is a general construction of tautological relations for $\overline{\mathcal{M}}_{g,n,\beta}(X)$ using the new double ramification cycle formula for target varieties of Janda-Pandharipande-Pixton-Zvonkine [20]. Because the construction essentially involves the geometry of X, the relations are not expected to be pull-backs. Examples and applications are provided in Section 4.

When X is a point, the relations constructed here specialize to the double ramification cycle relations for $\overline{\mathcal{M}}_{g,n}$ conjectured by Pixton [31] and proven by Clader and Janda in [8]. In fact, our proof follows the strategy of [8].

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2 X-valued Tautological Ring

2.1 X-valued stable graphs

Let X be a nonsingular projective variety over $\mathbb{C}$ and let $C(X)$ be the semigroup of effective curve classes in $H_2(X, \mathbb{Z})$. For each element β in $C(X)$, an Artin stack $\mathfrak{M}_{g,n,\beta}$ exists that parametrizes genus g, n marked prestable curves C together with

[1] For instance, if X is a K3 surface, the virtual fundamental class of the moduli space of stable maps is zero if β is nonzero. And it is not likely to be true that these relations come from tautological relations on $\overline{\mathcal{M}}_{g,n}$. In this note we will *not* consider such cases.

a labeling on each irreducible component of C by an element of $C(X)$. The labeling must satisfy the stability condition and the degrees must sum to β, see [10, Section 2] for details. This space is called the *moduli space of prestable curves with* $C(X)$-*structure*. It is a smooth Artin stack and is étale over the moduli space of marked prestable curves $\mathfrak{M}_{g,n}$.

Let us review the notion of X-valued stable graphs following [20].

Definition 2.1. An *X-valued stable graph* $\Gamma \in \mathcal{S}_{g,n,\beta}(X)$ consists of the data

$$\Gamma = (V,\, H,\, L,\, g : V \to \mathbb{Z}_{\geq 0},\, v : H \to V,\, \iota : H \to H,\, \beta : V \to C(X))$$

satisfying the properties:

(i) V is a vertex set with a genus function $g : V \to \mathbb{Z}_{\geq 0}$,

(ii) H is a half-edge set equipped with a vertex assignment $v : H \to V$ and an involution ι,

(iii) (V, H, ι) defines a connected graph satisfying the genus condition

$$\sum_{v \in V} g(v) + h^1(\Gamma) = g\,,$$

(iv) for each vertex $v \in V$, the stability condition holds: if $\beta(v) = 0$, then

$$2g(v) - 2 + n(v) > 0\,,$$

where $n(v)$ is the valence of Γ at v,

(v) the degree condition holds:

$$\sum_{v \in V} \beta(v) = \beta\,.$$

For a given graph Γ, denote E as the set of 2-cycles of ι corresponding to edges, and L as the set of fixed points of ι corresponding to the set of n markings. An automorphism of $\Gamma \in \mathcal{S}_{g,n,\beta}(X)$ consists of automorphisms of the sets V and H, which leaves invariant the structures L, g, v, ι, and β. Let $\mathrm{Aut}(\Gamma)$ denote the automorphism group of Γ.

For each X-valued stable graph Γ, consider a moduli space $\overline{\mathcal{M}}_\Gamma$ of stable maps of the prescribed degeneration together with the canonical morphism [20, Section 0.2]

$$j_\Gamma : \overline{\mathcal{M}}_\Gamma \to \overline{\mathcal{M}}_{g,n,\beta}(X)\,.$$

To construct j_Γ, a universal family of stable curves over $\overline{\mathcal{M}}_\Gamma$ is required [5]. The forgetful morphism from $\overline{\mathcal{M}}_\Gamma$ to $\mathfrak{M}_\Gamma$ carries a relative perfect obstruction theory. The universal curve

$$\pi : \mathcal{C}_{g,n,\beta}(X) \to \overline{\mathcal{M}}_{g,n,\beta}(X) \tag{2.1}$$

has n sections s_i and a universal evaluation morphism

$$f : \mathcal{C}_{g,n,\beta}(X) \to X\,. \tag{2.2}$$

Next, the notion of the strata algebra can be extended to X-valued stable graphs.

Definition 2.2. A *decorated X-valued stable graph* $[\Gamma, \gamma]$ is an X-valued stable graph $\Gamma \in \mathcal{S}_{g,n,\beta}(X)$ together with the following data γ:

(i) each leg $i \in L$ is decorated with $ev_i^*\alpha_i$, $\alpha_i \in A^*(X)\,,$

(ii) each half-edge $h \in H$ is decorated with a monomial $\psi_h^{y[h]}$ for some $y[h] \in \mathbb{Z}_{\geq 0}\,,$

(iii) each edge $e \in E$ is decorated with a monomial $ev_e^*\alpha_e$, $\alpha_e \in A^*(X)$,

(iv) each vertex $v \in V$ is decorated with a product of *twisted κ classes*

$$\kappa_{a_1,\cdots,a_m}(\alpha_1,\cdots,\alpha_m) = \pi_{m*}(\psi_{n+1}^{a_1+1} ev_{n+1}^*\alpha_1 \cdots \psi_{n+m}^{a_m+1} ev_{n+m}^*\alpha_n)$$

where $\pi_m : \overline{\mathcal{M}}_{g,n+m,\beta}(X) \to \overline{\mathcal{M}}_{g,n,\beta}(X)$ is the morphism forgetting the last m marked points, $\alpha_1, \ldots, \alpha_m \in A^*(X)$ and $a_1, \ldots, a_m \in \mathbb{Z}_{\geq -1}$.

We bound the degree of γ by the virtual dimension of the moduli space

$$\deg(\gamma) \leq \mathrm{vdim}\,\overline{\mathcal{M}}_{g,n,\beta}(X)\,.$$

Consider the $\mathbb{Q}$-vector space $\mathcal{S}_{g,n,\beta}(X)$ whose basis is given by the isomorphism classes of decorated X-valued stable graph $[\Gamma, \gamma]$. We give a product structure on $\mathcal{S}_{g,n,\beta}(X)$ as follows, see [16, 20]. Let

$$[\Gamma_A, \gamma_A], [\Gamma_B, \gamma_B] \in \mathcal{S}_{g,n,\beta}(X)$$

be two elements in $\mathcal{S}_{g,n,\beta}(X)$. The product of two elements is a finite linear combination of decorated X-valued graphs as follows:

1. Consider $\Gamma \in \mathcal{S}_{g,n,\beta}(X)$ with edges colored by A or B so that after contracting the edge not colored as A (resp. B), we obtain Γ_A (resp. Γ_B).

2. For each such Γ, we assign operational Chow classes by the following rules.

   • The operational Chow classes on legs are obtained by multiplying the corresponding leg monomials on γ_A and γ_B.

   • The operational Chow classes on a half edge colored A only or colored B only is determined by γ_A or γ_B respectively. On an edge $e = (h, h')$ colored both A and B, we multiply by

$$-(\psi_h + \psi_{h'}) \tag{2.3}$$

in addition to contributions from γ_A and γ_B.

- The factors of edges coming from γ_A and γ_B descend to Γ.
- The factors on the vertex v in Γ_A (resp. Γ_B) are split in all possible ways among the vertices which collapse to v as Γ is contracted to Γ_A and Γ_B. Then we multiply two vertex contributions.

The above product yields a $\mathbb{Q}$-algebra structure on $\mathcal{S}_{g,n,\beta}(X)$. Push-forward along j_Γ defines a $\mathbb{Q}$-linear map $q : \mathcal{S}_{g,n,\beta}(X) \to A_*(\overline{\mathcal{M}}_{g,n,\beta}(X))$,

$$q([\Gamma,\gamma]) = j_{\Gamma *}\left(\gamma \cap [\overline{\mathcal{M}}_\Gamma]^{vir}\right) \in A_*(\overline{\mathcal{M}}_{g,n,\beta}(X)) \,. \tag{2.4}$$

An element of the kernel of q is called a *tautological relation for the target variety* X. Unlike in [28], there is no intersection product on the Chow group of $A_*(\overline{\mathcal{M}}_{g,n,\beta}(X))$, so there is no obvious reason why the kernel of the map q is an ideal of $\mathcal{S}_{g,n,\beta}(X)$. However, the map q factors through the operational Chow ring of $\overline{\mathcal{M}}_{g,n,\beta}(X)$.

Definition 2.3. Let $\mathfrak{M}$ be an Artin stack over $\mathbb{C}$. For each scheme U and morphism $U \to \mathfrak{M}$, an operational Chow class $\alpha \in A^p(\mathfrak{M})$ is a collection of morphisms

$$\alpha_U : A_*(U) \to A_{*-p}(U)$$

which is compatible with proper push-forward, flat pull-back, locally complete intersection(l.c.i) pull-back, and Chern classes, see [13, Chapter 17.1].

For example, Chern classes of a vector bundle on a scheme are operational Chow classes. Because the pullback morphism is defined for operational Chow groups, operational Chow classes have well-defined product structure.

Lemma 2.4. *The kernel of q is an ideal in $\mathcal{S}_{g,n,\beta}(X)$.*

Proof. From the splitting axiom, the map q factors through the following diagram

$$\begin{tikzcd} \mathcal{S}_{g,n,\beta}(X) \arrow[r, "q_0"] \arrow[dr, "q"'] & A^*(\overline{\mathcal{M}}_{g,n,\beta}(X)) \arrow[d, "\cap[\overline{\mathcal{M}}_{g,n,\beta}(X)]^{vir}"] \\ & A_*(\overline{\mathcal{M}}_{g,n,\beta}(X)) \,. \end{tikzcd} \tag{2.5}$$

The map q_0 assigns $[\Gamma,\gamma]$ to an operational Chow class on $\overline{\mathcal{M}}_{g,n,\beta}(X)$ as follows. For each scheme U and a morphism $u : U \to \overline{\mathcal{M}}_{g,n,\beta}(X)$, consider the following fiber diagram

$$\begin{tikzcd} U' \arrow[r, "j_u"] \arrow[d, "u'"] & U \arrow[d, "u"] \\ \overline{\mathcal{M}}_\Gamma \arrow[r, "j_\Gamma"] \arrow[d] & \overline{\mathcal{M}}_{g,n,\beta}(X) \arrow[d] \\ \mathfrak{M}_\Gamma \arrow[r, "\xi_\Gamma"] & \mathfrak{M}_{g,n,\beta} \end{tikzcd}$$

where $\mathfrak{M}_\Gamma$ is the moduli space of prestable curves with prescribed degeneration defined by $\prod_{v\in V(\Gamma)} \mathfrak{M}_{g(v),n(v),\beta(v)}$ and ξ_Γ is the gluing morphism. The class

$$q_0([\Gamma,\gamma])(s) : A_*(U) \to A_{*-|E|-\deg(\gamma)}(U)$$

assigns $V \in A_*(U)$ to

$$j_{s*}(u'^*(\gamma) \cap \xi_\Gamma^! V)\,.$$

A parallel argument to that given in [16, Appendix] shows that the map q_0 is a ring homomorphism. Therefore the kernel of q is an ideal in $\mathcal{S}_{g,n,\beta}(X)$. $\square$

From the above, the image of the map q has a $\mathbb{Q}$-algebra structure. We call this ring as the *tautological ring of the moduli space of stable maps to* X and write $R^*(\overline{\mathcal{M}}_{g,n,\beta}(X))$.

2.2 Tautological relations from Pixton's 3-spin relations

We briefly review tautological relations on $\overline{\mathcal{M}}_{g,n,\beta}(X)$ coming from tautological relations on the moduli space of stable curves via the stabilization morphism (1.2). We assume $2g-2+n>0$ throughout this section. The following lemma illustrates the pull-back formula under the stabilization morphism.

Lemma 2.5. [2] *Let* $st : \overline{\mathcal{M}}_{g,n,\beta}(X) \to \overline{\mathcal{M}}_{g,n}$ *be the stabilization morphism. The following hold:*

1. *Let* $[\Gamma]$ *be a boundary class in* $R^*(\overline{\mathcal{M}}_{g,n})$. *Then*

$$st^* \frac{1}{|\mathrm{Aut}(\Gamma)|}[\Gamma] = \sum_{\Gamma'} \frac{1}{|\mathrm{Aut}(\Gamma')|}[\Gamma']\,.$$

   The sum is over all X*-valued stable graphs* Γ' *where the graph of* Γ' *is equal to* Γ *but the degree map* $\beta(v)$ *can vary.*

2. $st^*\psi_i = \psi_i - [D_i]$, *where* D_i *is an* X*-valued stable graph with one edge connecting two vertices* v_1 *and* v_2 *with* $g(v_1)=g$ *so that the only leg attached to* v_2 *is the* i*-th leg.*

3. *The pull-back of* κ *classes are tautological.*

Proof. It is sufficient to prove the lemma for the stabilization morphism $\mathfrak{M}_{g,n,\beta} \to \overline{\mathcal{M}}_{g,n}$. Proof of (1) and (2) are well known, see [5].

[2] This lemma was formulated with Johannes Schmitt.

Proof of (3). The pull-back formula follows from the fact that the log canonical sheaf $\omega_{\log}$ does not change under the stabilization for semistable curves. Consider the following diagram

$$\begin{tikzcd} \mathfrak{M}_{g,n+1,\beta} \arrow[r, "st"] \arrow[dr, "\pi_1"'] & \tilde{\mathfrak{C}} \arrow[d, "\pi_2"] \\ & \mathfrak{M}_{g,n,\beta} \end{tikzcd} \tag{2.6}$$

where $\tilde{\mathfrak{C}}$ is the pull-back of the universal curve over $\overline{\mathcal{M}}_{g,n}$ under the stabilization morphism and $st : \mathfrak{M}_{g,n+1,\beta} \to \tilde{\mathfrak{C}}$ is the fiberwise stabilization. Let $\omega_i = \omega_{\pi_i}^{\log}$, $i = 1, 2$, be the log canonical sheaves for each projections. Then,

$$st^* c_1(\omega_2) = c_1(\omega_1) - [D]\,,$$

where D is the sum of stable graphs with one edge connecting two vertex v_1 and v_2 where $g(v_1) = g, g(v_2) = 0$ and the only leg adjacent to v_2 is the $n+1$-th leg. Because the morphism st is birational,

$$st^* \kappa_n = \pi_{1*}(c_1(\omega_1) - D)^{n+1}\,.$$

After expanding the right hand side, we get the pull-back formula. □

Example 2.6. From the above lemma, we get

$$st^* \kappa_1 = \kappa_1 + [D]$$

in $R^1(\overline{\mathcal{M}}_{g,n,\beta}(X))$ where D is the sum of X-valued stable graph with one edge connecting two vertices v_1 and v_2 where $g(v_1) = g, g(v_2) = 0$ and no leg attached to v_2.

We can also define the notion of tautological relations on the stack of prestable curves $\mathfrak{M}_{g,n}$ [2]. It is natural to ask whether there are more relations on $\mathfrak{M}_{g,n}$ because the stabilization morphism factors through $\mathfrak{M}_{g,n}$. Unlike $\overline{\mathcal{M}}_{g,n}$, the stack $\mathfrak{M}_{g,n}$ involves difficulties to find relations. A full description of tautological relations on $\mathfrak{M}_{0,n}$ will appear in [2].

3 Tautological relations from double ramification cycles

3.1 Twisted double ramification relations

Let S be a line bundle over X and let $k \in \mathbb{Z}$ be an integer. A vector $A = (a_1, \cdots, a_n) \in \mathbb{Z}^n$ of *k-twisted double ramification data for a target variety* is defined by the condition

$$\sum_{i=1}^{n} a_i = \int_\beta c_1(S) + k(2g - 2 + n)\,. \tag{3.1}$$

Under this condition, we define k-twisted double ramification relations as follows. For each genus g, n pointed nodal curve $(C, x_1, \cdots, x_n)$, let

$$\omega_{\log} = \omega_C(x_1 + \cdots + x_n)$$

be the log canonical line bundle. In (2.1) and (2.2), we defined the universal curve and the universal evaluation map

$$\begin{array}{ccc} \mathcal{C}_{g,n,\beta}(X) & \xrightarrow{f} & X \\ \big\downarrow{\scriptstyle \pi} & & \\ \overline{\mathcal{M}}_{g,n,\beta}(X)\,. & & \end{array}$$

The following tautological classes are obtained from the above diagram
- $\xi_i = c_1(s_i^* f^* S)\,,$
- $\xi = f^* c_1(S)\,,$
- $\eta_{a,b} = \pi_*(c_1((\omega_{\log}))^a \xi^b)\,,$
- $\eta = \eta_{0,2} = \pi_*(\xi^2)\,.$

The subscript i corresponds to the i-th marked point and a, b are nonnegative integers. To state the double ramification (DR) vanishing formula, we recall the definition of weights for X-valued stable graphs.

Definition 3.1. Let $\Gamma \in \mathcal{S}_{g,n,\beta}(X)$ be an X-valued stable graph. A *k-weighting mod r* of Γ is a function on the set of half edges,

$$w : H(\Gamma) \to \{0, 1, \cdots, r-1\}$$

which satisfies:

(i) $\forall i \in L(\Gamma)$, corresponding to i-th marking,

$$w(i) = a_i \mod r\,,$$

(ii) $\forall e \in E(\Gamma)$, corresponding to half edges h, h',

$$w(h) + w(h') = 0 \mod r\,,$$

(iii) $\forall v \in V(\Gamma)$,

$$\sum_{v(h)=v} w(h) = \int_{\beta(v)} c_1(S) + k(2g(v) - 2 + n(v)) \mod r\,,$$

where the sum is over all half-edges incident to v.

We denote $\mathsf{W}_{\Gamma,r,k}$ be the set of all k-weightings mod r of Γ.

Let A be a vector of k-twisted ramification data for genus g. For each positive integer r, let $\mathsf{P}^{d,r}_{g,A,\beta,k}$ be the degree d component of the class

$$\sum_{\Gamma\in\mathcal{S}_{g,n,\beta}(X)}\sum_{w\in\mathsf{W}_{\Gamma,r,k}}\frac{r^{-h^1(\Gamma)}}{|\mathrm{Aut}(\Gamma)|}j_{\Gamma*}\Bigg[\prod_{i=1}^{n}\exp(\frac{1}{2}a_i^2\psi_i+a_i\xi_i)$$
$$\times\prod_{v}\exp(-\frac{1}{2}\eta(v)-k\eta_{1,1}(v)-\frac{1}{2}k^2\kappa_1(v))\prod_{e=(h,h')}\frac{1-\exp\big(-\frac{w(h)w(h')}{2}(\psi_h+\psi_{h'})\big)}{\psi_h+\psi_{h'}}\Bigg]. \tag{3.2}$$

The argument of [19, Appendix] can be directly applied to X-valued graph sums and the class $\mathsf{P}^{d,r}_{g,n,\beta,k}$ is polynomial in r for sufficiently large r [20, Proposition 1]. Denote $\mathsf{P}^{d}_{g,n,\beta,k}$ to be the constant term. In the next section, we will prove $\mathsf{P}^{d}_{g,n,\beta,k}$ vanishes if $d>g$.

3.2 The vanishing result

In this section, we closely follow Clader-Janda [8] and Janda-Pandharipande-Pixton-Zvonkine [20]. We first outline constructions in [20]. For each positive integer r, we consider the moduli space $\overline{\mathcal{M}}^r_{g,A,\beta}(X,S)$ of stable maps $(f:(C,x_1,\cdots,x_n)\to X)$ from twisted prestable curve[3] C with an r-th root of the line bundle

$$f^*S\otimes\omega_{\log}^{\otimes k}(-\sum_{i=1}^{n}a_ix_i).$$

Let $\mathfrak{M}^{r,L}_{g,n}$ be the stack of prestable twisted curves with a degree 0 line bundle without conditions on the stabilizers and let $\mathfrak{M}^{Z}_{g,n}$ be the stack of prestable curves with a degree 0 line bundle. We have a morphism

$$\overline{\mathcal{M}}_{g,n,\beta}(X)\to\mathfrak{M}^{Z}_{g,n},\ [f]\to(C,f^*S\otimes\omega_{\log}^{\otimes k}(-\sum_{i=1}^{n}a_ix_i)) \tag{3.3}$$

[3]Let μ_r be the group of r-th root of unity. On the étale neighborhood of a node, the orbifold structure of the family of twisted prestable curves

$$(x,y)\mapsto z\,,\ z=xy$$

is given by taking $\mu_r\times\mu_r$ quotient in the domain and μ_r quotient in the target. For the definition of the moduli space of prestable curves, see [20, Section 1.2].

and a fiber diagram

$$\begin{tikzcd}
\overline{\mathcal{M}}^r_{g,A,\beta}(X,S) \arrow[r, "\epsilon"] \arrow[d] & \overline{\mathcal{M}}_{g,n,\beta}(X) \arrow[d] \\
\mathfrak{M}^{r,L}_{g,n} \arrow[r, "\epsilon"] & \mathfrak{M}^{Z}_{g,n}.
\end{tikzcd} \tag{3.4}$$

Here, $\epsilon : \mathfrak{M}^{r,L}_{g,n} \to \mathfrak{M}^{Z}_{g,n}$ is the composition of two morphisms

$$\mathfrak{M}^{r,L}_{g,n} \to \mathfrak{M}^{r,Z,\mathrm{triv}}_{g,n} \to \mathfrak{M}^{Z}_{g,n}$$

where the first morphism maps (C,L) to $(C, Z = L^{\otimes r})$, and the second morphism comes from the r-th root construction on boundary divisors.

Theorem 3.2. *If $d > g$, and $n \geq 1$, then $\mathsf{P}^d_{g,A,\beta,k} = 0 \in A_{\mathrm{vdim}-d}(\overline{\mathcal{M}}_{g,n,\beta}(X))$.*

Proof. We follow four basic steps of the vanishing argument of [8].

Step 1. Reduction step. From the polynomiality of $\mathsf{P}^d_{g,A,\beta,k}$ in A, it is enough to prove the vanishing of (3.2) when exactly one a_i is negative, see [8, Section 5.3]. Assume that $a_1 < 0$ and $a_i \geq 0$ for all $i \geq 2$. Choose a positive number $r > \max\{|a_i|\}$ and take $A' = (a'_1, \cdots, a'_n) = (a_1 + r, a_2, \cdots, a_n)$. The constraint (3.1) is still valid modulo r and the constant term of (3.2) is invariant under the translation. Let $\mathfrak{M}^{Z'}_{g,n}$ be the stack of prestable curves together with a degree $-r$ line bundle, and $\mathfrak{M}^{r,L'}_{g,n}$ be the stack of prestable twisted curves together with a degree -1 line bundle. There is a universal twisted curve

$$\pi : \mathfrak{C}^{r,L'}_{g,n} \to \mathfrak{M}^{r,L'}_{g,n} \tag{3.5}$$

and the universal line bundle $\mathcal{L}_{A'} \to \mathfrak{C}^{r,L'}_{g,n}$.

Step 2. Use the polynomiality of $\mathsf{P}^d_{g,A,\beta,k}$. From the polynomiality of $\mathsf{P}^d_{g,A',\beta,k}$ in k [32], we may assume that k is a negative number. Since X is a projective variety, S can be written as a difference of two ample line bundles S_1 and S_2. Introduce new variables x, y and write $S = S_1^{\otimes x} \otimes S_2^{\otimes y}$. We can view the expression $\mathsf{P}^d_{g,A',\beta,k}(S)$ as a polynomial in $x, y, a'_1, \cdots, a'_{n-1}$, after substituting (3.1). The polynomiality ensures that it suffices to prove vanishing for $x, y \ll 0$.

Step 3. Use stability and the degree analysis of [8, Lemma 4.2]. Consider the morphism (3.5). The degree of the orbifold line bundle $\mathcal{L}_{A'}$ restricted to each fiber on the irreducible component C_v is

$$\frac{1}{r}\left[\int_{\beta_v} c_1(S) + k(2g_v - 2 + n_v) - \sum_{i \dashv v} a'_i\right].$$

If β_v is 0, $2g_v - 2 + n_v > 0$ and the degree of the line bundle is negative. If β_v is nonzero, $2g_v - 2 + n_v$ can be negative. However, we assumed that $x, y \ll 0$ so the degree of the line bundle is also negative. The proof of [8, Lemma 4.2] implies $-R\pi_*\mathcal{L}_{A'}$ is a locally free sheaf of rank g.

Step 4. Use the Grothendieck-Riemann-Roch formula in [20, Section 2]. From the computation in [20, Corollary 11], $r^{-2g+2d+1}\epsilon_* c_d(-R\pi_*\mathcal{L}_{A'})$ is obtained by substituting $r = 0$ into the degree d part of

$$\sum_{\Gamma \in \mathcal{S}_{g,n,\beta}(X)} \sum_{w \in \mathsf{W}_{\Gamma,r,k}} \frac{r^{-h^1(\Gamma)}}{|\mathrm{Aut}(\Gamma)|} j_{\Gamma *}\Bigg[\prod_v \exp(-\frac{1}{2}\eta(v) - k\eta_{1,1}(v) - \frac{1}{2}k^2\kappa_1(v)) \\ \prod_{e=(h,h')} \frac{1 - \exp\big(-\frac{w(h)w(h')}{2}(\psi_h + \psi_{h'})\big)}{\psi_h + \psi_{h'}}\Bigg]. \tag{3.6}$$

Twisting by $\omega_{\log}^{\otimes k}$ produces additional terms in (3.6). The above formula vanishes for all d greater than g. By the pull-back formula in [20, Lemma 4], we obtain the vanishing result on $\overline{\mathcal{M}}_{g,n,\beta}(X)$. □

In [20], the untwisted double ramification cycle formula for the target variety was proven via relative/orbifold Gromov-Witten theory (the cycle $\mathsf{DR}_{g,n,\beta}(X,S)$ was defined in [20, Definition 1]). In fact, a proof of the untwisted double ramification cycle relations was implicitly stated in [20].

Proof for untwisted classes. An alternative proof follows from the localization formula in [20, Section 3]. For $l \geq 1$, consider the class

$$\mathrm{Coeff}_{t^0}[\epsilon_*(t^l[\overline{\mathcal{M}}_{g,A,\beta}(\mathbb{P}(X,S)[r], D_\infty)]^{vir})] \tag{3.7}$$

in $A_*(\overline{\mathcal{M}}_{g,n,\beta}(X))$. From [20, Proposition 9] and the localization formula, (3.7) is a Laurent polynomial in r. The coefficient of r^{1-l} of (3.7), should vanish for all $l \geq 1$. If $l = 1$, we prove Theorem 3.3 and if $l \geq 2$, we get the vanishing result $\mathsf{P}^{g+l-1}_{g,A,\beta} = 0$. □

4 Examples

From the double ramification cycle relations, tautological relations on the moduli space of stable maps to a target X were constructed. Returning back to the questions in the introduction, it is natural to ask whether these relations can be obtained from tautological relations on $\mathfrak{M}_{g,n}$. Consider a system of ideals

$$I_{g,n,\beta} \subset \mathcal{S}_{g,n,\beta}(X)\,, \tag{4.1}$$

which is the smallest system satisfying three conditions:

(i) $I_{g,n,\beta}$ contains pull-back of every tautological relations on $\mathfrak{M}_{g,n}$[4],

[4] Tautological relations on $\mathfrak{M}_{g,n}$ can be defined similarly as tautological relations on $\overline{\mathcal{M}}_{g,n}$. The precise definitions and computations will be studied in [2].

(ii) $I_{g,n,\beta}$ is closed under the map $\mathcal{S}_{g,n+1,\beta}(X) \to \mathcal{S}_{g,n,\beta}(X)$ induced by forgetting the last marked point,

(iii) The system of ideals $\{I_{g,n,\beta}\}$ is closed under the map

$$\prod_{v \in V(\Gamma)} \mathcal{S}_{g_v,n_v,\beta_v}(X) \to \mathcal{S}_{g,n,\beta}(X)$$

induced by any X-valued stable graph Γ.

We say a class $R \in \mathcal{S}_{g,n,\beta}(X)$ *is obtained from moduli spaces of curves* if R is a class in $I_{g,n,\beta}$.

In many cases, it is difficult to prove that a tautological relation in $\overline{\mathcal{M}}_{g,n,\beta}(X)$ does not come from moduli spaces of curves. On the other hand, we will see that some DR relations are obtained from moduli spaces of curves in nontrivial ways.

4.1 Genus 0 case

[5] Let $S \to X$ be a line bundle over a nonsingular projective variety X. For each integer d greater than g, we have the vanishing result $\mathsf{P}^d_{g,A,\beta} = 0$ proven in the previous section. In this section, we extract tautological relations for the space X by using the polynomiality of $\mathsf{P}^d_{g,A,\beta}$. The class $\mathsf{P}^d_{g,A,\beta}$ is polynomial in the ramification data A. So, each coefficients of the polynomial in variables in A should vanish.

We can further separate each relations by considering the degree of the line bundle. The degree constraint

$$\int_\beta c_1(S) - \sum_{i=1}^n a_i = 0$$

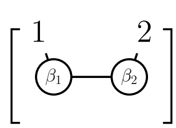
has a scale invariance

$$a_i \mapsto m a_i\,, \quad S \mapsto S^{\otimes m} \tag{4.2}$$

for $m \in \mathbb{Z}$. Considering a family of relations $\mathsf{P}^d_{g,mA,\beta} = 0$ for all integer m, each basic tautological class has a degree with respect to m. Therefore each relation breaks into smaller relations according to the degree m.

Notation 4.1. In the following examples, the symbol

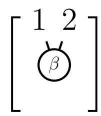
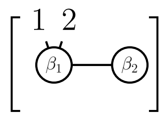
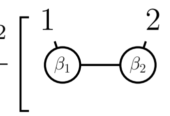
denotes a vertex with genus g with degree β. If the genus of the vertex is 0, we omit 0. We write



[5]The following genus 0 computation was done with Honglu Fan and Longting Wu.

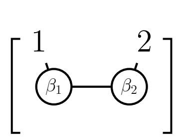
to indicate the sum over all possible stable splittings $\beta_1 + \beta_2 = \beta$. Also, we use the notation $b = \int_\beta c_1(S)$ and $b_i = \int_{\beta(v_i)} c_1(S)$.

Example 4.2. Let $g = 0$, $n = 2$, $d = 1$. We show that all relations can be obtained from the moduli space of prestable curves. There are three prestable graphs with at most one edge. The term $\mathsf{P}^1_{0,A,\beta}$ is equal to

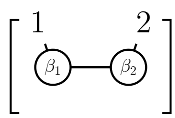
$(-\frac{1}{2}\eta + \frac{1}{2}a_1^2\psi_1 + \frac{1}{2}a_2^2\psi_2 + a_1\xi_1 + a_2\xi_2)$  $- \frac{b_2^2}{2}$  $- \frac{(b_1 - a_1)^2}{2}$ 

After substituting $a_2 = b - a_1$ in $\mathsf{P}^1_{0,A,\beta} = 0$, the following vanishing holds:

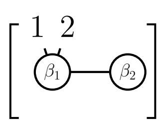
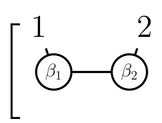
- Coefficient of a_1^2:

$\psi_1 + \psi_2 -$  $= 0\,,$

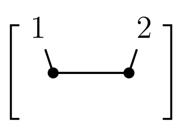
- Coefficient of a_1:

$\xi_1 - \xi_2 - b\psi_2 + b_1$  $= 0\,,$

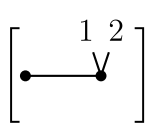
- Coefficient of a_1^0:

$-\eta + b^2\psi_2 + 2b\xi_2 - b_2^2$  $- b_1^2$  $= 0\,.$

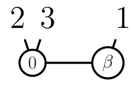
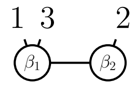
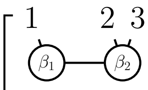
For more markings, relations can be obtained by pull-back via the morphism $\overline{\mathcal{M}}_{0,n,\beta}(X) \to \overline{\mathcal{M}}_{0,2,\beta}(X)$. The first two relations recover Lee-Pandharipande relations [22]. The first relation is independent of the target X. In fact, the relation

$\psi_1 + \psi_2 -$  $= 0$

holds in $A^1(\mathfrak{M}_{0,2})$. The second relation is a consequence of the following relation

$\psi_2 -$  $= 0$

in $A^1(\mathfrak{M}_{0,3})$. After pulling back the relation to $\overline{\mathcal{M}}_{0,3,\beta}(X)$, multiplying with ξ_3 and pushing-forward to $\overline{\mathcal{M}}_{0,2,\beta}$ by forgetting the third marked point, we get the second relation. The third relation is a consequence of the second relation. The pull-back of the second relation to $\overline{\mathcal{M}}_{0,3,\beta}(X)$ is

$\xi_1 - \xi_2 - b\psi_2 + b\Big[$  $\Big] + \sum_{\beta_2 \neq 0} b_1 \Big[$  $\Big] + \sum_{\beta_2 \neq 0} b_1^2 \Big[$  $\Big].$

Take a cup product this relation with ξ_1 and push-forward to $\overline{\mathcal{M}}_{0,2,\beta}(X)$ which forgets the first marking. After relabelling the third marking to the first marking, the relation is exactly the third relation.

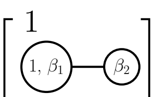
4.2 Genus 1 case

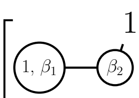
In genus 1 cases, it is useful to replace ψ classes with boundary strata.

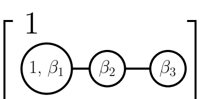
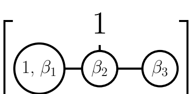
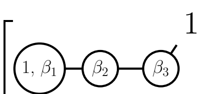
Example 4.3. Let $g = 1, n = 1, d = 2$. We substitute $a_1 = b$ in $\mathsf{P}^2_{1,A,\beta}$:

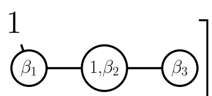
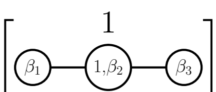
- Coefficient of m^4:

$0 = (\eta^2 - 4b^2\eta\psi_1 - 4b\eta\xi_1 + b^4\psi_1^2 + 4b^2\xi_1^2 + 4b^3\psi_1\xi_1)$

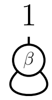
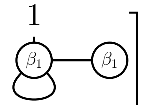
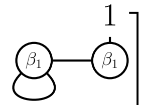
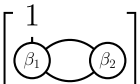
$+ (2b_2^2\eta_1 + 2b_2^2\eta_2 - 2b^2b_2^2\psi_1 - 4bb_2^2\xi_1)\Big[$  $\Big]$

$+ (2b_1^2\eta_1 + 2b_1^2\eta_2 - 2b^2b_1^2\psi_1 - 4bb_1^2\xi_1)\Big[$  $\Big]$

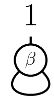
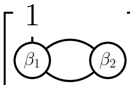
$+ 2(b - b_1)^2 b_3^2 \Big[$  $\Big] + 2b_1^2b_3^2\Big[$  $\Big] + 2b_1^2(b - b_3)^2\Big[$  $\Big]$

$+ 2b_3^2(b - b_2)^2\Big[$  $\Big] + 2b_2^2b_3^2\Big[$  $\Big],$

- Coefficient of m^2:

$(-\eta + 2b\xi_1)\Big[$  $\Big] - b_2^2\Big[$  $\Big] + (b^2 - b_1^2)\Big[$  $\Big] + 2b_2^2\Big[$  $\Big] = 0,$

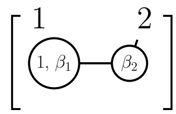
- Coefficient of m^0:

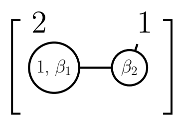
$(\psi_h + \psi_{h'})\Big[$  $\Big] - 2\Big[$  $\Big] = 0.$

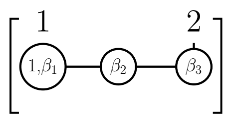
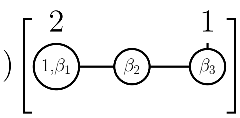
We could not show whether the first relation can be obtained from moduli spaces of curves. On the other hand, the second and the third relation comes from tautological relations on moduli spaces of curves.

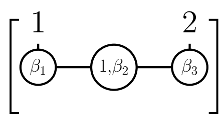
Example 4.4. Let $g = 1, n = 2, d = 2$. There are 26 prestable graphs with at most two edges. The degree of m can be either $0, 2,$ or 4. We prove that the relation which does not contain η classes can be obtained from tautological relations on genus 0 prestable curves. The coefficient of m^4 and a_1^3 is the most complicated example. For simplicity, let δ be the codimension one boundary stratum of $\overline{\mathcal{M}}_{1,2,\beta}(X)$ associated to the one loop X-valued stable graph. After simplification, the relation becomes

$$0 = b(\psi_1^2 - \psi_2^2) + 2(\psi_1 + \psi_2)(\xi_1 - \xi_2)$$

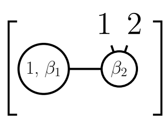
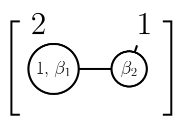
$$+ (-2b_2\psi_1 + 2b_1\psi_2 - 2\xi_1 + 2\xi_2 + (b_1 - b_2)(\psi_h + \psi_{h'}))$$ 

$$+ (-2b_1\psi_1 + 2b_2\psi_2 - 2\xi_1 + 2\xi_2 + (b_2 - b_1)(\psi_h + \psi_{h'}))$$ 

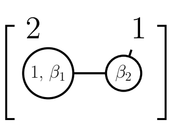
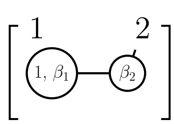
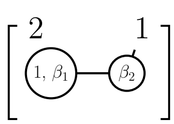
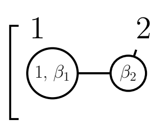
$$+ (-2b_1 + 2b_3)$$  $$+ (2b_1 - 2b_3)$$ 

$$+ (-2b_1 + 2b_3)$$  .

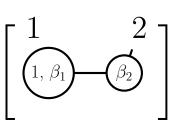
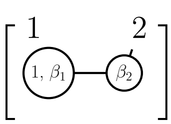
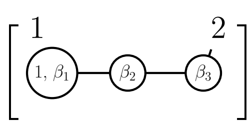
On $\overline{\mathcal{M}}_{1,n,\beta}(X)$, we have

$$\psi_1 = \frac{1}{12}\delta +$$  $+$  , (4.3)

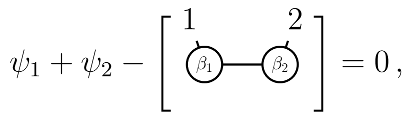
and similarly for ψ_2. The class $b(\psi_1^2 - \psi_2^2)$ is equal to

b  $^2 -$  $^2 + \frac{1}{6}\delta$  $- \frac{1}{6}\delta$  .

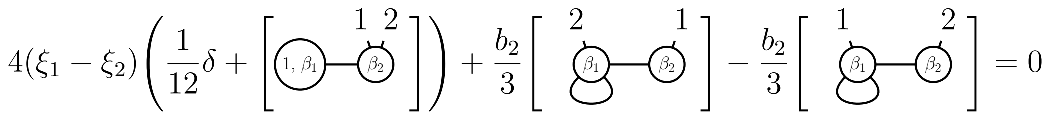
The excess intersection formula gives

 $^2 = -(\psi_h + \psi_{h'})$  $+ 2$  .

Using the $g = 0$, $n = 2$ relation



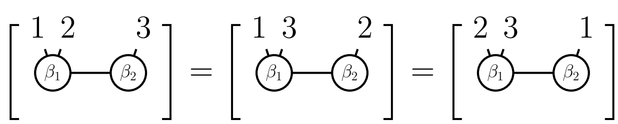
the equation is equivalent to



From the $g = 0$ DR relation, $\xi_1 - \xi_2$ can be written in terms of boundary strata

$$\xi_1 - \xi_2 = b\psi_2 - b_1 D(1|2)$$

where $D(1|2)$ is the sum over all codimension one boundary strata which splits the first and the second marked points. Using the relation



in $(g, n) = (0, 3)$, the left hand side is equal to zero. This computation shows that the original relation is obtained from curves.

4.3 Further directions

From the previous examples, we see that tautological relations are much richer than the relations pulled-back from tautological relations on the moduli spaces of stable curves. Since Theorem 3.3 produces relations uniformly for all target X, it is likely that, for a given X, further relations could also hold. The following questions are some possible directions for further studies.

1. *Lower genus cases.* From lower degree computations, Oprea conjectured that all tautological relations on $\overline{\mathcal{M}}_{0,n,\beta}(X)$ come from tautological relations on the moduli space of genus 0 curves [25]. However, tautological relations on $\mathfrak{M}_{0,n}$ have not been investigated much yet and we do not know if there is a finite number of relations which replace the role of the WDVV equation in $\overline{\mathcal{M}}_{0,n}$. In genus 1 cases, tautological relations are generated by the WDVV relation and the Getzler's relation on $\overline{\mathcal{M}}_{1,4}$ [15]. When the genus is equal to 0 or 1, it is unknown whether there exists a finite number of relations which generate all tautological relations on the moduli space of stable maps to X.

2. *Generalization of the Pixton's 3-spin relations.* In [30, 28], a system of tautological relations on $\overline{\mathcal{M}}_{g,n}$ was obtained from the study of Witten's 3-spin classes. In [18], the study of the equivariant GW theory of $\mathbb{P}^1$ also produces the 3-spin relations. The argument can be applied to produce tautological relations for a target space X as follows. Consider a split vector bundle V over X and its projectivization $\mathbb{P}(V)$. The torus localization formula of the equivariant virtual fundamental class of $\overline{\mathcal{M}}_{g,n,\beta}(\mathbb{P}(V))$ relative to the moduli space of stable maps to X was studied in [6, 9]. Then the pole cancellation technique applied to a variant of Givental's formalism [17, 18] gives variants of Pixton's 3-spin relations on $\overline{\mathcal{M}}_{g,n,\beta}(X)$ twisted by Chern characters of V.

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Pixton's formula and Abel-Jacobi theory on the Picard stack

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Abstract

This is a simplified version of [6] (under the same title).

0 Introduction

0.1 Double ramification cycles

Let $A = (a_1, \ldots, a_n)$ be a vector of n integers satisfying

$$\sum_{i=1}^{n} a_i = 0\,.$$

In the moduli space $\mathcal{M}_{g,n}$ of nonsingular curves of genus g with n marked points, consider the substack defined by the following classical condition:

$$\left\{ (C, p_1, \ldots, p_n) \in \mathcal{M}_{g,n} \,\middle|\, \mathcal{O}_C\Big(\sum_{i=1}^{n} a_i p_i\Big) \simeq \mathcal{O}_C \right\}\,. \tag{1}$$

From the point of view of relative Gromov-Witten theory, the most natural compactification of the substack (1) is the space $\overline{\mathcal{M}}^{\sim}_{g,A}$ of stable maps to *rubber*: stable maps to $\mathbb{CP}^1$ relative to 0 and ∞ modulo the $\mathbb{C}^*$-action on $\mathbb{CP}^1$. The rubber moduli space carries a natural virtual fundamental class $\big[\overline{\mathcal{M}}^{\sim}_{g,A}\big]^{\mathrm{vir}}$ of (complex) dimension $2g - 3 + n$. The pushforward via the canonical morphism

$$\epsilon : \overline{\mathcal{M}}^{\sim}_{g,A} \to \overline{\mathcal{M}}_{g,n}$$

is the *double ramification cycle*

$$\epsilon_* \big[\overline{\mathcal{M}}^{\sim}_{g,A}\big]^{\mathrm{vir}} \;=\; \mathsf{DR}_{g,A} \;\in \mathsf{CH}_{2g-3+n}(\overline{\mathcal{M}}_{g,n})\,.$$

Double ramification cycles have been studied intensively for the past two decades. Examples of early results can be found in [18, 19, 22, 29, 36, 38, 55]. A complete formula was conjectured by Pixton in 2014 and proven in [44]. For subsequent study and applications, see [4, 17, 21, 30, 32, 39, 40, 41, 58, 62, 66, 72, 75, 76]. Essential for our work is the formula for double ramification cycles for target varieties in [45].

We refer the reader to [44, Section 0] and [64, Section 5] for introductions to the subject. For a classical perspective from the point of view of Abel-Jacobi theory, see [39].

0.2 Twisted double ramification cycles

We develop here a theory which extends the study of double ramification cycles from the moduli space of stable curves $\overline{\mathcal{M}}_{g,n}$ to the Picard stack of curves with line bundles $\mathfrak{Pic}_{g,n}$. An object of $\mathfrak{Pic}_{g,n}$ over $\mathcal{S}$ is a flat family

$$\pi : \mathcal{C} \to \mathcal{S}$$

of prestable[1] n-pointed genus g curves together with a line bundle

$$\mathcal{L} \to \mathcal{C} \, .$$

The Picard stack $\mathfrak{Pic}_{g,n}$ is an algebraic (Artin) stack which is locally of finite type, see Section 1.3 for a treatment of foundational issues.

Since the degree of a line bundle is constant in flat families, there is a disjoint union

$$\mathfrak{Pic}_{g,n} = \bigcup_{d \in \mathbb{Z}} \mathfrak{Pic}_{g,n,d} \, ,$$

where $\mathfrak{Pic}_{g,n,d}$ is the Picard stack of curves with degree d line bundles. Let

$$A = (a_1, \ldots, a_n) \, , \qquad \sum_{i=1}^{n} a_i = d$$

be a vector of integers. The first result of the paper is the construction of a *universal twisted double ramification cycle* in the operational Chow theory[2] of $\mathfrak{Pic}_{g,n,d}$,

$$\mathsf{DR}^{\mathsf{op}}_{g,A} \in \mathsf{CH}^g_{\mathsf{op}}(\mathfrak{Pic}_{g,n,d}) \, .$$

The geometric intuition behind the construction is simple. Let

$$\pi : \mathcal{C} \to \mathcal{S} \, , \qquad p_1, \ldots, p_n : \mathcal{S} \to \mathcal{C} \, , \qquad \mathcal{L} \to \mathcal{C}$$

[1] A prestable n-pointed curve is a connected nodal curve with markings at distinct nonsingular points. For the entire paper, we avoid the $(g, n) = (1, 0)$ case because of non-affine stabilizers.

[2] All Chow theories in the paper will be taken with $\mathbb{Q}$-coefficients.

be an object of $\mathfrak{Pic}_{g,n,d}$. The class $\mathsf{DR}^{\mathsf{op}}_{g,A}$ should operate as the locus in the base $\mathcal{S}$ heuristically determined by the condition

$$\mathcal{O}_C\Big(\sum_{i=1}^n a_i p_i\Big) \simeq \mathcal{L}|_C\,.$$

To make the above idea precise, we do *not* use the virtual class of the moduli space of stable maps in Gromov-Witten theory, but rather an alternative approach by partially resolving the classical Abel-Jacobi map. The method follows the path of [39, 41] and may be viewed as a universal Abel-Jacobi construction over the Picard stack. Log geometry based on the stack of tropical divisors constructed in [56] plays a crucial role. Our construction is presented in Section 2.8.

The basic compatibility of our new operational class

$$\mathsf{DR}^{\mathsf{op}}_{g,A} \in \mathsf{CH}^g_{\mathsf{op}}(\mathfrak{Pic}_{g,n,d})$$

with the standard double ramification cycle is as follows. Let

$$A = (a_1, \ldots, a_n)\,, \qquad \sum_{i=1}^n a_i = 0\,,$$

be given. The universal data

$$\pi : \mathcal{C}_{g,n} \to \overline{\mathcal{M}}_{g,n}\,, \qquad \mathcal{O} \to \mathcal{C}_{g,n} \tag{2}$$

determine a map $\varphi_{\mathcal{O}} : \overline{\mathcal{M}}_{g,n} \to \mathfrak{Pic}_{g,n,0}$. The action of $\mathsf{DR}^{\mathsf{op}}_{g,A}$ on the fundamental class of $\overline{\mathcal{M}}_{g,n}$ corresponding to the family (2) then equals the previously defined double ramification cycle

$$\mathsf{DR}^{\mathsf{op}}_{g,A}(\varphi_{\mathcal{O}})\Big([\overline{\mathcal{M}}_{g,n}]\Big) = \mathsf{DR}_{g,A} \in \mathsf{CH}_{2g-3+n}(\overline{\mathcal{M}}_{g,n})\,.$$

More generally, for a vector $A = (a_1, \ldots, a_n)$ of integers satisfying

$$\sum_{i=1}^n a_i = k(2g-2)\,,$$

canonically twisted double ramification cycles,

$$\mathsf{DR}_{g,A,\omega^k} \in \mathsf{CH}_{2g-3+n}(\overline{\mathcal{M}}_{g,n})\,,$$

related to the classical loci

$$\left\{ (C, p_1, \ldots, p_n) \in \mathcal{M}_{g,n} \,\middle|\, \mathcal{O}_C\Big(\sum_{i=1}^n a_i p_i\Big) \simeq \omega_C^k \right\},$$

have been constructed in [37] for $k = 1$ and in [39, 40, 56] for all $k \geq 1$. The universal data

$$\pi : \mathcal{C}_{g,n} \to \overline{\mathcal{M}}_{g,n}\,, \qquad \omega_\pi^k \to \mathcal{C}_{g,n} \tag{3}$$

determine a map $\varphi_{\omega_\pi^k} : \overline{\mathcal{M}}_{g,n} \to \mathfrak{Pic}_{g,n,k(2g-2)}$. Here, ω_π is the relative dualizing sheaf of the morphism π.

The action of $\mathsf{DR}^{\mathsf{op}}_{g,A}$ on the fundamental class of $\overline{\mathcal{M}}_{g,n}$ corresponding to the family (3) is compatible with the constructions of [37, 39, 40, 56],

$$\mathsf{DR}^{\mathsf{op}}_{g,A}(\varphi_{\omega_\pi^k})\Big([\overline{\mathcal{M}}_{g,n}]\Big) = \mathsf{DR}_{g,A,\omega^k} \in \mathsf{CH}_{2g-3+n}(\overline{\mathcal{M}}_{g,n})$$

for all $k \geq 1$.

The above compatibilities of $\mathsf{DR}^{\mathsf{op}}_{g,A}$ with the standard and canonically twisted double ramification cycles are proven in Section 2.7.

Theorem 1. *Let $g \geq 0$ and $d \in \mathbb{Z}$. Let $A = (a_1, \ldots, a_n)$ be a vector of integers satisfying*

$$\sum_{i=1}^n a_i = d\,.$$

Logarithmic compactification of the Abel-Jacobi map yields a universal twisted double ramification cycle

$$\mathsf{DR}^{\mathsf{op}}_{g,A} \in \mathsf{CH}^g_{\mathsf{op}}(\mathfrak{Pic}_{g,n,d})$$

which is compatible with the standard double ramification cycle

$$\mathsf{DR}_{g,A,\omega^k} \in \mathsf{CH}_{2g-3+n}(\overline{\mathcal{M}}_{g,n})$$

in case $d = k(2g-2)$ for $k \geq 0$.

0.3 Pixton's formula

0.3.1 Prestable graphs

We define the set $\mathsf{G}_{g,n}$ of *prestable graphs* as follows. A prestable graph $\Gamma \in \mathsf{G}_{g,n}$ consists of the data

$$\Gamma = (\mathrm{V}\,,\ \mathrm{H}\,,\ \mathrm{L}\,,\ \mathrm{g} : \mathrm{V} \to \mathbb{Z}_{\geq 0}\,,\ v : \mathrm{H} \to \mathrm{V}\,,\ \iota : \mathrm{H} \to \mathrm{H})$$

satisfying the properties:

(i) V is a vertex set with a genus function $\mathrm{g} : \mathrm{V} \to \mathbb{Z}_{\geq 0}$,

(ii) H is a half-edge set equipped with a vertex assignment $v : \mathrm{H} \to \mathrm{V}$ and an involution ι,

(iii) E, the edge set, is defined by the 2-cycles of ι in H (self-edges at vertices are permitted),

(iv) L, the set of legs, is defined by the fixed points of ι and is placed in bijective correspondence with a set of n markings,

(v) the pair (V, E) defines a *connected* graph satisfying the genus condition

$$\sum_{v \in \mathrm{V}} \mathrm{g}(v) + h^1(\Gamma) = g\,.$$

To emphasize Γ, the notation $\mathrm{V}(\Gamma)$, $\mathrm{H}(\Gamma)$, $\mathrm{L}(\Gamma)$, and $\mathrm{E}(\Gamma)$ will also be used for the vertex, half-edges, legs, and edges of Γ.

An isomorphism between $\Gamma, \Gamma' \in \mathsf{G}_{g,n}$ consists of bijections $\mathrm{V} \to \mathrm{V}'$ and $\mathrm{H} \to \mathrm{H}'$ respecting the structures L, g, v, and ι. Let $\mathrm{Aut}(\Gamma)$ denote the automorphism group of Γ.

While the set of isomorphism classes of prestable graphs is infinite, the set of isomorphism classes of prestable graphs with prescribed bounds on the number of edges is finite.

Let $\mathfrak{M}_{g,n}$ be the algebraic (Artin) stack of prestable curves of genus g with n marked points. A prestable graph Γ determines an algebraic stack $\mathfrak{M}_\Gamma$ of curves with degenerations forced by the graph,

$$\mathfrak{M}_\Gamma = \prod_{v \in \mathrm{V}} \mathfrak{M}_{\mathrm{g}(v),\mathrm{n}(v)}$$

together with a canonical[3] map

$$j_\Gamma : \mathfrak{M}_\Gamma \to \mathfrak{M}_{g,n}\,.$$

Since $\mathfrak{M}_{g,n}$ is smooth and the morphism j_Γ is proper, representable, and lci, we obtain an operational Chow class on the algebraic stack of curves,

$$j_{\Gamma*}[\mathfrak{M}_\Gamma] \in \mathsf{CH}_{\mathsf{op}}^{|\mathrm{E}(\Gamma)|}(\mathfrak{M}_{g,n})\,.$$

Via the morphism of algebraic stacks,

$$\epsilon : \mathfrak{Pic}_{g,n,d} \to \mathfrak{M}_{g,n}\,,$$

$j_{\Gamma*}[\mathfrak{M}_\Gamma]$ also defines an operational Chow class on the Picard stack,

$$\epsilon^* j_{\Gamma*}[\mathfrak{M}_\Gamma] \in \mathsf{CH}_{\mathsf{op}}^{|\mathrm{E}(\Gamma)|}(\mathfrak{Pic}_{g,n,d})\,.$$

[3]To define the map, we choose an ordering on the half-edges at each vertex.

0.3.2 Prestable graphs with degrees

We will require a refinement of the prestable graphs of Section 0.3.1 which includes degrees of line bundles.

We define the set $\mathsf{G}_{g,n,d}$ of *prestable graphs of degree* d as follows:

$$\Gamma_\delta = (\Gamma, \delta) \in \mathsf{G}_{g,n,d}$$

consists of the data

- a prestable graph $\Gamma \in \mathsf{G}_{g,n}$,
- a function $\delta : \mathrm{V} \to \mathbb{Z}$ satisfying the degree condition

$$\sum_{v \in V} \delta(v) = d\,.$$

The function δ is often called the *multidegree*.

An automorphism of $\Gamma_\delta \in \mathsf{G}_{g,n,d}$ consists of an automorphism of Γ leaving δ invariant. Let $\mathrm{Aut}(\Gamma_\delta)$ denote the automorphism group of Γ_δ.

For $\Gamma_\delta \in \mathsf{G}_{g,n,k}$, let $\mathfrak{M}_\Gamma$ be the algebraic stack of curves defined in Section 0.3.1 with respect to the underlying prestable graph Γ. Let $\mathfrak{Pic}_{\Gamma_\delta}$ be the Picard stack,

$$\epsilon : \mathfrak{Pic}_{\Gamma_\delta} \to \mathfrak{M}_\Gamma\,,$$

parameterizing curves with degenerations forced by Γ and with line bundles which have degree $\delta(v)$ restriction to the components corresponding to the vertex $v \in \mathrm{V}$. We have a canonical map

$$j_{\Gamma_\delta} : \mathfrak{Pic}_{\Gamma_\delta} \to \mathfrak{Pic}_{g,n,d}\,.$$

Since $\mathfrak{Pic}_{g,n,d}$ is smooth and the morphism j_{Γ_δ} is proper, representable, and lci, we obtain an operational Chow class,

$$j_{\Gamma_\delta *}[\mathfrak{Pic}_{\Gamma_\delta}] \in \mathsf{CH}^{|\mathrm{E}(\Gamma)|}_{\mathsf{op}}(\mathfrak{Pic}_{g,n,d})\,.$$

As operational Chow classes, the following formula holds:

$$\epsilon^* j_{\Gamma *}[\mathfrak{M}_\Gamma] = \sum_{\delta} j_{\Gamma_\delta *}[\mathfrak{Pic}_{\Gamma_\delta}] \ \in \mathsf{CH}^{|\mathrm{E}(\Gamma)|}_{\mathsf{op}}(\mathfrak{Pic}_{g,n,d})\,, \tag{4}$$

where the sum[4] is over all functions $\delta : \mathrm{V} \to \mathbb{Z}$ satisfying the degree condition. Equivalently, we may write (4) as

$$\frac{1}{|\mathrm{Aut}(\Gamma)|}\,\epsilon^* j_{\Gamma *}[\mathfrak{M}_\Gamma] = \sum_{\Gamma_\delta \in \mathsf{G}_{g,n,d}} \frac{1}{|\mathrm{Aut}(\Gamma_\delta)|}\, j_{\Gamma_\delta *}[\mathfrak{Pic}_{\Gamma_\delta}] \ \in \mathsf{CH}^{|\mathrm{E}(\Gamma)|}_{\mathsf{op}}(\mathfrak{Pic}_{g,n,d})\,,$$

where the sum on the right side is now over all isomorphism classes of prestable graphs of degree d with underlying prestable graph Γ.

[4]The sum is infinite, but only finitely many terms are nonzero in any operation.

0.3.3 Tautological ψ, ξ, and η classes

The universal curve

$$\pi : \mathfrak{C}_{g,n} \to \mathfrak{Pic}_{g,n}$$

carries two natural line bundles: the relative dualizing sheaf ω_π and the universal line bundle

$$\mathfrak{L} \to \mathfrak{C}_{g,n} \, .$$

Let p_i be the ith section of the universal curve, let

$$\mathfrak{S}_i \subset \mathfrak{C}_{g,n}$$

be the corresponding divisor, and let

$$\omega_{\log} = \omega_\pi\Big(\sum_{i=1}^{n} \mathfrak{S}_i\Big)$$

be the relative log-canonical line bundle with first Chern class $c_1(\omega_{\log})$. Let

$$\xi = c_1(\mathfrak{L})$$

be the first Chern class of $\mathfrak{L}$.

Definition 2. The following operational classes on $\mathfrak{Pic}_{g,n}$ are obtained from the universal structures:

- $\psi_i = c_1(p_i^* \omega_\pi) \in \mathsf{CH}^1_{\mathsf{op}}(\mathfrak{Pic}_{g,n})$,
- $\xi_i = c_1(p_i^* \mathfrak{L}) \in \mathsf{CH}^1_{\mathsf{op}}(\mathfrak{Pic}_{g,n})$,
- $\eta_{a,b} = \pi_*\left(c_1(\omega_{\log})^a \xi^b\right) \in \mathsf{CH}^{a+b-1}_{\mathsf{op}}(\mathfrak{Pic}_{g,n})$.

For simplicity in the formulas, we will use the notation

$$\eta = \eta_{0,2} = \pi_*(\xi^2) \, .$$

The standard κ classes are defined by the π pushforwards of powers of $c_1(\omega_{\log})$,

$$\eta_{a,0} = \kappa_{a-1} \, .$$

Definition 3. A *decorated prestable graph* $[\Gamma_\delta, \gamma]$ *of degree* d is a prestable graph $\Gamma_\delta \in \mathsf{G}_{g,n,d}$ of degree d together with the following decoration data γ:

- each leg $i \in \mathrm{L}$ is decorated with a monomial $\psi_i^a \xi_i^b$,

- each half-edge $h \in \mathrm{H} \setminus \mathrm{L}$ is decorated with a monomial ψ_h^a,
- each edge $e \in \mathrm{E}$ is decorated with a monomial ξ_e^a,
- each vertex in V is decorated with a monomial in the variables $\{\eta_{a,b}\}_{a+b\geq 2}$.

In all four cases, the monomial may be be trivial.

Let $\mathsf{DG}_{g,n,d}$ be the set of decorated prestable graphs of degree d. To each decorated graph of degree d,

$$[\Gamma_\delta, \gamma] \in \mathsf{DG}_{g,n,d}\,,$$

we assign the operational class

$$j_{\Gamma_\delta *}[\gamma] \in \mathsf{CH}^*_{\mathsf{op}}(\mathfrak{Pic}_{g,n,d})$$

obtained via the proper representable morphism

$$j_{\Gamma_\delta} : \mathfrak{Pic}_{\Gamma_\delta} \to \mathfrak{Pic}_{g,n,d}$$

and the action of the decorations.

The action of decorations is described as follows. Given $\Gamma_\delta \in \mathsf{G}_{g,n,k}$, the stack $\mathfrak{Pic}_{\Gamma_\delta}$ admits a morphism[5]

$$\mathfrak{Pic}_{\Gamma_\delta} \to \prod_{v \in \mathrm{V}(\Gamma_\delta)} \mathfrak{Pic}_{\mathrm{g}(v),\mathrm{n}(v),\delta(v)}$$

which sends a line bundle $\mathcal{L}$ on a prestable curve $\mathcal{C}$ to its restrictions on the various components,

$$\mathcal{L}|_{\mathcal{C}_v}\,, \quad v \in \mathrm{V}(\Gamma_\delta)\,.$$

For $v \in \mathrm{V}(\Gamma_\delta)$, we define the operational class $\eta(v)$ on $\mathfrak{Pic}_{\Gamma_\delta}$ as the pullback of the operational class η on the factor $\mathfrak{Pic}_{\mathrm{g}(v),\mathrm{n}(v),\delta(v)}$ above. The operational classes ψ at the markings and ξ at the half-edges are defined similarly.

Definition 4. The *tautological classes* in $\mathsf{CH}^*_{\mathsf{op}}(\mathfrak{Pic}_{g,n,d})$ consist of the $\mathbb{Q}$-linear span of the operational classes associated to all $[\Gamma_\delta, \gamma] \in \mathsf{DG}_{g,n,d}$.

By standard analysis [33], the tautological classes have a natural ring structure. Our formula for $\mathsf{DR}^{\mathsf{op}}_{g,A}$ will be a sum of operational classes determined by decorated prestable graphs of degree $d = \sum_{i=1}^n a_i$ (and hence will be tautological).

[5]The fibers of the map are torsors under the group $\mathbb{G}_m^{h^1(\Gamma)}$.

0.3.4 Weightings mod r

Let $\Gamma_\delta \in \mathsf{G}_{g,n,d}$ be a prestable graph of degree d, and let r be a positive integer.

Definition 5. A *weighting mod* r of Γ_δ is a function on the set of half-edges,

$$w : \mathrm{H}(\Gamma_\delta) \to \{0, 1, \ldots, r-1\},$$

which satisfies the following three properties:

(i) $\forall i \in \mathrm{L}(\Gamma_\delta)$, corresponding to the marking $i \in \{1, \ldots, n\}$,

$$w(i) = a_i \mod r,$$

(ii) $\forall e \in \mathrm{E}(\Gamma_\delta)$, corresponding to two half-edges $h, h' \in \mathrm{H}(\Gamma_\delta)$,

$$w(h) + w(h') = 0 \mod r,$$

(iii) $\forall v \in \mathrm{V}(\Gamma_\delta)$,

$$\sum_{v(h)=v} w(h) = \delta(v) \mod r,$$

where the sum is taken over *all* $\mathsf{n}(v)$ half-edges incident to v.

We denote by $\mathsf{W}_{\Gamma_\delta,r}$ the finite set of all possible weightings mod r of Γ_δ. The set $\mathsf{W}_{\Gamma_\delta,r}$ has cardinality $r^{h^1(\Gamma_\delta)}$. We view r as a *regularization parameter*.

0.3.5 Calculation of the twisted double ramification cycle

We denote by $\mathsf{P}^{c,r}_{g,A,d} \in \mathsf{CH}^c_{\mathsf{op}}(\mathfrak{Pic}_{g,n,d})$ the codimension[6] c component of the tautological operational class

$$\sum_{\substack{\Gamma_\delta \in \mathsf{G}_{g,n,d}\\ w \in \mathsf{W}_{\Gamma_\delta,r}}} \frac{r^{-h^1(\Gamma_\delta)}}{|\mathrm{Aut}(\Gamma_\delta)|}\, j_{\Gamma_\delta *}\Bigg[\prod_{i=1}^{n} \exp\left(\frac{1}{2}a_i^2\psi_i + a_i\xi_i\right) \prod_{v \in \mathrm{V}(\Gamma_\delta)} \exp\left(-\frac{1}{2}\eta(v)\right)$$
$$\prod_{e=(h,h') \in \mathrm{E}(\Gamma_\delta)} \frac{1 - \exp\left(-\frac{w(h)w(h')}{2}(\psi_h + \psi_{h'})\right)}{\psi_h + \psi_{h'}}\Bigg].$$

Several remarks about the formula are required:

[6] Codimension here is usually called degree. But since we already have line bundle degrees, we use the term codimension for clarity.

(i) The sum is over all isomorphism classes of prestable graphs of degree d in the set $\mathsf{G}_{g,n,d}$. Only finitely many underlying prestable graphs $\Gamma \in \mathsf{G}_{g,n}$ can contribute in fixed codimension c. However, for each such prestable graph, the above formula has infinitely many summands corresponding to the infinitely many functions

$$\delta : \mathrm{V} \to \mathbb{Z}$$

which satisfy the degree condition. The operational Chow class $\mathsf{P}_{g,A,d}^{c,r}$ is nevertheless well-defined since only finitely many summands have nonvanishing operation on any given family of curves carrying a degree d line bundle over a base $\mathcal{S}$ of finite type.

(ii) Once the prestable graph Γ_δ is chosen, we sum over all $r^{h^1(\Gamma_\delta)}$ different weightings $w \in \mathsf{W}_{\Gamma_\delta,r}$.

(iii) Inside the pushforward in the above formula, the first product

$$\prod_{i=1}^{n} \exp\left(\frac{1}{2}a_i^2\psi_{h_i} + a_i\xi_{h_i}\right)$$

is over $h \in \mathrm{L}(\Gamma)$ via the correspondence of legs and markings.

(iv) The class $\eta(v)$ is the $\eta_{0,2}$ class of Definition 2 associated to the vertex.

(v) The third product is over all $e \in \mathrm{E}(\Gamma_\delta)$. The factor

$$\frac{1 - \exp\left(-\frac{w(h)w(h')}{2}(\psi_h + \psi_{h'})\right)}{\psi_h + \psi_{h'}}$$

is well-defined since

- the denominator formally divides the numerator,
- the factor is symmetric in h and h'.

No edge orientation is necessary.

The following fundamental polynomiality property of $\mathsf{P}_{g,A,d}^{c,r}$ is parallel to Pixton's polynomiality in [44, Appendix] and is a consequence of [44, Proposition 3″].

Proposition 6. *For fixed g, A, d, and c and a decorated graph $[\Gamma_\delta, \gamma]$ of degree d, the coefficient of $j_{\Gamma_\delta *}[\gamma]$ in the tautological class*

$$\mathsf{P}_{g,A,d}^{c,r} \in \mathsf{CH}_{\mathsf{op}}^{c}(\mathfrak{Pic}_{g,n,d})$$

is polynomial in r (for all sufficiently large r).

We denote by $\mathsf{P}^c_{g,A,d}$ the value at $r = 0$ of the polynomial associated to $\mathsf{P}^{c,r}_{g,A,d}$ by Proposition 6. In other words, $\mathsf{P}^c_{g,A,d}$ is the *constant* term of the associated polynomial in r.

The main result of the paper is a formula for the universal twisted double ramification cycle in operational Chow.[7]

Theorem 7. *Let $g \geq 0$ and $d \in \mathbb{Z}$. Let $A = (a_1, \ldots, a_n)$ be a vector of integers with $\sum_{i=1}^n a_i = d$. The universal twisted double ramification cycle is calculated by Pixton's formula:*

$$\mathsf{DR}^{\mathsf{op}}_{g,A} = \mathsf{P}^g_{g,A,d} \ \in \mathsf{CH}^g_{\mathsf{op}}(\mathfrak{Pic}_{g,n,d}) \,.$$

Theorem 7 is the most fundamental formulation of the relationship between Abel-Jacobi theory and Pixton's formula that we know. Certainly, Theorem 7 implies the double ramification cycle and X-valued double ramification cycle results of [44, 45] . But since we will use [45] in the proof of Theorem 7, we provide no new approach to these older results. However, the additional depth of Theorem 7 immediately allows new applications.

0.4 Vanishing

From his original double ramification cycle formula, Pixton conjectured an associated vanishing property in the tautological ring of the moduli space of curves which was proven by Clader and Janda [22]. The parallel vanishing statement in the tautological ring of the moduli space of stable maps to X was proven in [4]. The most general vanishing statement is the following result.

Theorem 8. *Let $g \geq 0$ and $d \in \mathbb{Z}$. Let $A = (a_1, \ldots, a_n)$ be a vector of integers with $\sum_{i=1}^n a_i = d$. Pixton's vanishing holds in operational Chow:*

$$\mathsf{P}^c_{g,A,d} \ = 0 \ \in \mathsf{CH}^c_{\mathsf{op}}(\mathfrak{Pic}_{g,n,d}) \quad \textit{for all} \ \ c > g \,.$$

Theorem 8 may be viewed as providing relations among tautological classes in the operational Chow ring of the Picard stack – a new direction of study with many open questions.[8] While Theorem 8 implies the vanishings of [4, 22], we will use these results in our proof.

[7]Our handling of the prefactor 2^{-g} in [44, Theorem 1] differs here. The factors of 2 are now placed in the definition of $\mathsf{P}^{c,r}_{g,A,d}$ as in [45]

[8]See [67, 77] for tautological relations on the Picard variety over the moduli space of smooth curves.

0.5 Twisted holomorphic and meromorphic differentials

0.5.1 Fundamental classes

Let $A = (a_1, \ldots, a_n)$ be a vector of zero and pole multiplicities satisfying

$$\sum_{i=1}^{n} a_i = 2g - 2\,.$$

Let $\mathcal{H}_g(A) \subset \mathcal{M}_{g,n}$ be the quasi-projective locus of pointed curves $(C, p_1, \ldots, p_n)$ satisfying the condition

$$\mathcal{O}_C\Big(\sum_{i=1}^{n} a_i p_i\Big) \simeq \omega_C\,.$$

In other words, $\mathcal{H}_g(A)$ is the locus of meromorphic differentials[9] with zero and pole multiplicities prescribed by A. In [32], a compact moduli space of twisted canonical divisors

$$\widetilde{\mathcal{H}}_g(A) \subset \overline{\mathcal{M}}_{g,n}$$

is constructed which contains $\mathcal{H}_g(A)$ as an open set.

In the strictly meromorphic case, where A contains at least one strictly negative part, $\widetilde{\mathcal{H}}_g(A)$ is of pure codimension g in $\overline{\mathcal{M}}_{g,n}$ by [32, Theorem 3]. A weighted fundamental cycle of $\widetilde{\mathcal{H}}_g(A)$,

$$\mathsf{H}_{g,A} \in \mathsf{CH}_{2g-3+n}(\overline{\mathcal{M}}_{g,n})\,, \tag{5}$$

is constructed in [32, Appendix A] with explicit nontrivial weights on the irreducible components. In the strictly meromorphic case, $\mathcal{H}_g(A) \subset \overline{\mathcal{M}}_{g,n}$ is also of pure codimension g. The closure

$$\overline{\mathcal{H}}_g(A) \subset \overline{\mathcal{M}}_{g,n}$$

contributes to the fundamental class $\mathsf{H}_{g,A}$ with multiplicity 1, but there are additional boundary contributions, see [32, Appendix A].

The universal family over the moduli space of stable curves together with the relative dualizing sheaf,

$$\pi : \mathcal{C}_{g,n} \to \overline{\mathcal{M}}_{g,n}\,, \qquad \omega_\pi \to \mathcal{C}_{g,n}\,, \tag{6}$$

determine an object of $\mathfrak{Pic}_{g,n,2g-2}$. By [42] and the compatibility of Theorem 1, the action of $\mathsf{DR}^{\mathsf{op}}_{g,A}$ on the fundamental class of $\overline{\mathcal{M}}_{g,n}$ equals the weighted fundamental class of $\widetilde{\mathcal{H}}_g(A)$,

$$\mathsf{DR}^{\mathsf{op}}_{g,A}(\varphi_\omega)\Big([\overline{\mathcal{M}}_{g,n}]\Big) = \mathsf{H}_{g,A} \in \mathsf{CH}_{2g-3+n}(\overline{\mathcal{M}}_{g,n})\,.$$

We can now apply Theorem 7 to prove the following result.

[9]If all the parts of A are non-negative, then $\mathcal{H}_g(A)$ is the locus of holomorphic differentials.

Theorem 9. *In the strictly meromorphic case,*

$$\mathsf{H}_{g,A} = \mathsf{P}^g_{g,A,2g-2}[\overline{\mathcal{M}}_{g,n}]$$

for the universal family

$$\pi : \mathcal{C}_{g,n} \to \overline{\mathcal{M}}_{g,n}\,, \qquad \omega_\pi \to \mathcal{C}_{g,n}\,.$$

Theorem 9 is exactly equivalent to Conjecture A of [32, Appendix A]. Since both the moduli space $\widetilde{\mathcal{H}}_g(A)$ and the weighted fundamental cycle $\mathsf{H}_{g,A}$ have explicit geometric definitions, the result provides a geometric representative of Pixton's cycle class in terms of twisted differentials. Theorem 9 is proven in Section 6.1 where the parallel conjectures [72] for higher differentials are also proven (by parallel arguments).

0.5.2 Closures

Let $A = (a_1, \ldots, a_n)$ be a vector of integers satisfying

$$\sum_{i=1}^{n} a_i = 2g - 2\,.$$

A careful investigation of the closure

$$\mathcal{H}_g(A) \subset \overline{\mathcal{H}}_g(A) \subset \overline{\mathcal{M}}_{g,n}$$

is carried out in [9] in both the holomorphic and meromorphic cases. By a simple procedure presented in [32, Appendix], Theorem 9 determines the cycle classes of the closures

$$[\overline{\mathcal{H}}_g(A)] \in \mathsf{CH}_*(\overline{\mathcal{M}}_{g,n})\,.$$

for A in both the holomorphic and meromorphic cases.

A similar procedure determines the corresponding classes for k-differentials, see [72, Section 3.4] for an explanation. In particular, our results imply that the cycle classes of the closures are tautological[10] for all k (as was previously known only for $k = 1$ due to [70]).

In the case of holomorphic differentials, another approach to the class of the closure $\overline{\mathcal{H}}_g(A) \subset \overline{\mathcal{M}}_{g,n}$ is provided by Conjecture A.1 of the Appendix of [65] via a limit of Witten's r-spin class. A significant first step in the proof of [65, Conjecture A.1] by Chen, Janda, Ruan, and Sauvaget can be found in [20]. Further progress requires a virtual localization analysis for moduli spaces of stable log maps. An approach to Theorem 9 using log stable maps, virtual localization in the log context, and the strategy of [44] also appears possible (once the required moduli spaces and localization formulas are established).

[10] The precise statement is given in Corollary 41 of Section 6.2.

0.6 Invariance properties

The universal twisted double ramification cycle has several basic invariance properties which play an important role in our study.

Recall that an object of $\mathfrak{Pic}_{g,n,d}$ over $\mathcal{S}$ is a flat family of prestable n-pointed genus g curves together with a line bundle of relative degree d,

$$\pi : \mathcal{C} \to \mathcal{S}\,, \quad p_1, \ldots, p_n : \mathcal{S} \to \mathcal{C}\,, \quad \mathcal{L} \to \mathcal{C}\,. \tag{7}$$

Let $\mathsf{DR}^{\mathsf{op}}_{g,A,\mathcal{L}} \in \mathsf{CH}^g_{\mathsf{op}}(\mathcal{S})$ be the twisted double ramification cycle associated to the above family (7) and the vector

$$A = (a_1, \ldots, a_n)\,, \quad d = \sum_{i=1}^{n} a_i\,.$$

Invariance I: Dualizing**.**

A new object of $\mathfrak{Pic}_{g,n,-d}$ over $\mathcal{S}$ is obtained from (7) by dualizing $\mathcal{L}$:

$$\pi : \mathcal{C} \to \mathcal{S}\,, \quad p_1, \ldots, p_n : \mathcal{S} \to \mathcal{C}\,, \quad \mathcal{L}^* \to \mathcal{C}\,. \tag{8}$$

Let $\mathsf{DR}^{\mathsf{op}}_{g,-A,\mathcal{L}^*} \in \mathsf{CH}^g_{\mathsf{op}}(\mathcal{S})$ be the twisted double ramification cycle associated to the new family (8) and the vector $-A = (-a_1, \ldots, -a_n)$. We have the invariance

$$\mathsf{DR}^{\mathsf{op}}_{g,-A,\mathcal{L}^*} = \epsilon^* \mathsf{DR}^{\mathsf{op}}_{g,A,\mathcal{L}}\,,$$

where $\epsilon : \mathfrak{Pic}_{g,n,-d} \to \mathfrak{Pic}_{g,n,d}$ is the natural map obtained via dualizing the line bundle.

Invariance II: Unweighted markings**.**

Assume we have an additional section $p_{n+1} : \mathcal{S} \to \mathcal{C}$ of π which yields an object of $\mathfrak{Pic}_{g,n+1,d}$,

$$\pi : \mathcal{C} \to \mathcal{S}\,, \quad p_1, \ldots, p_n, p_{n+1} : \mathcal{S} \to \mathcal{C}\,, \quad \mathcal{L} \to \mathcal{C}\,. \tag{9}$$

Let $A_0 \in \mathbb{Z}^{n+1}$ be the vector obtained by appending 0 (as the last coefficient) to A. Let $\mathsf{DR}^{\mathsf{op}}_{g,A_0,\mathcal{L}} \in \mathsf{CH}^g_{\mathsf{op}}(\mathcal{S})$ be the twisted double ramification cycle associated to the new family (9) and the vector A_0. We have the invariance

$$\mathsf{DR}^{\mathsf{op}}_{g,A_0,\mathcal{L}} = F^* \mathsf{DR}^{\mathsf{op}}_{g,A,\mathcal{L}}\,,$$

where $F : \mathfrak{Pic}_{g,n+1,d} \to \mathfrak{Pic}_{g,n,d}$ is the map forgetting the last marking.

Invariance III: Weight translation**.**

Let $B = (b_1, \ldots, b_n) \in \mathbb{Z}^n$ satisfy $\sum_{i=1}^n b_i = e$, then the family

$$\pi : \mathcal{C} \to \mathcal{S}\,, \quad p_1, \ldots, p_n : \mathcal{S} \to \mathcal{C}\,, \quad \mathcal{L}\Big(\sum_{i=1}^n b_i p_i\Big) \to \mathcal{C}\,. \tag{10}$$

defines an object of $\mathfrak{Pic}_{g,n,d+e}$. Let $\mathsf{DR}^{\mathsf{op}}_{g,A+B,\mathcal{L}(\sum_i b_i p_i)} \in \mathsf{CH}^g_{\mathsf{op}}(\mathcal{S})$ be the twisted double ramification cycle associated to the new family (10) and the vector $A + B$. We have the invariance

$$\mathsf{DR}^{\mathsf{op}}_{g,A+B,\mathcal{L}(\sum_i b_i p_i)} = \mathsf{DR}^{\mathsf{op}}_{g,A,\mathcal{L}}\,.$$

Invariance IV: Twisting by pullback.

Let $\mathcal{B} \to \mathcal{S}$ be any line bundle on the base. By tensoring (7) with $\pi^*\mathcal{B}$, we obtain a new object of $\mathfrak{Pic}_{g,n,d}$ over $\mathcal{S}$:

$$\pi : \mathcal{C} \to \mathcal{S}\,, \quad p_1, \ldots, p_n : \mathcal{S} \to \mathcal{C}\,, \quad \mathcal{L} \otimes \pi^*\mathcal{B} \to \mathcal{C}\,. \tag{11}$$

Let $\mathsf{DR}^{\mathsf{op}}_{g,A,\mathcal{L}\otimes\pi^*\mathcal{B}} \in \mathsf{CH}^g_{\mathsf{op}}(\mathcal{S})$ be the twisted double ramification cycle associated to the new family (11) and the vector A. We have the invariance

$$\mathsf{DR}^{\mathsf{op}}_{g,A,\mathcal{L}\otimes\pi^*\mathcal{B}} = \mathsf{DR}^{\mathsf{op}}_{g,A,\mathcal{L}}\,.$$

Invariance V: Vertical twisting**.**

Consider a partition of the genus, marking, and degree data,

$$g_1 + g_2 = g\,, \quad N_1 \sqcup N_2 = \{1, \ldots, n\}\,, \quad d_1 + d_2 = d\,, \tag{12}$$

which is *not* symmetric.[11] Such a partition defines a divisor

$$\Delta_1 \in \mathsf{CH}^1(\mathfrak{C}_{g,n,d})$$

in the universal curve over $\mathfrak{Pic}_{g,n,d}$ by twisting by the (g_1, N_1, d_1)-component of a curve with a separating node with separating data (12).

By tensoring (7) with $\mathcal{O}_{\mathcal{C}}(\Delta_1)$, we obtain a new object of $\mathfrak{Pic}_{g,n,d}$ over $\mathcal{S}$:

$$\pi : \mathcal{C} \to \mathcal{S}\,, \quad p_1, \ldots, p_n : \mathcal{S} \to \mathcal{C}\,, \quad \mathcal{L}(\Delta_1) \to \mathcal{C}\,. \tag{13}$$

[11]We require $(g_1, N_1, d_1) \neq (g_2, N_2, d_2)$ so that the two sides of a separating node with separating data (12) can be distinguished.

Let $\mathsf{DR}^{\mathsf{op}}_{g,A,\mathcal{L}(\Delta_1)} \in \mathsf{CH}^g_{\mathsf{op}}(\mathcal{S})$ be the twisted double ramification cycle associated to the new family (13) and the vector A. We have the invariance

$$\mathsf{DR}^{\mathsf{op}}_{g,A,\mathcal{L}(\Delta_1)} = \mathsf{DR}^{\mathsf{op}}_{g,A,\mathcal{L}}\,. \tag{14}$$

For *symmetric* separating data (12), equality (14) holds with $\Delta_1 \subset \mathfrak{C}_{g,n,d}$ defined as the full preimage of the locus $\Delta \subset \mathfrak{Pic}_{g,n,d}$ of curves with a separating node (12). Then, equality (14) follows from Invariance IV with $\mathcal{B} = \mathcal{O}(\Delta)$.

Invariance VI: Partial stabilization**.**

Consider a second family of prestable n-pointed genus g curves over $\mathcal{S}$,

$$\pi' : \mathcal{C}' \to \mathcal{S}\,, \quad p'_1, \ldots, p'_n : \mathcal{S} \to \mathcal{C}'\,,$$

together with a birational $\mathcal{S}$-morphism

$$f : \mathcal{C}' \to \mathcal{C}\,, \quad f \circ p'_i = p_i\,.$$

A line bundle of relative degree d is defined on $\mathcal{C}'$ by

$$f^*\mathcal{L} \to \mathcal{C}'\,.$$

We require the following property to hold: *if the section p'_i meets the exceptional locus of f, then $a_i = 0$.*

Let $\mathsf{DR}^{\mathsf{op}}_{g,A,f^*\mathcal{L}} \in \mathsf{CH}^g_{\mathsf{op}}(\mathcal{S})$ be the twisted double ramification cycle associated to the new family

$$\pi' : \mathcal{C}' \to \mathcal{S}\,, \quad p'_1, \ldots, p'_n : \mathcal{S} \to \mathcal{C}'\,, \quad f^*\mathcal{L} \to \mathcal{C}' \tag{15}$$

and the vector A. We have the invariance

$$\mathsf{DR}^{\mathsf{op}}_{g,A,f^*\mathcal{L}} = \mathsf{DR}^{\mathsf{op}}_{g,A,\mathcal{L}}\,.$$

Theorem 7 provides two paths to viewing the above invariance properties. The invariances can be seen either from formal properties of the geometric construction of the universal twisted double ramification cycle or from formal symmetries of Pixton's formula. In fact, all invariances hold not only for the codimension g part $\mathsf{P}^g_{g,A,d}$ which computes the double ramification cycle, but for the full mixed-degree class $\mathsf{P}^\bullet_{g,A,d}$.

For example, Invariance VI on the formula side says that for the maps

$$\varphi_{\mathcal{L}}, \varphi_{f^*\mathcal{L}} : \mathcal{S} \to \mathfrak{Pic}_{g,n,d}$$

obtained from the families (7) and (15), we have

$$\varphi^*_{f^*\mathcal{L}} \mathsf{P}^g_{g,A,d} = \varphi^*_{\mathcal{L}} \mathsf{P}^g_{g,A,d} \in \mathsf{CH}^g_{\mathsf{op}}(\mathcal{S})$$

for $\mathsf{P}^g_{g,A,d} \in \mathsf{CH}^g_{\mathsf{op}}(\mathfrak{Pic}_{g,n,d})$.

Proofs of all of the invariances are presented in [6, Section 7]. The above invariances (together with geometric definitions when transversality to the Abel-Jacobi map holds) do not characterize[12] $\mathsf{DR}^{\mathsf{op}}$.

0.7 Universal formula in degree 0

The most efficient statement of the double ramification cycle formula on the Picard stack of curves occurs in the degree $d = 0$ case with *no* markings. In order to avoid[13] the unpointed genus 1 case, let $g \neq 1$.

The specialization of Theorem 7 to $d = 0$ calculates $\mathsf{DR}^{\mathsf{op}}_{g,\emptyset}$ as the value at $r = 0$ of the degree g part of

$$\exp\left(-\frac{1}{2}\eta\right) \sum_{\substack{\Gamma_\delta \in \mathsf{G}_{g,0,0} \\ w \in \mathsf{W}_{\Gamma_\delta,r}}} \frac{r^{-h^1(\Gamma_\delta)}}{|\mathrm{Aut}(\Gamma_\delta)|} \, j_{\Gamma_\delta *}\left[\prod_{e=(h,h')\in \mathrm{E}(\Gamma_\delta)} \frac{1-\exp\left(-\frac{w(h)w(h')}{2}(\psi_h+\psi_{h'})\right)}{\psi_h+\psi_{h'}}\right]$$

as an operational Chow class on

$$\mathfrak{Pic}_g = \mathfrak{Pic}_{g,0,0}\,.$$

The full statement of Theorem 7 can be recovered from the above $d = 0$ specialization via pullback under the map

$$\mathfrak{Pic}_{g,n,d} \to \mathfrak{Pic}_g\,, \quad (C, p_1, \ldots, p_n, \mathcal{L}) \mapsto \left(C, \mathcal{L}\left(-\sum_{i=1}^n a_i p_i\right)\right).$$

Indeed, the above map is the composition of the morphism

$$\tau_{-A} : \mathfrak{Pic}_{g,n,d} \to \mathfrak{Pic}_{g,n,0}, (C, p_1, \ldots, p_n, \mathcal{L}) \mapsto \left(C, p_1, \ldots, p_n, \mathcal{L}\left(-\sum_{i=1}^n a_i p_i\right)\right)$$

with the morphism

$$F : \mathfrak{Pic}_{g,n,0} \to \mathfrak{Pic}_g$$

forgetting the markings $p_1, \ldots, p_n$.

[12] Further geometric properties are required, see Section 1.6 of [43] for a discussion.

[13] For $g = 1$, a parallel formula holds for $n = 1$ and $A = (0)$.

• For the $\mathsf{DR}^{\mathsf{op}}_{g,A}$ side of Theorem 7, Invariance II implies

$$F^*\mathsf{DR}^{\mathsf{op}}_{g,\emptyset} = \mathsf{DR}^{\mathsf{op}}_{g,\mathbf{0}}$$

for the zero vector $\mathbf{0} \in \mathbb{Z}^n$. Furthermore, Invariance III implies

$$\tau^*_{-A}\mathsf{DR}^{\mathsf{op}}_{g,\mathbf{0}} = \mathsf{DR}^{\mathsf{op}}_{g,A}\,.$$

• For the $\mathsf{P}^g_{g,A,d}$ side of Theorem 7, the corresponding invariance properties of Pixton's formula yield the parallel transformation

$$\tau^*_{-A}F^*\mathsf{P}^g_{g,\emptyset,0} = \mathsf{P}^g_{g,A,d}\,.$$

Therefore, the equality in Theorem 7 for general A and d follows from the specialization to $A = \emptyset$ and $d = 0$. For certain steps in our proof of Theorem 7, the $A = \emptyset$ and $d = 0$ geometry is advantageous and is used.

By restricting to suitable open subsets of $\mathfrak{Pic}_g$, we can simplify the $d = 0$ formula even further. Let

$$\mathfrak{Pic}_g^{\mathrm{ct}} \subset \mathfrak{Pic}_g$$

be the locus where the curve C is of compact type. We obtain

$$\mathsf{DR}^{\mathsf{op}}_{g,\emptyset}|_{\mathfrak{Pic}_g^{\mathrm{ct}}} = \frac{\theta^g}{g!}\,, \quad \text{for} \;\; \theta = -\frac{1}{2}\left(\eta + \sum_{\Delta} d_\Delta^2[\Delta]\right), \tag{16}$$

where the sum is over the boundary divisors $\Delta \subset \mathfrak{Pic}_g$ on which generically the curve splits into two components carrying line bundles of degrees d_Δ and $-d_\Delta$. The class θ here may be viewed as a universal theta divisor on $\mathfrak{Pic}_g^{\mathrm{ct}}$.

Formula (16) was first written on the moduli space of stable curves of compact type in [36, 38]. The operational Chow class $\mathsf{DR}^{\mathsf{op}}_{g,\emptyset}$ on $\mathfrak{Pic}_g$, however, is *not* the power of a divisor.

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1 Notation, conventions, and the plan

1.1 Ground field

In the Introduction, the ground field was the complex numbers $\mathbb{C}$. However, for the remainder of the paper, we will work more generally over a field K of characteristic zero. We will make essential use of the results of [45] which are stated over $\mathbb{C}$, but also hold over $\mathbb{Q}$ by the following standard argument:

(i) Both the DR cycle (via the b-Chow approach [39]) and Pixton's class are defined over $\mathbb{Q}$.

(ii) Rational equivalence of cycles uses finitely many subschemes and rational functions and hence descends to a finitely generated $\mathbb{Q}$-subalgebra of $\mathbb{C}$. A non-empty scheme of finite type over $\mathbb{Q}$ has points over some finite extension, hence the rational equivalence descends to a finite extension of $\mathbb{Q}$.

(iii) The rational equivalence descends (via a Galois argument) further to $\mathbb{Q}$ since we work in Chow with rational coefficients.

By similar arguments, our results are in fact true over $\mathbb{Z}[1/N]$ for a positive integer N depending on the ramification data. Understanding what happens at small primes (or integral Chow groups) is an interesting question.

1.2 Basics

Let K be a ground field of characteristic zero. When we work in the logarithmic category, we assume $\operatorname{Spec} K$ to be equipped with the trivial log structure.

We write $\overline{\mathcal{M}}$ for the stack of all stable (ordered) marked curves over K and $\mathfrak{M}$ for the stack of all prestable curves with ordered marked points. Both come with natural log structures, and the universal marked curves over these spaces are naturally log curves. For $\overline{\mathcal{M}}$, the log structure is described in [46]. The same construction applies unchanged to $\mathfrak{M}$, see [35, Appendix A]. The natural open immersion

$$\overline{\mathcal{M}} \to \mathfrak{M}$$

is strict (though, in contrast to [35, 46], we order the markings of our log curves). We use subscripts g and n to fix the genus and number of markings when necessary.

Let $\mathfrak{C}$ be the universal prestable curve over $\mathfrak{M}$. For efficiency of notation, we will also denote by $\mathfrak{C}$ the universal curve over the various other moduli stacks of curves with additional structure which will appear in the paper. These universal curves are always obtained by pulling-back $\mathfrak{C}$ over $\mathfrak{M}$.

1.3 The Picard stack and relative Picard space

Our stacks will be with respect to the fppf topology ([63, Definition 9.1.1]). We define the Picard stack $\mathfrak{Pic}_{g,n}$ as the fibred category over $\mathfrak{M}_{g,n}$ whose fibre over a scheme $T \to \mathfrak{M}_{g,n}$ is the groupoid of line bundles on $\mathfrak{C}_{g,n} \times_{\mathfrak{M}_{g,n}} T$ with morphisms given by isomorphisms of line bundles, see [50, 14.4.7]. We define the relative Picard space $\mathfrak{Pic}^{\mathrm{rel}}_{g,n}/\mathfrak{M}_{g,n}$ to be the quotient of $\mathfrak{Pic}_{g,n}$ by its relative inertia over $\mathfrak{M}_{g,n}$. Equivalently $\mathfrak{Pic}^{\mathrm{rel}}_{g,n}$ is the fppf-sheafification of the fibred category of *isomorphism classes* of line bundles on $\mathfrak{C}_{g,n} \times_{\mathfrak{M}_{g,n}} T$, see [15, Chapter 8] and [31, Chapter 9].

Relative representability of $\mathfrak{Pic}^{\mathrm{rel}}_{g,n}/\mathfrak{M}_{g,n}$ by smooth algebraic spaces can be checked locally on $\mathfrak{M}_{g,n}$. It then follows from [3, Appendix] as the curve

$$\mathfrak{C}_{g,n} \to \mathfrak{M}_{g,n}$$

is flat, proper, relatively representable by algebraic spaces, and cohomologically flat in dimension 0 (reduced and connected geometric fibers). The Picard stack $\mathfrak{Pic}_{g,n}$ is a $\mathbb{G}_m$-gerbe over $\mathfrak{Pic}^{\mathrm{rel}}_{g,n}$, hence is a (smooth) algebraic stack. In particular, $\mathfrak{Pic}_{g,n}$ is smooth over K of pure dimension $4g-4+n$, and $\mathfrak{Pic}^{\mathrm{rel}}_{g,n}$ is smooth over K of pure dimension $4g-3+n$.

1.4 Plan of the paper

The core of the paper starts in Section 2 where we give three equivalent definitions of the universal double ramification cycle on $\mathfrak{Pic}$. Our first definition is by taking a closure in the spirit of [40] which is simple, but rather difficult to work with. The second is via logarithmic geometry following [56]. The third is a b-Chow definition along the path of [41]. In Section 2.5, we give an explicit description of the set-theoretic image of the double ramification cycle in $\mathfrak{Pic}$. We prove Theorem 1 in Section 2.7.

In Section 3, we discuss properties of Pixton's cycle $\mathsf{P}^c_{g,A,d}$ defined Section 0.3.5. In particular, formal properties of Pixton's cycle parallel to the invariances of the double ramification cycles are proven. Compatibilities with definitions in previously studied cases are also proven.

Section 4 contains the proof of Theorem 7, the main result of the paper, by an eventual reduction to the formula of [45] in the case of target $\mathbb{P}^n$ for large n. A crucial step in the proof is the matching of the double ramification cycle defined in [45] via rubber maps with our new universal definition on $\mathfrak{Pic}$ in a suitable sense when the target is $\mathbb{P}^n$. The matching is verified in Section 5 where we follow the pattern of the proof given in [56] in case the target is a point.

2 The universal double ramification cycle

2.1 Overview

We fix a genus g, a number of markings n, and a vector $A = (a_1, \ldots, a_n) \in \mathbb{Z}^n$ of ramification data satisfying

$$\sum_{i=1}^{n} a_i = d\,.$$

We define here the associated universal twisted double ramification cycle class in the operational Chow group of the universal Picard stack $\mathfrak{Pic}_{g,n,d}$. The operational class is the class associated to a certain proper Deligne-Mumford type morphism

$$\mathbf{Div}_{g,A} \to \mathfrak{Pic}_{g,n,d}\,.$$

Our goal here is to define the stack $\mathbf{Div}_{g,A}$ over $\mathfrak{Pic}_{g,n,d}$.

We will present three essentially equivalent definitions of the universal twisted double ramification cycle in Sections 2.2–2.4 which yield the same operational class:

- a definition in Section 2.2 by closing the Abel-Jacobi section which is simple to state but difficult to handle,
- an intrinsic logarithmic definition in Section 2.3 following Marcus-Wise [56],
- a slight variation of the log definition in Section 2.4 which facilitates comparison to the spaces of rubber maps.

After analyzing the set-theoretic closure of the Abel-Jacobi section in Section 2.5, the equality of the three resulting classes will be shown in Section 2.6. In Section 2.8, we briefly discuss the lift of universal twisted double ramification cycle to operational b-Chow.

2.2 $\mathsf{DR}^{\mathsf{op}}_{g,A}$ by closure

We define the Abel-Jacobi section σ of $\mathfrak{Pic}_{g,n,d} \to \mathfrak{M}_{g,n}$ by

$$\sigma\colon \mathfrak{M}_{g,n} \to \mathfrak{Pic}_{g,n,d}\,, \qquad (C, p_1, \ldots, p_n) \mapsto \mathcal{O}_C\Big(\sum_{i=1}^{n} a_i p_i\Big)\,. \tag{17}$$

The section σ is not a closed immersion (both because of the $\mathbb{G}_m$-automorphism groups of line bundles and because the image is not closed). However, σ is quasi-compact and relatively representable by schemes, and hence admits a well-defined *schematic image* (we use that the formation of the schematic image is compatible with flat base-change, see [73, Tag 081I]). The schematic image is the smallest closed reduced substack through which σ factors.

Since the schematic image $\bar\sigma$ is a closed substack of pure dimension,

$$\iota : \bar\sigma \to \mathfrak{Pic}_{g,n,d}\,,$$

we obtain an operational class $\iota_{\mathsf{op}}[\bar\sigma]$ by [6, Definition 17]. Our first definition of the universal twisted double ramification cycles is via the schematic image of σ:

$$\mathsf{DR}^{\mathsf{op}}_{g,A} = \iota_{\mathsf{op}}[\bar\sigma] \,\in\, \mathsf{CH}^g_{\mathsf{op}}(\mathfrak{Pic}_{g,n,d})\,. \tag{18}$$

Let $\mathfrak{Pic}_{\underline{0}} \hookrightarrow \mathfrak{Pic}$ be the open substack consisting of line bundles having degree 0 on every irreducible component of every geometric fibre (*multidegree* $\underline{0}$), and let $\mathfrak{Pic}^{\mathrm{rel}}_{\underline{0}} \hookrightarrow \mathfrak{Pic}^{\mathrm{rel}}$ be defined analogously. We have a commutative diagram in which all squares are pullbacks:

$$\begin{tikzcd}
(B\mathbb{G}_m)_{\mathfrak{M}_{g,n}} \arrow[r] \arrow[d, equal] & \mathfrak{M}_{g,n} \arrow[d, equal] \arrow[ddrr, "e=\mathcal{O}_C"] & & \\
\bar\sigma^{\underline{0}} \arrow[r] \arrow[drr] \arrow[dd] & \bar\sigma^{\underline{0}}_{\mathsf{rel}} \arrow[drr] \arrow[dd] & & \\
& & \mathfrak{Pic}_{g,n,\underline{0}} \arrow[r] \arrow[dd] & \mathfrak{Pic}^{\mathrm{rel}}_{g,n,\underline{0}} \arrow[dd] \\
\bar\sigma \arrow[r] \arrow[drr] & \bar\sigma_{\mathsf{rel}} \arrow[drr] & & \\
& & \mathfrak{Pic}_{g,n,0} \arrow[r] & \mathfrak{Pic}^{\mathrm{rel}}_{g,n,0}\,.
\end{tikzcd} \tag{19}$$

Let $(C/B, p_1, \dots, p_n)$ be a prestable curve over a scheme B of finite type over K. Let L be a line bundle on C such that $L(-\sum_{i=1}^n a_i p_i)$ is of multidegree $\underline{0}$ for $A = (a_1, \dots, a_n) \in \mathbb{Z}^n$. The data

$$C \to B\,, \qquad L\Big(-\sum_{i=1}^n a_i p_i\Big) \to C$$

determine a map

$$\varphi\colon B \to \mathfrak{Pic}_{g,n,\underline{0}}\,,$$

and we form a pullback diagram

$$\begin{tikzcd} B' \arrow[r] \arrow[d, "\psi"] & \mathfrak{M}_{g,n} \arrow[d, "e"] \\ B \arrow[r, "\varphi^{\mathsf{rel}}"] & \mathfrak{Pic}^{\mathrm{rel}}_{g,n,\underline{0}}\,. \end{tikzcd} \tag{20}$$

Since $\mathfrak{M}_{g,n}$ is smooth and $\mathfrak{Pic}^{\mathrm{rel}}_{g,n,\underline{0}}$ is separated, the map e is a regular embedding.

Lemma 10. *In the multidegree $\underline{0}$ case, we have*

$$\mathsf{DR}^{\mathsf{op}}_{g,A}(\varphi)([B]) = \psi_* e^! [B]\,. \tag{21}$$

Proof. We begin by expanding the diagram (20) to

$$\begin{tikzcd} B' \arrow[r] \arrow[d, "\psi"] & \mathfrak{M}_{g,n} \times B \arrow[r, "f'"] \arrow[d] & \mathfrak{M}_{g,n} \arrow[d, "e"] \\ B \arrow[r, "\varphi'"] & \mathfrak{Pic}^{\mathrm{rel}}_{g,n,\underline{0}} \times B \arrow[r, "f"] & \mathfrak{Pic}^{\mathrm{rel}}_{g,n,\underline{0}}\,. \end{tikzcd} \tag{22}$$

Since $\mathfrak{Pic}_{g,n} \to \mathfrak{Pic}^{\mathrm{rel}}_{g,n}$ is smooth, we deduce from [6, Lemma 26] and diagram (19) that

$$\mathsf{DR}^{\mathsf{op}}_{g,A}(\varphi)([B]) = \psi_* \varphi'^! [\mathfrak{M}_{g,n} \times B]\,. \tag{23}$$

We then compute

$$\begin{aligned} \mathsf{DR}^{\mathsf{op}}_{g,A}(\varphi)([B]) &= \psi_* \varphi'^! [\mathfrak{M}_{g,n} \times B] \\ &= \psi_* \varphi'^! f'^* [\mathfrak{M}_{g,n}] \\ &= \psi_* \varphi'^! f'^* e^! [\mathfrak{Pic}^{\mathrm{rel}}_{g,n,\underline{0}}] \\ &= \psi_* \varphi'^! e^! f^* [\mathfrak{Pic}^{\mathrm{rel}}_{g,n,\underline{0}}] \\ &= \psi_* \varphi'^! e^! [\mathfrak{Pic}^{\mathrm{rel}}_{g,n,\underline{0}} \times B] \\ &= \psi_* e^! \varphi'^! [\mathfrak{Pic}^{\mathrm{rel}}_{g,n,\underline{0}} \times B] \\ &= \psi_* e^! [B]\,. \end{aligned} \tag{24}$$

$\lozenge$

In particular, if the intersection of B with the unit section in $\mathfrak{Pic}^{\mathrm{rel}}_{g,n,\underline{0}}$ is transversal, then we simply take the naive intersection in $\mathfrak{Pic}^{\mathrm{rel}}_{g,n,\underline{0}}$ and push it down to B.

2.3 Logarithmic definition of $\mathsf{DR}^{\mathsf{op}}$

2.3.1 Overview of log divisors

We begin by recalling various results and definitions from log geometry. We refer the reader to [47] for basics on log geometry and [56] for the details of what we do here. While log geometry will not play a substantial role elsewhere in the paper, it will reappear in Section 5.

Given a log scheme $S = (S, M_S)$, we write

$$\mathbb{G}_m^{log}(S) = \Gamma(S, M_S^{gp}) \quad \text{and} \quad \mathbb{G}_m^{trop}(S) = \Gamma(S, \bar{M}_S^{gp}),$$

which we call the logarithmic and tropical multiplicative groups. Both can naturally be extended to presheaves on the category $\mathbf{LSch}_S$ of log schemes over S, and both admit log smooth covers by log schemes (with subdivisions $\mathbb{P}^1$ and $[\mathbb{P}^1/\mathbb{G}_m]$ respectively). A *log (tropical)* line on S is a $\mathbb{G}_m^{log}$ ($\mathbb{G}_m^{trop}$) torsor on S, for the strict étale topology.

Definition 11. (See [56, Def. 4.6]) Let C be a logarithmic curve over a logarithmic scheme S. A *logarithmic divisor* on C is a tropical line P over S and an S-morphism $C \to P$.

Let $\mathbf{Div}_g^{\mathsf{rel}}$ be the stack[14] in the strict étale topology on logarithmic schemes whose S-points are triples (C, P, α) where C is a logarithmic curve of genus g over S, P is a tropical line over S, and

$$\alpha\colon C \to P$$

is an S-morphism.

If S is a geometric log point and C/S a log curve, then the set of isomorphism classes of $\mathbf{Div}_g^{\mathsf{rel}}(S)$ is given by $\pi_*\bar{M}_C^{gp}/\bar{M}_S^{gp}$. At the markings, an element of $\pi_*\bar{M}_C^{gp}/\bar{M}_S^{gp}$ determines an element of the groupified relative characteristic monoid $\mathbb{Z}$ (for those who prefer a tropical perspective, this can be viewed as the outgoing slope at the marking).

Definition 12. Let $\mathbf{Div}_{g,A}^{\mathsf{rel}}$ be the (open and closed) substack of $\mathbf{Div}_g^{\mathsf{rel}}$ consisting of those triples where the curve carries exactly n markings and where on each geometric fibre the outgoing slopes at the markings correspond to A (our log curves come with an ordering of their markings as explained in Section 1.2).

[14]The stack $\mathbf{Div}_g^{\mathsf{rel}}$ was denoted $\mathbf{Div}_g$ in [56], but we wish to reserve the latter notation for a certain $\mathbb{G}_m$-gerbe over $\mathbf{Div}_g^{\mathsf{rel}}$ which will play a much more prominent role in our paper.

Remark 13. It is natural to ask for a description of the functor of points of the underlying (non-logarithmic) stack of $\mathbf{Div}_{g,A}^{\mathsf{rel}}$ as a fibred category over $\mathfrak{M}_{g,n}$. However, we expect that such a description will not be simple. A closely related problem is solved in [13], and the intricacy of the resulting definition suggests that the path will not be easy.

2.3.2 Abel-Jacobi map

Given a log curve $\pi\colon C \to S$ of genus g, the right-derived pushforward to S of the standard exact sequence

$$1 \to \mathcal{O}_C^\times \to M_C^{gp} \to \bar{M}_C^{gp} \to 1\,, \tag{25}$$

yields a natural map

$$\pi_* \bar{M}_C^{gp} \to R^1\pi_*\mathcal{O}_C^\times\,,$$

which factors via the quotient

$$\pi_* \bar{M}_C^{gp}/\bar{M}_S^{gp} = \mathbf{Div}_g^{\mathsf{rel}}(S)\,.$$

We therefore obtain a *relative Abel-Jacobi* map

$$\mathsf{AJ}^{\mathsf{rel}}\colon \mathbf{Div}_g^{\mathsf{rel}} \to \mathfrak{Pic}_g^{\mathrm{rel}}\,,$$

which restricts to maps

$$\mathsf{AJ}^{\mathsf{rel}}\colon \mathbf{Div}_{g,A}^{\mathsf{rel}} \to \mathfrak{Pic}_{g,n,d}^{\mathrm{rel}}\,.$$

For a first example, suppose S is a geometric log point with $\bar{M}_S = \mathbb{N}$. The data then determines to first order a deformation of the curve over a DVR (which we take generically smooth), and the section of $\pi_*\bar{M}_C^{gp}/\bar{M}_S^{gp}$ gives the multiplicities of components in the special fibre and the twists by the markings.

For another example, consider what happens over the locus of (strict) smooth curves. Writing $N/\mathcal{M}_{g,n}^{\mathsf{log}}$ for the stack of markings (finite étale), we see $\mathbf{Div}_g^{\mathsf{rel}}$ is just the category of locally constant functions from N to $\mathbb{Z}$ – in other words, choices of outgoing slope/weight on each leg. The Abel-Jacobi map yields $\mathcal{O}_C(\sum_{i=1}^n a_i p_i)$ where the p_i are the markings and a_i are the weights. In particular, we see

$$\mathbf{Div}_{g,A}^{\mathsf{rel}} \to \mathcal{M}_{g,n}^{\mathsf{log}}$$

is birational (as we fixed an ordering of the markings) and log étale.

Definition 14. Let $\mathbf{Div}_g$ be the fibre product

$$\mathbf{Div}_g^{\mathsf{rel}} \times_{\mathfrak{Pic}_g^{\mathrm{rel}}} \mathfrak{Pic}_g\,. \tag{26}$$

More concretely, an S-point of $\mathbf{Div}_g$ is a quadruple $(C, P, \alpha, \mathcal{L})$ where (C, P, α) is an S-point of $\mathbf{Div}_g^{\mathsf{rel}}$ and $\mathcal{L}$ is a line bundle on C satisfying[15]

$$[\mathcal{L}] = \mathsf{AJ}^{\mathsf{rel}}(C, P, \alpha) \in \mathfrak{Pic}_g^{\mathrm{rel}}(S)\,.$$

We will denote by AJ the resulting Abel-Jacobi map

$$\mathbf{Div}_g \to \mathfrak{Pic}_g\,.$$

Observe that $\mathbf{Div}_g$ is a $\mathbb{G}_m$-gerbe over $\mathbf{Div}_g^{\mathsf{rel}}$, just as $\mathfrak{Pic}_{g,n,d}$ is a $\mathbb{G}_m$-gerbe over $\mathfrak{Pic}_{g,n,d}^{\mathrm{rel}}$. Analogously, we define

$$\mathbf{Div}_{g,A} = \mathbf{Div}_{g,A}^{\mathsf{rel}} \times_{\mathfrak{Pic}_{g,n,d}^{\mathrm{rel}}} \mathfrak{Pic}_{g,n,d} \quad \text{and} \quad \mathsf{AJ}\colon \mathbf{Div}_{g,A} \to \mathfrak{Pic}_{g,n,d}\,. \tag{27}$$

We summarise the key properties of the Abel-Jacobi map. These are proven in [56] for $\mathsf{AJ}^{\mathsf{rel}}$, and are stable under base-change.

Proposition 15. *The Abel-Jacobi map*

$$\mathsf{AJ}\colon \mathbf{Div}_{g,A} \to \mathfrak{Pic}_{g,n,d}$$

is proper, relatively representable by algebraic spaces, and is a monomorphism of log stacks.

We obtain an operational class $\mathsf{AJ}_{\mathsf{op}}[\mathbf{Div}_{g,A}]$ associated by [6, Definition 17] to the Abel-Jacobi map AJ. Our second definition of the universal twisted double ramification cycles is via AJ:

$$\mathsf{DR}_{g,A}^{\mathsf{op}} = \mathsf{AJ}_{\mathsf{op}}[\mathbf{Div}_{g,A}] \in \mathsf{CH}_{\mathsf{op}}^g(\mathfrak{Pic}_{g,n,d})\,. \tag{28}$$

The equivalence of definitions (18) and (28) will be proven in Section 2.6.

2.3.3 Description of $\mathbf{Div}_g$ with log line bundles

Our approach to $\mathbf{Div}_g$ in Definition 14 via a fiber product is indirect. While it will not be used in the paper, a more conceptual path is to consider the stack $\mathbf{Div}_g'$ whose objects are tuples

$$(C/S, \mathcal{P}, \alpha)$$

where C/S is a log curve, $\mathcal{P}$ is a *logarithmic* line on S (a $\mathbb{G}_m^{log}$ torsor), and α is a map from C to the *tropical* line P on S induced from $\mathcal{P}$ by the exact sequence (25). There is a natural map

$$\mathbf{Div}_g' \to \mathbf{Div}_g^{\mathsf{rel}}\,.$$

[15]Here, [] denotes the equivalence class under the relations of isomorphism and tensoring with the pullback of a line bundle from the base.

We can see α as a section of the tropicalisation of the pullback of $\mathcal{P}$ to C. As such, by the sequence (25), α induces a $\mathbb{G}_m$-torsor on C, giving us an Abel-Jacobi map $\mathbf{Div}'_g \to \mathfrak{Pic}_g$. Together these maps induce a map

$$\mathbf{Div}'_g \to \mathbf{Div}_g$$

to the fibre product, and a local computation verifies that this is an isomorphism.

The above discussion points[16] towards a definition of the double ramification cycle via the *logarithmic Picard functor* of [60] which we hope will be pursued in future.

2.4 Logarithmic rubber definition of $\mathsf{DR}^{\mathsf{op}}$

Marcus and Wise introduce a slight variant $\mathbf{Rub}_g^{\mathsf{rel}}$ of the stack $\mathbf{Div}_g^{\mathsf{rel}}$ which parametrises pairs (C, P, α) where P is a tropical line on S and $\alpha\colon C \to P$ is an S-morphism *such that on each geometric fibre over S the values taken by α on the irreducible components of C are totally ordered in* $(\bar{M}_S^{gp})_s$, with the ordering given by declaring the elements of $(\bar{M}_S)_s$ to be the non-negative elements.

The space $\mathbf{Rub}_{g,A}$, defined via pullback

$$\mathbf{Rub}_{g,A} \overset{\sim}{=} \mathbf{Div}_{g,A} \times_{\mathbf{Div}^{\mathsf{rel}}_{g,A}} \mathbf{Rub}^{\mathsf{rel}}_{g,A}\,,$$

is pure dimensional and comes with a proper birational map

$$\mathbf{Rub}_{g,A} \to \mathbf{Div}_{g,A}\,. \tag{29}$$

The stack $\mathbf{Rub}_{g,A}$ will play an important role in the comparison to classes coming from stable map spaces in Section 5.

We obtain an operational class $\mathsf{AJ}^{\mathsf{rub}}_{\mathsf{op}}[\mathbf{Rub}_{g,A}]$ associated by [6, Definition 17] to

$$\mathsf{AJ}^{\mathsf{rub}} : \mathbf{Rub}_{g,A} \to \mathfrak{Pic}_{g,n,d}$$

obtained by composing (29) with AJ. Our third defintion of the universal twisted double ramification cycles is via $\mathsf{AJ}^{\mathsf{rub}}$:

$$\mathsf{DR}^{\mathsf{op}}_{g,A} = \mathsf{AJ}^{\mathsf{rub}}_{\mathsf{op}}[\mathbf{Rub}_{g,A}] \in \mathsf{CH}^g_{\mathsf{op}}(\mathfrak{Pic}_{g,n,d})\,. \tag{30}$$

The equivalence with the first two definitions will be proven in Section 2.6.

[16]The *unit section* of $\mathfrak{Pic}$ is given by the stack of $\mathbb{G}_m$ torsors on the base. Similarly, the *unit section* of the *logarithmic* Picard stack $\mathfrak{LogPic}_g$ is given by the stack of $\mathbb{G}_m^{log}$ torsors on the base. The natural map $\mathfrak{Pic}_g \to \mathfrak{LogPic}_g$ is neither injective nor surjective: a logarithmic line bundle comes from a line bundle if and only if the associated tropical line bundle is trivial, and a choice of trivialisation of that tropical line bundle then determines a lift to a line bundle. Hence, we see that $\mathbf{Div}'_g$ is precisely the pullback of the unit section of $\mathfrak{LogPic}_g$ to $\mathfrak{Pic}_g$.

2.5 The image of the Abel-Jacobi map

The set theoretic image of the Abel-Jacobi map

$$\mathsf{AJ}\colon \mathbf{Div}_{g,A} \to \mathfrak{Pic}_{g,n,d}$$

can be characterized in terms of a condition on twisted divisors similar to the conditions of [32] for the moduli spaces $\widetilde{\mathcal{H}}_g(A)$ twisted canonical divisors.

Given a prestable graph Γ_δ of degree d, a *twist* on Γ_δ is a function $I : H(\Gamma) \to \mathbb{Z}$ satisfying

(i) $\forall j \in \mathrm{L}(\Gamma_\delta)$, corresponding to the marking $j \in \{1, \ldots, n\}$,

$$I(j) = a_j\,,$$

(ii) $\forall e \in \mathrm{E}(\Gamma_\delta)$, corresponding to two half-edges $h, h' \in \mathrm{H}(\Gamma_\delta)$,

$$I(h) + I(h') = 0\,,$$

(iii) $\forall v \in \mathrm{V}(\Gamma_\delta)$,

$$\sum_{v(h)=v} I(h) = \delta(v)\,,$$

where the sum is taken over *all* $\mathsf{n}(v)$ half-edges incident to v.

(iv) There is no *strict cycle*[17] in Γ.

Let $(C, p_1, \ldots, p_n)$ together with a line bundle $\mathcal{L} \to C$ of degree d be a geometric point of $\mathfrak{Pic}_{g,n,d}$. Let Γ_δ be the prestable graph of C decorated with the degrees $\delta(v)$ of the line bundle $\mathcal{L}$ restricted to the components C_v of C. Given a twist I on Γ_δ, let

$$\eta_I : C_I \to C$$

be the partial normalization of C at all nodes $q \in C$ corresponding to edges $e = (h, h')$ of Γ with

$$I(h) = -I(h') \neq 0\,.$$

Denote by $q_h, q_{h'} \in C_I$ the preimages of q corresponding to the half-edges h, h'. Denote by $\widehat{p}_i \in C_I$ the unique preimage of the ith marking $p_i \in C$.

[17] A strict cycle is a sequence $\vec{e}_i = (h_i, h_i')$, $i = 1, \ldots, \ell$ of directed edges in Γ forming a closed path in Γ such that $I(h_i) \geq 0$ for all i and there exists at least one i with $I(h_i) > 0$. Condition (iv) corresponds to the combination of the Vanishing, Sign, and Transitivity conditions for twists in [32, Section 0.3].

We say the point $(C, p_1, \ldots, p_n, \mathcal{L})$ of $\mathfrak{Pic}_{g,n,d}$ satisfies the *twisted divisor condition* for the integer vector A if and only if there exists a twist I on Γ_δ such that on the partial normalization C_I of C there exists an isomorphism of line bundles

$$\eta_I^* \mathcal{L} \cong \mathcal{O}_C \left(\sum_{i=1}^n a_i \widehat{p}_i + \sum_{h \in H(\Gamma)} I(h) q_h \right). \tag{31}$$

For $\mathcal{L} = \omega_C$ this exactly corresponds to the notion [32, Definition 1] of a twisted canonical divisor.

Proposition 16. *A geometric point $(C, p_1, \ldots, p_n, \mathcal{L})$ of $\mathfrak{Pic}_{g,n,d}$ lies in the image of the Abel-Jacobi map $\mathsf{AJ}\colon \mathbf{Div}_{g,A} \to \mathfrak{Pic}_{g,n,d}$ if and only if the twisted divisor condition for the vector A is satisfied.*

Proof. We may suppose that K is a separably closed field and $(C/S, p_1, \ldots, p_n)$ is a prestable curve over K. We must show that the twisted divisor condition is equivalent to the existence of a log structure on C/S satisfying the following property: *C/S is a log curve with markings given by the p_i which admits a global section α of $\bar{M}_C^{gp}$ with outgoing slope at p_i given by a_i.*

Suppose that such a log structure exists. From the log structure, we can determine a twist. To each leg we associate the outgoing slope of α on the corresponding leg. For an edge $\{h, h'\}$, we define $\ell(\{h, h'\})$ to be the element of $\bar{M}_S$ associated via the data of the log morphism $C \to S$ to the node of C corresponding to $\{h, h'\}$. If $\{h, h'\}$ is an edge with half-edge h attached to a vertex u and the opposite half-edge h' attached to v, we set the integer $I(h)$ to be the unique integer such that

$$\alpha(u) + I(h) \cdot \ell(\{h, h'\}) = \alpha(v) \in \bar{M}_S^{gp}. \tag{32}$$

That such an $I(h)$ exists follows from the structure of $\bar{M}_C^{gp}$.

Next, we verify that I is a twist. Conditions (i) and (ii) are immediate from the construction. We deduce condition (iv) because by following a strict cycle starting at some vertex u and applying (32) along each edge would yield $\alpha(u) < \alpha(u)$, which is impossible. Condition (iii) is immediate from the twisted divisor condition (31) and the fact that isomorphic line bundles have the same degree, so this will be proven once we have checked the latter condition.

For the latter condition, we must work a little harder. To start, we claim that there exists a morphism $\bar{M}_S \to \mathbb{N}$ which does not send the label of any edge to 0. Indeed, $\bar{M}_s^{gp}$ injects into its groupification which is a finitely generated torsion-free abelian group, hence isomorphic to $\mathbb{Z}^m$. Since $\bar{M}_S$ is sharp[18] and finitely generated, the non-zero elements of its image in $\mathbb{Z}^m$ land in some strict half-space of $\mathbb{Z}^m$ cut

[18] A monoid is sharp if 0 is indecomposable: $a + b = 0$ implies $a = b = 0$.

out by a linear equation with integral coefficients. Such a half-space admits a map to $\mathbb{N}$ such that the only preimage of 0 is 0.

After base changing over S along such a map, we may assume that $\bar{M}_S^{gp} = \mathbb{N}$ and that all edges have non-zero label. We obtain a first order map passing through our given point,

$$S = \operatorname{Spec} K[[t]] \to \mathbf{Div}_{g,A}$$

for which the induced prestable curve C/S is generically smooth. On the curve C, we define a Weil divisor D by assigning to an irreducible component v the integer $\alpha(v)$. The divisor D is then Cartier by (32), which still applies after base-change, and hence determines a line bundle $\mathcal{O}_C(-D)$, which is exactly the image of the Abel-Jacobi map. In particular, the bundle $\mathcal{L}$ is (up to isomorphism) given by restricting $\mathcal{O}_C(-D)$ to the central fibre, so it suffices to verify (31) for the latter bundle, which is a standard local calculation on a prestable curve over a discrete valuation ring.

Conversely, suppose the twisted divisor condition is satisfied. We must build a log structure and a suitable section $\alpha \in \bar{M}_C^{gp}(C)$. We could try equipping $(C/S, p_1, \ldots, p_n)$ with its minimal log structure (see Section 1.2), but then the section α is unlikely to exist – if there are no separating edges then there are no non-constant sections of $\bar{M}_C^{gp}$. Instead, we will construct a log structure by deforming the curve.

First, we claim that there exists an assignment of a positive integer $\ell(e) \in \mathbb{Z}_{>0}$ to each edge and of an integer $d(v) \in \mathbb{Z}$ to each vertex such that the following condition is satisfied:

$$\begin{gathered}\text{if } e = \{h, h'\} \text{ is an edge with } h \text{ attached to } u \text{ and } h' \text{ to } v, \text{ then} \\ d(u) + I(h) \cdot \ell(e) = d(v)\,. \end{gathered} \tag{*}$$

A twist I on Γ induces a binary relation $\preceq$ on $V(\Gamma)$ by

$$u \preceq v \iff \text{there is an edge } e = \{h, h'\} \text{ with } h \text{ at } u,\ h' \text{ at } v \text{ and } I(h) \geq 0\,.$$

The fact that Γ contains no strict cycles is equivalent by [74] to the existence of an extension of $\preceq$ to a total preorder on $V(\Gamma)$ (a reflexive, total, and transitive binary relation). Hence there exists a level function $d_0 : \mathrm{V}(\Gamma) \to \mathbb{Z}$ such that

$$u \preceq v \iff u, v \text{ connected by an edge and } d_0(u) \leq d_0(v)\,. \tag{33}$$

We define

$$L = \operatorname{lcm}(I(h) : h \in H(\Gamma), I(h) > 0)\,.$$

Then, $d(v) = L d_0(v)$ still has property (33), and, for any edge $e = \{h, h'\}$ with h attached to u and h' to v, we have two cases:

- $I(h) = 0$, in which case all edges $\{\tilde{h}, \tilde{h}'\}$ connecting u, v must satisfy $I(\tilde{h}) = 0$ (due to the strict cycle condition), so we can set $\ell(e) = 1$,

- $I(h) \neq 0$, in which case the number $\ell(e) = (d(v) - d(u))/I(h)$ is indeed a positive integer (since d has values in $L \cdot \mathbb{Z}$).

Clearly the functions d and ℓ thus constructed satisfy the condition above.

Such d and ℓ are far from unique, but we pick them. Consider then the space of all smoothings $\mathcal{C}$ of C over $K[[t]]$ such that the thickness[19] of $\mathcal{C}$ at the node corresponding to edge e is $\ell(e)$. Given such a smoothing $\mathcal{C}$, we construct a vertical Weil divisor D by assigning to the irreducible component corresponding to vertex v the weight $d(v)$. The divisor D is then Cartier by the condition (*). Set

$$\mathcal{L}_{\mathcal{C}} = \mathcal{O}_{\mathcal{C}}(D)|_C \otimes \mathcal{O}_C(\sum_i a_i p_i)\,.$$

The smoothing $\mathcal{C}$ also induces a log structure on C by taking the divisorial log structure of the special fibre. The twist I then determines an element α of $\pi_* \bar{M}_C^{gp}/\bar{M}_S^{gp}$. Applying the Abel-Jacobi map to α recovers $\mathcal{L}_{\mathcal{C}}$.

One can readily verify that $\mathcal{L}_{\mathcal{C}}$ satisfies (31) by a local computation, but we need to show more: the smoothing $\mathcal{C}$ can be chosen so that $\mathcal{L}_{\mathcal{C}}$ is isomorphic to the line bundle $\mathcal{L}$ that we started with. The space of such smoothings $\mathcal{C}$ naturally surjects onto

$$\bigoplus_{e=\{h,h'\}:I(h)\neq 0} \left(\mathcal{O}_{C_I}(h) \otimes \mathcal{O}_{C_I}(h')\right)^{\otimes \ell(e)}, \tag{34}$$

where the 1-dimensional K-vector space $\left(\mathcal{O}_{C_I}(h) \otimes \mathcal{O}_{C_I}(h')\right)^{\otimes \ell(e)}$ corresponds exactly to the ways to glueing the two branches of $\eta_I^*\mathcal{L}$ together at the points h and h'. In other words, by moving over the space $\bigoplus_{e=\{h,h'\}:I(h)\neq 0} \left(\mathcal{O}_{C_I}(h) \otimes \mathcal{O}_{C_I}(h')\right)^{\otimes \ell(e)}$, we can recover *all* ways of glueing $\eta_I^*\mathcal{L}$ to a line bundle on C. In particular, we can recover $\mathcal{L}$, hence we can realise $\mathcal{L}$ as $\mathcal{L}_{\mathcal{C}}$ for some smoothing $\mathcal{C}$, as required. $\lozenge$

2.6 Proof of the equivalence of the definitions

The equivalence of the classes coming from $\mathbf{Div}_{g,A}$ and from $\mathbf{Rub}_{g,A}$ is immediate by applying [6, Proposition 25]. We must compare the latter two with the class defined by (18) via the schematic image. We will require the following two easy results.

Lemma 17. *Let $U \hookrightarrow \mathbf{Div}_{g,A}$ denote the open locus where the log curve is classically smooth. Then U is schematically dense in $\mathbf{Div}_{g,A}$.*

Proof. Since $\mathbf{Div}_{g,A} \to \mathcal{M}_{g,n}^{\mathsf{log}}$ is log étale, we deduce that $\mathbf{Div}_{g,A}$ is log regular. In particular, $\mathbf{Div}_{g,A}$ is reduced, and the locus where the log structure is trivial is dense. $\lozenge$

[19] The local equation of the node is $xy = t^r$ for some positive integer r which we call the *thickness* of the node.

Lemma 18. *The Abel-Jacobi map* $\mathsf{AJ} : \mathbf{Div}_{g,A} \to \mathfrak{Pic}_{g,n,d}$ *factors through the inclusion* $\bar{\sigma} \to \mathfrak{Pic}_{g,n,d}$*, and the induced map*

$$\mathbf{Div}_{g,A} \to \bar{\sigma}$$

is proper and birational.

Proof. That $\mathbf{Div}_{g,a} \to \mathfrak{Pic}_{g,n,d}$ factors through $\bar{\sigma} \to \mathfrak{Pic}$ is immediate from Lemma 17 and the definition of the schematic image. The induced map $\mathbf{Div}_{g,A} \to \bar{\sigma}$ is proper since $\mathbf{Div}_{g,A}$ is proper over $\mathfrak{Pic}_{g,n,d}$ and is birational since it is an isomorphism over the locus of smooth curves. $\Diamond$

By another application of [6, Proposition 25], the definitions of $\mathsf{DR}^{\mathsf{op}}$ via $\mathbf{Div}_{g,A}$ and the schematic image are equivalent. $\Diamond$

2.7 Proof of Theorem 1

Let $k \geq 0$, and let $A = (a_1, \ldots, a_n)$ be a vector of integers satisfying

$$\sum_{i=1}^{n} a_i = k(2g-2)\,.$$

There are three definitions in the literature for the classical twisted double ramification cycle

$$\mathsf{DR}_{g,A,\omega^k} \in \mathsf{CH}_{2g-3+n}(\overline{\mathcal{M}}_{g,n})$$

on the moduli space of stable curves:

- via birational modifications of $\overline{\mathcal{M}}_{g,n}$ [39],
- via the closure of the image of the Abel-Jacobi section [40],
- via logarithmic geometry and the stack $\mathbf{Div}_g^{\mathsf{rel}}$ [56].

All three are shown to be equivalent in [39, 40]. For the proof of Theorem 1, we choose the definition of [56], as this will give the shortest path.

For $d = k(2g-2)$, let $\varphi : \overline{\mathcal{M}}_{g,n} \to \mathfrak{Pic}_{g,n,d}$ be the morphism associated to the data

$$\pi : \mathcal{C}_{g,n} \to \overline{\mathcal{M}}_{g,n}\,, \quad \omega_\pi^k \to \mathcal{C}_{g,n}\,. \tag{35}$$

To prove Theorem 1, we must show

$$\mathsf{DR}^{\mathsf{op}}_{g,A}(\varphi)([\overline{\mathcal{M}}_{g,n}]) = \mathsf{DR}_{g,A,\omega^k}\,,$$

where $[\overline{\mathcal{M}}_{g,n}]$ is the fundamental class.

We form the pullback diagram

$$\begin{tikzcd}
\mathbf{Div}_{g,A} \times_{\mathfrak{Pic}_{g,n,d}} \overline{\mathcal{M}}_{g,n} \arrow[r] \arrow[d, "\psi"] & \mathbf{Div}_{g,A} \times \overline{\mathcal{M}}_{g,n} \arrow[d, "a\times\mathrm{id}"] \\
\overline{\mathcal{M}}_{g,n} \arrow[r, "\varphi'=\varphi\times\mathrm{id}"] & \mathfrak{Pic}_{g,n,d} \times \overline{\mathcal{M}}_{g,n} \,.
\end{tikzcd} \tag{36}$$

Following [6, Definition 17], we have

$$\mathsf{DR}^{\mathsf{op}}_{g,A}(\varphi)([\overline{\mathcal{M}}_{g,n}]) = \psi_*(\varphi')^!([\mathbf{Div}_{g,A} \times \overline{\mathcal{M}}_{g,n}])\,.$$

The construction is equivalent to the definition of the class $\mathsf{DR}_{g,A,\omega^k}$ in [56] after making the standard translation between the Gysin pullback and the virtual fundamental class as in [12, Example 7.6].

2.8 The double ramification cycle in b-Chow

The construction of the double ramification cycle in [39] naturally yielded a more refined object: a b-cycle[20] on $\overline{\mathcal{M}}_{g,n}$ which pushes down to the usual double ramification cycle on $\overline{\mathcal{M}}_{g,n}$. The refined cycle was shown in [41] to have better properties with respect to intersection products than the usual double ramification cycle. By considering rational sections of the multidegree-zero relative Picard space over $\mathfrak{Pic}_{g,n,d}$, we can in an analogous way define a b-cycle on $\mathfrak{Pic}_{g,n,d}$ refining the universal twisted double ramification cycle introduced here. In future work, we will show that this refined universal cycle is compatible with intersection products in the sense of [41] and that the *toric contact cycles* of [68] can be obtained by pulling back these products.

3 Pixton's formula

3.1 Reformulation

Recall the cycle $\mathsf{P}^c_{g,A,d} \in \mathsf{CH}^c_{\mathsf{op}}(\mathfrak{Pic}_{g,n,d})$ defined in Section 0.3.5. We write

$$\mathsf{P}^\bullet_{g,A,d} = \sum_{c=0}^{\infty} \mathsf{P}^c_{g,A,d} \in \prod_{c=0}^{\infty} \mathsf{CH}^c_{\mathsf{op}}(\mathfrak{Pic}_{g,n,d})$$

[20] An element of the colimit of the Chow groups of smooth blowups of $\overline{\mathcal{M}}_{g,n}$ with transition maps given by pullback.

for the associated mixed dimensional class. We will rewrite the formula for $\mathsf{P}^\bullet_{g,A,d}$ in a more convenient form for computation.

Several factors in the formula of Section 0.3.5 can be pulled out of the sum over graphs and weightings. We require the following four definitions:

- Let $\mathsf{G}^{\mathrm{se}}_{g,n,d}$ be the set of graphs in $\mathsf{G}_{g,n,d}$ having exactly two vertices connected by a single edge. Such graphs are thus described by a partition
$$(g_1, I_1, \delta_1 \,|\, g_2, I_2, \delta_2) \tag{37}$$
of the genus, the marking set, and the degree of the universal line bundle.

- Given a vector $A = (a_1, \ldots, a_n) \in \mathbb{Z}^n$ satisfying
$$\sum_{i=1}^{n} a_i = d$$
and $\Gamma_\delta \in \mathsf{G}^{\mathrm{se}}_{g,n,d}$ corresponding to the partition (37), we define
$$c_A(\Gamma_\delta) = -(\delta_1 - \sum_{i \in I_1} a_i)^2 = -(\delta_2 - \sum_{i \in I_2} a_i)^2 \,.$$

- For $\Gamma_\delta \in \mathsf{G}^{\mathrm{se}}_{g,n,d}$, we write
$$[\Gamma_\delta] = \frac{1}{|\mathrm{Aut}(\Gamma_\delta)|} j_{\Gamma_\delta *}[\mathfrak{Pic}_{\Gamma_\delta}]$$
for the class of the boundary divisor of $\mathfrak{Pic}_{g,n,d}$ associated to Γ_δ.

- Let $\mathsf{G}^{\mathrm{nse}}_{g,n,d}$ be the set of graphs in $\mathsf{G}_{g,n,d}$ such that every edge is non-separating.

Proposition 19. *The class $\mathsf{P}^\bullet_{g,A,d}$ is the constant term in r of*
$$\exp\left(\frac{1}{2}\left(-\eta + \sum_{i=1}^{n} 2a_i\xi_i + a_i^2\psi_i + \sum_{\Gamma_\delta \in \mathsf{G}^{\mathrm{se}}_{g,n,d}} c_A(\Gamma_\delta)[\Gamma_\delta]\right)\right) \tag{38}$$
$$\sum_{\substack{\Gamma_\delta \in \mathsf{G}^{\mathrm{nse}}_{g,n,d}\\ w \in \mathsf{W}_{\Gamma_\delta, r}}} \frac{r^{-h^1(\Gamma_\delta)}}{|\mathrm{Aut}(\Gamma_\delta)|}\, j_{\Gamma_\delta *}\left[\prod_{e=(h,h') \in \mathrm{E}(\Gamma_\delta)} \frac{1 - \exp\left(-\frac{w(h)w(h')}{2}(\psi_h + \psi_{h'})\right)}{\psi_h + \psi_{h'}}\right],$$
for $r \gg 0$.

In the proof of Proposition 19, we will use the following computation which provides an interpretation for parts of the formula (38) and which will be used again in Section 6.

Lemma 20. *Let $A = (a_1, \dots, a_n) \in \mathbb{Z}^n$ with $\sum_{i=1}^n a_i = d$. For the line bundle $\mathcal{L}$ on the universal curve*

$$\pi : \mathfrak{C}_{g,n,d} \to \mathfrak{Pic}_{g,n,d}$$

with universal sections $p_1, \dots, p_n$, we define

$$\mathcal{L}_A = \mathcal{L}\Big(- \sum_{i=1}^n a_i [p_i] \Big) \, .$$

Then, we have

$$-\pi_* c_1(\mathcal{L}_A)^2 = -\eta + \sum_{i=1}^n 2a_i \xi_i + a_i^2 \psi_i \, .$$

Proof. The result follows from the definitions of the classes η and ξ_i:

$$\begin{aligned} -\pi_* c_1(\mathcal{L}_A)^2 &= -\pi_* \left(c_1(\mathcal{L})^2 + \sum_{i=1}^n -2a_i c_1(\mathcal{L})|_{[p_i]} + a_i^2 [p_i]^2 \right) \\ &= -\eta + \sum_{i=1}^n 2a_i \xi_i + a_i^2 \psi_i \, , \end{aligned}$$

where, for the self-intersection $[p_i]^2$, we have used that the first Chern class of the normal bundle of p_i is given by $-\psi_i$. $\Diamond$

Proof of Proposition 19. We denote by

$$\begin{aligned} \Phi_a(x) &= \frac{1 - \exp(-\frac{a}{2}x)}{x} \\ &= \sum_{m=0}^{\infty} (-1)^m (\frac{a}{2})^{m+1} \frac{1}{(m+1)!} x^m = \frac{a}{2} - \frac{a^2}{8} x + \dots \end{aligned}$$

the power series appearing in the edge-terms of Pixton's formula.

As a first step, we show that the constant term in r of

$$\exp\left(\frac{1}{2} \sum_{\Gamma_\delta \in \mathsf{G}^{\mathrm{se}}_{g,n,d}} c_A(\Gamma_\delta)[\Gamma_\delta] \right) \cdot \sum_{\substack{\Gamma_\delta \in \mathsf{G}^{\mathrm{nse}}_{g,n,d} \\ w \in \mathsf{W}_{\Gamma_\delta, r}}} \frac{r^{-h^1(\Gamma_\delta)}}{|\mathrm{Aut}(\Gamma_\delta)|} \, j_{\Gamma_\delta *} \prod_{e=(h,h') \in \mathrm{E}(\Gamma_\delta)} \Phi_{w(h)w(h')}(\psi_h + \psi_{h'}) \tag{39}$$

and the constant term in r of

$$\sum_{\substack{\Gamma_\delta \in \mathsf{G}_{g,n,d} \\ w \in \mathsf{W}_{\Gamma_\delta, r}}} \frac{r^{-h^1(\Gamma_\delta)}}{|\mathrm{Aut}(\Gamma_\delta)|} \, j_{\Gamma_\delta *} \prod_{e=(h,h') \in \mathrm{E}(\Gamma_\delta)} \Phi_{w(h)w(h')}(\psi_h + \psi_{h'}) \tag{40}$$

are equal. The formula (40) is a linear combination of boundary strata decorated by edge-terms $(\psi_h + \psi_{h'})^{m(e)}$ for nonnegative integers $m(e)$, $e \in \mathrm{E}(\Gamma_\delta)$ – terms of the form

$$j_{\Gamma_\delta *} \prod_{e=(h,h')\in \mathrm{E}(\Gamma_\delta)} (\psi_h + \psi_{h'})^{m(e)} \,. \tag{41}$$

A first consequence of the combinatorial rules for computing intersections in the tautological ring[21] of $\mathfrak{Pic}_{g,n,d}$ is that (39) is also a linear combination of such terms. The decorations $(\psi_h + \psi_{h'})^{m(e)}$ on separating edges $e = (h, h')$ appear naturally in the self-intersection formula for the boundary divisors $[\Gamma_\delta]$ since, for $\Gamma_\delta \in \mathsf{G}^{\mathrm{se}}_{g,n,d}$, the Chern class of the normal bundle of j_{Γ_δ} is given by $-(\psi_h + \psi_{h'})$.

We show that the coefficients of the term (41) in (39) and (40) have the same constant term in r. In (40), the coefficient is given by

$$\sum_{w\in \mathsf{W}_{\Gamma_\delta,r}} \frac{r^{-h^1(\Gamma_\delta)}}{|\mathrm{Aut}(\Gamma_\delta)|} \prod_{e=(h,h')\in \mathrm{E}(\Gamma_\delta)} (-1)^{m(e)} \left(\frac{w(h)w(h')}{2}\right)^{m(e)+1} \frac{1}{(m(e)+1)!} \,. \tag{42}$$

On the other hand, let $e_1, \ldots, e_\ell \in E(\Gamma_\delta)$ be the separating edges of Γ_δ, and let $\overline{\Gamma}_\delta \in \mathsf{G}^{\mathrm{nse}}_{g,n,d}$ be the graph obtained from Γ_δ by contracting these separating edges. Each separating edge e_j corresponds to a unique graph $(\Gamma_j)_{\delta_j} \in \mathsf{G}^{\mathrm{se}}_{g,n,d}$ obtained by contracting all edges of Γ_δ except for e_j.

In the product (39), the intersection rules of the tautological ring of $\mathfrak{Pic}_{g,n,d}$ imply that we obtain multiples of the term (41) by combining

- for $j = 1, \ldots, \ell$, a total of $m(e_j) + 1$ terms $[(\Gamma_j)_{\delta_j}]$ from expanding the power series
$$\exp\left(\frac{1}{2} \sum_{\Gamma_\delta \in \mathsf{G}^{\mathrm{se}}_{g,n,d}} c_A(\Gamma_\delta)[\Gamma_\delta]\right),$$
- the terms associated to $\overline{\Gamma}_\delta \in \mathsf{G}^{\mathrm{nse}}_{g,n,d}$ in the second factor.

Let $M = \sum_{j=1}^{\ell}(m(j)+1)$, then (41) appears in (39) with coefficient

$$\frac{1}{M!}\binom{M}{m(e_1)+1, \ldots, m(e_\ell)+1} \cdot \left(\prod_{j=1}^{\ell} \left(\frac{c_A((\Gamma_j)_{\delta_j})}{2}\right)^{m(e_j)+1} (-1)^{m(e_j)}\right) \tag{43}$$
$$\cdot \frac{|\mathrm{Aut}(\overline{\Gamma}_\delta)|}{|\mathrm{Aut}(\Gamma_\delta)|} \sum_{w\in \mathsf{W}_{\overline{\Gamma}_\delta,r}} \frac{r^{-h^1(\overline{\Gamma}_\delta)}}{|\mathrm{Aut}(\overline{\Gamma}_\delta)|} \prod_{e=(h,h')\in \mathrm{E}(\overline{\Gamma}_\delta)} (-1)^{m(e)} (\frac{w(h)w(h')}{2})^{m(e)+1} \frac{1}{(m(e)+1)!} \,.$$

[21] See [33] for the original treatment of the tautological ring of $\overline{\mathcal{M}}_{g,n}$. A corresponding treatment for $\mathfrak{M}_{g,n}$ will be given in [7, 8]. See also [45, Sections 1.1, 1.7].

To show the equality of (42) and (43), we combine a number of observations. First, for the multinomial coefficients, we have

$$\frac{1}{M!}\binom{M}{m(e_1)+1,\ldots,m(e_\ell)+1} = \prod_{j=1}^{\ell} \frac{1}{(m(e_j)+1)!}\,.$$

Second, for the graph morphism $\Gamma_\delta \to \overline{\Gamma}_\delta$ contracting the separating edges:

- we have an equality of Betti numbers $h^1(\Gamma_\delta) = h^1(\overline{\Gamma}_\delta)$,
- for the separating edges $e_j = (h_j, h_j')$ of Γ_δ, splitting the graph according to the partition $(g_1, I_1, \delta_1 \,|\, g_2, I_2, \delta_2)$, the value of every weighting $w \in \mathsf{W}_{\Gamma_\delta,r}$ is uniquely determined on h_j, h_j' since

$$w(h_j) = \delta_1 - \sum_{i\in I_1} a_i \mod r\,, \quad w(h_j') = \delta_2 - \sum_{i\in I_2} a_i \mod r.$$

  Hence, the constant term in r of $w(h_j)w(h_j')$ is precisely given by $c_A((\Gamma_j)_{\delta_j})$.
- concerning the non-separating edges for fixed Γ_δ with contraction $\Gamma_\delta \to \overline{\Gamma}_\delta$, the map $W_{\Gamma_\delta,r} \to W_{\overline{\Gamma}_\delta,r}$ given by restricting weightings $w \in W_{\Gamma_\delta,r}$ to the remaining half-edges $H(\overline{\Gamma}_\delta) \subset H(\Gamma_\delta)$ is a bijection.

The combination of these facts proves equality of (42) and (43) and hence the equality of (39) and (40).

To conclude the proof, we must show that the remaining part of the exponential term of (38) can be drawn into the graph sum. Using the projection formula, this identity is equivalent to showing

$$(j_{\Gamma_\delta})^* \exp\left(\frac{1}{2}\left(-\eta + \sum_{i=1}^n 2a_i\xi_i + a_i^2\psi_i\right)\right) = \prod_{v\in \mathrm{V}(\Gamma_\delta)} \exp\left(-\frac{1}{2}\eta(v)\right) \prod_{i=1}^n \exp\left(\frac{1}{2}a_i^2\psi_i + a_i\xi_i\right),$$

which immediately reduces to showing

$$(j_{\Gamma_\delta})^* \left(-\eta + \sum_{i=1}^n (2a_i\xi_i + a_i^2\psi_i)\right) = -\sum_{v\in \mathrm{V}(\Gamma_\delta)} \eta(v) + \sum_{i=1}^n (2a_i\xi_i + a_i^2\psi_i)\,.$$

By Lemma 20,

$$-\eta + \sum_{i=1}^n (2a_i\xi_i + a_i^2\psi_i) = -\pi_* c_1(\mathcal{L}_A)^2.$$

Now consider the diagram of universal curves

$$\begin{tikzcd}
\coprod_{v\in \mathrm{V}(\Gamma)} \mathfrak{C}_{\mathrm{g}(v),\mathrm{n}(v),\delta(v)} \arrow[d] & \mathfrak{C}'_{\Gamma_\delta} \arrow[l] \arrow[r, "G"] \arrow[d, "\pi'_{\Gamma_\delta}"] & \mathfrak{C}_{\Gamma_\delta} \arrow[r, "J_{\Gamma_\delta}"] \arrow[d, "\pi_{\Gamma_\delta}"] & \mathfrak{C}_{g,n,d} \arrow[d, "\pi"] \\
\prod_{v\in \mathrm{V}(\Gamma)} \mathfrak{Pic}_{\mathrm{g}(v),\mathrm{n}(v),\delta(v)} & \mathfrak{Pic}_{\Gamma_\delta} \arrow[l] \arrow[r, equal] & \mathfrak{Pic}_{\Gamma_\delta} \arrow[r, "j_{\Gamma_\delta}"] & \mathfrak{Pic}_{g,n,d}
\end{tikzcd}$$

where the left and right square are cartesian and the map G is the gluing map identifying sections of $\mathfrak{C}'_{\Gamma_\delta} \to \mathfrak{Pic}_{\Gamma_\delta}$ corresponding to pairs of half-edges forming an edge. This map G is proper, representable, and birational.

The space $\mathfrak{C}'_{\Gamma_\delta}$ is a disjoint union of universal curves

$$\pi'_{\Gamma_\delta,v} : \mathfrak{C}'_{\Gamma_\delta,v} \to \mathfrak{Pic}_{\Gamma_\delta}$$

for $v \in \mathrm{V}(\Gamma)$ and the bundle $G^* J^*_{\Gamma_\delta} \mathcal{L}_A$ restricted to the component $\mathfrak{C}'_{\Gamma_\delta,v}$ is equal to the pullback of the line bundle $\mathcal{L}_{v,A_v}$ from the factor $C_{\mathrm{g}(v),\mathrm{n}(v),\delta(v)}$ (where A_v is the vector formed by numbers a_i for i a marking at v, extended by 0 on the half-edges at v). Then using the projection formula together with [6, Proposition 25], we have

$$(j_{\Gamma_\delta})^* \pi_* c_1(\mathcal{L}_A)^2 = (\pi_{\Gamma_\delta})_* J^*_{\Gamma_\delta} c_1(\mathcal{L}_A)^2 = (G \circ \pi_{\Gamma_\delta})_* (G \circ \pi_{\Gamma_\delta})^* c_1(\mathcal{L}_A)^2$$

$$= \sum_{v \in \mathrm{V}(\Gamma_\delta)} (\pi'_{\Gamma_\delta,v})_* c_1(\mathcal{L}_{v,A_v})^2 = \sum_{v \in \mathrm{V}(\Gamma_\delta)} \eta(v) + \sum_{i=1}^n a_i^2 \psi_i + 2a_i \xi_i \,,$$

where for the last equality we again use Lemma 20. $\Diamond$

In the case $n = 0$ and $d = 0$, the formula $\mathsf{P}^\bullet_{g,\emptyset,0}$ takes a slightly simpler shape: it is the $r = 0$ term of the formula

$$\exp\left(-\frac{1}{2}\eta\right) \sum_{\substack{\Gamma_\delta \in \mathsf{G}_{g,0,0} \\ w \in \mathsf{W}_{\Gamma_\delta,r}}} \frac{r^{-h^1(\Gamma_\delta)}}{|\mathrm{Aut}(\Gamma_\delta)|}\, j_{\Gamma_\delta *}\left[\prod_{e=(h,h') \in \mathrm{E}(\Gamma_\delta)} \frac{1 - \exp\left(-\frac{w(h)w(h')}{2}(\psi_h + \psi_{h'})\right)}{\psi_h + \psi_{h'}} \right]. \tag{44}$$

As explained in Section 0.7, the full formula $\mathsf{P}^\bullet_{g,A,d}$ can be reconstructed from $\mathsf{P}^\bullet_{g,\emptyset,0}$.

3.2 Comparison to Pixton's k-twisted formula

Given $k \geq 0$ and a vector $A = (a_1, \ldots, a_n) \in \mathbb{Z}^n$ satisfying

$$\sum_i a_i = k(2g-2)\,,$$

let $\tilde{A} = (\tilde{a}_1, \ldots, \tilde{a}_n)$ be the vector with entries $\tilde{a}_i = a_i + k$. Denote by

$$P_g^{c,k}(\tilde{A}) \in \mathsf{CH}^c(\overline{\mathcal{M}}_{g,n})$$

Pixton's original formula defined in [44, Section 1.1].

In the $k = 0$ case, $A = \tilde{A}$, and $2^{-g} P_g^{g,0}(\tilde{A})$ is the class originally conjectured by Pixton to equal the double ramification cycle associated to the vector $\tilde{A}$. Compatibility with the formula for the universal twisted double ramification cycle is given by the following result.

Proposition 21. *Via the map $\varphi_{\omega_\pi^k} : \overline{\mathcal{M}}_{g,n} \to \mathfrak{Pic}_{g,n,k(2g-2)}$ associated to the universal data*

$$\pi : \mathcal{C}_{g,n} \to \overline{\mathcal{M}}_{g,n}, \qquad \omega_\pi^k \to \mathcal{C}_{g,n},$$

the class $\mathsf{P}^c_{g,A,k(2g-2)}$ acts as

$$\mathsf{P}^c_{g,A,k(2g-2)}(\varphi_{\omega_\pi^k})([\overline{\mathcal{M}}_{g,n}]) = 2^{-c} P_g^{c,k}(\tilde{A}) \tag{45}$$

for every $c \geq 0$.

Proof. The left-hand side of (45) is obtained from $\mathsf{P}^c_{g,A,k(2g-2)}$ by substituting

$$\mathcal{L} = \omega_\pi^{\otimes k} \tag{46}$$

in the formula and taking the action. A factor 2^{-c} arises on the left side since all terms in $\mathsf{P}^c_{g,A,k(2g-2)}$ increasing the codimension of the cycle naturally come with corresponding negative powers of 2 (which is placed as a prefactor on the right side in [44, Section 1.1]).

Under the substitution (46), the edge terms and weightings modulo r in the two formulas naturally correspond to each other. Using Proposition 19 and Lemma 20, we must show

$$\exp\left(-\frac{1}{2}\pi_* c_1(\omega_\pi^{\otimes k}(-\sum_{i=1}^n a_i[p_i]))^2\right) = \exp\left(-\frac{1}{2}\left(k^2\kappa_1 - \sum_{i=1}^n \tilde{a}_i^2 \psi_i\right)\right),$$

where again $[p_i]$ denotes the class of the image of the section $p_i : \overline{\mathcal{M}}_{g,n} \to \mathcal{C}_{g,n}$. Defining $\omega_\pi^{\log} = \omega_\pi(\sum_i p_i)$, we see

$$\begin{aligned} c_1(\omega_\pi^{\otimes k}(-\sum_{i=1}^n a_i[p_i]))^2 &= (kc_1(\omega_{\log}) - \sum_{i=1}^n \tilde{a}_i[p_i])^2 \\ &= k^2 c_1(\omega_{\log})^2 - 2k\sum_{i=1}^n \tilde{a}_i c_1(\omega_{\log})|_{[p_i]} + \sum_{i=1}^n \tilde{a}_i^2 [p_i]^2 \end{aligned}$$

After pushing forward, the first term gives $k^2\kappa_1$, the second vanishes (since $\omega_{\log}$ restricts to zero on the section p_i), and the third gives $-\sum_i \tilde{a}_i^2\psi_i$, as desired. $\Diamond$

3.3 Comparison to Pixton's formula with targets

Let X be a nonsingular projective variety over K. The moduli space $\overline{\mathcal{M}}_{g,n}(X,\beta)$ parametrizes stable maps

$$f\colon (C, p_1, \dots, p_n) \to X$$

from genus g, n-pointed curves C to X of degree $\beta \in H_2(X, \mathbb{Z})$. The moduli space carries a virtual fundamental class

$$[\overline{\mathcal{M}}_{g,n}(X,\beta)]^{\mathrm{vir}} \in \mathsf{CH}_{\mathrm{vdim}(g,n,\beta)}(\overline{\mathcal{M}}_{g,n}(X,\beta))$$

where

$$\mathrm{vdim}(g,n,\beta) = (\dim X - 3)(1-g) + \int_\beta c_1(X) + n\,.$$

See [12] for the construction of virtual fundamental classes.

Given the data of a line bundle $\mathcal{L}$ on X and a vector $A = (a_1, \dots, a_n) \in \mathbb{Z}^n$ satisfying

$$\int_\beta c_1(\mathcal{L}) = \sum_{i=1}^n a_i\,,$$

a double ramification cycle

$$\mathsf{DR}_{g,A}(X, \mathcal{L}) \in \mathsf{CH}_{\mathrm{vdim}(g,n,\beta)-g}(\overline{\mathcal{M}}_{g,n}(X,\beta))$$

virtually compactifying the locus of maps $f\colon (C, p_1, \cdots, p_n) \to X$ with

$$f^*\mathcal{L} \cong \mathcal{O}_C\left(\sum_{i=1}^n a_i p_i\right)$$

is defined in [45]. Furthermore, the authors define the notion of tautological classes inside the operational Chow ring $\mathsf{CH}^*_{\mathsf{op}}(\overline{\mathcal{M}}_{g,n}(X,\beta))$ of $\overline{\mathcal{M}}_{g,n}(X,\beta)$. The main result of [45] is a Pixton formula for a codimension g tautological class whose action on $[\overline{\mathcal{M}}_{g,n}(X,\beta)]^{\mathrm{vir}}$ yields $\mathsf{DR}_{g,A}(X, \mathcal{L})$.

We define a morphism

$$\varphi_{\mathcal{L}}\colon \overline{\mathcal{M}}_{g,n,\beta}(X) \to \mathfrak{Pic}_{g,A,d}\,, \quad f \mapsto (C, p_1, \dots, p_n, f^*\mathcal{L})\ .$$

The compatibility result here is

$$\mathsf{DR}_{g,A}(X,\mathcal{L}) = \varphi^*_{\mathcal{L}} \mathsf{P}^g_{g,A,d}\Big([\overline{\mathcal{M}}_{g,n}(X,\beta)]^{\mathrm{vir}}\Big)\,. \tag{47}$$

The equality follows by an exact matching of the definition of $\mathsf{P}^g_{g,\mathbf{A},d}$ in Section 0.3.5 (after pullback by $\varphi^*_{\mathcal{L}}$) with the Pixton formula in the main result of [45].

In fact, the compatibility (47) represented the starting point for our investigation of the universal twisted double ramification cycle here.

4 Proof of Theorem 7

4.1 Overview

We prove here the main result of the paper: for $A = (a_1, \ldots, a_n) \in \mathbb{Z}^n$ satisfying

$$\sum_{i=1}^{n} a_i = d\,,$$

the universal twisted double ramification cycle is calculated by Pixton's formula

$$\mathsf{DR}^{\mathsf{op}}_{g,A} = \mathsf{P}^g_{g,A,d} \ \in \mathsf{CH}^g_{\mathsf{op}}(\mathfrak{Pic}_{g,n,d})\,.$$

The result is an equality in the operational Chow group, and therefore an equality on every finite type family of prestable curves. Given $C \to B$ a prestable curve and a line bundle $\mathcal{L}$ on C of relative degree d, we obtain a map

$$\varphi_{\mathcal{L}} \colon B \to \mathfrak{Pic}_{g,n,d}\,.$$

We must prove

$$\mathsf{DR}^{\mathsf{op}}_{g,A}(\varphi_{\mathcal{L}}) = \mathsf{P}^g_{g,A,d}(\varphi_{\mathcal{L}}) : \mathsf{CH}_*(B) \to \mathsf{CH}_{*-g}(B)\,. \tag{48}$$

As explained in Section 0.7, the result for general $A \in \mathbb{Z}^n$ can be reduced to the case $n = 0, d = 0$, though the case of arbitrary A will be important in the proof as we proceed through a sequence of special cases. We recall that this reduction used the invariances II and III for the double ramification cycle and Pixton's formula. Note that these will be proved separately and independent of Theorem 7 in [6, Section 7], so no circular reasoning occurs.

4.2 On an open subset of $\overline{\mathcal{M}}_{g,n}(\mathbb{P}^l, \beta)$

As before, let $A = (a_1, \ldots, a_n) \in \mathbb{Z}^n$ with

$$\sum_{i=1}^{n} a_i = d\,.$$

We consider here the target $X = \mathbb{P}^l$. Let β be the class of d times a line in $\mathbb{P}^l$. Let

$$\mathcal{C} \to \overline{\mathcal{M}}_{g,n}(\mathbb{P}^l, \beta)$$

be the universal curve over the moduli of stable maps to $\mathbb{P}^l$, let

$$f \colon \mathcal{C} \to \mathbb{P}^l$$

be the universal map, and let $\mathcal{L} = f^*\mathcal{O}_X(1)$.

We have a tautological map

$$\varphi_{\mathcal{L}} \colon \overline{\mathcal{M}}_{g,n}(\mathbb{P}^l, \beta) \to \mathfrak{Pic}_{g,n,d}. \tag{49}$$

We would like to prove an equality of operational classes

$$\varphi_{\mathcal{L}}^* \mathsf{DR}_{g,A}^{\mathsf{op}} = \varphi_{\mathcal{L}}^* \mathsf{P}_{g,A,d}^g \in \mathsf{CH}_{\mathsf{op}}^g(\overline{\mathcal{M}}_{g,n}(\mathbb{P}^l, \beta)) .$$

We will apply the main result of [45] which relates the double ramification cycle there to Pixton's formula. However, only the action of $\varphi_{\mathcal{L}}^* \mathsf{P}_{g,A,d}^g$ on the virtual fundamental class $[\overline{\mathcal{M}}_{g,n}(\mathbb{P}^l, \beta)]^{\mathrm{vir}}$ is computed in [45]. Since we are interested here in the full operational class $\varphi_{\mathcal{L}}^* \mathsf{P}_{g,A,d}^g$, our first idea is to restrict to the open locus

$$\overline{\mathcal{M}}_{g,n}(\mathbb{P}^l, \beta)' \hookrightarrow \overline{\mathcal{M}}_{g,n}(\mathbb{P}^l, \beta)$$

where (on each geometric fibre) we have $H^1(C, \mathcal{L}) = 0$.

Lemma 22. *On the smooth Deligne-Mumford stack $\overline{\mathcal{M}}_{g,n}(\mathbb{P}^l, \beta)'$, the fundamental and virtual fundamental classes coincide.*

Proof. It suffices to show that $H^1(C, f^*T_{\mathbb{P}^l}) = 0$ on $\overline{\mathcal{M}}_{g,n}(\mathbb{P}^l, \beta)'$. Pulling back the Euler exact sequence on $\mathbb{P}^l$ via f yields

$$0 \to \mathcal{O}_C \to \oplus_1^{l+1} f^*\mathcal{O}_{\mathbb{P}^l}(1) \to f^*T_{\mathbb{P}^l} \to 0 . \tag{50}$$

Taking cohomology yields the exact sequence

$$\oplus_1^{l+1} H^1(C, f^*\mathcal{O}_{\mathbb{P}^l}(1)) \to H^1(C, f^*T_{\mathbb{P}^l}) \to H^2(C, \mathcal{O}_C) . \tag{51}$$

But $H^1(C, f^*\mathcal{O}_{\mathbb{P}^l}(1)) = 0$ by assumption, and $H^2(C, \mathcal{O}_C) = 0$ for dimension reasons. $\Diamond$

The next Lemma depends on a careful comparison of the logarithmic and rubber approaches to double ramification cycles, which will be postponed to Section 5.

Lemma 23. *Let $\varphi'_{\mathcal{L}}$ be the restriction of $\varphi_{\mathcal{L}}$ to $\overline{\mathcal{M}}_{g,n}(\mathbb{P}^l, \beta)'$. We have an equality of operational classes*

$$\varphi'^*_{\mathcal{L}} \mathsf{DR}_{g,A}^{\mathsf{op}} = \varphi'^*_{\mathcal{L}} \mathsf{P}_{g,A,d}^g \in \mathsf{CH}_{\mathsf{op}}^g(\overline{\mathcal{M}}_{g,n}(\mathbb{P}^l, \beta)') . \tag{52}$$

Proof. By [6, Lemma 15], the two sides of (52) are equal if and only if their actions on the fundamental class $[\overline{\mathcal{M}}_{g,n}(\mathbb{P}^l, \beta)']$ are equal in $\mathsf{CH}_*(\overline{\mathcal{M}}_{g,n}(\mathbb{P}^l, \beta)')$. By (47), the action of the right side of (52) on

$$[\overline{\mathcal{M}}_{g,n}(\mathbb{P}^l, \beta)'] = [\overline{\mathcal{M}}_{g,n}(\mathbb{P}^l, \beta)']^{\mathrm{vir}}$$

equals the restriction of $\mathsf{DR}_{g,A}(\mathbb{P}^l, \mathcal{L})$ to $\overline{\mathcal{M}}_{g,n}(\mathbb{P}^l, \beta)'$.

The cycle $\mathsf{DR}_{g,A}(\mathbb{P}^l, \mathcal{L})$ is defined in [45] as the pushforward of the virtual fundamental class of the space of rubber maps[22]. By Proposition 39 of Section 5.5, the restriction of $\mathsf{DR}_{g,A}(X, \mathcal{L})$ to $\overline{\mathcal{M}}_{g,n}(\mathbb{P}^l, \beta)'$ is equal to $\varphi'^*_{\mathcal{L}}\mathsf{DR}^{\mathsf{op}}_{g,A}([\overline{\mathcal{M}}_{g,n}(\mathbb{P}^l, \beta)'])$.

$\diamondsuit$

4.3 For sufficiently positive line bundles

Let $\pi\colon C \to B$ be an n-pointed prestable curve over a scheme of finite type over K. Let $\mathcal{L}$ on C be a line bundle of relative degree d. Let

$$A = (a_1, \dots, a_n) \in \mathbb{Z}^n$$

with $\sum_{i=1}^n a_i = d$. The line bundle $\mathcal{L}$ induces a map

$$\varphi_{\mathcal{L}}\colon B \to \mathfrak{Pic}_{g,n,d}\,.$$

We say $\mathcal{L}$ is *relatively sufficiently positive* if $\mathcal{L}$ is relatively base-point free and satisfies $R^1\pi_*\mathcal{L} = 0$.

Lemma 24. *Let $\mathcal{L}$ be a line bundle which is relatively sufficiently positive. Then we have an equality*

$$\mathsf{DR}^{\mathsf{op}}_{g,A}(\varphi_{\mathcal{L}}) = \mathsf{P}^g_{g,A,d}(\varphi_{\mathcal{L}})\colon \mathsf{CH}_*(B) \to \mathsf{CH}_{*-g}(B)\,. \tag{53}$$

Proof. For any finite-type scheme B the union of irreducible components of B maps properly and surjectively to B. Thus the pushforward from the Chow groups of the irreducible components to that of B is surjective, and hence it suffices to show the equality (53) of maps of Chow groups for B irreducible.

By relative sufficient positivity,

$$R\pi_*\mathcal{L} = \pi_*\mathcal{L}$$

is a vector bundle on B of rank N. For a positive integer l, we define

$$E_l = \bigoplus_1^{l+1} R\pi_*\mathcal{L}\,, \tag{54}$$

a vector bundle on B of rank $r = N(l+1)$. Let $U_l \subseteq E_l$ denote the open locus of linear systems which are base-point free. Via pullback along $\psi\colon U_l \to B$, we obtain a map

$$\psi^*\colon \mathsf{CH}_*(B) \to \mathsf{CH}_{*+r}(U_l)\,.$$

[22] Rubber maps will be discussed in Section 5.3.

We claim that for $l > \dim B$, the pullback (54) is injective. To prove the injectivity, we show that the boundary $E_l \setminus U_l$ has codimension in E_l greater than $\dim B$. Since $E_l \to B$ is flat with irreducible target, it suffices to bound the codimension on each geometric fibre over B: for a prestable curve C/K and a sufficiently positive line bundle $\mathcal{L}$ on C, we must show that the locus in $\bigoplus_1^{l+1} H^0(C, \mathcal{L})$ consisting of base point free linear systems has a complement of codimension greater than $\dim B$.

Since $\mathcal{L}$ is base point free on C, the dimension of the locus in $\bigoplus_1^{l+1} H^0(C, \mathcal{L})$ where the linear system has a base point at some given $p \in C$ is $(N-1)(l+1)$. Hence, as p varies, the complement of the base point free locus in $\bigoplus_1^{l+1} H^0(C, \mathcal{L})$ has dimension at most $1 + (N-1)(l+1)$. So the codimension is at least

$$N(l+1) - 1 - (N-1)(l+1) = l\,.$$

We have a canonical map $g\colon C \times_B U_l \to \mathbb{P}^l$ with $g^*\mathcal{O}_{\mathbb{P}^l}(1) = \mathcal{L}$ which induces a map

$$U_l \to \overline{\mathcal{M}}_{g,n}(\mathbb{P}^l, \beta)$$

which factors via the locus

$$\overline{\mathcal{M}}_{g,n}(\mathbb{P}^l, \beta)' \subset \overline{\mathcal{M}}_{g,n}(\mathbb{P}^l, d)$$

where $H^1(C, f^*\mathcal{O}_{\mathbb{P}^l}(1)) = 0$. By construction, the composition

$$U_l \xrightarrow{\psi} B \xrightarrow{\varphi_{\mathcal{L}}} \mathfrak{Pic}_{g,n,d}$$

then factors through the map $\overline{\mathcal{M}}_{g,n}(\mathbb{P}^l, \beta)' \to \mathfrak{Pic}_{g,n,d}$ induced by the line bundle $f^*\mathcal{O}_{\mathbb{P}^l}(1)$ as before. In other words, we have a commutative diagram

$$\begin{tikzcd}
U_l \arrow[r] \arrow[d, "\psi"] & \overline{\mathcal{M}}_{g,n}(\mathbb{P}^l, \beta)' \arrow[d, "\varphi_{f^*\mathcal{O}_{\mathbb{P}^l}(1)}"] \\
B \arrow[r, "\varphi_{\mathcal{L}}"] & \mathfrak{Pic}_{g,n,d}\,.
\end{tikzcd}$$

Lemma 23 then implies that

$$\left(\mathsf{DR}^{\mathsf{op}}_{g,A} - \mathsf{P}^g_{g,A,d}\right)(\varphi_{\mathcal{L}} \circ \psi)\colon \mathsf{CH}_*(U_l) \to \mathsf{CH}_{*-g}(U_l) \tag{55}$$

is the zero map, and we conclude the proof of the Lemma from the commutative diagram

$$\begin{tikzcd}
\mathsf{CH}_{*+r}(U) \arrow[r, "\left(\mathsf{DR}^{\mathsf{op}}_{g,A} - \mathsf{P}^g_{g,A,d}\right)(\varphi_{\mathcal{L}} \circ \psi)"] & \mathsf{CH}_{*+r-g}(U) \\
\mathsf{CH}_*(B) \arrow[u, "\psi^*"] \arrow[r, "\left(\mathsf{DR}^{\mathsf{op}}_{g,A} - \mathsf{P}^g_{g,A,d}\right)(\varphi_{\mathcal{L}})"] & \mathsf{CH}_{*-g}(B)\,. \arrow[u, "\psi^*"']
\end{tikzcd} \tag{56}$$

$\Diamond$

4.4 With sufficiently many sections

Let $\pi\colon C \to B$ be an n-pointed prestable curve with markings $p_1, \ldots, p_n$ over a scheme of finite type over K. Let $\mathcal{L}$ on C be a line bundle of relative degree d. Let

$$A = (a_1, \ldots, a_n) \in \mathbb{Z}^n$$

with $\sum_{i=1}^n a_i = d$. The line bundle $\mathcal{L}$ induces a map

$$\varphi_{\mathcal{L}}\colon B \to \mathfrak{Pic}_{g,n,d}\,.$$

Lemma 25. *For every geometric fibre of C/B, suppose the complement of the union of irreducible components which carry markings is a disjoint union of trees of nonsingular rational curves on which $\mathcal{L}$ is trivial. Then we have an equality*

$$\mathsf{DR}^{\mathsf{op}}_{g,A}(\varphi_{\mathcal{L}}) = \mathsf{P}^g_{g,A,d}(\varphi_{\mathcal{L}})\colon \mathsf{CH}_*(B) \to \mathsf{CH}_{*-g}(B)\,. \tag{57}$$

Proof. We can choose $A' = (a'_1, \ldots, a'_n)$ with entries

$$a'_i \gg 0\,, \quad \sum_{i=1}^n a'_i = d'$$

large enough so that

$$\mathcal{L}' = \mathcal{L}\Big(\sum_{i=1}^n a'_i p_i\Big)$$

is relatively sufficiently positive (by Riemann-Roch for singular curves).

We obtain an associated map

$$\varphi_{\mathcal{L}'}\colon B \to \mathfrak{Pic}_{g,n,d+d'}\,.$$

By Lemma 24,

$$\mathsf{DR}^{\mathsf{op}}_{g,A+A'}(\varphi_{\mathcal{L}'}) = \mathsf{P}^g_{g,A+A',d+d'}(\varphi_{\mathcal{L}'})\colon \mathsf{CH}_*(B) \to \mathsf{CH}_{*-g}(B). \tag{58}$$

Invariance III of Section 0.6 implies

$$\mathsf{DR}^{\mathsf{op}}_{g,A+A'}(\varphi_{\mathcal{L}'}) = \mathsf{DR}^{\mathsf{op}}_{g,A}(\varphi_{\mathcal{L}})\,,$$
$$\mathsf{P}^g_{g,A+A',d+d'}(\varphi_{\mathcal{L}'}) = \mathsf{P}^g_{g,A,d}(\varphi_{\mathcal{L}})\,,$$

which together with (58) finishes the proof. $\Diamond$

4.5 Proof in the general case

To conclude the proof of Theorem 7, will use the invariances of Section 0.6. As discussed in Section 4.1, we can reduce to showing the result in the case $n = 0, d = 0$.[23]

Let B be an irreducible scheme of finite type over K. Let $\pi\colon C \to B$ a prestable curve, and let $\mathcal{L}$ on C be a line bundle of relative degree 0. The line bundle $\mathcal{L}$ induces a map

$$\varphi_{\mathcal{L}}\colon B \to \mathfrak{Pic}_{g,0,0}\,.$$

Lemma 26. *There exists an alteration*[24] $B' \to B$ *and a destabilisation*

$$C' \to C \times_B B' \tag{59}$$

such that C' admits sections $p_1, \ldots, p_m$ and satisfies the following property:

$$(C'/B', p_1, \ldots, p_m)$$

is a family of m-pointed prestable curves and for every geometric fibre of C'/B', the complement of the union of irreducible components which carry markings is a disjoint union of trees of nonsingular rational curves which are contracted by the morphism (59).

Proof. We first claim, after an alteration $\widehat{B} \to B$, there exists a multisection[25]

$$Z \subset C_{\widehat{B}} = C \times_B \widehat{B} \to \widehat{B}$$

satisfying the following two conditions:

(i) Over the generic point of $\widehat{B}$, Z is contained in the smooth locus of $C_{\widehat{B}} \to \widehat{B}$.

(ii) Every component of every geometric fibre of $C_{\widehat{B}} \to \widehat{B}$ carries at least two distinct étale multisection points in the smooth locus. In other words, the étale locus of $Z \to \widehat{B}$ meets the smooth locus of every component of every geometric fibre of $C_{\widehat{B}} \to \widehat{B}$ in at least two points.

[23]In genus $g = 1$, we follow a slightly modified strategy since there we must avoid the case $n = 0$ for technical reasons. Instead, we can use the invariances to reduce to the case $g = 1$, $n = 1$, and $A = (0)$. All the proofs below generalize in a straightforward way since the vector $A = (0)$ does not affect the line bundles involved.

[24]An alteration here is a proper, surjective, generically finite morphism between irreducible schemes.

[25]By a multisection of $C_{\widehat{B}} \to \widehat{B}$, we mean a closed substack $Z \subset C_{\widehat{B}}$ such that $Z \to \widehat{B}$ is finite and flat.

To prove the above claim, we observe that for every geometric point b of B there exists an étale map $U_p \to B$ and a factorisation $U_p \to C$ whose image meets the smooth locus of every irreducible component of every geometric fibre in some Zariski neighbourhood $V_b \subseteq B$ of b at least twice. Choose a finite set of b such that the V_b cover B, define U to be the union of the U_b, and define Z' to be the closure of the image of U in C. Then $Z' \to B$ is proper and generically finite. Let

$$\widehat{B} \to B$$

be a modification which flattens Z' (see [69]), and let Z be the strict transform of Z' over $\widehat{B}$. Then

$$Z \to \widehat{B}$$

is proper, flat, and generically finite, and hence finite – so condition (i) is satisfied. Moreover, U already satisfies condition (ii), and the strict transform of a flat map is just the fibre product, hence Z also satisfies condition (ii).

Let $\widetilde{B} \to \widehat{B}$ be an alteration such that over $\widetilde{B}$ the multisection Z becomes a disjoint union of sections. In other words the pullback

$$C_{\widetilde{B}} = C \times_B \widetilde{B} \to \widetilde{B}$$

has sections $\widetilde{p}_1, \ldots, \widetilde{p}_m$ such that, as a set, the preimage of Z is given by the union of the images of sections $\widetilde{p}_1, \ldots, \widetilde{p}_m$. Such a $\widetilde{B}$ exists[26] by [24, Lemma 5.6]. We can assume that the sections $\widetilde{p}_i$ are pairwise disjoint over the generic point of $\widetilde{B}$.

By assumption (i) above, the family $C_{\widetilde{B}} \to \widetilde{B}$ with sections $\widetilde{p}_1, \ldots, \widetilde{p}_m$ is generically a stable m-pointed curve (since every component has at least *two* of the sections). We therefore obtain a rational map

$$\widetilde{B} \dashrightarrow \overline{\mathcal{M}}_{g,m}\,.$$

Let $B' \to \widetilde{B}$ be a blow-up resolving the indeterminacy of this map[27]

$$\begin{tikzcd} B' \arrow[r] \arrow[d] & \overline{\mathcal{M}}_{g,m} \\ \widetilde{B} \arrow[ur, dashed] & \end{tikzcd} \tag{60}$$

and let $C' \to B'$ with sections $p_1, \ldots, p_m : B' \to C'$ be the pullback of the universal curve over $\overline{\mathcal{M}}_{g,n}$ to B'. Let

$$C_{B'} = C \times_B B'$$

[26] The base $\widehat{B}$ is excellent since it is finite type over a field.

[27] As usual, this blowup is constructed by taking the closure of the graph and flattening. Then we check that this ensures the existence of the map to $C_{B'}$ as written below.

be the pullback of C/B under $B' \to \widetilde{B} \to \widehat{B} \to B$. Then we have a map $f : C' \to C_{B'}$ fitting in a commutative diagram

$$\begin{tikzcd} C' \arrow[rr, "f"] \arrow[dr] & & C_{B'} \arrow[dl] \\ & B' \arrow[ul, bend left, "p_i"] \arrow[ur, bend right, "\widetilde{p}_i"'] & \end{tikzcd} \tag{61}$$

such that f is a partial destabilization. On geometric fibers of $C' \to B'$, f collapses trees of rational curves to either nodes or coincident sections $\widetilde{p}_i$ on the geometric fibers of $C_{B'}$.

To conclude, we must show that for every geometric point $b \in B'$ and every irreducible component $D \subset C'_b$ which is not contracted by f, we can find a marking

$$p_i(b) \in D\,.$$

The image of D under f is a component of $(C_{B'})_b$. By condition (ii) above, $f(D)$ has at least one $\widetilde{p}_i(b)$ in the smooth locus of $f(D)$ pairwise distinct from all other $\widetilde{p}_j(b)$. Since there are no components of C'_b which collapse to $\widetilde{p}_i(b)$, we must have $p_i(b) \in D$. $\Diamond$

Lemma 27. *We have* $\mathsf{DR}^{\mathsf{op}}_{g,\emptyset}(\varphi_{\mathcal{L}}) = \mathsf{P}^g_{g,\emptyset,0}(\varphi_{\mathcal{L}}) : \mathsf{CH}_*(B) \to \mathsf{CH}_{*-g}(B)$.

Proof. We apply Lemma 26 to the family C/B to obtain

$$h\colon B' \to B\,, \qquad C' \to C_{B'}\,.$$

Let $\mathcal{L}'$ be the pullback of $\mathcal{L}$ to C'. After applying Lemma 25 with $A = \mathbf{0} \in \mathbb{Z}^m$, we obtain

$$\mathsf{DR}^{\mathsf{op}}_{g,\mathbf{0}}(\varphi_{\mathcal{L}'}) = \mathsf{P}^g_{g,\mathbf{0},d}(\varphi_{\mathcal{L}'}) : \mathsf{CH}_*(B') \to \mathsf{CH}_{*-g}(B')\,. \tag{62}$$

Since h is proper and surjective, for any $\alpha \in \mathsf{CH}_*(B)$ there exists $\alpha' \in \mathsf{CH}_*(B')$ satisfying $h_*\alpha' = \alpha$. If any operational class maps α' to 0, then it maps α to 0 because the operation commutes with h_*.

It therefore suffices to prove

$$\left(\mathsf{DR}^{\mathsf{op}}_{g,\emptyset} - \mathsf{P}^g_{g,\emptyset,0}\right)(\varphi_{\mathcal{L}}) \circ h_* \tag{63}$$

is the zero map on $\mathsf{CH}_*(B')$. By the compatibilities of operational classes we have

$$\left(\mathsf{DR}^{\mathsf{op}}_{g,\emptyset} - \mathsf{P}^g_{g,\emptyset,0}\right)(\varphi_{\mathcal{L}}) \circ h_* = h_*\left(\mathsf{DR}^{\mathsf{op}}_{g,\emptyset} - \mathsf{P}^g_{g,\emptyset,0}\right)(\varphi_{\mathcal{L}} \circ h)$$

and the proof below will in fact show $\left(\mathsf{DR}^{\mathsf{op}}_{g,\emptyset} - \mathsf{P}^g_{g,\emptyset,0}\right)(\varphi_{\mathcal{L}} \circ h) = 0$.

By (62), we need only show

$$\mathsf{DR}^{\mathsf{op}}_{g,\emptyset}(\varphi_{\mathcal{L}} \circ h) = \mathsf{DR}^{\mathsf{op}}_{g,\mathbf{0}}(\varphi_{\mathcal{L}'}) : \mathsf{CH}_*(B') \to \mathsf{CH}_{*-g}(B'), \tag{64}$$

$$\mathsf{P}^g_{g,\emptyset,0}(\varphi_{\mathcal{L}} \circ h) = \mathsf{P}^g_{g,\mathbf{0},0}(\varphi_{\mathcal{L}'}) : \mathsf{CH}_*(B') \to \mathsf{CH}_{*-g}(B')\,. \tag{65}$$

For the map $F : \mathfrak{Pic}_{g,n,0} \to \mathfrak{Pic}_{g,0,0}$ forgetting the markings, Invariance II from Section 0.6 for the double ramification cycle and the Pixton formula shows that we have

$$\mathsf{DR}^{\mathsf{op}}_{g,\mathbf{0}}(\varphi_{\mathcal{L}'}) = \mathsf{DR}^{\mathsf{op}}_{g,\emptyset}(F \circ \varphi_{\mathcal{L}'})$$
$$\mathsf{P}^g_{g,\mathbf{0},0}(\varphi_{\mathcal{L}'}) = \mathsf{P}^g_{g,\emptyset,0}(F \circ \varphi_{\mathcal{L}'})$$

So we are reduced to showing

$$\mathsf{DR}^{\mathsf{op}}_{g,\emptyset}(\varphi_{\mathcal{L}} \circ h) = \mathsf{DR}^{\mathsf{op}}_{g,\emptyset}(F \circ \varphi_{\mathcal{L}'}) \quad \text{and} \quad \mathsf{P}^g_{g,\emptyset,0}(\varphi_{\mathcal{L}} \circ h) = \mathsf{P}^g_{g,\emptyset,0}(F \circ \varphi_{\mathcal{L}'})\ . \tag{66}$$

The claims (66) follow from Invariance VI of Section 0.6. As before, let $C_{B'}$ be the pullback of C under h, and let $\mathcal{L}_{B'}$ be the pullback of $\mathcal{L}$ to $C_{B'}$. The map

$$\varphi_{\mathcal{L}} \circ h : B' \to \mathfrak{Pic}_{g,0,0}$$

is induced by the data

$$C_{B'} \to B'\,, \quad \mathcal{L}_{B'} \to C_{B'}\,,$$

whereas $F \circ \varphi_{\mathcal{L}'} : B' \to \mathfrak{Pic}_{g,0,0}$ is induced by

$$C' \to B'\,, \quad \mathcal{L}' \to C'\,.$$

By construction, we have a partial destabilization $C' \to C_{B'}$ over B', and the line bundle $\mathcal{L}'$ is the pullback of $\mathcal{L}_{B'}$ under this map. Hence the equalities (66) follow from Invariance VI of Section 0.6. ◊

5 Comparing rubber and log spaces

5.1 Overview

Our goal here is to compare the stack of stable rubber maps associated to a line bundle $\mathcal{L}$ on a target X (introduced by Li [53] and studied by Graber-Vakil [34]) to the stack $\mathbf{Rub}_{g,A}$ of Marcus-Wise (see Section 2.3) and our operational class $\mathsf{DR}^{\mathsf{op}}_{g,A}$. Rubber maps are reviewed in Section 5.3 and connected to the logarithmic space in Section 5.4. The relationship between the construction of Marcus-Wise and $\mathsf{DR}^{\mathsf{op}}_{g,A}$ is

[6, Lemma 55]. The comparison to the class of Graber-Vakil is carried out in [56] in the case where the target X is a point. We require the case where

$$X = \mathbb{P}^l \quad \text{and} \quad \mathcal{L} = \mathcal{O}(1)\,,$$

but only over the unobstructed locus

$$\overline{\mathcal{M}}_{g,n}(\mathbb{P}^l, d)' \subset \overline{\mathcal{M}}_{g,n}(\mathbb{P}^l, d)\,,$$

see Section 4.2. We will treat the case of a general nonsingular projective target X since restricting to $\mathbb{P}^l$ provides no simplification (though the unobstructed locus may be rather small for general X). The final comparison result is Proposition 39 in Section 5.5.3.

5.2 Refined definition of the logarithmic rubber space

As described in Section 2.4, Marcus and Wise define $\mathbf{Rub}_g^{\mathsf{rel}}$ to be the moduli space of pairs (C, P, α) where P is a tropical line on S and

$$\alpha\colon C \to P$$

is an S-morphism such that on each geometric fibre over S the values taken by α on the irreducible components of C are totally ordered in $(\bar{M}_S^{gp})_s$. However, with the above definition, certain key results of their paper (in particular concerning the comparison to spaces of rubber maps) are not correct as stated.

To explain the problem, we restrict to the case where the base C is a geometric log point. Subdividing P at the images of the vertices of C under the map α yields a *divided tropical line* Q (in the language of [56]). It is asserted in the discussion above [56, Proposition 5.5.2] that the fibre product $C \times_P Q$ is again a log curve over S, which, in general, is not true. For example, take $\bar{M}_S$ to be the sub-monoid of $\mathbb{Z}^2$ generated by $(1,1)$, $(1,0)$, and $(1,-1)$, and C, α to be as illustrated in Figure 1. In the fibre product, the edge with length $(1,0)$ must be subdivided into two shorter edges, but $(1,0)$ is an irreducible element of $\bar{M}_S$. In fact, *failure of divisibility* is the only thing that can go wrong.

Lemma 28. *Let (C, P, α) be a point of $\mathbf{Rub}_g^{\mathsf{rel}}$ over a geometric log point B, and let Q be obtained from P by subdividing at the image of α. Then the following are equivalent:*

(i) The fibre product $C \times_P Q$ is a log curve over B.

(ii) Let e be an edge of Γ_C between vertices u and v (satisfying $\alpha(v) \geq \alpha(u)$) with length $\ell_e \in \bar{M}_S$ and slope $\kappa_e = \frac{\alpha(v)-\alpha(u)}{\ell_e}$. Then, for every $y \in \mathrm{Image}(\alpha)$ with $\alpha(u) < y < \alpha(v)$, the monoid $\bar{M}_B$ contains the element $\frac{y-\alpha(u)}{\kappa_e}$.

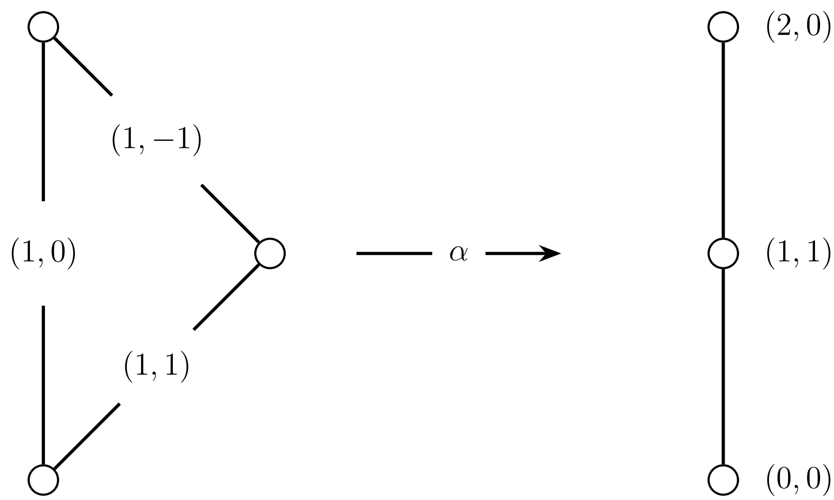

Figure 1: A point of **Rub**

Proof. The characteristic monoid at a singular point with length ℓ_e is given by the monoid

$$\{(a, b) \in \bar{M}_b^2 : \ell_e \mid a - b\}\,. \tag{67}$$

Taking the fibre product over P with Q subdivides the characteristic monoid at the element

$$\frac{y - \alpha(u)}{\kappa_e} \in \bar{M}_B^{\mathsf{gp}} \otimes_{\mathbb{Z}} \mathbb{Q}\,.$$

If $\frac{y-\alpha(u)}{\kappa_e}$ lies in $\bar{M}_B$, then the fiber product is easily seen to be a log curve. If not, then the subdivision is not even reduced. ◊

Definition 29. We define $\widetilde{\mathbf{Rub}}^{\mathsf{rel}}$ to be the full subcategory of $\mathbf{Rub}^{\mathsf{rel}}$ consisting of objects (C, P, α) which, on each geometric fibre over B, satisfy the equivalent conditions of Lemma 28. We define $\widetilde{\mathbf{Rub}}$ to be the fibre product of $\widetilde{\mathbf{Rub}}^{\mathsf{rel}}$ over $\mathfrak{Pic}^{\mathrm{rel}}$ with $\mathfrak{Pic}$.

Remark 30. The double ramification cycle $\mathsf{DR}^{\mathsf{op}}$ can be defined as the operational class induced by the map $\mathbf{Rub} \to \mathfrak{Pic}$ following [6, Definition 17]. Applying the same definition to the composite map $\widetilde{\mathbf{Rub}} \to \mathfrak{Pic}$ yields the same operational class, by [6, Proposition 25].

5.3 The stack of prestable rubber maps

Let $\mathfrak{M}(X)$ be the stack of maps from marked prestable curves to X. An S-point of $\mathfrak{M}(X)$ is a pair

$$(C/S,\ f\colon C \to X)$$

where C/S is prestable with markings. To simplify notation, we will often suppress the markings.

The space of rubber maps associated to a line bundle $\mathcal{L}$ on X is summarised in [45]: *a map to rubber with target X is a map to a rubber chain of $\mathbb{CP}^1$-bundles $\mathbb{P}(\mathcal{O}_X \oplus \mathcal{L})$ over X attached along their 0 and ∞ divisors.*

To facilitate our comparison, we begin by writing the definition explicitly. Let $\mathbf{P}$ denote the projective bundle $\mathbb{P}(\mathcal{O}_X \oplus \mathcal{L})$. The map collapsing the fibers,

$$\rho : \mathbf{P} \to X \tag{68}$$

admits two sections $r_0, r_\infty : X \to \mathbf{P}$ corresponding to $\mathcal{O}_X$ and $\mathcal{L}$ respectively.

Definition 31. An $(X, \mathcal{L})$-rubber target $(R/S, \rho, r_0, r_\infty)$ is flat, proper, and finitely presented

$$R \to S$$

and a collapsing map $\rho : R \to X_S$ with two sections

$$r_0, r_\infty : X_S \to R$$

satisfying the following properties:

(i) Every geometric fiber R_s is isomorphic over X_s to a finite chain

$$\mathbf{P} \cup \mathbf{P} \cup \ldots \cup \mathbf{P} \tag{69}$$

with the components attached successively along the respective 0 and ∞ divisors. The collapsing maps (68) on the components together define

$$\rho_s : R_s \to X_s\,.$$

The 0 and ∞ sections of ρ_s are determined by the 0 section of first component and the ∞ section of last components of the chain (69).

(ii) Étale locally near every point $s \in S$, the data of $(R/S, \rho, r_0, r_\infty)$ is pulled back from a versal deformation space described by Li [52] with one dimension for every component of the singular locus of (69).

Definition 32. The stack $\mathbf{Rub}^{\mathsf{pre}}(X, \mathcal{L})$ of prestable rubber maps to $\mathcal{L}$ is a fibred category over $\mathfrak{M}(X)$ whose fibre over a map $S \to \mathfrak{M}(X)$ consists of three pieces of data:

(i) a prestable curve $\tilde{C}/S$ and a partial stabilisation[28] map $\tau \colon \tilde{C} \to C_S$ which is allowed to contract genus 0 components with 2 special points,

[28] C_S is not necessarily a stable curve.

(ii) an $(X, \mathcal{L})$-rubber target $(R/S, \rho, r_0, r_\infty)$,

(iii) a map $\tilde{f} : \tilde{C} \to R$ for which the following diagram commutes:

$$\begin{tikzcd} \tilde{C} \arrow[r, "\tilde{f}"] \arrow[d, "\tau"] & R \arrow[d, "\rho"] \\ C_S \arrow[r, "f"] & X_S\,. \end{tikzcd} \tag{70}$$

The map $\tilde{f}$ in (iii) is finite over the singularities of R/X_S and predeformable[29]. Moreover, over each geometric point $s \in S$, the image $\tilde{f}(\tilde{C}_s)$ meets every component of R_s.

An isomorphism between two objects

$$(\tilde{C} \to C_S, R, r_0, r_\infty, \tilde{C} \to R) \quad \text{and} \quad (\tilde{C}' \to C_S, R', r'_0, r'_\infty, \tilde{C}' \to R')$$

over $S \to \mathfrak{M}(X)$ is given by the data of isomorphisms

$$\tilde{C}' \xrightarrow{\sim} \tilde{C}$$

over C_S and

$$R' \xrightarrow{\sim} R$$

over X_S, compatible with the markings and such that the diagram

$$\begin{tikzcd} \tilde{C}' \arrow[r, "\sim"] \arrow[d] & \tilde{C} \arrow[d] \\ R' \arrow[r, "\sim"] & R \end{tikzcd}$$

commutes. We leave the definition of the cartesian morphisms to the careful reader.

Suppose now that we fix a genus g and a vector of integers A of length n. We define the stack $\mathbf{Rub}^{\mathsf{pre}}_{g,A}(X, \mathcal{L})$ with objects being tuples

$$(\tau \colon (\tilde{C}, p_1, \ldots, p_n) \to C_S\,, R/X_S\,, \tilde{f} \colon \tilde{C} \to R) \tag{71}$$

where $(\tilde{C}, p_1, \ldots, p_n)$ is a prestable curve of genus g with n markings. The data (71) are as for $\mathbf{Rub}^{\mathsf{pre}}(X, \mathcal{L})$. Moreover,

- if $a_i > 0$, $p_i \in \tilde{C}$ is mapped to the 0-divisor with ramification degree a_i,
- if $a_i < 0$, $p_i \in \tilde{C}$ is mapped to the ∞-divisor with ramification degree $-a_i$,
- if $a_i = 0$, $p_i \in \tilde{C}$ is mapped to the smooth locus of R away from the 0 and ∞-divisors.

[29] See [52].

5.4 Comparison to the logarithmic space

The pullback of $\mathcal{L}$ from X to the universal curve over $\mathfrak{M}(X)$ induces a map $\mathfrak{M}(X) \to \mathfrak{Pic}$. The key comparison result is the following.

Proposition 33. *The stack* $\mathbf{Rub}^{\mathsf{pre}}_{g,A}(X, \mathcal{L})$ *is naturally isomorphic to the fibre product of* $\widetilde{\mathbf{Rub}}_{g,A}$ *over* $\mathfrak{Pic}$ *with* $\mathfrak{M}(X)$ *along the map induced by* $\mathcal{L}$*,*

$$\mathbf{Rub}^{\mathsf{pre}}_{g,A}(X, \mathcal{L}) \cong \widetilde{\mathbf{Rub}}_{g,A} \times_{\mathfrak{Pic}} \mathfrak{M}(X)\,.$$

Proof. The right hand side comes with a built-in log structure, but the left side does not. Our isomorphism will be between the underlying stacks. Our proof is based on the discussion above [56, Proposition 5.5.2], and we will use the language of *divided tropical lines* of [56].

We begin by building a map from the right to the left. We are given a log curve C/S, a tropical line $\mathcal{P}$ on S, a map $\alpha\colon C \to \mathcal{P}$ whose image is totally ordered, and a map $f\colon C \to X$, such that $f^*\mathcal{L}$ lies in the isomorphism class $\mathcal{O}_C(\alpha)$.

The $\mathbb{G}_m^{\mathsf{trop}}$-torsor $\mathcal{P}$ is rigidified by the least element among the images of the irreducible components of C (here we use the total ordering condition), and hence comes with a canonical $\mathbb{G}_m$-torsor $P \to \mathcal{P}$ (if we use the rigidification to identify $\mathcal{P} = \mathbb{G}_m^{\mathsf{trop}}$ then $P = \mathbb{G}_m^{\mathsf{log}}$). The pullback α^*P gives a *canonical* $\mathbb{G}_m$-torsor on C, which is isomorphic to $f^*\mathcal{L}^*$ up to pullback from S. In other words, the bundle $\alpha^*P \otimes f^*\mathcal{L}^\vee$ descends to a line bundle on S which we denote $\mathcal{M}$.

The images of the irreducible components of C yield a subdivision $\mathcal{Q}$ of $\mathcal{P}$, and we define a destabilisation $\tilde{C} = C \times_{\mathcal{P}} \mathcal{Q}$ of C, which is a log curve over S by Lemma 28. This $\mathcal{Q}$ comes with a canonical $\mathbb{G}_m$-torsor Q by pulling back P from $\mathcal{P}$; this Q is then a 2-marked semistable genus 0 curve by [56, Proposition 5.2.4]. We define an $(X, \mathcal{L})$-rubber target R over S by the formula

$$R = \mathrm{Hom}((\mathcal{L} \otimes \mathcal{M})^*, Q)\,.$$

Here, we pull back and take $\mathbb{G}_m$-equivariant homomorphisms over X_S.

Write $\tilde{f}\colon \tilde{C} \to X$. We need a predeformable map $\tilde{C} \to R$, equivalently an equivariant logarithmic map $\tilde{f}^*\mathcal{L}^* \to Q$ over X_S. It is enough to give a map $f^*\mathcal{L} \to P$ (since then we can tensor over $\mathcal{P}$ with $\mathcal{Q}$), which reduces to writing down an element of

$$\begin{aligned}\mathrm{Hom}_C(f^*(\mathcal{L} \otimes \mathcal{M}), \alpha^*P) &= \mathrm{Hom}_C(f^*\mathcal{L} \otimes f^*\mathcal{L}^\vee \otimes \alpha^*P, \alpha^*P) \\ &= \mathrm{Hom}_C(\alpha^*P, \alpha^*P),\end{aligned} \tag{72}$$

which contains the identity. The scheme-theoretic map is predeformable as it comes from a logarithmic map, see [48].

Finally we check that no component of $\tilde{C}$ is mapped to a non-smooth point of R and that every component is hit. The target R is constructed by subdividing $f^*\mathcal{L}$ at images of components of C, and then $\tilde{C}$ is constructed by subdividing C at points lying over these divisions, so both assertions are clear.

Now we construct a map from left to right. Given a prestable rubber map to $\mathcal{L}$ over a base S, we first need to equip S with a suitable log structure.

The curve R/X_S is a map $X_S \to \mathfrak{M}_{0,2}^{ss}$, giving a (minimal) log structure on X_S by pullback. Lemma 34 below shows that this log structure descends to S. The curve R/X_S now carries the structure of a log curve, and similarly the quotient $[R/\mathbb{G}_m]$ descends to S (again by the Lemma 34), determining our tropical line $\mathcal{P}$ — which evidently satisfies the divisibility condition in Lemma 28.

It remains to verify that the map $\tilde{C} \to R$ descends to a map $C \to \mathcal{P}$ and that the total ordering condition is satisfied. Write

$$\tau \colon \tilde{C} \to C\,.$$

By the proof of [56, Proposition 5.5.2], we see $R \to \tau_*\tau^*R$ is an isomorphism, hence the map descends as required. The condition that no components are mapped to the nodes implies that the values of α on the irreducible of C are a subset of the irreducible components of R, in particular are totally ordered. $\Diamond$

Lemma 34. *Let $(R/S, \rho, r_0, r_\infty)$ be an $(X, \mathcal{L})$-rubber target. Then there exists a (minimal) log structure on S such that R/X_S can be equipped with the structure of a log curve making X_S strict over S. The quotient log stack $[R/\mathbb{G}_m]$ descends to a divided tropical line on S.*

Proof. The curve R/X_S with markings r_i is prestable and hence admits a minimal log structure. We must verify that the resulting log structure on X_S descends to S. After a finite extension of K we may assume X has a K point, so that

$$\pi \colon X_S \to S$$

admits a section $x \colon S \to X_S$, and we can equip S with the pullback log structure. It remains to construct an isomorphism $\pi^*x^*M_{X_S} \to M_{X_S}$. We start by building a map from left to right.

We first build a map on the level of characteristic monoids. The characteristic monoid at a geometric point $t \in X_S$ is given by $\mathbb{N}^\ell$, where ℓ is the length of the chain of projective lines of R over t. Crucially, the irreducible elements of $\mathbb{N}^\ell$ come with a total order, given by proximity of the corresponding singularity to the r_0 marking. This rigidifies the characteristic monoid, so as we move along the fibre over $\pi(t)$ the characteristic monoids are *canonically* identified. We obtain canonical identifications

$$(\overline{x^*M_{X_S}})_{\pi(t)} \xrightarrow{\sim} (\bar{M}_{X_S})_t\,,$$

which give an isomorphism

$$\pi^* x^* \bar{M}_{X_S} \xrightarrow{\sim} \bar{M}_{X_S}\,.$$

To construct an isomorphism of log structures, we will use the perspective of [14, Section 3.1] that a log structure is a monoidal functor from the groupified characteristic monoid to the stack of line bundles. The rubber target is by definition pulled back from Li's versal deformation spaces, so it suffices to construct our map in that setting. We can therefore assume that S is regular and the locus of non-smooth curves is a reduced divisor in X_S. Since our map will be canonical, we may further shrink S to be atomic[30]. Then $\bar{M}_{X/S}$ is generated by its global sections, and there is a natural isomorphism of sheaves on X_S

$$\varphi\colon \mathbb{N}^\ell \xrightarrow{\sim} \bar{M}_{X/S}$$

where ℓ is the number of singular points in the fibre of C over any point of X_S lying over the closed stratum of S. Given $1 \le i \le \ell$, write D_i for the Cartier divisor in X_S where the singularity at distance i from the first marking persists. Then φ sends the ith generator of $\mathbb{N}^\ell$ to the section corresponding to the line bundle $\mathcal{O}_{X_S}(D_i)$. To build the required map of log structures

$$\pi^* x^* M_{X_S} \xrightarrow{\sim} M_{X_S}\,,$$

we must construct an isomorphism

$$\pi^* x^* \mathcal{O}_{X_S}(D_i) \xrightarrow{\sim} \mathcal{O}_{X_S}(D_i)\,.$$

Condition (i) of the Definition 31 implies that the underlying point set of D_i is a union of fibres of X_S/S. Since S is regular and X_S is smooth over S, it follows that

$$D_i = \pi^* x^* D_i$$

giving the required isomorphism.

The quotient log stack $[R/\mathbb{G}_m]$ is a divided tropical line on X_S with divisions coming from the divisors D_i. We can identify the underlying tropical line with $\mathbb{G}_m^{trop}$ by specifying that the smallest element in the sequence of divisions is mapped to 0. We have already established that these divisions D_i descend to S, hence so does the divided tropical line. $\lozenge$

After restriction to the locus where the infinitesimal automorphisms are trivial, we obtain a stable version of Proposition 33. Let $\overline{\mathcal{M}}_{g,n}(X,\beta)$ denote the stack of

[30] [1, Definition 2.2.4].

stable maps from n-pointed curves to X representing the class β. The line bundle $\mathcal{L}$ determines a map

$$\overline{\mathcal{M}}_{g,n}(X,\beta) \to \mathfrak{Pic}$$

and we can pullback $\widetilde{\mathbf{Rub}}_{g,A}$ as before. Let

$$\mathbf{Rub}_{g,A}(X,\mathcal{L}) \subset \mathbf{Rub}^{\mathsf{pre}}_{g,A}(X,\mathcal{L})$$

be the locus where the infinitesimal automorphisms are trivial.

Lemma 35. *The stack $\mathbf{Rub}_{g,A}(X,\mathcal{L})$ is the fibre product of $\widetilde{\mathbf{Rub}}_{g,A}$ over $\mathfrak{Pic}$ with $\overline{\mathcal{M}}_{g,n}(X,\beta)$ along the map given by $\mathcal{L}$.*

$$\mathbf{Rub}_{g,A}(X,\mathcal{L}) \overset{\sim}{=} \widetilde{\mathbf{Rub}}_{g,A} \times_{\mathfrak{Pic}} \overline{\mathcal{M}}_{g,n}(X,\beta)\,.$$

Next, we will compare the virtual fundamental classes on these spaces. We will carry out the comparison on a smaller open locus. We define

(i) $\overline{\mathcal{M}}_{g,n}(X,\beta)'$ is be the open locus of maps $(C, f\colon C \to X)$ in $\overline{\mathcal{M}}_{g,n}(X,\beta)$ where $H^1(C, f^*\mathcal{L}) = 0$.

(ii) $\mathbf{Rub}_{g,A}(X,\mathcal{L})' = \mathbf{Rub}_{g,A}(X,\mathcal{L}) \times_{\overline{\mathcal{M}}_{g,n}(X,\beta)} \overline{\mathcal{M}}_{g,n}(X,\beta)'$.

In Section 4.2, we considered the case $X = \mathbb{P}^l$ and showed that this unobstructed locus is large enough to control the cycles relevant to Theorem 7. For general X, the unobstructed locus might be very small (and possibly empty).

5.5 Comparing the virtual classes

5.5.1 Overview

We begin by briefly discussing of several spaces which will be relevant in setting up the obstruction theories. Let

$$\mathfrak{M}^{ss}_{g,n} \subset \mathfrak{M}_{g,n}$$

be the semistable locus (where every rational curve has at least two distinguished points). We write $\mathscr{T}$ for the algebraic stack with log structure which parametrises tropical lines with at least one division. There are natural maps

$$\mathfrak{M}^{ss}_{0,2} \to \mathscr{T} \quad \text{and} \quad \mathfrak{M}^{ss}_{0,2} \to B\mathbb{G}_m\,,$$

the former defined by dividing $\mathbb{G}_m^{\mathsf{trop}}$ at 1 and at the smoothing parameters of the nodes, and the latter defined by the normal bundle at the first marking. The induced map

$$\mathfrak{M}^{ss}_{0,2} \to \mathscr{T} \times B\mathbb{G}_m \tag{73}$$

is an isomorphism by [2, Proposition 3.3.3].

As $\mathbf{Rub}^{\mathsf{rel}}$ is the moduli stack of tuples $(C, \alpha\colon C \to \mathcal{P})$ where $\mathcal{P}$ is a tropical line and the images of the irreducible components of C are totally ordered, there is a natural map

$$\mathbf{Rub}^{\mathsf{rel}} \to \mathscr{T} \tag{74}$$

sending $(C, \alpha\colon C \to \mathcal{P})$ to the tropical line $\mathcal{P}$ with the division given by the images of the irreducible components of C.

We will construct a map

$$\mathbf{Rub}_{g,A}(X, \mathcal{L})' \to \mathfrak{M}^{ss}_{0,2} \tag{75}$$

lifting the morphism (74) by the following argument. A point of $\mathbf{Rub}_{g,A}(X, \mathcal{L})'$ is a tuple $(C, \alpha\colon C \to \mathcal{P}, f\colon C \to X)$ where $f^*\mathcal{L}$ lies in the class[31] $[\mathcal{O}_C(\alpha)]$. However, as $\mathcal{P}$ is divided, there is a unique isomorphism $\mathcal{P} \xrightarrow{\sim} \mathbb{G}_m^{trop}$ where the smallest division maps to 0. The universal $\mathbb{G}_m$ torsor $\mathbb{G}_m^{log} \to \mathbb{G}_m^{trop}$ pulls back to a well-defined $\mathbb{G}_m$-torsor $\mathcal{O}^*_C(\alpha)$ on C, and the difference $f^*\mathcal{L}^* \otimes_{\mathcal{O}^*_C} \mathcal{O}^*_C(-\alpha)$ descends to a $\mathbb{G}_m$-torsor on S by the construction of $\mathbf{Rub}_{g,A}(X, \mathcal{L})'$ as a fibre product. The $\mathbb{G}_m$-torsor on S induces a map $\mathbf{Rub}_{g,A}(X, \mathcal{L})' \to B\mathbb{G}_m$. Combined with the map $\mathbf{Rub}_{g,A}(X, \mathcal{L})' \to \mathscr{T}$ via (74), we obtain the map (75).

The space $\mathbf{Rub}_{g,A}(X, \mathcal{L})'$ carries three virtual fundamental classes by the following three constructions:

(i) The class $\mathsf{DR}^{\mathsf{op}}_{g,A}(\varphi_{\mathcal{L}})([\overline{\mathcal{M}}_{g,n}(X, \beta)'])$ obtained by applying $\mathsf{DR}^{\mathsf{op}}_{g,A}$ to the (virtual) fundamental class of $\overline{\mathcal{M}}_{g,n}(X, \beta)'$ via the map $\varphi_{\mathcal{L}}$.

(ii) The class obtained from a two-step obstruction theory described by Marcus and Wise [56] for the map $\mathbf{Rub}_{g,A}(X, \mathcal{L})' \to \mathfrak{M}_{g,n} \times \mathscr{T}$.

(iii) A class coming from a two-step obstruction theory studied by Graber and Vakil [34] for the map

$$\mathbf{Rub}_{g,A}(X, \mathcal{L})' \to \overline{\mathcal{M}}_{g,n}(X, \beta)' \times \mathfrak{M}^{ss}_{0,2}\,.$$

We will prove (i)-(iii) are all equal.

5.5.2 $\mathsf{DR}^{\mathsf{op}}$ and the obstruction theory of Marcus-Wise

The obstruction theory of Marcus-Wise is a *two-step* obstruction theory, a notion which we now recall. Unless otherwise stated, by *perfect obstruction theory* we mean an obstruction theory which is perfect in amplitude $[-1, 0]$.

[31] Here, $[\mathcal{O}_C(\alpha)]$ is an equivalence class under isomorphisms and tensoring with pullbacks from S.

Definition 36. A *two-step* obstruction theory for a map $f\colon X \to S$ consists of a factorisation

$$X \to Y \to S$$

together with prefect relative obstruction theories for X/Y and for Y/S.

A two-step obstruction theory induces a virtual pullback by composition.[32] If S has a fundamental class $[S]$, the virtual pullback of $[S]$ is the *virtual fundamental class* of X associated to the two-step obstruction theory.

We first recall the two-step obstruction theory of [56] in the case when X is a point. We have a diagram

$$\begin{tikzcd} \mathbf{Rub}_{g,A}(\mathsf{pt}, \mathcal{O})' \arrow[r] \arrow[dr] & \widetilde{\mathbf{Rub}}_{g,A} \arrow[d] \\ & \mathfrak{M}_{g,n} \times \mathscr{T}. \end{tikzcd} \tag{76}$$

A perfect relative obstruction theory for the horizontal map is given in [56, Section 5.6.3], and for the vertical map in [56, Proposition 5.6.5.3]; while the reader might expect that these arguments apply to $\mathbf{Rub}_{g,A}$ rather than the root stack $\widetilde{\mathbf{Rub}}_{g,A}$, Marcus and Wise in fact assume in both constructions the divisibility conditions of Lemma 28, hence their constructions in fact apply to $\widetilde{\mathbf{Rub}}_{g,A}$ (and not to $\mathbf{Rub}_{g,A}$). This two-step obstruction theory coincides with the rubber theory of Graber-Vakil, as shown in [56, Section 5.6.6]. Moreover, the virtual fundamental class obtained equals the operational class of $\mathbf{Div}_{g,A}$, see [56, Theorem 5.6.1]; again, these results all assume the divisibility condition of Lemma 28, and hence apply to $\widetilde{\mathbf{Rub}}_{g,A}$ in place of $\mathbf{Rub}_{g,A}$.

Returning to the case of arbitrary $(X, \mathcal{L})$, we can construct a similar commutative diagram

$$\begin{tikzcd} \mathbf{Rub}_{g,A}(X, \mathcal{L})' \arrow[r] \arrow[dr] & \widetilde{\mathbf{Rub}}_{g,A} \arrow[d] \\ & \mathfrak{M}_{g,n} \times \mathscr{T}. \end{tikzcd} \tag{77}$$

The vertical map is unchanged and so again has a perfect relative obstruction theory by [56, Proposition 5.6.5.3]; in fact the morphism is a local complete intersection, and the obstruction theory of [56, Proposition 5.6.5.3] is just the relative tangent complex.

---

[32] A two-step obstruction theory also induces a perfect obstruction theory for X/S in amplitude $[-2, 0]$, but we will not use the latter construction.

We need to supply a perfect obstruction theory for the horizontal map

$$\widetilde{\mathbf{Rub}}_{g,A}(X,\mathcal{L})' \to \mathbf{Rub}_{g,A}\,,$$

which we can factor as

$$\mathbf{Rub}_{g,A}(X,\mathcal{L})' \to \overline{\mathcal{M}}_{g,n}(X,\beta)' \times_{\mathfrak{M}_{g,n}} \widetilde{\mathbf{Rub}}_{g,A} \to \widetilde{\mathbf{Rub}}_{g,A}\,. \tag{78}$$

The second map is a base change of the unobstructed map $\overline{\mathcal{M}}_{g,n}(X,\beta)' \to \mathfrak{M}_{g,n}$, hence is unobstructed. For the first map, consider the pullback square

$$\begin{tikzcd}
\mathbf{Rub}_{g,A}(X,\mathcal{L})' \arrow[r] \arrow[d] & \overline{\mathcal{M}}_{g,n}(X,\beta)' \arrow[d] \\
\overline{\mathcal{M}}_{g,n}(X,\beta)' \times_{\mathfrak{M}_{g,n}} \widetilde{\mathbf{Rub}}_{g,A} \arrow[r] & \overline{\mathcal{M}}_{g,n}(X,\beta)' \times_{\mathfrak{M}_{g,n}} \mathfrak{Pic}_{g,n}\,.
\end{tikzcd} \tag{79}$$

The right vertical arrow is a section of a base change of the smooth morphism $\mathfrak{Pic}_{g,n} \to \mathfrak{M}_{g,n}$, and as such is lci and has a perfect relative obstruction theory given by the relative tangent complex $R^1\pi_*\mathcal{O}_C$. Pullback yields a corresponding perfect obstruction theory for the left vertical arrow. This gives a two-step obstruction theory for the composite map $\mathbf{Rub}_{g,A}(X,\mathcal{L})' \to \widetilde{\mathbf{Rub}}_{g,A}$, from which we obtain a virtual fundamental class following [54].

The discussion here is a very slight generalisation of the obstruction theory constructed in [56, Proposition 5.6.3.1].

Definition 37. The two-step obstruction theory for the diagonal map of (77),

$$\mathbf{Rub}_{g,A}(X,\mathcal{L})' \to \mathfrak{M}_{g,n} \times \mathscr{T}$$

is the *Marcus-Wise* obstruction theory.

Lemma 38. *The push forward along*

$$\psi\colon \mathbf{Rub}_{g,A}(X,\mathcal{L})' \to \overline{\mathcal{M}}_{g,n}(X,\beta)'$$

of the virtual fundamental class of the Marcus-Wise theory on $\mathbf{Rub}_{g,A}(X,\mathcal{L})'$ *equals the class* $\mathsf{DR}^{\mathsf{op}}_{g,A}(\varphi_{\mathcal{L}})([\overline{\mathcal{M}}_{g,n}(X,\beta)'])$ *obtained via the map*

$$\varphi_{\mathcal{L}}\colon \overline{\mathcal{M}}_{g,n}(X,\beta)' \to \mathfrak{Pic}_{g,n}\,.$$

Proof. From $\varphi_{\mathcal{L}}$, we obtain maps

$$\varphi'_{\mathcal{L}}\colon \overline{\mathcal{M}}_{g,n}(X,\beta)' \to \overline{\mathcal{M}}_{g,n}(X,\beta)' \times_{\mathfrak{M}_{g,n}} \mathfrak{Pic}_{g,n}\,,$$

$$\varphi''_{\mathcal{L}} \colon \overline{\mathcal{M}}_{g,n}(X,\beta)' \to \overline{\mathcal{M}}_{g,n}(X,\beta)' \times \mathfrak{Pic}_{g,n} \, .$$

Both are lci morphisms because $\mathfrak{Pic}_{g,n}/\mathfrak{M}_{g,n}$ is smooth. By [6, Definition 17] and Section 2.6 we have

$$\begin{aligned} \mathsf{DR}^{\mathsf{op}}_{g,A}(\varphi_{\mathcal{L}})([\overline{\mathcal{M}}_{g,n}(X,\beta)']) &= \psi_*(\varphi''_{\mathcal{L}})^![\overline{\mathcal{M}}_{g,n}(X,\beta)' \times \widetilde{\mathbf{Rub}}_{g,A}] \\ &= \psi_*(\varphi'_{\mathcal{L}})^![\overline{\mathcal{M}}_{g,n}(X,\beta)' \times_{\mathfrak{M}_{g,n}} \widetilde{\mathbf{Rub}}_{g,A}] \, . \end{aligned}$$

The virtual fundamental class of the Marcus-Wise theory is the virtual pullback of the fundamental class of $\mathfrak{M}_{g,n} \times \mathscr{T}$ along the composition

$$\mathbf{Rub}_{g,A}(X,\mathcal{L})' \xrightarrow{1} \overline{\mathcal{M}}_{g,n}(X,\beta)' \times_{\mathfrak{M}_{g,n}} \widetilde{\mathbf{Rub}}_{g,A} \xrightarrow{2} \widetilde{\mathbf{Rub}}_{g,A} \xrightarrow{3} \mathfrak{M}_{g,n} \times \mathscr{T} \, . \tag{80}$$

The map (3) is lci and the obstruction theory is the relative tangent complex, so the pullback of the fundamental class is the fundamental class of $\widetilde{\mathbf{Rub}}_{g,A}$. The map (2) is unobstructed, so the (virtual) pullback of the fundamental class is again the fundamental class. The obstruction theory of the map (1) is defined by pulling back the relative tangent complex of the lci morphism

$$\overline{\mathcal{M}}_{g,n}(X,\beta)' \to \overline{\mathcal{M}}_{g,n}(X,\beta)' \times_{\mathfrak{M}_{g,n}} \mathfrak{Pic}_{g,n}$$

via the pullback square

$$\begin{tikzcd} \mathbf{Rub}_{g,A}(X,\mathcal{L})' \arrow[r] \arrow[d] & \overline{\mathcal{M}}_{g,n}(X,\beta)' \times_{\mathfrak{M}_{g,n}} \widetilde{\mathbf{Rub}}_{g,A} \arrow[d] \\ \overline{\mathcal{M}}_{g,n}(X,\beta)' \arrow[r, "\varphi'_{\mathcal{L}}"] & \overline{\mathcal{M}}_{g,n}(X,\beta)' \times_{\mathfrak{M}_{g,n}} \mathfrak{Pic}_{g,n} \, , \end{tikzcd} \tag{81}$$

so the virtual pullback of the fundamental class of $\overline{\mathcal{M}}_{g,n}(X,\beta)' \times_{\mathfrak{M}_{g,n}} \widetilde{\mathbf{Rub}}_{g,A}$ is equal to the Gysin pullback $(\varphi'_{\mathcal{L}})^![\overline{\mathcal{M}}_{g,n}(X,\beta)' \times_{\mathfrak{M}_{g,n}} \widetilde{\mathbf{Rub}}_{g,A}]$. ◊

5.5.3 Marcus-Wise and Graber-Vakil

As recalled above, Marcus-Wise define a two-step obstruction theory for the map

$$\mathbf{Rub}_{g,A}(X,\mathcal{L})' \to \mathfrak{M}_{g,n} \times \mathscr{T} \, .$$

Graber and Vakil consider an obstruction theory for the map

$$\mathbf{Rub}_{g,A}(X,\mathcal{L})' \to \overline{\mathcal{M}}_{g,n}(X,\beta)' \times \mathfrak{M}^{ss}_{0,2} \, . \tag{82}$$

We wish to show an equality of the corresponding virtual fundamental classes on $\mathbf{Rub}_{g,A}(X,\mathcal{L})'$. Since

$$\mathfrak{M}^{ss}_{0,2} = \mathscr{T} \times B\mathbb{G}_m \, ,$$

and that the maps $\overline{\mathcal{M}}_{g,n}(X,\beta)' \to \mathfrak{M}_{g,n}$ and $B\mathbb{G}_m \to \operatorname{Spec} K$ are unobstructed, we have an unobstructed map

$$\overline{\mathcal{M}}_{g,n}(X,\beta)' \times \mathfrak{M}_{0,2}^{ss} \to \mathfrak{M}_{g,n} \times \mathfrak{M}_{0,2}^{ss} \to \mathfrak{M}_{g,n} \times \mathscr{T} .$$

Our final step is therefore to compare the obstruction theories (and thereby the corresponding virtual pullbacks) between Marcus-Wise and Graber-Vakil [34, 53]. We will match the obstruction spaces when the base S is a point. The full matching of deformation theories is similar and will be treated in [59]. The claims are also required for [56].

Suppose we are given the data of a point in $\mathbf{Rub}_{g,A}(X,\mathcal{L})'(S)$,

$$(\tau \colon \tilde{C} \to C_S, R/X_S, \varphi \colon \tilde{C} \to R, f \colon C_S \to X_S, p_1, \ldots, p_n, f^*\mathcal{L} \xrightarrow{\sim} \mathcal{O}_C(\alpha)) . \quad (83)$$

Proposition 39. *The restriction to $\overline{\mathcal{M}}_{g,n}(X,\beta)'$ of the class $\mathsf{DR}_{g,A}(X,\mathcal{L})$ of [45] is equal to the class obtained by letting $\mathsf{DR}^{\mathsf{op}}_{g,A}$ act on the fundamental class of $\overline{\mathcal{M}}_{g,n}(X,\beta)'$ via the map induced by $\mathcal{L}$.*

Proof. The primary obstruction of Graber-Vakil lies in

$$H^0(\tilde{C}, \varphi^{-1}\mathcal{E}xt^1(\Omega_{R/X_S}(\log D), \mathcal{O}_R)) . \quad (84)$$

Here, D is the divisor on R given by the sum of the two markings r_0 and r_∞, and $\Omega_{R/X_S}(\log D)$ is the sheaf of relative 1-forms on R/X_S allowed logarithmic poles along D (a coherent sheaf on R). The obstruction space (84) is isomorphic to the product of the deformation spaces of the nodes of R and coincides with the obstruction space for the map

$$\widetilde{\mathbf{Rub}}_{g,A} \to \mathfrak{M}_{g,n} \times \mathscr{T}$$

the vertical arrow in (77)), coming from [56, Proposition 5.6.5.3] (where they assume the divisibility conditions of Lemma 28, hence the results apply to $\widetilde{\mathbf{Rub}}_{g,A}$ and not to $\mathbf{Rub}_{g,A}$).

Suppose that the primary obstruction vanishes. Denote by

$$T_{R/X_S} = \mathcal{H}om_{\mathcal{O}_R}(\Omega_{R/X_S}, \mathcal{O}_R)$$

the relative tangent sheaf. There is a secondary obstruction in

$$H^1(\tilde{C}, \varphi^\dagger(T_{R/X_S})) , \quad (85)$$

where the $\varphi^\dagger$ is the torsion-free part of φ^*, see [34] and [56]. The obstruction space (85) is the image of the obstruction space $H^1(C_S, f^*T_R) = H^1(C_S, \mathcal{O}_{C_S})$ for the map

$$\mathbf{Rub}_{g,A}(X,\mathcal{L})' \to \widetilde{\mathbf{Rub}}_{g,A}$$

the horizontal arrow in (77), coming from [56, Proposition 5.6.3.1].

When both of these obstructions vanish, the deformations are a torsor under $H^0(\tilde{C}, \varphi^\dagger T_{R/X_S})$, an extension of the first term of the obstruction complex in [56, Proposition 5.6.5.3] by the first term of the obstruction complex of [56, Proposition 5.6.4.1]. The comparison of the obstruction theories is complete. $\Diamond$

6 Applications

6.1 Proofs of Theorem 9 and Conjecture A

We start by recalling notions presented in Section 0.5, but now in the more general setting of k-differentials. Let $A = (a_1, \ldots, a_n)$ be a vector of zero and pole multiplicities satisfying

$$\sum_{i=1}^{n} a_i = k(2g-2)\,.$$

Let $\mathcal{H}_g^k(A) \subset \mathcal{M}_{g,n}$ be the closed (generally non-proper) locus of pointed curves $(C, p_1, \ldots, p_n)$ satisfying the condition

$$\mathcal{O}_C\Big(\sum_{i=1}^{n} a_i p_i\Big) \simeq \omega_C^{\otimes k}\,.$$

In other words, $\mathcal{H}_g^k(A)$ is the locus of (possibly) meromorphic k-differentials with zero and pole multiplicities prescribed by A. In [32], a compact moduli space of twisted k-canonical divisors

$$\widetilde{\mathcal{H}}_g^k(A) \subset \overline{\mathcal{M}}_{g,n}$$

is constructed extending $\mathcal{H}_g^k(A) = \widetilde{\mathcal{H}}_g^k(A) \cap \mathcal{M}_{g,n}$ to the boundary of $\overline{\mathcal{M}}_{g,n}$.

For $k \geq 1$ and A not of the form $A = k \cdot A'$ with a vector A' of nonnegative integers, the locus $\widetilde{\mathcal{H}}_g^k(A)$ is of pure codimension g in $\overline{\mathcal{M}}_{g,n}$ by [32, Theorem 3] (for $k = 1$) and [72, Theorem 1.1] (for $k > 1$). A weighted fundamental cycle of $\widetilde{\mathcal{H}}_g^k(A)$,

$$\mathsf{H}_{g,A}^k \in \mathsf{CH}_{2g-3+n}(\overline{\mathcal{M}}_{g,n})\ , \tag{86}$$

is constructed in [32, Appendix A] and [72, Section 3.1] with explicit nontrivial weights on the irreducible components. The closure

$$\overline{\mathcal{H}}_{g,A}^k \subset \overline{\mathcal{M}}_{g,n}$$

contributes to the weighted fundamental class $\mathsf{H}_{g,A}^k$ with multiplicity 1, but there are additional boundary contributions, as described in the references above.

The weighted fundamental class $\mathsf{H}^k_{g,A}$ was conjectured in [32, 72] to equal the class given by Pixton's formula for the double ramification cycle. To state the conjecture, consider the shifted[33] vector $\widetilde{A} = (a_1 + k, \dots, a_n + k)$.

Conjecture A. *For $k \geq 1$ and A not of the form $A = k \cdot A'$ with a vector A' of nonnegative integers, we have an equality*

$$\mathsf{H}^k_{g,A} = 2^{-g} P_g^{g,k}(\widetilde{A}) \,,$$

where $P_g^{g,k}(\widetilde{A})$ is Pixton's cycle class defined in [44, Section 1.1].

By combining Theorem 7 with previous results of [42], we can now prove the conjecture.

Theorem 40. *Conjecture A is true.*

Proof. By Theorem 1.1 of [42], the weighted fundamental class $\mathsf{H}^k_{g,A}$ is equal to the double ramification cycle $\mathsf{DR}_{g,A,\omega^k}$ constructed in [39]. By Theorem 1, $\mathsf{DR}_{g,A,\omega^k}$ is in turn given by the action of $\mathsf{DR}^{\mathsf{op}}_{g,A}$ on the fundamental class of $\overline{\mathcal{M}}_{g,n}$ via the morphism $\varphi_{\omega^k_\pi} : \overline{\mathcal{M}}_{g,n} \to \mathfrak{Pic}_{g,n,k(2g-2)}$ associated to the family

$$\pi : \mathcal{C}_{g,n} \to \overline{\mathcal{M}}_{g,n} \,, \qquad \omega^k_\pi \to \mathcal{C}_{g,n} \,.$$

By Theorem 7, the class $\mathsf{DR}^{\mathsf{op}}_{g,A}$ is computed by the tautological class

$$\mathsf{P}^g_{g,A,d} \in \mathsf{CH}^g_{\mathsf{op}}(\mathfrak{Pic}_{g,n,k(2g-2)}) \,.$$

By Proposition 21, the action of $\mathsf{P}^g_{g,A,d}$ on $[\overline{\mathcal{M}}_{g,n}]$ is indeed given by Pixton's original formula $2^{-g} P_g^{g,k}(\widetilde{A})$, finishing the proof. ♢

The steps of the proof of Theorem 40 are summarized as follows:

$$\begin{aligned}
\mathsf{H}^k_{g,A} &= \mathsf{DR}_{g,A,\omega^k} && \text{[42, Theorem 1.1]} \\
&= \mathsf{DR}^{\mathsf{op}}_{g,A}(\varphi_{\omega^k_\pi})([\overline{\mathcal{M}}_{g,n}]) && \text{Theorem 1} \\
&= \mathsf{P}^g_{g,A,d}(\varphi_{\omega^k_\pi})([\overline{\mathcal{M}}_{g,n}]) && \text{Theorem 7} \\
&= 2^{-g} P_g^{g,k}(\tilde{A}) && \text{Proposition 21}\,.
\end{aligned}$$

The result provides a completely geometric representative of Pixton's cycle in terms of twisted k-differentials. Theorem 9 of Section 0.5 is the $k = 1$ case of Theorem 40.

[33] The shift is needed since Pixton's original formula worked with powers of the log-canonical line bundle $\omega_C^{\log} = \omega_C(\sum_{i=1}^n p_i)$ instead of ω_C.

6.2 Closures

Let $A = (a_1, \dots, a_n)$ be a vector of integers satisfying

$$\sum_{i=1}^{n} a_i = k(2g-2)\,.$$

A careful investigation of the closure

$$\mathcal{H}_g^k(A) \subset \overline{\mathcal{H}}_g^k(A) \subset \overline{\mathcal{M}}_{g,n}$$

is carried out in [9, 10]. By a simple method presented in [32, Appendix A] and [72, Section 3.4], Theorem 40 easily determines the cycle classes of the closures

$$[\overline{\mathcal{H}}_g^k(A)] \in \mathsf{CH}_*(\overline{\mathcal{M}}_{g,n})\,.$$

for the cases

- $k = 1$ and all a_i nonnegative, when $\overline{\mathcal{H}}_g^k(A)$ has pure codimension $g-1$ and
- $k \geq 1$ and A not of the form $A = k \cdot A'$ with a vector A' of nonnegative integers, when $\overline{\mathcal{H}}_g^k(A)$ has pure codimension g.

In particular, from the recursive formula for $[\overline{\mathcal{H}}_g^k(A)]$ and the fact that Pixton's cycle on $\overline{\mathcal{M}}_{g,n}$ is tautological, the following is immediate.

Corollary 41. *The cycles $[\overline{\mathcal{H}}_g^k(A)]$ are tautological classes in $\mathsf{CH}_*(\overline{\mathcal{M}}_{g,n})$.*

In the case $k = 1$, Corollary 41 was known by work of Sauvaget [70], who gave a different approach to $[\overline{\mathcal{H}}_g^1(A)]$ in terms of tautological classes. The recursive formulas for $[\overline{\mathcal{H}}_g^k(A)]$ from Corollary 41 have been implemented[34] in the software [26] for computations in the tautological ring of $\overline{\mathcal{M}}_{g,n}$.

Another application of Conjecture A is presented in the recent paper [71] by Sauvaget. The paper studies moduli spaces of flat surfaces of genus g with conical singularities at marked points $p_1, ..., p_n$. The singularities have fixed cone angles $2\pi\alpha_i$, for $\alpha_1, \dots, \alpha_n \in \mathbb{R}$, summing to $2g-2+n$. If all α_i are rational, the spaces of flat surfaces naturally contain $\mathcal{H}_g^k(kA)$, for

$$A = (\alpha_i - 1)_{i=1}^n\,,$$

as closed subsets (for k sufficiently divisibly). These subsets equidistribute (with respect to natural measures) as $k \to \infty$. Using the equidistribution, Sauvaget is able to apply the recursive expression for $\overline{\mathcal{H}}_g^k(kA)$ from Conjecture A to derive an explicit recursion for the volumes of the moduli spaces of flat surfaces.

[34] In the ongoing project [23], the authors study formulas for Euler characteristics of strata of differentials in terms of intersection numbers on the compactification of these strata constructed in [11]. The implementation of $[\overline{\mathcal{H}}_g^1(A)]$ has played a role in corroborating their formulas.

6.3 k-twisted DR cycles with targets

We define here *k-twisted double ramification cycles with targets* via the class $\mathsf{DR}^{\mathsf{op}}_{g,A}$.

Let X be a nonsingular projective variety with line bundle $\mathcal{L}$ and an effective curve class $\beta \in H_2(X, \mathbb{Z})$. Let

$$d_\beta = \int_\beta c_1(\mathcal{L}) \,.$$

Let $k \in \mathbb{Z}$ and $A = (a_1, \ldots, a_n) \in \mathbb{Z}^n$ satisfy

$$d_\beta + k(2g - 2 + n) = \sum_{i=1}^n a_i \,.$$

Consider the morphism

$$\varphi \colon \overline{\mathcal{M}}_{g,n}(X, \beta) \to \mathfrak{Pic}_{g,n,d_\beta + k(2g-2+n)}$$

defined by the universal data

$$\pi : \mathcal{C}_{g,n,\beta} \to \overline{\mathcal{M}}_{g,n}(X, \beta)\,, \qquad f^*\mathcal{L} \otimes \omega_{log}^{\otimes k} \to \mathcal{C}_{g,n,\beta}\,, \tag{87}$$

where $f : \mathcal{C}_{g,n,\beta} \to X$ is the universal map.

Definition 42. The *k-twisted X-valued double ramification cycle* is defined by

$$\mathsf{DR}^k_{g,n,\beta}(X, \mathcal{L}) = \mathsf{DR}^{\mathsf{op}}_{g,A}(\varphi)([\overline{\mathcal{M}}_{g,n}(X, \beta)]^{\mathrm{vir}}) \in \mathsf{CH}_{\mathrm{vdim}(g,n,\beta)-g}(\overline{\mathcal{M}}_{g,n}(X, \beta))\,.$$

In the notation of [45, Section 0.4], let $\mathsf{P}^{c,k,r}_{g,A,\beta}(X, \mathcal{L})$ be the codimension c part of the following expression

$$\sum_{\substack{\Gamma \in \mathsf{G}_{g,n,\beta}(X) \\ w \in \mathsf{W}_{\Gamma,r,k}(X)}} \frac{r^{-h^1(\Gamma)}}{|\mathrm{Aut}(\Gamma)|}\, j_{\Gamma_\delta *}\Bigg[\prod_{i=1}^n \exp\left(\frac{1}{2} a_i^2 \psi_i + a_i \xi_i\right)$$
$$\prod_{v \in \mathrm{V}(\Gamma)} \exp\left(-\frac{1}{2}\eta(v) - k\eta_{1,1}(v) - \frac{k^2}{2}\eta_{0,2}(v)\right)$$
$$\prod_{e=(h,h') \in \mathrm{E}(\Gamma)} \frac{1 - \exp\left(-\frac{w(h)w(h')}{2}(\psi_h + \psi_{h'})\right)}{\psi_h + \psi_{h'}}\Bigg]\,.$$

The definition of the admissible k-weightings $w \in \mathsf{W}_{\Gamma,r,k}(X)$ is similar to that in Section 0.3.4 but with the condition (iii) replaced by

$$k(2g(v) - 2 + n(v)) + \int_{\beta(v)} c_1(\mathcal{L}) = \sum_{v(h)=v} w(h) \ \text{ for } v \in \mathrm{V}(\Gamma)\,.$$

As in the case $k = 0$ discussed in [45, Proposition 1], the class $\mathsf{P}^{c,k,r}_{g,A,\beta}(X,\mathcal{L})$ is polynomial in r for all sufficiently large r. Denote by $\mathsf{P}^{c,k}_{g,A,\beta}(X,\mathcal{L})$ the value at $r = 0$ of this polynomial.

By Theorem 7 and a slight generalization of the procedure for pulling back Chow cohomology classes from $\mathfrak{Pic}_{g,n,d_\beta+k(2g-2+n)}$ to $\overline{\mathcal{M}}_{g,n}(X,\beta)$ described in [45, Section 1.5], we have

$$\begin{aligned}\mathsf{DR}^k_{g,n,\beta}(X,\mathcal{L}) &= \mathsf{DR}^{\mathsf{op}}_{g,A}(\varphi)([\overline{\mathcal{M}}_{g,n}(X,\beta)]^{\mathrm{vir}}) \\ &= \mathsf{P}^g_{g,A,d_\beta+k(2g-2+n)}(\varphi)([\overline{\mathcal{M}}_{g,n}(X,\beta)]^{\mathrm{vir}}) \\ &= \mathsf{P}^{g,k}_{g,A,\beta}(X,\mathcal{L})\,.\end{aligned}$$

6.4 Proof of Theorem 8

For all $c > g$, we will prove

$$\mathsf{P}^c_{g,A,d} \;= 0\; \in \mathsf{CH}^c_{\mathsf{op}}(\mathfrak{Pic}_{g,n,d})\,. \tag{88}$$

The path is parallel to the proof of Theorem 7.

By definition, the claim is equivalent to showing that the map

$$\mathsf{P}^c_{g,A,d}(\varphi) : \mathsf{CH}_*(B) \to \mathsf{CH}_{*-c}(B) \tag{89}$$

is zero for every morphism $\varphi : B \to \mathfrak{Pic}_{g,n,d}$ from an (irreducible) finite type scheme B corresponding to the data

$$C \to B\,, \quad \mathcal{L} \to C\,.$$

Retracing the steps of Section 4 (and using the invariance **??** for the codimension c part $\mathsf{P}^c_{g,A,d}$ of Pixton's formula), we can reduce to the situation where $\mathcal{L}$ on C is relatively sufficiently positive with respect to $C \to B$. As in Section 4.3, we can then find

$$\psi : U_l \to B$$

such that ψ^* is injective on Chow groups and such that the composition

$$U_l \to B \to \mathfrak{Pic}_{g,n,d}$$

factors through $\overline{\mathcal{M}}_{g,n}(\mathbb{P}^l,d)'$. By Theorem 3.2 of [4], we have the vanishing

$$\mathsf{P}^c_{g,A,d}(\mathbb{P}^l,\mathcal{O}(1)) = 0 \,\in\, \mathsf{CH}_{\mathrm{vdim}(g,n,d)-c}(\overline{\mathcal{M}}_{g,n}(\mathbb{P}^l,d))\,.$$

The same combination of [6, Lemma 15] and the injectivity of ψ^* then shows the desired vanishing of the map (89). $\Diamond$

6.5 Connections to past and future results

The relations of Theorem 8 generalize several previous results. For $g = 0$ and $c = 1$, the vanishing (88) was observed in [25, Proposition 1.2]. In fact, in genus 0, there are many connections to past equations, see [4, Section 4] for a full discussion with many examples including classical equations and the relations of [51].

Randal-Williams [67] proves a vanishing result in cohomology with integral coefficients on the locus $\mathfrak{Pic}^{\mathrm{sm}}_{g,0,d}$ of smooth curves for every $d \in \mathbb{Z}$. We can recover a version of Randal-William's vanishing in operational Chow with $\mathbb{Q}$-coefficients which extends to all of $\mathfrak{Pic}_{g,0,d}$. By Proposition 19 and Lemma 20, Pixton formula's on the locus $\mathfrak{Pic}^{\mathrm{sm}}_{g,0,0}$ takes the simple form

$$\mathsf{P}^c_{g,\emptyset,0} = \frac{1}{c!}(\mathsf{P}^1_{g,\emptyset,0})^c\,, \quad \mathsf{P}^1_{g,\emptyset,0} = -\frac{1}{2}\pi_*(c_1(\mathcal{L})^2)$$

for the universal curve and the universal line bundle

$$\pi : \mathfrak{C} \to \mathfrak{Pic}_{g,0,0}\,, \quad \mathcal{L} \to \mathfrak{C}\,.$$

We claim, up to scaling, the relation

$$\Omega^{g+1} = 0$$

of [67, Theorem A] is exactly the restriction of the pullback of the relation

$$(\mathsf{P}^1_{g,\emptyset,0})^{g+1} = (g+1)!\mathsf{P}^{g+1}_{g,\emptyset,0} = 0 \in \mathsf{CH}^{g+1}_{\mathsf{op}}(\mathfrak{Pic}_{g,0,0})$$

under the morphism

$$\mathfrak{Pic}_{g,0,d} \to \mathfrak{Pic}_{g,0,0}\,, \quad (C, \mathcal{L}) \mapsto (C, \mathcal{L}^{\otimes 2g-2} \otimes \omega_C^{\otimes(-d)})\,.$$

Indeed, over the locus of smooth curves, the pullback of $\mathsf{P}^1_{g,\emptyset,0}$ is given by

$$\begin{aligned} &-\frac{1}{2}\pi_*(c_1(\mathcal{L}^{\otimes 2g-2} \otimes \omega_C^{\otimes(-d)})^2) = \\ &\qquad -\frac{1}{2}\left((2g-2)^2\pi_*(c_1(\mathcal{L})^2) - 2d(2g-2)\pi_*(c_1(\mathcal{L})c_1(\omega_\pi)) + d^2\pi_*(c_1(\omega_\pi)^2)\right)\,, \end{aligned}$$

which matches the definition of Ω given in [67, Theorem A] up to scalars.

In Gromov-Witten theory, pulling back (88) under the morphisms

$$\overline{\mathcal{M}}_{g,n}(X, \beta) \to \mathfrak{Pic}_{g,n,d}$$

described in Section 6.3 and capping with the virtual class $[\overline{\mathcal{M}}_{g,n}(X, \beta)]^{\mathrm{vir}}$ simply recovers the known vanishing

$$\mathsf{P}^{c,k}_{g,A,\beta}(X, \mathcal{L}) = 0 \in \mathsf{CH}_{\mathrm{vdim}(g,n,\beta)-c}(\overline{\mathcal{M}}_{g,n}(X, \beta)) \tag{90}$$

for $c > g$ proven in [4]. However, there are new applications for *reduced* Gromov-Witten theory. Indeed, for a target X having a nondegenerate holomorphic 2-form, the virtual class of $\overline{\mathcal{M}}_{g,n}(X, \beta)$ vanishes when $\beta \neq 0$. To define invariants for such targets, the reduced class

$$[\overline{\mathcal{M}}_{g,n}(X, \beta)]^{\mathrm{red}} \in \mathsf{CH}_*(\overline{\mathcal{M}}_{g,n}(X, \beta))$$

is used instead, see [16, 57]. By pulling back (88) and capping with $[\overline{\mathcal{M}}_{g,n}(X, \beta)]^{\mathrm{red}}$, we obtain new relations among reduced Gromov-Witten invariants. An application to the Gromov-Witten theory of K3 surfaces will appear in [5] related to conjectures of [61].

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Curves on K3 surfaces in divisibility two

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0 Introduction

Let S be a complex nonsingular projective K3 surface and $\beta \in H_2(S, \mathbb{Z})$ an effective curve class. Gromov–Witten invariants of S are defined via intersection theory on the moduli space $\overline{M}_{g,n}(S, \beta)$ of stable maps from n-pointed genus g curves to S. This moduli space comes with a virtual fundamental class. However, the virtual class vanishes for $\beta \neq 0$ so, instead, we use the *reduced class*[1]

$$[\overline{M}_{g,n}(S, \beta)]^{red} \in A_{g+n}\big(\overline{M}_{g,n}(S, \beta), \mathbb{Q}\big)\,.$$

For integers $a_i \geq 0$ and cohomology classes $\gamma_i \in H^*(S, \mathbb{Q})$ we define

$$\big\langle \tau_{a_1}(\gamma_1) \ldots \tau_{a_n}(\gamma_n) \big\rangle^S_{g,\beta} = \int_{[\overline{M}_{g,n}(S,\beta)]^{red}} \prod_{i=1}^{n} \psi_i^{a_i} \cup \mathrm{ev}_i^*(\gamma_i)\,,$$

where $\mathrm{ev}_i \colon \overline{M}_{g,n}(S, \beta) \to S$ is the evaluation at i-th marking and ψ_i is the cotangent class at the i-th marking. By the deformation invariance of the reduced class, the invariant only depends on the norm $\langle \beta, \beta \rangle$ and the divisibility of the curve class β.

0.1 Quasimodularity

Gromov–Witten invariants of K3 surfaces for primitive curve classes are well-understood since the seminal paper by Maulik, Pandharipande, and Thomas [29]. The invariants are coefficients of weakly holomorphic[2] quasimodular forms with pole of order at most one [29, Theorem 4]. For imprimitive curve classes, the quasimodularity is conjectured with the level structure [29, Section 7.5].

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[1] We will identify this class with its image under the cycle class map $A_* \to H_{2*}$.

[2] Weakly holomorphic means holomorphic on the upper half plane with possible pole at the cusp $i\infty$.

The quasimodularity can be stated in a precise sense via elliptic K3 surfaces. Let

$$\pi\colon S \to \mathbb{P}^1$$

be an elliptic K3 surface with a section and denote by $B, F \in H_2(S, \mathbb{Z})$ the class of the section resp. a fiber. For any $m \geq 1$ one defines the descendent potential

$$\mathsf{F}_{g,m}\big(\tau_{a_1}(\gamma_1)\dots\tau_{a_n}(\gamma_n)\big) = \sum_{h\geq 0} \big\langle \tau_{a_1}(\gamma_1)\dots\tau_{a_n}(\gamma_n)\big\rangle^S_{g,mB+hF}\, q^{h-m}\,.$$

Note that this generating series involves curve classes $mB + hF$ of different divisibilities, bounded by m.

It is convenient to use the following homogenized insertions which will lead to quasimodular forms of pure weight. Let $\mathsf{1} \in H^0(S)$ and $\mathsf{p} \in H^4(S)$ be the identity resp. the point class. Denote

$$W = B + F \in H^2(S)$$

and let

$$U = \mathbb{Q}\langle F, W\rangle \subset H^2(S)$$

be the hyperbolic plane in $H^2(S)$ and let $U^\perp \subset H^2(S)$ be its orthogonal complement with respect to the intersection form. We only consider second cohomology classes which are pure with respect to the decomposition

$$H^2(S,\mathbb{Q}) \cong \mathbb{Q}\big\langle F\big\rangle \oplus \mathbb{Q}\big\langle W\big\rangle \oplus U^\perp\,.$$

Following [8, Section 4.6], define a modified degree function $\underline{\deg}$ by

$$\underline{\deg}(\gamma) = \begin{cases} 2 & \text{if } \gamma = W \text{ or } \mathsf{p}\,,\\ 1 & \text{if } \gamma \in U^\perp\,,\\ 0 & \text{if } \gamma = F \text{ or } \mathsf{1}\,. \end{cases}$$

For $m \geq 1$, consider the *Hecke congruence subgroup of level m*

$$\Gamma_0(m) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) \;\middle|\; c \equiv 0 \mod m \right\}$$

and let $\mathsf{QMod}(m)$ be the space of quasimodular forms for the congruence subgroup $\Gamma_0(m) \subset \mathrm{SL}_2(\mathbb{Z})$. Let $\Delta(q)$ be the modular discriminant

$$\Delta(q) = q\prod_{n\geq 1}(1-q^n)^{24}\,.$$

Our first main result proves level two quasimodularity of $\mathsf{F}_{g,2}$, previously conjectured by Maulik, Pandharipande, and Thomas [29, Section 7.5].

Theorem 1. Let $\gamma_1, \dots, \gamma_n \in H^*(S)$ be homogeneous on the modified degree function $\underline{\deg}$. Then $\mathsf{F}_{g,2}$ is the Fourier expansion of a quasimodular form

$$\mathsf{F}_{g,2}\big(\tau_{a_1}(\gamma_1)\dots\tau_{a_n}(\gamma_n)\big) \in \frac{1}{\Delta(q)^2}\mathsf{QMod}(2)$$

of weight $2g - 12 + \sum_i \underline{\deg}(\gamma_i)$ with pole at $q = 0$ of order at most 2.

0.2 Holomorphic anomaly equation

In the physics literature, the (conjectural) *holomorphic anomaly equation* [4, 5] predicts hidden structures of the Gromov–Witten partition function associated to Calabi–Yau varieties. For the past few years, there has been an extensive work to prove the holomorphic anomaly equation in many cases: local $\mathbb{P}^2$ [26], the quintic threefold [11, 16], K3 surface with primitive curve classes [33], elliptic fibration [34] and $\mathbb{P}^2$ relative to a smooth cubic [6].

Every quasimodular form for $\Gamma_0(m)$ can be written uniquely as a polynomial in C_2 with coefficients which are modular forms for $\Gamma_0(m)$ [18, Proposition 1]. Here,

$$C_2(q) = -\frac{1}{24}E_2(q)$$

is the renormalized second Eisenstein series. Assuming quasimodularity, the holomorphic anomaly equation fixes the non-holomorphic parameter of the Gromov–Witten partition function of K3 surfaces in terms of lower weight partition functions: it computes the derivative of $\mathsf{F}_{g,m}$ with respect to the C_2 variable. See [33] for the proof of holomorphic anomaly equation for K3 surfaces with primitive curve classes and [34] for the holomorphic anomaly equation associated to elliptic fibrations.

Define an endomorphism [33, Section 0.6]

$$\sigma\colon H^*(S^2) \to H^*(S^2)$$

by the following assignments:

$$\sigma(\gamma \boxtimes \gamma') = 0$$

if γ or $\gamma' \in H^0(S) \oplus \mathbb{Q}\langle F \rangle \oplus H^4(S)$, and for $\alpha, \alpha' \in U^\perp$,

$$\begin{aligned}
&\sigma(W \boxtimes W) = \Delta_{U^\perp},\ \sigma(W \boxtimes \alpha) = -\alpha \boxtimes F, \\
&\sigma(\alpha \boxtimes W) = -F \boxtimes \alpha,\ \sigma(\alpha, \alpha') = \langle \alpha, \alpha' \rangle F \boxtimes F\,,
\end{aligned}$$

where $\Delta_{U^\perp}$ denotes the diagonal class for the intersection pairing on $U^\perp$. We will view σ as the exterior product $\sigma_1 \boxtimes \sigma_2$ via Künneth decomposition.

Recall the virtual fundamental class for trivial curve classes which will play a role for the holomorphic anomaly equation. For $\beta = 0$ we have an isomorphism

$$\overline{M}_{g,n}(S, 0) \cong \overline{M}_{g,n} \times S$$

and the virtual class is given by

$$[\overline{M}_{g,n}(S,0)]^{vir} = \begin{cases} [\overline{M}_{0,n} \times S] & \text{if } g = 0\,, \\ c_2(S) \cap [\overline{M}_{1,n} \times S] & \text{if } g = 1\,, \\ 0 & \text{if } g \geq 2\,. \end{cases}$$

Also, consider the pullback under the morphism $\pi\colon S \to \mathbb{P}^1$ of the diagonal class of $\mathbb{P}^1$

$$\Delta_{\mathbb{P}^1} = 1 \boxtimes F + F \boxtimes 1 = \sum_{i=1}^{2} \delta_i \boxtimes \delta_i^\vee .$$

Define the generating series[3]

$$\begin{aligned}
&\mathsf{H}_{g,m}\big(\alpha; \gamma_1, \dots, \gamma_n\big) \qquad (1)\\
&= \mathsf{F}_{g-1,m}\big(\alpha; \gamma_1, \dots, \gamma_n, \Delta_{\mathbb{P}^1}\big)\\
&\quad + 2 \sum_{\substack{g=g_1+g_2\\ \{1,\dots,n\}=I_1\sqcup I_2\\ i\in\{1,2\}}} \mathsf{F}_{g_1,m}\big(\alpha_{I_1}; \gamma_{I_1}, \delta_i\big)\, \mathsf{F}^{vir}_{g_2}\big(\alpha_{I_2}; \gamma_{I_2}, \delta_i^\vee\big)\\
&\quad - 2 \sum_{i=1}^{n} \mathsf{F}_{g,m}\big(\alpha\psi_i; \gamma_1, \dots, \gamma_{i-1}, \pi^*\pi_*\gamma_i, \gamma_{i+1}, \dots, \gamma_n\big)\\
&\quad + \frac{20}{m} \sum_{i=1}^{n} \langle \gamma_i, F\rangle \mathsf{F}_{g,m}\big(\alpha; \gamma_1, \dots, \gamma_{i-1}, F, \gamma_{i+1}, \dots, \gamma_n\big)\\
&\quad - \frac{2}{m} \sum_{i<j} \mathsf{F}_{g,m}\big(\alpha; \gamma_1, \dots, \underbrace{\sigma_1(\gamma_i, \gamma_j)}_{i\text{th}}, \dots, \underbrace{\sigma_2(\gamma_i, \gamma_j)}_{j\text{th}}, \dots, \gamma_n\big)\,,
\end{aligned}$$

where F^{vir} denotes the generating series for virtual fundamental class. In most cases this term vanishes. The equation takes almost the same form for arbitrary m, only the last two terms acquire a factor of $\frac{1}{m}$. The appearance of these factors is explained in Section 3, see also Example 22. We conjecture that the holomorphic anomaly equation has the following form:

Conjecture 2.

$$\frac{d}{dC_2} \mathsf{F}_{g,m}\big(\alpha; \gamma_1, \dots, \gamma_n\big) = \mathsf{H}_{g,m}\big(\alpha; \gamma_1, \dots, \gamma_n\big)\,. \qquad (2)$$

For primitive curve classes, the holomorphic anomaly equation is proven in [33]. In higher divisiblity, it is precisely equation (2) that would be implied by the conjectural multiple cover formula for imprimitve Gromow–Witten invariants of K3 surfaces. We explain this in the following section. We prove Conjecture 2 unconditionally when $m = 2$:

Theorem 3. For any $g \geq 0$,

$$\frac{d}{dC_2} \mathsf{F}_{g,2}\big(\alpha; \gamma_1, \dots, \gamma_n\big) = \mathsf{H}_{g,2}\big(\alpha; \gamma_1, \dots, \gamma_n\big)\,. \qquad (3)$$

[3]Here, instead of descendent insertions we use a tautological class $\alpha \in R^*(\overline{M}_{g,n})$, see the comment in Section 2.2

0.3 Multiple cover formula

Motivated by the Katz–Klemm–Vafa (KKV) formula, Oberdieck and Pandharipande conjectured a formula which computes imprimitive invariants from the primitive invariants:

Conjecture 4. ([32, Conjecture C2]) For a primitive curve class β,

$$\begin{aligned}&\left\langle\tau_{a_1}(\gamma_1)\dots\tau_{a_n}(\gamma_n)\right\rangle_{g,m\beta} \\ &=\sum_{d|m} d^{2g-3+\deg}\left\langle\tau_{a_1}(\varphi_{d,m}(\gamma_1))\dots\tau_{a_n}(\varphi_{d,m}(\gamma_n))\right\rangle_{g,\varphi_{d,m}\left(\frac{m}{d}\beta\right)}.\end{aligned} \tag{4}$$

The invariants on the right hand side are with respect to primitive curve classes[4]. Assuming this formula, we can deduce the holomorphic anomaly equation:

Proposition 5. Let $m \geq 1$. Assume the multiple cover formula (4) holds for all curve classes of divisibility $d \mid m$ and all descendent insertions. Then the holomorphic anomaly equation (2) holds.

Given this proposition, it seems a natural strategy to prove the multiple cover formula in divisibility two and deduce, as a consequence, the holomorphic anomaly equation. Indeed, our method does follow this logic for $m = 2$ and for low genus: we verify the multiple cover formula for $g \leq 2$, see Example 35. For higher genus, however, our method does not seem suitable to achieve this. Instead, our proof of Theorem 1 provides an algorithm, based on the degeneration to the normal cone of a smooth elliptic fiber $E \subset S$, to reduce divisibility two invariants to low genus invariants for which the multiple cover formula is known[5]. The degeneration formula intertwines invariants of S with invariants of $\mathbb{P}^1 \times E$ in a non-trivial way. This phenomenon is illustrated in Example 35 for the genus 2 invariants

$$\left\langle\tau_0(\mathsf{p})^2\right\rangle_{2,2\beta}.$$

0.4 Hecke operator

In Section 2 we apply Conjecture 4 to an elliptic K3 surface to deduce a conjectural multiple cover formula for the descendent potentials $\mathsf{F}_{g,m}$. The multiple cover formula for any divisibility m is then simply a *Hecke operator of the wrong weight* acting on the primitive potential $\mathsf{F}_{g,1}$. Indeed, the weight of $\mathsf{F}_{g,1}$ (and conjecturally of $\mathsf{F}_{g,m}$) is $2g - 12 + \underline{\deg}$, whereas the Hecke operator has the weight of a descendent

[4]Section 2 contains all relevant definitions.

[5]The genus 0 and genus 1 cases are proved by Lee and Leung in [24, 25]. Their proof involves a degeneration formula in symplectic geometry which is not possible in algebraic geometry. We present an algebro-geometric approach using the KKV formula.

potential attached to elliptic curves, namely $2g-2+\underline{\deg}$. This operator can be expressed in terms of Hecke operators (of the correct weight) and translation $q\mapsto q^d$. Together with the holomorphic anomaly equation for primitive curve classes [33] this naturally leads to the above conjecture for the holomorphic anomaly equation for higher divisibility.

0.5 Plan of the paper

We prove the quasimodularity and the holomorphic anomaly equation by induction on the genus and the number of markings. In Section 1, we discuss Hecke theory for weakly holomorphic quasimodular forms. This leads to a natural formulation of the multiple cover formula in Section 2 and the imprimitive holomorphic anomaly equation in Section 3. In Section 4, compatibility of the holomorphic anomaly equation with the degeneration formula is presented. In Section 5, we derive the multiple cover formula, which implies the holomorphic anomaly equation, for genus 0, genus 1 and some genus 2 decendent invariants from the KKV formula. The genus 2 computation relies on double ramification relations with target variety. This result serves as the initial condition for our induction. In Section 6, we use previous results to prove Theorem 1 and 3. The property of the top tautological group $R^{g-1}(M_{g,n})$ reduces higher genus cases to lower genus invariants discussed in Section 5.

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1 Quasimodular forms and Hecke operators

We recall basic properties of quasimodular forms and Hecke operators, see [22, 39], in particular [22, pp. 156–163] and [22, Ch. 3, Section 3]. The Hecke theory for weakly holomorphic quasimodular forms however seems to be less well documented. We thus also include some proofs.

The following operators will play a central role. For any Laurent series

$$f(q) = \sum_{n=-\infty}^{\infty} a_n q^n \tag{5}$$

and $d \in \mathbb{Z}_{>0}$ we define

$$\mathsf{D}_q f = q\frac{d}{dq} f\,, \quad \mathsf{B}_d f = \sum_{n=-\infty}^{\infty} a_n q^{dn}\,, \quad \mathsf{U}_d f = \sum_{n=-\infty}^{\infty} a_{dn} q^n\,.$$

We will apply these operators to the Laurent series associated to certain modular functions. For this we briefly review the definition of modular forms.

1.1 Quasimodular forms

Let $\mathbb{H} = \{\tau \in \mathbb{C} \mid \mathrm{Im}(\tau) > 0\}$ be the upper half-plane. The group $\mathrm{GL}_2^+(\mathbb{R})$ of real 2×2-matrices with positive determinant acts on $\mathbb{H}$ via

$$A\tau = \frac{a\tau + b}{c\tau + d}\,, \quad A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2^+(\mathbb{R})\,.$$

Let $f \colon \mathbb{H} \to \mathbb{C}$ be a function and let

$$q = e^{2\pi i \tau}\,, \quad y = \mathrm{Im}(\tau)\,.$$

For $k \in \mathbb{Z}$ define the *k-th slash operator*

$$(f|_k A)(\tau) = \det(A)^{k/2} (c\tau + d)^{-k} f(A\tau)\,.$$

Definition 6. A *quasimodular form* of weight k for $\mathrm{SL}_2(\mathbb{Z})$ is a holomorphic function $f \colon \mathbb{H} \to \mathbb{C}$ admitting a Fourier expansion

$$f(q) = \sum_{n=0}^{\infty} a_n q^n\,, \quad |q| < 1\,, \tag{6}$$

such that there exist $p \geq 0$ and holomorphic functions f_r, $r = 0, \ldots, p$ satisfying the following conditions:

(i) the (non-holomorphic) function $\widehat{f} = \sum_{r=0}^{p} f_r y^{-r}$ satisfies the transformation law
$$\widehat{f}|_k \gamma = \widehat{f} \text{ for all } \gamma \in \mathrm{SL}_2(\mathbb{Z})\,,$$

(ii) $f = f_0$,

(iii) each f_r has an expansion of the form (6).

If $p = 0$ then f is called a *modular form*. We denote the space of modular resp. quasimodular forms by Mod and QMod.

Remark 7. If $\widehat{f} = \sum_{r=0}^{p} f_r y^{-r}$ as above with $f_p \neq 0$, then each f_r is a quasimodular form of weight $k - 2r$, see [39, Proposition 20]. Moreover, the last one, i.e. f_p is in fact modular (of weight $k - 2p$). The following structural results are well-known [39, Proposition 4, Proposition 20]

$$\mathsf{Mod} = \mathbb{C}[C_4, C_6]\,, \quad \mathsf{QMod} = \mathbb{C}[C_2, C_4, C_6]\,,$$

where

$$C_{2i}(q) = -\frac{B_{2i}}{2i \cdot (2i)!} E_{2i}(q)$$

is the renormalized $2i$-th Eisenstein series. The notion (i) defines the space AHM of *almost holomorphic modular forms* and the assignment $\widehat{f} \mapsto f$ is an isomorphism

$$\mathsf{AHM} \to \mathsf{QMod}\,.$$

Under this map, differentiation with respect to $\frac{1}{8\pi y}$ corresponds to differentiation with respect to C_2.

The modular functions considered in this paper will usually have poles at the cusp $\tau = i\infty$ corresponding to $q = 0$. We will refer to these functions as *weakly holomorphic* with pole of specified order. We want to clarify this terminology in the context of quasimodular forms.

Definition 8. A function f is said to be *weakly holomorphic quasimodular with pole of order at most* $m \geq 0$, if f satisfies the conditions in Definition 6 except that each f_r is allowed to have a pole at the cusp $i\infty$ of order at most m. If $p = 0$ then f is called a weakly holomorphic modular form with pole of order at most m.

By parallel arguments as in [39, Proposition 20], the assertions in Remark 7 hold analogously for weakly holomorphic quasimodular forms. In particular, f_p is weakly holomorphic modular with pole of order at most m. The space of weakly holomorphic modular forms is generated by $\frac{1}{\Delta}$ over Mod, where

$$\Delta(q) = q \prod_{n \geq 1} (1 - q^n)^{24}$$

is the modular discriminant.[6] As a consequence,

$$f_p \in \frac{1}{\Delta^m} \mathsf{Mod}$$

and since f_p is of weight $k - 2p$ (and there are no non-zero modular forms of negative weight) we have $k \geq 2p - 12m$.

For quasimodular forms we include the following observation.

[6]See [14] where the authors examine an explicit basis of the space of weakly holomorphic modular forms.

Lemma 9. The space of weakly holomorphic quasimodular forms with pole of order at most m is given by

$$\frac{1}{\Delta^m}\mathsf{QMod}\,.$$

Proof. Let f be a weakly holomorphic form with pole of order at most m and weight k and let

$$\widehat{f} = \sum_{r=0}^{p} f_r y^{-r}\,,$$

with $f = f_0$. Multiplying by Δ^m we have for all $\gamma \in \mathrm{SL}_2(\mathbb{Z})$

$$(\Delta^m \widehat{f})|_{k+12m}\gamma = (\Delta^m)|_{12m}\gamma \cdot (\widehat{f})|_k\gamma = \Delta^m \widehat{f}\,.$$

Since each $\Delta^m f_r$ is holomorphic at $i\infty$ this proves

$$f \in \frac{1}{\Delta^m}\mathsf{QMod}\,.$$

Analogous argument shows that the quotient of any quasimodular form by Δ^m defines a weakly holomorphic quasimodular form with pole of order at most m. □

1.2 Hecke operators

Let $m \in \mathbb{N}$ and consider the set of integral matrices of determinant m

$$H_m = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid a,b,c,d \in \mathbb{Z}\,, ad - bc = m \right\}.$$

The modular group $\mathrm{SL}_2(\mathbb{Z})$ acts on H_m by left multiplication. The classical *Hecke operators* T_m acting on modular forms f of weight k are defined by [39, Section 4.1]

$$\mathsf{T}_m f = m^{k/2-1} \sum_{\gamma \in \mathrm{SL}_2(\mathbb{Z})\backslash H_m} f|_k\gamma\,.$$

This definition is equivalent to [22, Ch. 3, Proposition 38]

$$\mathsf{T}_m = \sum_{ad=m} a^{k-1}\mathsf{B}_a\mathsf{U}_d\,. \tag{7}$$

The action of (7) naturally extends to the action of the q-expansion of weakly holomorphic quasimodular forms. We prove that the action again defines a weakly holomorphic quasimodular form. For simplicity (we will only use this case) we restrict to the case when f has a pole of order at most one.

Lemma 10. Let $f \in \frac{1}{\Delta}\mathsf{QMod}$ be of weight k. Then $\mathsf{T}_m f$ is a weakly holomorphic quasimodular form of weight k with pole of order at most m, i.e.

$$\mathsf{T}_m f \in \frac{1}{\Delta^m}\mathsf{QMod}\,.$$

Proof. In [31] it is shown that T_m defines a map $\mathsf{QMod} \to \mathsf{QMod}$ preserving the weight. We briefly recall the key arguments for $f \in \mathsf{QMod}$. The definition of quasimodular forms is equivalent to the condition[7]

$$(f|_k\gamma)(\tau) = \sum_{r=0}^{p}\left(\frac{c}{c\tau+d}\right)^r f_r(\tau) \text{ for all } \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})\,,$$

where f_r are as in Definition 6. Defining a modification of the slash operator for quasimodular forms[8]

$$(f||_kA)(\tau) = \sum_{r=0}^{p}(-c)^r(c\tau+d)^r(f_r|_kA)(\tau) \text{ for } A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2^+(\mathbb{R})\,,$$

then the quasimodularity is equivalent to

$$f||_k\gamma = f \text{ for all } \gamma \in \mathrm{SL}_2(\mathbb{Z})\,.$$

This leads to a parallel treatment of Hecke operators as in the classical context of modular forms. By [31, Proposition 2] we have

$$f||_k(\gamma A) = f||_kA\,, \text{ for all } \gamma \in \mathrm{SL}_2(\mathbb{Z})\,, A \in \mathrm{GL}_2^+(\mathbb{R})$$

and we define

$$\mathsf{T}_m f = m^{k/2-1}\sum_{A\in \mathrm{SL}_2(\mathbb{Z})\backslash H_m} f||_kA\,.$$

This definition is then independent of a choice of representatives of $\mathrm{SL}_2(\mathbb{Z}) \setminus H_m$. To conclude that $\mathsf{T}_m f$ is a quasimodular form, we would like to argue that it is invariant under $(-)||_k\gamma$ for all $\gamma \in \mathrm{SL}_2(\mathbb{Z})$. This statement, however, is not sensible at the moment[9] because the definition of $(-)||_k\gamma$ relies on the existence of associated functions f_r. This technicality is resolved in [31, Section 2.4, 2.5] by considering a certain period domain $\mathcal{P}$ and identifying quasimodular forms as holomorphic functions on $\mathcal{P}$, which are left $\mathrm{SL}_2(\mathbb{Z})$-invariant and satisfy a transformation property for a right action of the subgroup of upper triangular matrices. The domain $\mathcal{P}$ is

[7]This notion is called 'differential modular form' in [31]. As pointed out in [39, Section 5.3], this notion is equivalent to be a quasimodular form.

[8]This definition differs from [31, Equation 12] by a factor m^{-p}, where p is the depth of f. Our definition of the Hecke operator differs by the same factor.

[9]We are grateful to the referee for pointing out this subtle detail.

contained in $\mathrm{GL}_2(\mathbb{C})$ and it contains the upper-half plane $\mathbb{H}$. The actions are given by left resp. right multiplication. The argument carries over to weakly holomorphic quasimodular forms without change.

A particular set of representatives for $\mathrm{SL}_2(\mathbb{Z}) \setminus H_m$ is given by

$$\left\{\gamma_b = \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \mid a, d \in \mathbb{N}, ad = m, 0 \leq b < d\right\}.$$

Note that $(-)||_k\gamma_b = (-)|_k\gamma_b$ because the terms for $r > 0$ vanish. Since

$$\mathsf{U}_d f(\tau) = \frac{1}{d} \sum_{0 \leq b < d} f\left(\frac{\tau + b}{d}\right),$$

we thus recover equation (7):

$$\begin{aligned} \mathsf{T}_m f(\tau) &= m^{k/2-1} \sum_{\substack{ad=m \\ 0 \leq b < d}} d^{-k} m^{k/2} f\left(\frac{a\tau + b}{d}\right) \\ &= \sum_{ad=m} a^{k-1} \mathsf{B}_a \mathsf{U}_d f(\tau)\,. \end{aligned}$$

For weakly holomorphic quasimodular forms $f \in \frac{1}{\Delta}\mathsf{QMod}$ we follow the same proof. The difference here is that the functions f_r are allowed to have simple poles at $i\infty$. The slash operator $(-)||_k$ however may turn a simple pole into a pole of higher order. For $(-)||_k\gamma_b$ this order is bounded by m. As a consequence, $\mathsf{T}_m f$ is weakly holomorphic quasimodular with pole of order at most m. □

For our study of the multiple cover formula in Section 2 we will require a more flexible notion, where the exponent is not necessarily related to the weight. The action of this operator will preserve the weight of weakly holomorphic quasimodular forms, it will, however, introduce poles and level structure.

Definition 11. For $\ell \in \mathbb{Z}$, we define

$$\mathsf{T}_{m,\ell} = \sum_{ad=m} a^{\ell-1} \mathsf{B}_a \mathsf{U}_d\,.$$

The operator $\mathsf{T}_{m,\ell}$ is simply the m-th Hecke operator of weight ℓ, which we let act on functions of weight k. By Möbius inversion we may rewrite each of them in terms of the other (see [1, Section 2.7]). For this, let μ be the Möbius function.

Lemma 12. The action of $\mathsf{T}_{m,\ell}$ on weakly holomorphic quasimodular forms of weight k is given by

$$\mathsf{T}_{m,\ell} = \sum_{ad=m} c_{k,\ell}(a) \mathsf{B}_a \mathsf{T}_d\,,$$

where

$$c_{k,\ell}(a) = \sum_{r|a} r^{\ell-1} \mu\left(\frac{a}{r}\right) \left(\frac{a}{r}\right)^{k-1}.$$

Proof. The formula for $c_{k,\ell}$ above can be rewritten as

$$c_{k,\ell} = \mathrm{Id}_{\ell-1} \star (\mu \cdot \mathrm{Id}_{k-1})\,,$$

where $\mathrm{Id}_{\ell-1}(n) = n^{\ell-1}$ is the $(\ell-1)$-th power function and $\star$ denotes Dirichlet convolution, i.e. for functions g, h we have

$$(g \star h)(m) = \sum_{ad=m} g(a)h(d)\,.$$

Note also that B is multiplicative with respect to composition, i.e. for $e \mid a$ we have $\mathsf{B}_a = \mathsf{B}_e \mathsf{B}_{\frac{a}{e}}$ and therefore

$$\begin{aligned}
\mathsf{T}_{m,\ell} &= \sum_{ad=m} a^{\ell-1} \mathsf{B}_a \mathsf{U}_d \\
&= \sum_{ad=m} \big(\mathrm{Id}_{\ell-1} \star (\mu \cdot \mathrm{Id}_{k-1}) \star \mathrm{Id}_{k-1}\big)(a) \mathsf{B}_a \mathsf{U}_d \\
&= \sum_{ad=m} \left(\sum_{e|a} c_{k,\ell}(e) \left(\frac{a}{e}\right)^{k-1} \right) \mathsf{B}_a \mathsf{U}_d \\
&= \sum_{uw=m} c_{k,\ell}(u) \mathsf{B}_u \left(\sum_{v|w} v^{k-1} \mathsf{B}_v \mathsf{U}_{\frac{w}{v}} \right) \\
&= \sum_{uw=m} c_{k,\ell}(u) \mathsf{B}_u \mathsf{T}_w\,.
\end{aligned}$$

□

As a consequence we obtain the following result. Here, we let $\mathsf{Mod}(m)$ and $\mathsf{QMod}(m)$ be the space of modular resp. quasimodular forms for the congruence subgroup $\Gamma_0(m) \subset \mathrm{SL}_2(\mathbb{Z})$, see the introduction.

Proposition 13. Let $f \in \frac{1}{\Delta}\mathsf{QMod}$ be of weight k, then $\mathsf{T}_{m,\ell} f$ is a weakly holomorphic quasimodular of weight k with pole of order at most m for the congruence subgroup $\Gamma_0(m) \subset \mathrm{SL}_2(\mathbb{Z})$

$$\mathsf{T}_{m,\ell} f \in \frac{1}{\Delta^m}\mathsf{QMod}(m)\,.$$

Proof. We use the formula in Lemma 12 and treat each summand separately. By Lemma 9 each $\mathsf{T}_d f$ satisfies

$$\mathsf{T}_d f \in \frac{1}{\Delta^d}\mathsf{QMod}\,.$$

The action of B_a raises $q \mapsto q^a$, or equivalently $\tau \mapsto a\tau$, so it maps QMod to $\mathsf{QMod}(a)$, see [22, Ch. 3, Proposition 17]. Therefore

$$\mathsf{B}_a \mathsf{T}_d f \in \frac{1}{\Delta(q^a)^d}\mathsf{QMod}(a)\,.$$

Finally, the weakly holomorphic modular form for $\Gamma_0(a)$ defined by

$$\frac{\Delta(q)^a}{\Delta(q^a)}$$

is in fact holomorphic at $i\infty$, i.e. contained in $\mathsf{Mod}(a)$. Hence the same is true for its d-th power and we find

$$\mathsf{B}_a \mathsf{T}_d f \in \frac{1}{\Delta^m} \mathsf{QMod}(a)\,.$$

which concludes the proof since $\mathsf{QMod}(a) \subset \mathsf{QMod}(m)$. □

For later reference, we list the following basic commutator relations between the above operators acting on weakly holomorphic quasimodular forms f of weight k. Recall, that the algebra $\mathsf{QMod}(m)$ is freely generated by the Eisenstein series C_2 over the algebra $\mathsf{Mod}(m)$ of modular forms. Formal differentiation with respect to C_2 is therefore well-defined.

Lemma 14. Let d, $e \in \mathbb{N}$ and $\ell \in \mathbb{Z}$, then

(i) $\mathsf{B}_d \mathsf{B}_e = \mathsf{B}_{de} = \mathsf{B}_e \mathsf{B}_d\,,$

(ii) $\mathsf{U}_d \mathsf{U}_e = \mathsf{U}_{de} = \mathsf{U}_e \mathsf{U}_d\,,$

(iii) $\mathsf{D}_q \mathsf{B}_d = d\, \mathsf{B}_d \mathsf{D}_q\,, \quad \mathsf{U}_d \mathsf{D}_q = d\, \mathsf{D}_q \mathsf{U}_d\,,$

(iv) $\mathsf{T}_{m,\ell+2} \mathsf{D}_q = m\, \mathsf{D}_q \mathsf{T}_{m,\ell}\,,$

(v) $\frac{d}{dC_2} \mathsf{T}_{m,\ell+2} = m\, \mathsf{T}_{m,\ell} \frac{d}{dC_2}\,,$

(vi) $[\frac{d}{dC_2}, \mathsf{D}_q] = -2k.$

Proof. The proof for (i)-(iv) follows directly from the definition. For (v) one may use that under the isomorphism $\widehat{f} \mapsto f$ the differentiation $\frac{d}{dC_2}$ corresponds to differentiation with respect to $\frac{1}{8\pi y}$, see Remark 7. The statement (v) is then checked as an identity of Laurent series in q with polynomial coefficients in y^{-1}. The commutator relation (vi) is well-known, see e.g. [39, Section 5.3].

□

2 Multiple cover formula

This section contains a discussion of the multiple cover formula. We start by recalling the conjecture formulated in [32]. Then, we study the conjecture for the descendent potentials associated to elliptic K3 surfaces. The result is expressed in terms of Hecke operators. The discussion naturally leads to a candidate for the holomorphic anomaly equation in higher divisibility. We conclude with a proof of the multiple cover formula in fiber direction.

2.1 Multiple cover formula

Let S be a nonsingular projective K3 surface, $\beta \in H_2(S,\mathbb{Z})$ be a *primitive* effective curve class, $m \in \mathbb{N}$ and $d \mid m$ be a divisor of m. The proposed formula by Oberdieck and Pandharipande involves a choice of a real isometry

$$\varphi_{d,m}\colon \Big(H^2(S,\mathbb{R})\,,\langle\,,\,\rangle\Big) \to \Big(H^2(S_d,\mathbb{R})\,,\langle\,,\,\rangle\Big)$$

between two K3 surfaces such that

$$\varphi_{d,m}\left(\frac{m}{d}\beta\right) \in H_2(S_d,\mathbb{Z})$$

is a primitive effective curve class[10]. In [9] the second author proved that such an isometry can always be found and Gromov–Witten invariants are in fact independent of the choice of isometry.

Consider integers $a_i \in \mathbb{N}$, cohomology classes $\gamma_i \in H^*(S,\mathbb{Q})$ and let $\deg = \sum \deg(\gamma_i)$. Then, the conjectured multiple cover formula [32, Conjecture C2], identical to Conjecture 4 in Section 0, is

$$\begin{aligned}&\big\langle \tau_{a_1}(\gamma_1)\dots\tau_{a_n}(\gamma_n)\big\rangle_{g,m\beta}\\ &= \sum_{d|m} d^{2g-3+\deg}\big\langle \tau_{a_1}(\varphi_{d,m}(\gamma_1))\dots\tau_{a_n}(\varphi_{d,m}(\gamma_n))\big\rangle_{g,\varphi_{d,m}\left(\frac{m}{d}\beta\right)}\,.\end{aligned}$$

Let S be an elliptic K3 surface with a section[11]. The full (reduced) Gromov–Witten theory of K3 surfaces is captured by S with curve class $mB + hF$ via standard deformation arguments using the Torelli theorem. In fact, the multiple cover conjecture can be captured entirely via S as well: we may choose *the same* $S_d = S$ for any d dividing m and h. For $l \in \mathbb{Q}^*$ we define

$$\phi_l\colon H^*(S,\mathbb{Q}) \to H^*(S,\mathbb{Q})$$

acting on $U = \mathbb{Q}\langle F, W\rangle$ as

$$\phi_l(F) = \frac{1}{l}F\,, \qquad \phi_l(W) = lW\,,$$

and trivially on the orthogonal complement $U^\perp$. For $d \mid m$ and $d \mid h$ we may choose $\varphi_{d,m}$ as $\phi_{\frac{d}{m}}$:

$$\phi_{\frac{d}{m}}\left(\frac{m}{d}B + \frac{h}{d}F\right) = B + \left(\frac{m(h-m)}{d^2} + 1\right)F \ \text{ in } H_2(S,\mathbb{Z})$$

[10] We view curve classes also as cohomology classes under the natural isomorphism $H_2(S,\mathbb{Z}) \cong H^2(S,\mathbb{Z})$.

[11] Notations here are as in Section 0. In particular, we use the modified degree function $\underline{\deg}$.

which is a primitive curve class.

Altering the curve class via the isometry ϕ therefore results in additional factors of $\frac{d}{m}$ or $\frac{m}{d}$ while keeping the descendent insertions unchanged. This explains the change in exponents

$$2g-3+\deg \longleftrightarrow 2g-3+\underline{\deg}$$

and the factor $m^{\deg - \underline{\deg}}$ in the multiple cover formula below for the descendent potential. We use the operator $\mathsf{T}_{m,\ell}$ introduced in Definition 11. As pointed out in Section 0.4, this is the m-th Hecke operator for functions of weight ℓ, which we let act on $\mathsf{F}_{g,1}$ (which has weight $2g-12+\underline{\deg}$). Before stating the conjecture, we want to discuss the role of tautological classes and compatibility with respect to restriction to boundary strata.

2.2 Compatibility I

We will find it convenient to use pullbacks of tautological classes from $\overline{M}_{g,n}$ instead of ψ-classes on $\overline{M}_{g,n}(S,\beta)$. For $2g-2+n>0$, let

$$R^*(\overline{M}_{g,n}) \subseteq A^*(\overline{M}_{g,n})$$

be the tautological ring of $\overline{M}_{g,n}$. For a tautological class $\alpha \in R^*(\overline{M}_{g,n})$, we consider the invariants

$$\big\langle \alpha; \gamma_1, \ldots, \gamma_n \big\rangle = \int_{[\overline{M}_{g,n}(S,\beta)]^{red}} \pi^*\alpha \cup \prod_{i=1}^{n} \mathrm{ev}_i^*(\gamma_i)\,,$$

where $\pi \colon \overline{M}_{g,n}(S,\beta) \to \overline{M}_{g,n}$ is the stabilization morphism. We write

$$\mathsf{F}_{g,m}\big(\alpha; \gamma_1, \ldots, \gamma_n\big) = \sum_{h \geq 0} \big\langle \alpha; \gamma_1, \ldots, \gamma_n \big\rangle_{g, mB+hF}\, q^{h-m}$$

for the generating series in divisibility m. By the usual trading of cotangent line classes, these generating series are related to the ones defined via cotangent classes on $\overline{M}_{g,n}(S,\beta)$. Any monomial in ψ- and κ-classes can be written, after adding markings, as a product of ψ-classes. This procedure leaves deg and $\underline{\deg}$ unchanged. Before stating the multiple cover formula below, we explain the compatibility with respect to restriction to boundary strata in $\overline{M}_{g,n}(S,\beta)$.

A crucial point for this compatibility is the splitting behavior of the reduced class. Consider the pullback of the boundary divisor

$$\overline{M}_{g-1,n+2} \to \overline{M}_{g,n}$$

under the stabilization morphism π. Let α be the pushforward of a tautological class (we will omit pushforwards in the notation below). By the restriction property of the reduced class, we obtain

$$\mathsf{F}_{g,m}\big(\alpha; \gamma\big) = \mathsf{F}_{g-1,m}\big(\alpha; \gamma\Delta_S\big)\,.$$

Then, the compatibility follows from two facts. Firstly, for the diagonal class Δ_S we have

$$\big(\deg - \underline{\deg}\big)(\Delta_S) = 0\,,$$

thus the factor $m^{\deg - \underline{\deg}}$ in Conjecture 15 below remains unchanged. Secondly, we have $\underline{\deg}(\Delta_S) = 2$ which precisely offsets the genus reduction from g to $g-1$ in the formula

$$\ell = 2g - 2 + \underline{\deg}\,.$$

Next, consider the pullback of the boundary divisor

$$\overline{M}_{g_1,n_1+1} \times \overline{M}_{g_2,n_2+1} \to \overline{M}_{g,n}$$

under the stabilization morphism π. Let

$$\alpha = \alpha_1 \boxtimes \alpha_2\,, \quad \{1,\dots,n\} = I_1 \cup I_2\,, \quad \gamma = \gamma_1 \boxtimes \gamma_2$$

be the pushforward of the product of tautological classes, the splitting of markings, and the splitting of the insertions respectively. The Künneth decomposition of the class of the diagonal is denoted by

$$[\Delta_S] = \sum_j \Delta_j \boxtimes \Delta^j\,.$$

The splitting property implies that

$$\mathsf{F}_{g,m}\big(\alpha;\gamma\big) = \sum_{m_1+m_2=m} \sum_j \Bigg(\mathsf{F}_{g_1,m_1}\big(\alpha_1;\gamma_{I_1}\Delta_j\big)\cdot \mathsf{F}^{vir}_{g_2,m_2}\big(\alpha_1;\gamma_{I_1}\Delta^j\big) \\ + \mathsf{F}^{vir}_{g_1,m_1}\big(\alpha_1;\gamma_{I_1}\Delta_j\big)\cdot \mathsf{F}_{g_2,m_2}\big(\alpha_1;\gamma_{I_1}\Delta^j\big)\Bigg)\,.$$

The virtual class for non-zero curve classes vanishes, thus the contribution F^{vir} is a number. As a consequence, no non-trivial products of generating series appear when we use boundary expressions. By similar consideration as above, using the deg and $\underline{\deg}$ for the diagonal class, we find that the multiple cover formula is compatible with respect to this boundary divisor as well. We can now state the multiple cover formula for the generating series with tautological classes:

Conjecture 15. For $\underline{\deg}$-homogeneous $\gamma_i \in H^*(S,\mathbb{Q})$,

$$\mathsf{F}_{g,m}\big(\alpha;\gamma_1,\dots,\gamma_n\big) = m^{\deg - \underline{\deg}}\,\mathsf{T}_{m,\ell}\Big(\mathsf{F}_{g,1}\big(\alpha;\gamma_1,\dots,\gamma_n\big)\Big)\,,$$

where $\deg = \sum \deg(\gamma_i)$, $\underline{\deg} = \sum \underline{\deg}(\gamma_i)$ and $\ell = 2g - 2 + \underline{\deg}$.

Based on the discussion above, the same formula is conjectured for the potential

$$\mathsf{F}_{g,m}\big(\tau_{a_1}(\gamma_1)\dots\tau_{a_n}(\gamma_n)\big)\,.$$

We now show that our presentation of the multiple cover formula is equivalent to the original formula.

Lemma 16. Conjecture 4 for all $d \mid m$ is equivalent to Conjecture 15 for m.

Proof. By the deformation invariance of the reduced class, the Gromov–Witten invariants for arbitrary curve classes are fully captured by an elliptic K3 surface with a section. The primitive curve classes are $B + hF \in H_2(S, \mathbb{Z})$. Taking the coefficient of q^{mh-m} in Conjecture 15 gives a multiple cover formula for the curve class $mB + mhF$ which matches the formula in Conjecture 4. It is the other implication which we have to justify.

The generating series $\mathsf{F}_{g,m}$ involves curve classes $mB+hF$ of different divisibilities bounded by m. We apply Conjecture 4 to each invariant and use the isometries ϕ. Note that each appearance of $\gamma_i = F$ introduces a factor of $\frac{m}{d}$, while each appearance of $\gamma_i = W$ gives $\frac{d}{m}$. Moreover,

$$|\{i \mid \gamma_i = F\}| - |\{i \mid \gamma_i = W\}| = \deg - \underline{\deg}\,,$$

and therefore

$$\begin{aligned}
\mathsf{F}_{g,m}\big(\alpha;\gamma_1,\dots,\gamma_n\big) &= \sum_{h\geq 0} \big\langle \alpha;\gamma_1,\dots,\gamma_n\big\rangle_{g,mB+hF}\, q^{h-m} \\
&= \sum_{h\geq 0}\sum_{\substack{d\mid m\\ d\mid h}} d^{2g-3+\deg}\left(\frac{m}{d}\right)^{\deg-\underline{\deg}} \big\langle \alpha;\gamma_1,\dots,\gamma_n\big\rangle_{g,B+\left(\frac{m(h-m)}{d^2}+1\right)F}\, q^{h-m} \\
&= m^{\deg-\underline{\deg}} \sum_{d\mid m} d^{2g-3+\underline{\deg}} \left(\sum_{h\geq 0} \big\langle \alpha;\gamma_1,\dots,\gamma_n\big\rangle_{g,B+\left(\frac{m}{d}(h-\frac{m}{d})+1\right)F}\, \big(q^d\big)^{h-\frac{m}{d}}\right) \\
&= m^{\deg-\underline{\deg}} \sum_{d\mid m} d^{2g-3+\underline{\deg}} \left(\mathsf{B}_d \mathsf{U}_{\frac{m}{d}} \sum_{h\geq 0} \big\langle \alpha;\gamma_1,\dots,\gamma_n\big\rangle_{g,B+hF}\, q^{h-1}\right) \\
&= m^{\deg-\underline{\deg}} \sum_{d\mid m} d^{2g-3+\underline{\deg}}\, \mathsf{B}_d \mathsf{U}_{\frac{m}{d}} \mathsf{F}_{g,1}\big(\alpha;\gamma_1,\dots,\gamma_n\big) \\
&= m^{\deg-\underline{\deg}}\, \mathsf{T}_{m,\ell}\Big(\mathsf{F}_{g,1}\big(\alpha;\gamma_1,\dots,\gamma_n\big)\Big)\,.
\end{aligned}$$

□

As a direct consequence, the multiple cover formula implies level m quasimodularity.

Proposition 17. If the generating series $\mathsf{F}_{g,m}$ satisfies the multiple cover formula, it satisfies the quasimodularity conjecture. More precisely,

$$\mathsf{F}_{g,m} \in \frac{1}{\Delta(q)^m}\mathsf{QMod}(m)\,.$$

Proof. The descendent potentials for primitive curve classes are weakly holomorphic quasimodular with pole of order at most 1 and weight $2g-12+\underline{\deg}$, see [29, Theorem 4] and [8, Theorem 9]. The claim thus follows from Proposition 13. $\square$

2.3 Multiple cover formula in fiber direction

When the curve class is a multiple of the fiber class F, the multiple cover formula reduces to a property of the Gromov–Witten invariant of elliptic curves. Relevant properties are conjectured in [38].

Let $S \to \mathbb{P}^1$ be an elliptic K3 surface with section and let $\beta = mF$. By Section 6, Case 1, we may assume at least one of the insertions is the point class $\gamma_1 = \mathsf{p}$ and $g \geq 1$. Let

$$\iota \colon E \hookrightarrow S$$

be the inclusion of a fiber, representing the class F. Since the point class is represented by a transverse intersection of E and the section B, the Gromov–Witten theory of S localizes to the Gromov–Witten theory of E with the curve class mE. Computation of the obstruction bundle shows that the invariant is of the form

$$\big\langle \tau_{a_1}(\mathsf{p})\tau_{a_2}(\gamma_2)\dots\tau_{a_n}(\gamma_n)\big\rangle^S_{g,mF} = \big\langle \lambda_{g-1}; \tau_{a_1}(\omega)\tau_{a_2}(\iota^*\gamma_2)\dots\tau_{a_n}(\iota^*\gamma_n)\big\rangle^E_{g,mE}$$

where $\lambda_{g-1} = c_{g-1}(\mathbb{E}_g)$. In particular, if $\gamma_i \in \mathbb{Q}\langle F\rangle \oplus U^\perp \oplus \mathbb{Q}\langle \mathsf{p}\rangle$, the invariant vanishes. Consider the following generating series

$$\mathsf{F}^E_g\big(\tau_{a_1}(\gamma_1)\dots\tau_{a_n}(\gamma_n)\big) = \sum_{m\geq 0}\big\langle \lambda_{g-1}; \tau_{a_1}(\gamma_1)\dots\tau_{a_n}(\gamma_n)\big\rangle^E_{g,mE}\, q^m$$

where $\gamma_i = \mathsf{1}$ or ω and $\sum a_i + \sum \deg(\gamma_i) = g-1+n$.

The generating series F^E_g has a simple description in terms of Eisenstein series. The following formula is conjectured in [38].

Lemma 18. For $g \geq 1$,

$$\mathsf{F}^E_g\big(\tau_{g-1}(\omega)\big) = \frac{g!}{2^{g-1}}C_{2g}\,.$$

Proof. In [38, Proposition 4.4.7] this formula is given under assuming the Virasoro constraint for $\mathbb{P}^1 \times E$. The Virasoro constraint for any toric bundle over a nonsingular variety which satisfies the Virasoro constraint is proven in [13]. Combining this result with the Virasoro constraint for elliptic curves [35], the result follows. $\square$

When $\beta = mF$, Conjecture 4 is equivalent to the following proposition.

Proposition 19. There exists $c \in \mathbb{Q}$ such that

$$\mathsf{F}_g^E\big(\tau_{a_1}(\omega)\dots\tau_{a_r}(\omega)\tau_{a_{r+1}}(1)\dots\tau_{a_{r'}}(1)\big) = c\,\mathsf{D}_q^{r-1}\mathsf{F}_g^E\big(\tau_{g-1}(\omega)\big)\,.$$

Proof. Boundary strata with a vertex of genus less than g do not contribute because the invariants involve λ_h vanishes on $\overline{M}_{g,n}(E,m)$ when $h \geq g$. If $r' > r$, then $\sum a_i \geq g$ and we can reduce to the case when $r' = r$ by the topological recursion on the ψ-monomial in $R^{\geq g}(\overline{M}_{g,n})$ [23]. If $r' = r$, then $\sum a_i = g-1$ and similar argument as in Section 6, Case 3 can be applied. Therefore F_g^E is proportional to

$$\mathsf{F}_g^E\big(\tau_{g-1}(\omega)\tau_0(\omega)^{r-1}\big) = \mathsf{D}_q^{r-1}\mathsf{F}_g^E\big(\tau_{g-1}(\omega)\big)$$

where the equality comes from the divisor equation. □

Remark 20. One can find a closed formula for the constant $c \in \mathbb{Q}$ by integrating tautological classes on $\overline{M}_{g,n}$.

3 Holomorphic anomaly equation

This section contains a proof of Proposition 5. We derive the holomorphic anomaly equation for $m \geq 1$ from the conjectural multiple cover formula, such that both are compatible[12]. It turns out that the equation is almost identical to the one in the primitive case. Additional factors appear only in the last two terms, which are specific to K3 surfaces. We refer to [34, Section 7.3] for explanations on the appearance of these terms.

Proof of Proposition 5. Let $\gamma_1, \dots, \gamma_n \in H^*(S)$ with

$$\deg = \sum_i \deg(\gamma_i)\,, \quad \underline{\deg} = \sum_i \underline{\deg}(\gamma_i)\,.$$

We will simply write γ to denote $\gamma_1, \dots, \gamma_n$. Assume that the multiple cover formula (4) holds for all divisors $d \mid m$ and all descendent insertions. Using Lemma 16, also Conjecture 15 holds. By Proposition 17, the descendent potentials are quasi-modular forms of level m and we can consider the $\frac{d}{dC_2}$-derivative. We apply the $\frac{d}{dC_2}$-derivative to Conjecture 15 and use the commutator relations Lemma 14 to obtain:

$$\begin{aligned}\frac{d}{dC_2}\mathsf{F}_{g,m}\big(\alpha;\gamma\big) &= \frac{d}{dC_2}\Big(m^{\deg - \underline{\deg}}\mathsf{T}_{m,2g-2+\underline{\deg}}\mathsf{F}_{g,1}\big(\alpha;\gamma\big)\Big)\\ &= m^{\deg - \underline{\deg}+1}\mathsf{T}_{m,2g-4+\underline{\deg}}\frac{d}{dC_2}\mathsf{F}_{g,1}\big(\alpha;\gamma\big)\,.\end{aligned}$$

[12] We should point out that this derivation should be lifted to the cycle-valued holomorphic anomaly equation. Tautological classes play no role here.

We want to explain that the last row precisely recovers the definition of $\mathsf{H}_{g,m}$ in (1), after applying the holomorphic anomaly equation for the primitive series [33, Theorem 4]:

$$\frac{d}{dC_2}\mathsf{F}_{g,1}\big(\alpha;\gamma\big) = \mathsf{H}_{g,1}\big(\alpha;\gamma\big)\,.$$

We do so by explaining how each term of $\mathsf{H}_{g,1}\big(\alpha;\gamma\big)$ is affected:

(i) The degree deg of $\mathsf{F}_{g-1,1}\big(\alpha;\gamma\,\Delta_{\mathbb{P}^1}\big)$ has increased by one. The genus, however, dropped by 1. Thus, the first term precisely matches the multiple cover formula, i.e.

$$\mathsf{F}_{g-1,m}\big(\alpha;\gamma\Delta_{\mathbb{P}^1}\big) = m^{\deg-\underline{\deg}+1}\mathsf{T}_{m,2g-4+\underline{\deg}}\Big(\mathsf{F}_{g-1,1}\big(\alpha;\gamma\,\Delta_{\mathbb{P}^1}\big)\Big)\,.$$

(ii) The virtual class is non-zero only for curve class $\beta = 0$ and genus 0, 1, see Section 0. In these cases, the potential $\mathsf{F}^{vir}_{g_2}$ is simply a number and the operator $\mathsf{T}_{m,\ell}$ acts non-trivially only on $\mathsf{F}_{g_1,m}$. We distinguish the two cases:

$g_2 = 0$. The virtual class is given by the fundamental class and the integral is given by intersection pairing on S. Non-trivial terms are obtained from $\delta_i^\vee = 1$ or F. If $\delta_i^\vee = 1$ then

$$\deg(\gamma_{I_2}) = \underline{\deg}(\gamma_{I_2}) = 2\,.$$

The modified degree $\underline{\deg}$ of $\mathsf{F}_{g_1,1}\big(\alpha_{I_1};\gamma_{I_1}\delta_i\big)$ has decreased by 2, whereas deg decreased by 1 (the insertion $\delta_i = F$ contributes $\deg = 1$). The term thus matches the multiple cover formula:

$$\begin{aligned}&\mathsf{F}_{g_1,m}\big(\alpha_{I_1};\gamma_{I_1}\delta_i\big)\\ &= m^{\deg-\underline{\deg}+1}\mathsf{T}_{m,2g-4+\underline{\deg}}\Big(\mathsf{F}_{g_1,1}\big(\alpha_{I_1};\gamma_{I_1}\delta_i\big)\Big)\,.\end{aligned}$$

If $\delta_i^\vee = F$ then

$$\deg(\gamma_{I_2}) = 1\,,\quad \underline{\deg}(\gamma_{I_2}) = 2\,.$$

The modified degree $\underline{\deg}$ of $\mathsf{F}_{g_1,1}\big(\alpha_{I_1};\gamma_{I_1}\delta_i\big)$ has decreased by 2, whereas deg decreased by 1. The term matches the multiple cover formula.

$g_2 = 1$. The virtual class is given by $c_2(S)$ and the integral is given by intersection pairing on S. Non-trivial terms are obtained only from $\delta_i^\vee = 1$ and

$$\deg(\gamma_{I_2}) = \underline{\deg}(\gamma_{I_2}) = 0\,.$$

Analogously to case (i), the degree deg of $\mathsf{F}_{g_1,1}\big(\alpha_{I_1};\gamma_{I_1}\delta_i\big)$ has increased by 1, $\underline{\deg}$ remained unchanged, and the genus dropped by 1. The term matches the multiple cover formula.

(iii) The modified degree $\underline{\deg}$ of $\mathsf{F}_{g,1}\big(\alpha\psi_i; \gamma_1, \ldots, \pi^*\pi_*\gamma_i, \ldots, \gamma_n\big)$ has decreased by 2, whereas deg decreased by 1. Again we find that the term matches the multiple cover formula

$$\begin{aligned}&\mathsf{F}_{g,m}\big(\alpha\psi_i; \gamma_1, \ldots, \pi^*\pi_*\gamma_i, \ldots, \gamma_n\big)\\&= m^{\deg - \underline{\deg}+1}\mathsf{T}_{m,2g-4+\underline{\deg}}\Big(\mathsf{F}_{g,1}\big(\alpha\psi_i; \gamma_1, \ldots, \pi^*\pi_*\gamma_i, \ldots, \gamma_n\big)\Big)\,.\end{aligned}$$

(iv) The degree of $\langle \gamma_i, F\rangle \mathsf{F}_{g,1}\big(\alpha; \gamma_1, \ldots, F, \ldots, \gamma_n\big)$ remains unchanged, whereas $\underline{\deg}$ decreased by 2. An additional factor of $\frac{1}{m}$ therefore appears:

$$\begin{aligned}&\frac{1}{m}\langle \gamma_i, F\rangle \mathsf{F}_{g,m}\big(\alpha; \gamma_1, \ldots, F, \ldots, \gamma_n\big)\\&= m^{\deg - \underline{\deg}+1}\mathsf{T}_{m,2g-4+\underline{\deg}}\Big(\langle \gamma_i, F\rangle \mathsf{F}_{g,1}\big(\alpha; \gamma_1, \ldots, F, \ldots, \gamma_n\big)\Big)\,.\end{aligned}$$

(v) The term $\mathsf{F}_{g,1}\big(\ldots, \sigma_1(\gamma_i,\gamma_j), \ldots, \sigma_2(\gamma_i,\gamma_j), \ldots\big)$ is similar to the previous case: deg remains unchanged, whereas $\underline{\deg}$ decreases by 2, giving rise to an additional factor of $\frac{1}{m}$:

$$\begin{aligned}&\frac{1}{m}\mathsf{F}_{g,m}\big(\gamma_1, \ldots, \sigma_1(\gamma_i,\gamma_j), \ldots, \sigma_2(\gamma_i,\gamma_j), \ldots, \gamma_n\big)\\&= m^{\deg - \underline{\deg}+1}\mathsf{T}_{m,2g-4+\underline{\deg}}\Big(\mathsf{F}_{g,1}\big(\gamma_1, \ldots, \sigma_1(\gamma_i,\gamma_j), \ldots, \sigma_2(\gamma_i,\gamma_j), \ldots, \gamma_n\big)\Big)\end{aligned}$$

We arrive at the level m holomorphic anomaly equation (1) which appeared in Section 0. $\square$

3.1 Divisor equation

For primitive curve classes, it was pointed out in [33, Section 3.6, Case (i)] that the holomorphic anomaly equation in genus 0 is compatible with the divisor equation. For divisibility m, let

$$\frac{d}{d\gamma} = \langle \gamma, F\rangle \mathsf{D}_q + m\langle \gamma, W\rangle\,, \quad \gamma \in H^2(S)\,.$$

The divisor equation implies that

$$\begin{aligned}&\mathsf{F}_{g,m}\big(\tau_{a_1}(\gamma_1)\ldots\tau_{a_{n-1}}(\gamma_{n-1})\tau_0(\gamma_n)\big)\\&= \frac{d}{d\gamma_n}\mathsf{F}_{g,m}\big(\tau_{a_1}(\gamma_1)\ldots\tau_{a_{n-1}}(\gamma_{n-1})\big)\\&\quad+\sum_{i=1}^{n-1}\mathsf{F}_{g,m}\big(\tau_{a_1}(\gamma_1)\ldots\tau_{a_i-1}(\gamma_i\cup\gamma_n)\ldots\tau_{a_{n-1}}(\gamma_{n-1})\big)\,.\end{aligned}$$

The compatibility with the divisor equation corresponds to

$$\begin{aligned}
&\mathsf{H}_{g,m}\big(\tau_{a_1}(\gamma_1)\ldots\tau_{a_{n-1}}(\gamma_{n-1})\tau_0(\gamma_n)\big) \\
&= \frac{d}{d\gamma_n}\mathsf{H}_{g,m}\big(\tau_{a_1}(\gamma_1)\ldots\tau_{a_{n-1}}(\gamma_{n-1})\big) \\
&\quad - 2k\mathsf{F}_{g,m}\big(\tau_{a_1}(\gamma_1)\ldots\tau_{a_{n-1}}(\gamma_{n-1})\big) \\
&\quad + \sum_{i=1}^{n-1}\mathsf{H}_{g,m}\big(\tau_{a_1}(\gamma_1)\ldots\tau_{a_i-1}(\gamma_i\cup\gamma_n)\ldots\tau_{a_{n-1}}(\gamma_{n-1})\big)\,,
\end{aligned} \tag{8}$$

where k is the weight of $\mathsf{F}_{g,m}\big(\tau_{a_1}(\gamma_1)\ldots\tau_{a_{n-1}}(\gamma_{n-1})\big)$ and we have used the commutator relation

$$\Big[\frac{d}{dC_2}, \mathsf{D}_q\Big] = -2k\,.$$

The same check as in the primitive case works for arbitrary divisibility. This relies on the fact that the divisor equation for W is the same as applying the differential operator

$$\mathsf{D}_q = q\frac{d}{dq}$$

to the generating series. Indeed, for the curve class $\beta = mB + hF$,

$$\langle \beta, W\rangle = -2m + h + m = h - m\,,$$

which matches the exponent of q^{h-m} in the generating series $\mathsf{F}_{g,m}$. The divisor equation for F acts as multiplication by m on the generating series.

In Section 6, the refined induction reduces any generating series ultimately to genus 0 and 1. We thus have to justify compatibility of the holomorphic anomaly equation for generating series of the form

$$\mathsf{F}_{1,m}\big(\tau_0(\mathsf{p})\tau_0(\gamma_1)\ldots\tau_0(\gamma_n)\big)\,, \quad \gamma_i \in H^2(S)\,.$$

This compatibility however is true. By Proposition 28, the multiple cover formula, which is compatible with the divisor equation, holds in genus ≤ 1. Thus, we also find compatibility for the holomorphic anomaly equation.

Example 21. We consider $\mathsf{F}_{0,m}\big(\tau_0(W)^2\big)$ to illustrate the above compatibility. To compute $\mathsf{H}_{0,m}$, we use that $\sigma(W\boxtimes W) = U^\perp$, where the endomorphism σ is as defined in Section 0. Since the curve classes are contained in U, application of the divisor equation to a basis of $U^\perp$ implies

$$\mathsf{F}_{0,m}\big(\tau_0(U^\perp)\big) = 0\,.$$

We find that

$$\mathsf{H}_{0,m}\big(\tau_0(W)^2\big) = -4\mathsf{F}_{0,m}\big(\tau_1(1)\tau_0(W)\big) + \frac{40}{m}\mathsf{F}_{0,m}\big(\tau_0(F)\tau_0(W)\big)\,.$$

In the above notation, $\gamma_n = W$ is the second W and $k = -10$ is the weight of $\mathsf{F}_{0,m}\big(\tau_0(W)\big)$. We have to check that

$$\mathsf{H}_{0,m}\big(\tau_0(W)^2\big) = \mathsf{D}_q\mathsf{H}_{0,m}\big(\tau_0(W)\big) + 20\mathsf{F}_{0,m}\big(\tau_0(W)\big)\,.$$

By the dilaton equation, we can verify

$$\begin{aligned}
&\mathsf{H}_{0,m}\big(\tau_0(W)^2\big) - \mathsf{D}_q\mathsf{H}_{0,m}\big(\tau_0(W)\big)\\
&= -2\mathsf{D}_q\mathsf{F}_{0,m}\big(\tau_1(1)\big) - 4\mathsf{F}_{0,m}\big(\tau_0(W)\big) + \frac{20}{m}\mathsf{F}_{0,m}\big(\tau_0(F)\tau_0(W)\big)\\
&= 4\mathsf{D}_q\mathsf{F}_{0,m}\big(\emptyset\big) - 4\mathsf{D}_q\mathsf{F}_{0,m}\big(\emptyset\big) + 20\mathsf{F}_{0,m}\big(\tau_0(W)\big)\\
&= 20\mathsf{F}_{0,m}\big(\tau_0(W)\big)\,.
\end{aligned}$$

Example 22. The above example in genus 0 illustrates how the second last term in the holomorphic anomaly equation (2) plays a role. We consider

$$\mathsf{F}_{1,m}\big(\tau_1(W)\tau_0(W)\big)$$

to show how the last term, i.e. the term involving σ, interacts non-trivially with the other terms. The corresponding series $\mathsf{H}_{1,m}$ are

$$\begin{aligned}
\mathsf{H}_{1,m}\big(\tau_1(W)\tau_0(W)\big) &= 2\mathsf{F}_{0,m}\big(\tau_1(W)\tau_0(W)\tau_0(1)\tau_0(F)\big)\\
&\quad - 2\Big(\mathsf{F}_{1,m}\big(\tau_2(1)\tau_0(W)\big) + \mathsf{F}_{1,m}\big(\tau_1(W)\tau_1(1)\big)\Big)\\
&\quad + \frac{20}{m}\Big(\mathsf{F}_{1,m}\big(\tau_1(F)\tau_0(W)\big) + \mathsf{F}_{1,m}\big(\tau_1(W)\tau_0(F)\big)\Big)\\
&\quad - \frac{2}{m}\mathsf{F}_{1,m}\big(\psi_1;\Delta_{U^\perp}\big)\,,\\
\mathsf{H}_{1,m}\big(\tau_1(W)\big) &= 2\mathsf{F}_{0,m}\big(\tau_1(W)\tau_0(1)\tau_0(F)\big)\\
&\quad - 2\mathsf{F}_{1,m}\big(\tau_2(1)\big)\\
&\quad + \frac{20}{m}\mathsf{F}_{1,m}\big(\tau_1(F)\big)\,.
\end{aligned}$$

Let $k = -8$ be the weight of $\mathsf{F}_{1,m}\big(\tau_1(W)\big)$. Then (8) is equivalent to

$$\mathsf{H}_{1,m}\big(\tau_1(W)\tau_0(W)\big) = \mathsf{D}_q\mathsf{H}_{1,m}\big(\tau_1(W)\big) - 2k\mathsf{F}_{1,m}\big(\tau_1(W)\big)\,.$$

The term $\mathsf{F}_{1,m}\big(\psi_1;\Delta_{U^\perp}\big)$ can be computed using

$$\psi_1 = [\delta_1] + \frac{1}{24}[\delta_0] \in A^1(\overline{M}_{1,2})\,,$$

where $[\delta_0] \in A^1(\overline{M}_{1,2})$ is the class of the pushforward of the fundamental class under the map

$$\overline{M}_{0,4} \to \overline{M}_{1,2}$$

gluing the third and fourth markings and $[\delta_1]$ is the class of the boundary divisor of curves with a rational component carrying both markings. The genus 0 contribution vanishes by the divisor equation. Since the rank of $U^\perp$ is 20, we obtain the genus 1 contribution

$$\mathsf{F}_{1,m}\big(\psi_1; \Delta_{U^\perp}\big) = 20\mathsf{F}_{1,m}\big(\tau_0(\mathsf{p})\big)\,.$$

The divisor equation for F implies that

$$\frac{20}{m}\mathsf{F}_{1,m}\big(\tau_1(W)\tau_0(F)\big) = 20\mathsf{F}_{1,m}\big(\tau_1(W)\big) + \frac{20}{m}\mathsf{F}_{1,m}\big(\tau_0(\mathsf{p})\big)\,.$$

We can now verify the compatibility by a direct computation using divisor and dilaton equation:

$$\begin{aligned}
\mathsf{H}_{1,m}\big(\tau_1(W)\tau_0(W)\big) &= \mathsf{D}_q\mathsf{H}_{1,m}\big(\tau_1(W)\big) - 2\mathsf{F}_{1,m}\big(\tau_1(W)\big) - 2\mathsf{F}_{1,m}\big(\tau_1(W)\tau_1(\mathsf{1})\big)\\
&\quad + \frac{20}{m}\mathsf{F}_{1,m}\big(\tau_0(\mathsf{p})\big) + \frac{20}{m}\mathsf{F}_{1,m}\big(\tau_1(W)\tau_0(F)\big)\\
&\quad - \frac{2}{m}\mathsf{F}_{1,m}\big(\psi_1; \Delta_{U^\perp}\big)\\
&= \mathsf{D}_q\mathsf{H}_{1,m}\big(\tau_1(W)\big) - 4\mathsf{F}_{1,m}\big(\tau_1(W)\big)\\
&\quad + \frac{20}{m}\mathsf{F}_{1,m}\big(\tau_0(\mathsf{p})\big) + \frac{20}{m}\mathsf{F}_{1,m}\big(\tau_1(W)\tau_0(F)\big)\\
&\quad - \frac{40}{m}\mathsf{F}_{1,m}\big(\tau_0(\mathsf{p})\big)\\
&= \mathsf{D}_q\mathsf{H}_{1,m}\big(\tau_1(W)\big) + 16\mathsf{F}_{1,m}\big(\tau_1(W)\big)\,.
\end{aligned}$$

4 Relative holomorphic anomaly equation

In this section, we first state the degeneration formula for the reduced virtual class under the degeneration to the normal cone. For primitive curve class, the formula is proven in [29]. For sake of completeness, we summarize a proof for arbitrary divisibility in Appendix A. Then, we state the relative holomorphic anomaly equation and prove the compatibility with the degeneration formula.

4.1 Degeneration formula

Let $S \to \mathbb{P}^1$ be an elliptic K3 surface with a section. For $m \geq 1$, let $\beta = mB + hF$ be a curve class. Choose a smooth fiber E of $S \to \mathbb{P}^1$. Let $\epsilon\colon \mathcal{S} \to \mathbb{A}^1$ be the total space of the degeneration to the normal cone of E in S. This space corresponds to the degeneration

$$S \rightsquigarrow S \cup_E \mathbb{P}^1 \times E\,. \tag{9}$$

Over the center $\iota : 0 \hookrightarrow \mathbb{A}^1$, the fiber is $S \cup_E \mathbb{P}^1 \times E$ and over $t \neq 0$, the fiber is isomorphic to S. Let $\overline{M}_{g,n}(\epsilon, \beta)$ be the moduli space of stable maps to the degeneration $\mathcal{S}$. Over $t \neq 0$, this moduli space is isomorphic to $\overline{M}_{g,n}(S,\beta)$ and over $t = 0$, this moduli space parametrizes stable maps to the expanded target

$$\widetilde{\mathcal{S}}_0 = S \cup_E \mathbb{P}^1 \times E \cup_E \cdots \cup_E \mathbb{P}^1 \times E\,.$$

Let

$$\nu = (g_1, g_2, n_1, n_2, h_1, h_2)$$

be a splitting of the discrete data g, n, h and let $\beta_i = mB + h_iF$ be the splitting of the curve class. An ordered partition of m

$$\mu = (\mu_1, \ldots, \mu_l)$$

specifies the contact order along the relative divisor E.

Let $l = \mathrm{length}(\mu)$ and $\overline{M}_{g,n}(\mathcal{S}_0, \nu)_\mu$ be the fiber product

$$\overline{M}_{g,n}(\mathcal{S}_0, \nu)_\mu = \overline{M}_{g_1,n_1}(S/E, \beta_1)_\mu \times_{E^l} \overline{M}^{\bullet}_{g_2,n_2}(\mathbb{P}^1 \times E/E, \beta_2)_\mu \tag{10}$$

of the boundary evaluations at relative markings[13] and let

$$\iota_{\nu\mu} \colon \overline{M}_{g,n}(\mathcal{S}_0, \nu)_\mu \to \overline{M}_{g,n}(\mathcal{S}_0, \beta)$$

be the finite morphism. Let $\Delta_{E^l} \colon E^l \to E^l \times E^l$ be the diagonal embedding.

Theorem 23. The reduced virtual class of maps to the degeneration (9) satisfies the following properties.

(i) For $\iota_t \colon \{t\} \hookrightarrow \mathbb{A}^1$, the Gysin pullback of reduced class is given by

$$\iota_t^![\overline{M}_{g,n}(\epsilon, \beta)]^{red} = [\overline{M}_{g,n}(\mathcal{S}_t, \beta)]^{red}\,.$$

(ii) For the special fiber,

$$[\overline{M}_{g,n}(\mathcal{S}_0, \beta)]^{red} = \sum_{\nu,\mu} \frac{\prod_i \mu_i}{l!} \iota_{\nu\mu *}[\overline{M}_{g,n}(\mathcal{S}_0, \nu)_\mu]^{red}\,.$$

(iii) On the special fiber, we have the factorization

$$\begin{aligned}[\overline{M}_{g,n}(\mathcal{S}_0, \nu)_\mu]^{red} = \Delta^!_{E^l}\Big([\overline{M}_{g_1,n_1}(S/E, \beta_1)_\mu]^{red} \\ \times [\overline{M}^{\bullet}_{g_2,n_2}(\mathbb{P}^1 \times E/E, \beta_2)_\mu]^{vir}\Big)\,.\end{aligned}$$

[13] We put $\bullet$ to indicate (possibly) disconnected theory. Namely, for each connected component C of the domain curve, intersection of C with the relative divisor E is nontrivial.

Proof. When $m \geq 1$, the reduced class of the disconnected moduli space $\overline{M}^{\bullet}_{g,n}(S/E, \beta)$ vanishes on all components parameterizing maps with at least two connected components. Therefore, disconnected theory can only appear on the bubble $\mathbb{P}^1 \times E$. The proof is given in Appendix A. $\square$

Denote an ordered cohomology weighted partition by

$$\underline{\mu} = \big((\mu_1, \delta_1), \dots, (\mu_l, \delta_l)\big)\,,\ \delta_i \in H^*(E)$$

and let $\omega \in H^2(E)$ be the point class. The descendent potential for the pair (S, E) is defined analogously to the absolute case:

$$\mathsf{F}^{\mathrm{rel}}_{g,m}\big(\alpha; \gamma_1, \dots, \gamma_n \mid \underline{\mu}\big) = \sum_{h \geq 0} \big\langle \alpha; \gamma_1, \dots, \gamma_n \mid \underline{\mu} \big\rangle^{S/E}_{g, mB+hF}\, q^{h-m}\,.$$

The descendent potential for the pair $(\mathbb{P}^1 \times E, E)$ is defined by

$$\mathsf{G}^{\mathrm{rel},\bullet}_{g,m}\big(\alpha; \gamma_1, \dots, \gamma_n \mid \underline{\mu}\big) = \sum_{h \geq 0} \big\langle \alpha; \gamma_1, \dots, \gamma_n \mid \underline{\mu} \big\rangle^{\mathbb{P}^1 \times E/E, \bullet}_{g, mB+hF}\, q^{h}\,.$$

As a corollary, we get the degeneration formula of reduced Gromov–Witten invariants.

Corollary 24. Let $\gamma_1, \dots, \gamma_n \in H^*(S)$ and choose a lift of these cohomology classes to the total space $\mathcal{S}$. Then

$$\mathsf{F}_{g,m}\big(\tau_{a_1}(\gamma_1) \dots \tau_{a_n}(\gamma_n)\big) = \sum_{\nu} \sum_{\underline{\mu} \neq \underline{\mu_\omega}} \frac{\prod_i \mu_i}{l!} \mathsf{F}^{\mathrm{rel}}_{g_1,m}\big(\dots \mid \underline{\mu}\big) \cdot \mathsf{G}^{\mathrm{rel},\bullet}_{g_2,m}\big(\dots \mid \underline{\mu}^\vee\big)\,, \qquad (11)$$

where

$$\underline{\mu}^\vee = \big((\mu_1, \delta_1^\vee), \dots, (\mu_l, \delta_l^\vee)\big) \text{ and } \underline{\mu_\omega} = \big((\mu_1, \omega), \dots, (\mu_l, \omega)\big)\,.$$

Proof. By Theorem 23, we are left to prove that the relative profile $\underline{\mu_\omega}$ on S/E has vanishing contribution. Let x be the intersection of the section of the elliptic fibration and the fiber E. We consider (E, x) as an abelian variety. Let K be the kernel of the following morphism between abelian varieties

$$E^l \to \mathsf{Pic}^0(E)\,, (x_i)_i \mapsto \mathcal{O}_E\Big(\sum_i \mu_i(x_i - x)\Big)\,.$$

Consider a stable map f from a curve C to an expanded degeneration of S/E. The equality $f_*[C] = \beta_1$ (after pushforward to S) in $H_2(S, \mathbb{Z})$ lifts to a rational equivalence of line bundles on S because the cycle-class map

$$c_1 \colon \mathsf{Pic}(S) \to H^2(S, \mathbb{Z}) \cong H_2(S, \mathbb{Z})$$

is injective. Intersecting with the relative divisor, the two line bundles are, respectively, $\mathcal{O}_E(\sum \mu_i x_i)$ and $\mathcal{O}_E(mx)$. Thus, we see that the evaluation map $\overline{M}_{g_1,n_1}(S/E, \beta_1) \to E^l$ factors through K. Since $K \subset E^l$ has codimension 1 a generic point on E^l does not lie on K and thus the contribution from the relative profile $\underline{\mu_\omega}$ vanishes. $\square$

4.2 Relative holomorphic anomaly equations

Assuming quasimodularity, we have two ways to compute the derivative of $\mathsf{F}_{g,m}$ with respect to C_2:

(i) Apply the degeneration formula Corollary 24, together with the holomorphic anomaly equations for (S, E) and $(\mathbb{P}^1 \times E, E)$.

(ii) Apply the holomorphic anomaly equation (3) for S, followed by the degeneration formula for each term.

We argue that both ways yield the same result. This compatibility is parallel to the compatibility proved in [34, Section 4.6]. We first state the holomorphic anomaly equations for the relevant relative geometries.

Relative $(\mathbb{P}^1 \times E, E)$

Consider $\pi\colon \mathbb{P}^1 \times E \to \mathbb{P}^1$ as a trivial elliptic fibration over $\mathbb{P}^1$. For the pair $(\mathbb{P}^1 \times E, E)$ the holomorphic anomaly equation holds for cycle-valued generating series [34]. The equation for descendent potentials can thus be obtained by integrating against tautological classes $\alpha \in R^*(\overline{M}_{g,n})$. For insertions $\gamma_i \in H^*(\mathbb{P}^1 \times E, \mathbb{Q})$ we will simply write γ. Let $\underline{\mu} = \big((\mu_1, \delta_1), \ldots, (\mu_l, \delta_l)\big)$ and $\underline{\mu}'$ be ordered cohomology weighted partitions. We denote by

$$\mathsf{G}^{\sim,\bullet}_{g,m}\big(\underline{\mu} \mid \alpha; \gamma \mid \underline{\mu}'\big) = \sum_{h \geq 0} \big\langle \underline{\mu} \mid \alpha; \gamma \mid \underline{\mu}' \big\rangle^{\mathbb{P}^1 \times E, \sim, \bullet}_{g, m\mathbb{P}^1 + hE}\, q^h$$

the disconnected rubber generating series for $\mathbb{P}^1 \times E$ relative to divisors at 0 and ∞. Let $\Delta_E \subset E \times E$ be the class of the diagonal. Define the generating series

$$\begin{aligned}
&\mathsf{P}^{\mathrm{rel},\bullet}_{g,m}\big(\alpha; \gamma \mid \underline{\mu}\big) \\
&= \mathsf{G}^{\mathrm{rel},\bullet}_{g-1,m}\big(\alpha; \gamma, \Delta_{\mathbb{P}^1} \mid \underline{\mu}\big) \\
&\quad + 2 \sum_{\substack{g = g_1 + g_2 \\ \{1,\ldots,n\} = I_1 \sqcup I_2 \\ \forall i \in I_2 : \gamma_i \in H^2(E) \\ h \geq 0}} \sum_{\substack{b; b_1, \ldots, b_h \\ l_1, \ldots, l_h}} \frac{\prod_{i=1}^h b_i}{h!} \mathsf{G}^{\mathrm{rel},\bullet}_{g_1,m}\big(\alpha_{I_1}; \gamma_{I_1} \mid ((b,1), (b_i, \Delta_{E,\ell_i})_{i=1}^h)\big) \\
&\qquad\qquad\qquad \times \mathsf{G}^{\sim,\bullet}_{g_2,m}\big(((b,1), (b_i, \Delta^\vee_{E,\ell_i})_{i=1}^h) \mid \alpha_{I_2}; \gamma_{I_2} \mid \underline{\mu}\big) \\
&\quad - 2 \sum_{i=1}^n \mathsf{G}^{\mathrm{rel},\bullet}_{g,m}\big(\alpha\psi_i; \gamma_1, \ldots, \gamma_{i-1}, \pi^*\pi_*\gamma_i, \gamma_{i+1}, \ldots, \gamma_n \mid \underline{\mu}\big) \\
&\quad - 2 \sum_{i=1}^l \mathsf{G}^{\mathrm{rel},\bullet}_{g,m}\big(\alpha; \gamma \mid (\mu_1, \delta_1), \ldots, (\mu_i, \psi_i^{\mathrm{rel}} \pi^*\pi_*\delta_i), \ldots, (\mu_l, \delta_l)\big)
\end{aligned}$$

where ψ_i^{rel} is the cotangent line class at the i-th relative marking and $\Delta_E = \sum \Delta_{E,l_i} \otimes \Delta_{E,l_i}^\vee$ is the pullback of the Künneth decomposition of Δ_E at the corresponding relative marking. The holomorphic anomaly equation takes the form:

Proposition 25. ([34, Proposition 20]) $\mathsf{G}_{g,m}^{\mathrm{rel},\bullet}(\alpha;\gamma \mid \underline{\mu})$ is a quasimodular form and

$$\frac{d}{dC_2}\mathsf{G}_{g,m}^{\mathrm{rel},\bullet}(\alpha;\gamma \mid \underline{\mu}) = \mathsf{P}_{g,m}^{\mathrm{rel},\bullet}(\alpha;\gamma \mid \underline{\mu})\,.$$

Relative (S, E)

Since the log canonical bundle of (S, E) is nontrivial, relative moduli spaces in fiber direction have nontrivial virtual fundamental class. Define

$$\mathsf{F}_{g,0}^{vir-\mathrm{rel}}(\alpha;\gamma \mid \emptyset) = \sum_{h\geq 0} \big\langle \alpha;\gamma \mid \emptyset \big\rangle_{g,hF}^{S/E,vir} q^h\,.$$

Recall that we denote the pullback of the diagonal of $\mathbb{P}^1$ as

$$\Delta_{\mathbb{P}^1} = 1 \boxtimes F + F \boxtimes 1 = \sum_{i=1}^{2} \delta_i \boxtimes \delta_i^\vee\,.$$

Define a generating series

$$
\begin{aligned}
&\mathsf{H}^{\mathrm{rel}}_{g,m}\big(\alpha;\gamma \mid \underline{\mu}\big) \\
&= \mathsf{F}^{\mathrm{rel}}_{g-1,m}\big(\alpha;\gamma,\Delta_{\mathbb{P}^1} \mid \underline{\mu}\big) \\
&\quad + 2 \sum_{\substack{g=g_1+g_2 \\ \{1,\ldots,n\}=I_1\sqcup I_2 \\ i\in\{1,2\}}} \mathsf{F}^{\mathrm{rel}}_{g_1,m}\big(\alpha_{I_1};\gamma_{I_1},\delta_i \mid \underline{\mu}\big)\, \mathsf{F}^{vir-\mathrm{rel}}_{g_2,0}\big(\alpha_{I_2};\gamma_{I_2},\delta_i^\vee \mid \emptyset\big) \\
&\quad + 2 \sum_{\substack{g=g_1+g_2 \\ \{1,\ldots,n\}=I_1\sqcup I_2 \\ \forall i\in I_2:\gamma_i\in H^2(E) \\ h\geq 0}} \sum_{\substack{b;b_1,\ldots,b_h \\ l_1,\ldots,l_h}} \frac{\prod_{i=1}^h b_i}{h!} \mathsf{F}^{\mathrm{rel}}_{g_1,m}\big(\alpha_{I_1};\gamma_{I_1} \mid ((b,1),(b_i,\Delta_{E,\ell_i})_{i=1}^h)\big) \\
&\qquad\qquad\qquad\qquad \times \mathsf{G}^{\sim,\bullet}_{g_2,m}\big(((b,1),(b_i,\Delta^\vee_{E,\ell_i})_{i=1}^h) \mid \alpha_{I_2};\gamma_{I_2} \mid \underline{\mu}\big) \\
&\quad - 2\sum_{i=1}^n \mathsf{F}^{\mathrm{rel}}_{g,m}\big(\alpha\psi_i;\gamma_1,\ldots,\gamma_{i-1},\pi^*\pi_*\gamma_i,\gamma_{i+1},\ldots,\gamma_n \mid \underline{\mu}\big) \\
&\quad - 2\sum_{i=1}^l \mathsf{F}^{\mathrm{rel}}_{g,m}\big(\alpha;\gamma \mid ((\mu_1,\delta_1),\ldots,(\mu_i,\psi_i^{\mathrm{rel}}\pi^*\pi_*\delta_i),\ldots,(\mu_l,\delta_l))\big) \\
&\quad + \frac{20}{m}\sum_{i=1}^n \langle\gamma_i,F\rangle \mathsf{F}^{\mathrm{rel}}_{g,m}\big(\alpha;\gamma_1,\ldots,\gamma_{i-1},F,\gamma_{i+1},\ldots,\gamma_n \mid \underline{\mu}\big) \\
&\quad - \frac{2}{m}\sum_{i<j} \mathsf{F}^{\mathrm{rel}}_{g,m}\big(\alpha;\gamma_1,\ldots,\underbrace{\sigma_1(\gamma_i,\gamma_j)}_{i\text{th}},\ldots,\underbrace{\sigma_2(\gamma_i,\gamma_j)}_{j\text{th}},\ldots,\gamma_n \mid \underline{\mu}\big)\,.
\end{aligned}
$$

The conjectural holomorphic anomaly equation for (S,E) has the following form:

$$
\mathsf{F}^{\mathrm{rel}}_{g,m}(\alpha;\gamma \mid \underline{\mu}) \in \frac{1}{\Delta(q)^m}\mathsf{QMod}(m)
$$

and

$$
\frac{d}{dC_2}\mathsf{F}^{\mathrm{rel}}_{g,m}(\alpha;\gamma \mid \underline{\mu}) = \mathsf{H}^{\mathrm{rel}}_{g,m}(\alpha;\gamma \mid \underline{\mu})\,. \tag{12}
$$

Proposition 26. Let $m \geq 1$. Assuming quasimodularity for $\mathsf{F}_{g,m}$ and $\mathsf{F}^{\mathrm{rel}}_{g,m}$, the holomorphic anomaly equations are compatible with the degeneration formula in the above sense.

Proof. The proof given in [34, Proposition 21] treats virtual fundamental classes, not reduced classes. The splitting behavior of the reduced class with respect to restriction to boundary divisors [29, Section 7.3] calls for a slight adaptation of the proof. For this, we introduce a formal variable ε with $\varepsilon^2 = 0$. We can then interpret reduced Gromov–Witten invariants of the K3 surface as integrals against the class[14]

$$[\overline{M}_{g,n}(S,\beta)]^{vir} + \varepsilon\,[\overline{M}_{g,n}(S,\beta)]^{red}$$

followed by taking the $[\varepsilon]$-coefficient. We consider a similar class for S/E. This class has the advantage of satisfying the usual splitting behavior of virtual fundamental classes. Thus, for this class one can follow the proof of compatibility given in [34, Proposition 21]. All the terms appearing in the computation (ii) also appear in computation (i). We are left with proving the cancellation of the remaining terms in (i). This follows from comparing ψ_i^{rel}-class and the ψ-class pulled-back from the stack of target degeneration [34, Lemma 22]. In particular, we match the following terms: the third term of $\mathsf{H}^{\mathrm{rel}}$ times $\mathsf{G}^{\mathrm{rel},\bullet}$ with the fourth term of $\mathsf{F}^{\mathrm{rel}}$ times $\mathsf{P}^{\mathrm{rel},\bullet}$; and analogously for the fifth term of $\mathsf{H}^{\mathrm{rel}}$ times $\mathsf{G}^{\mathrm{rel},\bullet}$ with the second term of $\mathsf{F}^{\mathrm{rel}}$ times $\mathsf{P}^{\mathrm{rel},\bullet}$. □

The main advantage of the holomorphic anomaly equation is that it is compatible with the degeneration formula. Thus, the genus reduction from the degeneration formula connects the low genus results with arbitrary genus predictions. On the other hand, it is not even clear to say what should be the compatibility of the multiple cover formula and the degeneration formula.

5 Tautological relations and initial condition

This section contains a proof of the multiple cover formula in genus 0 and genus 1 for any divisibility m. It is a direct consequence of the KKV formula. However, as initial condition for our induction we also require a special case in genus 2, which cannot be easily deduced from the KKV formula. We treat this descendent potential separately, using double ramification relations [3] for K3 surfaces. This approach is likely to give *relations in any genus* and will be pursued in the future.

5.1 Double ramification relations

In this section we recall *double ramification relations with target variety* developed in [2, 3].

Let $\mathfrak{Pic}_{g,n}$ be the Picard stack for the universal curve over the stack of prestable curves $\mathfrak{M}_{g,n}$ of genus g with n markings. Let

$$\pi\colon \mathfrak{C} \to \mathfrak{Pic}_{g,n},\ s_i\colon \mathfrak{Pic}_{g,n} \to \mathfrak{C},\ \mathfrak{L} \to \mathfrak{C},\ \omega_\pi \to \mathfrak{C} \tag{13}$$

be the universal curve, the i-th section, the universal line bundle and the relative dualizing sheaf of π. The following operational Chow classes on $\mathfrak{Pic}_{g,n}$ are obtained from the universal structure (13):

[14]We thank G. Oberdieck for pointing this out.

- $\psi_i = c_1(s_i^*\omega_\pi) \in A^1_{\mathsf{op}}(\mathfrak{Pic}_{g,n})$,
- $\xi_i = c_1(s_i^*\mathfrak{L}) \in A^1_{\mathsf{op}}(\mathfrak{Pic}_{g,n})$,
- $\eta = \pi_*\left(c_1(\mathfrak{L})^2\right) \in A^1_{\mathsf{op}}(\mathfrak{Pic}_{g,n})$.

Let $A = (a_1, \ldots, a_n) \in \mathbb{Z}^n$ be a vector of integers satisfying

$$\sum_i a_i = d\,, \tag{14}$$

where d is the degree of the line bundle. We denote by $P^{c,r}_{g,A,d}$ the codimension c component of the class

$$\sum_{\substack{\Gamma\in\mathsf{G}_{g,n,d}\\ w\in\mathsf{W}_{\Gamma,r}}} \frac{r^{-h^1(\Gamma_\delta)}}{|\operatorname{Aut}(\Gamma_\delta)|}\, j_{\Gamma*}\Bigg[\prod_{i=1}^n \exp\left(\frac{1}{2}a_i^2\psi_i + a_i\xi_i\right)\prod_{v\in V(\Gamma_\delta)}\exp\left(-\frac{1}{2}\eta(v)\right)$$
$$\prod_{e=(h,h')\in E(\Gamma)} \frac{1-\exp\left(-\frac{w(h)w(h')}{2}(\psi_h+\psi_{h'})\right)}{\psi_h+\psi_{h'}}\Bigg].$$

We refer to [3] for details about the notations. This expression is polynomial in r when r is sufficiently large. Let $P^c_{g,A,d}$ be the constant part of $P^{c,r}_{g,A,d}$.

Theorem 27. ([3, Theorem 8]) $P^c_{g,A,d} = 0$ for all $c > g$ in $A^c_{\mathsf{op}}(\mathfrak{Pic}_{g,n})$.

After restricting $P^c_{g,A,d}$ to (14), this expression is a polynomial in $a_1, \ldots, a_{n-1}$. The polynomiality will be used to get refined relations.

Let L be a line bundle on S with degree

$$\int_\beta c_1(L) = d\,.$$

The choice of a line bundle L induces a morphism

$$\varphi_L\colon \overline{M}_{g,n}(S,\beta) \to \mathfrak{Pic}_{g,n},\ \ [f\colon C\to S] \mapsto (C, f^*L)\,.$$

Then Theorem 27 gives relations

$$P^c_{g,A,d}(L) = \varphi_L^* P^c_{g,A,d} \cap [\overline{M}_{g,n}(S,\beta)]^{red} = 0 \text{ for all } c > g \tag{15}$$

in $A_{g+n-c}\big(\overline{M}_{g,n}(S,\beta)\big)$.

5.2 Compatibility II

The relations among descendent potentials coming from tautological relations on $\overline{M}_{g,n}(S,\beta)$ are compatible with the multiple cover formula. This follows from two observations. Firstly, the splitting behavior of the reduced class, discussed in Section 2.2, is crucial. It is already crucial to justify compatibility with respect to boundary restriction for tautological classes pulled back from $\overline{M}_{g,n}$. For tautological relations on $\overline{M}_{g,n}(S,\beta)$, a second fact, which we want to explain below, is essential for the compatibility.

For $c > g > 0$, $A \in \mathbb{Z}^n$ and $b \in \mathbb{Z}$, consider the series of relations

$$P^c_{g,bA,db}(L^{\otimes b}) = 0$$

obtained by tensoring the line bundle L by b times. For each coefficient of a monomial in a_i-variables, this expression is polynomial in b and hence each of b-variable is a relation. As a consequence, each term of a relation $P^c_{g,A,m}(F)$ gives the same value of

$$m^{\deg - \underline{\deg}}\,,$$

where $\deg(\xi) = 1$ and $\underline{\deg}(\xi) = 0$, as in Definition 0.1. The same holds true with the roles of F and W interchanged. Thus, the relations are compatible with the operator

$$m^{\deg - \underline{\deg}} \mathsf{T}_{m,2g-2+\underline{\deg}}\,,$$

which gives the multiple cover formula in Conjecture 15.

5.3 Initial condition

The Katz–Klemm–Vafa (KKV) formula implies that the generating series of λ_g-integrals

$$\mathsf{F}_{g,m}(\lambda_g;\emptyset)$$

satisfy the multiple cover formula [36]. Here, $\lambda_g = c_g(\mathbb{E}_g)$ is the top Chern class of the rank g Hodge bundle $\mathbb{E}_g$ on $\overline{M}_g(S,\beta)$. The KKV formula will be the starting point of our genus induction.

The class λ_g is a tautological class by the Grothendieck–Riemann–Roch computation ([15]) but the formula is rather complicated. Instead we use an alternative expression of λ_g in terms of double ramification cycle, proven in [17]. We recall that the class $(-1)^g\lambda_g$ is equal to the double ramification cycle $\mathsf{DR}_g(\emptyset)$ with the empty condition. By [17, Theorem 1] the class $\mathsf{DR}_g(\emptyset)$ can be written as a graph sum of tautological classes *without* κ-classes.

Proposition 28. The multiple cover formula holds in genus 0 and genus 1 for all $m \geq 1$.

Proof. When $g = 0, 1$, the tautological ring $R^*(\overline{M}_{g,n})$ is additively generated by boundary strata ([19, 37]). Thus, one can replace descendents $\alpha \in R^*(\overline{M}_{g,n})$ by classes in $H^*(S)$. By the divisor equation and the dimension constraint, we can reduce to the case $\mathsf{F}_{0,m}(\emptyset)$ and $\mathsf{F}_{1,m}(\tau_0(\mathsf{p}))$. The genus 0 case is covered by the full Yau–Zaslow formula [21, 36]. The genus 1 case follows from the genus 2 KKV formula. Using the boundary expression of λ_2 on $\overline{M}_2$, we have

$$\begin{aligned} \mathsf{F}_{2,m}\big(\lambda_2; \emptyset\big) &= \frac{1}{240}\mathsf{F}_{1,m}\big(\psi_1; \Delta_S\big) + \frac{1}{1152}\mathsf{F}_{0,m}\big(\,; \Delta_S, \Delta_S\big) \\ &= \frac{1}{10}\mathsf{F}_{1,m}\big(\tau_0(\mathsf{p})\big) + \frac{1}{60}\mathsf{D}_q^2\mathsf{F}_{0,m}\big(\emptyset\big)\,, \end{aligned}$$

where $\Delta_S \subset S \times S$ is the diagonal class. Therefore, $\mathsf{F}_{1,m}\big(\tau_0(\mathsf{p})\big)$ satisfies Conjecture 15. □

In the argument below, we will use tautological relations on $\overline{M}_{g,n}$ which are recently obtained by r-spin relations. For convenience, we summarize the result.

Proposition 29. ([23]) Let $g \geq 2$ and $n \geq 1$. Consider tautological classes on $\overline{M}_{g,n}$.

(i) (Topological Recursion Relations) Any monomial of ψ-classes of degree at least g can be represented by a tautological class supported on boundary strata without κ-classes.

(ii) Any tautological class of degree $g-1$ can represented by a sum of a linear combination of $\psi_1^{g-1}, \ldots, \psi_n^{g-1}$ and a tautological class supported on boundary strata.

Proof. The proof of (i) follows from the proof of [23, Lemma 5.2] (see also [12, page 3]). By [23, Proposition 3.1] (or [10, Theorem 1.1]) the degree $g-1$ part $R^{g-1}(\mathcal{M}_{g,n})$ is spanned by $\psi_1^{g-1}, \ldots, \psi_n^{g-1}$. Since relations used in the proof are all tautological, the boundary expression is tautological and thus we obtain (ii). □

Together with the boundary expression for λ_{g+1} we obtain the following more general consequence of the KKV formula:

Proposition 30. Let $m \geq 1$ and $g \geq 1$. Assume the multiple cover formula Conjecture 15 holds for m and all descendents of genus $< g$. Then Conjecture 15 holds for

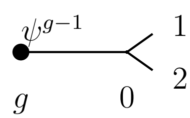
$$\mathsf{F}_{g,m}\big(\tau_{g-1}(\mathsf{p})\big)\,.$$

Proof. Let $\delta \in R^1(\overline{M}_g)$ be the boundary divisor corresponding to a curve with nonseparating node. Denote two half edges as h and h'. Recall that $(-1)^g\lambda_g$ is equal to the double ramification cycle $\mathsf{DR}_g(\emptyset)$ with the empty condition. We use this formula for genus $g+1$. By [17, Theorem 1],

$$\begin{aligned}(-1)^{g+1}\lambda_{g+1} &= \mathsf{DR}_{g+1}(\emptyset)\\ &= \frac{1}{2}\Big[-\frac{1}{(g+1)!}\sum_{w=0}^{r-1}\big(\frac{w^2}{2}(\psi_h+\psi_{h'})\big)^g\Big]_{r^1}\delta + \text{ lower genus}\,,\end{aligned}$$

where $[\cdots]_{r^1}$ is the coefficient of the linear part of a polynomial in r. The leading term is nonzero by Faulhaber's formula.

By Proposition 29 (i) any ψ-monomial in $R^{\geq g}(\overline{M}_{g,n})$ can be represented by a sum of tautological classes supported on boundary strata without κ classes. There is only one graph with a genus g vertex (with a rational component carrying both markings). The graph is decorated with a polynomial of degree $g-1$ in ψ- and κ-classes. By Proposition 29 (ii) this tautological class can be represented by a sum of a multiple of ψ^{g-1} and tautological classes supported on boundary strata. We find that [15]

$(\psi_1+\psi_2)^g = c$  $+$ lower genus

in $R^g(\overline{M}_{g,2})$ for some $c\in\mathbb{Q}$. Therefore, it suffices to prove that c is nonzero. Recall that $\lambda_g\lambda_{g-1}$ vanishes on $\overline{M}_{g,n}\setminus M^{rt}_{g,n}$, so

$$\int_{\overline{M}_{g,2}}(\psi_1+\psi_2)^g\lambda_g\lambda_{g-1} = c\int_{\overline{M}_{g,1}}\psi_1^{g-1}\lambda_g\lambda_{g-1}\,.$$

The left hand side of the equation is nonzero by [17, Lemma 8], which concludes the proof. □

We now consider the case of genus two. By the Getzler–Ionel vanishing on $\overline{M}_{2,n}$, the dimension constraint, and the divisor equation any descendent insertion reduces to the following three cases:

$$\mathsf{F}_{2,m}\big(\tau_1(\mathsf{p})\big)\,,\quad \mathsf{F}_{2,m}\big(\tau_0(\mathsf{p})^2\big)\,,\quad \mathsf{F}_{2,m}\big(\tau_1(\gamma)\tau_0(\mathsf{p})\big)\quad\text{with } \gamma\in H^2(S)\,.$$

The first case is treated in Proposition 30 and follows from the KKV formula in genus three and lower genus. The second case for $m=2$ is treated as part of the proof of Theorem 1 in Section 6. We use the double ramification relation (15) to prove the multiple cover formula for the third case. The point class p will be obtained as the product of F and W.

Proposition 31. For $\gamma\in H^2(S)$, the generating series $\mathsf{F}_{2,m}(\tau_1(\gamma)\tau_0(\mathsf{p}))$ satisfies Conjecture 15.

[15]The number under each vertex is the genus and legs correspond to markings.

Proof. We will use relations in $A_{2+n-3}\left(\overline{M}_{2,n}(S,\beta)\right)$:

$$P^3_{2,A,m}(F) = 0$$

associated to the line bundle $\mathcal{O}_S(F)$ on S. More precisely, we will distinguish two cases $\gamma \in U$ and $\gamma \in U^\perp$ and set respectively

$$A = (a_1\,, m - a_1)\,, \quad A = (a_1\,, a_2\,, m - a_1 - a_2)\,.$$

Refined relations are then obtained by considering particular monomials in the a_i, as outlined in the previous section. The η-class vanishes in this case because $\langle F, F\rangle = 0$ and, for the same reason, ξ_i^2 vanishes. Define

$$X = \mathsf{F}_{2,m}\big(\tau_1(\gamma)\tau_0(\mathsf{p})\big)\,.$$

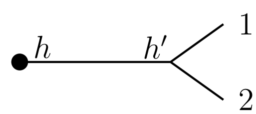
The case $\gamma = F$ is treated first. As explained in Section 5.1, the tautological relations are polynomial in a_i and we may obtain a refined relation by considering the $[a_1^4]$-coefficient of

$$P^3_{2,A,m}(F)|_{a_2=m-a_1}\,.$$

We will only need to consider boundary strata which both:

- contribute to X and
- contribute to the $[a_1^4]$-coefficient.

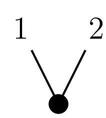
These two properties simplify the calculation significantly. By the splitting property of the reduced class, a relevant boundary stratum is a tree with one genus 2 vertex and contracted genus 0 components. The integrals are given by the intersection product of the corresponding insertions. In the case with only two markings, the only relevant stratum is[16]



The weight factor for this stratum is

$$\frac{w(h)w(h')}{2} = -\frac{m^2}{2}\,.$$

This stratum, therefore, cannot contribute to the $[a_1^4]$-coefficient, since ψ-classes on the genus 0 component vanish. It remains to determine the contributions from the trivial graph



We will order the terms by the total degree $\deg(\psi)$ in the ψ-classes.

[16]The genus 2 vertex is represented by a filled node and other nodes represent genus 0 vertices. Labeled half-edges correspond to markings.

0. $\deg(\psi) = 0$. The relation we consider is of codimension three. This case is therefore impossible by virtue of $\xi_i^2 = 0$.

1. $\deg(\psi) = 1$. This case results in non-trivial terms, discussed below.

2. $\deg(\psi) \geq 2$. We may apply Proposition 29 (i) to reduce to the descendent $\mathsf{F}_{2,m}\big(\tau_1(\mathsf{p})\big)$. This descendent is covered by Proposition 30.

Therefore, up to lower genus data, the $[a_1^4]$-coefficient is

$$-\frac{1}{2}\psi_1\xi_1\xi_2 - \frac{1}{2}\psi_2\xi_1\xi_2 \,.$$

Integrating

$$\mathrm{ev}_2^*(W)P^3_{2,A,m}(F)|_{a_2=m-a_1}$$

against the reduced class, we find (up to lower genus data)

$$-\frac{1}{2}X - \frac{m}{2}\mathsf{F}_{2,m}\big(\tau_1(\mathsf{p})\big)\,,$$

where the second term is obtained by application of the divisor equation. We thus find that X is a linear combination of terms which satisfy Conjecture 15. Switching the role of F and W, we obtain the same result for $\gamma = W$.

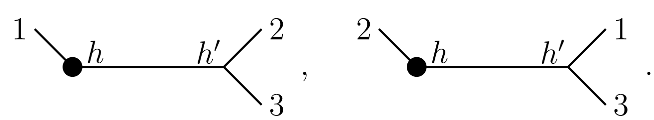
Next, we consider $\gamma \in U^\perp$. The following vanishing of intersection products will be used frequently:

$$\langle \gamma, F\rangle = 0\,, \quad \langle \gamma, W\rangle = 0\,, \quad \langle \gamma, \beta\rangle = 0\,.$$

We use a similar argument as above, this time, however, we use three markings and consider the $[a_1^3a_2]$-coefficient of[17]

$$\mathrm{ev}_1^*(\gamma)\,\mathrm{ev}_2^*(W)P^3_{2,A,m}(F)|_{a_3=m-a_1-a_2}\,. \tag{16}$$

By the above vanishing of intersection products, the only possible trees with non-trivial contribution are



[17]We are grateful to the referee for pointing out a mistake in an earlier version of the text. It has become clear that the choice of monomial, leading to non-trivial relations, is a very subtle one. Symmetry in the a_i and the insertions causes cancellation in many cases. We plan to come back to this in future work.

The weight factor for the right stratum is

$$\frac{w(h)w(h')}{2} = -\frac{(m-a_2)^2}{2}\,.$$

Since ψ-classes on the genus 0 component vanish, the power of a_1 in any monomial obtained from this stratum is bounded by two. The contribution to the $[a_1^3 a_2]$-coefficient is, therefore, zero.

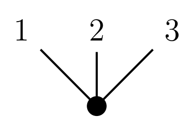
Next, we explain the contributions from the left stratum. Note that the left vertex is of genus 2 with two markings and we may apply the same reasoning as in the discussion for $\gamma = F$ above. Here, the $\deg(\psi) = 0$ term $\xi_1\xi_2$ has trivial contribution due to $\langle \gamma, F\rangle = 0$. The $\deg(\psi) = 1$ terms $\psi_h\xi_2$, $\psi_h\xi_3$ have vanishing contribution by application of the divisor equation for γ. Non-trivial contributions are obtained only from

$$\psi_1\xi_2\,, \quad \psi_1\xi_3\,.$$

These two terms have contributions

$$-\frac{(m-a_1)^2}{4}a_1^2 a_2 X\,, \quad -\frac{(m-a_1)^2}{4}a_1^2 a_3 X\,.$$

The $[a_1^3 a_2]$-coefficients, however, cancel due to $a_3 = m - a_1 - a_2$. It remains to determine the contributions from the trivial graph:



As above, we order the terms by the total degree $\deg(\psi)$ in the ψ-classes.

0. $\deg(\psi) = 0$. The relation we consider is of codimension three. Since $\xi_i^2 = 0$, the class ξ_1 must appear. This term, however, vanishes due to $\langle \gamma, F\rangle = 0$.

1. $\deg(\psi) = 1$. This case results in non-trivial terms corresponding to ψ_1 or ψ_3, discussed below. The choice of the monomial $[a_1^3 a_2]$ excludes the appearance of ψ_2.

2. $\deg(\psi) = 2$. This case results in non-trivial terms corresponding to $\psi_1\psi_3$ or ψ_3^2, discussed below. The choice of the monomial $[a_1^3 a_2]$ excludes the appearance of ψ_1^2.

3. $\deg(\psi) = 3$. As above, this case reduces to the descendent $\mathsf{F}_{2,m}\big(\tau_1(\mathsf{p})\big)$ which is covered already.

The contributions from $\deg(\psi) \in \{1, 2\}$ are:

$$\begin{aligned}
\psi_1\xi_2\xi_3 \quad &\to \quad \frac{1}{2}a_1^2a_2a_3\mathsf{F}_{2,m}\big(\tau_1(\gamma)\tau_0(\mathsf{p})\tau_0(F)\big) \\
&= \frac{1}{2}a_1^2a_2(m-a_1-a_2)mX\,, \\
\psi_3\xi_2\xi_3 \quad &\to \quad \frac{1}{2}a_3^2a_2a_3\mathsf{F}_{2,m}\big(\tau_0(\gamma)\tau_0(\mathsf{p})\tau_1(F)\big) \\
&= 0\,, \\
\psi_1\psi_3\xi_2 \quad &\to \quad \frac{1}{2}a_1^2\frac{1}{2}a_3^2a_2\mathsf{F}_{2,m}\big(\tau_1(\gamma)\tau_0(\mathsf{p})\tau_1(\mathsf{1})\big) \\
&= a_1^2a_2(m-a_1-a_2)^2X\,, \\
\psi_1\psi_3\xi_3 \quad &\to \quad \frac{1}{2}a_1^2\frac{1}{2}a_3^3\mathsf{F}_{2,m}\big(\tau_1(\gamma)\tau_0(W)\tau_1(F)\big) \\
&= \frac{1}{4}a_1^2(m-a_1-a_2)^3X + \text{ (lower genus)}\,, \\
\psi_3^2\xi_2 \quad &\to \quad \frac{1}{8}a_3^4a_2\mathsf{F}_{2,m}\big(\tau_0(\gamma)\tau_0(\mathsf{p})\tau_2(\mathsf{1})\big) \\
&= \frac{1}{8}a_2(m-a_1-a_2)^4X\,, \\
\psi_3^2\xi_3 \quad &\to \quad \frac{1}{8}a_3^4a_3\mathsf{F}_{2,m}\big(\tau_0(\gamma)\tau_0(W)\tau_2(F)\big) \\
&= 0\,.
\end{aligned}$$

The third calculation uses the dilaton equation. All of the other calculations are obtained by application of the divisor equation. Additionally, the fourth calculation involves Proposition 29. The only stratum with a genus 2 vertex (i.e. with both markings on a contracted genus 0 component) has vanishing contribution due to $\langle\gamma, F\rangle = 0$ and, therefore, the relation reduces to lower genus descendents. The total contribution to $[a_1^3a_2]$ is

$$-\frac{1}{2}mX - 2mX + \frac{3}{2}mX - \frac{1}{2}mX = -\frac{3}{2}mX\,.$$

We find that X is a linear combination of terms which satisfy Conjecture 15.

□

Remark 32. In fact, for $\gamma \in U^\perp$ the above generating series vanishes (and thus trivially satisfies the multiple cover formula). A proof in the primitive case is given in [9, Lemma 4].

6 Proof of Theorem 1 and 3

6.1 Proof of Theorem 1

The proof proceeds via induction on the pair (g, n) ordered by the lexicographic order: $(g', n') < (g, n)$ if

- $g' < g$ or
- $g' = g$ and $n' < n$.

Recall the dimension constraint of insertions:

$$g + n = \deg(\alpha) + \sum_i \deg(\gamma_i)\,.$$

We separate the proof into several steps.
Case 0. The genus 0 case is covered by Proposition 28. This serves as the start for our induction.
Case 1. If all cohomology classes γ_i satisfy $\deg(\gamma_i) \leq 1$, then $\deg(\alpha) \geq g$ and by the strong form of Getzler–Ionel vanishing [15, Proposition 2] we have $\alpha = \iota_* \alpha'$ with $\alpha' \in R^*(\partial \overline{M}_{g,n})$ and $\iota \colon \partial \overline{M}_{g,n} \to \overline{M}_{g,n}$. We are thus reduced to lower (g, n).
Case 2. Assume $\deg(\alpha) \leq g - 2$ or equivalently, there exist at least two descendents of the point class. We use the degeneration to the normal cone of a smooth elliptic fiber:

$$S \rightsquigarrow S \cup_E (\mathbb{P}^1 \times E)\,.$$

We specialize the point class to the bubble $\mathbb{P}^1 \times E$. Let $C = C' \cup C''$ be the splitting of a domain curve appearing in the degeneration formula in Theorem 23. Namely, C' is the component on S and C'' is the component on $\mathbb{P}^1 \times E$. We argue that this splitting has non-trivial contribution only for $g(C') < g$. If $g(C') = g$, this forces C'' to be a disconnected union of two rational curves. Since the degree of the curve class along the divisor is $\langle 2B + hF, F \rangle = 2$, the two descendents of the point class then force the cohomology weighted partition to be $(1, \mathsf{1})^2$ on the bubble or, equivalently, $(1, \omega)^2$ for (S, E). This contribution vanishes because there are no curves which can satisfy this condition (if $(1, \omega)^2$ is represented by a generic point in E^2, see Corollary 24).

Case 3. Assume $\deg(\alpha) = g - 1$ or equivalently, there exists only one desecendent of the point class. We may thus assume $\gamma_1 = \mathsf{p}$. If $n = 1, g \geq 2$, we can move $\tau_{g-1}(\mathsf{p})$ to the bubble and the genus on S drops.

When $n \geq 2$, moving the point class to the bubble as in Case 2 may not reduce the genus. In particular, moving $\tau_0(\mathsf{p})$ to the bubble has non-trivial contribution from rational curves on the bubble. On the other hand, if $a \geq 1$, moving $\tau_a(\mathsf{p})$ to the bubble reduces the genus on S because of the dimension constraint.

We use Buryak, Shadrin and Zvonkine's description of the top tautological group $R^{g-1}(M_{g,n})$ [10]. For any $\alpha \in R^{g-1}(\overline{M}_{g,n})$ the restriction of α to $M_{g,n}$ is a linear combination of

$$R^{g-1}(M_{g,n}) = \mathbb{Q}\big\langle \psi_1^{g-1}, \psi_2^{g-1}, \dots, \psi_n^{g-1} \big\rangle \tag{17}$$

and the boundary term is also tautological class in $R^{g-1}(\partial\overline{M}_{g,n})$. By the divisor equation and subsequent use of (17), we can reduce to cases for $\leq (g,2)$. When $g \geq 3$, (17) has a different basis

$$R^{g-1}(M_{g,2}) = \mathbb{Q}\big\langle \psi_1^{g-1}, \psi_1\psi_2^{g-2} \big\rangle$$

which is an easy consequence of the generalized top intersection formula. Therefore, we may assume the descendent of the point class is of the form $\tau_a(\mathsf{p})$ with $a \geq 1$. Now, specializing this insertion to the bubble $\mathbb{P}^1 \times E$ reduces the genus and hence the same argument as in Case 2 applies. The genus 2 case is covered by Proposition 31.

Relative vs. absolute. We reduced to invariants for (S,E) with genus $g' < g$. As explained in the proof of [29, Lemma 31] (see also [28]), the degeneration formula provides an upper triangular relation between absolute and relative invariants for all pairs $\leq (g', n')$. Thus, our induction applies.

6.2 Proof of Theorem 3

We argue by showing that each induction step in the proof of Theorem 1 is compatible with the holomorphic anomaly equation. Nontrivial step appears when the degeneration formula is used. From the compatibility result Proposition 26, we are reduced to proving the relative holomorphic anomaly equation for lower genus relative generating series $\mathsf{F}^{\mathrm{rel}}_{g',2}$ for (S,E) and relative generating series for $(\mathbb{P}^1 \times E, E)$. The holomorphic anomaly equation for $(\mathbb{P}^1 \times E, E)$ is established in [33]. Because of the relative vs. absolute correspondence [28], we are reduced to proving the holomorphic anomaly equation for $\mathsf{F}_{g',2}$ in genus $0,1$ and some genus 2 descendents. We proved the multiple cover formula for these cases in Section 5, which implies the holomorphic anomaly equation by Proposition 5.

Remark 33. Parallel argument shows that we can always reduce the proof for arbitrary descendent insertions to the case when the number of point insertions is less than or equal to $m-1$.

7 Examples

Example 34. We compute $\mathsf{F}_{1,2}\big(\tau_1(F)\big)$ via topological recursion in genus one and illustrate Conjecture 15. Let $[\delta_0] \in A^1(\overline{M}_{1,1})$ be the pushforward of the fundamental class under the gluing map

$$\overline{M}_{0,3} \to \overline{M}_{1,1}\,.$$

Since
$$\psi_1 = \frac{1}{24}[\delta_0] \in A^1(\overline{M}_{1,1})\,,$$
we obtain
$$\begin{aligned}\mathsf{F}_{1,1}\big(\tau_1(F)\big) &= \frac{1}{24}\mathsf{F}_{0,1}\big(\tau_0(F)\tau_0(\Delta_S)\big) = \frac{1}{12}\mathsf{F}_{0,1}\big(\tau_0(F)\tau_0(F\times W)\big)\\ &= \frac{1}{12}\mathsf{D}_q\mathsf{F}_{0,1}\,,\end{aligned}$$
where $\Delta_S \subset S\times S$ is the diagonal class. Analogously,
$$\mathsf{F}_{1,2}\left(\tau_1(F)\right) = \frac{1}{24}\,\mathsf{F}_{0,2}\big(\tau_0(F)\tau_0(\Delta_S)\big) = \frac{1}{3}\,\mathsf{D}_q\mathsf{F}_{0,2}\,.$$
Using the multiple cover formula in genus zero
$$\mathsf{F}_{0,2} = \mathsf{T}_2\mathsf{F}_{0,1} + \frac{1023}{8192}\,\mathsf{F}_{0,1}(q^2)\,,$$
we obtain
$$\begin{aligned}\mathsf{F}_{1,2}\big(\tau_1(F)\big) &= \frac{1}{3}\mathsf{D}_q\mathsf{F}_{0,2} = 2\,\mathsf{T}_2\frac{1}{12}\mathsf{D}_q\mathsf{F}_{0,1} + \frac{1023}{1024}\,\mathsf{B}_2\frac{1}{12}\mathsf{D}_q\mathsf{F}_{0,1}\\ &= 2\,\mathsf{T}_2\mathsf{F}_{1,1}\big(\tau_1(F)\big) + (2^0-2^{-10})\,\mathsf{B}_2\mathsf{F}_{1,1}\big(\tau_1(F)\big)\,,\end{aligned}$$
in perfect agreement with Conjecture 15 using the formula for $\mathsf{T}_{2,0}$ from Lemma 12.

Example 35. We compute $\mathsf{F}_{2,2}(\tau_0(\mathsf{p})^2)$ via degeneration formula and verify the multiple cover formula. The first two terms are computed by the classical geometry of K3 surface in [32]. For simplicity we write $\mathsf{F}_{1,2} = \mathsf{F}_{1,2}(\tau_0(\mathsf{p}))$. The relative invariants for (S,E) can be written in terms of absolute invariants:

Lemma 36. (i) $\mathsf{F}^{\mathrm{rel}}_{0,2}\big(\emptyset \mid (1,1)^2\big) = 2\mathsf{F}_{0,2}$,

(ii) $\mathsf{F}^{\mathrm{rel}}_{1,2}\big(\emptyset \mid (1,1),(1,\omega)\big) = \mathsf{F}_{1,2} - 2\mathsf{F}_{0,2}\mathsf{D}_qC_2$,

(iii) $\mathsf{F}^{\mathrm{rel}}_{1,2}\big(\emptyset \mid (2,1)\big) = \frac{1}{3}\mathsf{D}_q\mathsf{F}_{0,2} - 4C_2\mathsf{F}_{0,2}$.

Proof. It is a standard computation of the relative vs. absolute correspondence [28]. □

The relative invariants for $(\mathbb{P}^1\times E, E)$ can be computed by the Gromov–Witten invariants of E.

Lemma 37. (i) $\mathsf{G}^{\mathrm{rel}}_{0,1}\big(\tau_0(\mathsf{p}) \mid (1,1)\big) = 1$, $\mathsf{G}^{\mathrm{rel}}_{0,1}\big(\emptyset \mid (1,\omega)\big) = 1$,

(ii) $\mathsf{G}^{\mathrm{rel}}_{1,1}\big(\tau_0(\mathsf{p}) \mid (1,\omega)\big) = \mathsf{D}_q C_2, \quad \mathsf{G}^{\mathrm{rel}}_{1,1}\big(\tau_0(\mathsf{p})^2 \mid (1,1)\big) = 2\mathsf{D}_q C_2,$

(iii) $\mathsf{G}^{\mathrm{rel}}_{2,1}\big(\tau_0(\mathsf{p})^2 \mid (1,\omega)\big) = (\mathsf{D}_q C_2)^2,$

(iv) $\mathsf{G}^{\mathrm{rel}}_{1,2}\big(\tau_0(\mathsf{p})^2 \mid (2,\omega)\big) = \mathsf{D}_q^2 C_2, \quad \mathsf{G}^{\mathrm{rel}}_{1,2}\big(\tau_0(\mathsf{p})^2 \mid (1,\omega)^2\big) = \mathsf{D}_q^3 C_2.$

Consider the degeneration where two point insertions move to the bubble $\mathbb{P}^1 \times E$. By Theorem 23,

$$\begin{aligned}\mathsf{F}_{2,2}\big(\tau_0(\mathsf{p})^2\big) &= \big(\mathsf{F}_{1,2} - 2\mathsf{F}_{0,2}\mathsf{D}_q C_2\big)4\mathsf{D}_q C_2 + \big(\frac{1}{3}\mathsf{D}_q\mathsf{F}_{0,2} - 4C_2\mathsf{F}_{0,2}\big)2\mathsf{D}_q^2 C_2 \\ &\quad + (2\mathsf{F}_{0,2})\frac{1}{2}\big(\mathsf{D}_q^3 C_2 + 4(\mathsf{D}_q C_2)^2\big) \\ &= 36q + 8760q^2 + 754992q^3 + 36694512q^4 + \cdots .\end{aligned}$$

On the other hand, the primitive generating series

$$\mathsf{F}_{2,1}\big(\tau_0(\mathsf{p})^2\big) = \frac{\big(\mathsf{D}_q C_2\big)^2}{\Delta(q)}$$

is computed in [7] and one can apply the multiple cover formula to obtain a candidate for $\mathsf{F}_{2,2}\big(\tau_0(\mathsf{p})^2\big)$. The first few terms of the two generating series match. It is enough to conclude that the two generating series are indeed equal because the space of quasimodular forms with given weight is finite dimensional. However, it seems non-trivial to match the above formula from the degeneration with the formula provided by Conjecture 15.

A A proof of degeneration formula

For a self-contained exposition, we present a proof of the degeneration formula which is parallel to the proof in [29, 30]. When $m = 1, 2$, a proof using symplectic geometry was presented in [24].

Perfect obstruction theory

For simplicity assume $n = 0$. General cases easily follow from this case. Let $\epsilon\colon \mathcal{S} \to \mathbb{A}^1$ be the total family of the degeneration and

$$\overline{M}_g(\epsilon, \beta) \to \mathbb{A}^1$$

be the moduli space of stable maps to the expanded target $\widetilde{\mathcal{S}}$. For the relative profile μ, the embedding

$$\iota_\mu\colon \overline{M}_g(\mathcal{S}_0, \mu) \hookrightarrow \overline{M}_g(\epsilon, \beta)$$

can be realized as a Cartier pseudo-divisor (L_μ, s_μ).

Let $\mathrm{E}_\epsilon \to \mathbb{L}_{\overline{\mathrm{M}}_\mathrm{g}(\epsilon,\beta)}$ be the perfect obstruction theory constructed in [27]. Then the perfect obstruction theories E_0 and E_μ of $\overline{M}_g(\mathcal{S}_0, \beta)$ and $\overline{M}_g(\mathcal{S}_0, \mu)$ sit in exact triangles

$$\begin{aligned} L_0^\vee \to \iota_0^*\mathrm{E}_\epsilon \to \mathrm{E}_0 \xrightarrow{[1]} \\ L_\mu^\vee \to \iota_\mu^*\mathrm{E}_\epsilon \to \mathrm{E}_\mu \xrightarrow{[1]} . \end{aligned}$$

On $\overline{M}_g(\mathcal{S}_0, \mu)$, the perfect obstruction theory splits as follows. Let E_1 and E_2 be the perfect obstruction theory of relative stable map spaces $\overline{M}_g(S/E, \beta_1)_\mu$ and $\overline{M}_g(\mathbb{P}^1 \times E/E, \beta_2)_\mu$ respectively. There exists an exact triangle

$$\bigoplus_{i=1}^{l(\mu)} (N^\vee_{\Delta_E/E\times E})_i \to \mathrm{E}_1 \boxplus \mathrm{E}_2 \to \mathrm{E}_\mu \xrightarrow{[1]} \tag{18}$$

where $(N^\vee_{\Delta_E/E\times E})_i$ is the pullback of the conormal bundle of the diagonal $\Delta_E \subset E \times E$ along the i-th relative marking.

Reduced class

Let $\rho\colon \widetilde{\mathcal{S}} \to S \times \mathbb{A}^1 \to S$ be the projection. By pulling back the holomorphic symplectic form on S via ρ, one can define a cosection of the obstruction sheaf of E_ϵ

$$Ob_{\overline{M}_g(\epsilon,\beta)} \to \mathcal{O}\,,$$

see [20, Section 5]. Dualizing the cosection gives a morphism

$$\gamma\colon \mathcal{O}[1] \to \mathrm{E}_\epsilon\,.$$

Let $\mathrm{E}_\epsilon^{\mathrm{red}}$ be the cone of γ which gives the reduced class on $\overline{M}_g(\epsilon, \beta)$. Similarly we can construct

$$\gamma_{\mathrm{rel}}\colon \mathcal{O}[1] \to \mathrm{E}_1$$

for the moduli space of relative stable maps $\overline{M}_g(S/E, \beta)$.

Degeneration formula for reduced class

Restricting γ to $\overline{M}_g(\mathcal{S}_0, \beta)$ and $\overline{M}_g(\mathcal{S}_0, \mu)$, we get

$$\begin{aligned} \gamma_0\colon \mathcal{O}[1] \to \iota_0^*\mathrm{E}_\epsilon \to \mathrm{E}_0 \\ \gamma_\mu\colon \mathcal{O}[1] \to \iota_\mu^*\mathrm{E}_\epsilon \to \mathrm{E}_\mu \end{aligned}$$

where the compositions induce reduced classes. The exact triangles

$$L_0^\vee \to \iota_0^* \mathrm{E}_\epsilon^{\mathrm{red}} \to \mathrm{E}_0^{\mathrm{red}} \xrightarrow{[1]},$$
$$L_\mu^\vee \to \iota_\mu \mathrm{E}_\epsilon^{\mathrm{red}} \to \mathrm{E}_\mu^{\mathrm{red}} \xrightarrow{[1]},$$

still hold.

Lemma 38. We have an exact triangle

$$N^\vee_{\Delta_{E^l}/E^l\times E^l} \to \mathrm{E}_1^{\mathrm{red}} \boxplus \mathrm{E}_2 \to \mathrm{E}_\mu^{\mathrm{red}} \xrightarrow{[1]}$$

on $\overline{M}_g(\mathcal{S}_0,\mu)$ compatible with the structure maps to the cotangent complex.

Proof. Consider the diagram of complexes

$$\begin{tikzcd}
 & \mathcal{O}[1]\boxplus 0 \arrow[r, equal] \arrow[d, "\gamma_{\mathrm{rel}}\boxplus 0"] & \mathcal{O}[1] \arrow[d, "\gamma_\mu"] \\
\bigoplus_{i=1}^{l(\mu)}(N^\vee_{\Delta_E/E\times E})_i \arrow[r] \arrow[d, equal] & \mathrm{E}_1\boxplus \mathrm{E}_2 \arrow[r] \arrow[d] & \mathrm{E}_\mu \arrow[d] \\
\bigoplus_{i=1}^{l(\mu)}(N^\vee_{\Delta_E/E\times E})_i \arrow[r] & \mathrm{E}_1^{\mathrm{red}}\boxplus \mathrm{E}_2 \arrow[r] & \mathrm{E}_\mu^{\mathrm{red}}
\end{tikzcd}$$

where the middle horizontal morphisms are the exact triangle from (18). The square on the top commutes because the cosections for $\widetilde{\mathcal{S}}$ and (S,E) are both coming from the holomorphic symplectic form on S. The vertical morphisms are exact triangles and hence induces a map between cones. $\square$

Now Theorem 23 is a direct consequence of Lemma 38.

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Chow rings of stacks of prestable curves I

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Abstract

This is a simplified version of [6] (under the same title).

1 Introduction

Let $\overline{\mathcal{M}}_{g,n}$ be the moduli space of stable curves. It parameterizes tuples $(C, p_1, \ldots, p_n)$ of a nodal curve C of arithmetic genus g with n distinct smooth marked points such that C has only finitely many automorphisms fixing the points p_i. After Mumford's seminal paper [24], there has been a substantial study of the structure of the tautological rings

$$\mathrm{R}^*(\overline{\mathcal{M}}_{g,n}) \subseteq \mathrm{CH}^*(\overline{\mathcal{M}}_{g,n})_{\mathbb{Q}} \,.$$

The tautological rings form a system of subrings of $\mathrm{CH}^*(\overline{\mathcal{M}}_{g,n})_{\mathbb{Q}}$ with explicit generators defined using the universal curve and the boundary gluing maps of the spaces $\overline{\mathcal{M}}_{g,n}$, see [19].

A natural extension of $\overline{\mathcal{M}}_{g,n}$ is the moduli stack $\mathfrak{M}_{g,n}$ of marked prestable curves, in which we drop the condition of having only finitely many automorphisms. It is a smooth algebraic stack, locally of finite type over the base field k and containing $\overline{\mathcal{M}}_{g,n}$ as an open substack. However, by allowing infinite automorphism groups, the stacks of prestable curves are no longer Deligne-Mumford stacks and not of finite type[1].

A recent application of Chow groups of such non-finite type algebraic stacks appeared in the paper [4], which studied cycle classes and tautological rings for the universal Picard stack $\mathfrak{Pic}_g$ over the stack $\mathfrak{M}_g$. The stack $\mathfrak{Pic}_g$ parameterizes pairs $(C, \mathcal{L})$ of a prestable curve C and a line bundle $\mathcal{L}$ on C. In [4], results from [20] are used to prove a formula for the fundamental class of the closure of the zero section $\{(C, \mathcal{O}_C)\} \subseteq \mathfrak{Pic}_g$. By pulling back this equality under natural morphisms

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[1] In fact, the stack $\mathfrak{M}_{0,0}$ contains a finite type open substack which is not even a quotient stack, see [22, Proposition 5.2].

$\overline{\mathcal{M}}_{g,n} \to \mathfrak{Pic}_g$, new results about the classical double ramification cycles on the moduli of stable curves are established.

In the paper [4], the intersection theory of $\mathfrak{Pic}_g$ is studied using a definition of operational Chow groups modelled on [17, Chapter 17]. In our paper, we follow the approach [21] by Kresch, who developed a cycle theory for algebraic stacks of finite type over a field k. This theory has many structural advantages over the operational theory of [4], such as projective pushforwards and an excision sequence, and for a smooth stack always admits a natural map to this operational theory.[2]

We extend Kresch's theory from the case of finite-type stacks to the case of algebraic stacks $\mathfrak{X}$ locally of finite type over k (such as $\mathfrak{M}_{g,n}$) by defining their Chow groups[3] as the limit

$$\mathrm{CH}_*(\mathfrak{X}) = \varprojlim_{i \in I} \mathrm{CH}_*(\mathcal{U}_i),$$

for $(\mathcal{U}_i)_{i \in I}$ a directed system of finite-type open substacks covering $\mathfrak{X}$. Using this definition, we define the tautological ring $\mathrm{R}^*(\mathfrak{M}_{g,n}) \subseteq \mathrm{CH}^*(\mathfrak{M}_{g,n})$, extending the definition [19] for the moduli spaces of stable curves.[4]

Proper pushforwards of Chow groups of algebraic stacks

When extending the definition of the tautological ring to the stacks of prestable curves, we immediately encounter a problem: for the spaces $\overline{\mathcal{M}}_{g,n}$ of stable curves, these rings can be defined as the smallest system of subrings of $\mathrm{CH}^*(\overline{\mathcal{M}}_{g,n})$ closed under pushforwards by gluing morphisms and the morphisms $\overline{\mathcal{C}}_{g,n} = \overline{\mathcal{M}}_{g,n+1} \to \overline{\mathcal{M}}_{g,n}$ giving the universal curve over $\overline{\mathcal{M}}_{g,n}$. However, for the stacks $\mathfrak{M}_{g,n}$ of prestable curves, the analogous universal curve morphisms are in general proper, but not projective (see [15, Example 2.3]). Thus Kresch's Chow theory, which a priori only has projective pushforwards, cannot be applied immediately. Historically, this has been a major obstruction in the study of the Chow groups $\mathrm{CH}^*(\mathfrak{M}_{g,n})$ and made it necessary to give many ad-hoc constructions of classes that are traditionally defined by proper pushforwards (see [16, 14]).

To overcome this obstacle, joint with Skowera, we define proper pushforwards for cycle groups of algebraic stacks in [6, Appendix B].

Theorem 1.1. (see [6, Theorem B.17] and Proposition B.1) Let Y be a stack stratified by global quotient stacks, and let $f : X \to Y$ be a proper, representable morphism. Then there is a proper pushforward $f_* : \mathrm{CH}_d(X, \mathbb{Z}) \to \mathrm{CH}_d(Y, \mathbb{Z})$ for all d which is functorial (with respect to compositions) and compatible with flat pullbacks and refined Gysin pullbacks.

[2]See Appendix C for more details on the comparison of the definitions.

[3]Unless stated otherwise, all Chow groups in the paper will be taken with $\mathbb{Q}$-coefficients.

[4]There is a small caveat: the intersection theory of $\mathrm{CH}^*(\mathfrak{M}_{1,0})$ is not covered by [21] because the stabilizer group at the general point is not a linear algebraic group. In this paper, we exclude this case.

If, instead, f is proper and of relative Deligne-Mumford type, then there is a proper pushforward $f_* : \mathrm{CH}_d(X, \mathbb{Q}) \to \mathrm{CH}_d(Y, \mathbb{Q})$ for all d, with the properties above.

The universal curve over the stack of prestable curves

A second problem that we encounter when generalizing the definition of the tautological ring of $\overline{\mathcal{M}}_{g,n}$ to $\mathfrak{M}_{g,n}$ is that the universal curve $\mathfrak{C}_{g,n} \to \mathfrak{M}_{g,n}$ is *not* given by the forgetful map $\mathfrak{M}_{g,n+1} \to \mathfrak{M}_{g,n}$. In particular, since the forgetful maps are in general not proper, we cannot define $(\mathrm{R}^*(\mathfrak{M}_{g,n}))_{g,n}$ as the smallest system of subrings of $(\mathrm{CH}^*(\mathfrak{M}_{g,n}))_{g,n}$ closed under gluing and forgetful pushforwards.

To overcome this issue (and give a modular interpretation of $\mathfrak{C}_{g,n}$ as a stack of $(n+1)$-pointed curves), we use the notion of prestable curves *with values in a semigroup* $\mathcal{A}$ from [8, 10]. Given a suitable (commutative) semigroup[5] $\mathcal{A}$ and an element $a \in \mathcal{A}$, these references define a stack $\mathfrak{M}_{g,n,a}$ parameterizing tuples $(C, p_1, \ldots, p_n, (a_{C_v})_v)$ of a prestable curve $(C, p_1, \ldots, p_n)$ together with a value $a_{C_v} \in \mathcal{A}$ for each component C_v of C such that all a_{C_v} sum up to a in $\mathcal{A}$. Moreover, in contrast to the stack $\mathfrak{M}_{g,n}$, the definition of $\mathfrak{M}_{g,n,a}$ includes a stability condition: any component C_v such that $a_{C_v} = \mathbb{0} \in \mathcal{A}$ is the neutral element of $\mathcal{A}$ must actually be stable, i.e. have a finite group of automorphisms fixing all markings and nodes on C_v. The advantage of this stability condition is that the natural forgetful map $\pi : \mathfrak{M}_{g,n+1,a} \to \mathfrak{M}_{g,n,a}$, which forgets the last marking and contracts the component containing it if it becomes unstable, does define the universal curve over $\mathfrak{M}_{g,n,a}$.

Applying this machinery to a particularly simple semigroup, we obtain the desired modular interpretation of $\mathfrak{C}_{g,n}$. For this consider the semigroup

$$\mathcal{A} = \{\mathbb{0}, \mathbb{1}\} \text{ with } \mathbb{0} + \mathbb{0} = \mathbb{0}, \mathbb{0} + \mathbb{1} = \mathbb{1} + \mathbb{0} = \mathbb{1} + \mathbb{1} = \mathbb{1}.$$

Then we show the following.

Proposition 1.2 (see Proposition 2.7, Corollary 2.8)**.** Let $g, n \geq 0$ and consider the semigroup $\mathcal{A} = \{\mathbb{0}, \mathbb{1}\}$ above. Then the stack $\mathfrak{M}_{g,n}$ is naturally contained inside $\mathfrak{M}_{g,n,\mathbb{1}}$ as the open substack of $(C, p_1, \ldots, p_n, (a_{C_v})_v)$ such that $a_{C_v} = \mathbb{1}$ for all v. Thus the universal curve $\mathfrak{C}_{g,n}$ is naturally contained as an open substack of $\mathfrak{M}_{g,n+1,\mathbb{1}}$ sitting in the cartesian diagram

$$\begin{tikzcd} \mathfrak{C}_{g,n} \arrow[r, phantom, "\subseteq"] \arrow[d] & \mathfrak{M}_{g,n+1,\mathbb{1}} \arrow[d, "\pi"] \\ \mathfrak{M}_{g,n} \arrow[r, phantom, "\subseteq"] & \mathfrak{M}_{g,n,\mathbb{1}} \end{tikzcd}$$

In particular, this proposition indeed gives an interpretation of $\mathfrak{C}_{g,n}$ as a stack of $(n+1)$-pointed prestable curves together with some additional structure (see the paragraph below Corollary 2.8 for more details).

[5] See Section 2.2 for the precise definition and technical conditions we require for these semigroups.

Tautological rings of stacks of prestable curves

Having solved both the issues with proper pushforwards and the modular interpretation of the universal curve, we are now ready to define the tautological rings. Since the discussion in the last section shows that the spaces $\mathfrak{M}_{g,n,a}$ appear naturally, we will in fact define the tautological rings for these spaces and obtain the rings for $\mathfrak{M}_{g,n}$ by restriction. To write down the definition, we note that in addition to the forgetful maps

$$\pi : \mathfrak{M}_{g,n+1,a} \to \mathfrak{M}_{g,n,a} \tag{1}$$

mentioned above, there also exist gluing maps

$$\xi_\Gamma : \mathfrak{M}_\Gamma = \prod_{v \in V(\Gamma)} \mathfrak{M}_{g(v),n(v),a(v)} \to \mathfrak{M}_{g,n,a} \tag{2}$$

for every prestable graph Γ together with an $\mathcal{A}$-valuation $a : V(\Gamma) \to \mathcal{A}$ satisfying $\sum_{v \in V(\Gamma)} a(v) = a$. Here $V(\Gamma)$ is the set of vertices of the graph Γ.

Definition 1.3. The tautological rings $(\mathrm{R}^*(\mathfrak{M}_{g,n,a}))_{g,n,a}$ are defined as the smallest system of $\mathbb{Q}$-subalgebras with unit of the Chow rings $(\mathrm{CH}^*(\mathfrak{M}_{g,n,a}))_{g,n,a}$ closed under taking pushforwards by the natural forgetful and gluing maps (1) and (2).

The tautological ring $\mathrm{R}^*(\mathfrak{M}_{g,n}) \subseteq \mathrm{CH}^*(\mathfrak{M}_{g,n})$ is defined as the image of the restriction of $\mathrm{R}^*(\mathfrak{M}_{g,n,\mathbb{1}})$ to the open substack $\mathfrak{M}_{g,n} \subseteq \mathfrak{M}_{g,n,\mathbb{1}}$ from Proposition 1.2.

Just as for the moduli spaces of stable curves, we define ψ and κ-classes: given $1 \leq i \leq n$ we set

$$\psi_i = c_1(\sigma_i^* \omega_\pi) \in \mathrm{CH}^1(\mathfrak{M}_{g,n,a})\,,$$

where $\sigma_i : \mathfrak{M}_{g,n,a} \to \mathfrak{M}_{g,n+1,a}$ is the i-th universal section and ω_π is the relative dualizing sheaf of π. Similarly, given $m \geq 0$ we set

$$\kappa_m = \pi_* \left(\psi_{n+1}^{m+1}\right) \in \mathrm{CH}^m(\mathfrak{M}_{g,n,a})\,.$$

It is easy to see that both types of classes are in fact tautological. Given any $\mathcal{A}$-valued prestable graph Γ, consider the products

$$\alpha = \prod_{v \in V} \left(\prod_{i \in H(v)} \psi_{v,i}^{a_i} \prod_{a=1}^{m_v} \kappa_{v,a}^{b_{v,a}} \right) \in \mathrm{CH}^*(\mathfrak{M}_\Gamma). \tag{3}$$

of ψ and κ-classes on the space $\mathfrak{M}_\Gamma$ above. Then we define the *decorated stratum class* $[\Gamma, \alpha]$ as the pushforward

$$[\Gamma, \alpha] = (\xi_\Gamma)_* \alpha \in \mathrm{R}^*(\mathfrak{M}_{g,n,a}).$$

The following result (generalizing [19, Proposition 11]) show that these classes additively generate the tautological rings.

Theorem 1.4. The tautological ring $\mathrm{R}^*(\mathfrak{M}_{g,n,a})$ is generated as a $\mathbb{Q}$-vector space by the decorated strata classes $[\Gamma, \alpha]$. In addition to being closed under pushforwards by gluing and forgetful maps, the tautological rings are likewise closed under pullbacks by these maps, with explicit formulas describing all these operations on the generators $[\Gamma, \alpha]$.[6]

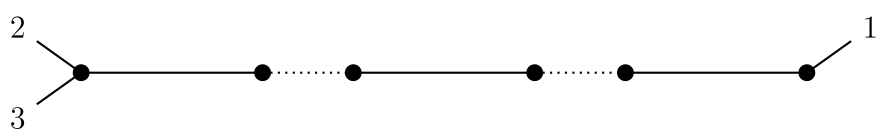
This result gives an effective method to perform computations in the Chow rings of the stacks $\mathfrak{M}_{g,n,a}$. Moreover, it shows that while both the Chow and the tautological rings of these stacks are in general infinite-dimensional, the individual graded pieces of $\mathrm{R}^*(\mathfrak{M}_{g,n,a})$ always have a finite set of generators.

Relations to other work

In this section we explain how our results relate to previous results on the intersection theory of the stacks $\mathfrak{M}_{g,n}$.

As a first example, in [18] Gathmann used the pullback formula of ψ-classes along the stabilization morphism $\mathrm{st} \colon \mathfrak{M}_{g,1} \to \overline{\mathcal{M}}_{g,1}$ to prove certain properties of the Gromov-Witten potential. In Section 3.2 we compute arbitrary pullbacks of tautological classes under the stabilization map, in particular recovering Gathmann's result.

In [25], Oesinghaus computed the Chow rings of the open locus $\mathcal{T} \subset \mathfrak{M}_{0,3}$ of curves with dual graph of the shape

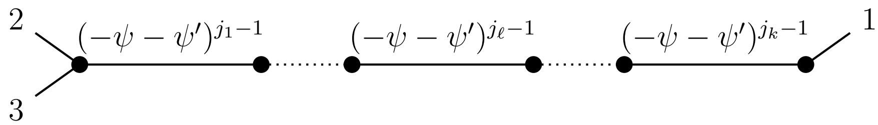


This stack has a natural interpretation as the stack of *expanded pairs* appearing in [1]. Oesinghaus showed that the Chow ring of $\mathcal{T}$ is given by the known algebra of *quasi-symmetric functions* QSym (see [23] for an overview). The ring QSym has a natural basis M_J (as a $\mathbb{Q}$-vector space) indexed by positive integer vectors $J = (j_1, \ldots, j_k) \in \mathbb{Z}_{\geq 1}^k$ of some length $k \geq 0$, and the product $M_J \cdot M_{J'}$ can be defined in terms of a certain *shuffle rule* on the vectors J, J' (see [25, Proposition 2]).

Oesinghaus' proof worked by writing down an open exhaustion of $\mathcal{T}$ by quotient stacks, allowing to write the Chow ring as a certain projective limit of polynomial rings which is known to produce the algebra QSym. However, due to the nature of this proof, a geometric interpretation for the generators M_J was not immediately clear (see [25, Remark 7]). Using the techniques of our paper, we can now answer this question, showing that the generators M_J have a concrete interpretation as tautological classes.

[6] For the precise statement and formulas from the theorem, we refer the reader to Section 3.2.

Proposition 1.5 (see Example 4.3)**.** For $J = (j_1, \ldots, j_k) \in \mathbb{Z}^k_{\geq 1}$, the generator $M_J \in \mathrm{QSym} \cong \mathrm{CH}^*(\mathcal{T})$ is given by the restriction of the tautological class



(4)

on $\mathfrak{M}_{0,3}$.

Furthermore, it is straightforward to see that the shuffle rule describing products $M_J \cdot M_{J'}$ is an immediate consequence of the product formula for the tautological classes (4). Oesinghaus also computes the Chow rings of the loci $\mathfrak{M}^{\mathrm{ss}}_{0,2}$ and $\mathfrak{M}^{\mathrm{ss}}_{0,3}$ of semistable curves in $\mathfrak{M}_{0,2}$ and $\mathfrak{M}_{0,3}$, giving a description in terms of tensor products involving the rings QSym. Again we give a description in terms of tautological classes in Example 4.3.

Tautological relations in genus zero

The present paper lays down the foundations of the theory of the Chow rings $\mathrm{CH}^*(\mathfrak{M}_{g,n})$. In the second part [5] we use results of this paper to fully determine the Chow rings of $\mathfrak{M}_{0,n}$ for all n: we prove that all classes are tautological and we compute all relations among generators of the tautological ring.

Structure of the paper

In Section 2 we establish basic properties of the stacks $\mathfrak{M}_{g,n}$. We discuss boundary gluing maps in Section 2.1, introduce the stacks of prestable curves with values in a semigroup in Section 2.2. In Section 3 we establish basic properties of the Chow group of $\mathfrak{M}_{g,n}$. In Section 3.1 we define Chow groups and tautological rings of such stacks. In Section 3.2 we compute formulas for intersection products and pullbacks and pushforwards of tautological classes under natural maps. In Section 4 we compare our result with previous works by Gathmann [18], Pixton [26] and Oesinghaus [25].

In Appendix A we give some general treatment of Chow groups of locally finite type algebraic stacks. We give a definition of such Chow groups based on [21] and show various basic properties. In Appendix B we show basic compatibility properties of these pushforwards and explain how they extend to the setting of algebraic stacks locally of finite type. Finally, in Appendix C we give a definition and establish basic properties of operational Chow groups on locally finite type stacks, a technical tool needed for some of the computations in Section 3.2.

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2 The stack $\mathfrak{M}_{g,n}$ of prestable curves

Throughout the paper we work over an arbitrary base field k. Let $\mathfrak{M}_{g,n}$ be the moduli stack of prestable curves of genus g with n marked points. An object of $\mathfrak{M}_{g,n}$ over a scheme S is a tuple

$$(\pi : C \to S,\ \ p_1, \ldots, p_n : S \to C)\,,$$

where C is an algebraic space, the map π is a flat, proper morphism of finite presentation and relative dimension 1. The geometric fibers of π are connected, reduced curves of arithmetic genus g with at worst nodal singularities. The morphisms $p_1, \ldots, p_n$ are disjoint sections of π with image in the smooth locus of π, see [29, 0E6S].

This stack is quasi-separated, smooth and locally of finite type over k ([2]) and of dimension $3g-3+n$ ([7]). For $2g-2+n>0$ there is a natural stabilization morphism

$$\mathrm{st} : \mathfrak{M}_{g,n} \to \overline{\mathcal{M}}_{g,n}$$

which contracts unstable rational components. This morphism is flat by [7, Proposition 3].

2.1 Boundary gluing maps

A prestable graph Γ of genus g with n markings consists of the data

$$\Gamma = (V,\, H,\, \ell : L \to \{1, \ldots, n\},\ g\colon V \to \mathbb{Z}_{\geq 0}\,,\ v\colon H \to V\,,\ \iota : H \to H)$$

satisfying the properties:

(i) V is a vertex set with a genus function $g\colon V \to \mathbb{Z}_{\geq 0}$,

(ii) H is a half-edge set equipped with a vertex assignment $v : H \to V$ and an involution ι,

(iii) E, the edge set, is defined by the 2-cycles of ι in H (self-edges at vertices are permitted),

(iv) $L \subseteq H$, the set of legs, is defined by the fixed points of ι and corresponds to n markings via the bijection $\ell : L \to \{1, \ldots, n\}$,

(v) the pair (V, E) defines a connected graph satisfying the genus condition

$$\sum_{v \in V} g(v) + h^1(\Gamma) = g\,,$$

where $h^1(\Gamma) = |E| - |V| + 1$ is the first Betti number of the graph Γ.

A prestable graph Γ is called *stable* if the following additional condition is satisfied:

(vi) for each vertex $w \in V$, we have

$$2g(w) - 2 + n(w) > 0\,,$$

where $n(w) = |v^{-1}(w)|$ is the valence of w in Γ, i.e. the number of half-edges incident to w.

Given a second graph $\Gamma' = (V', H', \ell', g', v', \iota')$ an *isomorphism* $\varphi : \Gamma \to \Gamma'$ is the data of bijective maps

$$\varphi_V : V \to V'\,,\ \ \varphi_H : H \to H'$$

which are compatible with the remaining data of the prestable graphs, in the sense that

$$\ell' \circ \varphi_H|_L = \ell\,,\ g' \circ \varphi_V = g\,,\ v' \circ \varphi_H = \varphi_V \circ v\,,\ \iota' \circ \varphi_H = \varphi_H \circ \iota\,.$$

For every vertex $v \in V(\Gamma)$ let $H(v)$ be the set of half-edges at v, with cardinality $n(v)$. Then there exists a natural gluing morphism

$$\xi_\Gamma : \mathfrak{M}_\Gamma = \prod_{v \in V(\Gamma)} \mathfrak{M}_{g(v),n(v)} \to \mathfrak{M}_{g,n}\,,$$

which assigns to a collection $((C_v, (p_h)_{h \in H(v)})$ the curve $(C, p_1, \ldots, p_n)$ obtained by identifying the markings $p_h, p_{h'}$ for each pair (h, h') forming an edge of Γ. [7] Restricting to the preimage of the open substack $\overline{\mathcal{M}}_{g,n} \subset \mathfrak{M}_{g,n}$ we get the usual gluing maps

$$\xi_\Gamma : \overline{\mathcal{M}}_\Gamma = \prod_{v \in V(\Gamma)} \overline{\mathcal{M}}_{g(v),n(v)} \to \overline{\mathcal{M}}_{g,n}\,,$$

Note that unless Γ is stable, the left hand side is empty.

[7] Note that while it is customary in the field to write the factors of the domain of ξ_Γ as $\mathfrak{M}_{g(v),n(v)}$, it would maybe be more appropriate to define prestable curves with markings indexed by the set $H(v)$ and write $\mathfrak{M}_{g(v),H(v)}$, since otherwise we need to implicitly choose an ordering on the half-edges at v to define the map ξ_Γ. This does not affect the arguments presented below and the reader may assume that an arbitrary such an ordering is chosen.

On the other hand, given $m \geq 0$ we have the forgetful morphism

$$F_m : \mathfrak{M}_{g,n+m} \to \mathfrak{M}_{g,n}, (C, p_1, \ldots, p_n, q_1, \ldots, q_m) \mapsto (C, p_1, \ldots, p_n) .$$

Since the curve C remains prestable after forgetting a subset of the markings, there is no stabilization procedure in the morphism F_m and the underlying curve remains unchanged.

Lemma 2.1. The morphism F_m is smooth and representable of relative dimension m and the collection

$$\left(F_m|_{\overline{\mathcal{M}}_{g,n+m}} : \overline{\mathcal{M}}_{g,n+m} \to \mathfrak{M}_{g,n} \right)_{m \in \mathbb{Z}_{\geq 0}}$$

forms a smooth and representable cover of $\mathfrak{M}_{g,n}$. The complement of the image of $\overline{\mathcal{M}}_{g,n+m}$ under F_m in $\mathfrak{M}_{g,n}$ has codimension $\lfloor \frac{m}{2} \rfloor + 1$, except for finitely many m in the unstable setting $2g - 2 + n \leq 0$.[8]

Proof. Except for the statement about the codimension of the complement of the image, this is [7, Proposition 2]. To show the formula for the codimension, observe on the one hand that in a prestable graph Γ, every unstable vertex can be stabilized by adding at most two legs. Conversely, consider the prestable graph Γ_0 formed by a central vertex of genus g with all n legs, connected via single edges to c outlying vertices of genus 0 with no legs. Then Γ_0 belongs to a codimension c stratum and we need precisely $2c$ additional legs to stabilize it. Thus the stratum of $\mathfrak{M}_{g,n}$ associated to Γ_0 lies in the complement of $F(\overline{\mathcal{M}}_{g,n+m})$ if and only if $c \geq \lfloor \frac{m}{2} \rfloor + 1$. The finitely many exceptions in the unstable range arise from the fact that the central vertex of Γ_0 is not stable if $2g - 2 + n + c < 0$. □

Let $\mathrm{st}_m(\Gamma)$ be the set of stable graphs Γ' in genus g with $n+m$ markings obtained from a prestable graph Γ of genus g with n legs by adding m additional legs, labeled $n+1, \ldots, n+m$, at vertices of Γ. As explained above, for a fixed prestable graph Γ the set $\mathrm{st}_m(\Gamma)$ starts being nonempty for m sufficiently large.

Given $\Gamma' \in \mathrm{st}_m(\Gamma)$ there is a natural map

$$F_{\Gamma' \to \Gamma} : \overline{\mathcal{M}}_{\Gamma'} \to \mathfrak{M}_{\Gamma}$$

which is just the product of forgetful maps $F_{m_v} : \overline{\mathcal{M}}_{g(v),n(v)+m_v} \to \mathfrak{M}_{g(v),n(v)}$ for each $v \in V(\Gamma) = V(\Gamma')$.

[8]The exceptions occur for $(g, n, m) = (1, 0, 0)$ and for $g = 0, n \leq 2, m \leq 4$. We leave it as an exercise to the reader to work out the codimension in these cases.

Lemma 2.2. For every prestable graph Γ in genus g with n markings and every $m \geq 0$ there is a fibre diagram

$$\begin{tikzcd} \coprod_{\Gamma' \in \mathrm{st}_m(\Gamma)} \overline{\mathcal{M}}_{\Gamma'} \arrow[r, "\coprod \xi_{\Gamma'}"] \arrow[d, "\coprod F_{\Gamma' \to \Gamma}"] & \overline{\mathcal{M}}_{g,n+m} \arrow[d, "F_m"] \\ \mathfrak{M}_\Gamma \arrow[r, "\xi_\Gamma"] & \mathfrak{M}_{g,n} \, . \end{tikzcd} \tag{5}$$

In particular the map $\xi_\Gamma : \mathfrak{M}_\Gamma \to \mathfrak{M}_{g,n}$ is representable, proper and a local complete intersection.

Proof. An object of the fibre product of $\mathfrak{M}_\Gamma$ with $\overline{\mathcal{M}}_{g,n+m}$ over a (connected) scheme S is given by

- a collection of families $(C_v, (p_h)_{h \in H(v)})$ of prestable curves over S for each $v \in V(\Gamma)$,
- a family $(C', p'_1, \ldots, p'_n, q'_1, \ldots, q'_m)$ of stable curves over S,
- an isomorphism (of families of prestable curves)

  $$\varphi : C = \coprod_v C_v / (p_h \sim p_{h'}, (h, h') \in E(\Gamma)) \to C'$$

  satisfying $\varphi(p_i) = p'_i$.

By the assumption that S is connected, for each $j = 1, \ldots, m$ there exists a unique $v = v(j) \in V(\Gamma)$ such that $q'_j \in \varphi(C_v)$ at each point of S. This uses that via φ, the smooth unmarked points of C_v ($v \in V(\Gamma)$) form a disjoint open cover of the smooth unmarked points of $(C', p_1, \ldots, p_n)$ in which q'_j is always contained.

But for j, v as above, we obtain a section $q_j = \varphi^{-1} \circ q'_j : S \to C_v$ landing in the smooth unmarked locus of C_v. Thus for every $v \in V(\Gamma)$ this allows us to define a family

$$\widehat{C}_v = (C_v, (p_h)_{h \in H(v)}, (q_j)_{v(j)=v}) \to S \tag{6}$$

of prestable curves over S. From the fact that $(C', p'_1, \ldots, p'_n, q'_1, \ldots, q'_m)$ is a family of stable curves, it follows that the family (6) is actually a family of stable curves. Then one sees that the collection $(\widehat{C}_v)_{v \in V(\Gamma')}$ is exactly an S-point of one of the spaces $\overline{\mathcal{M}}_{\Gamma'}$ for the suitable $\Gamma' \in \mathrm{st}_m(\Gamma)$ for which the marking q_j is added at the vertex $v(j) \in V(\Gamma') = V(\Gamma)$.

The above operations defines a map from $\mathfrak{M}_\Gamma \times_{\mathfrak{M}_{g,n}} \overline{\mathcal{M}}_{g,n+m}$ to the disjoint union of the $\overline{\mathcal{M}}_{\Gamma'}$ and clearly this disjoint union also maps to the fibre product using the maps $F_{\Gamma' \to \Gamma}$ and $\xi_{\Gamma'}$. One verifies that these are inverse isomorphisms.

Since being proper and being a local complete intersection is local on the target and since the maps F_m form a smooth cover of $\mathfrak{M}_{g,n}$, these properties of ξ_Γ follow from the corresponding properties of the maps $\xi_{\Gamma'}|_{\overline{\mathcal{M}}_{\Gamma'}}$. $\square$

Later we will need some stronger statements about the locus of curves whose prestable graph is exactly a given graph Γ. This locus is a locally closed substack $\mathfrak{M}^\Gamma$ of $\mathfrak{M}_{g,n}$ whose geometric points are precisely the curves $(C, p_1, \ldots, p_n)$ with prestable graph isomorphic to Γ. However, since a family of prestable curves over an arbitrary base does not in general have a well-defined prestable graph, this definition is slightly tricky to write down in a functorial way. Thus we approach the definition from a different angle and then show that it defines the desired locus.

Definition 2.3. Let Γ be a prestable graph in genus g with n markings and let e be the number of edges of Γ. Then we define

$$\mathfrak{M}^\Gamma = \mathrm{im}(\xi_\Gamma) \setminus \bigcup_{\Gamma' : |E(\Gamma')| = e+1} \mathrm{im}(\xi_{\Gamma'}),$$

where im denotes the stack theoretic image and the union goes over prestable graphs Γ' with precisely $e+1$ edges.

By definition, we have that $\mathfrak{M}^\Gamma$ is a locally closed substack of $\mathfrak{M}_{g,n}$. In the following lemma, we check that its geometric points are as desired.

Lemma 2.4. The geometric points of $\mathfrak{M}^\Gamma$ are precisely the $(C, p_1, \ldots, p_n)$ with prestable graph isomorphic to Γ.

Proof. First we note that since ξ_Γ is proper, it is surjective onto its image. Then on the one hand each $(C, p_1, \ldots, p_n)$ with prestable graph isomorphic to Γ is in $\mathfrak{M}^\Gamma$, since it is in the image of ξ_Γ but cannot be in the image of a gluing map for a graph Γ' with more than e edges (since its number of nodes is precisely e). Conversely, let $(C_v)_v = (C_v, (p_h)_{h \in H(v)})_{v \in V(\Gamma)} \in \mathfrak{M}_\Gamma$ be a geometric point. Then if all C_v are smooth, its image $\xi_\Gamma((C_v)_v)$ has prestable graph Γ. On the other hand, if any of the C_v are not smooth, then the prestable graph of $\xi_\Gamma((C_v)_v)$ has at least $e+1$ edges. By contracting all but $e+1$ of them, we obtain one of the prestable graphs Γ' in the definition of $\mathfrak{M}^\Gamma$, and it is easy to see that $\xi_\Gamma((C_v)_v)$ is then in the image of $\xi_{\Gamma'}$. $\square$

We have the following neat description of $\mathfrak{M}^\Gamma$ which is a generalization of [15, Lemma 5.1]. For the statement, let

$$\mathfrak{M}_{g,n}^{\mathrm{sm}} \subset \mathfrak{M}_{g,n}$$

be the open substack where the curve C is smooth. For g, n in the stable range, this is the usual stack $\mathcal{M}_{g,n}$ of smooth curves, but since the latter might be defined to be empty for $2g-2+n<0$ we use the notation $\mathfrak{M}_{g,n}^{\mathrm{sm}}$ for clarity.

Proposition 2.5. For a prestable graph Γ, consider the open substack

$$\mathfrak{M}_\Gamma^{\mathrm{sm}} = \prod_{v \in V(\Gamma)} \mathfrak{M}_{g(v),n(v)}^{\mathrm{sm}} \subset \mathfrak{M}_\Gamma.$$

Then the restriction of the gluing map ξ_Γ to $\mathfrak{M}_\Gamma^{\mathrm{sm}}$ factors through $\mathfrak{M}^\Gamma$ and it is invariant under the natural action of $\mathrm{Aut}(\Gamma)$. The induced map

$$\mathfrak{M}_\Gamma^{\mathrm{sm}}/\mathrm{Aut}(\Gamma) \xrightarrow{\xi_\Gamma} \mathfrak{M}^\Gamma \tag{7}$$

from the quotient stack[9] of $\mathfrak{M}_\Gamma$ by $\mathrm{Aut}(\Gamma)$ is an isomorphism.

Proof. For each point $(C_v)_v = (C_v, (p_h)_{h\in H(v)})_{v\in V(\Gamma)} \in \mathfrak{M}_\Gamma^{\mathrm{sm}}$, the stabilizer $\mathrm{Aut}(\Gamma)_{(C_v)_v}$ under the action of $\mathrm{Aut}(\Gamma)$ is the set of automorphisms of Γ such that there exist compatible isomorphisms of the curves $(C_v, (p_h)_{h\in H(v)})$. The stabilizer group of $[(C_v)_v] \in \mathfrak{M}_\Gamma^{\mathrm{sm}}/\mathrm{Aut}(\Gamma)$ is then an extension of the product of the automorphism groups of the $(C_v, (p_h)_{h\in H(v)})$ by the group $\mathrm{Aut}(\Gamma)_{(C_v)_v}$.

On the other hand, for the curve $(C, p_1, \ldots, p_n)$ obtained from $(C_v)_v$ by gluing and an element $\sigma \in \mathrm{Aut}(\Gamma)_{(C_v)_v}$, the isomorphisms between the curves C_v that are compatible with σ can be glued to an automorphism of $(C, p_1, \ldots, p_n)$. From this it follows that there exists an exact sequence

$$1 \to \prod_{v\in V(\Gamma)} \mathrm{Aut}(C_v, (p_h)_{h\in H(v)}) \to \mathrm{Aut}(C, p_1, \ldots, p_n) \to \mathrm{Aut}(\Gamma)_{(C_v)_v} \to 1\,.$$

From this sequence we see that $\mathrm{Aut}(C, p_1, \ldots, p_n)$ is precisely the group extension defining the stabilizer of $[(C_v)_v] \in \mathfrak{M}_\Gamma^{\mathrm{sm}}/\mathrm{Aut}(\Gamma)$ and hence ξ_Γ induces an isomorphism of each stabilizer. Thus the morphism ξ_Γ in (7) is representable. It is easy to check that it is bijective on geometric points and it is separated by similar argument as in Lemma 2.2. So by [29, 0DUD] it is enough to show that ξ_Γ is an étale morphism to conclude that it is an isomorphism.

Consider the atlas F_m restricted to $\mathfrak{M}^\Gamma$. Since being étale is local on the target, it is enough to show that ξ_Γ is étale on each atlas. On each atlas, the dimension of the fiber is constantly zero. The domain of ξ_Γ is smooth because it can be written as a quotient of a smooth algebraic space by a group scheme ([29, 0DLS]). Following a slight variation of the proof of [3, Proposition 10.11], the stack $\mathfrak{M}^\Gamma$ is also smooth. Since the domain and the target of ξ_Γ are smooth, the 'miracle flatness' ([29, 00R3]) implies that ξ_Γ is flat. Furthermore the morphism is smooth because it is flat and each geometric fiber is smooth. Smooth and quasi-finite morphisms are étale and hence ξ_Γ is an isomorphism. □

2.2 $\mathcal{A}$-valued prestable curves

For each g, n there exists the universal curve $\mathfrak{C}_{g,n} \to \mathfrak{M}_{g,n}$. For later applications, it will be necessary to compute with tautological classes on $\mathfrak{C}_{g,n}$ (and tautological

[9] Since $\mathfrak{M}_\Gamma^{\mathrm{sm}}$ is not an algebraic space, one can either use the notion of group actions and quotients for algebraic stacks defined by Romagny [27] to make sense of the quotient $\mathfrak{M}_\Gamma^{\mathrm{sm}}/\mathrm{Aut}(\Gamma)$ or one observes that $\mathfrak{M}_\Gamma^{\mathrm{sm}}$ is itself a quotient stack and that the action of $\mathrm{Aut}(\Gamma)$ can be lifted compatibly to write $\mathfrak{M}_\Gamma^{\mathrm{sm}}/\mathrm{Aut}(\Gamma)$ again as a quotient stack. See [15, Section 5] for more details.

classes on the universal curve over $\mathfrak{C}_{g,n}$, etc). For the moduli spaces of stable curves, a separate theory is not necessary because the universal curve over $\overline{\mathcal{M}}_{g,n}$ is given by the forgetful map $\overline{\mathcal{M}}_{g,n+1} \to \overline{\mathcal{M}}_{g,n}$. The same is *not* true for $\mathfrak{M}_{g,n}$. Indeed, in Lemma 2.1 we saw that the forgetful morphism $\mathfrak{M}_{g,n+1} \to \mathfrak{M}_{g,n}$ is smooth, so it cannot be the universal curve over $\mathfrak{M}_{g,n}$. In this section we put an additional structure on prestable curves, called the $\mathcal{A}$-value, which allows to give a modular interpretation of the universal curve as a stack of $(n+1)$-pointed curves with additional structure. This realization will be convenient to compute tautological classes on $\mathfrak{C}_{g,n}$.

So let us start by recalling the notion of prestable curves with values in a semigroup $\mathcal{A}$ from [10]. In what follows let $\mathcal{A}$ be a commutative semigroup with unit $\mathbb{0} \in \mathcal{A}$ such that

- $\mathcal{A}$ has indecomposable zero, i.e. for $x, y \in \mathcal{A}$ we have $x + y = \mathbb{0}$ implies $x = \mathbb{0}$, $y = \mathbb{0}$,
- $\mathcal{A}$ has finite decomposition, i.e. for $a \in \mathcal{A}$ the set

$$\{(a_1, a_2) \in \mathcal{A} \times \mathcal{A} : a_1 + a_2 = a\}$$

  is finite.

Classical examples include $\mathcal{A} = \{\mathbb{0}\}$ or $\mathcal{A} = \mathbb{N}$, but later we are going to work with

$$\mathcal{A} = \{\mathbb{0}, \mathbb{1}\} \text{ with } \mathbb{0} + \mathbb{0} = \mathbb{0}, \mathbb{0} + \mathbb{1} = \mathbb{1} + \mathbb{0} = \mathbb{1} + \mathbb{1} = \mathbb{1}.$$

Fixing $\mathcal{A}$ and an element $a \in \mathcal{A}$, Behrend-Manin [8] and Costello [10] define an algebraic stack $\mathfrak{M}_{g,n,a}$. A geometric point corresponds to a prestable curve $(C, p_1, \ldots, p_n)$ together with a map $C_v \mapsto a_{C_v}$ from the set of irreducible components C_v of the normalization of C to $\mathcal{A}$ such that the sum of all a_{C_v} equals a. The curve must satisfy the stability condition that for each C_v either $a_{C_v} \neq \mathbb{0}$ or that C_v is stable, in the sense that for $g(C_v) = 0$ it carries three special points and for $g(C_v) = 1$ it carries at least one special point. Over an arbitrary base scheme the definition of $\mathcal{A}$-valued stable curves needs extra care, see [10, p.569] for details. As an example, for any $\mathcal{A}$ as above and $a = \mathbb{0}$ we obtain $\mathfrak{M}_{g,n,\mathbb{0}} = \overline{\mathcal{M}}_{g,n}$.

Our main motivation for considering the moduli spaces $\mathfrak{M}_{g,n,a}$ is the fact that we have a forgetful morphism $\pi : \mathfrak{M}_{g,n+1,a} \to \mathfrak{M}_{g,n,a}$ making $\mathfrak{M}_{g,n+1,a}$ the universal curve over $\mathfrak{M}_{g,n,a}$. The image of a point

$$(C, p_1, \ldots, p_n, p_{n+1}, (a_{C_v})_v) \in \mathfrak{M}_{g,n+1,a}$$

under π is formed by first forgetting the marked point p_{n+1}. Then, if the component C_v of C containing p_{n+1} then becomes unstable[10], the component C_v of C is contracted. With this notation in place, we summarize the relevant properties of $\mathfrak{M}_{g,n,a}$ from [10].

[10]This happens precisely for $a_{C_v} = \mathbb{0}$ and C_v being of genus 0 with at most two special points apart from p_{n+1}.

Proposition 2.6. The stack $\mathfrak{M}_{g,n,a}$ is a smooth, algebraic stack, locally of finite type and the morphism $\mathfrak{M}_{g,n,a} \to \mathfrak{M}_{g,n}$ forgetting the values in $\mathcal{A}$ is étale and relatively a scheme of finite type. The universal curve over $\mathfrak{M}_{g,n,a}$ is given by the forgetful morphism $\pi : \mathfrak{M}_{g,n+1,a} \to \mathfrak{M}_{g,n,a}$.

Proof. See Proposition 2.0.2 and 2.1.1 from [10]. □

The fact that the universal curve is given by a moduli space of curves with an extra marked point turns out to be very convenient. Indeed, as discussed above this is not the case for the forgetful morphism $\mathfrak{M}_{g,n+1} \to \mathfrak{M}_{g,n}$. Indeed, it is easy to identify $\mathfrak{M}_{g,n+1}$ as the open substack $\mathfrak{M}_{g,n+1} \subset \mathfrak{C}_{g,n}$ given as the complement of the set of markings and nodes.

Many other constructions we saw for prestable curves work in the $\mathcal{A}$-valued setting. For instance, for $g_1 + g_2 = g$, $n_1 + n_2 = n$ and $a_1, a_2 \in \mathcal{A}$ with $a_1 + a_2 = a$, we have a gluing morphism

$$\xi : \mathfrak{M}_{g_1,n_1+1,a_1} \times \mathfrak{M}_{g_2,n_2+1,a_2} \to \mathfrak{M}_{g,n,a}.$$

These gluing maps are again representable, proper and local complete intersections. Indeed, we have a fibre diagram

$$\begin{tikzcd} \coprod_{a_1+a_2=a} \mathfrak{M}_{g_1,n_1+1,a_1} \times \mathfrak{M}_{g_2,n_2+1,a_2} \arrow[r] \arrow[d] & \mathfrak{M}_{g,n,a} \arrow[d] \\ \mathfrak{M}_{g_1,n_1+1} \times \mathfrak{M}_{g_2,n_2+1} \arrow[r] & \mathfrak{M}_{g,n} \end{tikzcd}$$

and the map at the bottom has all these properties by Lemma 2.2. More generally, one defines the notion of an $\mathcal{A}$-valued stable graph and the corresponding gluing map has all the desired properties.

The following result allows us to apply the machinery of Costello to the moduli spaces of prestable curves.

Proposition 2.7. Let $\mathcal{A} = \{\mathbb{0}, \mathbb{1}\}$ with $\mathbb{1} + \mathbb{1} = \mathbb{1}$, then given g, n the subset $\mathfrak{Z}_{g,n} \subset \mathfrak{M}_{g,n,\mathbb{1}}$ of $\mathcal{A}$-valued curves $(C, p_1, \ldots, p_n; (a_{C_v})_v)$ such that one of the values a_{C_v} equals $\mathbb{0}$ is closed. Let $\mathfrak{U}_{g,n} = \mathfrak{M}_{g,n,\mathbb{1}} \setminus \mathfrak{Z}_{g,n}$ be its complement. Then the composition

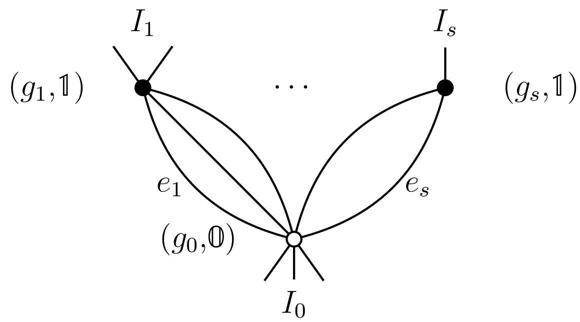
$$\mathfrak{U}_{g,n} \hookrightarrow \mathfrak{M}_{g,n,\mathbb{1}} \to \mathfrak{M}_{g,n}$$

of the inclusion of $\mathfrak{U}_{g,n}$ with the morphism $\mathfrak{M}_{g,n,\mathbb{1}} \to \mathfrak{M}_{g,n}$ forgetting the $\mathcal{A}$-values defines an isomorphism $\mathfrak{U}_{g,n} \cong \mathfrak{M}_{g,n}$.

Proof. The underlying reason why $\mathfrak{Z}_{g,n}$ is closed is that $\mathbb{0}$ is indecomposable in $\mathcal{A}$: given a curve $(C, p_1, \ldots, p_n; (a_{C_v})_v)$ such that some $a_{C_v} = \mathbb{0}$, any degeneration of this curve still has some component with value $\mathbb{0}$ since in a degeneration of C_v, a_{C_v} must distribute to the components to which C_v degenerates.

More concretely, we can write $\mathfrak{Z}_{g,n}$ as the union of images of gluing maps ξ_Γ for suitable $\mathcal{A}$-valued prestable graphs Γ. Indeed, we exactly have to remove the images of ξ_Γ for Γ of the form

$\Gamma =$



(8)

where $s \geq 1$, $e_1, \ldots, e_s \in \mathbb{Z}_{>0}$,

$$g_0 + g_1 + \ldots + g_s = g + \sum_{i=1}^{s} e_i - 1$$

and $I_0 \coprod I_1 \coprod \ldots \coprod I_s = \{1, \ldots, n\}$. Note that for this locus to be nonempty, we must require $g_0 > 0$ or $|I_0| + \sum_i e_i > 2$.

While the image of each ξ_Γ is closed, we use infinitely many of them. But in the open exhaustion of $\mathfrak{M}_{g,n,\mathbb{1}}$ by the substacks of curves with at most ℓ nodes, each of these open substacks only intersects finitely many of the images of ξ_Γ nontrivially, so still the union of their images is closed.

The fact that $\mathfrak{U}_{g,n} \to \mathfrak{M}_{g,n}$ is an isomorphism can be seen in different ways: its inverse is just given by the functor sending each prestable curve $(C, p_1, \ldots, p_n)$ to itself with value $a_{C_v} = \mathbb{1}$ on each component, i.e.

$$\mathfrak{M}_{g,n} \to \mathfrak{U}_{g,n}, (C, p_1, \ldots, p_n) \mapsto (C, p_1, \ldots, p_n; (\mathbb{1})_v).$$

Alternatively one observes that $\mathfrak{U}_{g,n} \to \mathfrak{M}_{g,n}$ is étale, representable and a bijection on geometric points. $\square$

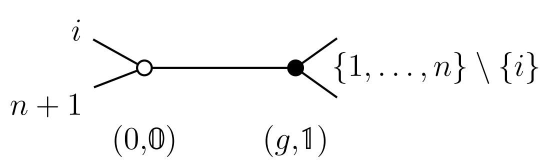
Corollary 2.8. The universal curve $\mathfrak{C}_{g,n} \to \mathfrak{M}_{g,n}$ is given by the morphism

$$\pi : \mathfrak{M}_{g,n+1,\mathbb{1}} \setminus \pi^{-1}(\mathfrak{Z}_{g,n}) \to \mathfrak{M}_{g,n,\mathbb{1}} \setminus \mathfrak{Z}_{g,n}$$

forgetting the marking $n+1$ and contracting the component containing it if this component becomes unstable. The $\mathcal{A}$-valued prestable graphs Γ appearing in $\mathfrak{Z}_{g,n+1}$ but *not* contained in $\pi^{-1}(\mathfrak{Z}_{g,n})$ are exactly of one of the three following forms

- for $i = 1, \ldots, n$ the graphs

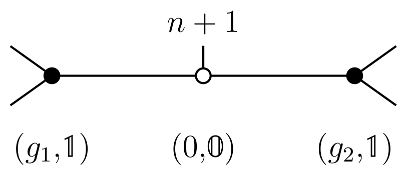
$\Gamma =$



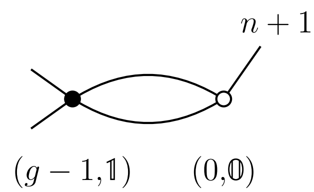
(9)

corresponding to the n sections of the universal curve $\pi : \mathfrak{C}_{g,n} \to \mathfrak{M}_{g,n}$,

- boundary divisors with edge subdivided, inserting a genus zero, value $\mathbb{0}$ vertex carrying $n+1$



where $g_1 + g_2 = g$ and



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corresponding to the locus of nodes inside the universal curve $\pi : \mathfrak{C}_{g,n} \to \mathfrak{M}_{g,n}$.

Corollary 2.8 shows that in order to develop the intersection theory of $\mathfrak{M}_{g,n}$ and $\mathfrak{C}_{g,n}$, it suffices to consider the general case of the intersection theory of $\mathfrak{M}_{g,n,\mathbb{1}}$ (or even more generally, $\mathfrak{M}_{g,n,a}$ for any semigroup $\mathcal{A}$ and $a \in \mathcal{A}$).

3 Chow groups and the tautological ring of $\mathfrak{M}_{g,n}$

3.1 Definitions

In this paper, we want to study the Chow groups (with $\mathbb{Q}$-coefficients) of the stacks $\mathfrak{M}_{g,n}$ (and more generally, the stacks $\mathfrak{M}_{g,n,a}$ for some element $a \in \mathcal{A}$ in a semigroup $\mathcal{A}$).

To define these Chow groups, recall that in [21] Kresch constructed Chow groups $\mathrm{CH}_*(\mathcal{X})$ for algebraic stacks $\mathcal{X}$ of finite type over a field k. Moreover, there is an intersection product on $\mathrm{CH}_*(\mathcal{X})$ when $\mathcal{X}$ is smooth and *stratified by global quotient stacks*[11], see [21, Theorem 2.1.12]. This last condition can be checked point-wise: a reduced stack $\mathcal{X}$ is stratified by global quotient stacks if and only if the stabilizers of geometric points of $\mathcal{X}$ are affine ([21, Proposition 3.5.9]).

Now the spaces $\mathfrak{M}_{g,n,a}$ are in general not of finite type (only locally of finite type) and so we need to extend the definition of Chow groups above. Assume that $\mathfrak{M}$ is an algebraic stack, locally of finite type over a field k. Choose a directed system[12]

[11] This means that there exists a stratification by locally closed substacks which are each isomorphic to a global quotient of an algebraic space by a linear algebraic group.

[12] Recall that this means that for all $\mathcal{U}_i, \mathcal{U}_j$ there exists a $\mathcal{U}_k$ containing both of them.

$(\mathcal{U}_i)_{i\in I}$ of finite type open substacks of $\mathfrak{M}$ whose union is all of $\mathfrak{M}$. Then we set

$$\mathrm{CH}_*(\mathfrak{M}) = \varprojlim_{i\in I} \mathrm{CH}_*(\mathcal{U}_i),$$

where for $\mathcal{U}_i \subseteq \mathcal{U}_j$ the transition map $\mathrm{CH}_*(\mathcal{U}_j) \to \mathrm{CH}_*(\mathcal{U}_i)$ is given by the restriction to $\mathcal{U}_i$. In other words, we have

$$\mathrm{CH}_*(\mathfrak{M}) = \{(\alpha_i)_{i\in I} : \alpha_i \in \mathrm{CH}_*(\mathcal{U}_i), \alpha_j|_{\mathcal{U}_i} = \alpha_i \text{ for } \mathcal{U}_i \subseteq \mathcal{U}_j\}.$$

We give the details of this definition in Appendix A and show that the Chow groups of locally finite type stacks inherit all the usual properties (e.g. flat pullback, projective pushforward, Chern classes of vector bundles and Gysin pullbacks) of the Chow groups from [21]. Moreover, if $\mathfrak{M}$ is smooth and has affine stabilizer groups at geometric points, the intersection products on the groups $\mathrm{CH}_*(\mathcal{U}_i)$ give rise to an intersection product on $\mathrm{CH}_*(\mathfrak{M})$. In this case, for $\mathfrak{M}$ equidimensional we often use the cohomological degree convention

$$\mathrm{CH}^*(\mathfrak{M}) = \mathrm{CH}_{\dim \mathfrak{M} - *}(\mathfrak{M}).$$

Proposition 3.1. Let $g, n \geq 0$, let $\mathcal{A}$ be a semigroup with indecomposable zero and finite decomposition as in Section 2.2 and $a \in \mathcal{A}$. Then the stacks $\mathfrak{M}_{g,n}$ and $\mathfrak{M}_{g,n,a}$ have well-defined Chow groups $\mathrm{CH}_*(\mathfrak{M}_{g,n})$ and $\mathrm{CH}_*(\mathfrak{M}_{g,n,a})$. For $(g,n) \neq (1,0)$ the stabilizer groups of all geometric points of $\mathfrak{M}_{g,n}$ and $\mathfrak{M}_{g,n,a}$ are affine and so the Chow groups have an intersection product.

Proof. The stacks $\mathfrak{M}_{g,n}$ and $\mathfrak{M}_{g,n,a}$ are locally of finite type (and smooth) by Proposition 2.6 and thus satisfy the conditions of Definition A.1 from the appendix. For the existence of intersection products, we need to check that geometric points have affine stabilizers. The stabilizer group of such a prestable curve is a finite extension of the automorphism groups of its components. The only non-finite automorphism groups that can occur here are in genus 0 (where they are subgroups of PGL_2 and thus affine) and in genus 1 with no special points. Since the prestable curves are assumed to be connected, the last case can only occur for $(g,n) = (1,0)$. ◻

Now recall from Definition 1.3 that the tautological rings $(\mathrm{R}^*(\mathfrak{M}_{g,n,a}))_{g,n,a}$ are defined as the smallest system of $\mathbb{Q}$-subalgebras with unit of the Chow rings $(\mathrm{CH}^*(\mathfrak{M}_{g,n,a}))_{g,n,a}$ closed under taking pushforwards by the natural forgetful and gluing maps.

We recall the following particular examples of tautological classes:

Definition 3.2. Let $\pi : \mathfrak{M}_{g,n+1,a} \to \mathfrak{M}_{g,n,a}$ be the universal curve over $\mathfrak{M}_{g,n,a}$ and for $i = 1, \ldots, n$ let $\sigma_i : \mathfrak{M}_{g,n,a} \to \mathfrak{M}_{g,n+1,a}$ be the section corresponding to the i-th marked points. Let ω_π be the relative canonical line bundle on $\mathfrak{M}_{g,n+1,a}$. Then we define

$$\psi_i = \sigma_i^* c_1(\omega_\pi) \in \mathrm{CH}^1(\mathfrak{M}_{g,n,a}) \text{ for } i = 1, \ldots, n \tag{10}$$

and

$$\kappa_m = \pi_*\big(\psi_{n+1}^{m+1}\big) \in \mathrm{CH}^m(\mathfrak{M}_{g,n,a})\,. \tag{11}$$

Definition 3.3. Let Γ be an $\mathcal{A}$-valued prestable graph in genus g with n markings with total value $a \in \mathcal{A}$. For $\mathfrak{M}_\Gamma = \prod_{v \in V(\Gamma)} \mathfrak{M}_{g(v),n(v),a(v)}$ a decoration α on Γ is an element of $\mathrm{CH}^*(\mathfrak{M}_\Gamma)$ given by a product of κ and ψ-classes on the factors $\mathfrak{M}_{g(v),n(v),a(v)}$ of $\mathfrak{M}_\Gamma$. Thus it has the form

$$\alpha = \prod_{v \in V} \left(\prod_{i \in H(v)} \psi_{v,i}^{a_i} \prod_{a=1}^{m_v} \kappa_{v,a}^{b_{v,a}} \right) \in \mathrm{CH}^*(\mathfrak{M}_\Gamma) \tag{12}$$

where $a_i, b_{v,a} \geq 0$ and $m_v \geq 0$ are some integers. We define the decorated stratum class $[\Gamma, \alpha]$ as the pushforward

$$[\Gamma, \alpha] = (\xi_\Gamma)_* \alpha \in \mathrm{CH}^*(\mathfrak{M}_{g,n,a}).$$

One of the main goals of this section is to show that the set of tautological classes $\mathrm{R}^*(\mathfrak{M}_{g,n,a}) \subseteq \mathrm{CH}^*(\mathfrak{M}_{g,n,a})$ is the $\mathbb{Q}$-linear span of all classes $[\Gamma, \alpha]$.

Remark 3.4. We define tautological classes on the spaces $\mathfrak{M}_{g,n}$ and $\mathfrak{C}_{g,n}$ by seeing these stacks as open subsets of $\mathfrak{M}_{g,n,\mathbb{1}}$ and $\mathfrak{M}_{g,n+1,\mathbb{1}}$ for $\mathcal{A} = \{\mathbb{0}, \mathbb{1}\}$ as in Corollary 2.8. Then tautological classes on $\mathfrak{M}_{g,n}$ and $\mathfrak{C}_{g,n}$ are given by the restrictions of tautological classes on $\mathfrak{M}_{g,n,\mathbb{1}}$ and $\mathfrak{M}_{g,n+1,\mathbb{1}}$.

From the point of view of decorated strata classes, note that for $\mathfrak{M}_{g,n}$ only $\mathcal{A}$-valued prestable graphs where all values are $\mathbb{1}$ can contribute (and these are in natural bijections with prestable graphs without valuation). On the other hand, for $\mathfrak{C}_{g,n}$ we can have vertices v with value $\mathbb{0}$ contributing nontrivial classes. This happens exactly for the graphs shown in Corollary 2.8, corresponding to the universal sections of $\mathfrak{C}_{g,n} \to \mathfrak{M}_{g,n}$ and the loci of nodes inside $\mathfrak{C}_{g,n}$ over boundary strata of $\mathfrak{M}_{g,n}$.

3.2 Intersections and functoriality of tautological classes

In this section we describe how the classes $[\Gamma, \alpha]$ behave under taking intersections as well as pullbacks and pushforwards under natural gluing, forgetful and stabilization maps.

Pushforwards by gluing maps

Pushing forward by gluing maps is by far the easiest operation: given an $\mathcal{A}$-valued graph Γ_0 and classes $[\Gamma_v, \alpha_v] \in \mathrm{R}^*(\mathfrak{M}_{g(v),n(v),a(v)})$ for $v \in V(\Gamma_0)$, the pushforward of the class

$$\prod_{v \in V(\Gamma)} [\Gamma_v, \alpha_v] \in \mathrm{CH}^*(\mathfrak{M}_\Gamma)$$

is given by $[\Gamma, \alpha]$ where Γ is obtained by gluing the Γ_i into the vertices of the outer graph Γ_0 and α is obtained by combining the decorations α_v using that $V(\Gamma) = \coprod_{v \in V(\Gamma_0)} V(\Gamma_v)$.

Pullbacks by gluing maps and intersection products

The next natural question is how a class $[B, \beta]$ pulls back along a gluing morphism ξ_A for an $\mathcal{A}$-valued graph A. This operation allows a purely combinatorial description, generalizing the description in $\overline{\mathcal{M}}_{g,n}$ from [19] (and already discussed for graphs A with exactly one edge in [10, Section 4]). As combinatorial preparation, we recall the notion of morphisms of $\mathcal{A}$-valued stable graphs.

Definition 3.5. An A-*structure* on an $\mathcal{A}$-valued prestable graph Γ (write $\Gamma \to A$) is a choice of subgraphs Γ_v of Γ such that Γ can be constructed by replacing each vertex v of A by the corresponding $\mathcal{A}$-valued graph Γ_v. More precisely, the data of $\Gamma \to A$ is given by maps

$$V(\Gamma) \to V(A) \text{ and } H(A) \to H(\Gamma).$$

They must satisfy that $V(\Gamma) \to V(A)$ is surjective, such that the preimage of $v \in V(A)$ are the vertices of a subgraph Γ_v of Γ with total $\mathcal{A}$-value a_v. The map $H(A) \to H(\Gamma)$ of half-edges in the opposite direction is required to be injective, and allows one to see identify half-edges $h \in H(v)$ of A with legs of the graph Γ_v. These maps must respect the incidence relation of half-edges and vertices and the pairs of half-edges forming edges. In particular, the injection of half-edges allows to see the set of edges $E(A)$ of A as a subset of the set of edges $E(\Gamma)$ of Γ (see e.g. [28, Definition 2.5] for more details in the case of stable graphs).

Given an A-structure $\Gamma \to A$, there exists a gluing morphism

$$\xi_{\Gamma \to A} \colon \mathfrak{M}_\Gamma \to \mathfrak{M}_A\,.$$

For a decoration α on $\mathfrak{M}_A$ as in (12), it is easy to describe $\xi^*_{\Gamma \to A}\alpha$ using that

- $\xi^*_{\Gamma \to A}\psi_{v,i} = \psi_{w,j}$ if $\Gamma \to A$ maps half-edge i in A to half-edge j in Γ,
- $\xi^*_{\Gamma \to A}\kappa_{v,\ell} = \sum_{w \mapsto v} \kappa_{w,\ell}$, where the sum goes over vertices w of Γ mapping to the vertex v of A on which $\kappa_{v,\ell}$ lives.

Both these properties follow immediately from the definitions[13] of κ and ψ-classes.

Let $f_A : \Gamma \to A$, $f_B : \Gamma \to B$ be A and B-structures on the prestable graph Γ. The pair $f = (f_A, f_B)$ is called a *generic* (A, B)-*structure* $f = (f_A, f_B)$ on Γ if every half-edge of Γ corresponds to a half-edge of A or a half-edge of B. Given a second

[13] For the pullback of κ-classes the proof also uses Proposition 3.8 below.

(A,B)-structure $f' = (f'_A : \Gamma' \to A, f'_B : \Gamma' \to B)$, an isomorphism from f to f' is an isomorphism $\Gamma \to \Gamma'$ commuting with the maps to A, B. Let $\mathcal{G}_{A,B}$ be the set of isomorphism classes of prestable graphs Γ together with a generic (A,B)-structures on Γ.

Proposition 3.6. Let A, B be $\mathcal{A}$-valued prestable graphs for $\mathfrak{M}_{g,n,a}$, then the fibre product of the gluing maps $\xi_A : \mathfrak{M}_A \to \mathfrak{M}_{g,n,a}$ and $\xi_B : \mathfrak{M}_B \to \mathfrak{M}_{g,n,a}$ is given by a disjoint union

$$\begin{tikzcd} \coprod_{\Gamma \in \mathcal{G}_{A,B}} \mathfrak{M}_\Gamma \arrow[r, "\xi_{\Gamma \to B}"] \arrow[d, "\xi_{\Gamma \to A}"] & \mathfrak{M}_B \arrow[d, "\xi_B"] \\ \mathfrak{M}_A \arrow[r, "\xi_A"] & \mathfrak{M}_{g,n,a} \end{tikzcd} \tag{13}$$

of spaces $\mathfrak{M}_\Gamma$ for the set of isomorphism classes of generic (A,B)-structures on prestable graphs Γ. The top Chern class of the excess bundle

$$E_\Gamma = \xi^*_{\Gamma \to A} \mathcal{N}_{\xi_A} / \mathcal{N}_{\xi_{\Gamma \to B}} \tag{14}$$

is given by

$$c_{\mathrm{top}}(E_\Gamma) = \prod_{e=(h,h') \in E(A) \cap E(B) \subset E(\Gamma)} -\psi_h - \psi_{h'}, \tag{15}$$

where the product is over the edges of Γ coming both from edges of A and edges of B in the generic (A,B)-structure.

Proof. The proof from [19, Proposition 9] of the analogous result for the moduli spaces of stable curves goes through verbatim (see also [28, Section 2] for a more detailed version of the argument). □

Using the projection formula, we can then also intersect tautological classes.

Corollary 3.7. Given decorated stratum classes $[A, \alpha]$, $[B, \beta]$ on $\mathfrak{M}_{g,n,a}$, their product is given by

$$[A, \alpha] \cdot [B, \beta] = \sum_{\Gamma \in \mathcal{G}_{A,B}} (\xi_\Gamma)_* \left(\xi^*_{\Gamma \to A} \alpha \cdot \xi^*_{\Gamma \to B} \beta \cdot c_{\mathrm{top}}(E_\Gamma) \right). \tag{16}$$

Pushforwards and pullbacks by forgetful maps of points

In this section, we look at the behaviour of tautological classes under the forgetful map $\pi : \mathfrak{M}_{g,n+1,a} \to \mathfrak{M}_{g,n,a}$, which is the universal curve over $\mathfrak{M}_{g,n,a}$. As such, it is both flat and proper, so we can compute pullbacks as well as pushforwards. We will start with pullbacks.

Proposition 3.8. Given an $\mathcal{A}$-valued prestable graph Γ for $\mathfrak{M}_{g,n,a}$, we have a commutative diagram

$$\begin{tikzcd} \coprod_{v\in V(\Gamma)} \mathfrak{M}_{\widehat{\Gamma}_v} \arrow[r, "\xi_{\widehat{\Gamma}_v}"] \arrow[d, "\pi_v"] & \mathfrak{M}_{g,n+1,a} \arrow[d, "\pi"] \\ \mathfrak{M}_\Gamma \arrow[r, "\xi_\Gamma"] & \mathfrak{M}_{g,n,a} \end{tikzcd} \tag{17}$$

where the graph $\widehat{\Gamma}_v$ is obtained from Γ by adding the marking $n+1$ at vertex v and the map π_v is the identity on the factors of $\mathfrak{M}_{\widehat{\Gamma}_v}$ for vertices $w \neq v$ and the forgetful map of marking $n+1$ at the vertex v. The induced map

$$\coprod_{v\in V(\Gamma)} \mathfrak{M}_{\widehat{\Gamma}_v} \to \mathfrak{M}_\Gamma \times_{\mathfrak{M}_{g,n,a}} \mathfrak{M}_{g,n+1,a} \tag{18}$$

satisfies that the fundamental class on the left pushes forward to the fundamental class on the right.

Proof. This follows from the definition of the gluing map ξ_Γ: giving the map ξ_Γ is the same as giving the universal curve over $\mathfrak{M}_\Gamma$ and this curve is obtained by gluing the universal curves $\mathfrak{M}_{\widehat{\Gamma}_v}$ over the various factors along the half-edges connected in Γ. The map (18) is obtained by taking, for each edge $\{h_1, h_2\} \in E(\Gamma)$ the loci inside $\mathfrak{M}_{\widehat{\Gamma}_{v_i}}$ where marking p_{n+1} and marking q_{h_i} are on a contracted component and identifying them. Thus, if p_{n+1} is not on a contracted component, the map is an isomorphism in a neighborhood. Therefore the map (18) is an isomorphism at the general point of each component of the right hand side and the fundamental class pushes forwards to the fundamental class. □

Corollary 3.9. Given a tautological class $[\Gamma, \alpha]$ write $\alpha = \prod_{v\in V(\Gamma)} \alpha_v$ with α_v the factors of α located at vertex v of Γ. Then we have

$$\pi^*[\Gamma, \alpha] = \sum_{v\in V(\Gamma)} [\widehat{\Gamma}_v, (\pi_v^*\alpha_v) \cdot \prod_{w\neq v} \alpha_w].$$

Proof. The class $[\Gamma, \alpha]$ is represented by $\xi_{\Gamma*}(\alpha \cap [\mathfrak{M}_\Gamma])$ where α is an operational Chow class in $\mathrm{CH}^*_{\mathrm{OP}}(\mathfrak{M}_\Gamma)$. We refer the reader to Appendix C for definitions and properties of these operational classes. By Proposition 3.8 the diagram (17) together with the map (18) satisfies assumptions in Lemma C.8. Therefore the equality follows from Lemma C.8. □

The above corollary shows that to finish our understanding of pullbacks of tautological classes, it suffices to understand how κ and ψ-classes pull back.

Proposition 3.10. For the universal curve morphism $\pi : \mathfrak{M}_{g,n+1,a} \to \mathfrak{M}_{g,n,a}$ we have

$$\pi^*\psi_i = \psi_i - D_{i,n+1}\,, \tag{19}$$
$$\pi^*\kappa_a = \kappa_a - \psi_{n+1}^a\,, \tag{20}$$

where $D_{i,n+1} \subset \mathfrak{M}_{g,n+1,a}$ is the image of the section σ_i of π corresponding to the i-th marked point. It can be seen as the tautological class corresponding to the (undecorated) graph (9) above.

Proof. The statement is a generalization of the classical pullback formulas for $\overline{\mathcal{M}}_{g,n}$ (which are the case $\mathcal{A} = \{\mathbb{0}\}$). A convenient way to prove it is to use that

$$\psi_i = -\pi_*(D_{i,n+1}^2)\,. \tag{21}$$

To show this, we note that σ_i can be identified with the gluing map

$$\sigma_i \colon \mathfrak{M}_{g,n,a} \times \overline{\mathcal{M}}_{0,\{\bullet,i,n+1\},\mathbb{0}} \to \mathfrak{M}_{g,n+1,a}\,,$$

where we glue the i-th marking on $\mathfrak{M}_{g,n,a}$ with the marking $\bullet$ on $\overline{\mathcal{M}}_{0,\{\bullet,i,n+1\},\mathbb{0}}$. Then indeed the locus $D_{i,n+1}$ is the image of the above gluing map (similar to the usual case of stable maps) and equation (21) follows from Corollary 3.7. On the other hand, it can also be seen directly from the fact that σ_i is a closed embedding with normal bundle $\sigma_i^*(\omega_\pi^\vee)$.

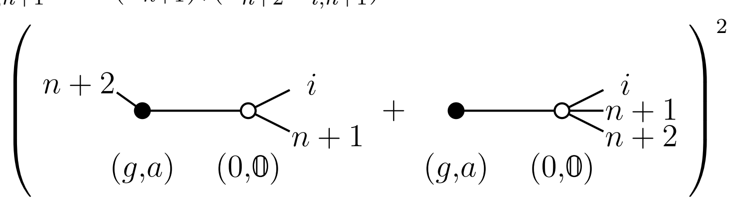
Now we have a commutative diagram

$$\begin{tikzcd} \mathfrak{M}_{g,n+2,a} \arrow[r, "\pi_{n+1}"] \arrow[d, "\pi_{n+2}"] & \mathfrak{M}_{g,n+1,a} \arrow[d, "\pi"] \\ \mathfrak{M}_{g,n+1,a} \arrow[r, "\pi"] & \mathfrak{M}_{g,n,a} \end{tikzcd} \tag{22}$$

and the space in the upper left maps birationally to the fibre product of the two forgetful maps. Then

$$\pi^*\psi_i = -\pi^*\pi_* D_{i,n+1}^2 = -(\pi_{n+1})_*(\pi_{n+2}^* D_{i,n+1}^2)$$

$$= -(\pi_{n+1})_*$$



$$= \psi_i - 2D_{i,n+1} + D_{i,n+1} = \psi_i - D_{i,n+1}.$$

Similarly, using the same diagram, the definition of κ_a and the pullback formula for ψ, one concludes the pullback formula for κ_a. $\square$

We now turn to the question how to push forward tautological classes $[\Gamma, \alpha] \in \mathrm{R}^*(\mathfrak{M}_{g,n+1,a})$ under the map π.

Proposition 3.11. Let $[\Gamma, \alpha] \in \mathrm{R}^*(\mathfrak{M}_{g,n+1,a})$ with $\alpha = \prod_{v \in V(\Gamma)} \alpha_v$. Let $v \in V(\Gamma)$ be the vertex incident to $n+1$ and let Γ' be the graph obtained from Γ by forgetting the marking $n+1$ and stabilizing if the vertex v becomes unstable. There are two cases:

- if the vertex v remains stable, then
$$\pi_*[\Gamma, \alpha] = (\xi_{\Gamma'})_* \left((\pi_v)_* \alpha_v \cdot \prod_{w \neq v} \alpha_w \right),$$
where π_v is the forgetful map of marking $n+1$ of vertex v.
- if the vertex v becomes unstable, then $g(v) = 0$, $n(v) = 3$ and $a(v) = \mathbb{0}$. If $\alpha_v \neq 1$ then $[\Gamma, \alpha] = 0$. Otherwise, we have
$$\pi_*[\Gamma, \alpha] = [\Gamma', \prod_{w \neq v} \alpha_w] \,.$$

Proof. The result follows from the fact that the composition of the gluing map ξ_Γ and the forgetful map π factors through the gluing map $\xi_{\Gamma'}$ downstairs. In the second part, we use that $\mathfrak{M}_{0,3,\mathbb{0}} = \overline{\mathcal{M}}_{0,3} = \operatorname{Spec} k$, so any nontrivial decoration by κ and ψ-classes on this space vanishes. $\square$

The proposition allows us to reduce to computing forgetful pushforwards of products of κ and ψ-classes. As in the case of $\overline{\mathcal{M}}_{g,n}$, these can be computed using the projection formula. Indeed, given a product
$$\alpha = \prod_a \kappa_a^{e_a} \cdot \prod_{i=1}^n \psi_i^{\ell_i} \cdot \psi_{n+1}^{\ell_{n+1}} \in \mathrm{R}^*(\mathfrak{M}_{g,n+1,a})$$
we can use Proposition 3.10 and the known intersection formulas on $\mathfrak{M}_{g,n+1,a}$ to write it as
$$\alpha = \pi^* \left(\prod_a \kappa_a^{e_a} \cdot \prod_{i=1}^n \psi_i^{\ell_i} \right) \cdot \psi_{n+1}^{\ell_{n+1}} + \text{boundary terms.}$$
Using the projection formula we conclude
$$\pi_*(\alpha) = \left(\prod_a \kappa_a^{e_a} \cdot \prod_{i=1}^n \psi_i^{\ell_i} \right) \cdot \kappa_{\ell_{n+1}-1} + \pi_*(\text{boundary terms}),$$
where $\kappa_0 = 2g-2+n$ and $\kappa_{-1} = 0$. The boundary terms are handled by induction on the degree together with Proposition 3.11.

Together with the previous results of this section, this shows that the $\mathbb{Q}$-linear span of the decorated strata classes $[\Gamma, \alpha]$ in $\mathrm{CH}^*(\mathfrak{M}_{g,n,a})$ is closed under intersections as well as pushforwards under gluing and forgetful maps. Thus, by definition, it equals the tautological ring of $\mathfrak{M}_{g,n,a}$, so that we finished the proof of Theorem 1.4.

Pullbacks by forgetful maps of $\mathcal{A}$-values

Proposition 3.12. For the map $F_{\mathcal{A}} : \mathfrak{M}_{g,n,a} \to \mathfrak{M}_{g,n}$ forgetting the $\mathcal{A}$-values on all components of the curve, without stabilizing, we have

$$F_{\mathcal{A}}^*[\Gamma, \alpha] = \sum_{\substack{(a_v)_{v \in V(\Gamma)} \\ \sum_v a_v = a}} [\Gamma_{(a_v)_v}, \alpha] \,,$$

where the sum is over tuples $(a_v)_v$ of elements of $\mathcal{A}$ summing to a, such that the $\mathcal{A}$-valuation $v \mapsto a_v$ on the vertices v of Γ gives a well-defined $\mathcal{A}$-valued graph $\Gamma_{(a_v)_v}$.

Proof. The fibre product of $F_{\mathcal{A}}$ and a gluing map ξ_Γ is the disjoint union of the gluing maps $\xi_{\Gamma_{(a_v)_v}}$. A short computation shows that $F_{\mathcal{A}}^* \psi_i = \psi_i$ and $F_{\mathcal{A}}^* \kappa_a = \kappa_a$. □

Pullback by stabilization map

We saw before that the stabilization morphism $\mathrm{st} : \mathfrak{M}_{g,n} \to \overline{\mathcal{M}}_{g,n}$ is flat, so we can ask how to pull back tautological classes along this morphism. We start by computing the pullback of gluing maps under st.

Proposition 3.13. Given a stable graph Γ in genus g with n marked points (for $2g - 2 + n > 0$), we have a commutative diagram

$$\begin{tikzcd} \mathfrak{M}_\Gamma \arrow[r, "\prod_v \mathrm{st}_v"] \arrow[d, "\xi_\Gamma"] & \overline{\mathcal{M}}_\Gamma \arrow[d, "\xi_\Gamma"] \\ \mathfrak{M}_{g,n} \arrow[r, "\mathrm{st}"] & \overline{\mathcal{M}}_{g,n} \end{tikzcd} \tag{23}$$

where $\mathrm{st}_v : \mathfrak{M}_{g(v),n(v)} \to \overline{\mathcal{M}}_{g(v),n(v)}$ is the stabilization morphism at vertex v. Moreover, the induced map

$$\mathfrak{M}_\Gamma \to \mathfrak{M}_{g,n} \times_{\overline{\mathcal{M}}_{g,n}} \overline{\mathcal{M}}_\Gamma \tag{24}$$

is proper and birational. In particular

$$\mathrm{st}^* \left[\Gamma, \prod_v \alpha_v \right] = (\xi_\Gamma)_* \left(\prod_v \mathrm{st}_v^* \alpha_v \right) . \tag{25}$$

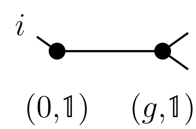
Proof. The commutativity of the diagram (23) follows from the definition of the stabilization. The map (24) is easily seen to be birational and its properness follows from the diagram

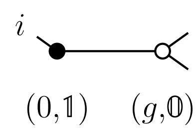
$$\begin{tikzcd} \mathfrak{M}_\Gamma \arrow[rr] \arrow[dr, "\xi_\Gamma"'] & & \mathfrak{M}_{g,n} \times_{\overline{\mathcal{M}}_{g,n}} \overline{\mathcal{M}}_\Gamma \arrow[dl, "\mathrm{pr}_1"] \\ & \mathfrak{M}_{g,n} & \end{tikzcd}$$

and the cancellation property of proper morphisms (in the diagram, the map ξ_Γ and pr_1 are proper). Equation (25) again follows by an application of Lemma C.8. $\square$

The proposition above reduces the pullback of tautological classes under st to computing the pullback of κ and ψ-classes.

Proposition 3.14. Let g, n with $2g-2+n > 0$ and let $\mathcal{A} = \{\mathbb{0}, \mathbb{1}\}$. Then for $1 \le i \le n$ and the stabilization map $\mathrm{st}_{\mathcal{A}} : \mathfrak{M}_{g,n,\mathbb{1}} \to \overline{\mathcal{M}}_{g,n}$ we have

$$\mathrm{st}_{\mathcal{A}}^* \psi_i = \psi_i - $$ 

$$+$$ 

. (26)

Proof. Consider the commutative diagram

$$\begin{tikzcd} \mathfrak{M}_{g,n+1,\mathbb{1}} \arrow[r, "c"] \arrow[dr, "\pi_{\mathcal{A}}"] & \mathfrak{M}_{g,n,\mathbb{1}} \times_{\overline{\mathcal{M}}_{g,n}} \overline{\mathcal{M}}_{g,n+1} \arrow[r, "\widehat{\mathrm{st}}_{\mathcal{A}}"] \arrow[d] & \overline{\mathcal{M}}_{g,n+1} \arrow[d, "\pi"] \\ & \mathfrak{M}_{g,n,\mathbb{1}} \arrow[r, "\mathrm{st}_{\mathcal{A}}"] & \overline{\mathcal{M}}_{g,n} \end{tikzcd} \tag{27}$$

where the right square is cartesian and the map c is the map contracting the unstable components of the universal curve $\mathfrak{M}_{g,n+1,\mathbb{1}} \to \mathfrak{M}_{g,n,\mathbb{1}}$. By the cancellation property of proper morphisms, the map c is proper and easily seen to be birational.

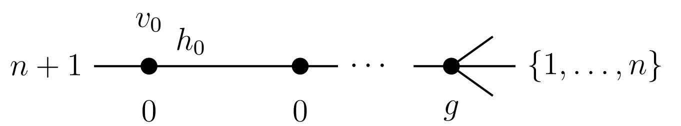
For computing the pullback of ψ_i under $\mathrm{st}_{\mathcal{A}}$, we use that $\psi_i = -\pi_*(D_{i,n+1}^2)$ on $\overline{\mathcal{M}}_{g,n}$, where $D_{i,n+1} \subset \overline{\mathcal{M}}_{g,n+1}$ is the image of the i-th section. By Lemma C.8, we have

$$\mathrm{st}_{\mathcal{A}}^* \psi_i = -\mathrm{st}_{\mathcal{A}}^* \pi_*(D_{i,n+1}^2) = (\pi_{\mathcal{A}})_* \left((c \circ \widehat{\mathrm{st}}_{\mathcal{A}})^* D_{i,n+1} \right)^2 .$$

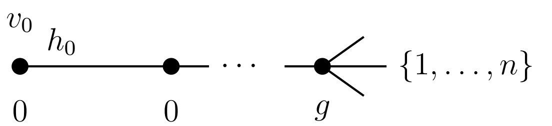
The composition $c \circ \widehat{\mathrm{st}}_{\mathcal{A}}$ is just the usual stabilization map and the pullback of $D_{i,n+1}$ under this map is the sum of three boundary divisors of $\mathfrak{M}_{g,n+1,\mathbb{1}}$: their underlying graph is the same as for $D_{i,n+1}$ and the $\mathcal{A}$-values correspond to the three different ways $\mathbb{0} + \mathbb{1} = \mathbb{1} + \mathbb{0} = \mathbb{1} + \mathbb{1}$ to distribute the value $\mathbb{1}$ to the two vertices. A short computation using the rules for intersection and pushforward presented earlier gives the formula (26). $\square$

The formula for the pullback of κ-classes is more involved and we need to introduce a bit of notation to state it. Fix g, n with $2g-2+n>0$, then for $k \geq 0$ let $\widehat{G}_k, G_k$ be the following $(n+1)$ and n-pointed prestable graphs in genus g with k edges

$\widehat{G}_k =$



$G_k =$



Here v_0 is the leftmost vertex and, for $k \geq 1$, h_0 is the unique half-edge incident to this vertex. For $k=0$, the graphs $\widehat{G}_k, G_k$ are the trivial graphs, respectively.

Also, in the proposition below we consider the power series

$$\Phi(t) = \frac{\exp(t)-1}{t} = 1 + \frac{t}{2} + \frac{t^2}{6} + \ldots.$$

We use the notation $[\Phi(t)]_{t^a \mapsto \kappa_a}$ to indicate that in the power series Φ the term t^a is substituted with the class κ_a, getting the mixed-degree Chow class

$$[\Phi(t)]_{t^a \mapsto \kappa_a} = 1 + \frac{\kappa_1}{2} + \frac{\kappa_2}{6} + \ldots.$$

Proposition 3.15. For g, n with $2g-2+n>0$ the stabilization morphism $\mathrm{st} : \mathfrak{M}_{g,n} \to \overline{\mathcal{M}}_{g,n}$ satisfies the following equality of mixed-degree Chow classes on $\mathfrak{M}_{g,n}$:

$$\begin{aligned} &\mathrm{st}^* \left[\Phi(t)\right]_{t^a \mapsto \kappa_a} \\ &= [\Phi(t)]_{t^a \mapsto \kappa_a} + \sum_{k \geq 1} (\xi_{G_k})_* \left(\left([\Phi(t)]_{t^a \mapsto \kappa_{v_0,a}} + \psi_{h_0}^{-1} \right) \cdot \mathrm{Cont}_{E(G_k)} \right). \end{aligned} \tag{28}$$

Here $\mathrm{Cont}_{E(G_k)}$ is the mixed-degree class

$$\mathrm{Cont}_{E(G_k)} = \prod_{(h,h') \in E(G_k)} -\Phi(\psi_h + \psi_{h'})$$

on $\mathfrak{M}_{G_k}$. In the formula above, the term $\psi_{h_0}^{-1}$ is understood to vanish unless it pairs with a term of $\mathrm{Cont}_{E(G_k)}$ containing a positive power of ψ_{h_0} and we have $\kappa_{v_0,0} = 2 \cdot 0 - 2 + 1 = -1$.

To obtain the pullback of an individual class κ_a under st we take the degree a part of (28) and obtain a formula of the form

$$\mathrm{st}^* \kappa_a = \kappa_a + \text{ boundary corrections.}$$

As an example, for $a = 1, 2$ we obtain

$$\mathrm{st}^*\kappa_1 = \kappa_1 + [G_1]$$

and

$$\mathrm{st}^*\kappa_2 = \kappa_2 - 3[G_1, \kappa_{v_0,1}] + 2[G_1, \psi_{h_0}] + [G_1, \psi_{h_1}] - 3[G_2]$$

where $e = (h_0, h_1)$ is the unique edge of the graph G_1.

Proof of Proposition 3.15. Consider the following commutative diagram

$$\begin{tikzcd} \mathfrak{M}_{g,n+1,\mathbb{1}} \arrow[d] & \supset & \mathfrak{C}_{g,n} \arrow[r, "\widehat{\mathrm{st}}"] \arrow[d, "\pi'"] & \overline{\mathcal{M}}_{g,n+1} \arrow[d, "\pi"] \\ \mathfrak{M}_{g,n,\mathbb{1}} & \supset & \mathfrak{M}_{g,n} \arrow[r, "\mathrm{st}"] & \overline{\mathcal{M}}_{g,n} . \end{tikzcd} \tag{29}$$

Then as $\mathfrak{C}_{g,n}$ maps proper and birationally to the fibre product in the right diagram, we have

$$\mathrm{st}^*\kappa_a = \mathrm{st}^*\pi_*\psi_{n+1}^{a+1} = \pi'_*\widehat{\mathrm{st}}^*\psi_{n+1}^{a+1} = \pi'_*\left(\psi_{n+1} - [\widehat{G}_1]\right)^{a+1} . \tag{30}$$

Here we use that computations in $\mathfrak{C}_{g,n}$ can be performed in $\mathfrak{M}_{g,n+1,\mathbb{1}}$ together with the pullback formula from Proposition 3.14 (noting that the third term in (26) vanishes since it lies in the complement of the open substack $\mathfrak{C}_{g,n} \subset \mathfrak{M}_{g,n+1,\mathbb{1}}$).

From now on it will be more convenient working with mixed-degree classes and exponentials. In this language, equation (30) translates to

$$\mathrm{st}^*[\Phi(t)]_{t^a \mapsto \kappa_a} = \pi'_* \exp(\psi_{n+1} - [\widehat{G}_1]) = \pi'_*\left(\exp(\psi_{n+1}) \cdot \exp(-[\widehat{G}_1])\right) . \tag{31}$$

The occurrence of the power series Φ is due to the discrepancy between the degree a of κ_a on the left of (30) and the degree $a + 1$ of the term on the right. Using the rules for intersections of tautological classes, one shows

$$\exp(-[\widehat{G}_1]) = \sum_{k \geq 1} (\xi_{\widehat{G}_k})_* \left(\prod_{(h,h') \in E(\widehat{G}_k)} -\Phi(\psi_h + \psi_{h'}) \right) . \tag{32}$$

Now in the pushforward (31) the only terms supported on the trivial graph are those from

$$\pi'_*(\exp(\psi_{n+1})) = [\Phi(t)]_{t^a \mapsto \kappa_a} ,$$

explaining the first term of the answer. All other terms of the product of the exponentials are supported on some $\widehat{G}_k$ for $k \geq 1$, where marking $n + 1$ is on a rational component with just one other half-edge h_0. Using the formulas for the pushforward by the forgetful map π' from Proposition 3.11, the only nontrivial pushforward we have to compute is the one by the universal curve $\pi_{0,1} : \mathfrak{C}_{0,1} \to \mathfrak{M}_{0,1}$, corresponding to forgetting $n+1$ on the 2-marked genus 0 component v_0 of $\widehat{G}_k$. Here, a short computation shows

$$(\pi_{0,1})_* \psi_1^a \psi_2^b = \psi_1^a \kappa_{b-1} + \delta_{a,0} \psi_1^{a-1} \tag{33}$$

where $\delta_{a,0}$ is the Kronecker delta and we have the convention $\kappa_{-1} = \psi_1^{-1} = 0$. Applying this formula for the pushforward, the first term in (33) gives rise to the term of the result involving $[\Phi(t)]_{t^a \mapsto \kappa_{v_0,a}}$, where again Φ appears due to the shift of degree from b to $b-1$ in (33). The second term of (33) gives rise to the term involving $\psi_{h_0}^{-1}$, where due to the Kronecker delta $\delta_{a,0}$ only the constant term of $\exp(\psi_{n+1})$ survives in the pushforward. □

Remark 3.16. The following is a nontrivial check and application of the computations from the last sections: for g, n, m with $2g-2+n>0$, consider the diagram

$$\begin{tikzcd} \overline{\mathcal{M}}_{g,n+m} \arrow[d, "F_m"] \arrow[dr, "\pi"] & \\ \mathfrak{M}_{g,n} \arrow[r, "\mathrm{st}"] & \overline{\mathcal{M}}_{g,n} \end{tikzcd} \tag{34}$$

where F_m is the map forgetting the last m markings (without stabilizing the curve), the map st is the stabilization map and their composition π is the "usual" forgetful map between moduli spaces of stable curves. The pullback of tautological classes under π is known classically and the pullback by the two other maps has been computed in the previous sections. Since the pullbacks must be compatible, this gives rise to tautological relations, which we can verify in examples.

For instance, for the class $\kappa_1 \in \mathrm{CH}^1(\overline{\mathcal{M}}_{g,n})$ we have

$$\pi^* \kappa_1 = \kappa_1 - \sum_{i=n+1}^{n+m} \psi_i + \sum_{\substack{I \subset \{n+1,\ldots,n+m\} \\ |I| \geq 2}} D_{0,I}$$

where $D_{0,I} \subset \overline{\mathcal{M}}_{g,n+m}$ is the boundary divisor of curves with a rational component carrying markings I. On the other hand we have

$$\begin{aligned} \mathrm{st}^* \kappa_1 &= \kappa_1 + [G_1] = \kappa_1 + D_{0,\emptyset} \\ \pi^* \mathrm{st}^* \kappa_1 &= \kappa_1 - \sum_{i=n+1}^{n+m} \psi_i + \sum_{\substack{I \subset \{n+1,\ldots,n+m\} \\ |I| \geq 2}} D_{0,I} \,. \end{aligned}$$

So indeed we get the same answer.

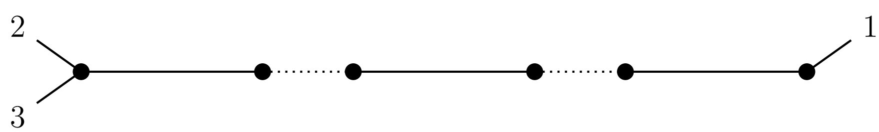
4 Relation to previous works

In this section we review several results in the literature relating to our study of the intersection theory of the stacks $\mathfrak{M}_{g,n}$.

Example 4.1. In [18, Lemma 1], Gathmann used the pullback formula of ψ-classes along the stabilization morphism $\mathrm{st}\colon \mathfrak{M}_{g,1} \to \overline{\mathcal{M}}_{g,1}$ to prove certain properties of the Gromov–Witten potential. It coincides with our calculation in Proposition 3.14.

Example 4.2. In [26], Pixton introduces classes $[\Gamma] \in \mathrm{R}^*(\overline{\mathcal{M}}_{g,n})$ indexed by *prestable* graphs of genus g with n legs. In his construction, chains of unstable vertices encode insertions of κ and ψ-classes in such a way that the formula for products $[\Gamma] \cdot [\Gamma']$ takes a particularly simple shape. While it is not a priori obvious how to relate his classes to the corresponding boundary strata classes $[\Gamma] \in \mathrm{R}^*(\mathfrak{M}_{g,n})$ in the moduli stack of prestable curves, this is a question which we plan to investigate in future work.

Example 4.3. In [25], Oesinghaus computes the Chow ring (with integral coefficients) of a certain open substack $\mathcal{T}$ of $\mathfrak{M}_{0,3}$, defined by the condition that the curve is *semistable* (i.e. every component of the curve has at least two distinguished points) and that the markings $2, 3$ are on a stable component of the curve. As a consequence, the prestable graphs of geometric points of $\mathcal{T}$ are all of the form



where we denote by Γ_k the graph of the shape above with k edges (for $k \geq 0$). The stack $\mathcal{T}$ has an atlas given by

$$\pi_n : \mathcal{A}^n = [\mathbb{A}^n/\mathbb{G}_m^n] \to \mathcal{T} \text{ for } n \geq 1.$$

Since $\mathcal{A}^n$ is a vector bundle over $B\mathbb{G}_m^n$, its Chow group[14] is given by

$$\mathrm{CH}^*(\mathcal{A}^n) = \mathbb{Q}[\alpha_1, \ldots, \alpha_n],$$

where α_ℓ is the class of the ℓ-th coordinate hyperplane

$$\iota_\ell : [V(x_i)/\mathbb{G}_m^n] \hookrightarrow \mathcal{A}^n.$$

From a computation in [25, Lemma 1] it follows that the first Chern class of the normal bundle of ι_ℓ is given by the restriction

$$c_1(\mathcal{N}_{\iota_\ell}) = \iota_\ell^* \alpha_\ell$$

[14]For the comparison with Oesinghaus' results we formulate everything in terms of $\mathbb{Q}$-coefficients since this is the convention of the present paper.

of α_ℓ to this hyperplane. Using the charts π_n, Oesinghaus shows that the Chow ring $\mathrm{CH}^*(\mathcal{T})$ is given by the ring QSym of *quasi-symmetric functions* on the index set $\mathbb{Z}_{>0}$. QSym can be seen as the subring of $\mathbb{Q}[\alpha_1, \alpha_2, \ldots]$ with additive basis given by

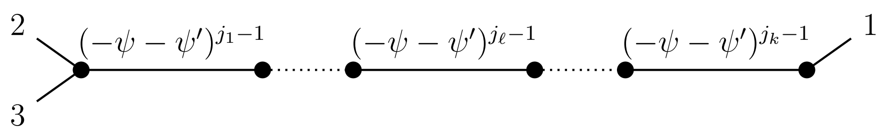
$$M_J = \sum_{i_1 < \ldots < i_k} \alpha_{i_1}^{j_1} \cdots \alpha_{i_k}^{j_k} \text{ for } k \geq 1, J = (j_1, \ldots, j_k) \in \mathbb{Z}_{\geq 1}^k. \tag{35}$$

Under the isomorphism $\mathrm{CH}^*(\mathcal{T}) \cong \mathrm{QSym}$, the element M_J is a basis element of degree $\sum_\ell j_\ell$ in the Chow group of $\mathcal{T}$. The pullback

$$\pi_n^* : \mathrm{CH}^*(\mathcal{T}) \to \mathrm{CH}^*(\mathcal{A}^n) = \mathbb{Q}[\alpha_1, \ldots, \alpha_n]$$

is induced by the map sending α_m to zero for $m > n$. In particular, it is easy to see that it is injective in Chow-degree at most n.

With these preparations in place, we can now identify the generators M_J of $\mathrm{CH}^*(\mathcal{T})$ with tautological classes. Indeed, we claim that M_J corresponds to the class supported on the stratum $\mathfrak{M}^{\Gamma_k}$ given by



(36)

To see this, we note that from the definition of the charts π_n in [25, Section 3.3] one can show that we have a fibre diagram

$$\begin{array}{ccc} \bigsqcup_{1 \leq i_1 < \cdots < i_k \leq n} V(x_{i_1}, \ldots, x_{i_k}) & \longrightarrow & \mathbb{A}^n \\ \Big\downarrow & & \Big\downarrow \\ & & [\mathbb{A}^n / \mathbb{G}_m^n] \\ & & \Big\downarrow{\scriptstyle \pi_n} \\ \mathfrak{M}_{\Gamma_k} & \xrightarrow{\xi_{\Gamma_k}} & \mathfrak{M}_{0,3} \end{array} \tag{37}$$

As a first example, this implies that $[\Gamma_k, 1] = (\xi_{\Gamma_k})_* [\mathfrak{M}_{\Gamma_k}]$ corresponds to the class

$$\sum_{1 \leq i_1 < \cdots < i_k \leq n} \alpha_{i_1} \cdots \alpha_{i_k}$$

on $\mathcal{A}^n$, which indeed is equal to $\pi_n^*(M_{(1,\ldots,1)})$. For the comparison of more complicated classes, we observe that the decorations $(-\psi - \psi')^{j_\ell - 1}$ are the $(j_\ell - 1)$-st powers of the Chern class of the normal bundle associated to the ℓ-th edge of Γ_k. On the other hand, in the diagram (37) the function x_{i_ℓ} around the linear subspace $V(x_{i_1}, \ldots, x_{i_k})$ is the smoothing parameter for the ℓ-th node of the curve and the first Chern class of the normal bundle to the locus $V(x_{i_\ell})$ where the ℓ-th node persists is given by (the restriction of) α_{i_ℓ}. Since $V(x_{i_1}, \dots, x_{i_k})$ has class $\alpha_{i_1} \cdots \alpha_{i_k}$ in $\mathcal{A}_n$, we conclude that the element (36) of $\mathrm{CH}^*(\mathcal{T})$ pulls back via π_n to

$$\sum_{1 \le i_1 < \cdots < i_k \le n} \alpha_{i_1} \cdots \alpha_{i_k} \cdot \alpha_{i_1}^{j_1 - 1} \cdots \alpha_{i_k}^{j_k - 1} = M_J|_{\mathcal{A}_n} .$$

This shows the desired correspondence because this holds for all n and π_n^* is injective in degree at most n.

Using the correspondence, it is straightforward to see that the product formula for expressing $M_J \cdot M_{J'}$ in terms of basis elements M_{J_i} discussed in [25, Proposition 2] follows from the product formula for decorated strata classes discussed in Section 3.2.

In [25], Oesinghaus also computes the Chow group of the semistable loci $\mathfrak{M}_{0,2}^{ss}$ and $\mathfrak{M}_{0,3}^{ss}$ and there are correspondence results to the tautological generators of these spaces closely parallel to the above discussion. We leave the details to the interested reader.

A Chow groups of locally finite type algebraic stacks

The Chow group of a finite type algebraic stack over a field k is defined in [21]. We extend this notion to an algebraic stack which is not necessarily of finite type over k.

Definition A.1. Let $\mathfrak{M}$ be an algebraic stack, locally of finite type over a field k. Choose $(\mathcal{U}_i)_{i\in I}$ a directed system[15] of finite type open substacks of $\mathfrak{M}$ whose union is all of $\mathfrak{M}$. Then we define

$$\mathrm{CH}_*(\mathfrak{M}) = \varprojlim_{i\in I} \mathrm{CH}_*(\mathcal{U}_i),$$

where for $\mathcal{U}_i \subseteq \mathcal{U}_j$ the transition map $\mathrm{CH}_*(\mathcal{U}_j) \to \mathrm{CH}_*(\mathcal{U}_i)$ is given by the restriction to $\mathcal{U}_i$. In other words, we have

$$\mathrm{CH}_*(\mathfrak{M}) = \{(\alpha_i)_{i\in I} : \alpha_i \in \mathrm{CH}_*(\mathcal{U}_i), \alpha_j|_{\mathcal{U}_i} = \alpha_i \text{ for } \mathcal{U}_i \subseteq \mathcal{U}_j\}.$$

For the existence of a system $(\mathcal{U}_i)_{i\in I}$ as above, observe that since $\mathfrak{M}$ is locally of finite type, we can simply take the system of *all* finite type substacks $\mathcal{U} \subset \mathfrak{M}$. Moreover, given any two systems $(\mathcal{U}_i)_{i\in I}$, $(\mathcal{U}'_i)_{i\in I'}$, one uses Noetherian induction to show that they mutually dominate each other. By standard arguments, the Chow group $\mathrm{CH}_*(\mathfrak{M})$ is independent of the choice of $(\mathcal{U}_i)_{i\in I}$.

From the definition as a limit one sees that the Chow groups $\mathrm{CH}_*(\mathfrak{M})$ inherit all the usual properties (e.g. flat pullback, projective pushforward, Chern classes of vector bundles and Gysin pullbacks) of the Chow groups from [21]. Moreover, if $\mathfrak{M}$ is smooth and has affine stabilizer groups at geometric points, the intersection products on the groups $\mathrm{CH}_*(\mathcal{U}_i)$ give rise to an intersection product on $\mathrm{CH}_*(\mathfrak{M})$. In this case, if $\mathfrak{M}$ is equidimensional we often use the cohomological degree convention

$$\mathrm{CH}^*(\mathfrak{M}) = \mathrm{CH}_{\dim \mathfrak{M} - *}(\mathfrak{M})\,.$$

The Chow group of a locally finite type algebraic stack is defined as taking a projective limit. Since taking projective limits is not an exact functor and does not commute with tensor products, some of the properties of Chow groups of finite type algebraic stacks do not (obviously) extend. In the following definition we present two finiteness assumptions on locally finite type stacks, which guarantee that the Chow groups we define continue to have some nice properties (like having an excision sequence).

Definition A.2. Let $\mathfrak{M}$ be an equidimensional algebraic stack, locally finite type over a field k.

a) We say $\mathfrak{M}$ is *Lindelöf* if every cover of $\mathfrak{M}$ by open substacks has a countable subcover.

[15] Recall that this means that for all $\mathcal{U}_i, \mathcal{U}_j$ there exists a $\mathcal{U}_\ell$ containing both of them.

b) We say that $\mathfrak{M}$ has a *good filtration by finite type substacks*[16] if there exists a collection $(\mathcal{U}_m)_{m\in\mathbb{N}}$ of open substack of finite type on $\mathfrak{M}$ which is *increasing* (i.e. $\mathcal{U}_m \subset \mathcal{U}_\ell$ for $m < \ell$) and such that $\dim(\mathfrak{M} \setminus \mathcal{U}_m) < \dim \mathfrak{M} - m$.

Lemma A.3. A locally finite type algebraic stack $\mathfrak{M}$ over k is Lindelöf if and only if it has a countable cover $(\mathcal{U}_i)_{i\in\mathbb{N}}$ by finite type open substacks $\mathcal{U}_i \subseteq \mathfrak{M}$. In this case the cover $\mathcal{U}_i$ can be chosen to be increasing. In particular, if $\mathfrak{M}$ has a good filtration by finite type substacks it is automatically Lindelöf.

Proof. If $\mathfrak{M}$ is Lindelöf, its cover by the system of *all* finite type substacks has a countable subcover. Conversely assume that $(\mathcal{U}_i)_{i\in\mathbb{N}}$ is a countable cover of $\mathfrak{M}$ by finite type open substacks. Given any open cover $(\mathcal{V}_j)_{j\in J}$ of $\mathfrak{M}$, each single open $\mathcal{U}_i$ is covered by finitely many elements $\mathcal{V}_{j_{i,\ell}}$ of the second cover via Noetherian induction. Then the system $(\mathcal{V}_{j_{i,\ell}})_{i,\ell}$ is a countable subcover. □

Example A.4. a) The stacks $\mathfrak{M}_{g,n}$ of prestable curves have a good filtration by finite type substacks, given by the loci $\mathfrak{M}_{g,n}^{\leq e}$ of curves having at most e nodes.

b) The universal Picard stack $\mathfrak{Pic}_{g,n}$ over $\mathfrak{M}_{g,n}$ parameterizing tuples

$$(C, p_1, \ldots, p_n, \mathcal{L})$$

of a prestable marked curve and a line bundle $\mathcal{L}$ on C is Lindelöf, but does not have a good filtration by finite type substacks.

Indeed, we do get a countable cover $(\mathcal{U}_m)_{m\in\mathbb{N}}$ by finite type substacks, where $\mathcal{U}_m$ is the set of $(C, p_1, \ldots, p_n, \mathcal{L})$ such that C has at most m nodes and such that the absolute value of the degree of $\mathcal{L}$ on any component of C is at most m. This cover is increasing, but does not satisfy that $\dim(\mathfrak{Pic}_{g,n} \setminus \mathcal{U}_m)$ goes to $-\infty$.

The fact that no good filtration can exist follows from the observation that $\mathfrak{Pic}_{g,n}$ has infinitely many boundary divisors (corresponding to ways that the degree of $\mathcal{L}$ can split up on the components of a curve with two components) and no finite type stack $\mathcal{U}_1$ can contain all generic points of these divisors.

The paper [4] studied cycles and relations in the operational Chow ring $\mathrm{CH}^*_{\mathrm{op}}(\mathfrak{Pic}_{g,n})$, see Section C for details. While we do not pursue this direction of study in the current paper, the Picard stack is one of our main motivations for introducing the property of being Lindelöf.

c) For completeness, let us mention that an example of an irreducible scheme which is not Lindelöf is the *line with uncountably many origins*, obtained from the disjoint union of uncountably many copies of the affine line by identifying them away from the origin.

[16] This definition is taken from [25, Definition 5].

When $\mathfrak{M}$ has a good filtration $(\mathcal{U}_m)_{m\in\mathbb{N}}$ by finite type substacks, for fixed d we have

$$\mathrm{CH}^d(\mathfrak{M}) = \mathrm{CH}^d(\mathcal{U}_m)$$

for $m > d$. This implies that, as long as we are interested in a fixed codimension, all computations can be carried out on a finite type stack and thus essentially all results for the Chow groups of such stacks carry over (e.g. the excision sequence, including the version extended on the left by one higher Chow group from [21, Proposition 4.2.1]).

For stacks which are Lindelöf, we get at least the first three terms of the excision sequence.

Proposition A.5. Let $\mathfrak{M}$ be an equidimensional algebraic stack, locally finite type over a field k which is Lindelöf. Let $j : \mathfrak{Z} \to \mathfrak{M}$ be a closed substack with complement $i : \mathfrak{V} = \mathfrak{M} \setminus \mathfrak{Z} \to \mathfrak{M}$. Then there exists a complex

$$\mathrm{CH}_*(\mathfrak{Z}) \xrightarrow{j_*} \mathrm{CH}_*(\mathfrak{M}) \xrightarrow{i^*} \mathrm{CH}_*(\mathfrak{V}) \to 0 \tag{38}$$

which is exact at $\mathrm{CH}_*(\mathfrak{V})$, i.e. i^* is surjective. If moreover the stack $\mathfrak{V}$ is a *countable* (finite or infinite) disjoint union of quotient stacks, the sequence is also exact at $\mathrm{CH}_*(\mathfrak{M})$.

Note that while the condition on $\mathfrak{V}$ being a countable union of quotient stacks might sound far-fetched, this *is* the situation that we would encounter e.g. by taking $\mathfrak{M}$ to be the boundary of the Picard stack $\mathfrak{Pic}_{g,n}$ and taking $\mathfrak{Z} \subset \mathfrak{M}$ the closed substack where the curve has at least 2 nodes.

Proof of Proposition A.5. Let $(\mathcal{U}_m)_{m\in\mathbb{N}}$ be an increasing cover of $\mathfrak{M}$ by finite type substacks. Denote by $\mathfrak{M}_m, \mathfrak{Z}_m, \mathfrak{V}_m$ the intersections of $\mathfrak{M}, \mathfrak{Z}, \mathfrak{V}$ with $\mathcal{U}_m$ and by j_m, i_m the restrictions of j, i. Then by the usual excision sequence for finite-type stacks we have that

$$0 \to \mathrm{CH}_*(\mathfrak{Z}_m)/\ker((j_m)_*) \xrightarrow{(j_m)_*} \mathrm{CH}_*(\mathfrak{M}_m) \xrightarrow{i_m^*} \mathrm{CH}_*(\mathfrak{V}_m) \to 0\,. \tag{39}$$

are exact sequences. From another application of the excision sequence we see that the restriction maps

$$\mathrm{CH}_*(\mathfrak{Z}_m) \to \mathrm{CH}_*(\mathfrak{Z}_{m'}) \text{ for } m' < m$$

are surjective. This implies that the system $(\mathrm{CH}_*(\mathfrak{Z}_m)/\ker((j_m)_*))_m$ is Mittag-Leffler (see [29, Tag 0596]). Then it follows from [29, Tag 0598] that we obtain an exact sequence

$$0 \to \varprojlim_m(\mathrm{CH}_*(\mathfrak{Z}_m)/\ker((j_m)_*)) \to \mathrm{CH}_*(\mathfrak{M}) \to \mathrm{CH}_*(\mathfrak{U}) \to 0\,.$$

This finishes the proof of exactness of (38) at $\mathrm{CH}_*(\mathfrak{U})$.

If $\mathfrak{V}$ is a countable disjoint union of quotient stacks, we can modify the exact sequence (39), extending it to the left to obtain

$$0 \to \mathrm{CH}_*(\mathfrak{V}_m, 1)/\ker(\partial_m) \xrightarrow{\partial_m} \mathrm{CH}_*(\mathfrak{Z}_m) \xrightarrow{(j_m)_*} \ker(i_m^*) \to 0\,. \tag{40}$$

We claim that the directed system

$$(K_m)_m = (\mathrm{CH}_*(\mathfrak{V}_m, 1)/\ker(\partial_m))_m$$

is Mittag-Leffler, i.e. that for m fixed, the images of the restriction maps $K_{m'} \to K_m$ stabilize for $m' \gg m$. This follows from two easy observations:

- Since by construction $\mathfrak{V}_m = \mathfrak{V} \cap \mathcal{U}_m$ is of finite type, it is supported on a finite number of connected components $\mathfrak{V}^1, \ldots, \mathfrak{V}^{e_m}$ of $\mathfrak{V}$.
- Since the stacks $\mathcal{U}_m$ cover $\mathfrak{M}$, we know by Noetherian induction that for $m' \gg m$ the stack $\mathfrak{V}_{m'}$ contains the union of $\mathfrak{V}^1, \ldots, \mathfrak{V}^{e_m}$, so that we have

$$K_{m'} = \bigoplus_{i=1}^{e_m} \mathrm{CH}_*(\mathfrak{V}^i, 1) \oplus \mathrm{CH}_*\left(\mathfrak{V}_{m'} \setminus \bigcup_{i=1}^{e_m} \mathfrak{V}^i, 1\right)\,.$$

Under the map $K_{m'} \to K_m$, the second direct summand always maps to zero (since it is supported on a different connected component). Thus indeed for $m' \gg m$, the image of $K_{m'} \to K_m$ stabilizes to the image of the restriction morphism

$$\bigoplus_{i=1}^{e_m} \mathrm{CH}_*(\mathfrak{V}^i, 1) \to \mathrm{CH}_*(\mathfrak{V}_m, 1) = K_m\,.$$

We conclude that the system $(K_m)_m$ is Mittag-Leffler, so again using [29, Tag 0598] we obtain an exact sequence

$$0 \to \varprojlim_m (K_m) \to \mathrm{CH}_*(\mathfrak{Z}) \to \varprojlim_m \ker(i_m^*) = \ker(i^*) \to 0\,,$$

where we use that taking directed inverse limits is left-exact to identify the limit of $\ker(i_m^*)$ as $\ker(i^*)$. Thus we conclude that $\mathrm{CH}_*(\mathfrak{Z})$ surjects onto the kernel of i^*, obtaining exactness of (38) at $\mathrm{CH}_*(\mathfrak{M})$. □

Remark A.6. In the context of algebraic spaces, a different definition of Chow groups for locally finite type spaces is presented in [29, Tag 0EDZ]. It works directly with formal linear combinations of cycles where locally only finitely many of the coefficients are nonzero. We expect that adapting the definitions of [21], one can give a similar definition in the case of algebraic stacks locally of finite type. It seems likely that for stacks which are far away from being finite type (e.g. stacks not being Lindelöf) these alternative Chow groups have better formal properties than the groups we construct, since e.g. in the setting of algebraic spaces they satisfy an excision sequence without such assumptions on the ambient space (see [29, Tag 0EP9]). The advantage of Definition A.1 is that it does not require us to reprove the standard constructions and properties of Chow groups since they descend easily to the projective limit. We do expect that the alternative definition of Chow groups modeled on [29, Tag 0EDZ] coincides with our definition for algebraic stacks locally of finite type over a field which have a good filtration by finite type substacks.

B Properties of proper pushforward

The proper pushforward satisfies the following expected properties.

Proposition B.1. Let Y be an algebraic stack stratified by global quotient stacks and let $f\colon X \to Y$ be a proper, representable morphism as above. Then proper pushforward is compatible with representable flat pullback and refined Gysin maps for representable regular local immersions. The same compatibilities hold for Chow groups with $\mathbb{Q}$-coefficients when f is of relative Deligne-Mumford type.

Proof. We will prove the case when f is a representable morphism. The proof for Deligne-Mumford type is similar. We first prove the compatibility of proper pushforward with flat pullbacks. Consider a cartesian diagram

$$\begin{array}{ccc} W & \xrightarrow{v} & X \\ \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle f} \\ Z & \xrightarrow{u} & Y \end{array} \tag{41}$$

where $u\colon Z \to Y$ is a representable flat morphism of relative constant dimension ℓ. Since u is representable, Z is stratified by global quotient stacks (see [21, Proposition 3.5.5]). By property (iii) mentioned above [6, Proposition B.8] the natural isomorphism $\mathrm{CH}_*^f \to \mathrm{CH}_*$ is compatible with flat pullbacks. A class (ρ, α) in $\mathrm{CH}_*^f(X)$ is represented by the following cartesian diagram

$$\begin{array}{ccccc} E' & \longrightarrow & X' & \longrightarrow & X \\ \downarrow{\scriptstyle f''} & & \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\ E & \xrightarrow{\pi} & Y' & \xrightarrow{\rho} & Y \end{array} \tag{42}$$

where ρ is a projective morphism, E is a vector bundle on Y' and α is a class in

$\mathrm{CH}^{\circ}_{d+\mathrm{rk}E'}(E')$. By pulling back (42) along (41), we get

$$\begin{tikzcd}
\mathrm{CH}^{\circ}_{d+\mathrm{rk}E'}(E') \arrow[r, "f''_*"] \arrow[d, "v^*"] & \mathrm{CH}^{\circ}_{d+\mathrm{rk}E'}(E) \arrow[d, "u^*"] \\
\mathrm{CH}^{\circ}_{d+\mathrm{rk}E'+\ell}(F') \arrow[r, "g''_*"] & \mathrm{CH}^{\circ}_{d+\mathrm{rk}E'+\ell}(F)
\end{tikzcd}$$

where F is the pullback of E to $Z \times_Y Y'$ and F' is the pullback of E' to $W \times_X X'$. This diagram commutes because of the corresponding statement for naive Chow groups. Therefore the following diagram

$$\begin{tikzcd}
\mathrm{CH}^{f}_{d}(X) \arrow[r, "f_*"] \arrow[d, "v^*"] & \mathrm{CH}_{d}(Y) \arrow[d, "u^*"] \\
\mathrm{CH}^{g}_{d+\ell}(W) \arrow[r, "g_*"] & \mathrm{CH}_{d+\ell}(Z)
\end{tikzcd}$$

commutes.

The compatibility with regular local immersions states that for a Cartesian diagram (41) with u a regular local immersion, we have $u^! f_* = g_* v^!$. Note that here the pushforward by g is well-defined since the assumption that Y is stratified by global quotient stacks together with the representability of u implies that Z likewise is stratified by global quotient stacks, using [21, Proposition 3.5.5]. Then the desired compatibility also follows from a formal argument as above (see [4, Proposition 18]). □

The following proposition is a generalization of [30, Lemma 3.8] to algebraic stacks.

Proposition B.2. Let X and Y be algebraic stacks, finite type over a field k and stratified by global quotient stacks. Let $f\colon X \to Y$ be a proper, surjective morphism of relative Deligne-Mumford type. Then the pushforward

$$f_*\colon \mathrm{CH}_d(X, \mathbb{Q}) \to \mathrm{CH}_d(Y, \mathbb{Q})$$

is surjective for all d.

Proof. By the assumption of being stratified by global quotient stacks, there exists a nonempty open substack $U \subset Y$ isomorphic to a quotient $U \cong [T/G]$ of a quasi-projective scheme T by a smooth, connected linear algebraic group G acting linearly on T. Let $Z = Y \setminus U$ be the complement. Consider a commutative diagram

$$\begin{tikzcd}
\mathrm{CH}_d(f^{-1}(Z), \mathbb{Q}) \arrow[r] \arrow[d] & \mathrm{CH}_d(X, \mathbb{Q}) \arrow[r] \arrow[d] & \mathrm{CH}_d(f^{-1}(U), \mathbb{Q}) \arrow[r] \arrow[d] & 0 \\
\mathrm{CH}_d(Z, \mathbb{Q}) \arrow[r] & \mathrm{CH}_d(Y, \mathbb{Q}) \arrow[r] & \mathrm{CH}_d(U, \mathbb{Q}) \arrow[r] & 0
\end{tikzcd} \tag{43}$$

with exact rows from the excision sequences and vertical arrows from the proper pushforwards by f as defined above. Commutativity of the squares follows from the compatibility of proper pushforward with compositions of proper maps and pullbacks by flat maps. The pushforward f_* is surjective over Y if it is surjective over U and Z by the four lemma. Since the assumptions of the proposition hold for Z and since Y is of finite type, we can reduce by Noetherian induction to showing the statement over U.

By [21, Proposition 3.5.10], the global quotient U admits a vector bundle $\pi : E \to U$ such that E is represented by a scheme off a locus of arbitrarily high codimension. Moreover, the pullback π^* induces an isomorphism of Chow groups. Let $V = f^{-1}(U)$ and $E' = (f|_V)^* E \to V$ be the pullback of E under f. It suffices to show that the pushforward under $E' \to E$ is surjective because of the compatibility of proper pushforward with flat pullback. This computation can be done away from the locus where E is not a scheme because we work in a fixed dimension d. Thus we have reduced to the case where the target is a scheme.

Now assume that $f : X \to Y$ is as in the proposition and Y is a scheme. Then the domain is a Deligne-Mumford stack because f is assumed to be of relative Deligne-Mumford type. By [12, Theorem 2.7], such a stack admits a finite surjective morphism from a scheme. This means that the machinery of Chow groups as developed by Vistoli in [30] is applicable. By [30, Lemma 3.8], the pushforward f_* is surjective on naive Chow groups (with $\mathbb{Q}$-coefficients). Since Y is a scheme, the naive Chow groups agree with those defined in [21], so for every cycle on Y there exists a naive cycle on X pushing forward to it. Finally, for naive cycles on X, the definition of proper pushforward agrees with the naive pushforward, so we are done. □

Remark B.3. Even in very good situations, the pushforward by proper, surjective maps of finite type stacks is *not* surjective on naive Chow groups. Indeed, for $n \geq 1$ consider the map

$$f : [\mathbb{P}^n/\mathrm{PGL}_{n+1}] \to B\mathrm{PGL}_{n+1},$$

where PGL_{n+1} acts in the usual way on $\mathbb{P}^n$. Then f is a representable, proper surjective morphism of quotient stacks and in fact we claim that it is also projective. To see the latter, note that the line bundle $\mathcal{O}_{\mathbb{P}^n}(n+1)$ on $\mathbb{P}^n$ is PGL_{n+1}-linearizable (see e.g. [9, Example 3.2.7]). Thus it descends to a line bundle on $[\mathbb{P}^n/\mathrm{PGL}_{n+1}]$ which is relatively very ample for f. Therefore, the proper morphism f is indeed projective. However, even though f has all these nice properties, the pushforward f_* still vanishes on naive Chow groups. Let $V \to [\mathbb{P}^n/\mathrm{PGL}_{n+1}]$ be an integral closed substack. Let $V_{\mathbb{P}^n}$ be the fiber product

$$\begin{tikzcd} V_{\mathbb{P}^n} \arrow[r] \arrow[d] & V \arrow[d] \\ \mathbb{P}^n \arrow[r] & {[\mathbb{P}^n/\mathrm{PGL}_{n+1}]} \end{tikzcd}.$$

Then $V_{\mathbb{P}^n}$ is a closed subscheme of $\mathbb{P}^n$ which is invariant under PGL_{n+1} so $V_{\mathbb{P}^n} = \mathbb{P}^n$ and thus also $V = [\mathbb{P}^n/\mathrm{PGL}_{n+1}]$. On the other hand f is relative dimension n hence $f_*[V]=0$. We are grateful to Andrew Kresch, who pointed out this example to us.

Remark B.4. Since the proper pushforward constructed in this section is compatible with flat pullbacks, it follows immediately from the definition of Chow groups of locally finite type stacks in the previous section that these inherit the proper pushforward construction. In particular, [6, Theorem B.17] and Proposition B.1 remain true for stacks only locally of finite type.

On the other hand, Proposition B.2 remains true for algebraic stacks X, Y which are Lindelöf and stratified by global quotient stacks (as is the case for the stacks $\mathfrak{M}_{g,n,a}$ described in Section 2.2).

Indeed, let U_i be a finite type cover of Y, then $V_i = f^{-1}(U_i)$ is also finite type since f is proper. Let K_i be the kernel of the proper pushforward $f_* : \mathrm{CH}_*(V_i) \to \mathrm{CH}_*(U_i)$, then applying the proposition we have a commutative diagram

$$\begin{tikzcd}
0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
K_{ij} \arrow[r] \arrow[d] & K_i \arrow[r] \arrow[d] & K_j \arrow[d] & \\
\mathrm{CH}_*(V_i \setminus V_j) \arrow[r] \arrow[d] & \mathrm{CH}_*(V_i) \arrow[r] \arrow[d] & \mathrm{CH}_*(V_j) \arrow[r] \arrow[d] & 0 \\
\mathrm{CH}_*(U_i \setminus U_j) \arrow[r] \arrow[d] & \mathrm{CH}_*(U_i) \arrow[r] \arrow[d] & \mathrm{CH}_*(U_j) \arrow[r] \arrow[d] & 0 \\
0 & 0 & 0 &
\end{tikzcd} \tag{44}$$

The columns are exact by Proposition B.2 and the middle and the bottom rows are exact by [21, Proposition 2.3.6]. Applying a small variant of the Snake lemma[17] we see that the maps $K_i \to K_j$ of the directed system $(K_i)_i$ are surjective, so this system is Mittag-Leffler (see [29, Tag 0596]). Then it follows from [29, Tag 0598] that the induced map

$$\mathrm{CH}_*(X) = \varprojlim_i \mathrm{CH}_*(V_i) \longrightarrow \varprojlim_i \mathrm{CH}_*(U_i) = \mathrm{CH}_*(Y)$$

is indeed surjective.

[17] Note that in comparison to the usual situation of the Snake lemma, we don't have injectivity of the map $i_* : \mathrm{CH}_*(U_i \setminus U_j) \to \mathrm{CH}_*(U_i)$. This can be repaired by replacing $\mathrm{CH}_*(U_i \setminus U_j)$ with $\mathrm{CH}_*(U_i \setminus U_j)/\ker(i_*)$ and observing that the map from $\mathrm{CH}_*(V_i \setminus V_j)$ is still surjective.

C Operational Chow groups for algebraic stacks

In this section we give a definition of operational Chow classes for algebraic stacks which we assume throughout to be locally finite type over k.

Definition C.1. An operational class c in the p-th operational Chow group $\mathrm{CH}^p_{\mathrm{OP}}(X)$ is a collection of homomorphisms

$$c(g)\colon \mathrm{CH}_m(B) \to \mathrm{CH}_{m-p}(B)$$

for all morphisms $g\colon B \to X$ where B is an algebraic stack of finite type over k, stratified by global quotient stacks and for all integers m, compatible with representable proper pushforward, flat pullback, and refined Gysin pullback along representable lci morphisms (see [17, Section 17.1]). In particular the compatibility with refined Gysin pullback means the following compatibility condition: consider a diagram

$$\begin{array}{ccc} B' & \longrightarrow & Z' \\ \big\downarrow{\scriptstyle f'} & & \big\downarrow{\scriptstyle f} \\ B & \longrightarrow & Z \\ \big\downarrow{\scriptstyle g} & & \\ X & & \end{array}$$

where $g : B \to X$ is a morphism from an algebraic stack B of finite type over k, stratified by global quotient stacks, $f : Z' \to Z$ is a representable lci morphism and Z' is stratified by global quotient stacks and the square in the diagram is Cartesian. Then we require that for all $c \in \mathrm{CH}^*_{\mathrm{OP}}(X)$ and $\alpha \in \mathrm{CH}_*(B)$ we have

$$f^!(c(g) \cap \alpha) = c(gf') \cap f^!\alpha \ \text{ in } \mathrm{CH}_*(B')\,.$$

This notion of operational Chow group of algebraic stacks shares the following formal properties of operational Chow group of schemes: it is a contravariant functor for all morphisms and has the structure of an associative $\mathbb{Q}$-algebra coming from composing two operations. Note that for the functoriality under all morphisms, it is important that we did not restrict the morphisms $g : B \to X$ in the definition above e.g. to be representable. Otherwise we would only get functoriality under representable morphisms.

Example C.2. Let E be a vector bundle on X, then its *r-th Chern class*

$$c_r(E) \in \mathrm{CH}^r_{\mathrm{OP}}(X)$$

is naturally an operational class on X. Given $g : B \to X$ it acts by

$$(c_r(E))(g)\colon \mathrm{CH}_m(B) \to \mathrm{CH}_{m-p}(B), \alpha \mapsto c_r(g^*E) \cap \alpha$$

where the Chern class of g^*E and its action on the cycle α on B are as defined in [21].

We start with a small observation: the operational Chow group of X can be computed on a suitable finite-type cover of X.

Lemma C.3. Let X be a locally finite type algebraic stack over k. Let $(\mathcal{U}_i)_{i\in I}$ be a directed system of finite type open substacks of X whose union is all of X. Then the flat pullbacks j_i^* by the inclusions $j_i : \mathcal{U}_i \to X$ induce a map

$$\Phi : \mathrm{CH}^*_{\mathrm{OP}}(X) \to \varprojlim_{i\in I} \mathrm{CH}^*_{\mathrm{OP}}(\mathcal{U}_i)$$

and this map Φ is an isomorphism.

Proof. The proof, which uses the fact that each map $B \to X$ from a finite-type stack B must factor through one of the $\mathcal{U}_i$ by Noetherian induction, goes verbatim as the proof presented in [4, Corollary 15]. □

Lemma C.4. Let $f\colon X \to Y$ be a representable, proper, flat morphism of relative dimension d. Then there is a pushforward map

$$f_*\colon \mathrm{CH}^*_{\mathrm{OP}}(X) \to \mathrm{CH}^{*-d}_{\mathrm{OP}}(Y)$$

defined as follows. For a cartesian diagram

$$\begin{tikzcd} C \arrow[r, "f'"] \arrow[d, "g'"] & B \arrow[d, "g"] \\ X \arrow[r, "f"] & Y \end{tikzcd} \tag{45}$$

and $c \in \mathrm{CH}^*_{\mathrm{OP}}(X)$,

$$(f_*c)(g)\cdot\alpha = f'_*(c(g')\cdot(f')^*\alpha), \text{ for } \alpha \in \mathrm{CH}_*(B)\,.$$

If $f\colon X \to Y$ is a representable, proper, lci morphism of relative dimension d, the pushforward map is similarly defined by the formula

$$(f_*c)(g)\cdot\alpha = f'_*(c(g')\cdot f^!\alpha), \text{ for } \alpha \in \mathrm{CH}_*(B)$$

using the refined Gysin pullback. For a morphism f which is representable, flat and lci, the two definitions coincide.

Proof. We check that the collection of maps f_*c defines an operational Chow class. We will only give a proof for the case that f is lci, the proof for flat morphisms is similar. The fact that the two notions coincide for f both flat an lci follows from the formula and the fact that the flat pullback and the lci pullback of cycles coincide.

Let $h\colon B' \to B$ be a representable proper morphism and consider the following cartesian diagram

$$\begin{tikzcd} C' \arrow[r, "f''"] \arrow[d, "h'"] & B' \arrow[d, "h"] \\ C \arrow[r, "f'"] \arrow[d, "g'"] & B \arrow[d, "g"] \\ X \arrow[r, "f"] & Y. \end{tikzcd} \tag{46}$$

For $c \in \mathrm{CH}^*_{\mathrm{OP}}(X)$ and $\alpha \in \mathrm{CH}_*(B')$ we have

$$\begin{aligned} h_*(f_*c)(g \circ h)(\alpha) &= h_*\left(f''_*(c(g' \circ h') \cdot f^!\alpha)\right) \\ &= f'_*\left(h'_*(c(g' \circ h') \cdot f^!\alpha)\right) \\ &= f'_*\left(c(g') \cdot h'_* f^!\alpha\right) \\ &= f'_*\left(c(g') \cdot f^! h_*\alpha\right) = (f_*c)(g)(h_*\alpha), \end{aligned}$$

where the third equality uses the compatibility with proper pushforward for the operational class c and the fourth equality uses compatibility of Gysin pullback and proper pushforward in Proposition B.1.

Similarly let $h\colon B' \to B$ be a flat morphism and $\beta \in \mathrm{CH}_*(B)$, then we have

$$\begin{aligned} (f_*c)(g \circ h)(h^*\beta) &= f''_*(c(g' \circ h') \cdot f^! h^*\beta) \\ &= f''_*(c(g' \circ h') \cdot (h')^* f^!\beta) \\ &= f''_*((h')^*(c(g') \cdot f^!\beta)) \\ &= h^* f'_*(c(g') \cdot f^!\beta) = h^*(f_*c)(g)(\beta), \end{aligned}$$

where the second equality uses compatibility of Gysin maps with flat pullbacks, the third equality uses that c is compatible with flat pullbacks and the fourth equality uses that pushforwards are compatible with flat pullbacks.

Let $j\colon B \to Z$ be a morphism and $h\colon Z' \to Z$ be a (representable) regular local immersion where Z' is stratified by global quotient stacks. Consider the fiber diagram

$$\begin{tikzcd} C' \arrow[r, "f''"] \arrow[d, "h''"] & B' \arrow[r, "j'"] \arrow[d, "h'"] & Z' \arrow[d, "h"] \\ C \arrow[r, "f'"] \arrow[d, "g'"] & B \arrow[r, "j"] \arrow[d, "g"] & Z \\ X \arrow[r, "f"] & Y & \end{tikzcd} .$$

For $\alpha \in \mathrm{CH}_*(B)$ we have

$$\begin{aligned} h^!(f_*c)(g)(\alpha) &= h^!\left(f'_*(c(g') \cdot f^!\alpha)\right) \\ &= f''_*\left(h^!(c(g') \cdot f^!\alpha)\right) \\ &= f''_*\left(c(g' \circ h'') \cdot h^! f^!(\alpha)\right) \\ &= f''_*\left(c(g' \circ h'') \cdot f^! h^!(\alpha)\right) = (f_*c)(g \circ h')(h^!\alpha) \end{aligned}$$

where the fourth equality comes from the commutativity of refined Gysin pullback ([21, Section 3.1]) and the second equality comes from Proposition B.1. $\square$

Lemma C.5. Let X be an equidimensional algebraic stack of dimension n which is stratified by global quotient stacks. Then there exists a well-defined map

$$\cap\,[X] \colon \mathrm{CH}^*_{\mathrm{OP}}(X) \to \mathrm{CH}_{n-*}(X)\,. \tag{47}$$

Proof. Let $(\mathcal{U}_i)_{i\in I}$ be a directed system of finite type open substacks of X whose union is all of X. Given $c \in \mathrm{CH}^*_{\mathrm{OP}}(X)$, for each open embedding $\iota_i \colon \mathcal{U}_i \to X$ we consider the cycle $c(\iota_i)\cdot[\mathcal{U}_i] \in \mathrm{CH}_*(\mathcal{U}_i)$. For a given $i \in I$ let $\ell \in I$ be an element such that $\mathcal{U}_\ell$ contains $\mathcal{U}_i$. Let $\iota_{i\ell} : \mathcal{U}_i \to \mathcal{U}_\ell$ be the open embedding. Since an operational Chow class commutes with the flat pullback, we have

$$\iota^*_{i\ell}(c(\iota_\ell) \cdot [\mathcal{U}_\ell]) = c(\iota_i) \cdot (\iota^*_{i\ell}[\mathcal{U}_\ell]) = c(\iota_i) \cdot [\mathcal{U}_i]\,.$$

Therefore the collection of cycles $c(\iota_i)\cdot[\mathcal{U}_i]$ gives a well-defined element in $\varprojlim_{i\in I} \mathrm{CH}_*(\mathcal{U}_i)$. $\square$

The following theorem is an analogy of the Poincaré duality for smooth stacks.

Theorem C.6. Let X be a smooth equidimensional algebraic stack of dimension n stratified by global quotient stacks. Then the canonical map (47) with $\mathbb{Q}$-coefficients is an isomorphism of associative $\mathbb{Q}$-algebras.

Proof. In the following proof all Chow groups are with $\mathbb{Q}$-coefficients. By Lemma C.3 we see that both sides of (47) can be defined as the inverse limit over a cover of X by finite-type open substacks $\mathcal{U}_i$ and the map (47) is the map induced by the compatible system of maps

$$\cap\,[\mathcal{U}_i] \colon \mathrm{CH}^*_{\mathrm{OP}}(\mathcal{U}_i) \to \mathrm{CH}_{n-*}(\mathcal{U}_i),\ c \mapsto c(\mathrm{id}) \cdot [\mathcal{U}_i]\,.$$

Thus it suffices to prove the result for X of finite type over k.

To start, there exists a map

$$\Phi \colon \mathrm{CH}_{n-*}(X) \to \mathrm{CH}^*_{\mathrm{OP}}(X) \tag{48}$$

constructed in [4, Section 2.3]. Given $\beta \in \mathrm{CH}_{n-*}(X)$ and $\varphi : B \to X$ from an algebraic stack finite type over k, stratified by global quotient stacks, consider the graph morphism $\varphi_B : B \to B \times X$. Then φ_B is representable and regular local immersion because X is smooth over k. The map Φ is defined by

$$\Phi(\beta)(\varphi) : \mathrm{CH}_*(B) \to \mathrm{CH}_*(B), \alpha \mapsto \varphi_B^! \left(\alpha \times \beta\right) .$$

We show that Φ is the inverse of $\cap [X]$ following the parallel argument in [17, Chapter 17]. We note that from the definition of Φ it is easy to see that it is multiplicative, using that the product in $\mathrm{CH}_*(X)$ is defined by $\beta_1 \cdot \beta_2 = \Delta^!(\beta_1 \times \beta_2)$, where $\Delta\colon X \to X \times X$ is the diagonal morphism.

Let $p_2\colon X \times X \to X$ be the projection to the second factor. For all $\beta \in \mathrm{CH}_*(X)$, we have

$$\Delta^! \left([X] \times \beta\right) = \Delta^! p_2^* \beta = \Delta^! p_2^! \beta = \beta$$

by the functoriality of the Gysin pullback. It shows that $\Phi(\beta) \cap [X] = \beta$.

We prove the other direction. For $c \in \mathrm{CH}^*_{\mathrm{OP}}(X)$ and $\alpha \in \mathrm{CH}_*(B)$ it is sufficient to prove that

$$\varphi_B^! \left(\alpha \times (c(\mathrm{id}) \cdot [X])\right) = c(\varphi) \cdot \alpha \,.$$

Let $p_2\colon B \times X \to X$ be the projection to the second factor. We first prove

$$\alpha \times (c(\mathrm{id}) \cdot [X]) = c(p_2) \cdot (\alpha \times [X]) \text{ in } \mathrm{CH}_*(B \times X)\,. \tag{49}$$

When B is equidimensional and α is the class of the fundamental class $[B]$, the equality follows from the compatibility of c with flat pullback by p_2:

$$[B] \times (c(\mathrm{id}) \cdot [X]) = p_2^* \left(c(\mathrm{id}) \cdot [X]\right) = c(p_2) \cdot [B \times X]\,. \tag{50}$$

A general class α can be represented as (f, α_0) where $f\colon Y \to B$ is a projective morphism and E is a vector bundle on Y and $\alpha_0 \in \mathrm{CH}^\circ_*(E)$. Adding a trivial component to Y we may assume f is surjective. By Proposition B.2 and the homotopy invariance property it is enough to check this equality for a class in $\mathrm{CH}^\circ_*(E)$. A class in $\mathrm{CH}^\circ_*(E)$ can be written as a linear combination of classes $[V]$ where $j : V \to E$ are integral closed substacks. Thus it suffices to show the statement for $\alpha_0 = [V]$. For this, consider the composition of maps

$$V \times X \xrightarrow{J = j \times \mathrm{id}} E \times X \xrightarrow{p_2} X\,.$$

By the definition of exterior product we have

$$J_*([V] \times (c(\mathrm{id}) \cdot [X])) = j_*[V] \times (c(\mathrm{id}) \cdot [X])$$

in $\mathrm{CH}_*(E \times X)$. Then we get

$$\begin{aligned} j_*[V] \times (c(\mathrm{id}) \cdot [X]) &= (J)_*([V] \times c(\mathrm{id}) \cdot [X]) \\ &= (J)_*(c(p_2 \circ J) \cdot [V \times X]) \\ &= c(p_2) \cdot J_*[V \times X] \\ &= c(p_2) \cdot (j_*[V] \times [X]) \end{aligned}$$

where the second equality follows from the proven case (50) and the third equality follows from the compatibility of proper pushforward. Using (49) we then have

$$\begin{aligned} \varphi_B^! \left(\alpha \times (c(\mathrm{id}) \cdot [X])\right) &= \varphi_B^! \left(c(p_2) \cdot (\alpha \times [X])\right) \\ &= c(\varphi) \cdot \varphi_B^! \left(\alpha \times [X]\right) = c(\varphi) \cdot \alpha \end{aligned}$$

which proves the theorem. □

The above theorem gives the commutativity of operational Chow group under assumptions.

Corollary C.7. Let X be an algebraic stack stratified by global quotient stacks.

a) When X is smooth over k, $\mathrm{CH}^*_{\mathrm{OP}}(X)_{\mathbb{Q}}$ is a commutative ring.

b) When $\mathrm{char}(k) = 0$, $\mathrm{CH}^*_{\mathrm{OP}}(X)_{\mathbb{Q}}$ is a commutative ring.

Proof. We adapt the proof of [17, Example 17.4.4]. Part a) is a direct consequence of Theorem C.6 since the intersection product of [21] is commutative:

$$\alpha \cdot \beta = \Delta^!(\alpha \times \beta) = \Delta^!(\beta \times \alpha) = \beta \cdot \alpha$$

for $\alpha, \beta \in \mathrm{CH}_*(X)$ and where $\Delta : X \to X \times X$ is the diagonal, which is invariant under switching the factors of $X \times X$.

The proof of b) relies on functorial resolution of singularities. When $\mathrm{char}(k) = 0$, the resolution of singularities can be done functorially with respect to smooth morphisms, see [13]. Therefore we can find a smooth stack $\widetilde{X}$ and a representable, surjective, birational morphism $p : \widetilde{X} \to X$. Since X is stratified by global quotient stacks and p is representable, $\widetilde{X}$ is also stratified by global quotient stacks. Therefore the commutativity of $\mathrm{CH}^*_{\mathrm{OP}}(X)_{\mathbb{Q}}$ follows because the pullback p^* on $\mathrm{CH}^*_{\mathrm{OP}}(-)_{\mathbb{Q}}$ is injective. □

When k is a perfect field, the commutativity of rational operational Chow groups of schemes follows from de Jong's alteration ([11]). However, the authors do not know whether a functorial (with respect to smooth morphisms) construction of alteration is possible. Hence we cannot prove for now the commutativity of operational Chow groups for algebraic stacks over a perfect field of positive characteristic.

However, we note that nonetheless all results and formulas concerning tautological classes discussed in the main text are valid over arbitrary fields. Indeed, the operational classes only appear in intermediate steps of some of the computations and while these contain some examples of non-smooth spaces (like the universal curve over the space $\mathfrak{M}_\Gamma$), we are never in the position of having to exchange orders of multiplication of operational classes on these singular spaces.

In the main part of the paper we use Theorem C.6 above to realize tautological classes in $\mathrm{CH}_*(\mathfrak{M}_{g,n,a})$ as operational Chow classes. The following lemma is then useful when for doing calculus between tautological classes.

Lemma C.8. Consider the following commutative diagram

$$\begin{tikzcd}
U \arrow[dr, "p"] \arrow[drr, bend left, "r"] \arrow[ddr, bend right, "s"'] & & \\
& W \arrow[r, "f'"] \arrow[d, "\pi'"] & Z \arrow[d, "\pi"] \\
& X \arrow[r, "f"] & Y
\end{tikzcd}$$

where all stacks are locally of finite type over k, equidimensional, stratified by global quotient stacks and the square in the middle is a cartesian square. Suppose f is representable, proper and π is representable, proper, flat and p is representable, proper, birational. For $\alpha \in \mathrm{CH}^*_{\mathrm{OP}}(X)$

$$\pi^* f_*(\alpha \cap [X]) = r_*(s^*\alpha \cap [U]) \quad \text{in } \mathrm{CH}_*(Z, \mathbb{Q})\,.$$

Proof. Using the compatibility of proper pushforward and flat pullback of the cycle $\alpha \cap [X]$ in $\mathrm{CH}_*(X, \mathbb{Q})$ and the fact that, by definition, α is compatible with flat pullback by π', we have

$$\begin{aligned}
\pi^* f_*(\alpha \cap [X]) &= f'_*(\pi')^*(\alpha \cap [X]) \\
&= f'_*\left(((\pi')^*\alpha) \cap (\pi')^*[X]\right) \\
&= f'_*\left(((\pi')^*\alpha) \cap [W]\right)\,.
\end{aligned}$$

Since $p_*[U] = W$, we use the projection formula for the proper morphism p and obtain

$$\begin{aligned}
f'_*\left(((\pi')^*\alpha) \cap [W]\right) &= f'_*\left(((\pi')^*\alpha) \cap p_*[U]\right) \\
&= f'_*\left(p_*\left((p^*(\pi')^*\alpha) \cap [U]\right)\right) \\
&= r_*(s^*\alpha) \cap [U].
\end{aligned}$$

This proves the identity. $\square$

We conclude the section by comparing the approach to operational classes above to the one taken in the paper [4]. This paper studies the intersection theory of the universal Picard stack $\mathfrak{Pic}_{g,n}$ of the universal curve $\mathfrak{C}_{g,n} \to \mathfrak{M}_{g,n}$. However, instead of studying the operational Chow classes defined above, this paper considers operational Chow groups $\mathrm{CH}^*_{\mathrm{op}}(\mathfrak{Pic}_{g,n})$ where the test spaces $B \to X$ are restricted to be finite type *schemes*. A class $c \in \mathrm{CH}^p_{\mathrm{op}}(X)$ on a locally finite type algebraic stack X over k is a collection of operations

$$c(\varphi) : \mathrm{CH}_*(B) \to \mathrm{CH}_{*-p}(B)$$

for every morphism $\varphi : B \to X$ where B is a scheme of finite type over k, satisfying compatibility conditions as in Definition C.1, see [4, Definition 10] for details. We have comparison maps between $\mathrm{CH}^*_{\mathrm{OP}}, \mathrm{CH}^*_{\mathrm{op}}$ and CH^*.[18] As explained in [4, Section 2.3], for X smooth, equidimensional and admitting a stratification by global quotient stacks, there exists a natural map

$$\mathrm{CH}^*(X) \to \mathrm{CH}^*_{\mathrm{op}}(X)\,. \tag{51}$$

On the other hand for any algebraic stack X, there exists a natural map

$$\mathrm{CH}^*_{\mathrm{OP}}(X) \to \mathrm{CH}^*_{\mathrm{op}}(X) \tag{52}$$

defined by the restriction. The following statement is a direct consequence of Theorem C.6.

Corollary C.9. When X is an equidimensional smooth Deligne-Mumford stack over k, the comparison maps (51) and (52) are isomorphisms.

Proof. Indeed, for the two maps

$$\mathrm{CH}^*(X) \to \mathrm{CH}^*_{\mathrm{OP}}(X) \to \mathrm{CH}^*_{\mathrm{op}}(X)$$

we have that the first is an isomorphism by Theorem C.6 and the composition is an isomorphism by [4, Lemma 15] and thus also the second map must be an isomorphism. $\square$

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Chow rings of stacks of prestable curves II

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Abstract

We continue the study of the Chow ring of the moduli stack $\mathfrak{M}_{g,n}$ of prestable curves begun in [5]. In genus 0, we show that the Chow ring of $\mathfrak{M}_{0,n}$ coincides with the tautological ring and give a complete description in terms of (additive) generators and relations. This generalizes earlier results by Keel and Kontsevich-Manin for the spaces of stable curves. Our argument uses the boundary stratification of the moduli stack together with the study of the first higher Chow groups of the strata, in particular providing a new proof of the results of Kontsevich and Manin.

1 Introduction

The tautological ring of the moduli stack of prestable curves

Let $\mathfrak{M}_{g,n}$ be the moduli stack of prestable curves of genus g with n markings. It is a natural extention of the Deligne-Mumford space $\overline{\mathcal{M}}_{g,n}$ of stable curves. In the paper [5], we studied the rational Chow ring[1] $\mathrm{CH}^*(\mathfrak{M}_{g,n})$ and its subring

$$\mathrm{R}^*(\mathfrak{M}_{g,n}) \subseteq \mathrm{CH}^*(\mathfrak{M}_{g,n})$$

of *tautological classes*, naturally extending the corresponding notion on $\overline{\mathcal{M}}_{g,n}$.

To describe the elements of $\mathrm{R}^*(\mathfrak{M}_{g,n})$, let

$$\pi\colon \mathfrak{C}_{g,n} \to \mathfrak{M}_{g,n}$$

be the universal curve and let ω_π be the relative dualizing sheaf. Let

$$\sigma_i\colon \mathfrak{M}_{g,n} \to \mathfrak{C}_{g,n}$$

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[1]We assume (g,n) is different from $(1,0)$.

be the i-th universal section and $\mathfrak{S}_i \subset \mathfrak{C}_{g,n}$ be the corresponding divisor. We define ψ and κ-classes: given $1 \leq i \leq n$ we set

$$\psi_i = c_1(\sigma_i^* \omega_\pi) \in \mathrm{CH}^1(\mathfrak{M}_{g,n}) , \tag{1}$$

and for given $m \geq 0$ we set

$$\kappa_m = \pi_* \left(c_1 \left(\omega_\pi \left(\sum_{i=1}^n \mathfrak{S}_i \right) \right)^{m+1} \right) \in \mathrm{CH}^m(\mathfrak{M}_{g,n}) . \tag{2}$$

Let Γ be a prestable[2] graph in genus g with n markings. Each prestable graph defines a gluing map

$$\xi_\Gamma : \mathfrak{M}_\Gamma = \prod_{v \in V(\Gamma)} \mathfrak{M}_{g(v),n(v)} \to \mathfrak{M}_{g,n}$$

(see e.g. [5, Section 2.1]). Given any prestable graph Γ, consider the products

$$\alpha = \prod_{v \in V} \left(\prod_{i \in H(v)} \psi_{v,i}^{a_i} \prod_{a=1}^{m_v} \kappa_{v,a}^{b_{v,a}} \right) \in \mathrm{CH}^*(\mathfrak{M}_\Gamma) . \tag{3}$$

of ψ and κ-classes on the space $\mathfrak{M}_\Gamma$ above. Then we define the *decorated stratum class* $[\Gamma, \alpha]$ as the pushforward

$$[\Gamma, \alpha] = (\xi_\Gamma)_* \alpha \in \mathrm{R}^*(\mathfrak{M}_{g,n}).$$

Definition 1.1. *The tautological ring* $\mathrm{R}^*(\mathfrak{M}_{g,n})$ *is the* $\mathbb{Q}$*-subspace of* $\mathrm{CH}^*(\mathfrak{M}_{g,n})$ *additively generated by decorated strata classes.*[3]

The paper [5] then develops a calculus of decorated stratum classes. Below, such results from [5] are frequently referred.

In full generality, a description of the tautological ring $\mathrm{R}^*(\mathfrak{M}_{g,n})$ is hard to approach. In this paper we specialize our attention to the moduli space of genus zero prestable curves.

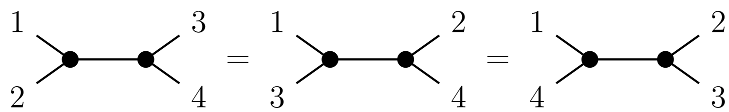
The tautological ring in genus zero

In Section 2, we give a complete description of the Chow groups of $\mathfrak{M}_{0,n}$ in terms of explicit generators and relations.

[2]A prestable graph is given by the same data as a stable graph, except that one removes the condition that every vertex v should be stable, i.e. satisfy $2g(v) - 2 + n(v) > 0$.

[3]In [5, Definition 1.3] the tautological ring of $\mathfrak{M}_{g,n}$ is defined in a much more conceptual way, but we show that it is equivalent to the above presentation ([5, Theorem 1.4]).

For the moduli spaces $\overline{\mathcal{M}}_{0,n}$ of stable curves, Keel [22] proved that the tautological ring of $\overline{\mathcal{M}}_{0,n}$ coincides with the Chow ring. Moreover, he showed that this ring is generated as an *algebra* by the boundary divisors of $\overline{\mathcal{M}}_{0,n}$ and that the *ideal* of relations is generated by the *WDVV relations*, the pullbacks of the relations



(4)

in $\mathrm{CH}^1(\overline{\mathcal{M}}_{0,4})$ under the forgetful maps $\overline{\mathcal{M}}_{0,n} \to \overline{\mathcal{M}}_{0,4}$, together with the relations $D_1 \cdot D_2 = 0$ for D_1, D_2 disjoint boundary divisors.

Later, Kontsevich and Manin [26, 27] showed that the Chow groups of $\overline{\mathcal{M}}_{0,n}$ are generated as a $\mathbb{Q}$-*vector space* by the classes of the closures of boundary strata of $\overline{\mathcal{M}}_{0,n}$. Moreover, the set of *linear* relations between such strata classes are generated by the pushforwards of WDVV relations under boundary gluing maps. Our treatment of the Chow groups of $\mathfrak{M}_{0,n}$ will be closer in spirit to the one by Kontsevich and Manin, since we provide *additive* generators and relations.

Generators

A first new phenomenon we see for $\mathfrak{M}_{0,n}$ is that its Chow group is no longer generated by boundary strata. This comes from the fact that for $n = 0, 1, 2$, the loci $\mathfrak{M}^{\mathrm{sm}}_{0,n} \subset \mathfrak{M}_{0,n}$ of smooth curves already have non-trivial Chow groups. They are given by polynomial algebras

$$\mathrm{CH}^*(\mathfrak{M}^{\mathrm{sm}}_{0,0}) = \mathbb{Q}[\kappa_2], \quad \mathrm{CH}^*(\mathfrak{M}^{\mathrm{sm}}_{0,1}) = \mathbb{Q}[\psi_1], \quad \mathrm{CH}^*(\mathfrak{M}^{\mathrm{sm}}_{0,2}) = \mathbb{Q}[\psi_1]. \tag{5}$$

generated by the class κ_2 on $\mathfrak{M}_{0,0}$ and the classes ψ_1 on $\mathfrak{M}_{0,1}$ and $\mathfrak{M}_{0,2}$.[4] So we see that the Chow group can no longer be generated by boundary strata because all strata contained in the boundary restrict to zero on the locus $\mathfrak{M}^{\mathrm{sm}}_{0,n}$ of smooth curves. For $n \geq 3$, the complement $\mathfrak{M}_{0,n} \setminus \mathfrak{M}^{\mathrm{sm}}_{0,n}$ contains strata of the form $\mathfrak{M}^{\mathrm{sm}}_{0,1} \times \mathfrak{M}^{\mathrm{sm}}_{0,n}$. Then one can combine the excision sequence for $\mathfrak{M}^{\mathrm{sm}}_{0,n} \subset \mathfrak{M}_{0,n}$, and the description of the Chow group of $\mathfrak{M}^{\mathrm{sm}}_{0,1}$ to show that $\mathrm{CH}^*(\mathfrak{M}_{0,n})$ is not generated as a vector space by the boundary strata.[5]

Instead, we prove that $\mathrm{CH}^*(\mathfrak{M}_{0,n})$ is generated by strata of $\mathfrak{M}_{0,n}$ decorated by κ and ψ-classes. More precisely, the generators are indexed by the data $[\Gamma, \alpha]$, where Γ is a prestable graph (describing the shape of the generic curve inside the boundary stratum) and α is a product of ψ-classes at vertices of Γ with 1 or 2 outgoing half-edges (or Γ is the trivial graph for $\mathfrak{M}_{0,0}$ and $\alpha = \kappa_2^a$). We call such a class

[4]These κ and ψ-classes are defined similarly to the corresponding classes on the moduli space of stable curves, see [5, Definition 3.2].

[5]In particular, this statement follows from our full description of the tautological relations in $\mathfrak{M}_{0,n}$ given in Theorem 1.4.

$[\Gamma, \alpha]$ a *decorated stratum class in normal form.* The allowed decorations α precisely reflect the non-trivial Chow groups (5) above. We illustrate some of the generators that appear in Figure 1, for the precise construction of the corresponding classes $[\Gamma, \alpha] \in \mathrm{CH}^*(\mathfrak{M}_{0,n})$ see [5, Definition 3.3].

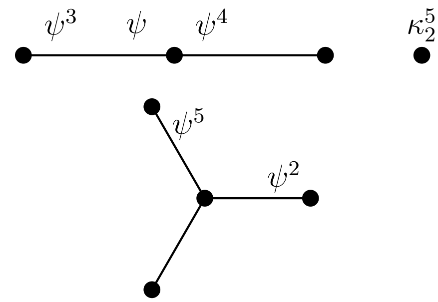


Figure 1: Some decorated strata classes $[\Gamma, \alpha]$ in normal form, giving generators of $\mathrm{CH}^{10}(\mathfrak{M}_{0,0})$

In particular, since all such classes are contained in the tautological ring, we generalize Keel's result that all Chow classes on $\overline{\mathcal{M}}_{0,n}$ are tautological.

Theorem 1.2. *For $n \geq 0$ we have the equality $\mathrm{CH}^*(\mathfrak{M}_{0,n}) = \mathrm{R}^*(\mathfrak{M}_{0,n})$.*

The idea of proof for this first theorem is easy to describe: consider the excision sequence of Chow groups for the open substack $\mathfrak{M}_{0,n}^{\mathrm{sm}} \subset \mathfrak{M}_{0,n}$ with complement $\partial\, \mathfrak{M}_{0,n}$:

$$\mathrm{CH}^{*-1}(\partial\, \mathfrak{M}_{0,n}) \to \mathrm{CH}^*(\mathfrak{M}_{0,n}) \to \mathrm{CH}^*(\mathfrak{M}_{0,n}^{\mathrm{sm}}) \to 0. \tag{6}$$

From (5) for $n = 0, 1, 2$ and the classical statement

$$\mathrm{CH}^*(\mathfrak{M}_{0,n}^{\mathrm{sm}}) = \mathrm{CH}^*(\mathcal{M}_{0,n}) = \mathbb{Q} \cdot [\mathcal{M}_{0,n}] \text{ for } n \geq 3 \tag{7}$$

we see that all classes in $\mathrm{CH}^*(\mathfrak{M}_{0,n}^{\mathrm{sm}})$ have tautological representatives. It follows that it suffices to prove that all classes supported on $\partial\, \mathfrak{M}_{0,n}$ are tautological. But $\partial\, \mathfrak{M}_{0,n}$ is parameterized (via the union of finitely many gluing morphisms) by products of spaces $\mathfrak{M}_{0,n_i}$. This allows us to set up a recursive proof.

One thing to verify in this last part of the argument is that the Chow group of a product of spaces $\mathfrak{M}_{0,n_i}$ is generated by cycles coming from the factors $\mathfrak{M}_{0,n_i}$. In fact, we can show more, namely that the stacks of prestable curves in genus 0 satisfy a certain *Chow-Künneth property.* To formulate it, we need to introduce two technical properties of locally finite type stacks Y: we say that Y has a *good filtration by finite type stacks* if Y is the union of an increasing sequence $(\mathcal{U}_j)_j$ of finite type open substacks such that the codimension of the complement of $\mathcal{U}_j$ becomes arbitrarily large as j increases. We say that Y has a *stratification by quotient stacks* if there exists a stratification of Y by locally closed substacks which are each isomorphic to a global quotient of an algebraic space by a linear algebraic group. All stacks $\mathfrak{M}_{g,n}$ for $(g, n) \neq (1, 0)$ satisfy both of these properties.

Proposition 1.3 (Proposition 2.6, Corollary 2.22)**.** *Consider the stack $\mathfrak{M}_{0,n}$ (for $n \geq 0$) and let Y be a locally finite type stack. Then the map*[6]

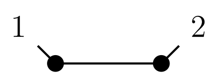
$$\mathrm{CH}_*(\mathfrak{M}_{0,n}) \otimes_{\mathbb{Q}} \mathrm{CH}_*(Y) \to \mathrm{CH}_*(\mathfrak{M}_{0,n} \times Y), \alpha \otimes \beta \mapsto \alpha \times \beta$$

is surjective if Y has a good filtration by finite type stacks and a stratification by quotient stacks. The map is an isomorphism if Y is a quotient stack.

In the proposition above, the technical conditions (like Y being a quotient stack or having a stratification by quotient stacks) are currently needed since some of the results we cite in our proof have them as assumptions. We expect that these conditions can be relaxed, but do not pursue this since Proposition 1.3 is sufficient for the purpose of our paper.

Relations

Returning to the stacks $\mathfrak{M}_{0,n}$ themselves, we also give a full description of the set of linear relations between the generators $[\Gamma, \alpha]$ above. An important example is the degree one relation

$\psi_1 + \psi_2 =$ 

$\in \mathrm{CH}^1(\mathfrak{M}_{0,2})$ (8)

on $\mathfrak{M}_{0,2}$. What we can show is that *all* tautological relations in genus 0 are implied by the relation (8) together with the natural extension of the relation (4) to $\mathrm{CH}^1(\mathfrak{M}_{0,4})$.

Theorem 1.4 (informal version, see Theorems 2.31 and 2.33)**.** *For $n \geq 0$, the system of all linear relations in $\mathrm{CH}^*(\mathfrak{M}_{0,n})$ between the decorated strata classes $[\Gamma, \alpha]$ in normal form is generated by the WDVV relation (4) on $\mathfrak{M}_{0,4}$ and the relation (8) on $\mathfrak{M}_{0,2}$.*

We give a precise description of what we mean by the system of relations "generated" by (4) and (8) in Definition 2.27, but roughly the allowed operations are as follows:

- For $n \geq 4$ we can pull back the WDVV relation (4) under the morphism $\mathfrak{M}_{0,n} \to \mathfrak{M}_{0,4}$ forgetting $n - 4$ of the marked points.
- We can multiply the relation (8) by an arbitrary polynomial in ψ_1, ψ_2.

[6] Below, the notation $\alpha \times \beta$ denotes the exterior product of cycles constructed in [28, Proposition 3.2.1].

- Given a decorated stratum $[\Gamma_0, \alpha_0]$ in normal form, a vertex $v \in V(\Gamma_0)$ at which α_0 is the trivial decoration and a known relation R_0 in the Chow group $\mathrm{CH}^*(\mathfrak{M}_{0,n(v)})$ associated to the vertex v, we can create a new relation by gluing R_0 into the vertex v of $[\Gamma, \alpha]$. In other words, for $\pi_v : \mathfrak{M}_\Gamma \to \mathfrak{M}_{0,n(v)}$ the projection on the factor associated to v, the new relation is given by

$$[\Gamma, \alpha] = (\xi_{\Gamma_0})_* (\alpha_0 \cdot \pi_v^* R_0) = 0 \in \mathrm{CH}^*(\mathfrak{M}_{0,n}) .$$

  See Example 2.28 for an illustration.

Again, our proof strategy for Theorem 1.4 begins by looking at the excision sequence (6), but now extended on the left using the first higher Chow group of $\mathfrak{M}_{0,n}^{\mathrm{sm}}$, again defined by the work of [28]

$$\mathrm{CH}^*(\mathfrak{M}_{0,n}^{\mathrm{sm}}, 1) \xrightarrow{\partial} \mathrm{CH}^{*-1}(\partial\, \mathfrak{M}_{0,n}) \to \mathrm{CH}^*(\mathfrak{M}_{0,n}) \to \mathrm{CH}^*(\mathfrak{M}_{0,n}^{\mathrm{sm}}) \to 0 . \tag{9}$$

To illustrate how we can compute tautological relations using this sequence, consider the set of prestable graphs Γ_i with exactly one edge. The associated decorated strata classes $[\Gamma_i] = [\Gamma_i, 1]$ are supported on $\partial\mathfrak{M}_{0,n}$ and in fact form a basis of $\mathrm{CH}^0(\partial\mathfrak{M}_{0,n})$. Then from (9) we see that the linear relations between the classes $[\Gamma_i] \in \mathrm{CH}^1(\mathfrak{M}_{0,n})$ are exactly determined by the image of ∂.

For this purpose, we compute the first higher Chow groups of the space $\mathfrak{M}_{0,0}^{\mathrm{sm}}$ and finite products of spaces $\mathfrak{M}_{0,n_i}^{\mathrm{sm}}$ ($n_i \geq 1$), which parametrize strata in the boundary of $\mathfrak{M}_{0,n}$. The corresponding results are given in Propositions 2.14 and 2.16. The proof of Theorem 1.4 then proceeds by an inductive argument using, again, the stratification of $\mathfrak{M}_{0,n}$ according to dual graphs.

Restricting our argument to the moduli spaces $\overline{\mathcal{M}}_{0,n}$ of stable curves, our approach to tautological relations via higher Chow groups gives a new proof that the relations between classes of strata are additively generated by boundary pushforwards of WDVV relations. As mentioned before, this result was originally stated by Kontsevich and Manin in [26, Theorem 7.3] together with a sketch of proof which was expanded in [27].

The proof relied on Keel's result [22] that the WDVV relations generate the ideal of relations multiplicatively and thus required an explicit combinatorial analysis of the product structure of $\mathrm{CH}^*(\overline{\mathcal{M}}_{0,n})$. In turn, the original proof by Keel proceeded by constructing $\overline{\mathcal{M}}_{0,n}$ as an iterated blowup of $(\mathbb{P}^1)^{n-3}$, carefully keeping track how the Chow group changes in each step.

In comparison, our proof is more conceptual, since we can trace each WDVV-relation on $\overline{\mathcal{M}}_{0,n}$ to a generator of a higher Chow group $\mathrm{CH}^*(\mathcal{M}^\Gamma, 1)$ of some stratum $\mathcal{M}^\Gamma \subseteq \overline{\mathcal{M}}_{0,n}$ of the moduli space. A very similar approach appears in [40], where Petersen used the mixed Hodge structure of $\mathcal{M}_{0,n}$ and the spectral sequence associated to stratification of $\overline{\mathcal{M}}_{0,n}$ to reproduce [22, 26, 27].

A non-trivial consequence of our proof is the following result, stating that in codimension at least two the Chow groups of $\overline{\mathcal{M}}_{0,n}$ agree with the Chow groups of its boundary $\partial\overline{\mathcal{M}}_{0,n}$ (up to a degree shift).

Corollary 1.5 (see Corollary 2.37)**.** *Let $n \geq 4$, then the inclusion*

$$\iota : \partial\overline{\mathcal{M}}_{0,n} \to \overline{\mathcal{M}}_{0,n}$$

of the boundary of $\overline{\mathcal{M}}_{0,n}$ induces an isomorphism

$$\iota_* : \mathrm{CH}^\ell(\partial\overline{\mathcal{M}}_{0,n}) \to \mathrm{CH}^{\ell+1}(\overline{\mathcal{M}}_{0,n})$$

for $\ell > 0$.

This result follows easily using higher Chow groups: we have the exact sequence

$$\mathrm{CH}^{\ell+1}(\mathcal{M}_{0,n}, 1) \xrightarrow{\partial} \mathrm{CH}^\ell(\partial\overline{\mathcal{M}}_{0,n}) \xrightarrow{\iota_*} \mathrm{CH}^{\ell+1}(\overline{\mathcal{M}}_{0,n}) \to 0\,.$$

Using that $\mathcal{M}_{0,n}$ can be seen as a hyperplane complement in $\mathbb{A}^{n-3}$, it is easy to show that the group $\mathrm{CH}^{\ell+1}(\mathcal{M}_{0,n}, 1)$ vanishes for $\ell > 0$. Thus for $\ell > 0$ the map ι_* is an isomorphism by the exact sequence. In Remark 2.38 we explain how, alternatively, the corollary follows from the results [26, 27] of Kontsevich and Manin.

Relation to other work

Gromov-Witten theory

Gromov-Witten theory studies intersection numbers on the moduli spaces $\overline{\mathcal{M}}_{g,n}(X,\beta)$ of stable maps to a nonsingular projective variety X. Since the spaces of stable maps admit forgetful morphisms

$$\overline{\mathcal{M}}_{g,n}(X,\beta) \to \mathfrak{M}_{g,n},\ (f : (C, p_1, \ldots, p_n) \to X) \mapsto (C, p_1, \ldots, p_n), \tag{10}$$

results about the Chow groups of $\mathfrak{M}_{g,n}$ can often be translated to results about Gromov-Witten invariants of *arbitrary* target varieties X.[7]

As an example, in [17] Gathmann used the pullback formula of ψ-classes along the stabilization morphism $\mathrm{st}\colon \mathfrak{M}_{g,1} \to \overline{\mathcal{M}}_{g,1}$ to prove certain properties of the Gromov-Witten potential. Similarly, the paper [32] proved degree one relations on the moduli space $\overline{\mathcal{M}}_{0,n}(\mathbb{P}^N, d)$ of stable maps to a projective space and used them to reduce two pointed genus 0 potentials to one pointed genus 0 potentials. As we explain in Example 2.40, the relations used in [32] are the pullback of the tautological relation (8) on $\mathfrak{M}_{0,2}$ and a similar relation on $\mathfrak{M}_{0,3}$ under forgetful morphisms (10).

[7] Note that a priori it is not possible to directly pull back classes in $\mathrm{CH}^*(\mathfrak{M}_{g,n})$ under the map $\overline{\mathcal{M}}_{g,n}(X,\beta) \to \mathfrak{M}_{g,n}$, since this map is in general neither flat nor lci. However, there exists an isomorphism

$$\mathrm{CH}^*(\mathfrak{M}_{g,n}) \to \mathrm{CH}^*_{\mathrm{OP}}(\mathfrak{M}_{g,n})$$

from the Chow group of $\mathfrak{M}_{g,n}$ to its operational Chow group, and operational Chow classes are functorial under arbitrary morphisms. Then, any operational Chow class acts on the Chow group of $\overline{\mathcal{M}}_{g,n}(X,\beta)$, see Section [5, Appendix C.].

Chow rings of open substacks of $\mathfrak{M}_{0,n}$

Several people have studied Chow rings with rational coefficients of open substacks of $\mathfrak{M}_{0,n}$, and we explain how their results relate to ours.

In [36], Oesinghaus computed the Chow rings of the loci $\mathfrak{M}_{0,2}^{\mathrm{ss}}$ and $\mathfrak{M}_{0,3}^{\mathrm{ss}}$ of semistable curves in $\mathfrak{M}_{0,2}$ and $\mathfrak{M}_{0,3}$. His proof identified the rings in terms of the known algebra of quasi-symmetric functions QSym (see [33] for an overview). However, for many generators of QSym it remained unclear which (geometric) cycle classes on $\mathfrak{M}_{0,2}^{\mathrm{ss}}$ and $\mathfrak{M}_{0,3}^{\mathrm{ss}}$ they corresponded to. In [5] we answered this question, identifying an additive basis of QSym with explicit decorated strata classes in the tautological rings of $\mathfrak{M}_{0,2}^{\mathrm{ss}}$ and $\mathfrak{M}_{0,3}^{\mathrm{ss}}$. In Example 2.42 below we continue this argument by showing how Theorem 1.2 and 1.4 can be used to give a new proof of Oesinghaus' results, showing that the decorated strata classes above are indeed linearly independent generators of the Chow group.

On the other hand, in [14, 15, 13] Fulghesu gave a computation of the Chow ring $\mathrm{CH}^*(\mathfrak{M}_{0,3}^{\leq 3})$ of the locus $\mathfrak{M}_{0,0}^{\leq 3}$ of curves with at most three nodes inside $\mathfrak{M}_{0,0}$. Using a computer program, we compare his results to ours and find that our results *almost* agree, except for the fact that in [13] there is a missing tautological relation in the final step of the proof. This is explained in detail in Example 2.43.

Outlook and open questions

We want to finish the introduction with a discussion of some conjectures and questions about the Chow groups of $\mathfrak{M}_{g,n}$.

The first concerns the relation to the Chow groups of the moduli spaces $\overline{\mathcal{M}}_{g,n}$ of stable curves. Since $\overline{\mathcal{M}}_{g,n}$ is an open substack of $\mathfrak{M}_{g,n}$, the Chow groups of $\mathfrak{M}_{g,n}$ determine those of $\overline{\mathcal{M}}_{g,n}$. The following conjecture would imply that the converse holds as well.

Conjecture (Conjecture 3.1)**.** *Let $(g,n) \neq (1,0)$, then for a fixed $d \geq 0$ there exists $m_0 \geq 0$ such that for any $m \geq m_0$, the forgetful morphism*[8]

$$F_m : \overline{\mathcal{M}}_{g,n+m} \to \mathfrak{M}_{g,n}, (C, p_1, \ldots, p_n, p_{n+1}, \ldots, p_{n+m}) \mapsto (C, p_1, \ldots, p_n)$$

satisfies that the pullback

$$F_m^* : \mathrm{CH}^d(\mathfrak{M}_{g,n}) \to \mathrm{CH}^d(\overline{\mathcal{M}}_{g,n+m})$$

is injective.

It is easy to see that the system of morphisms $(F_m)_{m \geq 0}$ forms an atlas of $\mathfrak{M}_{g,n}$ and that the complement of the image of F_m has arbitrarily large codimension as m

[8]Note that, importantly, the morphism F_m does not stabilize the curve C, it simply forgets the last m markings and returns the corresponding prestable curve.

increases. Thus for a fixed degree d, the Chow groups $\mathrm{CH}^d(F_m(\overline{\mathcal{M}}_{g,n+m}))$ converge to $\mathrm{CH}^d(\mathfrak{M}_{g,n})$, but it remains to verify that the pullback by F_m indeed becomes injective. In Section 3.1 we provide some additional motivation and a number of cases (n,d) in genus zero where the conjecture holds.

Since the map F_m^* sends tautological classes on $\mathfrak{M}_{g,n}$ to tautological classes in $\overline{\mathcal{M}}_{g,n+m}$, the conjecture would also imply that knowing all tautological rings of moduli spaces of stable curves would uniquely determine the tautological rings of the stacks of prestable curves. In [41], Pixton proposed a set of relations between tautological classes on the moduli spaces of stable curves, proven to hold in cohomology [38] and in Chow [20], and he conjectured that these are *all* tautological relations. Combined with the conjecture above, this would then determine all tautological rings of the stacks $\mathfrak{M}_{g,n}$. It is an interesting question if Pixton's set of relations can also be generalized directly to the stacks of prestable curves to give a conjecturally complete set of relations.

Finally, recall that Theorems 1.2 and 1.4 completely determine the Chow rings of $\mathfrak{M}_{0,n}$. Given an open substack $U \subseteq \mathfrak{M}_{0,n}$ which is a union of strata, it is easy to see that $\mathrm{CH}^*(U)$ is the quotient of $\mathrm{CH}^*(\mathfrak{M}_{0,n})$ by the span of all tautological classes supported on the complement of U, so the Chow rings of such U are likewise determined.

For such open substacks U we can ask some more refined questions. The first concerns the structure of $\mathrm{CH}^*(U)$ as an algebra.

Question 1 (Question 2.44)**.** Is it true that for $U \subset \mathfrak{M}_{0,n}$ an open substack of finite type which is a union of strata, the Chow ring $\mathrm{CH}^*(U)$ is a finitely generated $\mathbb{Q}$-algebra?

Supporting evidence for this question is that it has an affirmative answer for all stacks $\mathfrak{M}_{0,n}^{\mathrm{sm}}$ by (5) and (7), and by the computations in [13] also for the substacks $U = \mathfrak{M}_{0,0}^{\leq e}$, $e = 0,1,2,3$, of unmarked rational curves with at most e nodes. Similar to the proof technique in [13], a possible approach to Question 1 for arbitary U is to gradually enlarge U, adding one stratum of the moduli stack $\mathfrak{M}_{0,n}$ at a time and showing in each step that only finitely many additional generators are necessary.

Note that for U not of finite type, Question 1 will have a negative answer in general: from [36] it is easy to see that the Chow ring $\mathrm{CH}^*(\mathfrak{M}_{0,2}^{\mathrm{ss}})$ of the semistable locus in $\mathfrak{M}_{0,2}$ is not finitely generated as an algebra.

Our second question concerns the Hilbert series

$$H_U = \sum_{d \geq 0} \dim_{\mathbb{Q}} \mathrm{CH}^d(U) t^d$$

of the Chow ring of U.

Question 2 (Question 2.45)**.** Is it true that for $U \subset \mathfrak{M}_{0,n}$ any open substack which is a union of strata, the Hilbert series H_U is the expansion of a rational function at $t = 0$?

First note that a positive answer to Question 1 would imply Question 2 for all finite type substacks $U \subset \mathfrak{M}_{0,n}$, since the Hilbert series of a finitely generated graded algebra is a rational function, all of whose poles are at roots of unity ([34, Theorem 13.2]). However, Question 2 *also* has a positive answer for the non-finite type stacks $U = \mathfrak{M}_{0,2}^{\mathrm{ss}}$ and $\mathfrak{M}_{0,3}^{\mathrm{ss}}$ studied in [36]. In Figure 2 we collect some examples of Hilbert series for different U, computed in Example 2.43 and Section 2.6. Note how for $U = \mathfrak{M}_{0,2}^{\mathrm{ss}}$ or $\mathfrak{M}_{0,3}^{\mathrm{ss}}$ the rational function H_U has poles at $1/2$, which is not a root of unity (thus giving one way to see that the Chow rings are not finitely generated).

U	H_U
$\mathfrak{M}_0^{\leq 0}$	$\dfrac{1}{1-t^2}$
$\mathfrak{M}_0^{\leq 1}$	$\dfrac{1}{(1-t^2)(1-t)}$
$\mathfrak{M}_0^{\leq 2}$	$\dfrac{t^4+1}{(1-t^2)^2(1-t)}$
$\mathfrak{M}_0^{\leq 3}$	$\dfrac{t^6+t^5+2t^4+t^3+1}{(1-t^2)^2(1-t)(1-t^3)}$
$\mathfrak{M}_{0,2}^{\mathrm{ss}}$	$\dfrac{1}{1-2t}$
$\mathfrak{M}_{0,3}^{\mathrm{ss}}$	$\dfrac{(1-t)^3}{(1-2t)^3}$

Figure 2: The Hilbert series of the Chow rings of open substacks U of $\mathfrak{M}_{0,n}$

Structure of the paper

In Section 2 we treat the Chow groups of the stacks $\mathfrak{M}_{0,n}$ of prestable curves of genus zero. We start in Section 2.1 by computing the Chow groups of the loci $\mathfrak{M}_{0,n}^{\mathrm{sm}}$ of smooth curves and explaining how (most) κ and ψ-classes on $\mathfrak{M}_{0,n}$ can be expressed in terms of cycles supported on the boundary. In Section 2.2 we show that every class in the Chow ring of $\mathfrak{M}_{0,n}$ is tautological. In Section 2.3 we compute the first higher Chow groups of the strata of $\mathfrak{M}_{0,n}$ and use this in Section 2.4 to classify the tautological relations on $\mathfrak{M}_{0,n}$. We finish this part of the paper by discussing the relation to earlier work in Section 2.5 and including some observations and questions about Chow groups of open substacks of $\mathfrak{M}_{0,n}$ in Section 2.6.

In Section 3 we compare the Chow rings of the stacks $\mathfrak{M}_{g,n}$ of prestable curves and the stacks $\overline{\mathcal{M}}_{g,n}$ of stable curves. We present a conjectural relation between these in Section 3.1. We extend the known results about divisor classes on $\overline{\mathcal{M}}_{g,n}$ to $\mathfrak{M}_{g,n}$ in Section 3.2 and discuss how the study of zero cycles extends in Section 3.3.

Finally, Appendix A summarizes a construction of a Gysin pullback for higher Chow groups following [9, 24].

Notations and conventions

We work over an arbitrary base field k. For the convenience of the reader, we provide an overview of notations used in the paper in Table 1.

$\mathfrak{M}_{g,n}$	moduli space of prestable curves
$\mathfrak{M}_{g,n,a}$	moduli space of prestable curves with values in a semigroup
$\mathfrak{M}_\Gamma$	$\prod_{v\in V(\Gamma)} \mathfrak{M}_{g(v),n(v)}$, where Γ is a prestable graph
$\mathfrak{M}^\Gamma$	moduli space of curves with dual graph precisely Γ
$\mathcal{R}_{\mathrm{WDVV}}$	set of WDVV relations
$\mathcal{R}_{\kappa,\psi}$	set of ψ and κ relations

Table 1: Notations

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2 The Chow ring in genus 0

In this section we prove Theorem 1.2 and Theorem 1.4. These results completely describe the rational Chow group of $\mathfrak{M}_{0,n}$.

2.1 ψ and κ classes in genus 0

In [26, 27], Kontsevich and Manin described the Chow groups of $\overline{\mathcal{M}}_{0,n}$ via generators given by boundary strata and additive relations, called the *WDVV relations*. Their approach relies on the fact that every class on $\overline{\mathcal{M}}_{0,n}$ can be represented by boundary classes without ψ or κ classes. This is because the locus of smooth n pointed rational curves $\mathcal{M}_{0,n}$ has a trivial Chow group for $n \geq 3$.

However the Chow group of the locus of smooth curves $\mathfrak{M}_{0,n}^{\mathrm{sm}}$ is no longer trivial when $n = 0, 1, 2$ and hence not all tautological classes on $\mathfrak{M}_{0,n}$ can be represented by boundary classes. We first summarize what is known about the Chow groups of $\mathfrak{M}_{0,n}^{\mathrm{sm}}$. For a smooth group scheme G over k we write $BG := [\operatorname{Spec} k/G]$ for the classifying stack of G, whose S-points are G-torsors over S.

Lemma 2.1. *For the moduli spaces of prestable curves in genus* 0 *we have*

(a) $\mathfrak{M}_{0,0}^{\mathrm{sm}} = B\mathrm{PGL}_2$ *and* $\mathrm{CH}^*(\mathfrak{M}_{0,0}^{\mathrm{sm}}) = \mathbb{Q}[\kappa_2]$,

(b) $\mathfrak{M}_{0,1}^{\mathrm{sm}} = B\mathbb{U}$ *for*

$$\mathbb{U} = \left\{ \begin{bmatrix} a & b \\ 0 & d \end{bmatrix} \in \mathrm{PGL}_2 \right\} \cong \mathbb{G}_a \rtimes \mathbb{G}_m$$

and $\mathrm{CH}^*(\mathfrak{M}_{0,1}^{\mathrm{sm}}) = \mathbb{Q}[\psi_1]$,

(c) $\mathfrak{M}_{0,2}^{\mathrm{sm}} \cong B\mathbb{G}_m$ *and* $\mathrm{CH}^*(\mathfrak{M}_{0,2}^{\mathrm{sm}}) = \mathbb{Q}[\psi_1]$,

(d) $\mathfrak{M}_{0,n}^{\mathrm{sm}} = \mathcal{M}_{0,n}$ *and* $\mathrm{CH}^*(\mathfrak{M}_{0,n}^{\mathrm{sm}}) = \mathbb{Q} \cdot [\mathfrak{M}_{0,n}^{\mathrm{sm}}]$ *for* $n \geq 3$.

Proof. The first three statements are proved in [15]. The last statement comes from the fact that $\mathcal{M}_{0,n}$ is an open subscheme of $\mathbb{A}^{n-3}$. □

Note that for part (a) of the Lemma above, it is important that we work with $\mathbb{Q}$-coefficients. Indeed, the Chow groups with integral coefficients of $B\mathrm{PGL}_2 \cong B\mathrm{SO}(3)$ have been computed in [37] as

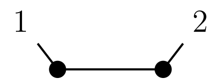
$$\mathrm{CH}^*(B\mathrm{PGL}_2)_{\mathbb{Z}} = \mathbb{Z}[c_1, c_2, c_3]/(c_1, 2c_3)\,,$$

so we see that there exists a non-trivial 2-torsion element in codimension 3.

By Lemma 2.1 we know that any monomial in κ and ψ-classes on $\mathfrak{M}_{0,n}$ can be written as a multiple of our preferred generators above (a power of κ_2 for $n = 0$ or a power of ψ_1 for $n = 1, 2$) plus a contribution from the boundary. Next we give explicit formulas how to do this.

We start with the ψ-classes. For $n = 0$ there is no marking and for $n = 1$ the class ψ_1 is our preferred generator. For $n = 2$ we have the following useful tautological relation.

Lemma 2.2. *There is a codimension one relation*

$$\psi_1 + \psi_2 =$$ 

in $\mathrm{CH}^1(\mathfrak{M}_{0,2})$.

Proof. Consider the $\mathbb{G}_m$-action on $\mathbb{P}^1$ given by

$$t.[x_0 : x_1] = [t \cdot x_0 : x_1] \text{ for } t \in \mathbb{G}_m(k).$$

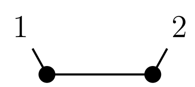
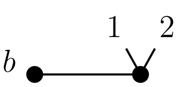
For the identification $\mathfrak{M}_{0,2}^{\mathrm{sm}} \cong B\mathbb{G}_m = [\operatorname{Spec} k/\mathbb{G}_m]$, the universal family over $\mathfrak{M}_{0,2}^{\mathrm{sm}}$ is given by

$$\begin{tikzcd} {[\mathbb{P}^1/\mathbb{G}_m]} \arrow[d] \\ {[\operatorname{Spec} k/\mathbb{G}_m]} \arrow[u, bend left, "p_1=0"] \arrow[u, bend right, "p_2=\infty"'] \end{tikzcd}.$$

We have that $-\psi_1, -\psi_2$ are the first Chern classes of the normal bundles of p_1, p_2. We have $\psi_1 + \psi_2 = 0$ in $\mathrm{CH}^1(\mathfrak{M}_{0,2}^{\mathrm{sm}})$ because the $\mathbb{G}_m$-action on $\mathbb{P}^1$ has opposite weights at $0, \infty$. Thus, from the excision sequence

$$\mathrm{CH}^0(\partial\,\mathfrak{M}_{0,2}) \to \mathrm{CH}^1(\mathfrak{M}_{0,2}) \to \mathrm{CH}^1(\mathfrak{M}_{0,2}^{\mathrm{sm}}) \to 0\,,$$

it follows that $\psi_1 + \psi_2$ can be written as a linear combination of fundamental class of two boundary strata

$$\psi_1 + \psi_2 = a$$  $$+\, b$$  .

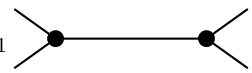
Consider the morphism $F_3 : \overline{\mathcal{M}}_{0,5} \to \mathfrak{M}_{0,2}$ forgetting the last three markings. Denote by $D(A|B)$ the boundary divisor with markings splitting to the two vertices as $A \sqcup B$ (see below for an illustration). It follows from [5, Section 3.2] that

$$\begin{aligned} F_3^* \psi_i &= \psi_i \\ F_3^*\, D(\{1\}|\{2\}) &= \sum_{\substack{I_1 \sqcup I_2 = \{3,4,5\} \\ |I_1|,|I_2| \geq 1}} D(\{1\} \cup I_1 | \{2\} \cup I_2) \\ F_3^*\, D(\emptyset|\{1,2\}) &= \sum_{\substack{I_1 \sqcup I_2 = \{3,4,5\} \\ |I_1| \geq 2}} D(I_1 | \{1,2\} \cup I_2)\,. \end{aligned}$$

On $\overline{\mathcal{M}}_{0,5}$ there is a unique linear relation between the pullbacks of $\psi_1 + \psi_2$, $D(\{1\}|\{2\})$ and $D(\emptyset|\{1,2\})$ under F_3, from which the coefficients a, b can be read off as $a = 1, b = 0$. $\square$

For $n = 2$ we can express ψ_2 as the multiple $-\psi_1$ of our preferred generator plus a term supported in the boundary.

Now let $n \geq 3$. For $\{1, \ldots, n\} = I_1 \sqcup I_2$, we denote by

$D(I_1 \mid I_2) =$ I_1  I_2

the class of the boundary divisor in $\mathrm{CH}^1(\mathfrak{M}_{0,n})$ associated to the splitting $I_1 \sqcup I_2$ of the marked points. The following lemma shows how to write ψ-classes on $\mathfrak{M}_{0,n}$ via boundary strata.

Lemma 2.3. *For $n \geq 3$ and $1 \leq i \leq n$, we have*

$$\psi_i = \sum_{\substack{I_1 \sqcup I_2 = \{1,\ldots,n\} \\ i \in I_1; j,\ell \in I_2}} D(I_1 \mid I_2)$$

in $\mathrm{CH}^1(\mathfrak{M}_{0,n})$ for any choice of $1 \leq j, \ell \neq i \leq n$.

Proof. When $n = 3$, we have $\mathrm{CH}^1(\mathfrak{M}_{0,3}^{\mathrm{sm}}) = 0$, so ψ_i can be written as a linear combination of the four boundary divisors of $\mathfrak{M}_{0,3}$. Again, the coefficients can be determined via the pullback under $F_2 : \overline{\mathcal{M}}_{0,5} \to \mathfrak{M}_{0,3}$. The relation for $n \geq 4$ follows by pulling back the relation on $\mathfrak{M}_{0,3}$ via the morphism $F \colon \mathfrak{M}_{0,n} \to \mathfrak{M}_{0,3}$ forgetting all markings except $\{i, j, \ell\}$. This pullback can be computed as

$$F^*\psi_m = \psi_m \text{ and } F^*D(I|J) = \sum_{I' \sqcup J' = \{4,\ldots,n\}} D(I \sqcup I' | J \sqcup J') \text{ for } I \sqcup J = \{1, 2, 3\}$$

via [5, Corollary 3.9]. □

For the κ-classes on $\mathfrak{M}_{0,n}$ we have the following boundary expressions.

Lemma 2.4. *Let a be a nonnegative integer and consider $\kappa_a \in \mathrm{CH}^a(\mathfrak{M}_{0,n})$.*

(a) When $n \geq 1$, the class κ_a can be written as a linear combination of monomials in ψ-classes and boundary classes $[\Gamma_i, \alpha_i]$ for nontrivial prestable graphs Γ_i.

(b) When $n = 0$, the class κ_a can be written as a linear combination of monomials in κ_2 and ψ-classes and boundary classes $[\Gamma_i, \alpha_i]$ for nontrivial prestable graphs Γ_i.

In the calculation below, we use the notion of tautological classes on the moduli stack $\mathfrak{M}_{0,n,\mathbb{1}}$ of $\mathcal{A}$-valued prestable curves when $\mathcal{A}$ is a semigroup with two elements $\{\mathbb{0}, \mathbb{1}\}$ so that $\mathbb{1} + \mathbb{1} = \mathbb{1}$. This stack parametrizes prestable curves

$$\big(C, p_1, \ldots, p_n, (a_v)_{v \in V(\Gamma(C))}\big)$$

with additional decoration of $a_v \in \mathcal{A}$ at each component C_v of C, with $\sum_{v \in V(\Gamma(C))} a_v = \mathbb{1}$. They must satisfy the stability condition that any component C_v with fewer than three special points must have $a_v = \mathbb{1}$. The reason why this stack is useful is that unlike the moduli space of stable curves $\overline{\mathcal{M}}_{g,n}$, the space $\mathfrak{M}_{g,n+1}$ is not the universal curve of $\mathfrak{M}_{g,n}$. On the other hand, for moduli spaces $\mathcal{A}$-valued prestable curves, the map $\mathfrak{M}_{g,n+1,\mathbb{1}} \to \mathfrak{M}_{g,n,\mathbb{1}}$ which forgets the last marked point and contracts the component C_v containing it if it becomes unstable, *is* the universal curve. Since $\mathfrak{M}_{0,n,\mathbb{1}}$ contains $\mathfrak{M}_{0,n}$ as the locus of $\mathcal{A}$-valued curves satisfying $a_v = \mathbb{1}$ for all v, it is useful to develop the theory of tautological classes on the space $\mathfrak{M}_{0,n,\mathbb{1}}$ and then simply restrict to $\mathfrak{M}_{0,n}$ in the end. We refer to [5, Section 2.2] for details.

Proof. It is enough to prove the corresponding statement on $\mathfrak{M}_{0,n,\mathbb{1}}$ because the restriction to the open substack $\mathfrak{M}_{0,n} \subset \mathfrak{M}_{0,n,\mathbb{1}}$ does not create additional κ classes.

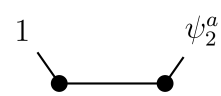
(a) Consider the universal curve

$$\pi \colon \mathfrak{M}_{0,n+1,\mathbb{1}} \to \mathfrak{M}_{0,n,\mathbb{1}}$$

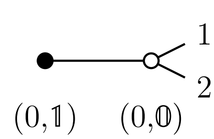
so that $\kappa_a = \pi_*(\psi_{n+1}^{a+1})$. We prove the claim by induction on a, where the induction start $a = 0$ is trivial since $\kappa_0 = n - 2$. In the computation, we repeatedly use the formula for π_* proven in [5, Proposition 3.11].

If $n \geq 2$ and $a \geq 1$, we claim that $\psi_{n+1} \in \mathrm{CH}^1(\mathfrak{M}_{0,n+1,\mathbb{1}})$ can be written as a sum of boundary divisors. Indeed, by Lemma 2.3 this is true for $\psi_{n+1} \in \mathrm{CH}^1(\mathfrak{M}_{0,n+1})$ and so the statement on $\mathfrak{M}_{0,n+1,\mathbb{1}}$ follows by pulling back under the forgetful map $F_{\mathcal{A}} : \mathfrak{M}_{0,n+1,\mathbb{1}} \to \mathfrak{M}_{0,n+1}$ of $\mathcal{A}$-values using [5, Proposition 3.12]. Thus replacing one of the factors ψ_{n+1} in ψ_{n+1}^{a+1} with this boundary expression, we get a sum of boundary divisors in $\mathfrak{M}_{0,n+1,\mathbb{1}}$ decorated with ψ_{n+1}^a. After pushing forward to $\mathfrak{M}_{0,n,\mathbb{1}}$, this class can be written as a tautological class without κ classes by the induction hypothesis.

When $n = 1$, we also conclude by induction on a. Pulling back the relations of Lemma 2.2 along the morphism $\mathfrak{M}_{0,2,\mathbb{1}} \to \mathfrak{M}_{0,2}$ forgetting $\mathcal{A}$-values, we have

$$\kappa_a = \pi_*(\psi_2^{a+1}) = \pi_*\Big(- \psi_2^a \psi_1 + $$  $$\Big),$$

where implicitly we sum over all $\mathcal{A}$-valued graphs where the sum of degrees is equal to $\mathbb{1}$. By [5, Proposition 3.10] we have

$$\psi_1 = \pi^* \psi_1 + $$  .

Using the projection formula, the expression [5, Proposition 3.11] for the pushforward π_* and the induction hypothesis we get the result.

(b) When a is an odd number, this statement follows from the Grothendieck-Riemann-Roch computation in [12, Proposition 1]. Namely,

$$0 = \mathrm{ch}_{2a-1}(\pi_* \omega_\pi) = \frac{B_{2a}}{(2a)!}\Big(\kappa_{2a-1} + \frac{1}{2}\sum_{\Gamma}\sum_{i=0}^{2a-2}(-1)^i \psi_h^i \psi_{h'}^{2a-a-i}[\Gamma]\Big)$$

where the sum is over $\mathcal{A}$-valued graphs Γ with one edge $e = (h, h')$ and degree $\mathbb{1}$. Here B_{2a} is the $2a$-th Bernoulli number.

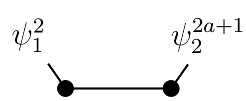
We prove the statement for κ_{2a} by the induction on a. Consider the forgetful morphism

$$\pi_2 \colon \mathfrak{M}_{0,2,\mathbb{1}} \to \mathfrak{M}_{0,0,\mathbb{1}} \,.$$

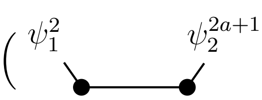
which is a composition of two forgetful maps $\mathfrak{M}_{0,2,\mathbb{1}} \xrightarrow{\pi_1} \mathfrak{M}_{0,1,\mathbb{1}} \xrightarrow{\pi_0} \mathfrak{M}_{0,0,\mathbb{1}}$. By the projection formula and [5, Proposition 3.10], one computes

$$\begin{aligned}\pi_{2*}(\psi_1^3\psi_2^{2a+1}) &= \pi_{0*}\pi_{1*}(\psi_2^{2a+1}(\pi_1^*\psi_1 + D_{1,2})^3)\\ &= \pi_{0*}\pi_{1*}(\psi_2^{2a+1}\pi_1^*\psi_1^3)\\ &= \pi_{0*}(\psi_1^3\kappa_{2a}) = \pi_{0*}(\psi_1^3\pi_0^*\kappa_{2a} + \psi_1^{2a+3}) = \kappa_{2a+2} + \kappa_2\kappa_{2a}\,,\end{aligned}$$

where $D_{1,2}$ is the divisor class on $\mathfrak{M}_{0,2,\mathbb{1}}$ which is the image of the universal section of π_1 associated to the first marking, which satisfies $\psi_2 \cdot D_{1,2} = 0$. On the other hand,

$$\pi_{2*}(\psi_1^3\psi_2^{2a+1}) = \pi_{2*}\Big(- \psi_1^2\psi_2^{2a+2} +$$


$$\Big)$$

$$= -\kappa_{2a+2} - \kappa_1\kappa_{2a+1} + \pi_{2*}\Big($$


$$\Big)\,.$$

by Lemma 2.2. By the induction hypothesis, comparing the two equalities ends the proof. □

2.2 Generators of $\mathrm{CH}^*(\mathfrak{M}_{0,n})$

The goal of this subsection is to prove Theorem 1.2. The basic idea is simple: by Lemma 2.1 we know that classes on the smooth locus of $\mathfrak{M}_{0,n}$ have tautological representatives. By an excision argument, it suffices to show that classes supported on the boundary are tautological. But the boundary is parametrized under the gluing maps by products of $\mathfrak{M}_{0,n_i}$. Then we want to conclude using an inductive argument.

The two main technical steps to complete are as follows:

- The boundary of $\mathfrak{M}_{0,n}$ is covered by a finite union of boundary gluing maps, which are proper and representable. We want to show that the direct sum of pushforwards by the gluing maps is surjective on the Chow group of the boundary.

- Knowing that classes on $\mathfrak{M}_{0,n_1}$ and $\mathfrak{M}_{0,n_2}$ are tautological up to a certain degree d, we want to conclude that classes of degree at most d on the product $\mathfrak{M}_{0,n_1} \times \mathfrak{M}_{0,n_2}$ are tensor products of tautological classes.

The first issue is resolved by the fact that the pushforward along a proper surjective morphism of relative Deligne-Mumford type is surjective on the rational Chow group. We prove this statement in [5, Appendix B.4].

We now turn to the second issue, understanding the Chow group of products of spaces $\mathfrak{M}_{0,n_i}$. We make the following general definition, extended from [36, Definition 6].

Definition 2.5. *Let X, Y be algebraic stacks with X locally of finite type over k and Y of finite type over k. We say that X has the* Chow Künneth generation property *(CKgP) for Y if the natural morphism*

$$\mathrm{CH}_*(X) \otimes \mathrm{CH}_*(Y) \to \mathrm{CH}_*(X \times Y) \tag{11}$$

is surjective, and we say that it has the Chow Künneth property *(CKP) for Y if the map (11) is an isomorphism. Similarly, we define that X has the CKgP (or CKP) if it has the CKgP (or CKP) for all algebraic stacks Y of finite type over k.*

Recall that a locally of finite type stack X has a *good filtration by finite type stacks* if X is the union of an increasing sequence $(\mathcal{U}_j)_j$ of finite type open substacks such that the codimension of the complement of $\mathcal{U}_j$ becomes arbitrarily large as j increases. It is immediate that if X has the CKgP (or the CKP) and in addition has a good filtration, then the map (11) is surjective (or an isomorphism) for all Y locally finite type over k admitting a good filtration. The additional assumption of the good filtration is added since in general tensor products and right exact sequences are not compatible with inverse limits.

We now turn to showing the following result, resolving the second issue mentioned at the beginning of the section.

Proposition 2.6. *For all $n \geq 0$, the stacks $\mathfrak{M}_{0,n}$ have the CKgP for finite type stacks Y having a stratification by quotient stacks.*

For the proof we start with the smooth part of $\mathfrak{M}_{0,n}$.

Proposition 2.7. *For all $n \geq 0$, the stacks $\mathfrak{M}_{0,n}^{\mathrm{sm}}$ have the CKP.*

Proof. Starting with the easy cases, for $n = 2$ we have $\mathfrak{M}_{0,2}^{\mathrm{sm}} \cong B\mathbb{G}_m$ by Lemma 2.1 and it was shown in [36, Lemma 2] that this satisfies the CKP. On the other hand, for $n \geq 3$ we have $\mathfrak{M}_{0,n}^{\mathrm{sm}} = \mathcal{M}_{0,n}$, which is an open subset of $\mathbb{A}^{n-3}$. Then for any finite type stack Y we have $\mathrm{CH}_*(\mathcal{M}_{0,n}) \otimes \mathrm{CH}_*(Y) \cong \mathrm{CH}_*(Y)$ and the map (11) is just the pullback under the projection $\mathcal{M}_{0,n} \times Y \to Y$. Combining [28, Corollary

2.5.7] and the excision sequence, we see that this pullback is surjective. On the other hand, composing it with the Gysin pullback by an inclusion

$$Y \cong \{C_0\} \times Y \subset \mathcal{M}_{0,n} \times Y$$

for some $C_0 \in \mathcal{M}_{0,n}$ we obtain the identity on $\mathrm{CH}_*(Y)$, so it is also injective.

Next we consider the case $n = 1$. By Lemma 2.1 we have $\mathfrak{M}_{0,1}^{\mathrm{sm}} \cong B\mathbb{U}$ for $\mathbb{U} = \mathbb{G}_a \rtimes \mathbb{G}_m$. The group $\mathbb{U}$ contains $\mathbb{G}_m$ as a subgroup and we claim that the natural map $B\mathbb{G}_m \to B\mathbb{U}$ is an affine bundle with fibre $\mathbb{A}^1$. Indeed, the fibres are $\mathbb{U}/\mathbb{G}_m \cong \mathbb{A}^1$ and the structure group is $\mathbb{U} = \mathrm{Aff}(1)$ acting by affine transformations on $\mathbb{A}^1$. Of course also for any finite type stack Y it is still true that $Y \times B\mathbb{G}_m \to Y \times B\mathbb{U}$ is an affine bundle. Then by [28, Corollary 2.5.7] we have that the two vertical maps in the diagram

$$\begin{tikzcd}
\mathrm{CH}_*(Y) \otimes \mathrm{CH}_*(B\mathbb{G}_m) \arrow[r] & \mathrm{CH}_*(Y \times B\mathbb{G}_m) \\
\mathrm{CH}_*(Y) \otimes \mathrm{CH}_*(B\mathbb{U}) \arrow[r] \arrow[u] & \mathrm{CH}_*(Y \times B\mathbb{U}) \arrow[u]
\end{tikzcd}$$

induced by pullback of the affine bundles are isomorphisms. The top arrow in the diagram is also an isomorphism since, as seen above, $B\mathbb{G}_m$ has the CKP. Thus the bottom arrow is an isomorphism as well.

We are left with the case $n = 0$. The forgetful map

$$\pi \colon \mathfrak{M}_{0,1}^{\mathrm{sm}} \to \mathfrak{M}_{0,0}^{\mathrm{sm}} \tag{12}$$

gives the universal curve over $\mathfrak{M}_{0,0}^{\mathrm{sm}}$. The map (12) can be thought of as the morphism between quotient stacks

$$\pi : [\mathbb{P}^1/\mathrm{PGL}_2] \to [\operatorname{Spec} k/\mathrm{PGL}_2]$$

induced by the PGL_2-equivariant map $\mathbb{P}^1 \to \operatorname{Spec} k$. By [5, Remark B.20], the map π is projective and the line bundle $\mathcal{O}_{\mathbb{P}^1}(2)$ on $\mathbb{P}^1$ descends to a π-relatively ample line bundle on $[\mathbb{P}^1/\mathrm{PGL}_2]$.

Now for any finite type stack Y consider a commutative diagram

$$\begin{tikzcd}
\mathrm{CH}_*(Y) \otimes \mathrm{CH}_*(B\mathrm{PGL}_2) \arrow[r] & \mathrm{CH}_*(Y \times B\mathrm{PGL}_2) \\
\mathrm{CH}_*(Y) \otimes \mathrm{CH}_*(B\mathbb{U}) \arrow[r, "\cong"] \arrow[u, "\mathrm{id}\otimes\pi_*"] & \mathrm{CH}_*(Y \times B\mathbb{U}) \arrow[u, "(\mathrm{id}\times\pi)_*"']
\end{tikzcd}$$

induced by the projective pushforward π_*. Note that the map $(\mathrm{id} \times \pi)_*$ is surjective. Indeed, a small computation[9] shows that for $\alpha \in \mathrm{CH}_*(Y \times B\mathrm{PGL}_2)$ we have

$$(\mathrm{id} \times \pi)_* \left(\frac{1}{2} c_1(\mathcal{O}_{\mathbb{P}^1}(2)) \cap (\mathrm{id} \times \pi)^* \alpha \right) = \alpha\,.$$

[9] See the proof of Proposition 2.14 for a variant of this computation.

Then the surjectivity of the top arrow follows.

To prove injectivity of the top arrow consider the diagram

$$\begin{tikzcd} \mathrm{CH}_*(Y) \otimes \mathrm{CH}_*(B\mathrm{PGL}_2) \arrow[r] \arrow[d, "\mathrm{id}\otimes\pi^*"] & \mathrm{CH}_*(Y \times B\mathrm{PGL}_2) \arrow[d, "(\mathrm{id}\times\pi)^*"] \\ \mathrm{CH}_*(Y) \otimes \mathrm{CH}_*(B\mathbb{U}) \arrow[r, "\cong"] & \mathrm{CH}_*(Y \times B\mathbb{U}) \end{tikzcd}$$

induced by the flat pullback π^*. Similar to above, we see that for $\alpha \in \mathrm{CH}_*(B\mathrm{PGL}_2)$ we have

$$\pi_* \left(\frac{1}{2} c_1(\mathcal{O}_{\mathbb{P}^1}(2)) \cap \pi^* \alpha \right) = \alpha\,.$$

Thus the map $\mathrm{id} \otimes \pi^*$ is injective and thus the top arrow must be injective as well, finishing the proof. $\square$

For the next results, we say that an equidimensional, locally finite type stack X has the *Chow Künneth generation property up to codimension d* if (11) is surjective in all codimensions up to d.

Lemma 2.8. *Let X, X' be equidimensional algebraic stacks, locally of finite type over k and admitting good filtrations. Then for X, X' having the CKgP (up to codimension d), also $X \times X'$ has the CKgP (up to codimension d).*

Proof. Fixing $d \geq 0$ and $U \subseteq X$ a finite type open substack with complement of codimension at least $d+1$, one has $\mathrm{CH}^d(X) \cong \mathrm{CH}^d(U)$, and similarly $\mathrm{CH}^d(X \times Y) \cong \mathrm{CH}^d(U \times Y)$ for any finite type algebraic stack Y. This follows from the definition of the Chow groups as a limit (see the discussion above [5, Proposition A.5]). Using this we can reduce the proof of the lemma to the case where X (and similarly X') are of finite type over k, where it then follows from simple diagram chasing. $\square$

Lemma 2.9. *Let Z be an algebraic stack, locally finite type over k, stratified by quotient stacks and with a good filtration by finite type substacks. Let $\widehat{Z} \to Z$ be a proper, surjective map representable by Deligne-Mumford stacks such that $\widehat{Z}$ has the CKgP. Then Z has the CKgP for stacks Y stratified by quotient stacks.*

If both Z and $\widehat{Z}$ are equidimensional with $\dim \widehat{Z} - \dim Z = e \geq 0$ then if $\widehat{Z}$ has the CKgP up to codimension d, Z has the CKgP (for stacks Y stratified by quotient stacks) up to codimension $d - e$.

Proof. Let Y be an algebraic stack of finite type over k, stratified by quotient stacks. Then in the diagram

$$\begin{tikzcd} \mathrm{CH}_*(\widehat{Z} \times Y) \arrow[r, two heads] & \mathrm{CH}_*(Z \times Y) \\ \mathrm{CH}_*(\widehat{Z}) \otimes \mathrm{CH}_*(Y) \arrow[r] \arrow[u, two heads] & \mathrm{CH}_*(Z) \otimes \mathrm{CH}_*(Y) \arrow[u] \end{tikzcd} \tag{13}$$

the top arrow is surjective by [5, Proposition B.19] (and [5, Remark B.21]) applied to $\widehat{Z} \times Y \to Z \times Y$ and the left arrow is surjective since $\widehat{Z}$ has the CKgP. It follows that

$$\mathrm{CH}_*(Z) \otimes \mathrm{CH}_*(Y) \to \mathrm{CH}_*(Z \times Y)$$

is surjective, so Z has the CKgP for stacks Y stratified by quotient stacks. The statement with bounds on codimensions follows by looking at the correct graded parts of the above diagram and noting that codimension d' cycles on $\widehat{Z}$ push forward to codimension $d' - e$ cycles on Z. □

Proposition 2.10. *Let X be an algebraic stack over k with a good filtration by finite type substacks and let $U \subset X$ be an open substack with complement $Z = X \setminus U$ such that U and Z have the CKgP. Then X has the CKgP.*

If X is equidimensional and Z has pure codimension e, U has the CKgP up to codimension d and Z has the CKgP up to codimension $d - e$, then X has the CKgP up to codimension d.

Proof. For Y a finite type stack, using excision exact sequences on X and $X \times Y$ we obtain a commutative diagram

$$\begin{tikzcd}
\mathrm{CH}_*(Z \times Y) \arrow[r] & \mathrm{CH}_*(X \times Y) \arrow[r] & \mathrm{CH}_*(U \times Y) \arrow[r] & 0 \\
\mathrm{CH}_*(Z) \otimes \mathrm{CH}_*(Y) \arrow[r] \arrow[u, two heads] & \mathrm{CH}_*(X) \otimes \mathrm{CH}_*(Y) \arrow[r] \arrow[u] & \mathrm{CH}_*(U) \otimes \mathrm{CH}_*(Y) \arrow[r] \arrow[u, two heads] & 0
\end{tikzcd}$$

with exact rows. The vertical arrows for U, Z are surjective since U, Z have the CKgP. By the four lemma, the middle arrow is surjective as well, so X has the CKgP. Again, the variant with bounds on the codimension follows by looking at the correct graded parts of the above diagram, noting that codimension d' cycles on Z push forward to codimension $d' + e$ cycles on X. □

Combining these ingredients, we are now ready to prove Proposition 2.6.

Proof of Proposition 2.6. We will show that for all $d \geq 0$, all spaces $\mathfrak{M}_{0,n}$ have the CKgP for finite type stacks Y having a stratification by quotient stacks up to codimension d by induction on d. Every stack has the CKgP up to codimension $d = 0$, so the induction start is fine. Let now $d \geq 1$, then we want to apply Proposition 2.10 for $X = \mathfrak{M}_{0,n}$ with $U = \mathfrak{M}_{0,n}^{\mathrm{sm}}$. Then U has the CKgP by Proposition 2.7. Its complement $Z = \partial\,\mathfrak{M}_{0,n}$ admits a proper, surjective, representable cover

$$\widehat{Z} = \coprod_{I \subset \{1,\ldots,n\}} \mathfrak{M}_{0,I \cup \{p\}} \times \mathfrak{M}_{0,I^c \cup \{p'\}} \to Z = \partial\,\mathfrak{M}_{0,n} \subset \mathfrak{M}_{0,n} \tag{14}$$

by gluing maps. Note that $\widehat{Z}$ and Z are both equidimensional of the same dimension. By induction the spaces $\mathfrak{M}_{0,I\cup\{p\}}$ and $\mathfrak{M}_{0,I^c\cup\{p'\}}$ have the CKgP up to codimension

$d-1$ (note that they both have at least one marking). So by Lemma 2.8 their product has the CKgP up to codimension $d-1$. The stabilizer group of each geometric points of $Z = \partial\mathfrak{M}_{0,n}$ is affine and hence by [28, Proposition 3.5.9] the stack Z is stratified by quotient stacks. By Lemma 2.9 we have that Z has the CKgP up to codimension $d-1$. This is sufficient to apply Proposition 2.10 to conclude that $\mathfrak{M}_{0,n}$ has the CKgP for finite type stacks Y having a stratification by quotient stacks up to codimension d as desired. $\square$

Proof of Theorem 1.2. We show $\mathrm{CH}^d(\mathfrak{M}_{0,n}) = \mathrm{R}^d(\mathfrak{M}_{0,n})$ (for all $n \geq 0$) by induction on $d \geq 0$. The induction start $d = 0$ is trivial. So let $d \geq 1$ and assume the statement holds in codimensions up to $d-1$. By excision we have an exact sequence

$$\mathrm{CH}^{d-1}(\partial\,\mathfrak{M}_{0,n}) \to \mathrm{CH}^d(\mathfrak{M}_{0,n}) \to \mathrm{CH}^d(\mathfrak{M}^{\mathrm{sm}}_{0,n}) \to 0.$$

By Lemma 2.1 all elements of $\mathrm{CH}^d(\mathfrak{M}^{\mathrm{sm}}_{0,n})$ have tautological representatives, so it suffices to show that this is also true for elements coming from $\mathrm{CH}^{d-1}(\partial\,\mathfrak{M}_{0,n})$. Using the parametrization (14) it suffices to show that codimension $d-1$ classes on products $\mathfrak{M}_{0,n_1} \times \mathfrak{M}_{0,n_2}$ are tautological (where $n_1, n_2 \geq 1$). By Proposition 2.6 we have a surjection

$$\mathrm{CH}^*(\mathfrak{M}_{0,n_1}) \otimes \mathrm{CH}^*(\mathfrak{M}_{0,n_2}) \to \mathrm{CH}^*(\mathfrak{M}_{0,n_1} \times \mathfrak{M}_{0,n_2})$$

and by the induction hypothesis, all classes on the left side are (tensor products of) tautological classes up to degree $d-1$. Since tensor products of tautological classes map to tautological classes under gluing maps, this finishes the proof. $\square$

2.3 Higher Chow-Künneth property

The goal of this section is to give a background to compute the higher Chow group of $\mathfrak{M}^{\mathrm{sm}}_{\Gamma}$ for prestable graphs Γ. Computing higher Chow groups of $\mathfrak{M}^{\mathrm{sm}}_{0,n}$ has two different flavors. When $n = 0, 1, 2$ or 3, we use the projective bundle formula and its consequences. When $n \geq 4$, $\mathfrak{M}^{\mathrm{sm}}_{0,n}$ is a hyperplane complement inside affine space and we use the motivic decomposition from [8].

Below we study the Chow-Künneth property for higher Chow groups. Unlike the Chow-Künneth property for Chow groups, formulating the Chow-Künneth property for higher Chow groups in general is rather complicated, see [44, Theorem 7.2]. Below, we focus on the case of the first higher Chow group $\mathrm{CH}^*(X, 1)$ defined in [28][10]. To simplify the notation, we write $\mathrm{CH}^*(k, \bullet) := \mathrm{CH}^*(\operatorname{Spec} k, \bullet)$.

Definition 2.11. *A quotient stack X over k is said to have the* higher Chow Künneth property *(hCKP) if for all algebraic stacks Y of finite type over k the natural morphism*

$$\mathrm{CH}^*(X, \bullet) \otimes_{\mathrm{CH}^*(k,\bullet)} \mathrm{CH}^*(Y, \bullet) \to \mathrm{CH}^*(X \times Y, \bullet) \tag{15}$$

[10]In [28], this group is denoted by $\underline{A}_*(X)$.

is an isomorphism in total degree $\bullet = 1$*. A quotient stack* X *over* k *is said to have the* higher Chow Künneth generating property *(hCKgP) if the above morphism is surjective.*

Expanding this definition slightly, the degree $\bullet = 1$ part of the left hand side of (15) is given by the quotient

$$\frac{(\mathrm{CH}^*(X,1) \otimes_{\mathbb{Q}} \mathrm{CH}^*(Y,0)) \oplus (\mathrm{CH}^*(X,0) \otimes_{\mathbb{Q}} \mathrm{CH}^*(Y,1))}{\mathrm{CH}^*(k,1) \otimes_{\mathbb{Q}} \mathrm{CH}^*(X,0) \otimes_{\mathbb{Q}} \mathrm{CH}^*(Y,0)}, \tag{16}$$

where

$$\alpha \otimes \beta_X \otimes \beta_Y \in \mathrm{CH}^*(k,1) \otimes_{\mathbb{Q}} \mathrm{CH}^*(X,0) \otimes_{\mathbb{Q}} \mathrm{CH}^*(Y,0)$$

maps to

$$((\alpha \cdot \beta_X) \otimes \beta_Y, -\beta_X \otimes (\alpha \cdot \beta_Y))$$

in the numerator of (16). The cokernel of the following map

$$\mathrm{CH}^1(k,1) \otimes \mathrm{CH}^{*-1}(X) \to \mathrm{CH}^*(X,1)$$

is called the *indecomposable part* $\overline{\mathrm{CH}}^*(X,1)$ of $\mathrm{CH}^*(X,1)$. For example $\overline{\mathrm{CH}}^1(\operatorname{Spec} k,1) = 0$.

We summarize some properties for higher Chow groups of quotient stacks $X = [U/G]$. In this case, the definition of the first higher Chow group of X from [28] coincides with the definition using Bloch's cycle complex of the finite approximation of $U_G = U \times_G EG$ from [11]. For the properties of higher Chow groups presented below, many of the proofs follow from this presentation.

Lemma 2.12. *Let* X *be a quotient stack and* $E \to X$ *be a vector bundle of rank* $r+1$*, and let* $\pi\colon \mathbb{P}(E) \to X$ *be the projectivization. Let* $\mathcal{O}(1)$ *be the hyperplane line bundle on* $\mathbb{P}(E)$*. Then the map*

$$\theta_E(\bullet)\colon \bigoplus_{i=0}^{r} \mathrm{CH}_{*+i}(X,1) \to \mathrm{CH}_{*+r}(\mathbb{P}(E),1)$$

given by

$$(\alpha_0, \ldots, \alpha_r) \mapsto \sum_{i=0}^{r} c_1(\mathcal{O}(1))^i \cap \pi^*\alpha_i$$

is an isomorphism.

Proof. Let $X = [U/G]$ be a quotient stack. Choose a G-representation V and an open subspace $W \subset V$ on which G acts freely. We can take a representation V so that the codimension of $V \setminus W$ in V has arbitrary large codimension. By [11, Section 2.7], the group $\mathrm{CH}_*(X,1)$ is isomorphic to $\mathrm{CH}_*(U \times W/G, 1)$ and the similar formula holds for $\mathrm{CH}_*(\mathbb{P}(E),1)$. Now the property follows from the projective bundle formula [6, Theorem 7.1].[11] □

[11] See also [21, Theorem 4.2.2].

An affine bundle of rank r over X is a morphism $B \to X$ such that locally (in the smooth topology) on X, B is a trivial affine r plane over X [28, Section 2.5]. We assume that the structure group of an affine bundle of rank r is the group of affine transformations $\mathrm{Aff}(r)$ in $\mathrm{GL}(r+1)$. Therefore there exists an associated vector bundle E of rank $r+1$ and an exact sequence of vector bundles

$$0 \to F \to E \to \mathcal{O}_X \to 0\,.$$

The complement of $\mathbb{P}(F) \hookrightarrow \mathbb{P}(E)$ is the affine bundle B.

We have a homotopy invariance property of higher Chow groups for affine bundles (see also [31, Proposition 2.3]).

Corollary 2.13. *Let X be a quotient stack and $\varphi\colon B \to X$ be an affine bundle of rank r. Then*

$$\varphi^*\colon \mathrm{CH}_*(X,1) \to \mathrm{CH}_{*+r}(B,1)$$

is an isomorphism.

Proof. Let p and q be projections from $\mathbb{P}(E)$ and $\mathbb{P}(F)$ to X respectively. There exists an excision sequence

$$\mathrm{CH}_*(\mathbb{P}(F),1) \xrightarrow{i_*} \mathrm{CH}_*(\mathbb{P}(E),1) \xrightarrow{j^*} \mathrm{CH}_*(B,1) \xrightarrow{\partial} \mathrm{CH}_*(\mathbb{P}(F)) \xrightarrow{i_*} \mathrm{CH}_*(\mathbb{P}(E))$$

because all stacks are quotient stacks [11]. Since $\mathbb{P}(F)$ is the vanishing locus of the canonical section of $\mathcal{O}_{\mathbb{P}(E)}(1)$ we have

$$i_*q^*\alpha = c_1(\mathcal{O}_{\mathbb{P}(E)}(1)) \cap p^*\alpha \text{ for } \alpha \in \mathrm{CH}_*(X)$$

by [16, Lemma 3.3]. As α runs through a basis of $\mathrm{CH}_*(X)$, the classes

$$c_1(\mathcal{O}_{\mathbb{P}(F)}(1))^\ell \cap q^*\alpha \text{ for } 0 \le \ell \le r-1$$

run through a basis of $\mathrm{CH}_*(\mathbb{P}(F))$ by Lemma 2.12. Pushing them forward via i, the classes

$$i_*\left(c_1(\mathcal{O}_{\mathbb{P}(F)}(1))^\ell \cap q^*\alpha\right) = c_1(\mathcal{O}_{\mathbb{P}(E)}(1))^{\ell+1} \cap p^*\alpha \text{ for } 0 \le \ell \le r-1, \alpha \qquad (17)$$

form part of a basis of $\mathrm{CH}_*(\mathbb{P}(E))$. In particular, the map $i_*\colon \mathrm{CH}_*(\mathbb{P}(F)) \to \mathrm{CH}_*(\mathbb{P}(E))$ is injective and furthermore, we see that

$$p^* : \mathrm{CH}_*(X,1) \to \mathrm{CH}_*(\mathbb{P}(E),1)/\mathrm{CH}_*(\mathbb{P}(F),1) \qquad (18)$$

gives an isomorphism.

The injectivity of i_* implies (via the excision sequence above) that j^* is surjective. Using Lemma A.2, the formula (17) holds verbatim for higher Chow classes $\alpha \in$

$\mathrm{CH}^*(X,1)$ so $i_*\colon \mathrm{CH}_*(\mathbb{P}(F),1) \to \mathrm{CH}_*(\mathbb{P}(E),1)$ is injective. Thus the excision sequence implies that j^* induces an isomorphism

$$j^* : \mathrm{CH}_*(\mathbb{P}(E),1)/\mathrm{CH}_*(\mathbb{P}(F),1) \to \mathrm{CH}_*(B,1). \tag{19}$$

But since $\varphi^* = j^*p^*$, we know that φ is an isomorphism as the composition of the two isomorphisms (19) and (18). □

Proposition 2.14. *For $n = 0, 1, 2$ or 3, the stacks $X = \mathfrak{M}^{\mathrm{sm}}_{0,n}$ satisfy the hCKP for quotient stacks Y. Moreover we have $\overline{\mathrm{CH}}^*(X,1) = 0$ and the natural morphism*

$$\mathrm{CH}_*(X) \otimes_{\mathbb{Q}} \mathrm{CH}_*(Y,1) \to \mathrm{CH}_*(X \times Y,1) \tag{20}$$

is an isomorphism. In particular, setting $Y = \operatorname{Spec} k$ we find

$$\mathrm{CH}_*(\mathfrak{M}^{\mathrm{sm}}_{0,n},1) \cong \mathrm{CH}_*(\mathfrak{M}^{\mathrm{sm}}_{0,n}) \otimes_{\mathbb{Q}} \mathrm{CH}_*(k,1)\,.$$

Proof. When $n = 3$, $\mathfrak{M}^{\mathrm{sm}}_{0,3} = \operatorname{Spec} k$, so there is nothing to prove.

When $n = 2$, we use finite dimensional approximation of $B\mathbb{G}_m$ via projective spaces $\mathbb{P}^N$, similar to the proof of [36, Lemma 2]. Indeed, for the vector bundle $[\mathbb{A}^{N+1}/\mathbb{G}_m] \to B\mathbb{G}_m$, pullback induces an isomorphism of Chow groups and $[\mathbb{A}^{N+1}/\mathbb{G}_m]$ is isomorphic to $\mathbb{P}^N$ away from codimension $N+1$. This shows the known identity

$$\mathrm{CH}^\ell(B\mathbb{G}_m) \cong \mathrm{CH}^\ell(\mathbb{P}^N) \text{ for } \ell \le N.$$

Similarly, for Y a quotient stack, $[\mathbb{A}^{N+1} \times Y/\mathbb{G}_m]$ is a vector bundle over $B\mathbb{G}_m \times Y$. By [11, Proposition 5], the higher Chow group of $[\mathbb{A}^{N+1} \times Y/\mathbb{G}_m]$ and $\mathbb{P}^N \times Y$ is isomorphic up to degree $\ell \le N$. One can use the homotopy invariance of higher Chow groups proven in [28, Proposition 4.3.1] to show that we have

$$\mathrm{CH}^\ell(B\mathbb{G}_m \times Y,1) \cong \mathrm{CH}^\ell(\mathbb{P}^N \times Y,1) \text{ for } \ell \le N.$$

On the other hand, the natural morphism

$$\mathrm{CH}^*(\mathbb{P}^N) \otimes \mathrm{CH}^*(Y,1) \to \mathrm{CH}^*(\mathbb{P}^N \times Y,1)$$

is an isomorphism by Lemma 2.12. Combining with the equalities above, this shows that the map

$$\mathrm{CH}^*(B\mathbb{G}_m) \otimes \mathrm{CH}^*(Y,1) \to \mathrm{CH}^*(B\mathbb{G}_m \times Y,1) \tag{21}$$

is an isomorphism. This shows the hCKP of $B\mathbb{G}_m$.

When $n = 1$, we have $\mathfrak{M}^{\mathrm{sm}}_{0,1} \cong B\mathbb{U}$ for $\mathbb{U} = \mathbb{G}_a \rtimes \mathbb{G}_m$ by Lemma 2.1. We already saw that for any finite type stack Y the map $B\mathbb{G}_m \times Y \to B\mathbb{U} \times Y$ is an affine bundle. By Corollary 2.13 we have the homotopy invariance

$$\mathrm{CH}^*(B\mathbb{U} \times Y,1) \cong \mathrm{CH}^*(B\mathbb{G}_m \times Y,1)$$

for all quotient stacks Y. Then the hCKP and the vanishing $\overline{\mathrm{CH}}^*(B\mathbb{U},1)=0$ for $B\mathbb{U}$ follow from the corresponding properties of $B\mathbb{G}_m$ proven above.

We are left with the case $n=0$. For any quotient stack Y consider a commutative diagram

$$\begin{tikzcd}
\mathrm{CH}_*(Y,1)\otimes\mathrm{CH}_*(B\mathrm{PGL}_2) \arrow[r] & \mathrm{CH}_*(Y\times B\mathrm{PGL}_2,1) \\
\mathrm{CH}_*(Y,1)\otimes\mathrm{CH}_*(B\mathbb{U}) \arrow[r, "\cong"] \arrow[u, "\mathrm{id}\otimes\pi_*"] & \mathrm{CH}_*(Y\times B\mathbb{U},1) \arrow[u, "(\mathrm{id}\times\pi)_*"']
\end{tikzcd}$$

induced by the projective pushforward π_*. We start by proving surjectivity of $(\mathrm{id}\times\pi)_*$. By [5, Remark B.20] the morphism $\mathrm{id}\times\pi$ can be factorized as

$$\begin{tikzcd}
Y\times B\mathbb{U} \arrow[r, hook, "i"] \arrow[d, "\mathrm{id}\times\pi"] & Y\times\mathbb{P}(E) \arrow[dl, "p"] \\
Y\times B\mathrm{PGL}_2 &
\end{tikzcd}$$

where E is a rank 3 vector bundle E on $B\mathrm{PGL}_2$ associated to $H^0(\mathbb{P}^1,\mathcal{O}_{\mathbb{P}^1}(2))$. Let

$$\xi=c_1(\mathcal{O}_{\mathbb{P}(E)}(1))$$

be the relative hyperplane class. For any class α in $\mathrm{CH}(Y\times B\mathrm{PGL}_2,1)$, we have

$$\begin{aligned}
(\mathrm{id}\times\pi)_*((i^*\xi)\cdot(\mathrm{id}\times\pi)^*\alpha) &= p_*i_*((i^*\xi)\cdot i^*p^*\alpha)\\
&= p_*\left(\xi\cdot i_*(i^*(p^*\alpha))\right)\\
&= 2p_*(\xi^2\cdot p^*\alpha)\\
&= 2\alpha
\end{aligned}$$

where the first equality comes from the functoriality of pushforward and Gysin pullback for higher Chow groups and the second equality is the projection formula (52) from Appendix A. The third equality comes from Lemma A.2 and the factor of two comes from the fact that the map $B\mathbb{U}\to\mathbb{P}(E)$ is the second Veronese embedding of fiberwise degree two. The fourth equality comes from [30, Proposition 4.6]. Therefore $(\mathrm{id}\times\pi)_*$ is surjective.

To prove injectivity of the top arrow consider the diagram

$$\begin{tikzcd}
\mathrm{CH}_*(Y,1)\otimes\mathrm{CH}_*(B\mathrm{PGL}_2) \arrow[r] \arrow[d, "\mathrm{id}\otimes\pi^*"] & \mathrm{CH}_*(Y\times B\mathrm{PGL}_2,1) \arrow[d, "(\mathrm{id}\times\pi)^*"] \\
\mathrm{CH}_*(Y,1)\otimes\mathrm{CH}_*(B\mathbb{U}) \arrow[r, "\cong"] & \mathrm{CH}_*(Y\times B\mathbb{U},1)
\end{tikzcd}$$

induced by the flat pullback π^*. As seen in the proof of Proposition 2.7, the map

$$\pi^*:\mathrm{CH}_*(B\mathrm{PGL}_2)\to\mathrm{CH}_*(B\mathbb{U})$$

is injective and thus the left arrow of the above diagram is likewise injective. Hence the top arrow is an isomorphism, finishing the proof. □

The language of motives is a convenient way to state the higher Chow-Künneth property for $\mathcal{M}_{0,n}$ in the case $n \geq 4$. For simplicity, let k be a perfect field[12]. Let $\mathrm{DM}(k;\mathbb{Q})$ be the Voevodsky's triangulated category of motives over k with $\mathbb{Q}$-coefficients. Let Sch/k be the category of separated schemes of finite type over k. Then there exists a functor

$$\mathsf{M}\colon \mathrm{Sch}/k \to \mathrm{DM}(k;\mathbb{Q})$$

which sends a scheme to its motive. The category $\mathrm{DM}(k;\mathbb{Q})$ is a tensor triangulated category, with a symmetric monoidal product $\otimes$ and M preserves the monoidal structure, namely $\mathsf{M}(X \times_k Y) = \mathsf{M}(X) \otimes \mathsf{M}(Y)$. See [35] for the basic theory of motives.

There is an invertible object, called the *Tate motive*

$$\mathbb{Q}(1)[2] \in \mathrm{DM}(k;\mathbb{Q}),$$

and by taking its shifting and tensor product we have $\mathbb{Q}(a)[n]$ for any integers a and n. Define the motivic cohomology of a scheme X (in $\mathbb{Q}$-coefficient) as

$$H^i(X, \mathbb{Q}(j)) = \mathrm{Hom}_{\mathrm{DM}(k;\mathbb{Q})}(\mathsf{M}(X), \mathbb{Q}(j)[i]).$$

The motivic cohomology is a bi-graded module over the motivic cohomology of the base field k. The motivic cohomology of k is related to Milnor's K-theory of fields.

In [45] Voevodsky proved that for any smooth scheme X over k, the higher Chow group and the motivic cohomology have the following comparison isomorphism

$$H^i(X, \mathbb{Z}(j)) \cong \mathrm{CH}^j(X, 2j - i)_{\mathbb{Z}} \tag{22}$$

where the right hand side is Bloch's higher Chow group introduced in [6]. Bloch's definition of higher Chow groups will be used to compute the connecting homomorphism of the localization sequence. When X is a smooth scheme over k, the higher Chow group and the motivic cohomology have product structure

$$\mathrm{CH}^a(X, p) \otimes \mathrm{CH}^b(X, q) \to \mathrm{CH}^{a+b}(X, p + q)$$

and the comparison isomorphism (22) is a ring isomorphism ([25]).

Now we summarize results from [8]. For a hyperplane complement $U \subset \mathbb{A}^N$, there is a finite index set I and $n_i \geq 0$ such that

$$\mathsf{M}(U) \cong \bigoplus_{i \in I} \mathbb{Q}(n_i)[n_i].$$

As a corollary, $\mathrm{CH}^*(U, \bullet)$ is a finitely generated free module over $\mathrm{CH}^*(k, \bullet)$ and

$$\mathrm{CH}^\ell(U, 1)_{\mathbb{Z}} = \begin{cases} H^0(U, \mathcal{O}_U^\times) & , \text{if } \ell = 1 \\ 0 & , \text{otherwise.} \end{cases} \tag{23}$$

[12] This assumption can be removed by the work of Cisinski and Déglise, see [44, Theorem 5.1]

The isomorphism $\mathrm{CH}^{\ell}(U,1)_{\mathbb{Z}} \cong H^1(U,\mathbb{Z}(1)) \cong H^0(U,\mathcal{O}_U^{\times})$ follows from [35, Corollary 4.2]. There exists an isomorphism

$$\mathrm{CH}^1(U,1) \cong \overline{\mathrm{CH}}^1(U,1) \oplus \mathrm{CH}^1(k,1)\,.$$

Example 2.15. *Let $U = \operatorname{Spec} k[x,x^{-1}]$ be the complement of the origin in the affine line. Then*

$$\mathrm{CH}^1(U,1)_{\mathbb{Z}} \cong \bigsqcup_{a\in\mathbb{Z}} k^{\times}\langle x^a\rangle \cong \mathbb{Z}\oplus k^{\times}$$

and the element $m \in \mathrm{CH}^0(k)_{\mathbb{Z}} = \mathbb{Z}$ acts by $x^a \mapsto x^{ma}$ and $\lambda \in \mathrm{CH}^1(k,1)_{\mathbb{Z}} = k^{\times}$ acts by $x^a \mapsto \lambda x^a$. In fact $\mathrm{CH}^*(U,\bullet)$ is generated by the fundamental class and $\langle x\rangle$ over $\mathrm{CH}^*(k,\bullet)$.

Proposition 2.16. *Let $U \subset \mathbb{A}^N$ be a hyperplane complement as above. Then the hCKP holds for quotient stacks.*

Proof. Let Y be a quotient stack and hence $Y \times U$ is also a quotient stack. Y admits a vector bundle E such that the vector bundle is represented by a scheme off a locus of arbitrarily high codimension. Since U is a scheme, the pullback of E to $Y\times U$ also satisfies the same property. For higher Chow groups of quotient stacks, the homotopy invariance for vector bundle and the extended localization sequence is proven in [30]. Therefore we may assume that Y is a scheme. When Y is a scheme, there exist isomorphisms

$$\begin{aligned}
\mathrm{CH}^l(Y\times U,1) &= \mathrm{Hom}(\mathsf{M}(Y\times U),\mathbb{Q}(l)[2l-1]) \\
&= \mathrm{Hom}(\mathsf{M}(Y)\otimes\mathsf{M}(U),\mathbb{Q}(l)[2l-1]) \\
&= \mathrm{Hom}(\bigoplus_{i\in I}\mathsf{M}(Y)(n_i)[n_i],\mathbb{Q}(l)[2l-1]) \\
&= \bigoplus_{i\in I}\mathrm{Hom}(\mathsf{M}(Y)(n_i)[n_i],\mathbb{Q}(l)[2l-1]) \\
&= \bigoplus_{i\in I}\mathrm{Hom}(\mathsf{M}(Y),\mathbb{Q}(l-n_i)[2l-n_i-1]) \\
&= \bigoplus_{i\in I}\mathrm{CH}^{l-n_i}(Y,1-n_i) \\
&= \bigoplus_{n_i\leq 1}\mathrm{CH}^{l-n_i}(Y,1-n_i)
\end{aligned}$$

where the fifth equality comes from the cancellation theorem. In the proof of [8, Proposition 1.1], the index $n_i = 0$ corresponds to $\mathrm{CH}^0(U,0)$ and the indices $n_i = 1$ corresponds to generators of $\overline{\mathrm{CH}}^1(U,1)$ over $\mathbb{Q}$. Therefore we get the isomorphism. $\square$

After identifying

$$\mathcal{M}_{0,n} = \{(x_1, \ldots, x_n) \in \mathbb{A}^{n-3} \colon x_i \neq x_j, \text{for } i \neq j,\ x_i \neq 0, x_j \neq 1\} \subset \mathbb{A}^{n-3}\,,$$

Proposition 2.14 and 2.16 compute the higher Chow group of $\prod_{v\in V(\Gamma)} \mathfrak{M}^{\mathrm{sm}}_{0,n(v)}$ for any prestable graph Γ.

Now we revisit the CKP for the stack $\mathfrak{M}_{0,n}$. We recall the definition of Bloch's higher Chow groups [6]. Let

$$\Delta^m = \operatorname{Spec}\left(k[t_0, \ldots, t_m]/(t_0 + \ldots + t_m - 1)\right)$$

be the algebraic m simplex. For $0 \leq i_1 < \ldots < i_a \leq m$, the equation $t_{i_1} = \ldots = t_{i_a} = 0$ defines a face $\Delta^{m-a} \subset \Delta^n$. Let X be an equidimensional quasi-projective scheme over k. Let $z^i(X, m)$ be the free abelian group generated by all codimension i subvarieties of $X \times \Delta^m$ which intersect all faces $X \times \Delta^l$ properly for all $l < m$. Taking the alternating sum of restriction maps to $i+1$ faces of $X \times \Delta^i$, we get a chain complex $(z^*(X, m), \delta)$. The higher Chow group $\mathrm{CH}^i(X, m)$ is the i-th cohomology of the complex $z^*(X, m)$.

When $m = 1$, the proper intersection is equivalent to saying that cycles are not contained in any of the (strict) faces. Let $R = \Delta^1 \setminus \{[0], [1]\}$. Then the group $z^*(X, 1)$ is equal to $z^*(X \times R)$ and the differential

$$\ldots \to z^*(X \times R) \xrightarrow{\delta} z^*(X) \to 0$$

is given by specialization maps. If $\sum a_i W_i$ is a cycle in $X \times R$,

$$\delta\left(\sum a_i W_i\right) = \sum a_i \overline{W_i} \cap X \times [0] - \sum a_i \overline{W_i} \cap X \times [1] \tag{24}$$

where $\overline{W_i}$ is the closure of W_i in $X \times \Delta^1$. In the following, given a product $X \times Y$ and classes $\alpha \in \mathrm{CH}_*(X, a)$, $\beta \in \mathrm{CH}_*(Y, b)$, we write $\alpha \times \beta \in \mathrm{CH}_*(X \times Y, a + b)$ for their exterior product.

Lemma 2.17. *Let X_1, X_2 be algebraic stacks stratified by quotient stacks and $j_1 \colon Z_1 \hookrightarrow X_1$ and $j_2 : Z_2 \hookrightarrow X_2$ be closed substacks with complements*

$$i_1 \colon U_1 = X_1 \setminus Z_1 \hookrightarrow X_1, \quad i_2 : U_2 = X_2 \setminus Z_2 \hookrightarrow X_2$$

where U_1, U_2 are quotient stacks. Let $Z_{12} = X_1 \times X_2 \setminus U_1 \times U_2$. Denote by

$$\begin{aligned} \partial_1 \colon \mathrm{CH}_*(U_1, 1) &\to \mathrm{CH}_*(Z_1)\,, \\ \partial_2 \colon \mathrm{CH}_*(U_2, 1) &\to \mathrm{CH}_*(Z_2)\,, \\ \partial \colon \mathrm{CH}_*(U_1 \times U_2, 1) &\to \mathrm{CH}_*(Z_{12}) \end{aligned}$$

the boundary maps for the inclusions $U_1 \subset X_1$, $U_2 \subset X_2$, $U_1 \times U_2 \subset X_1 \times X_2$.

(a) For $\alpha \in \mathrm{CH}_*(U_1, 1)$ and $\beta \in \mathrm{CH}_*(U_2)$,

$$\partial(\alpha \times \beta) = \partial_1(\alpha) \times \overline{\beta} \ \text{ in } \mathrm{CH}_*(Z_{12})$$

where $\overline{\beta} \in \mathrm{CH}_*(X_2)$ is any extension of β.

(b) The following diagram commutes

$$\begin{array}{ccc} \begin{matrix} \mathrm{CH}_*(U_1,1)\otimes \mathrm{CH}_*(X_2) \\ \oplus \mathrm{CH}_*(X_1)\otimes \mathrm{CH}_*(U_2,1) \end{matrix} & \xrightarrow{(\partial_1\otimes \mathrm{id})\oplus(\mathrm{id}\otimes\partial_2)} & \begin{matrix} \mathrm{CH}_*(Z_1)\otimes \mathrm{CH}_*(X_2) \\ \oplus \mathrm{CH}_*(X_1)\otimes \mathrm{CH}_*(Z_2) \end{matrix} \\ \Big\downarrow {\scriptstyle (\mathrm{id}\otimes i_2^*)\oplus(i_1^*\otimes \mathrm{id})} & & \Big\downarrow \\ \mathrm{CH}_*(U_1\times U_2,1) & \xrightarrow{\quad\partial\quad} & \mathrm{CH}_*(Z_{12})\,, \end{array} \tag{25}$$

where the arrow on the right is induced by the natural map

$$Z_1 \times X_2 \sqcup X_1 \times Z_2 \to Z_{12}.$$

Proof. (a) We first prove that the right hand side is well-defined. For a different choice of extension $\overline{\beta}'$ of β, there exists $\gamma \in \mathrm{CH}_*(Z_2)$ such that $j_{2*}\gamma = \overline{\beta} - \overline{\beta}'$. Therefore, the class $\partial_1(\alpha) \times (\overline{\beta} - \overline{\beta}')$ on Z_{12} is supported on $Z_1 \times Z_2$. In particular, it is a class pushed forward from $\mathrm{CH}_*(X_1 \times Z_2)$. Consider the following commutative diagram

$$\begin{array}{ccc} Z_1\times Z_2 & \xrightarrow{j_1\times \mathrm{id}} & X_1\times Z_2 \\ \Big\downarrow{\scriptstyle \mathrm{id}\times j_2} & & \Big\downarrow{\scriptstyle g} \\ Z_1\times X_2 & \xrightarrow{\quad f\quad} & Z_{12}\,. \end{array}$$

Then we have

$$\partial_1(\alpha) \times (\overline{\beta} - \overline{\beta}') = f_*(\mathrm{id} \times j_2)_*(\partial_1(\alpha) \times \gamma) = g_*(j_1 \times \mathrm{id})_*(\partial_1(\alpha) \times \gamma)\,.$$

This class vanishes because $(j_1)_*\partial_1\alpha$ vanishes as a class in $\mathrm{CH}_*(X_1)$.

We first prove the equality when X_1, X_2 are schemes. The proof follows from diagram chasing. Recall that the connecting homomorphism $\partial\colon \mathrm{CH}_*(U_1 \times U_2, 1) \to \mathrm{CH}_*(Z_{12})$ is defined using the following diagram

$$\begin{array}{ccccc} z^*(Z_{12},1) & \longrightarrow & z^*(Z_{12}) & \longrightarrow & 0 \\ \Big\downarrow & & \Big\downarrow{\scriptstyle j_*} & & \\ z^*(X_1\times X_2,1) & \xrightarrow{\ \delta\ } & z^*(X_1\times X_2) & \longrightarrow & 0 \\ \Big\downarrow{\scriptstyle i^*} & & \Big\downarrow & & \\ z^*(U_1\times U_2,1) & \xrightarrow{\ \delta\ } & z^*(U_1\times U_2) & \longrightarrow & 0\,. \end{array}$$

For each class in $\mathrm{CH}_*(U_1 \times U_2, 1)$ take a representative in $z^*(U_1 \times U_2, 1)$. By taking a preimage under i^*, applying the map δ and taking a preimage under j_*, we get a class in $\mathrm{CH}_*(Z_{12})$ which corresponds to the image of ∂. Fix a representative of α in $z^*(U_1 \times R)$ and β in $z^*(U_2)$. Let $\overline{\alpha}$ be the closure of α in $X_1 \times R$ and $\overline{\beta}$ be the closure of β in X_2. Let $\widetilde{\alpha}$ be the closure of $\overline{\alpha}$ in $X_1 \times \mathbb{A}^1$. Then to compute $\partial(\alpha \times \beta)$ we observe that $\alpha \times \beta = i^*(\overline{\alpha} \times \overline{\beta})$. Applying δ, we have

$$\begin{aligned}\delta(\overline{\alpha} \times \overline{\beta}) &= \widetilde{\alpha} \times \overline{\beta} \cap X_1 \times [0] \times X_2 - \widetilde{\alpha} \times \overline{\beta} \cap X_1 \times [1] \times X_2 \\ &= j_* \left(\widetilde{\alpha} \cap X_1 \times [0] - \widetilde{\alpha} \cap X_1 \times [1]\right) \times \overline{\beta} \\ &= j_*(\partial_1(\alpha) \times \overline{\beta})\end{aligned}$$

and this proves the equality.

In general, let U_1 be a quotient stack by assumption. For a projective morphism $S_1 \to U_1$ from a reduced stack S_1, there exists a projective morphism $T_1 \to X_1$ such that $S_1 \cong U_1 \times_{X_1} T_1$ ([28, Corollary 2.3.2]). Let E_1 be a vector bundle on S_1. By [28, Proposition 2.3.3], there exists a projective modification $T_1' \to T_1$ and a vector bundle E_1' which restricts to E_1. We perform a similar construction for the quotient stack U_2. The image of $\mathrm{CH}_*(U_1, 1) \otimes \mathrm{CH}_*(U_2)$ under the boundary map

$$\partial \colon \mathrm{CH}_*(U_1 \times U_2, 1) \to \mathrm{CH}_*(Z_{12})$$

is defined by the limit of boundary maps for naive higher Chow groups of $E_1 \boxtimes E_2 \subset E_1' \boxtimes E_2'$ (see [28, (4.2.2)]). The corresponding computation is precisely equal to the case above. Therefore the same formula holds for stacks X_1 and X_2.

(b) Let $\alpha \otimes \beta \in \mathrm{CH}_*(U_1, 1) \otimes \mathrm{CH}_*(X_2)$. We take a natural extension β of $i_2^*\beta$. Then by (a), we have

$$\begin{aligned}\partial \circ (\mathrm{id} \otimes i_2^*)\big(\alpha \otimes \beta\big) &= \partial\big(\alpha \times i_2^*\beta\big) \\ &= \partial_1(\alpha) \times \beta \\ &= \partial_1 \otimes \mathrm{id}\big(\alpha \otimes \beta\big)\,.\end{aligned}$$

The same computation holds for $\mathrm{CH}_*(X_1) \otimes \mathrm{CH}_*(U_2, 1)$ and we get the commutativity of (25). $\square$

Remark 2.18. Applying Lemma 2.17 to $X_1 = U_1 = \operatorname{Spec} k$ (so that $Z_1 = \emptyset$) and $Z = Z_2 \subseteq X = X_2$ with $U = X \setminus Z$, we find that the composition

$$\mathrm{CH}_*(k, 1) \otimes \mathrm{CH}_*(U) \to \mathrm{CH}_*(U, 1) \xrightarrow{\partial} \mathrm{CH}_*(Z)$$

vanishes since for $\alpha \in \mathrm{CH}_*(k, 1)$ and $\beta \in \mathrm{CH}_*(U)$ we have $\partial(\alpha \otimes \beta) = \partial_1(\alpha) \otimes \overline{\beta} = 0$ as $\partial_1(\alpha)$ lives in $\mathrm{CH}_*(Z_1) = \mathrm{CH}_*(\emptyset) = 0$. This implies that ∂ factors through the indecomposable part $\overline{\mathrm{CH}}_*(U, 1)$ of $\mathrm{CH}_*(U, 1)$.

To prove the CKP for $\mathfrak{M}_{0,n}$, we want to use that via the boundary gluing morphisms, the space $\mathfrak{M}_{0,n}$ is stratified by (finite quotients of) products of spaces $\mathfrak{M}^{\mathrm{sm}}_{0,n_i}$, for which we know the CKP. The following proposition tells us that indeed the CKP for such a stratified space can be checked on the individual strata.

Proposition 2.19. *Let X be an algebraic stack, locally of finite type over k with a good filtration and stratified by quotient stacks $X = \cup X_i$. Suppose each stratum X_i has the CKP and the hCKgP for quotient stacks. Then X has the CKP for quotient stacks.*

Proof. Since X has a good filtration, the Chow groups of X and $X \times Y$ of a fixed degree can be computed on a sufficiently large finite-type open substack. This allows us to reduce to the case where X has finite type.

Now by assumption, there exists a nonempty open substack $U \subset X$ which is a quotient stack and has the CKP. Let $Z = X \setminus U$ be the complement. For a quotient stack Y consider a commutative diagram

$$\begin{tikzcd}
\mathrm{CH}_*(U,1)\otimes \mathrm{CH}_*(Y) \arrow[r] \arrow[d, "\gamma_1"] & \mathrm{CH}_*(Z)\otimes \mathrm{CH}_*(Y) \arrow[r] \arrow[d, "\gamma_2"] & \mathrm{CH}_*(X)\otimes \mathrm{CH}_*(Y) \arrow[r] \arrow[d, "\gamma_3"] & \mathrm{CH}_*(U)\otimes \mathrm{CH}_*(Y) \arrow[r] \arrow[d, "\gamma_4"] & 0 \\
\mathrm{CH}_*(U\times Y,1) \arrow[r] & \mathrm{CH}_*(Z\times Y) \arrow[r] & \mathrm{CH}_*(X\times Y) \arrow[r] & \mathrm{CH}_*(U\times Y) \arrow[r] & 0
\end{tikzcd} \tag{26}$$

where the rows are exact by the excision sequence. Since U has the CKP, the arrow γ_4 is an isomorphism and by Noetherian induction, the same is true for γ_2. We extend the domain of the map γ_1 by inserting an extra component $\mathrm{CH}_*(U) \otimes \mathrm{CH}_*(Y,1)$. Then the following diagram

$$\begin{tikzcd}
\begin{array}{r}\mathrm{CH}_*(U,1)\otimes \mathrm{CH}_*(Y)\\ \oplus\mathrm{CH}_*(U)\otimes \mathrm{CH}_*(Y,1)\end{array} \arrow[r] \arrow[d, "\gamma_1'"] & \mathrm{CH}_*(Z)\otimes \mathrm{CH}_*(Y) \arrow[d] \\
\mathrm{CH}_*(U\times Y,1) \arrow[r] & \mathrm{CH}_*(Z\times Y)
\end{tikzcd}$$

commutes by applying Lemma 2.17 to $U \times Y \subset X \times Y$. Note that the new factor $\mathrm{CH}_*(U) \otimes \mathrm{CH}_*(Y,1)$ maps to $\mathrm{CH}_*(X) \otimes \mathrm{CH}_*(Y \setminus Y) = 0$ under the top arrow, and so in particular the top row of (26) remains exact after the modification. Furthermore, the modified map γ_1' is surjective because U has the hCKgP for quotient stacks. Therefore γ_3 is an isomorphism by applying the five lemma to the modified version of (26). □

To apply this to the stratification of $\mathfrak{M}_{0,n}$ by prestable graphs, we need a small further technical lemma, due to the fact that the strata of $\mathfrak{M}_{0,n}$ are quotients of products of $\mathfrak{M}_{0,n_i}$ by finite groups. The notion of taking a quotient of an algebraic stack by a finite group action is defined in [42]. See also footnote 9 of [5].

Lemma 2.20. *Let $\mathfrak{M}$ be an algebraic stack of finite type over k and stratified by quotient stacks with an action of a finite étale group G over k. Then the quotient map $\pi : \mathfrak{M} \to \mathfrak{M}/G$ induces an isomorphism*

$$\pi^* : \mathrm{CH}_*(\mathfrak{M}/G) \to \mathrm{CH}_*(\mathfrak{M})^G \tag{27}$$

from the Chow group of the quotient $\mathfrak{M}/G$ to the G-invariant part of the Chow group of $\mathfrak{M}$. On the other hand, the map

$$\pi_* : \mathrm{CH}_*(\mathfrak{M}) \to \mathrm{CH}_*(\mathfrak{M}/G) \tag{28}$$

is a surjection.

Proof. The map π is representable and a principal G-bundle, hence in particular it is finite and étale. Thus we can both pull back cycles and push forward cycles under π. For $g \in G$ let $\sigma_g : \mathfrak{M} \to \mathfrak{M}$ be the action of g on $\mathfrak{M}$. Then the relation $\pi \circ \sigma_g \cong \pi$ shows that σ_g^* acts as the identity on the image of π^* and thus π^* has image in the G-invariant part of $\mathrm{CH}_*(\mathfrak{M})$. The equality

$$\pi_* \circ \pi^* = |G| \cdot \mathrm{id} : \mathrm{CH}_*(\mathfrak{M}/G) \to \mathrm{CH}_*(\mathfrak{M}/G)$$

shows that π^* is injective and that π_* is surjective (since we work with $\mathbb{Q}$-coefficients). On the other hand, we have

$$\pi^* \circ \pi_* = \sum_{g \in G} \sigma_g^* : \mathrm{CH}_*(\mathfrak{M}) \to \mathrm{CH}_*(\mathfrak{M})$$

thus restricted on the G-invariant part, we again have

$$\pi^* \circ \pi_*|_{\mathrm{CH}_*(\mathfrak{M})^G} = |G| \cdot \mathrm{id} : \mathrm{CH}_*(\mathfrak{M})^G \to \mathrm{CH}_*(\mathfrak{M})^G ,$$

showing π^* is surjective. □

In the following, we typically apply the above lemma to the action of $\mathrm{Aut}(\Gamma)$ on the stack $\mathfrak{M}_\Gamma$ (and open substacks of $\mathfrak{M}_\Gamma$). Here $\mathrm{Aut}(\Gamma)$ is the constant group scheme over k associated to the abstract group of automorphisms of Γ, which thus is finite and étale over k.

Remark 2.21. The above lemma is also true for the first higher Chow groups with $\mathbb{Q}$-coefficients.

Corollary 2.22. *For all $n \geq 0$, the stacks $\mathfrak{M}_{0,n}$ have the CKP for quotient stacks.*

Proof. Recall that for a prestable graph Γ of genus 0 with n markings, there exists the locally closed substack $\mathfrak{M}^\Gamma \subset \mathfrak{M}_{0,n}$ of curves with dual graph exactly Γ. By Proposition 2.19, it suffices to show that the stacks $\mathfrak{M}^\Gamma$ have the CKP and the hCKgP for quotient stacks. Now from [5, Proposition 2.4] we know that the restriction of the gluing map ξ_Γ induces an isomorphism

$$\left(\prod_{v\in V(\Gamma)} \mathfrak{M}^{\mathrm{sm}}_{0,n(v)}\right)/\mathrm{Aut}(\Gamma) \xrightarrow{\xi_\Gamma} \mathfrak{M}^\Gamma\,.$$

The product of spaces $\mathfrak{M}^{\mathrm{sm}}_{0,n(v)}$ has the CKP by Proposition 2.7 and the hCKgP for quotient stacks by Proposition 2.14 and Proposition 2.16. From Lemma 2.20 and Remark 2.21 it follows that the quotient of a space with the CKP (or hCKgP) under an action of a finite étale group still has the CKP (or hCKgP), so by the above isomorphism all $\mathfrak{M}^\Gamma$ have the CKP and hCKgP for quotient stacks. This finishes the proof. □

We proved the Chow-Künneth property of $\mathfrak{M}_{0,n}$ with respect to quotient stacks. This assumption comes from technical assumptions in [28]. For example, the extended excision sequence is only proven when the open substack is a quotient stack. Such assumptions are not necessary for a different cycle theory of algebraic stacks constructed in [24]. Therefore, the following remark could remove the technical assumptions in the above Chow-Künneth property.

Remark 2.23. Let X be an algebraic stack, locally of finite type over k. Let $\mathrm{H}^{\mathrm{BM}}_*$ be the rational motivic Borel–Moore homology theory defined in [24]. There exists a cycle class map

$$\mathrm{cl}\colon \mathrm{CH}_*(X)_{\mathbb{Q}} \to \mathrm{H}^{\mathrm{BM}}_*(X)$$

which is compatible with projective pushforward, Chern classes and lci pullbacks. In [4], we will show that the cycle class map cl is an isomorphism when X is stratified by quotient stacks.

2.4 Tautological relations

In this section, we formulate and prove a precise form of Theorem 1.4, see Theorem 2.31. Recall that for a prestable graph Γ, a decoration α is an element of $\mathrm{CH}^*(\mathfrak{M}_\Gamma)$ given as a product $\alpha = \prod_v \alpha_v$ where $\alpha_v \in \mathrm{CH}^*(\mathfrak{M}_{0,n(v)})$ are monomials in κ and ψ-classes on the factors $\mathfrak{M}_{0,n(v)}$ of $\mathfrak{M}_\Gamma$.

Definition 2.24. *Define the* strata space $\mathcal{S}_{g,n}$ *to be the free* $\mathbb{Q}$*-vector space with basis given by isomorphism classes of decorated prestable graphs* $[\Gamma, \alpha]$.

By definition, the image of the map

$$\mathcal{S}_{g,n} \to \mathrm{CH}^*(\mathfrak{M}_{g,n})\,,\ \ [\Gamma,\alpha] \mapsto \xi_{\Gamma *}\alpha$$

is the tautological ring $\mathrm{R}^*(\mathfrak{M}_{g,n})$[13].

For the proofs below, it is convenient to allow decorations α_v at vertices of Γ which are combinations of monomials in κ and ψ-classes as follows.

Definition 2.25. *Given a prestable graph Γ in genus 0 with n markings, an element*

$$\alpha = \prod_{v \in V(\Gamma)} \alpha_v \in \prod_{v \in V(\Gamma)} \mathrm{CH}^*(\mathfrak{M}_{0,n(v)})$$

is said to be in normal form *if*

a) for vertices $v \in V(\Gamma)$ with $n(v) = 0$, we have $\alpha_v = \kappa_2^a$ for some $a \geq 0$,

b) for vertices $v \in V(\Gamma)$ with $n(v) = 1$, we have $\alpha_v = \psi_h^b$, where h is the unique half-edge at v and $b \geq 0$,

c) for vertices $v \in V(\Gamma)$ with $n(v) = 2$, we have $\alpha_v = \psi_h^c + (-\psi_{h'})^c$, where h, h' are two half-edges at v and $c \geq 0$,

d) for vertices $v \in V(\Gamma)$ with $n(v) \geq 3$, we have that $\alpha_v = 1$ is trivial.

Note that because of the terms $\psi_h^c + (-\psi_{h'})^c$ in case c) above, the element α is not strictly speaking a decoration, since the α_v are not monomials. However, given Γ, α as in Definition 2.25, we write $[\Gamma, \alpha]$ for the element in $\mathcal{S}_{0,n}$ obtained by expanding α in terms of monomial decorations.

Definition 2.26. *For $g = 0$ let $\mathcal{S}_{0,n}^{\mathrm{nf}} \subset \mathcal{S}_{0,n}$ be the subspace additively generated by $[\Gamma, \alpha]$ for α in normal form*[14].

Definition 2.27. *Let $R_0 \in \mathcal{S}_{g_0,n_0}$ be a tautological relation. Given g, n, we say that* the set of relations in $\mathcal{S}_{g,n}$ generated by R_0 *is the subspace of the $\mathbb{Q}$-vector space $\mathcal{S}_{g,n}$ generated by elements of $\mathcal{S}_{g,n}$ obtained by*

- *choosing a prestable graph Γ in genus g with n markings and a vertex $v \in V(\Gamma)$ with $g(v) = g_0, n(v) = n_0$,*
- *choosing an identification of the n_0 half-edges incident to v with the markings $1, \ldots, n_0$ for $\mathcal{S}_{g_0,n_0}$,*
- *choosing decorations $\alpha_w \in \mathrm{CH}^*(\mathfrak{M}_{g(w),n(w)})$ for all vertices $w \in V(\Gamma) \setminus \{v\}$,*

[13] From [5, Corollary 3.7] we see that there is a $\mathbb{Q}$-algebra structure on $\mathcal{S}_{g,n}$ which makes this map into a $\mathbb{Q}$-algebra homomorphism, since products of decorated strata classes are given by explicit combinations of further decorated strata classes.

[14] Note that at vertices $v \in V(\Gamma)$ with $n(v) = 2$ we have a choice of ordering of the two half-edges h, h', and the possible decorations $\alpha_v = \psi_h^c + (-\psi_{h'})^c$ differ by a sign for c odd. Still they generate the same subspace of $\mathcal{S}_{0,n}$ and the independence of this subspace from the choice of ordering of half-edges will be important in a proof below.

- *gluing the relation R_0 into the vertex v of Γ, putting decorations α_w in the other vertices and expanding as an element of $\mathcal{S}_{g,n}$.*

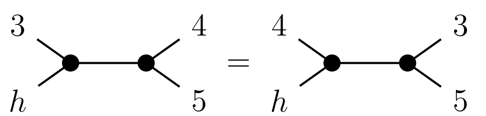
More generally, given any family $(R_0^i \in \mathcal{S}_{g_0^i,n_0^i})_{i\in I}$ of tautological relations, we define the relations in $\mathcal{S}_{g,n}$ generated by this family to be the sum of the spaces of relations generated by the R_0^i.

On the level of Chow groups, the relations in Definition 2.27 are of the form

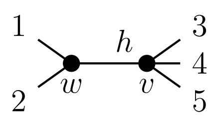
$$R = (\xi_\Gamma)_* \left(\pi_v^* R_0 \cdot \prod_{w \in V(\Gamma)\setminus\{v\}} \pi_w^* \alpha_w \right) = 0 \in \mathrm{CH}^*(\mathfrak{M}_{g,n})\,, \tag{29}$$

where π_v, π_w are the projections from $\mathfrak{M}_\Gamma$ to the factors associated to v, w. The only additional observation needed to make sense of the definition is that the left hand side of (29) also makes sense as an element of the strata algebra $\mathcal{S}_{g,n}$ if R is in $\mathcal{S}_{g_0,n_0}$ and the α_w are monomials in κ- and ψ-classes.

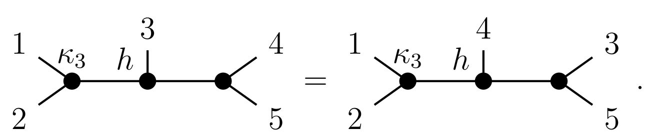
Example 2.28. *Let $n_0 = 4$ with markings labelled $\{3,4,5,h\}$ and let $R_0 \in \mathcal{S}_{0,n_0}$ be the WDVV relation*



For a prestable graph

$\Gamma =$  *in $\mathcal{S}_{0,5}$*

and a decoration $\alpha_w = \kappa_3$, the corresponding relation is



Definition 2.29. *Consider the family $\mathcal{R}^0_{\kappa,\psi}$ of relations obtained by multiplying the relations of κ and ψ-classes from Lemmas 2.2, 2.3 and 2.4 with an arbitrary monomial in κ and ψ-classes. Define the space $\mathcal{R}_{\kappa,\psi} \subset \mathcal{S}_{0,n}$ as the space of relations generated by $\mathcal{R}^0_{\kappa,\psi}$.*

Define $\mathcal{R}_{\mathrm{WDVV}} \subset \mathcal{S}^{\mathrm{nf}}_{0,n}$ as the space of relations obtained by gluing some WDVV relation into a decorated prestable graph $[\Gamma,\alpha]$ in normal form at a vertex v with $n(v) \geq 4$. In other words, it is the space of relations generated by the WDVV relation as in Definition 2.27 where we restrict to $\Gamma, (\alpha_w)_{w\neq v}$ such that $[\Gamma,\alpha]$ (with $\alpha_v = 1$) is in normal form[15].

[15] Again we relax the condition of the α_w being monomials and allow α_w of the form $\psi_h^c + (-\psi_{h'})^c$ at vertices of valence 2.

Remark 2.30. Let us comment on the role of the sets of relations appearing above. The relations $\mathcal{R}^0_{\kappa,\psi}$ allow to write any monomial α in κ and ψ-classes on $\mathfrak{M}_{0,n}$ as a sum $\alpha = \alpha_0 + \beta$ of

- a (possibly zero) monomial term α_0 in κ and ψ-classes such that the trivial prestable graph with decoration α_0 is in normal form (implying that α_0 restricts to a basis element of $\mathrm{CH}^*(\mathfrak{M}^{\mathrm{sm}}_{0,n})$ as computed in Lemma 2.1),
- a sum β of generators $[\Gamma_i, \alpha_i]$ supported in the boundary (i.e. with Γ_i nontrivial).

The relations $\mathcal{R}_{\kappa,\psi}$ allow to do the above at each of the vertices of a decorated stratum class $[\Gamma, \alpha]$ and by a recursive procedure allow to write $[\Gamma, \alpha]$ as a sum of decorated strata classes in normal form. The relations in $\mathcal{R}_{\mathrm{WDVV}}$ then encode the remaining freedom to express relations among these classes in normal form generated by the WDVV relation. The following theorem and the course of its proof make precise the statement that these processes describe tautological relations on $\mathfrak{M}_{0,n}$.

Theorem 2.31. *The kernel of the surjection $\mathcal{S}_{0,n} \to \mathrm{CH}^*(\mathfrak{M}_{0,n})$ is given by*

$$\mathcal{R}_{\kappa,\psi} + \mathcal{R}_{\mathrm{WDVV}}\,.$$

In particular, we have

$$\mathrm{CH}^*(\mathfrak{M}_{0,n}) = \mathcal{S}_{0,n}/(\mathcal{R}_{\kappa,\psi} + \mathcal{R}_{\mathrm{WDVV}})\,.$$

We split the proof of the above theorem into two parts.

Proposition 2.32. *The map $\mathcal{S}^{\mathrm{nf}}_{0,n} \hookrightarrow \mathcal{S}_{0,n} \to \mathcal{S}_{0,n}/\mathcal{R}_{\kappa,\psi}$ is surjective.*

Proof. The statement says that we can use κ, ψ relations on each vertex to express any decorated stratum class as a linear combination of stratum classes in normal form. This follows from Lemma 2.2, 2.3 and 2.4 as described in Remark 2.30. In particular, for vertices v of valency $n(v) = 2$ and adjacent half-edges h, h', we know that any class α_v on $\mathfrak{M}_{0,2}$ of codimension c can be written as a multiple of ψ_h^c plus an element of $\mathcal{R}_{\kappa,\psi}$. Up to such relations, we have $\psi_h^c = (-\psi_{h'})^c$ by Lemma 2.2, and so we obtain a more symmetric decoration by averaging and writing

$$\alpha_v \in \mathbb{Q} \cdot (\psi_h^c + (-\psi_{h'})^c) + \mathcal{R}_{\kappa,\psi}. \qquad \square$$

Theorem 2.33. *The kernel of the surjection $\mathcal{S}^{\mathrm{nf}}_{0,n} \to \mathrm{CH}^*(\mathfrak{M}_{0,n})$ is given by $\mathcal{R}_{\mathrm{WDVV}}$.*

The proof is separated into several steps. The overall strategy is to stratify $\mathfrak{M}_{0,n}$ by the number of edges of the prestable graph Γ and use an excision sequence argument. For $p \geq 0$ we denote by $\mathfrak{M}^{\geq p}_{0,n}$ the closed substack of $\mathfrak{M}_{0,n}$ of curves with at least

p nodes. Similarly, we denote by $\mathfrak{M}_{0,n}^{=p}$ the open substack of $\mathfrak{M}_{0,n}^{\geq p}$ of curves with exactly p nodes. It is clear that

$$\mathfrak{M}_{0,n}^{\geq p} \setminus \mathfrak{M}_{0,n}^{=p} = \mathfrak{M}_{0,n}^{\geq p+1}$$

and also

$$\mathfrak{M}_{0,n}^{=p} = \coprod_{\Gamma \in \mathcal{G}_p} \mathfrak{M}^{\Gamma} ,$$

where $\mathcal{G}_p$ is the set of prestable graphs of genus 0 with n markings having exactly p edges. For the strata space $\mathcal{S}_{0,n}$, consider the decomposition

$$\mathcal{S}_{0,n} = \bigoplus_{p \geq 0} \mathcal{S}_{0,n}^{p}$$

according to the number p of edges of graph Γ.[16] This descends to decompositions

$$\mathcal{S}_{0,n}^{\mathrm{nf}} = \bigoplus_{p \geq 0} \mathcal{S}_{0,n}^{\mathrm{nf},p}, \ \mathcal{R}_{\mathrm{WDVV}} = \bigoplus_{p \geq 0} \mathcal{R}_{\mathrm{WDVV}}^{p}$$

for $\mathcal{S}_{0,n}^{\mathrm{nf}}$ and $\mathcal{R}_{\mathrm{WDVV}}$. We note that $\mathcal{R}_{\mathrm{WDVV}}^{p}$ is exactly the space of relations obtained by taking a prestable graph Γ with $p-1$ edges, a decoration α on Γ in normal form and inserting a WDVV relation at a vertex $v_0 \in V(\Gamma)$ with $n(v_0) \geq 4$.

From Proposition [5, Proposition 2.4] and Lemma 2.20 it follows that

$$\mathrm{CH}^*(\mathfrak{M}_{0,n}^{=p}) = \bigoplus_{\Gamma \in \mathcal{G}_p} \mathrm{CH}^*(\mathfrak{M}^{\Gamma}) = \bigoplus_{\Gamma \in \mathcal{G}_p} \mathrm{CH}^*(\mathfrak{M}_{\Gamma}^{\mathrm{sm}})^{\mathrm{Aut}(\Gamma)} , \tag{30}$$

$$\mathrm{CH}^*(\mathfrak{M}_{0,n}^{=p}, 1) = \bigoplus_{\Gamma \in \mathcal{G}_p} \mathrm{CH}^*(\mathfrak{M}^{\Gamma}, 1) = \bigoplus_{\Gamma \in \mathcal{G}_p} \mathrm{CH}^*(\mathfrak{M}_{\Gamma}^{\mathrm{sm}}, 1)^{\mathrm{Aut}(\Gamma)} . \tag{31}$$

Note that we have a natural map $\mathcal{S}_{0,n}^{\mathrm{nf},p} \to \mathrm{CH}^*(\mathfrak{M}_{0,n}^{\geq p})$.

Lemma 2.34. *The composition $\mathcal{S}_{0,n}^{\mathrm{nf},p} \to \mathrm{CH}^*(\mathfrak{M}_{0,n}^{\geq p}) \to \mathrm{CH}^*(\mathfrak{M}_{0,n}^{=p})$ is an isomorphism.*

Proof. First we note that $\mathcal{S}_{0,n}^{\mathrm{nf},p}$ decomposes into a direct sum of subspaces $\mathcal{S}_{0,n}^{\mathrm{nf},\Gamma}$ indexed by prestable graphs $\Gamma \in \mathcal{G}_p$, according to the underlying prestable graph of the generators. The analogous decomposition of $\mathrm{CH}^*(\mathfrak{M}_{0,n}^{=p})$ is given by (30). Now for two non-isomorphic prestable graphs Γ and Γ' with the same number p of edges, the induced map $\mathcal{S}_{0,n}^{\mathrm{nf},\Gamma} \to \mathrm{CH}^*(\mathfrak{M}^{\Gamma'})$ vanishes. Indeed, the locally closed substack $\mathfrak{M}^{\Gamma'}$ is disjoint from the image of the gluing map ξ_Γ and all generators of $\mathcal{S}_{0,n}^{\mathrm{nf},\Gamma}$ are pushforwards under ξ_Γ. Thus we are reduced to showing that $\mathcal{S}_{0,n}^{\mathrm{nf},\Gamma} \to \mathrm{CH}^*(\mathfrak{M}^{\Gamma})$

[16] This decomposition is not equal to the standard decomposition of $\mathcal{S}_{0,n}$ via degree of a class.

is an isomorphism. The image of a generator $[\Gamma, \alpha]$ under this map is obtained by pushing forward $\alpha \in \mathrm{CH}^*(\mathfrak{M}_\Gamma)$ to $\mathfrak{M}_{0,n}^{\geq p}$ under ξ_Γ and restricting to the open subset $\mathfrak{M}^\Gamma$. From the cartesian diagram

$$\begin{tikzcd} \mathfrak{M}_\Gamma \arrow[r, "\xi_\Gamma"] & \mathfrak{M}_{0,n}^{\geq p} \\ \mathfrak{M}_\Gamma^{\mathrm{sm}} \arrow[u, hook] \arrow[r, "\xi_\Gamma^{\mathrm{sm}}"] & \mathfrak{M}^\Gamma = \mathfrak{M}_\Gamma^{\mathrm{sm}}/\mathrm{Aut}(\Gamma) \arrow[u, hook] \end{tikzcd}$$

in which the vertical arrows are open embeddings, it follows that this is equivalent to first restricting α to $\mathfrak{M}_\Gamma^{\mathrm{sm}}$ and then pushing forward to $\mathfrak{M}^\Gamma$. As we saw in (30), we can identify $\mathrm{CH}^*(\mathfrak{M}^\Gamma)$ with the $\mathrm{Aut}(\Gamma)$-invariant part of $\mathrm{CH}^*(\mathfrak{M}_\Gamma^{\mathrm{sm}})$ via pullback under ξ_Γ^{sm}. But clearly

$$(\xi_\Gamma^{\mathrm{sm}})^*[\Gamma, \alpha] = (\xi_\Gamma^{\mathrm{sm}})^*(\xi_\Gamma^{\mathrm{sm}})_*\alpha = \sum_{\sigma \in \mathrm{Aut}(\Gamma)} \sigma^*\alpha\,, \tag{32}$$

where the automorphisms σ act on $\mathfrak{M}_\Gamma^{\mathrm{sm}}$ by permuting the factors.

Now by Proposition 2.7 we have

$$\mathrm{CH}^*(\mathfrak{M}_\Gamma^{\mathrm{sm}}) = \bigotimes_{v \in V(\Gamma)} \mathrm{CH}^*(\mathfrak{M}_{g(v),n(v)}^{\mathrm{sm}})\,.$$

Thus it follows from Lemma 2.1 that the set of all possible α such that $[\Gamma, \alpha]$ is in normal form is a basis of $\mathrm{CH}^*(\mathfrak{M}_\Gamma^{\mathrm{sm}})$. Now given such an $\alpha \in \mathrm{CH}^*(\mathfrak{M}_\Gamma^{\mathrm{sm}})$, consider the orbit of α under $\mathrm{Aut}(\Gamma)$, recalling that the projection of α to the $\mathrm{Aut}(\Gamma)$-invariant part of $\mathrm{CH}^*(\mathfrak{M}_\Gamma^{\mathrm{sm}})$ is given by $\sum_{\sigma \in \mathrm{Aut}(\Gamma)} \sigma^*\alpha$. Then there are two possibilities:

- either the orbit contains $-\alpha$, in which case $\sum_{\sigma \in \mathrm{Aut}(\Gamma)} \sigma^*\alpha = 0$, and likewise we have $[\Gamma, \alpha] = [\Gamma, -\alpha] = 0$ in the strata algebra. This can happen if there exists a vertex v of Γ of valency 2 and an automorphism σ of Γ switching the two half-edges adjacent to v, if α has a decoration $\psi_h^c + (-\psi_{h'})^c$ at v with c odd.

- or the distinct elements in the orbit are linearly independent in $\mathrm{CH}^*(\mathfrak{M}_\Gamma^{\mathrm{sm}})$. This follows from the fact that α is already determined up to sign by the distribution of its degree to the factors $\mathfrak{M}_{g(v),n(v)}$ of $\mathfrak{M}_\Gamma$, so any two elements of the orbit which are not equal have pairwise distinct such multidegrees.

Basis elements $\alpha \in \mathrm{CH}^*(\mathfrak{M}_\Gamma^{\mathrm{sm}})$ of the first type neither contribute to $\mathcal{S}_{0,n}^{\mathrm{nf,p}}$ nor to $\mathrm{CH}^*(\mathfrak{M}_{0,n}^{=p})$. For basis elements of the second type, the automorphisms σ of Γ act on them by permutation. Hence a basis of the $\mathrm{Aut}(\Gamma)$-invariant part of $\mathrm{CH}^*(\mathfrak{M}_\Gamma^{\mathrm{sm}})$ is given by the sums of orbits of these basis elements (with the dimension of $\mathrm{CH}^*(\mathfrak{M}_\Gamma^{\mathrm{sm}})^{\mathrm{Aut}(\Gamma)}$ being the number of such orbits). Now recall that we chose the basis of $\mathcal{S}_{0,n}^{\mathrm{nf},\Gamma}$ to be the set of $[\Gamma,\alpha]$ in normal form *up to isomorphism*. In other words, one can fix some ordering on the half-edges of Γ, look at all decorations α in normal form, and choose a representative in each $\mathrm{Aut}(\Gamma)$-orbit. Then this chosen basis maps via (32) to the basis of $\mathrm{CH}^*(\mathfrak{M}_\Gamma^{\mathrm{sm}})^{\mathrm{Aut}(\Gamma)}$, by sending the representative α of an $\mathrm{Aut}(\Gamma)$-orbit to the sum $\sum_{\sigma\in\mathrm{Aut}(\Gamma)}\sigma^*\alpha$ of the elements of the orbit. The fact that distinct elements of an orbit are linearly independent implies that the map $\alpha\mapsto\sum_{\sigma\in\mathrm{Aut}(\Gamma)}\sigma^*\alpha$ is injective, since there can be no cancellation between different entries of the orbit. □

Next we realize the WDVV relation as the image of the connecting homomorphism ∂ of the excision sequence

$$\mathrm{CH}^*(\mathfrak{M}_{0,n}^{=p},1)\xrightarrow{\partial}\mathrm{CH}^{*-1}(\mathfrak{M}_{0,n}^{\geq p+1})\to\mathrm{CH}^*(\mathfrak{M}_{0,n}^{\geq p})\to\mathrm{CH}^*(\mathfrak{M}_{0,n}^{=p})\to 0\,. \tag{33}$$

By[5, Proposition 2.4], the stack $\mathfrak{M}_{0,n}^{=p}$ is a quotient stack and hence (33) is exact by [28, Proposition 4.2.1].

Before we study the map ∂ in the sequence (33), we consider an easier situation: we show that in the setting of the moduli spaces $\overline{\mathcal{M}}_{0,n}$ of stable curves, we can explicitly compute the connecting homomorphism ∂, see Proposition 2.36 below. In the proof, we will need the following technical lemma about the connecting homomorphisms of excision sequences.

Lemma 2.35. *Let X be an equidimensional scheme and let*

$$Z'\xrightarrow{j'}Z\xrightarrow{j}X$$

be two closed immersions. Consider the open embedding

$$U=X\setminus Z\xrightarrow{i}U'=X\setminus Z'.$$

Then we have a commutative diagram

$$\begin{tikzcd}
\mathrm{CH}_n(U',1) \arrow[r, "\partial'"] \arrow[d, "i^*"] & \mathrm{CH}_n(Z') \arrow[d, "j'_*"] \\
\mathrm{CH}_n(U,1) \arrow[r, "\partial"] & \mathrm{CH}_n(Z)
\end{tikzcd}\quad, \tag{34}$$

where ∂ and ∂' are the connecting homomorphisms for the inclusions of U and U' in X.

Proof. Elements of $\mathrm{CH}_n(U',1)$ are represented by cycles $\sum a_iW_i$ on $U'\times\Delta^1$ with the W_i of dimension $n+1$ intersecting the faces of $U'\times\partial\Delta^1$ properly. On the one hand, to evaluate the connecting homomorphism ∂', we form the closures $\overline{W_i}$ in $X\times\Delta^1$

and take alternating intersections with faces. This is a sum of cycles of dimension n supported on $Z' \times \partial\Delta^1$ and via j'_* we regard it as a sum of cycles on $Z \times \partial\Delta^1$.

On the other hand, to evaluate $\partial \circ i^*$ we first restrict all W_i to $U \times \Delta^1$, take the closure $\overline{W_i \cap U \times \Delta^1}$ and take alternating intersection with faces. But the only way that this closure can be different from $\overline{W_i}$ is when W_i has generic point in $Z \times \Delta^1$. But then it defines an element of $z_n(Z,1)$ and thus it maps to zero in $\mathrm{CH}_*(Z)$. $\square$

Proposition 2.36. *For $n \geq 4$, the image of the connecting homomorphism ∂ of*

$$\mathrm{CH}^1(\mathcal{M}_{0,n},1) \xrightarrow{\partial} \mathrm{CH}^0(\overline{\mathcal{M}}_{0,n}^{\geq 1}) \xrightarrow{\iota_*} \mathrm{CH}^1(\overline{\mathcal{M}}_{0,n}) \to 0 = \mathrm{CH}^1(\mathcal{M}_{0,n})\,.$$

is spanned by the set of WDVV relations, where we identify $\mathrm{CH}^0(\overline{\mathcal{M}}_{0,n}^{\geq 1})$ as the $\mathbb{Q}$-vector space with basis given by boundary divisors of $\overline{\mathcal{M}}_{0,n}$.

Proof. First, we prove this proposition when $n = 4$. Identify

$$\overline{\mathcal{M}}_{0,4} \cong \mathbb{P}^1 \text{ and } \mathcal{M}_{0,4} \cong \mathbb{A}^1 - \{0,1\}\,.$$

Then $\mathrm{CH}^*(\mathcal{M}_{0,4},\bullet)$ is a $\mathrm{CH}^*(k,\bullet)$-algebra generated by two elements f_0 and f_1 corresponding to two points in $\mathbb{A}^1$ ([8]). Fix a (non-canonical) isomorphism $\Delta^1 \cong \mathbb{A}^1$ and set the two faces as 0 and 1. Consider a line L_0 through $(0,0)$ and $(1,1)$ in $\mathbb{P}^1 \times \Delta^1$ restricted to $(\mathbb{P}^1 - \{0,1,\infty\}) \times \Delta^1$ as illustrated in Figure 3.

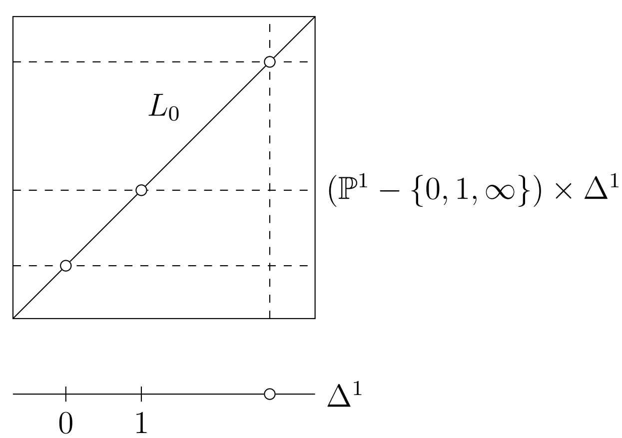


Figure 3: The line L_0 in $(\mathbb{P}^1 - \{0,1,\infty\}) \times \Delta^1$

Then $f_0 = [L_0]$ and

$$\partial(L_0) = [0] - [1] \in \mathrm{CH}^0(\overline{\mathcal{M}}_{0,4} \setminus \mathcal{M}_{0,4})\,.$$

This is one of the WDVV relations on $\overline{\mathcal{M}}_{0,4}$ after identifying $[0], [1]$ and $[\infty]$ with three boundary strata in (4). The second one is obtained from the generator f_1 in an analogous way, finishing the proof for $n = 4$.

For the case of general $n \geq 4$, the space $\mathcal{M}_{0,n}$ is a hyperplane complement with hyperplanes associated to pairs of points that collide and there is a correspondence between generators of $\mathrm{CH}^1(\mathcal{M}_{0,n}, 1)$ and hyperplanes. On the one hand, the action of the symmetric group S_n on $\mathcal{M}_{0,n}$ is transitive on the hyperplanes (and thus on the generators). On the other hand, we can obtain one of the hyperplanes as the pullback of a boundary point in $\mathcal{M}_{0.4}$ under the forgetful morphism $\pi : \overline{\mathcal{M}}_{0,n} \to \overline{\mathcal{M}}_{0,4}$ and thus, via the action of S_n, any hyperplane can be obtained under a suitable forgetful morphism to $\overline{\mathcal{M}}_{0,4}$ (varying the subset of four points to remember).

Note that the morphism π is flat and that we have an open embedding

$$i : \mathcal{M}_{0,n} \to \pi^{-1}(\mathcal{M}_{0,4})$$

and a closed embedding $j' : \pi^{-1}(\partial\overline{\mathcal{M}}_{0,4}) \to \partial\overline{\mathcal{M}}_{0,n}$. Combining the compatibility of the connecting homomorphism ∂ with flat pullback and Lemma 2.35 above, we obtain a commutative diagram

$$\begin{tikzcd}
\mathrm{CH}^1(\mathcal{M}_{0,n}, 1) \arrow[r, "\partial"] & \mathrm{CH}^0(\partial\overline{\mathcal{M}}_{0,n}) \\
\mathrm{CH}^1(\pi^{-1}(\mathcal{M}_{0,4}), 1) \arrow[u, "i^*"] \arrow[r, "\partial"] & \mathrm{CH}^0(\pi^{-1}(\partial\overline{\mathcal{M}}_{0,4})) \arrow[u, "j'_*"'] \\
\mathrm{CH}^1(\mathcal{M}_{0,4}, 1) \arrow[u, "\pi^*"] \arrow[r, "\partial"] & \mathrm{CH}^0(\partial\overline{\mathcal{M}}_{0,4}) \arrow[u, "\pi^*"']
\end{tikzcd}$$

Therefore, since (under a suitable permutation of the markings) every generator of $\mathrm{CH}^1(\mathcal{M}_{0,n}, 1)$ can be obtained as the image of one of the generators of $\mathrm{CH}^1(\mathcal{M}_{0,4}, 1)$, the image of

$$\mathrm{CH}^1(\mathcal{M}_{0,n}, 1) \to \mathrm{CH}^0(\partial\overline{\mathcal{M}}_{0,n})$$

is generated by WDVV relations on $\mathrm{CH}^0(\partial\overline{\mathcal{M}}_{0,4})$ pulled back via π. ☐

We extend the above computation to $\mathfrak{M}_{0,n}$.

Corollary 2.37. *For $n \geq 4$, the image of the connecting homomorphism ∂ of*

$$\mathrm{CH}^{\ell+1}(\mathcal{M}_{0,n}, 1) \xrightarrow{\partial} \mathrm{CH}^{\ell}(\mathfrak{M}_{0,n}^{\geq 1}) \xrightarrow{\iota_*} \mathrm{CH}^{\ell+1}(\mathfrak{M}_{0,n}) \to 0$$

is the set of WDVV relations for $\ell = 0$ and is zero for $\ell > 0$.

Proof. By (23), we have $\mathrm{CH}^{\ell+1}(\mathcal{M}_{0,n}, 1) = 0$ when $\ell > 0$ and hence ∂ is trivial in this range.

For the statement in degree $\ell = 0$ we in fact ignore the definition of ∂ and the machinery of higher Chow groups and simply use that here the image of ∂ is given by the kernel of the map

$$\mathrm{CH}^0(\mathfrak{M}_{0,n}^{\geq 1}) \xrightarrow{\iota_*} \mathrm{CH}^1(\mathfrak{M}_{0,n}), \tag{35}$$

in other words by linear combinations of boundary divisors adding to zero in $\mathrm{CH}^1(\mathfrak{M}_{0,n})$. Given such a relation, restricting to the open substack $\overline{\mathcal{M}}_{0,n}$ simply kills all unstable boundary divisors and by Proposition 2.36 (or classical theory) the result is a combination of WDVV relations. After subtracting those from the original relation, we obtain a combination of unstable boundary divisors forming a relation. The proof is finished if we can show that this must be the trivial linear combination, i.e. that the unstable boundary divisors are linearly independent.

There are exactly $n+1$ strictly prestable graphs with one edge. Let Γ_0 be the prestable graph with a vertex of valence 1 and Γ_i be the semistable graph with the i-th leg on the semistable vertex. Suppose there is a linear relation

$$R = a_0[\Gamma_0] + a_1[\Gamma_1] + \ldots + a_n[\Gamma_n] = 0,\ a_i \in \mathbb{Q}$$

in $\mathrm{CH}^1(\mathfrak{M}_{0,n})$, then we want to show that all $a_i = 0$.

To see this we can simply construct test curves $\sigma_i : \mathbb{P}^1 \to \mathfrak{M}_{0,n}$ intersecting precisely the divisor $[\Gamma_i]$ and none of the others. To obtain σ_i, start with the trivial family $\mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^1$ and a tuple of n disjoint constant sections $p_1, \ldots, p_n$. Let σ_0 be defined by the family of prestable curves obtained by blowing up a point on $\mathbb{P}^1 \times \mathbb{P}^1$ away from any of the sections. For $1 \leq i \leq n$ we similarly obtain σ_i by blowing up a point on the image of p_i and taking the strict transform of the old section p_i. Then we have $0 = \sigma_i^* R = a_i$ for all i, finishing the proof. □

Remark 2.38. As a consequence of the above result, for $n \geq 3$, the map

$$\iota_* : \mathrm{CH}^\ell(\mathfrak{M}_{0,n}^{\geq 1}) \to \mathrm{CH}^{\ell+1}(\mathfrak{M}_{0,n})$$

is an isomorphism in degree $\ell \geq 1$. Restricting to the locus of stable curves, the same proof implies that

$$\iota_* : \mathrm{CH}^\ell(\partial\overline{\mathcal{M}}_{0,n}) \to \mathrm{CH}^{\ell+1}(\overline{\mathcal{M}}_{0,n})$$

is an isomorphism for $\ell \geq 1$.

The surjectivity of ι_* comes from the excision sequence that we discussed. The injectivity can be explained from the results of Kontsevich and Manin ([26]). Indeed, by Proposition [5, Proposition B.19], the vector space $\mathrm{CH}^\ell(\partial\overline{\mathcal{M}}_{0,n})$ is generated by boundary strata of $\overline{\mathcal{M}}_{0,n}$ with at least one edge. To show injectivity of ι_*, it is enough to show that any relation among boundary strata in $\mathrm{CH}^{\ell+1}(\overline{\mathcal{M}}_{0,n})$ is a pushforward of a relation holding already in the Chow group of $\partial\overline{\mathcal{M}}_{0,n}$. By [26, Theorem 7.3] the set of relations between boundary strata in $\mathrm{CH}^{\ell+1}(\overline{\mathcal{M}}_{0,n})$ is spanned by the relations obtained from gluing the WDVV relation into a vertex v_0 of a stable graph Γ with at least ℓ edges. When $\ell \geq 1$, this relation is a pushforward of a class

$$\prod_{v \neq v_0} [\overline{\mathcal{M}}_{0,n(v)}] \times \mathrm{WDVV} \in \mathrm{CH}^1(\overline{\mathcal{M}}_\Gamma)$$

where WDVV $\in \mathrm{CH}^1(\overline{\mathcal{M}}_{0,n(v_0)})$ is the WDVV relation corresponding to the choice of four half edges at v_0. Under the gluing map, this class is a relation on $\partial\overline{\mathcal{M}}_{0,n}$. Therefore we get the injectivity of ι_*.

This corollary is enough to compute the connecting homomorphism in arbitrary degree.

Proposition 2.39. *The image of* $\partial\colon \mathrm{CH}^*(\mathfrak{M}_{0,n}^{=p},1) \to \mathrm{CH}^{*-1}(\mathfrak{M}_{0,n}^{\geq p+1})$ *in* (33) *is equal to the image of the composition*

$$\mathcal{R}_{\mathrm{WDVV}}^{p+1} \to \mathcal{S}_{0,n}^{\mathrm{nf},p+1} \to \mathrm{CH}^*(\mathfrak{M}_{0,n}^{\geq p+1})\,.$$

Thus we can write

$$\mathrm{CH}^*(\mathfrak{M}_{0,n}^{\geq p+1})/\mathrm{CH}^{*+1}(\mathfrak{M}_{0,n}^{=p},1) = \mathrm{CH}^*(\mathfrak{M}_{0,n}^{\geq p+1})/\mathcal{R}_{\mathrm{WDVV}}^{p+1}\,. \tag{36}$$

Proof. From the functoriality of higher Chow groups, it follows that we have a commutative diagram

$$\begin{tikzcd}
\bigoplus_\Gamma \mathrm{CH}^*(\mathfrak{M}_\Gamma^{\mathrm{sm}},1) \arrow[r, "\bigoplus(\xi_\Gamma^{\mathrm{sm}})_*"] \arrow[d, "\bigoplus\partial_\Gamma"] & \bigoplus_\Gamma \mathrm{CH}^*(\mathfrak{M}^\Gamma,1) \arrow[r, equal] & \mathrm{CH}^*(\mathfrak{M}_{0,n}^{=p},1) \arrow[d, "\partial"] \\
\bigoplus_\Gamma \mathrm{CH}^{*-1}(\mathfrak{M}_\Gamma \setminus \mathfrak{M}_\Gamma^{\mathrm{sm}}) \arrow[rr, "\sum(\xi_\Gamma)_*"] & & \mathrm{CH}^{*-1}(\mathfrak{M}_{0,n}^{\geq p+1})
\end{tikzcd} \tag{37}$$

where the sums run over prestable graphs with exactly p edges. By Remark 2.21, the maps $(\xi_\Gamma^{\mathrm{sm}})_*$ are surjective. Thus the image of ∂ is given by the sum of the images of the maps $(\xi_\Gamma)_* \circ \partial_\Gamma$.

From Remark 2.18 we know that ∂_Γ vanishes on the image of the map

$$\mathrm{CH}_*(k,1) \otimes \mathrm{CH}_*(\mathfrak{M}_\Gamma^{\mathrm{sm}}) \to \mathrm{CH}_*(\mathfrak{M}_\Gamma^{\mathrm{sm}},1), \tag{38}$$

and thus factors through its cokernel. On the other hand, it follows from Propositions 2.14 and 2.16 that the cokernel of (38) is generated by classes coming from the direct sum

$$\bigoplus_{v\in V(\Gamma),n(v)\geq 4} \left(\mathrm{CH}^1(\mathfrak{M}_{0,n(v)}^{\mathrm{sm}},1) \otimes \bigotimes_{v'\in V(\Gamma),v\neq v'} \mathrm{CH}^*(\mathfrak{M}_{0,n(v')}^{\mathrm{sm}})\right).$$

For an element $\alpha_v \otimes \bigotimes_{v\neq v'} \alpha_{v'}$ in $\mathrm{CH}^*(\mathfrak{M}_\Gamma^{\mathrm{sm}},1)$, choose an extension $\overline{\alpha}_{v'}$ of each $\alpha_{v'}$ from $\mathfrak{M}_{0,n(v')}^{\mathrm{sm}}$ to $\mathfrak{M}_{0,n(v')}$. By Lemma 2.17 (a), the boundary map ∂_Γ has the form

$$\partial_\Gamma\left(\alpha_v \otimes \bigotimes_{v\neq v'} \alpha_{v'}\right) = \partial(\alpha_v) \otimes \bigotimes_{v\neq v'} \overline{\alpha}_{v'}\,.$$

By Corollary 2.37, the elements $\partial(\alpha_v)$ at vertices v with $n(v) \geq 4$ are precisely the WDVV relations on $\mathfrak{M}_{0,n(v)}$, whereas the classes $\overline{\alpha}_{v'}$ at other vertices v' are exactly the types of decorations allowed in decorated strata classes in normal form. After pushing forward via ξ_Γ this is precisely our definition of the relations $\mathcal{R}_{\mathrm{WDVV}}^{p+1}$. $\square$

Proof of Theorem 2.33. Recall that by Lemma 2.34, the composition

$$\mathcal{S}_{0,n}^{\mathrm{nf},p} \to \mathrm{CH}^*(\mathfrak{M}_{0,n}^{\geq p}) \to \mathrm{CH}^*(\mathfrak{M}_{0,n}^{=p})$$

is an isomorphism. Thus in the diagram

$$\begin{tikzcd}
 & & & \mathcal{S}_{0,n}^{\mathrm{nf},p} \arrow[dl] \arrow[d, "\cong"] & \\
\mathrm{CH}^*(\mathfrak{M}_{0,n}^{=p}, 1) \arrow[r, "\partial"] & \mathrm{CH}^{*-1}(\mathfrak{M}_{0,n}^{\geq p+1}) \arrow[r] & \mathrm{CH}^*(\mathfrak{M}_{0,n}^{\geq p}) \arrow[r] & \mathrm{CH}^*(\mathfrak{M}_{0,n}^{=p}) \arrow[r] & 0
\end{tikzcd}$$

we obtain a canonical splitting of the excision exact sequence (33) and thus we have

$$\mathrm{CH}^*(\mathfrak{M}_{0,n}^{\geq p}) = \mathcal{S}_{0,n}^{\mathrm{nf},p} \oplus \mathrm{CH}^{*-1}(\mathfrak{M}_{0,n}^{\geq p+1})/\mathrm{CH}^*(\mathfrak{M}_{0,n}^{=p}, 1)\,. \tag{39}$$

Combining (39) and equation (36) from Proposition 2.39, we see

$$\mathrm{CH}^*(\mathfrak{M}_{0,n}^{\geq p}) = \mathcal{S}_{0,n}^{\mathrm{nf},p} \oplus \mathrm{CH}^{*-1}(\mathfrak{M}_{0,n}^{\geq p+1})/\mathcal{R}_{\mathrm{WDVV}}^{p+1}\,. \tag{40}$$

Applying (40) for $p = 0, 1, 2, \ldots$ we obtain

$$\begin{aligned}
\mathrm{CH}^*(\mathfrak{M}_{0,n}) &= \mathrm{CH}^*(\mathfrak{M}_{0,n}^{\geq 0}) \\
&= \mathcal{S}_{0,n}^{\mathrm{nf},0} \oplus \mathrm{CH}^{*-1}(\mathfrak{M}_{0,n}^{\geq 1})/\mathcal{R}_{\mathrm{WDVV}}^{1} \\
&= \mathcal{S}_{0,n}^{\mathrm{nf},0} \oplus \left(\mathcal{S}_{0,n}^{\mathrm{nf},1}/\mathcal{R}_{\mathrm{WDVV}}^{1}\right) \oplus \mathrm{CH}^{*-2}(\mathfrak{M}_{0,n}^{\geq 2})/\mathcal{R}_{\mathrm{WDVV}}^{2} \\
&= \ldots \\
&= \bigoplus_{p\geq 0} \mathcal{S}_{0,n}^{\mathrm{nf},p}/\mathcal{R}_{\mathrm{WDVV}}^{p} \\
&= \mathcal{S}_{0,n}^{\mathrm{nf}}/\mathcal{R}_{\mathrm{WDVV}}\,,
\end{aligned}$$

finishing the proof. □

Finally we end the proof of the main theorem.

Proof of Theorem 2.31. We know that the kernel of $\mathcal{S}_{0,n} \to \mathrm{CH}^*(\mathfrak{M}_{0,n})$ contains $\mathcal{R}_{\kappa,\psi}$. We define

$$\mathcal{R}_{\mathrm{res}} = \ker\left(\mathcal{S}_{0,n}/\mathcal{R}_{\kappa,\psi} \to \mathrm{CH}^*(\mathfrak{M}_{0,n})\right)\,.$$

Likewise, by Theorem 2.33 we know that the kernel of $\mathcal{S}_{0,n}^{\mathrm{nf}} \to \mathrm{CH}^*(\mathfrak{M}_{0,n})$ is equal to $\mathcal{R}_{\mathrm{WDVV}}$. We then obtain a diagram of morphisms with exact rows

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{R}_{\mathrm{WDVV}} \arrow[r] \arrow[d, dashed] & \mathcal{S}_{0,n}^{\mathrm{nf}} \arrow[r] \arrow[d, two heads] & \mathrm{CH}^*(\mathfrak{M}_{0,n}) \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \mathcal{R}_{\mathrm{res}} \arrow[r] & \mathcal{S}_{0,n}/\mathcal{R}_{\kappa,\psi} \arrow[r] & \mathrm{CH}^*(\mathfrak{M}_{0,n}) \arrow[r] & 0
\end{tikzcd} \tag{41}$$

where the arrow $\mathcal{S}_{0,n}^{\mathrm{nf}} \to \mathcal{S}_{0,n}/\mathcal{R}_{\kappa,\psi}$ is surjective by Proposition 2.32. A short diagram chase shows that $\mathcal{R}_{\mathrm{WDVV}}$ factors through $\mathcal{R}_{\mathrm{res}}$ via the dashed arrow. By the four-lemma, the map $\mathcal{R}_{\mathrm{WDVV}} \to \mathcal{R}_{\mathrm{res}}$ is surjective. This clearly implies the statement of the theorem. □

2.5 Relation to previous works

Let us start by pointing out several results in Gromov–Witten theory, studying intersection numbers on moduli spaces of stable maps, which can be seen as coming from results about the tautological ring of $\mathfrak{M}_{g,n}$.

Example 2.40. *In [32], degree one relations on the* moduli space of stable maps to a projective space $\overline{\mathcal{M}}_{0,n}(\mathbb{P}^N, d)$ *are used to reduce two pointed genus* 0 *potentials to one pointed genus* 0 *potentials. Theorem 1.(2) from [32] can be obtained by the pullback of the relation in Lemma 2.2 along the forgetful morphism*

$$\overline{\mathcal{M}}_{0,n}(\mathbb{P}^N, d) \to \mathfrak{M}_{0,2}\,.$$

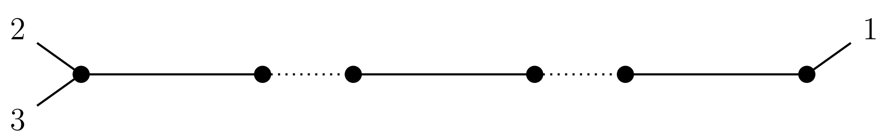
Similarly, Theorem 1.(1) of [32] can be obtained from Lemma 2.3 on $\mathfrak{M}_{0,3}$. *The relevant computations are given explicitly in [3].*

From Theorem 2.31 we see that any universal relation in the Gromov–Witten theory of genus 0 obtained from tautological relations on $\mathfrak{M}_{0,n}$ must follow either from the WDVV relation (4) or the relation (8) between ψ and boundary classes on $\mathfrak{M}_{0,2}$.

Apart from applications to Gromov–Witten theory, there are several results in the literature which compute the Chow groups of some strict open subloci of $\mathfrak{M}_{0,n}$.

Example 2.41. *Restricting to the locus* $\overline{\mathcal{M}}_{0,n} \subset \mathfrak{M}_{0,n}$ *of stable curves, Theorem 2.31 specializes to the classical result in [22, 26] that all relations between undecorated strata of* $\overline{\mathcal{M}}_{0,n}$ *are additively generated by the WDVV relations.*

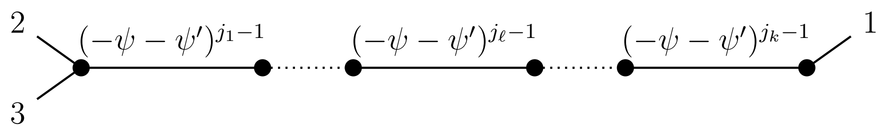
Example 2.42. *In [36], Oesinghaus computes the Chow ring (with integer coefficients) of the open substack* $\mathcal{T}$ *of* $\mathfrak{M}_{0,3}$ *of curves with prestable graph of the form*



where we denote by Γ_k *the graph of the shape above with* k *edges (for* $k \geq 0$*). Oesinghaus shows that the Chow ring* $\mathrm{CH}^*(\mathcal{T})$ *is given by the ring QSym of* quasi-symmetric functions *on the index set* $\mathbb{Z}_{>0}$. *QSym can be seen as the subring of* $\mathbb{Q}[\alpha_1, \alpha_2, \ldots]$ *with additive basis given by*

$$M_J = \sum_{i_1 < \ldots < i_k} \alpha_{i_1}^{j_1} \cdots \alpha_{i_k}^{j_k} \text{ for } k \geq 1, J = (j_1, \ldots, j_k) \in \mathbb{Z}_{\geq 1}^k. \tag{42}$$

Under the isomorphism $\mathrm{CH}^*(\mathcal{T}) \cong$ *QSym, the element* M_J *is a basis element of degree* $\sum_\ell j_\ell$ *in the Chow group of* $\mathcal{T}$*. As we explain in [5, Example 4.3], the cycle* M_J *corresponds to the tautological class supported on the stratum* $\mathfrak{M}^{\Gamma_k}$ *given by*



(43)

Using the correspondence, we can verify several of the results of our paper in this particular example. Indeed, one can use Theorem 2.33 to verify that the classes (43) form a basis of $\mathrm{CH}^*(\mathcal{T})$*. For this, one observes that decorated strata in normal form generically supported on* $\mathcal{T}$ *must have underlying graph* Γ_k *for some* k*, with trivial decoration on the valence* 3 *vertex and decorations* $(-\psi_h - \psi_{h'})^{c_\ell}$ *on the valence* 2 *vertices. Since every term appearing in a WDVV relation has at least two vertices of valence at least* 3*, all these relations restrict to zero on* $\mathcal{T}$ *and thus the above generators form a basis by Theorem 2.33. Note that the form of these generators in normal form is not quite the same as the one shown in (43), but a small combinatorial argument shows that the two bases can be converted to each other by using the relation (8) between* ψ*-classes and the boundary divisor on* $\mathfrak{M}_{0,2}$*.*

Note that [36] also computes the Chow group of the semistable loci $\mathfrak{M}_{0,2}^{ss}$ *and* $\mathfrak{M}_{0,3}^{ss}$*. By a straightforward generalization of the discussion above, a correspondence of the generators in [36] to the tautological generators on these spaces, as well as a comparison of relations can be established.*

Example 2.43. *In a series of papers [14, 15, 13], Fulghesu presented a computation of the Chow ring of the open substack* $\mathfrak{M}_0^{\leq 3} \subset \mathfrak{M}_0$ *of rational curves with at most* 3 *nodes, as an explicit algebra with* 10 *generators and* 11 *relations. Some of the generators are given by* κ*-classes, some are classes of strata and others are decorated classes supported on strata.*

Establishing a precise correspondence to the generators and relations discussed in our paper is challenging due to the complexity of the involved combinatorics. However, as a nontrivial check of our results we can compare the dimensions $\dim \mathrm{CH}^d(\mathfrak{M}_0^{\leq 3})$ *of the graded pieces of the Chow ring. Given any open substack* $U \subseteq \mathfrak{M}_0$*, we package the ranks of the Chow groups of* U *in the generating function*

$$H_U = \sum_{d \geq 0} \dim \mathrm{CH}^d(U) t^d,$$

which is the Hilbert series *of the graded ring* $\mathrm{CH}^*(U)$*.*

In [13], Fulghesu computes the Chow rings of the open substacks $U = \mathfrak{M}_0^{\leq e}$ *for* $e = 0, 1, 2, 3$ *in terms of generators and relations. Using the software Macaulay2 [19] we can compute*[17] *the Hilbert functions* H_U^F *of the graded algebras given by Fulghesu. We list them in Figure 4.*

[17]The output of the relevant computation can be found here.

U	H_U^F	
$\mathfrak{M}_0^{\leq 0}$	$\frac{1}{1-t^2}$	$= 1 + t^2 + t^4 + \ldots$
$\mathfrak{M}_0^{\leq 1}$	$\frac{1}{(1-t^2)(1-t)}$	$= 1 + t + 2t^2 + 2t^3 + 3t^4 + \ldots$
$\mathfrak{M}_0^{\leq 2}$	$\frac{t^4+1}{(1-t^2)^2(1-t)}$	$= 1 + t + 3t^2 + 3t^3 + 7t^4 + 7t^5 + 13t^6 + 13t^7 + 21t^8 + \ldots$
$\mathfrak{M}_0^{\leq 3}$	H'	$= 1 + t + 3t^2 + 5t^3 + 10t^4 + 15t^5 + 26t^6 + 36t^7 + \mathbf{55}t^8 + \ldots$

Figure 4: The Hilbert series of the Chow rings of open substacks U of $\mathfrak{M}_0$, as computed by Fulghesu; for space reasons we don't write the full formula for the rational function H', only giving the expansion

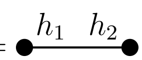
On the other hand, since for stable graphs with at most three edges no WDVV relations can appear, Theorem 2.33 implies that the Chow group $\mathrm{CH}^*(\mathfrak{M}_0^{\leq e})$ *is equal to the subspace of the strata algebra* $\mathcal{S}_{0,0}$ *spanned by decorated strata in normal form with at most* e *nodes (for* $e \leq 3$*). By some small combinatorial arguments, this allows us to compute the Hilbert functions* H_U *of the spaces* $U = \mathfrak{M}_0^{\leq e}$*:*

$\mathbf{e = 0}$ *For* $U = \mathfrak{M}_0^{\leq 0}$ *the only generators in normal form are the classes* κ_2^a*, existing in every even degree* $d = 2a$*, so that the generating function is given by*

$$H_{\mathfrak{M}_0^{\leq 0}} = 1 + t^2 + t^4 + \ldots = \frac{1}{1-t^2},$$

recovering the formula from Figure 4.

$\mathbf{e = 1}$ *On* $U = \mathfrak{M}_0^{\leq 0}$ *we get additional generators*

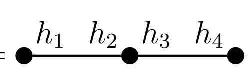
$$[\Gamma_1, \psi_{h_1}^a \psi_{h_2}^b] \text{ for } \Gamma_1 = $$


Since the automorphism group of Γ_1 *exchanges* h_1, h_2*, the numbers* a, b *above are only unique up to ordering. We get a canonical representative by requiring* $a \leq b$*. Overall, we obtain the generating series*

$$\begin{aligned} H_{\mathfrak{M}_0^{\leq 1}} = H_{\mathfrak{M}_0^{\leq 0}} + \sum_{0 \leq a \leq b} t^{a+b+1} &= \frac{1}{1-t^2} + t \sum_{a \geq 0} \sum_{c \geq 0} t^{a+(a+c)} \\ &= \frac{1}{1-t^2} + t \left(\sum_{a \geq 0} t^{2a} \right) \left(\sum_{c \geq 0} t^c \right) \\ &= \frac{1}{1-t^2} + t \frac{1}{1-t^2} \frac{1}{1-t} = \frac{1}{(1-t^2)(1-t)}, \end{aligned}$$

where we used the substitution $b = a + c$*. Again we recover the formula from Figure 4.*

$\mathbf{e = 2}$ *The additional generators for* $U = \mathfrak{M}_0^{\leq 2}$ *are given by*

$$[\Gamma_2, \psi_{h_1}^a (\psi_{h_2}^b + (-\psi_{h_3})^b) \psi_{h_4}^c] \text{ for } \Gamma_2 = $$


The automorphism group of Γ_2 exchanges h_1, h_4 and h_2, h_3, so a, c are only well-defined up to ordering. Moreover, for $a = c$ and $b = 2\ell + 1$ odd, this symmetry implies

$$[\Gamma_2, \psi_{h_1}^a(\psi_{h_2}^b + (-\psi_{h_3})^b)\psi_{h_4}^a] = -[\Gamma_2, \psi_{h_1}^a(\psi_{h_2}^b + (-\psi_{h_3})^b)\psi_{h_4}^a],$$

so the corresponding generator vanishes. Overall, the numbers of basis elements supported on Γ_2 have generating series

$$t^2 \cdot \Big(\underbrace{\sum_{0 \le a \le c} \sum_{b \ge 0} t^{a+b+c}}_{= \frac{1}{(1-t^2)(1-t)^2}} - \underbrace{\sum_{a \ge 0} \sum_{\ell \ge 0} t^{2a+2\ell+1}}_{= \frac{t}{(1-t^2)^2}} \Big) = \frac{t^2(t^2+1)}{(1-t)(1-t^2)^2}.$$

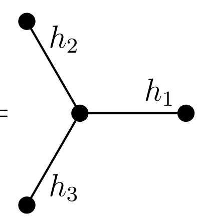
Adding this to the generating series for $\mathfrak{M}_0^{\le 1}$ we obtain the formula

$$H_{\mathfrak{M}_0^{\le 2}} = H_{\mathfrak{M}_0^{\le 1}} + \frac{t^2(t^2+1)}{(1-t)(1-t^2)^2} = \frac{t^4+1}{(1-t^2)^2(1-t)},$$

again obtaining the same formula as in Figure 4.

$\mathbf{e = 3}$ *For the full locus $U = \mathfrak{M}_0^{\le 3}$ a discrepancy between our results and Fulghesu's computations appears. There are two new types of generators appearing: firstly we have*

$$[\Gamma_3', \psi_{h_1}^a \psi_{h_2}^b \psi_{h_3}^c] \text{ for } \Gamma_3' =$$

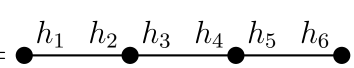


giving a contribution of

$$t^3 \sum_{0 \le a \le b \le c} t^{a+b+c} = \frac{t^3}{(1-t^3)(1-t^2)(1-t)}$$

to the generating series. The second type of generator is

$$[\Gamma_3'', \psi_{h_1}^a(\psi_{h_2}^b + (-\psi_{h_3})^b)((-\psi_{h_4})^c + \psi_{h_5}^c)\psi_{h_6}^d] \text{ for } \Gamma_3'' =$$



Since Γ_3'' again has an automorphism of order 2, we count such generators using a trick: if the vertices of Γ_3'' were ordered, the generating series would be

$$t^3 \sum_{a,b,c,d \ge 0} t^{a+b+c+d} = \frac{t^3}{(1-t)^4}.$$

Due to the automorphism, we counted almost all the generators twice, except *those fixed by the automorphism, for which* $(a,b,c,d) = (a,b,b,a)$ *and whose generating series is*

$$t^3 \sum_{a,b\geq 0} t^{2a+2b} = \frac{t^3}{(1-t^2)^2}.$$

Adding these two series, we count every generator twice, so we obtain the correct count after dividing by two. Overall we get

$$\begin{aligned} H_{\mathfrak{M}_0^{\leq 3}} &= H_{\mathfrak{M}_0^{\leq 2}} + \frac{t^3}{(1-t^3)(1-t^2)(1-t)} + \frac{1}{2}\left(\frac{t^3}{(1-t)^4} + \frac{t^3}{(1-t^2)^2}\right) \\ &= \frac{t^6+t^5+2t^4+t^3+1}{(1-t^2)^2(1-t)(1-t^3)}. \end{aligned}$$

However, expanding this series we obtain

$$\frac{t^6+t^5+2t^4+t^3+1}{(1-t^2)^2(1-t)(1-t^3)} = 1+t+3t^2+5t^3+10t^4+15t^5+26t^6+36t^7+\mathbf{54}t^8+\ldots \quad (44)$$

Comparing with the expansion of the corresponding function H' *in Figure 4 we see that the coefficient of* t^8 *is* 55 *for Fulghesu and* 54 *for us. We used a modified version of the software package admcycles [10] for the open-source software SageMath [43] to verify the number* 54 *above.*

After revisiting Fulghesu's proof, we think we can explain this discrepancy from a relation that was missed in [13]. In the notation of this paper, we claim that there is a relation

$$r \cdot q - \gamma_3'' \cdot s + 2u \cdot \gamma_2 - \gamma_3'' \cdot q \cdot \kappa_2 - s \cdot \gamma_2 \cdot \kappa_1 = 0 \in \mathrm{CH}^8(\mathfrak{M}_0^{\leq 3}). \quad (45)$$

Here the classes r, s, u, γ_3'' *are supported on the closed stratum* $\mathfrak{M}^{\Gamma''_3} \subset \mathfrak{M}_0^{\leq 3}$. *The relation* (45) *follows from the description of the Chow ring* $\mathrm{CH}^*(\mathfrak{M}^{\Gamma''_3})$ *and the formulas for restrictions of classes* $q, \gamma_3'', \gamma_2, \kappa_2, \kappa_1$ *to* $\mathfrak{M}^{\Gamma''_3}$ *computed in [13, Section 6.2]. On the other hand, using Macaulay2 we verified that the relation* (45) *is* not *contained in the ideal of relations given in [13, Theorem 6.3]. Adding this missing relation, we obtain the correct rank* 54 *for* $\mathrm{CH}^8(\mathfrak{M}_0^{\leq 3})$.

Our numerical experiments indicated that there are further relations missing in degrees $d > 9$. *So while the general proof strategy of [13] seems sound, more care needed is needed in the final step of the computation.*

2.6 Chow rings of open substacks of $\mathfrak{M}_{0,n}$ - finite generation and Hilbert series

In the previous section, we saw some explicit computations for Chow groups $\mathrm{CH}^*(U)$ *of of open substacks* $U \subset \mathfrak{M}_{0,n}$ *and their Hilbert series*

$$H_U = \sum_{d\geq 0} \dim_{\mathbb{Q}} \mathrm{CH}^d(U) t^d.$$

For $U = \mathfrak{M}_0^{\leq e}$ and $e = 0, 1, 2$ we have that $\mathrm{CH}^*(U)$ is a finitely generated graded algebra by the results of [13]. But recall that any such algebra, having generators in degrees $d_1, \ldots, d_r$ has a Hilbert series which is the expansion (at $t = 0$) of a rational function $H(t)$ of the form

$$H(t) = \frac{Q(t)}{\prod_{i=1}^r (1 - t^{d_i})} \text{ for some } Q(t) \in \mathbb{Z}[t]$$

(see [34, Theorem 13.2]). This explains the shape of the Hilbert functions of $\mathfrak{M}_0^{\leq e}$ from Figure 4. We remark here, that all functions $H(t)$ of the above form have poles only at roots of unity.

On the other hand, for the open substack $\mathcal{T} \subset \mathfrak{M}_{0,3}$ studied by Oesinghaus, we saw $\mathrm{CH}^*(\mathcal{T}) \cong QSym$, where the algebra QSym had an additive basis element M_J in degree d for each composition J of d. Since for $d \geq 1$ the number of compositions of d is 2^{d-1}, the Hilbert series of the Chow ring of $\mathcal{T}$ is given by

$$H_{\mathcal{T}} = 1 + \sum_{d \geq 0} 2^{d-1} t^d = 1 + \frac{t}{1 - 2t} = \frac{1 - t}{1 - 2t}.$$

From this we can see two things:

- *The Chow ring $\mathrm{CH}^*(\mathcal{T})$ cannot be a finitely generated algebra, since the function $H_{\mathcal{T}}$ has a pole at $1/2$, which is not a root of unity.*
- *On the other hand, we still have that $H_{\mathcal{T}}$ is the expansion of a rational function, even though $\mathcal{T}$ is not even of finite type.*

The above observations lead to the following two questions.

Question 2.44. *Is it true that for $U \subset \mathfrak{M}_{0,n}$ an open substack of finite type, the Chow ring $\mathrm{CH}^*(U)$ is a finitely generated algebra?*

Question 2.45. *Is it true that for $U \subset \mathfrak{M}_{0,n}$ any open substack which is a union of strata $\mathfrak{M}^\Gamma$, the Hilbert series H_U is the expansion of a rational function at $t = 0$?*

For the first question, we note that by Theorem 1.2 we know that $\mathrm{CH}^*(U)$ is additively generated by possibly infinitely many decorated strata $[\Gamma, \alpha]$, supported on finitely many *prestable graphs Γ. It is far from obvious whether we can obtain all of them multiplicatively from a finite collection of $[\Gamma_i, \alpha_i]$.*

For the second question, we observe that it would be implied for all finite type open substacks U of $\mathfrak{M}_{0,n}$ assuming a positive answer to the first question. Further evidence is provided by the results from [36]: as we saw above, the open substack $\mathcal{T} \subset \mathfrak{M}_{0,3}$ has a rational generating series $H_{\mathcal{T}}$. In fact, as mentioned above Oesinghaus computes *the Chow ring for the the entire semistable locus in* $\mathfrak{M}_{0,2}$ *and* $\mathfrak{M}_{0,3}$ *(see [36, Corollary 2,3]) and obtains*

$$\mathrm{CH}^*(\mathfrak{M}_{0,2}^{\mathrm{ss}}) = \mathrm{QSym} \otimes_{\mathbb{Q}} \mathbb{Q}[\beta],\ \mathrm{CH}^*(\mathfrak{M}_{0,3}^{\mathrm{ss}}) = \mathrm{QSym} \otimes_{\mathbb{Q}} \mathrm{QSym} \otimes_{\mathbb{Q}} \mathrm{QSym}.$$

Since we know that the Hilbert series of QSym *is* $(1-t)/(1-2t)$ *and the Hilbert series of* $\mathbb{Q}[t]$ *is* $1/(1-t)$ *and that Hilbert series are multiplicative under tensor products, we easily see that*

$$H_{\mathfrak{M}_{0,2}^{\mathrm{ss}}} = \frac{1}{1-2t},\ H_{\mathfrak{M}_{0,3}^{\mathrm{ss}}} = \frac{(1-t)^3}{(1-2t)^3}.$$

So Question 2.45 has a positive answer for the non-finite type substacks of semistable points in $\mathfrak{M}_{0,2}$ *and* $\mathfrak{M}_{0,3}$.

To finish this section, we want to record some numerical data about the Chow groups of the full stacks $\mathfrak{M}_{0,n}$. *Using Theorems 1.2 and 2.31 these groups have a completely combinatorial description. This has been implemented in a modified version of the software package* `admcycles` *[10], which can enumerate prestable graphs, decorated strata in normal form and the relations* $\mathcal{R}_{\kappa,\psi}, \mathcal{R}_{\mathrm{WDVV}}$ *between them. Thus, from linear algebra we can compute the ranks of Chow groups of* $\mathfrak{M}_{0,n}$ *in many cases. We record the results in Figure 5.*

d	n=0	n=1	n=2	n=3	n=4	n=5	n=6	n=7	n=8
0	1	1	1	1	1	1	1	1	1
1	1	2	3	4	6	11	23	50	108
2	3	5	9	16	33	80	215	621	1900
3	5	12	27	62	162	481	1572		
4	13	32	84	235	739	2594			
5	27	84	263	875	3219				
6	70	234	837	3219					
7	166	656	2683						
8	438	1892							
9	1135								
10	3081								

Figure 5: The rank of the Chow groups $\mathrm{CH}^d(\mathfrak{M}_{0,n})$

3 Comparison with the tautological ring of the moduli of stable curves

3.1 Injectivity of pullback by forgetful charts

Assume we are in the stable range $2g-2+n>0$ so that the moduli space $\overline{\mathcal{M}}_{g,n}$ is nonempty. Since $\overline{\mathcal{M}}_{g,n} \subset \mathfrak{M}_{g,n}$ is an open substack, the Chow groups of $\mathfrak{M}_{g,n}$ determine those of $\overline{\mathcal{M}}_{g,n}$: we have that $\mathrm{CH}^*(\overline{\mathcal{M}}_{g,n})$ is the quotient of $\mathrm{CH}^*(\mathfrak{M}_{g,n})$ by the span of classes supported on the strictly unstable locus.

Restricting to the subrings of tautological classes, we note that the tautological ring $\mathrm{R}^*(\overline{\mathcal{M}}_{g,n})$ of $\overline{\mathcal{M}}_{g,n}$ is the subring of $\mathrm{CH}^*(\overline{\mathcal{M}}_{g,n})$ given by the restriction of $\mathrm{R}^*(\mathfrak{M}_{g,n})$ inside $\mathrm{CH}^*(\mathfrak{M}_{g,n})$ under the open embedding $i\colon \overline{\mathcal{M}}_{g,n} \hookrightarrow \mathfrak{M}_{g,n}$. Thus the tautological ring $\mathrm{R}^*(\mathfrak{M}_{g,n})$ determines $\mathrm{R}^*(\overline{\mathcal{M}}_{g,n})$ since the composition

$$\mathrm{R}^*(\overline{\mathcal{M}}_{g,n}) \xrightarrow{\mathrm{st}^*} \mathrm{R}^*(\mathfrak{M}_{g,n}) \xrightarrow{i^*} \mathrm{R}^*(\overline{\mathcal{M}}_{g,n})$$

is the identity and thus $\mathrm{R}^*(\overline{\mathcal{M}}_{g,n}) \xrightarrow{\mathrm{st}^*} \mathrm{R}^*(\mathfrak{M}_{g,n})$ is injective.

It is an interesting question whether the converse is true: do the Chow (or tautological) rings of the moduli spaces of stable curves determine the Chow (or tautological) ring of $\mathfrak{M}_{g,n}$? The following conjecture gives a precise way in which this could be true.

Conjecture 3.1. *Let $(g,n) \neq (1,0)$, then for a fixed $d \geq 0$ there exists $m_0 \geq 0$ such that for any $m \geq m_0$, the forgetful morphism*

$$F_m : \overline{\mathcal{M}}_{g,n+m} \to \mathfrak{M}_{g,n}$$

satisfies that the pullback

$$F_m^* : \mathrm{CH}^d(\mathfrak{M}_{g,n}) \to \mathrm{CH}^d(\overline{\mathcal{M}}_{g,n+m})$$

is injective.

We have seen in [5, Lemma 2.1] that (for m sufficiently large) the image of F_m is open with complement of codimension $\lfloor \frac{m}{2} \rfloor + 1$. So for $m \geq 2d$, we have

$$\mathrm{CH}^d(F_m(\overline{\mathcal{M}}_{g,n+m})) \cong \mathrm{CH}^d(\mathfrak{M}_{g,n})\,,$$

so certainly the image of F_m is sufficiently large to capture the Chow group of codimension d cycles. Still, it is not true that a surjective, smooth morphism has injective pullback in Chow (if the fibres are not proper, as is the case for F_m), so this does not suffice to prove the conjecture.

One aspect of the conjecture we can prove so far is the statement that if $F_{m_0}^*$ is injective, it is true that for any $m \geq m_0$ the map F_m^* remains injective.

Proposition 3.2. *For* $(g,n) \neq (1,0)$ *and* $0 \leq m \leq m'$ *with* $2g-2+n+m > 0$ *we have* $\ker F_{m'}^* \subseteq \ker F_m^*$. *In other words, the subspaces* $(\ker F_\ell^*)_\ell$ *form a* non-increasing *sequence of subspaces of* $\mathrm{CH}^d(\mathfrak{M}_{g,n})$.

Proof. It suffices to show the statement for $m' = m+1$. Consider the following *non-commutative* diagram.

$$\begin{tikzcd} \overline{\mathcal{M}}_{g,n+m+1} \arrow[d, "\pi"] \arrow[dr, "F_{m+1}"] & \\ \overline{\mathcal{M}}_{g,n+m} \arrow[r, "F_m"] & \mathfrak{M}_{g,n} \end{tikzcd} \tag{46}$$

Here π is the usual map forgetting the marking p_{n+m+1} and stabilizing the curve. For this reason, the diagram is only commutative on the complement of the locus

$$Z = \left\{ (C, p_1, \ldots, p_{n+m+1}) : \begin{array}{c} p_{n+m+1} \text{ contained in} \\ \text{rational component of } C \\ \text{with 3 special points} \end{array} \right\} \subseteq \overline{\mathcal{M}}_{g,n+m+1} \,.$$

Let $i : Z \to \overline{\mathcal{M}}_{g,n+m+1}$ be the inclusion of Z and let $\alpha \in \mathrm{CH}^*(\mathfrak{M}_{g,n})$ be any class. By the commutativity of the diagram (46) away from Z, we know that the class $F_{m+1}^*\alpha - \pi^* F_m^* \alpha$ restricts to zero on the complement of Z and thus, by the usual excision sequence, there exists a class $\beta \in \mathrm{CH}^*(Z)$ such that

$$F_{m+1}^*\alpha - \pi^* F_m^*\alpha = i_*\beta \,.$$

We want to transport this to an equality of classes on $\overline{\mathcal{M}}_{g,n+m}$ by intersecting with ψ_{n+m+1} and pushing forward via π. But notice that $\psi_{n+m+1}|_Z = 0$ since on Z the component of C containing p_{n+m+1} is parametrized by $\overline{\mathcal{M}}_{0,3}$ and thus the psi-class of p_{n+m+1} vanishes here. Thus

$$\psi_{n+m+1} \cdot F_{m+1}^*\alpha - \psi_{n+m+1} \cdot \pi^* F_m^*\alpha = (i_*\beta)\psi_{n+m+1} = i_*(i^*\psi_{n+m+1} \cdot \beta) = 0 \,. \tag{47}$$

Pushing forward by π and using $\pi_*\psi_{n+m+1} = (2g-2+n+m) \cdot [\overline{\mathcal{M}}_{g,n+m}]$ we obtain

$$\pi_* \left(\psi_{n+m+1} \cdot F_{m+1}^*\alpha \right) = (2g-2+n+m) \cdot F_m^*\alpha \,.$$

Thus, since $2g-2+n+m \geq 2g-2+n > 0$, any class α with $F_{m+1}^*\alpha = 0$ also satisfies $F_m^*\alpha = 0$, finishing the proof. $\square$

Again, for $g = 0$ we can give some numerical evidence for the above conjecture. In Figure 6 we compare ranks of the Chow groups $\mathrm{CH}^d(\mathfrak{M}_{0,n})$ to (lower bounds on) the ranks of $F_m^*(\mathrm{CH}^d(\mathfrak{M}_{0,n}))$. We see that the bounds for $F_m^*(\mathrm{CH}^d(\mathfrak{M}_{0,n}))$ increase monotonically in m (as predicted by Proposition 3.2) and, in all cases which we could handle computationally, stabilize at the rank of $\mathrm{CH}^d(\mathfrak{M}_{0,n})$, implying that the corresponding pullbacks F_m^* are indeed injective.

(n,d)	$\mathrm{CH}^d(\mathfrak{M}_{0,n})$	$F_m^*(\mathrm{CH}^d(\mathfrak{M}_{0,n}))$									
		m=0	1	2	3	4	5	6	7	8	9
(0,0)	1				1	1	1	1	1	1	1
(0,1)	1					1	1	1	1	1	1
(0,2)	3						1	2	3	3	3
(0,3)	5							1	2	4	5
(0,4)	13								1	2	7
(1,0)	1			1	1	1	1	1	1	1	1
(1,1)	2				1	2	2	2	2	2	2
(1,2)	5					1	3	5	5	5	5
(1,3)	12						1	4	7	12	12
(2,0)	1		1	1	1	1	1	1	1		
(2,1)	3			1	3	3	3	3	3		
(2,2)	9				1	5	9	9	9		
(2,3)	27					1	7	11	27		
(3,0)	1	1	1	1	1	1	1	1			
(3,1)	4		1	4	4	4	4	4			
(3,2)	16			1	5	15	16	16			
(3,3)	62				1	5	16	62			

Figure 6: The ranks of the Chow groups $\mathrm{CH}^d(\mathfrak{M}_{0,n})$ compared to (lower bounds on) the ranks of $F_m^*(\mathrm{CH}^d(\mathfrak{M}_{0,n}))$; in many cases it was not feasible to obtain the precise rank of $F_m^*(\mathrm{CH}^d(\mathfrak{M}_{0,n}))$, but a lower bound could be achieved by computing the rank of the intersection pairing of $F_m^*(\mathrm{CH}^d(\mathfrak{M}_{0,n}))$ with a selection of tautological classes on $\overline{\mathcal{M}}_{0,n+m}$

3.2 The divisor group of $\mathfrak{M}_{g,n}$

As for the moduli space of stable curves, the group of divisor classes on $\mathfrak{M}_{g,n}$ can be fully understood in terms of tautological classes and relations. For $g = 0$ we already saw that all divisor classes are tautological and we explicitly described the relations, so below we can restrict to $g \geq 1$. As before, we also want to exclude the case $g = 1, n = 0$ since $\mathfrak{M}_{1,0}$ does not have a stratification by quotient stacks.

Thus we can restrict to the range $2g - 2 + n > 0$, where the space $\overline{\mathcal{M}}_{g,n}$ is nonempty. Then we have an exact sequence

$$\begin{tikzcd} \mathrm{CH}_*(\overline{\mathcal{M}}_{g,n}, 1) \arrow[r] & \mathrm{CH}_*(\mathfrak{M}_{g,n}^{\mathrm{us}}) \arrow[r] & \mathrm{CH}_*(\mathfrak{M}_{g,n}) \arrow[r] & \mathrm{CH}_*(\overline{\mathcal{M}}_{g,n}) \arrow[r] \arrow[l, bend right, dotted, "\mathrm{st}^*"'] & 0 \end{tikzcd} \tag{48}$$

where $\mathfrak{M}_{g,n}^{\mathrm{us}}$ is the *unstable locus* of $\mathfrak{M}_{g,n}$, i.e. the complement of the open substack $\overline{\mathcal{M}}_{g,n} \subset \mathfrak{M}_{g,n}$. Using this sequence, we can completely understand $\mathrm{CH}^1(\mathfrak{M}_{g,n})$ from the explicit description of $\mathrm{CH}^1(\overline{\mathcal{M}}_{g,n})$ in [1, Theorem 2.2].

Proposition 3.3. *For $(g, n) \neq (1, 0)$ we have $\mathrm{R}^1(\mathfrak{M}_{g,n}) = \mathrm{CH}^1(\mathfrak{M}_{g,n})$. Furthermore, for $2g - 2 + n > 0$, all tautological relations in $\mathrm{R}^1(\mathfrak{M}_{g,n})$ are pulled backed from relations in $\mathrm{R}^1(\overline{\mathcal{M}}_{g,n})$ via the stabilization morphism.*

Proof. As discussed before, for the statement that all divisor classes are tautological we can restrict to the stable range $2g - 2 + n > 0$, since the case of $g = 0$ was treated before. For the moduli spaces of stable curves it holds that $\mathrm{R}^1(\overline{\mathcal{M}}_{g,n})$ and $\mathrm{CH}^1(\overline{\mathcal{M}}_{g,n})$ coincide by [1]. Since the locus $\mathfrak{M}^{\mathrm{us}}_{g,n}$ is a union of boundary divisors, whose fundamental classes are pushforwards of appropriate gluing maps, the image of the pushforward map $\mathrm{CH}^0(\mathfrak{M}^{\mathrm{us}}_{g,n}) \to \mathrm{CH}^1(\mathfrak{M}_{g,n})$ is contained in $\mathrm{R}^1(\mathfrak{M}_{g,n})$. Therefore the excision sequence (48) gives the conclusion.

To compute the set of relations, we observe that

$$\mathrm{CH}^1(\overline{\mathcal{M}}_{g,n}, 1) \cong H^0(\overline{\mathcal{M}}_{g,n}, \mathcal{O}^{\times}_{\overline{\mathcal{M}}_{g,n}}) = k^{\times} \tag{49}$$

because $\overline{\mathcal{M}}_{g,n}$ is smooth and projective over k. For smooth projective varieties X, the isomorphism $\mathrm{CH}^1(X, 1) \cong k^{\times}$ is proven in [6, Theorem 6.1]. For connected smooth projective Deligne-Mumford stacks $\mathcal{X}$ over k which are quotient stacks, the corresponding isomorphism also holds after tensoring with $\mathbb{Q}$. Indeed, let $q : \mathcal{X} \to \operatorname{Spec} k$ be the structure morphism. By [29, Theorem 1] there exists a finite flat surjective morphism $p : X \to \mathcal{X}$ from a smooth projective scheme over k. Then we have

$$k^{\times} \otimes_{\mathbb{Z}} \mathbb{Q} \cong \mathrm{CH}^1(k, 1)_{\mathbb{Q}} \xrightarrow{q^*} \mathrm{CH}^1(\mathcal{X}, 1)_{\mathbb{Q}} \xrightarrow{p^*} \mathrm{CH}^1(X, 1) \cong k^{\times} \otimes_{\mathbb{Z}} \mathbb{Q}\,,$$

where the composition is an isomorphism. This shows that p^* is surjective, and on the other hand, by the projection formula (see e.g. [24, Section 2.3.4]) the pullback p^* is injective. Thus $\mathrm{CH}^1(\mathcal{X}, 1)_{\mathbb{Q}} \cong k^{\times} \otimes_{\mathbb{Z}} \mathbb{Q}$. Applying this to $\mathcal{X} = \overline{\mathcal{M}}_{g,n}$ we obtain (49), where we use that $\overline{\mathcal{M}}_{g,n}$ is a quotient stack (see [2, Chapter XII, Theorem 5.6]). Therefore, in the degree 1 part of the sequence (48), the image of the connecting homomorphism is trivial. Thus, since the pullback st^* by the stabilization morphism defines a splitting of (48) on the right, we have that

$$\mathrm{CH}^1(\mathfrak{M}_{g,n}) = \bigoplus_{\substack{\Gamma \text{ unstable} \\ |E(\Gamma)|=1}} \mathbb{Q} \cdot [\Gamma] \oplus \mathrm{CH}^1(\overline{\mathcal{M}}_{g,n}).$$

Thus all relations between decorated strata classes in codimension 1 are pulled back from $\overline{\mathcal{M}}_{g,n}$. □

3.3 Zero cycles on $\mathfrak{M}_{g,n}$

After treating the case of codimension 1 cycles in the previous section, we want to make some remarks about cycles of *dimension* 0. For the moduli spaces of stable curves, these exhibit many interesting properties:

- In [18], Graber and Vakil showed that the group $\mathrm{R}_0(\overline{\mathcal{M}}_{g,n})$ of tautological zero cycles on $\overline{\mathcal{M}}_{g,n}$ is always isomorphic to $\mathbb{Q}$, even though the full Chow group $\mathrm{CH}_0(\overline{\mathcal{M}}_{g,n})$ can be infinite-dimensional (e.g. for $(g,n) = (1,11)$, see [18, Remark 1.1]).

- In [39], Pandharipande and the second author presented geometric conditions on stable curves $(C, p_1, \ldots, p_n)$ ensuring that the zero cycle $[(C, p_1, \ldots, p_n)]$ in $\overline{\mathcal{M}}_{g,n}$ is tautological.

We want to note here, that for the moduli stacks of prestable curves, the behaviour of tautological zero cycles becomes more complicated:

- For $\mathfrak{M}_{0,n}$ with $n = 0, 1, 2$, we have $\mathrm{R}_0(\mathfrak{M}_{0,n}) = \mathrm{CH}_0(\mathfrak{M}_{0,n}) = 0$ for dimension reasons.

- As visible from Figure 5, the group $\mathrm{R}_0(\mathfrak{M}_{0,n})$ is no longer one-dimensional for $n \geq 4$. Indeed, looking at the example of $n = 4$ we note that the boundary divisor of curves with one component having no marked points is a nonvanishing zero cycle (since it pulls back to an effective boundary divisor under the forgetful map $F_2 : \overline{\mathcal{M}}_{0,6} \to \mathfrak{M}_{0,4}$), but it restricts to zero on $\overline{\mathcal{M}}_{0,4} \subset \mathfrak{M}_{0,4}$ and is thus linearly independent of the generator of $\mathrm{R}_0(\overline{\mathcal{M}}_{0,4})$.

This indicates that for the moduli stacks of prestable curves, the group of zero cycles plays less of a special role than for the moduli spaces of stable curves.

A Gysin pullback for higher Chow groups

In [9], Déglise, Jin and Khan generalized Gysin pullback along a regular imbedding to motivic homotopy theories. We summerize the construction in the language of higher Chow groups. For a moment, let X be a quasi-projective scheme over k and we consider higher Chow groups with $\mathbb{Z}$-coefficients. For simplicity, we write $\mathbb{G}_m X = X \times \mathbb{G}_m$. Let $[t]$ be a generator of

$$\mathrm{CH}_0(\mathbb{G}_m, 1) \cong \big(k[t, t^{-1}]\big)^{\times}$$

and let

$$\gamma_t \colon \mathrm{CH}_*(X, m) \to \mathrm{CH}_*(\mathbb{G}_m X, m+1)\,,\ \alpha \mapsto \alpha \times [t]$$

be the morphism defined by the exterior product. Let $i \colon Z \to X$ be a regular imbedding of codimension r and let $q : N_Z X \to Z$ be the normal bundle. Let $D_Z X$ be the Fulton–MacPherson's deformation space defined by

$$D_Z X = \mathrm{Bl}_{Z\times 0}(X \times \mathbb{A}^1) - \mathrm{Bl}_{Z\times 0}(X \times 0)$$

which fits into the cartesian diagram

$$\begin{tikzcd} N_Z X \arrow[r] \arrow[d] & D_Z X \arrow[d] & \mathbb{G}_m X \arrow[l] \arrow[d] \\ \{0\} \arrow[r] & \mathbb{A}^1 & \mathbb{G}_m\,. \arrow[l] \end{tikzcd}$$

By [6, 7] we have the localization sequence

$$\ldots \to \mathrm{CH}_d(\mathbb{G}_m X, m+1) \xrightarrow{\partial} \mathrm{CH}_d(N_Z X, m) \to \mathrm{CH}_d(D_Z X, m) \to \ldots\,. \tag{50}$$

Definition A.1. *For a regular imbedding $i \colon Z \to X$, we define*

$$i^* \colon \mathrm{CH}_d(X, m) \to \mathrm{CH}_{d-r}(Z, m)$$

as a composition of following morphisms

$$\mathrm{CH}_d(X, m) \xrightarrow{\gamma_t} \mathrm{CH}_d(\mathbb{G}_m X, m+1) \xrightarrow{\partial} \mathrm{CH}_d(N_Z X, m) \xrightarrow[\cong]{(q^*)^{-1}} \mathrm{CH}_{d-r}(Z, m) \tag{51}$$

where ∂ is the boundary map in (50) and the flat pullback q^* is an isomorphism by [6].

This definition extends the Gysin pullback for Chow groups in the sense that it coincides with the Gysin pullback defined in [16] when $m = 0$. This construction extends to all lci morphism and satisfies functoriality, transverse base change and excess intersection formula, see [9].

Given a line bundle $q : L \to X$ with the zero section $0 : X \to L$, the action of the first Chern class on higher Chow groups can be defined by

$$c_1(L)\cap\colon \mathrm{CH}_d(X,m) \xrightarrow{0_*} \mathrm{CH}_d(L,m) \xrightarrow[\cong]{(q^*)^{-1}} \mathrm{CH}_{d-1}(X,m)\,.$$

We want to note two basic compatibilities of this operation: firstly, given a proper morphism $f : X' \to X$ and the line bundle $L \to X$, a short computation shows the projection formula

$$f_*\left(c_1(f^*L) \cap \alpha\right) = c_1(L) \cap f_*\alpha \text{ for } \alpha \in \mathrm{CH}_*(X',m)\,. \tag{52}$$

Secondly, intersecting higher Chow cycles with a Cartier divisor has the same formula as for ordinary Chow groups.

Lemma A.2. *Let $i\colon D \to X$ be an effective divisor and let $q\colon \mathcal{O}(D) \to X$ be the associated line bundle. Then*

$$i_*i^*\alpha = c_1(\mathcal{O}(D)) \cap \alpha\,,\ \alpha \in \mathrm{CH}_d(X,m)\,. \tag{53}$$

Proof. Consider the following cartesian diagram

$$\begin{tikzcd} D \arrow[r, "i"] \arrow[d, "i"] & X \arrow[d, "s"] \\ X \arrow[r, "0"] & \mathcal{O}(D) \end{tikzcd}$$

where $s\colon X \to \mathcal{O}(D)$ be the regular section defining D and 0 is the zero section. Recall that the action of the first Chern class can be defined by

$$c_1(\mathcal{O}(D)) \cap -\colon \mathrm{CH}_d(X,m) \xrightarrow{0_*} \mathrm{CH}_d(\mathcal{O}(D),m) \xrightarrow[\cong]{(q^*)^{-1}} \mathrm{CH}_{d-1}(X,m)\,.$$

By the transverse base change formula [9, Proposition 2.4.2],

$$i_*i^*\alpha = s^*0_*\alpha\,.$$

Now we can conclude the result because s^* is an inverse of q^*. □

The above construction can be extended to global quotient stacks without any difficulty. Let X be an equidimensional quasi-projective scheme with a linearized G-action. Applying the Borel construction developed in [11] yields the definition of higher Chow groups for $[X/G]$, see [30]. For an arbitrary algebraic stack, the authors do not know whether a direct generalization of [16] is possible. Relying on the recent development of motivic homotopy theories, Khan used the six-operator formalism ([23]) to construct motivic Borel–Moore homology for derived algebraic stacks in [24].

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Relations on the moduli space of stable maps from equivariant projective bundle

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Abstract

We find tautological relations on the moduli space of stable maps to a target X depending on the choice of a line bundle on X. Our main result is a proof of certain set of relations on $\overline{\mathcal{M}}_{g,n}(X,\beta)$ which is a natural generalization of Pixton's relations,. We use the equivariant Gromov-Witten theory of the projectivization of the line bundle over X.

1 Introduction

1.1 Overview

After Mumford [24] initiated the study of relations in the cohomology of the moduli space of curves in the 1980s, the subject has developed rapidly in the last three decades. In 1990s, Faber and Zagier systematically studied the algebra of κ classes on the moduli space $\mathcal{M}_g$ of nonsingular genus g curves and conjectured a concise set FZ of relations among κ classes. These relations were proven by Pandharipande and Pixton using the geometry of stable quotients in 2010 [26]. In 2012, Pixton conjectured a set P of tautological relations on the moduli spaces $\overline{\mathcal{M}}_{g,n}$ of stable curves. The set P recovers FZ when restricted to $\mathcal{M}_g \subset \overline{\mathcal{M}}_g$. Pandharipande, Pixton and Zvonkine proved the Pixton conjecture via the geometry of 3-spin curves [27]. We refer the reader to [25] for an introduction to tautological classes and Pixton's relations.

In [1], the first author extended the notion of tautological classes to the moduli space $\overline{\mathcal{M}}_{g,n}(X,\beta)$ of stable maps to a target variety X. Those tautological classes are defined by a natural perfect obstruction theory on $\overline{\mathcal{M}}_{g,n}(X,\beta)$ [5]. It is natural to ask the following question: Is there a set of tautological relations P_X on $\overline{\mathcal{M}}_{g,n}(X,\beta)$ which depends on the geometry of X and naturally generalizes the Pixton relations such that the set P_X recovers P if X is a point? We give an answer to this question

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which depends on the choice of line bundle S on X. We consider the stable maps to the projective bundle $\mathbb{P}(\mathcal{O}_X \oplus S)$ and use the Givental's quantization formalism over the moduli space of stable maps to X. Our method is motivated from Coates-Givental-Tseng [11] and Janda [17]. As a consequence, we prove a set of relations on $\overline{\mathcal{M}}_{g,n}(X,\beta)$ which naturally generalize Pixton's relations.

1.2 X-valued stable graphs

Let X be a nonsingular projective variety over $\mathbb{C}$ and let $\beta \in H_2(X,\mathbb{Z})$ be an effective curve class. We review the X-valued stable graphs introduced in [18]. Boundary strata of the moduli space of stable maps to X correspond to *X-valued stable graphs*

$$\Gamma = (V, H, L, g : V \to \mathbb{Z}_{\geq 0}, \beta : V \to H_2(X,\mathbb{Z}), v : H \to V, \iota : H \to H)$$

satisfying the following properties:

(i) V is a vertex set with a genus function $g : V \to \mathbb{Z}_{\geq 0}$ and a degree function $d : V \to H_2(X,\mathbb{Z})$,

(ii) H is a half-edge set equipped with a vertex assignment $v : H \to V$ and an involution ι,

(iii) E, the edge set, is defined by the 2-cycle of ι in H (self-edges at vertices are allowed),

(iv) L, the set of legs, is defined by the fixed points of ι and endowed with a bijective correspondence with a set of markings,

(v) the pair (V, E) defines a *connected* graph,

(vi) for each vertex v, the stability condition holds:

$$2g(v) - 2 + n(v) > 0 \text{ if } \beta(v) = 0\,,$$

where $n(v)$ is the valence of Γ at v including both edges and legs,

(vii) the degree condition holds

$$\sum_v \beta(v) = \beta.$$

An automorphism of Γ consist of automorphisms of the sets V and H which leave invariant the structures g, β, ι, and v (and hence respect E and L). Let $\mathrm{Aut}(\Gamma)$ denote the automorphism group of Γ.

The genus of a stable graph Γ is defined by

$$g(\Gamma) = \sum_{v \in V} g(v) + h^1(\Gamma)\,.$$

A boundary stratum of the moduli space $\overline{\mathcal{M}}_{g,n}(X,\beta)$ of stable maps naturally determines a stable graph of genus g, degree d with n legs by considering the dual graph of a generic pointed domain curve parameterized by the stratum. Let $\mathsf{G}_{g,n,\beta}(X)$ be the set of isomorphism classes of X-valued stable graphs of genus g and degree β with n legs.

To each stable graph Γ, we associate the moduli space $\overline{\mathcal{M}}_\Gamma$ which is the substack of the product

$$\prod_{v\in V}\overline{\mathcal{M}}_{g(v),n(v)}(X,\beta(v))$$

cut out by the inverse image of the diagonal $\Delta_X \subset X\times X$ under the evaluation maps associated to all edges $e=(h,h')\in E$,

$$\prod_{v\in V}\overline{\mathcal{M}}_{g(v),n(v)}(X,\beta(v)) \xrightarrow{\mathrm{ev}_e} X\times X\,.$$

Let π_v be the projection from $\overline{\mathcal{M}}_\Gamma$ to $\overline{\mathcal{M}}_{g(v),n(v)}(X,\beta(v))$ associated to the vertex v. There is a canonical morphism

$$\xi_\Gamma:\overline{\mathcal{M}}_\Gamma\to\overline{\mathcal{M}}_{g,n}(X,\beta)$$

with the image equal to the boundary stratum associated to the graph Γ. To construct ξ_Γ, a family of stable maps over $\overline{\mathcal{M}}_\Gamma$ is required. Such a family is easily obtained by gluing pull-backs of the universal families over each of the $\overline{\mathcal{M}}_{g(v),n(v)}(X,\beta(v))$ along the sections corresponding to half-edges. The moduli space $\overline{\mathcal{M}}_\Gamma$ carries a natural virtual fundamental class $[\overline{\mathcal{M}}_\Gamma]^{\mathrm{vir}}$ induced by the Gysin pull-back along diagonals

$$[\overline{\mathcal{M}}_\Gamma]^{\mathrm{vir}}=\prod_{e\in E}\mathrm{ev}_e^{-1}(\Delta)\cap\prod_{v\in V}[\overline{\mathcal{M}}_{g(v),n(v)}(X,\beta(v))]^{\mathrm{vir}}\,. \tag{1}$$

1.3 Strata algebra

For any target X, we can associate a $\mathbb{Q}$-algebra, called the X-valued strata algebra [1], which represents tautological classes on $\overline{\mathcal{M}}_{g,n}(X,\beta)$. In this paper, we will restrict to the subalgebra of X-valued strata algebra associated to a fixed line bundle on X. Let S be a line bundle over X. There are two canonical line bundles on the universal curve

$$\pi:\mathcal{C}_{g,n,\beta}(X)\to\overline{\mathcal{M}}_{g,n}(X,\beta)\,.$$

The first one is the relative dualizing sheaf ω_π and the second one is the pull-back f^*S of the line bundle S via the universal map,

$$f:\mathcal{C}_{g,n,\beta}(X)\to X\,.$$

Let s_i be the i-th section of π, and let

$$D_i\subset\mathcal{C}_{g,n,\beta}(X)$$

be the corresponding divisor. Denote by $\omega_{\log}$ the relative logarithmic line bundle

$$\omega_\pi\Big(\sum_i^n D_i\Big)$$

with the first Chern class $c_1(\omega_{\log})$. Let $\xi = c_1(f^*S)$ be the first Chern class of the pull-back of S. Tautological classes ψ, ξ, and η classes on $\overline{\mathcal{M}}_{g,n}(X,\beta)$ are defined as follows:

$$\psi_i := c_1(s_i^*\omega_\pi)\,,\ \ \xi_i := s_i^*\xi\,,\ \ \eta_{a,b} = \pi_*(c_1(\omega_{\log})^a\xi^b).$$

Definition 1. *A* decorated X-valued stable graph $[\Gamma,\gamma]$ *is an X-valued stable graph $\Gamma \in \mathsf{G}_{g,n,\beta}(X)$ together with the following decoration data γ:*

(i) each leg $i \in L$ is decorated with a monomial $\psi_i^a\xi_i^b$,

(ii) each half-edge $h \in H \setminus L$ is decorated with a monomial ψ_h^a,

(iii) each edge $e \in E$ is decorated with a monomial ξ_e^a,

(iv) each vertex in V is decorated with a monomial in the variables $\{\eta_{a,b}\}_{a+b\geq 2}$.

Consider the $\mathbb{Q}$-vector space $\mathcal{S}_{g,n,\beta}(X)$[1] whose basis consists of the isomorphism classes of a decorated X-valued stable graph $[\Gamma,\gamma]$. There is a product structure on $\mathcal{S}_{g,n,\beta}(X)$ which generalizes the intersection product on the strata algebra $\mathcal{S}_{g,n}$ of $\overline{\mathcal{M}}_{g,n}$ ([1, 18]). If we assign a grading

$$\deg[\Gamma,\gamma] = |E| + \deg_{\mathbb{C}}(\gamma)\,,$$

to each basis element $[\Gamma,\gamma]$, $\mathcal{S}_{g,n,\beta}(X)$ is a graded $\mathbb{Q}$-algebra

$$\mathcal{S}_{g,n,\beta}(X) = \bigoplus_{k=0}^{\mathrm{vdim}} \mathcal{S}^k_{g,n,\beta}(X)\,.$$

Via this intersection product, $\mathcal{S}_{g,n,\beta}(X)$ is a finite dimensional $\mathbb{Q}$-algebra which we call the *strata algebra* following [1, 18, 27].

To each element $[\Gamma,\gamma] \in \mathcal{S}_{g,n,\beta}(X)$, we assign a cycle class $\xi_{\Gamma*}[\gamma]$ obtained by the push-forward via

$$\overline{\mathcal{M}}_\Gamma \to \overline{\mathcal{M}}_{g,n}(X,\beta)$$

of the action of the product of the ψ,ξ and η decorations on $[\overline{\mathcal{M}}_\Gamma]^{\mathrm{vir}}$

$$\xi_{\Gamma*}[\gamma] := \xi_{\Gamma*}\Big(\gamma \cap [\overline{\mathcal{M}}_\Gamma]^{\mathrm{vir}}\Big) \in A_*(\overline{\mathcal{M}}_{g,n}(X,\beta))_{\mathbb{Q}}\,.$$

[1] This vector space depends on the choice of a line bundle S. For simplicity, we suppress the dependence of S in the notation.

Then ξ_Γ defines a $\mathbb{Q}$-linear map

$$\mathsf{q} : \mathcal{S}_{g,n,\beta}(X) \to A_*(\overline{\mathcal{M}}_{g,n}(X,\beta)) \,, \;\; \mathsf{q}([\Gamma,\gamma]) = \xi_{\Gamma*}[\gamma]$$

and it is known that the kernel of q is an ideal. An element of the kernel is called a *tautological relation for the target variety* X. We denote by $R^*(\overline{\mathcal{M}}_{g,n}(X,\beta))$ the image of q.

1.4 X-valued Pixton's relations

Let $P = c_1(S) \in A^1(X)$ be the first Chern class of the line bundle S. The following hypergeometric series are required to describe tautological relations

$$A(z) = \sum_{i\geq 0} \frac{(6i)!}{(3i)!(2i)!}\left(\frac{-z}{576}\right)^i, \qquad B(z) = \sum_{i\geq 0} \frac{(6i)!}{(3i)!(2i)!}\frac{1+6i}{4}\left(\frac{-z}{576}\right)^i,$$

$$C(z) = \sum_{i\geq 0} \frac{(6i)!}{(3i)!(2i)!}\frac{1+6i}{1-6i}\left(\frac{-z}{576}\right)^i, \qquad D(z) = \sum_{i\geq 0} \frac{(6i)!}{(3i)!(2i)!}\frac{1+6i}{-4}\left(\frac{-z}{576}\right)^i.$$

Define new power series

$$\mathsf{B}_0(z) = A(z) + \widetilde{P}B(z) \,, \;\; \mathsf{B}_1(z) = C(z) + \widetilde{P}D(z) \,,$$

where $\widetilde{P}$ is a degree one formal variable with the condition

$$\widetilde{P}^2 = 0 \,.$$

For a power series with vanishing constant and linear terms in T,

$$f(T,\widetilde{P}) \in (T^2, T\widetilde{P})\mathbb{Q}[\widetilde{P}][[T]] \,,$$

we define

$$\kappa(f) = \sum_{m\geq 0} \frac{1}{m!} p_{m*}\Big(f(\psi_{n+1}, \mathrm{ev}_{n+1}^*(P)) \dots f(\psi_{n+m}, \mathrm{ev}_{n+m}^*(P))\Big) \tag{2}$$

in $A^*(\overline{\mathcal{M}}_{g,n}(X,\beta))$. Here

$$p_m : \overline{\mathcal{M}}_{g,n+m}(X,\beta) \to \overline{\mathcal{M}}_{g,n}(X,\beta) \tag{3}$$

is the map which forgets the last m marked points and contract unstable components and ev_j is the j-th evaluation map. The sum (2) is finite because of the degree reason.

Let $\mathsf{G}_{g,n,\beta}(X)$ be the (finite) set of stable graphs of genus g, degree β with n legs (up to isomorphism). For each vertex $v \in V$, we introduce an auxiliary variable ζ_v and impose conditions

$$\zeta_v \zeta_{v'} = \zeta_{v'} \zeta_v \,, \ \ \zeta_v^2 = 1 \,.$$

The variables ζ_v will keep track of local parity condition at each vertex.

In order to describe the tautological relations, we require the following map

$$\mathsf{K} : \mathcal{S}_{g,n,\beta}(X) \otimes \mathbb{Q}[\zeta_v] \to \mathcal{S}_{g,n,\beta}(X) \tag{4}$$

defined by

$$\mathsf{K}\Big(\prod_{v\in V} \zeta_v^{\alpha_v}[\Gamma,\gamma]\Big) = \begin{cases} [\Gamma,\gamma] & \text{if } \ \alpha_v \equiv g_v - 1 (\text{mod } 2) \text{ for all}\, v \in V \\ & \text{and } P \text{ appears at most once in}\, \gamma\,, \\ \Big(\int_{\beta(w)} P\Big) \cdot [\Gamma,\gamma] & \text{if } \ \alpha_v \equiv g_v - 1 (\text{mod } 2) \text{ for all}\, v \in V \\ & \text{except at exactly one vertex } w \text{ and} \\ & P \text{ does not appear in}\, \gamma\,, \\ 0 & \text{otherwise.} \end{cases}$$

The formula for $\mathcal{P}^k_{g,M,\beta}(X)$ is given by a sum over graphs in $\mathsf{G}_{g,n,\beta}(X)$. The contribution of $\Gamma \in \mathsf{G}_{g,n,\beta}(X)$ is a product of vertex, leg, and edge factors:

- For $v \in V$, let $\kappa_v = \kappa(T - T\, \mathsf{B}_0(-\zeta_v T))$.
- For $l \in L$, let $\eta_l = \mathsf{B}_{m_l}(-\zeta_{v(l)}\psi_l)$, where $v(l) \in V$ is the vertex to which the leg is attached.
- For $e \in E$, *let*

  $$\Delta_e = \frac{\zeta' + \zeta'' - \mathsf{B}_0(-\zeta'\psi')\zeta''\mathsf{B}_1(-\zeta''\psi'') - \zeta'\mathsf{B}_1(-\zeta'\psi')\mathsf{B}_0(-\zeta''\psi'')}{\psi' + \psi''}$$

  where ζ', ζ'' are the ζ-variables assigned to two vertices adjacent to the edge e and ψ', ψ'' are the ψ-classes corresponding to two half-edges $e = (h', h'')$.

The numerator of Δ_e is divisible by the denominator due to the identity

$$\mathsf{B}_0(T)\mathsf{B}_1(-T) + \mathsf{B}_0(-T)\mathsf{B}_1(T) = 2\,.$$

This equality extends a similar identity first discovered in [28]. We note that the edge contribution Δ_e is symmetric in the half-edges.

Definition 2. *For $M = (m_1, \ldots, m_n) \in \{0,1\}^n$, let $\mathcal{P}^k_{g,M,\beta}(X) \in \mathcal{S}_{g,n,\beta}(X)$ be the degree k component of a strata algebra class*

$$\mathsf{K}\Big(\sum_{\Gamma \in \mathsf{G}_{g,n,\beta}(X)} \frac{1}{|\mathrm{Aut}(\Gamma)|} \frac{1}{2^{h^1(\Gamma)}} \Big[\Gamma, \prod \kappa_v \prod \eta_l \prod \Delta_e\Big]\Big) \in \mathcal{S}_{g,n,\beta}(X)$$

where the products are taken over all vertices, all legs, and all edges of the graph Γ.

Definition 3. *We denote by* $\widetilde{\mathsf{P}}(X)$ *the set of classes* $\mathcal{P}^k_{g,M,\beta}(X)$ *for all*

$$k > \frac{(g-1)+\sum_i^n m_i}{3}\,. \tag{5}$$

We present the main result of this paper.

Theorem 4. *For a smooth projective variety* X *and a line bundle* S *on* X*, each element* $\mathcal{P}^k_{g,M,\beta}(X)$ *in* $\widetilde{\mathsf{P}}(X)$ *lies in the kernel of the homomorphism*

$$\mathsf{q} : \mathcal{S}^k_{g,n,\beta}(X) \to R^k(\overline{\mathcal{M}}_{g,n}(X,\beta))\,.$$

1.5 Plan of the paper

Our proof relies on the Givental's formalism of torus localization developed in [11, 14]. The main difference from previous works is that we consider the Gromov–Witten theory of the total space relative to the base. After reviewing the genus 0 computations in Section 2 and 3, the higher genus Gromov–Witten theory of a $\mathbb{P}^1$-bundle over X is presented in Section 4. Then we adopt Janda [17] to get the result. Our main theorem on tautological relations is proven in Section 5.

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2 Localization graphs

2.1 Equivariant projective bundle

Let X be a smooth projective variety with a line bundle S. Consider the projective bundle

$$\pi : E = \mathbb{P}(\mathcal{O}_X \oplus S) \to X \tag{6}$$

over X. The two dimensional torus $\mathsf{T} = (\mathbb{C}^*)^2$ acts on the fiber of (6) with weights $-\lambda_0$ and $-\lambda_1$. The T-equivariant Chow ring of E is

$$A^*_{\mathsf{T}}(E) \cong A^*(X)[H,\lambda_0,\lambda_1]/\langle (H-\lambda_0)(H-\lambda_1-P)\rangle$$

where H is the hyperplane class of π and $P = c_1(S)$ is the first Chern class of S. Let $\mathsf{f} \in H_2(E, \mathbb{Z})$ be the fiber class. We choose a (non-canonical) splitting

$$H_2(E, \mathbb{Z}) \cong \mathbb{Z}\langle \mathsf{f} \rangle \oplus H_2(X, \mathbb{Z})$$

where classes in $H_2(X, \mathbb{Z})$ maps to classes in $H_2(E, \mathbb{Z})$ under the push-forward along the infinity section $X \hookrightarrow E = \mathbb{P}(\mathcal{O}_X \oplus S)$.

The zero and the infinity section of π corresponds to two fixed loci X_0 and X_1 of the action T on E. For each T-fixed locus, let $e_i = e(N_{X_i/E})$ be the equivariant Euler class of the normal bundle of X_i in E. Consider an idempotent basis

$$\phi_0 = \frac{H - \lambda_1 - P}{e_0}, \ \phi_1 = \frac{H - \lambda_0}{e_1} \in A^*_{\mathsf{T}}(E) \otimes \mathbb{C}(\lambda_0, \lambda_1) \tag{7}$$

and we use $\phi^i = e_i \phi_i$ as the basis of $A^*_{\mathsf{T}}(E)$.

Following the convention in [22], we will use the following specialization of equivariant parameters

$$\lambda_0 = 1\,, \quad \lambda_1 = -1$$

in Section 4.2 and Section 5.

2.2 Torus fixed loci

For an effecitive curve class $\beta \in H_2(E, \mathbb{Z})$, let $\overline{\mathcal{M}}_{g,n}(E, \beta)$ be the moduli space of stable maps from prestable curves of genus g with n marked points in class β. The topological data g, n, β should lie in the stable range

$$2g - 2 + n > 0 \ \text{ if } \beta = 0\,. \tag{8}$$

For our purpose we take $\beta = d\mathsf{f} + \mathsf{s}$.

Consider the torus action T on E in Section 2.1. This torus action induces an action of T on $\overline{\mathcal{M}}_{g,n}(E, \beta)$. We organize T-fixed loci of $\overline{\mathcal{M}}_{g,n}(E, d\mathsf{f} + \mathsf{s})$ organized according to decorated graphs [11]. A *decorated graph* $\Gamma \in \mathsf{G}_{g,n,\mathsf{s}}(E)$ consists of the data $(\mathsf{V}, \mathsf{E}, \mathsf{v}, \mathsf{s}, \mathsf{g}, \mathsf{p})$ satisfying the following properties:

(i) V is a vertex set,

(ii) E is the edge set (including possible self-edges),

(iii) $\mathsf{v} : \mathsf{L} = \{1, 2, \ldots, n\} \to \mathsf{V}$ is the marking assignment,

(iv) $\mathsf{s} : \mathsf{V} \to H_2(X, \mathbb{Z})$ is a degree assignment satisfying

$$\widetilde{\mathsf{s}} = \sum_{v \in \mathsf{V}} \mathsf{s}(v)\,,$$

(v) $\mathsf{g} : \mathsf{V} \to \mathbb{Z}_{\geq 0}$ is a genus assignment satisfying

$$g = \sum_{v \in \mathsf{V}} \mathsf{g}(v) + h^1(\Gamma)$$

and for which $(\mathsf{V}, \mathsf{E}, \mathsf{v}, \mathsf{s}, \mathsf{g}, \mathsf{p})$ satisfies the condition (8),

(vi) $\mathsf{p} : \mathsf{V} \to \{0, 1\}$ is an assignment of a T-fixed locus $X_{\mathsf{p}(v)}$ to each vertex $v \in \mathsf{V}$.

The markings $\mathsf{L} = \{1, 2, \ldots, n\}$ are often called *legs*.

2.3 Virtual localization

Consider T-equivariant the morphism

$$\pi : \overline{\mathcal{M}}_{g,n}(E, d\mathsf{f} + \mathsf{s}) \to \overline{\mathcal{M}}_{g,n}(X, \mathsf{s}) \tag{9}$$

induced by (6). The moduli space of stable maps to E carries a virtual fundamental class [5]. We use virtual localization formula to compute the equivariant pushforward of the virtual fundamental class along (9).

The T-invariant locus of $\overline{\mathcal{M}}_{g,n}(E, d\mathsf{f} + \mathsf{s})$ whose associated graph is $\Gamma \in \mathsf{G}_{g,n,\mathsf{s}}(E)$ has the following description.

(a) For each $v \in \mathsf{V}$, we associate $\overline{\mathcal{M}}_{\mathsf{g}(v),\mathsf{n}(v)}(X_{\mathsf{p}(v)}, \mathsf{d}(v))$.

(b) There is a bijective correspondence between the connected components of $C \setminus D$ and the set of edges and legs of Γ respecting vertex incidence where C is domain curve and D is union of all subcurves of C which appear in (a).

For $1 \leq i \leq n$, consider the evaluation morphism

$$\mathrm{ev}_i : \overline{\mathcal{M}}_{g,n}(E, d\mathsf{f} + \mathsf{s}) \to E \tag{10}$$

which is T-equivariant.

Proposition 5. *For $\gamma_1, \cdots, \gamma_n \in A^*_{\mathsf{T}}(E)$, we have*

$$\pi_*\Big(\prod_{i=1}^{n} \mathrm{ev}_i^* \gamma_i \cap [\overline{\mathcal{M}}_{g,n}(E, d\mathsf{f} + \mathsf{s})]^{\mathrm{vir}}\Big) \in R^*(\overline{\mathcal{M}}_{g,n}(X, \mathsf{s})) \otimes \mathbb{C}(\lambda_0, \lambda_1)\,.$$

Proof. By the virtual localization formula [16], each T-fixed locus corresponds carries the virtual fundamental class which matches with (1). Moreover the inverse of equivariant virtual normal bundle contributes to tautological classes on each leg, edge and vertex. Therefore the equivariant pushforward of the virtual fundamental class is tautological. □

In the following, we consider the cycle valued generating series

$$\sum_{d\geq 0} \pi_*\Big([\overline{\mathcal{M}}_{g,n}(E, d\mathsf{f}+\mathsf{s})]^{\mathrm{vir}}\Big) q^d \in R^*(\overline{\mathcal{M}}_{g,n}(X,\mathsf{s})) \otimes \mathbb{C}(\lambda_0,\lambda_1)[[q]]\,,$$

where q denotes a formal variable. By Proposition 5, we get a graph sum formula

$$\sum_{d\geq 0} \pi_*\Big([\overline{\mathcal{M}}_{g,n}(E, d\mathsf{f}+\mathsf{s})]^{\mathrm{vir}}\Big) q^d = \sum_{\Gamma \in \mathsf{G}_{g,n,\mathsf{s}}(E)} \mathrm{Cont}_\Gamma$$

where each Cont_Γ is tautological. While $\mathsf{G}_{g,n,\mathsf{s}}(E)$ is a finite set, each contribution Cont_Γ is a series in q obtained from an infinite sum over all edge contributions (b).

3 Local contributions

3.1 First correlators

The virtual localization contribution for stable maps to E can be encoded into certain set of generating series. Let $K = \mathbb{C}(\lambda_0,\lambda_1)$. For $\gamma_i \in A^*_{\mathsf{T}}(E)$, $i=1,\ldots,n$, consider the cycle-valued series

$$\begin{aligned}
\langle \gamma_1\psi^{a_1},\ldots,\gamma_n\psi^{a_n}\rangle_{g,n,d,\mathsf{s}} &= \pi_*\Big(\prod_{i=1}^{n} \mathrm{ev}_i^*(\gamma_i)\psi_i^{a_i} \cap [\overline{\mathcal{M}}_{g,n}(E,d\mathsf{f}+\mathsf{s})]^{\mathrm{vir}}\Big)\,,\\
\langle\langle \gamma_1\psi^{a_1},\ldots,\gamma_n\psi^{a_n}\rangle\rangle_{g,n,\mathsf{s}} &= \sum_{d\geq 0} \langle \gamma_1\psi^{a_1},\ldots,\gamma_n\psi^{a_n}\rangle_{g,n,d,\mathsf{s}}\, q^d\,,
\end{aligned}$$

in $R^*(\overline{\mathcal{M}}_{g,n}(X,\mathsf{s})) \otimes_{\mathbb{C}} K[[q]]$. For unstable ranges, we interpret $\overline{\mathcal{M}}_{0,1}(X,0)$ and $\overline{\mathcal{M}}_{0,2}(X,0)$ as X.

The generating series of genus zero contributions for fiber curve classes can be obtained by the $\mathbb{S}$-operator for the Gromov-Witten theory of equivariant $\mathbb{P}^1$ [14, 21, 20]. This observation will be crucial to compute contracted terms under the pushforward (9). We follow convention in [22, Section 3].

We consider moduli space of stable maps with curve class equal to a multiple of the fiber class. For curve classes $d\mathsf{f} \in H_2(E,\mathbb{Z})$, there exists a morphism

$$\pi : \overline{\mathcal{M}}_{g,n}(E, d\mathsf{f}) \to X\,, \tag{11}$$

given by evaluating general point of the domain curve and projecting along π. A fiber of π is isomorphic to $\overline{\mathcal{M}}_{g,n}(\mathbb{P}^1,h)$. The evaluation morphisms (10) factors through projections π. We consider the $A^*(X)\otimes K$-valued power series

$$\mathbb{S}_i(\gamma) := e_i\langle\langle \frac{\phi_i}{z-\psi}, \gamma\rangle\rangle_{0,2,0} \;\text{ and }\; \mathbb{V}_{ij} := \langle\langle \frac{\phi_i}{x-\psi}, \frac{\phi_j}{y-\psi}\rangle\rangle_{0,2,0}$$

where the unstable degree 0 terms are included by hand in the above equations. The unstable degree 0 term is $\gamma|_{X_i}$ for $\mathbb{S}_i(\gamma)$ and $\frac{\delta_{ij}}{e_i(x+y)}$ for $\mathbb{V}_{ij}$. These generating series take value in the subring generated by P and 1 by the virtual localization formula. We denote $\mathbb{S}(\gamma) = \sum_{i=0}^{1} \phi^i \cdot \mathbb{S}_i(\gamma)$.

We compute the X-valued $\mathbb{S}$-operator via the $\mathbb{S}$-operator for $X = \operatorname{Spec}\mathbb{C}$. To prevent confusion we denote T' by the 2-dimensional torus acting on $\mathbb{P}^1$ with weights s_0, s_1. The associated $\mathbb{S}$ operator is denoted by $\mathbb{S}^\bullet$. Consider a ring homomorphism

$$\epsilon : \mathbb{C}(s_0, s_1) \to A^*(X) \otimes K\,, (s_0, s_1) \mapsto (\lambda_0, \lambda_1 + P) \tag{12}$$

where we expand the rational function in λ_0, λ_1 in λ_1. This is well-defined since P is a nilpotent element in $A^*(X) \otimes K$.

Proposition 6. *Let $\mathbb{S}$ be the X-valued $\mathbb{S}$-operator for the equivariant $\mathbb{P}^1$ bundle E and $\mathbb{S}^\bullet$ be the $\mathbb{S}$-operator for the equivariant $\mathbb{P}^1$. Under the ring homomorphism (12), we have*

$$\mathbb{S}^\bullet \otimes_{\mathbb{C}(s_0,s_1),\epsilon} (A^*(X) \otimes K) = \mathbb{S}\,.$$

Proof. For genus 0, the morphism (11) is a smooth fibration and the smooth pullback of the fundamental class of X is the virtual fundamental class of $\overline{\mathcal{M}}_{0,2}(E, d\mathsf{f})$. We compare the localization contribution of $\overline{\mathcal{M}}_{0,2}(\mathbb{P}^1, d)$ and $\overline{\mathcal{M}}_{0,2}(E, d\mathsf{f})$. The connected components of T-fixed loci match and the equivariant normal bundle contribution coincides under the change of variable. Therefore we get the result. $\square$

Corollary 7. *The series $\mathbb{S}_i$ and $\mathbb{V}_{ij}$ satisfy the basic relation*

$$e_i \mathbb{V}_{ij}(x, y) e_j = \frac{\sum_{k=0}^{1} \mathbb{S}_i(\phi_k)|_{z=x} \mathbb{S}_j(\phi^k)|_{z=y}}{x + y}\,.$$

Proof. When $X = \operatorname{Spec}\mathbb{C}$, this relation follows from the string equation and the WDVV equation [13]. For general X, this follows from Proposition 6. $\square$

3.2 Asymptotic expansions

In [15] the asymptotic expansion of the $\mathbb{S}$-operator for equivariant $\mathbb{P}^N$ is obtained by Givental. For equivariant $\mathbb{P}^1$ bundles, Proposition 6 gives a parallel result.

For equivariant $\mathbb{P}^1$, geometric constructions of the $\mathbb{I}$-function can be obtained by the quasi-map theory [7, 6, 8]. Consider a function

$$\mathbb{I}^\bullet = \sum_{d \geq 0} \frac{q^d}{\prod_{k=1}^{d} (H - s_0 + kz)(H - s_1 + kz)} \tag{13}$$

valued in $A^*_{\mathsf{T}'}(\mathbb{P}^1)[[q, z^{-1}]]$. Let $\mathbb{S}^\bullet$ be the $\mathbb{S}$-operator for equivariant $\mathbb{P}^1$. Via the Birkhoff factorization procedure [7] (see also [20, Section 2.3]), $\mathbb{S}^\bullet$ can be explicitly evaluated by the $\mathbb{I}$-function

$$\begin{aligned} \mathbb{S}^{\bullet}(1) &= \mathbb{I}\,, \\ \mathbb{S}^{\bullet}(H) &= \mathsf{M}\,\mathbb{S}^{\bullet}(1)\,, \\ \mathsf{M}\,\mathbb{S}^{\bullet}(H) &= (1+q)\,\mathbb{S}^{\bullet}(1)\,. \end{aligned} \tag{14}$$

Here M is the operator $H + q\frac{\partial}{\partial q}$.

For an equivariant $\mathbb{P}^1$ bundle E over X, the X-valued $\mathbb{I}$-function is defined by the extension of scalars along (12)

$$\mathbb{I} := \mathbb{I}^{\bullet} \otimes_{\epsilon} (A^*(X) \otimes K) = \sum_{d=0}^{\infty} \frac{q^d}{\prod_{k=1}^{d}(H-\lambda_0+kz)(H-\lambda_1-P+kz)}\,. \tag{15}$$

Lemma 8. *The X-valued $\mathbb{S}$-operator satisfies the following equalities*

$$\begin{aligned} \mathbb{S}(1) &= \mathbb{I}\,, \\ \mathbb{S}(H) &= \mathsf{M}\,\mathbb{S}(1)\,, \\ \mathsf{M}\,\mathbb{S}(H) &= (1+q-P)\,\mathbb{S}(1) + P\,\mathbb{S}(H)\,. \end{aligned}$$

Proof. By Proposition 6 the X-valued $\mathbb{S}$-operator can be obtained by $\mathbb{S}^{\bullet}$ using (12). Since we have

$$\mathbb{I} = 1 + \frac{q}{z^2} + \frac{qP - 2qH}{z^3} + \mathcal{O}(z^{-4})$$

from the Birkhoff factorization, the equalities (14) give the result. □

The functions $\mathbb{S}_i(1)$ and $\mathbb{S}_i(H)$ admit the following asymptotic forms

$$\begin{aligned} \mathbb{S}_i(1) &= e^{\frac{\sum_{l\geq 0} u_{il}P^l}{z}}\Big(\mathbb{A}_{i0}(z) + \mathbb{A}_{i1}(z)P + \mathbb{A}_{i2}(z)P^2 + \dots\Big)\,, \\ \mathbb{S}_i(H) &= e^{\frac{\sum_{l\geq 0} u_{il}P^l}{z}}\Big(\mathbb{C}_{i0}(z) + \mathbb{C}_{i1}(z)P + \mathbb{C}_{i2}(z)P^2 + \dots\Big)\,, \end{aligned} \tag{16}$$

with series

$$u_{i0} = \int_0^q \frac{L_i(x) - (-1)^i}{x}dx \in \mathbb{C}[[q]]\,, \quad u_{i1} = (-1)^{i+1}\mathrm{Log}\left(\frac{2}{1+(-1)^i L_i}\right) \in \mathbb{C}[[q]]\,,$$

$$\mathbb{A}_{il}(z) := \sum_{k=0}^{\infty} a_{kil}(q) z^k \in \mathbb{C}[[q]][z]\,, \qquad \mathbb{C}_{il}(z) := \sum_{k=0}^{\infty} c_{kil}(q) z^k \in \mathbb{C}[[q]][z]\,,$$

where $L_i(q) := (-1)^i(1+q)^{\frac{1}{2}}$. The function $\mathbb{V}_{ij}$ admits the following asymptotic form

$$\mathbb{V}_{ij} = e^{\sum_{l\geq 0} u_{il}\cdot P^l/x + \sum_{l\geq 0} u_{jl}\cdot P^l/y}\Big(\mathbb{E}_{ij}(x,y)\Big)\,,$$

with series

$$\mathbb{E}_{ij}(x,y) := \sum_{k,l\geq 0} e_{klijh}(q)P^h x^k y^l \ \in \mathbb{C}[[q]][P][x,y].$$

Proposition 9. *For all $k, l, i, j, h \geq 0$, we have*

$$a_{kil}\,, c_{kil}\,, e_{klijh} \ \in \mathbb{C}[L_0^{\pm 1/2}]\,.$$

Proof. By Lemma 8 the structural constants has expansion in terms of the $\mathbb{I}$ function (15). Since (15) is obtained by $\mathbb{I}^\bullet$, it is enough to check whether $\mathbb{I}^\bullet$ has a Laurent series expansion in $L_0^{1/2}$. This follows from the recursive structure of the Picard–Fuchs equation for studied in [29, Theorem 4] (see also [23, Appendix A.3]). □

The following lemma will be used in Section 5.2 for the analysis of the lowest order term in the localization formula.

Lemma 10. *For $l \geq 2$, we have*

$$\begin{aligned}
LO_{L_0}(u_{il}) &= 3 - 2l\,,\\
LO_{L_0}(a_{kil}) &= -\frac{1}{2} - 3k - 2l\,,\\
LO_{L_0}(c_{kil}) &= \frac{1}{2} - 3k - 2l\,,
\end{aligned}$$

where $LO_{L_0}(\cdot)$ denotes the lowest order term with respect to L_0.

Proof. By Proposition 9 the lowest order term with respect to L_0 is well-defined. The assertion follows from the asymptotic forms (16) applied to Lemma 8. □

4 Reconstruction theory

4.1 Cycle-valued localization series

We review the Givental's quantization [14, 21, 22] which expresses genus g Gromov-Witten classes in terms of genus 0 data and its generalization in Coates-Givental-Tseng [11]. Cycle-valued formula of [11, Theorem 3.1] has the following graph sum expansion. We refer Section 1.3 for our notations for decorated strata classes.

Proposition 11. *For $m_1, \ldots, m_n \in \{0,1\}$, the genus g correlator has the form*

$$\langle\langle H^{m_1}, \ldots, H^{m_n}\rangle\rangle_{g,n,\mathsf{s}} = \sum_{\Gamma \in \mathsf{G}_{g,n,\mathsf{s}}(E)} \frac{1}{\mathrm{Aut}(\Gamma)} \xi_{\Gamma *}\Big[\prod \kappa_v \prod \eta_l \prod \Delta_e\Big]$$

in $R^*(\overline{\mathcal{M}}_{g,n}(X,\mathsf{s}))$ where

- *for* $v \in \mathsf{V}$, *let*

$$\kappa_v = \mathrm{Obs}_v \cdot \mathrm{Exp}\Big(\int_{\mathsf{s}(v)} P \cdot u_{\mathsf{p}(v)1}\Big) q^{\mathsf{p}(v)\int_{\mathsf{s}(v)} P} (\frac{1}{a_{0\mathsf{p}(v)0}})^{2\mathsf{g}(v)-2+\mathsf{n}(v)}$$
$$\cdot \kappa \left(T - T\left(\frac{\mathbb{A}_{\mathsf{p}(v)0}(-T)}{a_{0\mathsf{p}(v)0}} + \tilde{P}\frac{\mathbb{A}_{\mathsf{p}(v)1}(-T)}{a_{0\mathsf{p}(v)0}} + \tilde{P}^2 \frac{\mathbb{A}_{\mathsf{p}(v)2}(-T)}{a_{0\mathsf{p}(v)0}} + \dots \right)\right),$$

- *for* $l \in \mathsf{L}$, *let*

$$\eta_l = \begin{cases} \sum_{k\geq 0} \mathbb{A}_{\mathsf{p}(\mathsf{v}(l))k}(-\psi_l) \cdot P^k & \text{if } m_l = 0, \\ \sum_{k\geq 0} \mathbb{C}_{\mathsf{p}(\mathsf{v}(l))k}(-\psi_l) \cdot P^k & \text{if } m_l = 1, \end{cases}$$

  where $\mathsf{v}(l) \in \mathsf{V}$ *is the vertex to which the leg is attached,*

- *for* $e \in \mathsf{E}$, *let*

$$\Delta_e = \mathbb{E}_{\mathsf{p}(v_1),\mathsf{p}(v_2)}(-\psi', -\psi'')$$

  where v_1, v_2 *are two vertices connected by* e *and* ψ', ψ'' *are two* ψ*-classes at* $e = (h', h'')$.

Here the term Obs_v at the vertex $v \in \mathsf{V}$ is the inverse of the T-equivariant Euler class of the virtual vector bundle

$$N^{\mathsf{p}(v)}_{\mathsf{g}(v),\mathsf{n}(v),\mathsf{s}(v)}$$

on $\overline{\mathcal{M}}_{\mathsf{g}(v),\mathsf{n}(v)}(X_{\mathsf{p}(v)}, \mathsf{s}(v))$. The fiber of $N^{\mathsf{p}(v)}_{\mathsf{g}(v),\mathsf{n}(v),\mathsf{s}(v)}$ at the stable map $f : C \to X_{\mathsf{p}(v)}$ is given by

$$H^0(C, f^*N^{\mathsf{p}(v)}) \ominus H^1(C, f^*N^{\mathsf{p}(v)}),$$

where $N^{\mathsf{p}(v)}$ is the normal bundle of $X_{\mathsf{p}(v)}$ in E.

4.2 Vertex terms

The vertex term Obs_v in Proposition 11 can be computed by the Grothendieck–Riemann–Roch(GRR) theorem. In [10] the effect of GRR to Givental's quantization formalism is given by the twisted dilaton shift [10, eq.(6)]. Applying this result, the factor Obs_v at each vertex $v \in \mathsf{V}$ can be removed via modifying the series $\mathbb{S}_i$.

Define new series by

$$\sum_{l\geq 0} \widetilde{\mathbb{A}}_{il}(z) \cdot P^l :=$$
$$\sqrt{\frac{e_i}{(-1)^i 2}} \cdot \mathrm{Exp}\left(\sum_{k\geq 1} \frac{B_{2k}}{2k(2k-1)} \Big(\frac{(-1)^{i+1} z}{2-P}\Big)^{2k-1}\right)$$
$$\cdot \left[\mathrm{Exp}\Big(\frac{\sum_{l\geq 2} u_{il} P^l}{z}\Big)\Big(\sum_{l\geq 0} \mathbb{A}_{il}(z) \cdot P^l\Big)\right]_+$$

and

$$\sum_{l\geq 0}\widetilde{\mathbb{C}}_{il}(z)\cdot P^l :=$$

$$\sqrt{\frac{e_i}{(-1)^i 2}}\cdot \mathrm{Exp}\left(\sum_{k\geq 1}\frac{B_{2k}}{2k(2k-1)}\Big(\frac{(-1)^{i+1}z}{2-P}\Big)^{2k-1}\right)$$
$$\cdot\left[\mathrm{Exp}\Big(\frac{\sum_{l\geq 2}u_{il}P^l}{z}\Big)\Big(\sum_{l\geq 0}\mathbb{C}_{il}(z)\cdot P^l\Big)\right]_+$$

for $i=0,1$. Here B_{2k} is the $2k$-th Bernoulli number

$$\frac{1}{1-e^{-x}}=\frac{x}{2}+\sum_{k\geq 1}B_{2k}\frac{x^{2m}}{(2m)!}$$

and the notation $[\cdot]_+$ means the non-negative term with respect to z. $[\cdot]_+$ is well-defined because P is a nilpotent element in $A^*(X)$. We write

$$\widetilde{\mathbb{A}}_{il}(z):=\sum_{k=0}^{\infty}\widetilde{a}_{kil}(q)z^k\in\mathbb{C}[[q]][z]\,, \tag{17}$$
$$\widetilde{\mathbb{C}}_{il}(z):=\sum_{k=0}^{\infty}\widetilde{c}_{kil}(q)z^k\in\mathbb{C}[[q]][z]\,.$$

Now Proposition 11 can be written as the following expression.

Proposition 12. *For $m_1,\ldots,m_n\in\{0,1\}$, we have*

$$\langle\langle H^{m_1},\ldots,H^{m_n}\rangle\rangle_{g,n,\mathsf{s}}=\sum_{\Gamma\in\mathsf{G}_{g,n,\mathsf{s}}(E)}\frac{1}{|\mathrm{Aut}(\Gamma)|}\xi_{\Gamma*}\Big[\prod\kappa_v\prod\eta_l\prod\Delta_e\Big] \tag{18}$$

in $R^*(\overline{\mathcal{M}}_{g,n}(X,\mathsf{s}))[[q]]$ where

- *for $v\in\mathsf{V}$, let* $\kappa_v=\mathrm{Vert}_v\cdot\kappa\Big(T-T\Big(\frac{\widetilde{\mathbb{A}}_{\mathsf{p}(v)0}(-T)}{a_{0\mathsf{p}(v)0}}+\tilde{P}\frac{\widetilde{\mathbb{A}}_{\mathsf{p}(v)1}(-T)}{a_{0\mathsf{p}(v)0}}+\tilde{P}^2\frac{\widetilde{\mathbb{A}}_{\mathsf{p}(v)2}(-T)}{a_{0\mathsf{p}(v)0}}+\ldots\Big)\Big)$ *with*

$$\mathrm{Vert}_v=\Big[\mathrm{Exp}\Big(u_{\mathsf{p}(v)1}+(-1)^{\mathsf{p}(v)}\mathrm{Log}\Big((-1)^{\mathsf{p}(v)}2\Big)\Big)\Big]^{\int_{\mathsf{s}(v)}P}$$
$$\cdot\, q^{\mathsf{p}(v)\int_{\mathsf{s}(v)}P}\Big(\frac{1}{a_{0\mathsf{p}(v)0}}\Big)^{2\mathsf{g}(v)-2+\mathsf{n}(v)}\Big(\frac{1}{(-1)^{\mathsf{p}(v)}2}\Big)^{\mathsf{g}(v)-1},$$

- *for $l\in\mathsf{L}$, let*

$$\eta_l=\begin{cases}\sum_{k\geq 0}\widetilde{\mathbb{A}}_{\mathsf{p}(\mathsf{v}(l))k}(-\psi_l)\cdot P^k & \text{if } m_l=0,\\ \sum_{k\geq 0}\widetilde{\mathbb{C}}_{\mathsf{p}(\mathsf{v}(l))k}(-\psi_l)\cdot P^k & \text{if } m_l=1\,,\end{cases}$$

where $\mathsf{v}(l)\in\mathsf{V}$ is the vertex to which the leg is attached,

- *for $e \in \mathsf{E}$, let*

$$\Delta_e = \frac{-1}{\psi' + \psi''}\Big((\sum_{l\geq 0} \widetilde{\mathbb{A}}_{\mathsf{p}(v_1)l}(-\psi') \cdot P_e^l)(\sum_{l\geq 0} \widetilde{\mathbb{C}}_{\mathsf{p}(v_2)l}(-\psi'') \cdot P_e^l) + (\sum_{l\geq 0} \widetilde{\mathbb{C}}_{\mathsf{p}(v_1)l}(-\psi') \cdot P_e^l)(\sum_{l\geq 0} \widetilde{\mathbb{A}}_{\mathsf{p}(v_2)l}(-\psi'') \cdot P_e^l) + P(\sum_{l\geq 0} \widetilde{\mathbb{A}}_{\mathsf{p}(v_1)l}(-\psi') \cdot P_e^l)(\sum_{l\geq 0} \widetilde{\mathbb{A}}_{\mathsf{p}(v_2)l}(-\psi'') \cdot P_e^l)\Big)$$

where ψ', ψ'' are the ψ-classes corresponding to the half-edges and $P_e := \mathrm{ev}_e^* P$.

Proof. We apply [10, Theorem 1] at each vertex $v \in \mathsf{V}$. Then the proof immediately follows from the argument in [14, Section 2.3]. We use Corollary 7 for the explicit form of Δ_e in terms of series (17). □

5 X-valued tautological relations

5.1 Genus zero mirror symmetry for the equivariant projective line

We review Givental's the genus 0 mirror theorem for equivariant $\mathbb{P}^1$ [12]. We follow exposition in [17]. The power series expansion of the $\mathbb{I}$-function (13) for the equivariant $\mathbb{P}^1$ can be written in terms of the oscillating integrals on the mirror manifold.

The mirror manifold of equivariant $\mathbb{P}^1$ is defined by

$$\{(T_0, T_0) \subset \mathbb{C}^2 | e^{T_0+T_1} = q\},$$

together with the superpotential

$$\mathsf{F}(T_0, T_1) = e^{T_0} + e^{T_1} + s_0 T_0 + s_1 T_1 .$$

The oscillating integrals attached to this mirror manifold is given by

$$\mathcal{I}_i = e^{-\mathrm{Log}(q)\, s_i/z}(-2\pi z)^{-\frac{1}{2}} \int_{\Gamma_i \subset \{\sum T_j = \mathrm{Log}\, q\}} e^{\mathsf{F}(T_0,T_1)/z}\omega, \quad i = 1, 2$$

where the integrals are along 1-cycles Γ_i through a specific critical point of the superpotential F The 2-form ω is the standard 2-form $dT_0 \wedge dT_1$ on $\mathbb{C}^2$ restricted to the cycle Γ_i.

There are two critical points of F at which integrals $\mathcal{I}_i$ admit stationary phase expansions. Let Z_i be two solutions of the equation

$$(Z - s_0)(Z - s_1) = q$$

with limit s_i as $q \to 0$. If we choose the critical point $T_j = \mathrm{Log}(Z_i - s_j)$ of F, the factor

$$e^{\frac{u_i}{z}} := \mathrm{Exp}\left(\Big(\sum_{j=0}^{1}(Z_i - s_j + s_j\mathrm{Log}(Z_i - s_j)) - s_i\mathrm{Log}\, q\Big)/z\right)$$

is well defined in the limit as $q \to 0$. Via the shift of the integral to the critical point and re-scaling of coordinates by $\sqrt{z}$, we have

$$\mathcal{I}_i = e^{\frac{u_i}{z}} \int \mathrm{Exp}\Big(- \sum_j (Z_j - s_j) \sum_{k=3}^{\infty} \frac{T_j^k (-z)^{(k-2)/2}}{k!}\Big) d\mu_i \,, \tag{19}$$

where $d\mu_i$ is the Gaussian distribution

$$\frac{1}{\sqrt{2\pi}} \cdot \mathrm{Exp}\Big(- \sum_j (Z_i - s_j)\frac{T_j^2}{2}\Big)\omega \,.$$

To get an asymptotic expansion, we formally expand the exponential in (19) and integrate over the cycle Γ_i. Let

$$F(x,y) = 1 - (\frac{5}{24} - \frac{1}{8}y)\frac{x}{2} + (\frac{385}{1152} - \frac{77}{192}y + \frac{9}{128}y^2)\big(\frac{x}{2}\big)^2 + \cdots \in \mathbb{Q}[y][[x]]$$

be the formal power series appears in the stationary phase asymptotics for certain oscillating integral. See [17, Section 3.3].

Lemma 13. *For the choice* $(s_0, s_1) = (1, -1)$*, we have*

$$\mathcal{I}_i = e^{\frac{u_i}{z}} \cdot L_0^{-\frac{1}{2}} F\Big(\frac{z}{2L_0^3}, L_0^2\Big) \,.$$

Proof. The asymptotic expansion of (19) after specializing the equivariant parameters to $(s_0, s_1) = (0, 1)$ is given in [17, Section 3.3]. Computation for $(s_0, s_1) = (0, 1)$ can be done similarly. $\square$

When we expand $F(x,y)$ in the y variable

$$F(x,y) = F_0(x) + F_1(x)y + F_2(x)y^2 + \cdots \,,$$

the constant term has an explicit formula

$$F_0(x) = F(x,0) = \sum_{i \geq 0} \frac{(6i)!}{(3i)!(2i)!}\Big(\frac{-x}{576}\Big)^i \tag{20}$$

([17, Section 3.3]). The power series $A(z)$ in Section 1.4 coincides with $F(z,0)$.

After specializing equivariant parameters to

$$(s_0, s_1) = (1, -1 - \widetilde{P})$$

where $\widetilde{P}$ is a formal variable with[2] $\widetilde{P}^2 = 0$, we obtain the following results from the genus 0 mirror theorem [12] for equivariant $\mathbb{P}^1$.

Corollary 14. *For $l = 0, 1$, we have the following explicit formula for the lowest order terms of the series $\widetilde{\mathbb{A}}_{il}$ and $\widetilde{\mathbb{C}}_{il}$ with respect to L_0*

$$\sum_{k\geq 0} \mathrm{Low}_{L_0}(\widetilde{a}_{ki0}) z^k = L_0^{-\frac{1}{2}} A\Big(\frac{(-1)^i z}{L_0^3}\Big), \quad \sum_{k\geq 0} \mathrm{Low}_{L_0}(\widetilde{a}_{ki1}) z^k = L_0^{-\frac{5}{2}} B\Big(\frac{(-1)^i z}{L_0^3}\Big),$$
$$\sum_{k\geq 0} \mathrm{Low}_{L_0}(\widetilde{c}_{ki0}) z^k = -\, L_0^{\frac{1}{2}} C\Big(\frac{(-1)^i z}{L_0^3}\Big), \quad \sum_{k\geq 0} \mathrm{Low}_{L_0}(\widetilde{c}_{ki1}) z^k = -\, L_0^{-\frac{3}{2}} D\Big(\frac{(-1)^i z}{L_0^3}\Big),$$

where $\mathrm{Low}_{L_0}(\cdot)$ is the lowest order term with respect to L_0.

Proof. This follows from the explicit presentation of the $\mathbb{I}$-function in Lemma 13. □

5.2 Proof of Theorem 4

The left hand side of (18) is a formal power series in q. We check that it is indeed polynomial in q. For any $m_1, \ldots, m_n \in \{0, 1\}$, the coefficient of q^d in $\langle\langle H^{m_1}, \ldots, H^{m_n}\rangle\rangle_{g,n,\mathsf{s}}$ has degree

$$\begin{aligned}
&\Big((1-g)(\mathrm{dim}X - 3) + \int_{\widetilde{\mathsf{s}}} c_1(X) + n\Big) \\
&- \Big((1-g)(\mathrm{dim}E - 3) + 2d + \int_{\mathsf{s}} \big(c_1(S) + c_1(X)\big) + n\Big) + \sum_{i=1}^n m_i \\
&= g - 1 - 2d - \int_{\mathsf{s}} c_1(S) + \sum_{i=1}^n m_i .
\end{aligned}$$

Since the degree is negative for large enough d, the sum over d in $\langle\langle H^{m_1}, \ldots, H^{m_n}\rangle\rangle_{g,n,\mathsf{s}}$ is finite and defines an element in $A_*^{\mathsf{T}}(\overline{\mathcal{M}}_{g,n}(X, \mathsf{s})) \otimes_{\mathbb{C}} \mathbb{C}[q]$.

On the other hand, structural constants in (17) are Laurent polynomials in $L_0 = (1+q)^{\frac{1}{2}}$ by Proposition 9 and hence the right hand side of (18) is a Laurent polynomial in L_0. By the polynomiality of q this implies that each coefficient of poles of L_0 in the right side of (18) is equal to zero.

[2]We only need to consider the linear term of P in the formula (18).

We concentrate the lowest order pole of L_0 which gives the tautological relations in Theorem 4. By Lemma 10 only the constant and linear terms with respect to P appear in the coefficient of the lowest order term of L_0 in the formula (18). In other words, we may assume $P^2 = 0$ since we only consider coefficient of the lowest order term of L_0. By Corollary 14, the coefficient of the lowest degree term with respect to L_0 can be written in term of the hypergeometric series (20).

The degree condition (5) follows from the analysis on the order of the lowest degree of L_0 in the right side of (18). Since the order of the lowest degree of L_0 is independent of the degree of stable maps, this analysis is parallel to that of [17, Section 3.4].

We explain the parity condition at each vertex. The function K in Definition 2 does the role of picking only non-trivial contributions from the right side of (18). The non-triviality of each contribution is determined by the parity condition which is imposed at each vertex $v \in \mathsf{V}$ by ζ_v. When the degree of the vertex is trivial, this argument is parallel to [27, Section 3.4]. For the vertex with positive degree, we need few more explanations. Since the parity condition is a local property on each vertex, it is enough to consider the case where Γ has a single vertex. From the definition of $\mathbb{S}_i(\gamma)$ and the argument of [17, Section 3.3], we can easily obtain the following results:

(i) $\widetilde{\mathbb{A}}_{00}(z) = \widetilde{\mathbb{A}}_{10}(-z)\,,\ \widetilde{\mathbb{A}}_{01}(z) = \widetilde{\mathbb{A}}_{11}(-z)\,, \widetilde{\mathbb{C}}_{00}(z) = -\widetilde{\mathbb{C}}_{10}(-z)\,,$

(ii) $\frac{\widetilde{a}_{ki0}}{\widetilde{a}_{0i0}}\,,\ \frac{\widetilde{a}_{ki1}}{\widetilde{a}_{0i0}} \in L_0^k\,\mathbb{C}[L_0^{\pm 2}]\,,\ \ \frac{\widetilde{c}_{ki0}}{\widetilde{a}_{0i0}} \in L_0^{k+1}\,\mathbb{C}[L_0^{\pm 2}]\,,$

(iii) $\frac{\widetilde{c}_{k01}}{\widetilde{a}_{000}} + (-1)^j \frac{\widetilde{c}_{k11}}{\widetilde{a}_{010}} \in L_0^j\,\mathbb{C}[L_0^{\pm 2}]$ for $j = 0,\, 1\,.$

For a decorated graph Γ with the single vertex v, the vanishing condition of the contribution of Γ in Proposition 12 is induced by above results. Moreover the following equations

$$\Big[\mathrm{Exp}\Big(u_{01} + \mathrm{Log}\,(2)\Big)\Big]^d + \Big[\mathrm{Exp}\Big(u_{11} - \mathrm{Log}\,(-2)\Big)\Big]^d q^d = 2 + \mathsf{O}(L_0^2)\,,$$
$$\Big[\mathrm{Exp}\Big(u_{01} + \mathrm{Log}\,(2)\Big)\Big]^d - \Big[\mathrm{Exp}\Big(u_{11} - \mathrm{Log}\,(-2)\Big)\Big]^d q^d = 2d \cdot L_0 + \mathsf{O}(L_0^3)\,.$$

coincides with the vanishing condition of the corresponding contribution of Γ in Definition 2 induced by the parity condition via ζ_v.

5.3 More relations.

By the definition of K in (4), the factor P appears at most once at each term in the relation $\mathcal{P}^k_{g,M,\widetilde{\mathsf{s}}}(X)$. This is related to the fact that $\mathcal{P}^k_{g,M,\widetilde{\mathsf{s}}}(X)$ comes from the coefficient of lowest order of L_0 in the sum in Proposition 12. If we consider the coefficient of k-th lowest order of L_0, we get relations such that P appears at most k times. For concise form of these relations, we require a closed formula of the full asymptotic form $F(x, y)$ of the oscillating integral in Lemma 13.

5.4 Examples

We give a few examples of Theorem 4 when X is $\mathbb{P}^1$. For a polynomial $f_i(T, \widetilde{P}) \in \mathbb{Q}[T, \widetilde{P}]$, we use the notation

$$\langle f_1(\psi, P), \ldots, f_m(\psi, P) \rangle = p_{m*}\Big(f_1(\psi_{n+1}, \mathrm{ev}^*_{n+1}(P)), \ldots,\ f_m(\psi_{n+m}, \mathrm{ev}^*_{n+m}(P)) \Big)\,,$$

for each vertex of the graph. Here p_m is the map forgetting the last m marked points (3). A vector $(g, d) \in \mathbb{Z}^2_{\geq 0}$ at each vertex indicates the genus and the degree of the vertex.

Example 15. *Let $g = 0$, $n = 0$, $d = 2$. We get degree one relations in $R^1(\overline{\mathcal{M}}_{0,0}(\mathbb{P}^1, 2))$:*

$$2\langle \psi^2 \rangle + \Big[(0,1) \text{---} (0,1) \Big] = 0,$$

$$4\langle \psi P \rangle + \Big[(0,1) \text{---} (0,1) \Big] = 0\,.$$

Alternately, two relations can be obtained as follows. Consider the moduli space $\mathfrak{M}_{0,0}$ of prestable curves of genus 0 with no markings. The first relation can be obtained by the pull-back of a degree 1 relation

$$2\kappa_1 + [\bullet\!\text{---}\!\bullet] = 0 \ \text{ in } A^1(\mathfrak{M}_{0,0}).$$

in $A^1(\mathfrak{M}_{0,0})$ appears in [4]. The second relation can be obtained from the following genus 0 DR-relation on $R^1(\overline{\mathcal{M}}_{0,1}(\mathbb{P}^1, 2))$

$$4\psi_1 + 4\xi_1 + \Big[\overset{1}{(0,1)} \text{---} (0,1) \Big] = 0.$$

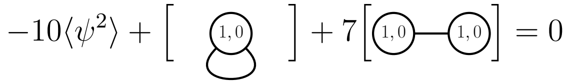
After multiplying the above with ξ_1 and pushforward to $\overline{\mathcal{M}}_{0,0}(\mathbb{P}^1, 2)$, we get the second relation.

We obtain the following relation of degree 2 in $R^2(\overline{\mathcal{M}}_{0,0}(\mathbb{P}^1, 2))$:

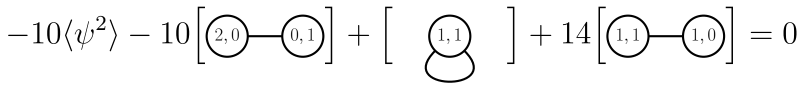
$$\begin{aligned} &-77\langle \psi^3 \rangle - 84\langle \psi^2 P \rangle + 5\langle \psi^2, \psi^2 \rangle + 12\langle \psi^2, \psi P \rangle - 2\Big[\overset{\langle \psi^2 \rangle}{(0,1)} \text{---} (0,1) \Big] \\ &+ 12\Big[\overset{\langle \psi P \rangle}{(0,1)} \text{---} (0,1) \Big] + 7\Big[(0,1) \overset{\psi}{\text{---}} (0,1) \Big] - 36\Big[(0,1) \overset{P}{\text{---}} (0,1) \Big] = 0. \end{aligned}$$

Example 16. *Let $g = 2$, $n = 0$. We obtain the relations of degree 1 in $R^1(\overline{\mathcal{M}}_{2,0}(\mathbb{P}^1, d))$.*

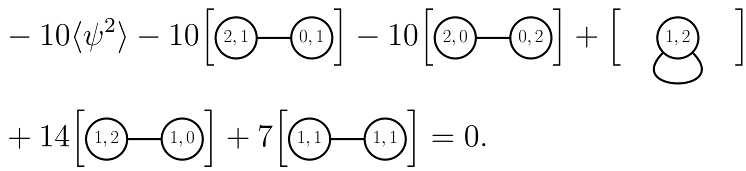
- $d = 0$:



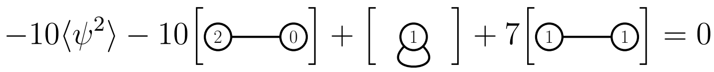
- $d = 1$:



- $d = 2$:



The last two relations coincide with the relations which can be obtained by pull-back of the following relation



in $A^1(\mathfrak{M}_{2,0})$.

5.5 Relation to Pixton's relations

Pixton conjectured a set P which conjecturally generates all tautological relations on $\overline{\mathcal{M}}_{g,n}$. When X is a nonsingular toric variety, the push-forward of the virtual fundamental class via the stabilization morphism

$$p : \overline{\mathcal{M}}_{g,n}(X, \beta) \to \overline{\mathcal{M}}_{g,n}$$

is a tautological class. Therefore it induces a map

$$p_* : \mathcal{S}_{g,n,\beta}(X) \to \mathcal{S}_{g,n}{}^{3}$$

between strata algebras. After pushing forward our relations, we obtain relations

$$p_* \mathcal{P}^k_{g,M,\beta}(X) = 0$$

in $R^*(\overline{\mathcal{M}}_{g,n})$. Whether these relations can be obtained by the Pixton relation is an interesting question.

Question 17. *Are the relations* $p_* \mathcal{P}^k_{g,M,\beta}(X)$ *implied by* P ?

[3] For an arbitrary target, $p_*[\overline{\mathcal{M}}_{g,n}(X, \beta)]^{vir}$ is not a tautological class in general.

6 Further generalizations

6.1 Higher order poles

6.2 Tautological relations on the universal Picard stack

Tautological relations in Theorem 4 does not depend on specific geometry of (X, S). Therefore it is natural to ask whether the relations come from a universal space. A natural candidate is the *universal Picard stack* $\mathfrak{Pic}_{g,n}$ parametrizing line bundles on marked prestable curves. A relevant approach for X-valued double ramification relations [1] is pursued in [2].

We write a possible lift of Theorem 4 to tautological relations on $\mathfrak{Pic}_{g,n}$. For a given pair (X, S), consider the morphism

$$\varphi_S \colon \overline{\mathcal{M}}_{g,n}(X,\beta) \to \mathfrak{Pic}_{g,n}\,, (f\colon C \to X) \mapsto (C, f^*S)\,.$$

Generalizing the definition of strata algebra of $\overline{\mathcal{M}}_{g,n}(X,\beta)$, there exists a natural tautological relation $\mathcal{P}^k_{g,M,d} = 0$[4] in the operational Chow group of $\mathfrak{Pic}_{g,n}$ such that

$$\varphi_S^*\mathcal{P}^k_{g,M,d} \cap [\overline{\mathcal{M}}_{g,n}(X,\beta)]^{vir} = \mathcal{P}^k_{g,M,\beta}(X)$$

and hence it recovers our relations.

Question 18. *Does $\mathcal{P}^k_{g,M,d} = 0$ hold in $A^*(\mathfrak{Pic}_{g,n})$?*

In general, it is hard to check Question 18 from Theorem 4. In [3] we will give an alternative approach for Question 18 by constructing a moduli space of stable Picard quotients over universal Picard scheme.

6.3 Toric bundles

We can consider the equivariant Gromov–Witten theories of a general toric bundles over X where $\mathbb{P}^1$-bundles in our paper are special cases of more general result in [11]. For example, for rank r split vector bundle $L_1 \oplus \cdots \oplus L_r$ on X, our method produces relations with depends on Chern classes of L_i. We speculate that our method can be generalized to toric bundles over X.

6.4 Non-split vector bundle

When the Chow ring of X is generated by divisor classes, our tautological relations give universal relations on the Gromov–Witten theory of X. One can ask a similar question to universal relations involving arbitrary class in $A^*(X)$. In order to get

[4] d is the degree of line bundle $\int_\beta c_1(S)$.

such relations, we should consider arbitrary rank vector bundle V on X because any class in $A^*(X)$ can be represented by Chern classes of some vector bundle on X. However when V is a non-split vector bundle, localization formula used in this paper for the stable maps to $\mathbb{P}(\mathcal{O}_X \oplus V)$ does not give tautological relations on $\overline{\mathcal{M}}_{g,n}(X, \beta)$. It would be an interesting question whether we can extend localization technique for non-split vector bundles.

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Tautological relations on the relative Picard scheme

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Abstract

We prove a system of tautological relations on the relative Picard scheme by introducing certain relative Quot scheme over the relative Picard scheme.

1 Introduction

1.1 Tautological ring of the universal Picard stack

Let $\mathfrak{M}_{g,m}$ be the moduli stack of prestable curves of genus g with m markings over the base field k and let $\pi : \mathfrak{C}_{g,m} \to \mathfrak{M}_{g,m}$ be the universal curve. Over $\mathfrak{M}_{g,m}$ there exists the universal Picard stack $\mathfrak{Pic}_{g,m}$ parametrizing prestable curves together with line bundles. There exists the universal curve, m-sections and the universal line bundle

$$\pi : \mathfrak{C} \to \mathfrak{Pic}_{g,m},\ p_1, \cdots, p_m : \mathfrak{Pic}_{g,m} \to \mathfrak{C},\ \mathcal{L} \to \mathfrak{C}\,. \tag{1.1}$$

Using the universal structure (1.1) one can define tautological classes. In particular the *twisted κ-classes* are define by push forward

$$\kappa_{a,b} := \pi_*(c_1(\omega_\pi(p_1 + \cdots + p_m))^{a+1} c_1(\mathcal{L})^b) \text{ for } a \geq -1, b \geq 0\,.$$

The smallest subring inside the rational Chow ring

$$\mathsf{R}^*(\mathfrak{Pic}_{g,m}) \subset \mathsf{CH}^*(\mathfrak{Pic}_{g,m})_{\mathbb{Q}}$$

generated by tautological classes is called the *tautological ring* of the universal Picard stack. See Section 2 for the precise definition of the tautological ring.

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1.2 Overview

Our goal is to find structure of tautological relations on the universal Picard stack or it proper open substacks. We give several evidences to expect interesting structural results for the tautological ring of the universal Picard stack.

In [5] authors lift the target double ramification cycle formula [33] and relations [4] to the universal Picard stack. This leads to a system of tautological relations on the universal Picard stack which does not follow from tautological relations on the moduli space of stable curves.

In [6] we find a system of tautological relations on the moduli space of stable maps which can be viewed as twisted version of Pixton's relations [43, 31]. For a smooth projective variety X and a line bundle L on X, the equivariant pushforward of the virtual fundamental class along the morphism

$$\overline{\mathcal{M}}_{g,m}(\mathbb{P}(L \oplus \mathcal{O}_X)) \to \overline{\mathcal{M}}_{g,m}(X)$$

produces tautological relations on $\overline{\mathcal{M}}_{g,m}(X)$. We expect the relations should come from $\mathfrak{Pic}_{g,m}$ because the tautological relations obtained in this way only depends on $[L] : X \to B\mathbb{G}_m$.

In [19, 23] the Mumford's conjecture on the stability of rational cohomology is generalized to Picard stacks

$$\lim_{g\to\infty} H^*(\mathsf{Pic}^0(\mathcal{M}_g), \mathbb{Q}) = \mathbb{Q}[\kappa_{a,b} : a \geq -1, b \geq 0]\,.$$

This stability can be used to generalize the Givental-Teleman classification [25, 47] for the cohomological field theory (CohFT) on the universal Picard stack [42]. For the moduli space of stable curves, Pixton's relations are obtained by semi-simple CohFTs [43, 32] and hence one can hope that a suitable Picard CohFT can produce tautological relations in $H^*(\mathfrak{Pic}_{g,m}, \mathbb{Q})$.

We realize these ideas by constructing moduli spaces of *stable Picard quotients* over the relative Picard scheme $\mathsf{Pic}^0_{g,m}$ of multi-degree zero line bundles over the moduli space of stable curves. The moduli space of stable maps, or its variants, is a powerful tool to study the tautological ring of $\overline{\mathcal{M}}_{g,m}$. Similarly, the moduli space of stable Picard quotients will be an important method to study the tautological ring of $\mathsf{Pic}^0_{g,m}$. As an application we will show that virtual geometry of stable Picard quotients produces a system of tautological relations on the relative Picard scheme. The resulting relations can be viewed as a generalization of Pixton's relations on the relative Picard scheme.

1.3 Moduli space of stable Picard quotients

Our main new input is to consider certain relative Quot schemes over the Picard scheme. Let $\mathsf{Pic}^0_{g,m}$ be the Picard scheme of *multi-degree zero* line bundles over $\overline{\mathcal{M}}_{g,m}$.

It is a smooth separated Deligne-Mumford stack of finite type of dimension $4g-3+m$ over the base field k. We study tautological ring of $\mathsf{Pic}^{\underline{0}}_{g,m}$ using the virtual geometry over $\mathsf{Pic}^{\underline{0}}_{g,m}$.

Let $A = (a_1, \cdots, a_n) \in \mathbb{Z}^n$ be a tuple of integer and $d, r \geq 0$ be non-negative integers. Consider the moduli stack $\mathsf{P}_{g,m}(r, A, d)$ of stable quotient parametrizing a tuple (C, L, q) where C is a genus g nodal curve with m markings, L is a multi-degree zero line bundle on C and q is a quotient

$$0 \to S \to L^{a_1} \oplus \cdots \oplus L^{a_n} \to Q \to 0$$

which satisfies a stability condition similar to [38]. There exists a natural morphism

$$\mu : \mathsf{P}_{g,m}(r, A, d) \to \mathsf{Pic}^{\underline{0}}_{g,m} \tag{1.2}$$

defined by forgetting stable Picard quotients. Our first main result is the properness of $\mathsf{P}_{g,m}(r, A, d)$ over $\mathsf{Pic}^{\underline{0}}_{g,m}$ and the existence of perfect obstruction theory.

Theorem 1.3.1. For each numerical data g, m, r, A, d the moduli stack of stable Picard quotients $\mathsf{P}_{g,m}(r, A, d)$ is a finite type separated Deligne-Mumford stack which is proper over $\mathsf{Pic}^{\underline{0}}_{g,m}$. Moreover the moduli space $\mathsf{P}_{g,m}(r, A, d)$ carries a perfect obstruction theory.

Our construction generalizes the moduli space of stable quotients [38] to over the relative Picard scheme. We also show that the system of pushforwards of virtual cycles along (1.2) satisfies the splitting and the unit axiom similar to CohFT.

1.4 Faber-Zagier type relations

We apply the moduli space of stable Picard quotients to get tautological relations on the relative Picard scheme $\mathsf{Pic}^{\underline{0}}(\mathcal{M}_g)$ over the moduli space of smooth genus g curves. For our purpose, we only consider moduli spaces of stable Picard quotient when $A = (1, 0)$. Consider an element $\widetilde{\gamma}$ in $\mathbb{Q}[\{\kappa_{a,b}\}_{a \geq -1. b \geq 0}][[u, t, x]]$

$$\begin{aligned}
\widetilde{\gamma} := &\sum_{w=2}^{\infty} \frac{1}{w(w-1)} \kappa_{-1,w} u^w t^{w-1} + \sum_{w=1}^{\infty} \frac{1}{2w} \kappa_{0,w} u^w t^w \\
&+ \sum_{w=0}^{\infty} \sum_{s=1}^{\infty} \binom{2s-2+w}{w} \frac{B_{2s}}{2s(2s-1)} \kappa_{2s-1,w} u^w t^{2s+w-1} \\
&+ \sum_{d=1}^{\infty} \sum_{r=-1}^{\infty} \sum_{w=0}^{\infty} \tilde{C}_{r,d,w} \kappa_{r,w} u^w t^{r+w} \frac{x^d}{d!},
\end{aligned}$$

where $\widetilde{C}_{r,w,d}$ are defined by the coefficients of

$$\sum_{d=1}^{\infty} \sum_{r} \sum_{w} \tilde{C}_{r,w,d} t^r u^w \frac{x^d}{d!} = \log\left(1 + \sum_{d=1}^{\infty} \prod_{i=1}^{d} \frac{1}{1 - it - ut} \frac{(-1)^d}{t^d} \frac{x^d}{d!}\right).$$

Our second main result is an extension of Faber-Zagier type relations [24] on $\mathsf{Pic}^{\underline{0}}(\mathcal{M}_g)$.

Theorem 1.4.1. For $g-2d-1<r$ and $g\equiv r+1+w \mod 2$, we have

$$[\exp(-\widetilde{\gamma}))]_{u^w t^r x^d} = 0 \text{ in } \mathsf{R}^r(\mathsf{Pic}^{\underline{0}}(\mathcal{M}_g))\,.$$

Moreover when the parity condition does not hold the expression is trivially zero.

This set of relations follow from the equivariant pushforward of the virtual fundamental class along (1.2). In Theorem 5.3.4 we get more tautological relations by capping tautological classes with the virtual fundamental class. Our use of relative Quot scheme to prove Faber-Zagier type relations is motivated from Pandharipande-Pixton [41].

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2 Tautological ring of the Picard stack

2.1 Picard stacks

Over a scheme B, a curve C/B is called *prestable* if C is an algebraic space, the morphism is proper, flat of finite presentation and geometric fibers are reduced connected with at worst nodal singularities. Let $\mathfrak{M}_{g,m}$ be the moduli stack of prestable curves of genus g with m markings. It is an algebraic stack, smooth, locally of finite type over the base field k. Let $\mathfrak{C}_{g,m}\to\mathfrak{M}_{g,m}$ be the universal curve. The stack $\mathcal{M}_{g,m}$ has open substacks

$$\mathcal{M}_{g,m}\subset\mathcal{M}^{\mathrm{rt}}_{g,m}\subset\mathcal{M}^{\mathrm{ct}}_{g,m}\subset\overline{\mathcal{M}}_{g,m}\subset\mathfrak{M}_{g,m}$$

where $\mathcal{M}_{g,m}$ is the locus where curves are smooth, $\mathcal{M}^{\mathrm{rt}}_{g,m}$ is the locus where curves have rational tails, $\mathcal{M}^{\mathrm{ct}}_{g,m}$ is the locus of compact type curves and $\overline{\mathcal{M}}_{g,m}$ is the locus of stable curves.

We consider several relative Picard stacks of $\mathfrak{C}_{g,m}/\mathfrak{M}_{g,m}$:

- $\mathfrak{Pic}_{g,m}=\mathfrak{Pic}(\mathfrak{C}_{g,m}/\mathfrak{M}_{g,m})$ be the universal Picard stack.

- $\mathsf{Pic}_{g,m} = \mathfrak{Pic}_{g,m}/B\mathbb{G}_m$ be the rigidification.

- $\mathsf{Pic}^{\underline{0}}_{g,m} \subset \mathsf{Pic}(\mathcal{C}_{g,m}/\overline{\mathcal{M}}_{g,m})$ be the identity locus i.e. locus where the line bundle has degree zero on each irreducible component of the curve.

- $\mathsf{Jac}_g := \mathsf{Pic}(\mathcal{C}_{g,1}/\mathcal{M}_{g,1})$.

When the family of curves C over the base B is clear from the context we often abbreviate $\mathfrak{Pic}(B) := \mathfrak{Pic}(C/B)$. The above stacks are all locally of finite type and smooth over k. For construction of Picard stack, we refer [37, 13]. The universal Picard stack decomposes as the total degree of the line bundle

$$\mathfrak{Pic}_{g,m} = \bigsqcup_{d\in\mathbb{Z}} \mathfrak{Pic}_{g,m,d}\,.$$

We give a geometric description of rigidifed Picard scheme. Let C/B be a nodal curve. The rigidified Picard scheme $\mathsf{Pic}(C/B)$ is an algebraic space smooth over B. The stack Pic is not separated nor finite type over B. The identity component $\mathsf{Pic}^{\underline{0}}(C/B)$ is a scheme smooth and separated over B by [21, Prop. 4.3] and it is an open substack of Pic by [2, VIB, Theoreme 3.10]. When there exists a section, the rigidified Picard stack has a nice description. Let $\pi : C \to B$ be a nodal curve with a section ε. For any scheme T/B, we have

$$\mathsf{Pic}(C/B)(T) = \{\text{line bundles on } C \times_T B \text{ rigidified along the induced section}\}\,.$$

The universal line bundle $\mathcal{L}$ (rigidified along ε) on the universal curve $\mathfrak{C}$ over $\mathsf{Pic}(C/B)$ has an expected universal property [13, (8.2.4)]. When there is no confusion, we always consider the case when $m \geq 1$.

2.2 Chow group of Picard stack

In this paper, all Chow groups are tensored with $\mathbb{Q}$. For algebraic stacks finite type over k, integral Chow theory is developed by Kresch [36]. For algebraic stacks locally of finite type over k, we extend the definition by taking the inverse limit of Chow groups of all finite type open substacks [8, Appendix A]. When $(g, m) \neq (1, 0)$, the stack $\mathfrak{Pic}_{g,m}$ has affine stabilizers and hence the Chow group $\mathsf{CH}^*(\mathfrak{Pic}_{g,m})$ has a well-defined ring structure with respect to Gysin pullback along the diagonal [36].

The Chow ring of Picard stack has additional structure coming from the relative group structure of $\mathfrak{Pic}_{g,m}/\mathfrak{M}_{g,m}$. For $r, s \in \mathbb{Z}$ and $W = (w_1, \cdots, w_m) \in \mathbb{Z}^m$, there exists a morphism

$$\phi_{r,s,W} : \mathfrak{Pic}_{g,m} \to \mathfrak{Pic}_{g,m} \tag{2.1}$$

induced by

$$(C, (p_i)_i, L) \mapsto (C, (p_i)_i, L^{\otimes r} \otimes \omega_{\log}^{\otimes s}(w_1p_1 + \cdots + w_mp_m))$$

where $\omega_{\log} = \omega_\pi(\sum_{i=1}^m p_i)$ is the relative log canonical line bundle. Since $\mathfrak{Pic}_{g,m}$ is smooth, the (lci) pullback $\phi_{r,s,W}^*$ is well-defined and it induces an endomorphism on the tautological ring of $\mathfrak{Pic}_{g,m}$.

Tensoring the universal line bundle has a special role. For $N \in \mathbb{Z}$, let

$$[N] := \phi_{N,0,0} : \mathfrak{Pic}_{g,m} \to \mathfrak{Pic}_{g,m}$$

be the morphism given by the multiplication by N. We denote

$$\mathsf{CH}^i_{(j)}(\mathfrak{Pic}) := \{\alpha \in \mathsf{CH}^i(\mathfrak{Pic}) : [N]^*(\alpha) = N^j\alpha, \forall N \in \mathbb{Z}\}.$$

Proposition 2.2.1. When $2g-2+m>0$, we have a decomposition

$$\mathsf{CH}^i(\mathsf{Pic}^{\underline{0}}_{g,m}) = \bigoplus_{j\in\mathbb{Z}} \mathsf{CH}^i_{(j)}(\mathsf{Pic}^{\underline{0}}_{g,m}).$$

For a Chow class $\alpha \in \mathsf{CH}^i(\mathsf{Pic}^{\underline{0}}_{g,m})$, we write $\alpha = \sum \alpha_{(j)}$ the weight decomposition

Proof. When $m \geq 1$, this follows from the decomposition of relative motive for smooth commutative group scheme over a regular base [1, Theorem 4.9]. In [1] they assumed that the base is a scheme. For $\overline{\mathcal{M}}_{g,m}$ the same statement holds by taking scheme cover $\widetilde{\mathcal{M}}_{g,m} \to \overline{\mathcal{M}}_{g,m}$ and use Kimura sequence. When $m=0$ we use the fact that the pullback $\mathsf{Pic}^{\underline{0}}_{g,1} \to \mathsf{Pic}^{\underline{0}}_{g}$ is injective. $\square$

Over $\mathcal{M}^{\mathrm{ct}}_{g,m}$, the Fourier-Mukai transformation of $\mathsf{Pic}^{\underline{0}}(\mathcal{M}^{\mathrm{ct}}_{g,m})$ produces additional vanishing of the Chow ring [9, 22]. This can be extended to $\mathsf{Pic}^{\underline{0}}_{g,m}$.

Proposition 2.2.2. For $\alpha \in \mathsf{CH}^i(\mathsf{Pic}^{\underline{0}}_{g,m})$ we have $\alpha_{(j)} = 0$ unless $\max\{i-(3g-3+m), 0\} \leq j \leq \min\{g+i, 2g\}$.

Proof. Over $\mathcal{M}_{g,1}$, this vanishing is proven in [22, Theorem 2.19] using the Fourier-Mukai transformation between dual abelian schemes. Over $\overline{\mathcal{M}}_{g,m}$, we prove by induction on the genus. It is enough to consider the case when $m=1$. Let $\mathcal{M}^{\mathrm{ct}}_{g,1} \subset \overline{\mathcal{M}}_{g,1}$ be the open substack of compact type curves and let $\Delta_g = \overline{\mathcal{M}}_{g,1} \setminus \mathcal{M}_{g,1}$ be the complement. Consider the excision sequence

$$\mathsf{CH}^{*-1}(\mathsf{Pic}^{\underline{0}}_{\Delta_g}) \to \mathsf{CH}^*(\mathsf{Pic}^{\underline{0}}(\overline{\mathcal{M}}_{g,1})) \to \mathsf{CH}^*(\mathsf{Pic}^{\underline{0}}(\mathcal{M}^{\mathrm{ct}}_{g,1})) \to 0$$

where $\mathsf{Pic}^{\underline{0}}_{\Delta_g}$ denotes the restriction $\mathsf{Pic}^{\underline{0}}(\overline{\mathcal{M}}_{g,1})|_{\Delta_g}$. Let $\overline{\mathcal{M}}_{g-1,3} \to \Delta_g$ be the morphism associated to the normalization of a node. Then the morphism $f : \mathsf{Pic}^{\underline{0}}(\Delta_g) \to \mathsf{Pic}^{\underline{0}}(\overline{\mathcal{M}}_{g-1,3})$ is a $\mathbb{G}_m$-torsor. Therefore the flat pullback f^* is surjective on Chow groups. Since f^* preserves both grading, the vanishing follows from the induction. $\square$

Remark 2.2.3. One can ask whether similar properties hold for the rigidified Picard scheme $\mathsf{Pic}^0_{g,m}$ of *total degree* 0 line bundles. They sit in the short exact sequence of group spaces

$$0 \to \mathsf{Pic}^{\underline{0}}_{g,m} \to \mathsf{Pic}^0_{g,m} \to \pi_0(\mathsf{Pic}^0_{g,m}/\overline{\mathcal{M}}_{g,m}) \to 0 \tag{2.2}$$

where $\pi_0(\mathsf{Pic}^0_{g,m}/\overline{\mathcal{M}}_{g,m})$ is the quotient étale sheaf [44] (see also [1, Lemma 7.5]). Both Proposition 2.2.1 and 2.2.2 do not hold for the Picard scheme of total degree 0 line bundles. For example, when $g = 0, m \geq 4$, $\mathsf{Pic}^{\underline{0}}_{g,m}$ is trivial group scheme over $\overline{\mathcal{M}}_{0,m}$ and the failure can be seen easily from the divisor classes on $\pi_0(\mathsf{Pic}^0_{0,m}/\overline{\mathcal{M}}_{0,m})$. See also Remark 7.1.3.

We end this subsection by comparing the Chow group of $\mathfrak{Pic}_{g,m}$ and its rigidification $\mathsf{Pic}_{g,m}$. When $m \geq 1$ the morphism $\mathfrak{Pic}_{g,m} \to \mathsf{Pic}_{g,m}$ has a section. Therefore this is a trivial $\mathbb{G}_m$-gerbe which induces an isomorphism (depending on a choice of the universal line bundle)

$$\mathsf{CH}^*(\mathfrak{Pic}_{g,m}) \cong \mathsf{CH}^*(\mathsf{Pic}_{g,m}) \otimes \mathsf{CH}^*(B\mathbb{G}_m) \tag{2.3}$$

where $B\mathbb{G}_m$ is the quotient stack $[\mathrm{Spec}(k)/\mathbb{G}_m]$. When there is no marking, the tautological ring can be defined as follows. For $g \geq 2, m = 0$, we note that $\mathfrak{C}_g \to \mathsf{Pic}^{\underline{0}}_g$ has no universal line bundle [39]. Consider the fibre diagram

$$\begin{array}{ccc}
\mathfrak{C}'_g & \xrightarrow{f} & \mathfrak{C}_g \\
\downarrow{\scriptstyle p} & & \downarrow{\scriptstyle \pi} \\
\mathfrak{Pic}^{\underline{0}}(\overline{\mathcal{M}}_g) & \xrightarrow{g} & \mathsf{Pic}^{\underline{0}}_g
\end{array}$$

where f, g are the $B\mathbb{G}_m$-rigidification.

Proposition 2.2.4. The pullback $g^* : \mathsf{CH}^*(\mathsf{Pic}^{\underline{0}}_g) \to \mathsf{CH}^*(\mathfrak{Pic}^{\underline{0}}(\overline{\mathcal{M}}_g))$ is injective.

Proof. f^* is injective because it is a trivial $\mathbb{G}_m$-gerbe. π^* is injective because it is proper and surjective. Since the diagram is cartesian, g^* is injective. □

2.3 Tautological classes

In this paper we are mostly interested in tautological classes on the relative Jacobian. However, it is convenient to consider tautological classes on the universal Picard stack $\mathfrak{Pic}_{g,m}$. The tautological ring of $\mathfrak{Pic}_{g,m}$ for operational Chow group is defined in [5]. A straightforward generalization can be made to define the tautological ring of $\mathfrak{Pic}_{g,m}$.

Definition 2.3.1. Let $\pi : \mathfrak{C} \to \mathfrak{Pic}_{g,m}$ be the universal curve and for $i = 1, \cdots, m$ let $p_i : \mathfrak{Pic}_{g,m} \to \mathfrak{C}$ be the section corresponding to i-th section. Let $\mathcal{L}$ be the universal line bundle on $\mathfrak{C}$. Denote ω_π by the relative canonical line bundle on $\mathfrak{C}$. Then we define

$$\psi_i := p_i^* c_1(\omega_\pi)\,,\ \xi_i := p_i^* c_1(\mathcal{L}) \in \mathsf{CH}^1(\mathfrak{Pic}_{g,m}) \text{ for } i = 1, \cdots, m$$

and *twisted κ-classes*

$$\kappa_{a,b} := \pi_*(c_1(\omega_{\log})^{a+1} c_1(\mathcal{L})^b) \in \mathsf{CH}^{a+b}(\mathfrak{Pic}_{g,m}), \ \text{ for } a \geq -1, b \geq 0\,,$$

where $\omega_{\log} := \omega_\pi(\sum_{i=1}^m p_i)$ is the relative log canonical line bundle.

Note that there is a degree shift on twisted κ-classes compared to [5].

Boundary strata of $\mathfrak{Pic}_{g,m}$ can be described by prestable graphs. Denote $\mathsf{G}_{g,m}$ the set of prestable curves of genus g with m markings ([5, Sec. 0.3.1]). For a prestable graph $\Gamma \in \mathsf{G}_{g,m}$, let $\mathfrak{M}_\Gamma := \prod_{v \in V(\Gamma)} \mathfrak{M}_{g(v),n(v)}$ and $\xi_\Gamma : \mathfrak{M}_\Gamma \to \mathfrak{M}_{g,m}$ be the gluing morphism [5, Sec. 0.3]. Consider a fiber diagram

$$\begin{tikzcd}
\mathfrak{Pic}_\Gamma \arrow[r, "\xi_\Gamma"] \arrow[d] & \mathfrak{Pic}_{g,m} \arrow[d] \\
\mathfrak{M}_\Gamma \arrow[r] & \mathfrak{M}_{g,m}.
\end{tikzcd}$$

The image of the morphism

$$\xi_\Gamma : \mathfrak{Pic}_\Gamma \to \mathfrak{Pic}_{g,m} \tag{2.4}$$

is a boundary strata of $\mathfrak{Pic}_{g,m}$ associated to Γ. Moreover $\mathfrak{Pic}_\Gamma$ admits a morphism

$$\rho_\Gamma : \mathfrak{Pic}_\Gamma \to \prod_{v \in V(\Gamma)} \mathfrak{Pic}_{g(v),n(v)} \tag{2.5}$$

which sends a line bundle L on a prestable curve C to its restrictions on the various components of the normalization of C. The fibers of the map are torsors under the group $(\mathbb{G}_m)^{h^1(\Gamma)}$. In fact $\mathfrak{Pic}_\Gamma$ can be further decomposed by specifying the degree of each vertex. A *prestable graph of degree* $\Gamma_\delta = (\Gamma, \delta)$ consists of a prestable graph Γ together with a function

$$\delta : V(\Gamma) \to \mathbb{Z}\,.$$

For each Γ_δ there exists a Picard stack

$$\mathfrak{Pic}_{\Gamma_\delta} \to \mathfrak{M}_\Gamma$$

parameterizing curves with degeneration forced by Γ and with line bundles which have degree $\delta(v)$ restriction to the components corresponding to the vertex $v \in V(\Gamma)$.

Definition 2.3.2. Let $\Gamma_\delta \in \mathsf{G}_{g,m}$ be a prestable graph with degrees. A *decoration* α on Γ_δ is an element in $\mathsf{CH}^*(\prod_{v \in V(\Gamma)} \mathfrak{Pic}_{g(v),m(v),\delta(v)})$ given by a product of κ, ψ and ξ-classes on the factors $\mathfrak{Pic}_{g(v),m(v),\delta(v)}$. We define the *decorated stratum class* $[\Gamma_\delta, \alpha]$ as

$$\xi_{\Gamma_\delta *} \rho^*_{\Gamma_\delta} \alpha \in \mathsf{CH}^*(\mathfrak{Pic}_{g,m}) \,.$$

Definition 2.3.3. The tautological ring $\mathsf{R}^*(\mathfrak{Pic}_{g,m})$ is a $\mathbb{Q}$-subvector space of $\mathsf{CH}^*(\mathfrak{Pic}_{g,m})_{\mathbb{Q}}$ additively generated by locally finite[1] $\mathbb{Q}$-linear sum of decorated strata classes $[\Gamma_\delta, \alpha]$.

On the relative Picard stack Pic when a universal line bundle exists, it is only defined up to a tensor product with a line bundle pulled back from the base. When $m \geq 1$, we choose a universal line bundle which is trivialized along the first section.

Definition 2.3.4. Let $m \geq 1$. Take a universal line bundle $\mathcal{L}$ on $\mathsf{Pic}_{g,m}$ which is trivialized along the first marking. It induces a morphism $\iota : \mathsf{Pic}_{g,m} \to \mathfrak{Pic}_{g,m}$. Define the *tautological ring* of $\mathsf{Pic}_{g,m}$ as $\mathsf{R}^*(\mathsf{Pic}_{g,m}) = \iota^*\mathsf{R}^*(\mathfrak{Pic}_{g,m})$.

A different choice of a universal line bundle on $\mathsf{Pic}_{g,m}$ gives a different presentation of the tautological ring of $\mathsf{Pic}_{g,m}$. For example when the universal line bundle $\mathcal{L}$ is trivialized along i-th marking, then ξ_i vanishes. This shows that the tautological ring $\mathsf{R}^*(\mathsf{Pic}_{g,m})$ is not equivariant with respect to permuting m-markings.

Remark 2.3.5. When $m \geq 1$ we have an isomorphism (2.3) and the tautological relations on $\mathsf{Pic}_{g,m}$ has a canonical lift to tautological relations on $\mathfrak{Pic}_{g,m}$. When $m = 0$, the pullback along the rigidification morphism $\mathfrak{Pic}_{g,m} \to \mathsf{Pic}_{g,m}$ is injective by Proposition 2.2.4. Therefore we can identify the Chow ring of $\mathsf{Pic}_{g,m}$ as a subring of the Chow ring of $\mathfrak{Pic}_{g,m}$. For explicit presentation of the tautological classes on Pic we consider the following normalization of twisted κ-classes. Let $\mathcal{L}$ be the universal line bundle on $\mathfrak{C}'_g$. Then the class $p^*p_*(c_1(\mathcal{L})c_1(\omega_p)) + \frac{1}{2-2g}c_1(\mathcal{L})$ lies in $\mathsf{CH}^1(\mathfrak{C}_g)$. This class can be used the define κ-classes on $\mathsf{Pic}_{g,m}$.

The notion of tautological classes can be extended to the moduli space of compactififed Picard stack. We follow notion of stability condition in [35]. Let θ be a degree d stability condition for $\mathcal{C}_{g,m}/\overline{\mathcal{M}}_{g,m}$. Let $\mathsf{P}^\theta_{g,m}$ be the moduli space of θ-stable line bundles on a quasi-stable model of stable curves. Choose a universal line bundle which is trivialized along the first marking. By the choice of a universal line bundle, there exists a morphism

$$\iota : \mathsf{P}^\theta_{g,m} \to \mathfrak{Pic}^d_{g,m} \,.$$

Definition 2.3.6. The tautological ring of $\mathsf{P}^\theta_{g,m}$ is defined by $\mathsf{R}^*(\mathsf{P}^\theta_{g,m}) := \iota^*\mathsf{R}^*(\mathfrak{Pic}^d_{g,m})$.

[1]Here locally finite means that for any finte type open substack of $\mathfrak{Pic}_{g,m}$, the (possibly) infinite sum becomes finite.

We can define the tautological ring of $\mathfrak{Pic}_{g,m}$ inside the rational cohomology of $\mathfrak{Pic}_{g,m}$. For an algebraic stack $\mathfrak{X}$ over $\mathbb{C}$ and a choice of a scheme $U \to \mathfrak{X}$ on can associate a simplicial scheme $\mathfrak{X}_\bullet$ [20]. We define rational cohomology H^* and Borel-Moore homology $\mathsf{H}^{\mathrm{BM}}_*$ of $\mathfrak{X}$ by $\mathfrak{X}_\bullet$. There exists a cycle class map

$$\mathsf{cl} : \mathsf{CH}_d(\mathfrak{X})_\mathbb{Q} \to \mathsf{H}^{\mathrm{BM}}_{2d}(\mathfrak{X}_\bullet)_\mathbb{Q}\,.$$

When $\mathfrak{X}$ is smooth of pure dimension d we have the duality $\mathsf{H}^{\mathrm{BM}}_*(\mathfrak{X}_\bullet) \cong \mathsf{H}^{2d-*}(\mathfrak{X}_\bullet)$ which is independent of the choice of the cover.

Definition 2.3.7. *Tautological cohomology ring* is defined by $\mathsf{RH}^*(\mathfrak{Pic}_{g,m}) := \mathsf{cl}\big(\mathsf{R}^*(\mathfrak{Pic}_{g,m})\big)$.

Remark 2.3.8. For Jac_g, the definition of tautological ring appears in the literature [48]. Definition in [48] is different from our definition.

We give basic properties of tautological classes along pullback and pushforward.

Proposition 2.3.9. For $r, s \in \mathbb{Z}$ and $W = (w_1, \cdots, w_m) \in \mathbb{Z}^m$, let $\phi_{r,s,W} : \mathfrak{Pic}_{g,m} \to \mathfrak{Pic}_{g,m}$ be the morphism defined in (2.1). Then we have

(i) $\phi^*_{r,s,W}\psi_i = \psi_i$,

(ii) $\phi^*_{r,s,W}\xi_i = r\xi_i - w_i\psi_i$,

(iii) $\phi^*_{r,s,W}\kappa_{a,b} = \sum_{k=0}^{b}\binom{b}{k} r^k s^{b-k}\kappa_{a+b-k,k}$ for $a \geq 0$ and

$$\phi^*_{r,s,W}\kappa_{-1,b} = r^b\kappa_{-1,b} + \sum_{k=0}^{b-1}\binom{b}{k} r^k s^{b-k}\kappa_{k-1,b-k} - \sum_{i=1}^{m}\sum_{k=1}^{b}\binom{b}{k}(-w_i)^k r^{b-k}\psi_i^{k-1}\xi_i^{b-k}$$

for $a = -1$,

(iv) $\phi^*_{r,s,W}[\Gamma_\delta] = [\Gamma_{\delta'}]$ where the degree function $\delta' : V(\Gamma) \to \mathbb{Z}$ is defined by

$$r\delta'(v) + s(2g-2+m) + \sum_{i=1}^{m} w_i = \delta(v) \text{ for all } v \in V(\Gamma)^2\,.$$

If such δ' does not exists, we have $\phi^*_{r,s,W}[\Gamma_\delta] = 0$.

Proof. It follows from the straightforward computation on the universal curve over $\mathfrak{Pic}_{g,m}$. ☐

[2]When $r = 0$, the pullback becomes an infinite sum of boundart strata. For any finite-type open substack of $\mathfrak{Pic}_{g,m}$, this infinite sum restricts to a finite sum and hence the expression is well defined.

Corollary 2.3.10. The tautological ring $\mathsf{R}^*(\mathfrak{Pic}_{g,m})$ is stable under the pullback $\phi^*_{r,s,W}$. In particular, when $2g-2+m>0$ we have $\mathsf{R}^*(\mathsf{Pic}^{\underline{0}}_{g,m}) = \oplus_{0\le j\le 2g}\mathsf{R}^*_{(j)}(\mathsf{Pic}^{\underline{0}}_{g,m})$.

Proof. For a decorated strata class $[\Gamma,\alpha]$, Proposition 2.3.9 shows that $\phi^*_{r,s,W}[\Gamma,\alpha]$ can be written as a linear combination of decorated strata classes. Therefore $\mathsf{R}^*(\mathfrak{Pic}_{g,m})$ us stable under the pullback $\phi^*_{r,s,W}$. In particular, $\mathsf{R}^*(\mathsf{Pic}^{\underline{0}}_{g,m})$ is stable under $\phi^*_{N,0,0}$. Therefore we have the weight decomposition for $\mathsf{R}^*(\mathsf{Pic}^{\underline{0}}_{g,m})$. □

Lemma 2.3.11. Let $\pi:\mathfrak{C}_{g,m}\to\mathfrak{Pic}_{g,m}$ be the universal curve. For $a\ge 0$, we have $\pi^*\kappa_{a,b}=\kappa_{a,b}-\psi^a\xi^b$. For $a=-1$, we have $\pi^*\kappa_{-1,b}=\kappa_{-1,b}$.

Corollary 2.3.12. Let R be a $\mathbb{Q}$-algebra. For $i\ge 0$, $i+j\ge 1$, let $a_{i,j}\in R$. Let

$$T(z,\tau)=z\Big(1-\exp\big(-\sum_{\substack{i\ge 0\\ i+j\ge 1}} a_{i,j}z^i\tau^j\big)\Big)\in R[[z,\tau]]$$

be a formal power series. Then

$$\exp\Big(\sum a_{i,j}\kappa_{i,j}\Big)=\sum_{m=0}\frac{1}{m!}p_{m*}\Big(T(\psi_{n+1},\xi_{n+1}),\cdots,T(\psi_{n+m},\xi_{n+m})\Big)$$

in $\mathsf{CH}^*(\mathsf{Pic}^{\underline{0}}_{g,n})\otimes_{\mathbb{Q}} R$. Here, $p_m:\mathsf{Pic}^{\underline{0}}_{g,n+m}\to\mathsf{Pic}^{\underline{0}}_{g,n}$ is the morphism forgetting the last m markings.

Proof. This follows from Lemma 2.3.11 and the equality $\log(1-(1-\exp x))=x$. □

3 Moduli space of stable Picard quotients

3.1 Construction of moduli spaces

We construct moduli spaces of stable quotients over the relative rigidified Picard stack of multi-degree zero. Let $A=(a_1,\ldots,a_n)\in\mathbb{Z}^n$ be a tuple of integers. For fixed $(C,L)\in\mathsf{Pic}^{\underline{0}}_{g,m}(k)$[3], consider a short exact sequence

$$0\to S\to L^{a_1}\oplus\cdots\oplus L^{a_n}\to Q\to 0\,.$$

Let $\mathbb{Q}_+$ be the set of (strictly) positive rational numbers. For a nodal curve C let ω_C be its dualizing sheaf and $\omega_{\log}:=\omega_C(p_1+\cdots+p_m)$ be the log dualizing sheaf. When there is no ambiguity, the above short exact sequence is denoted as (L,q).

Definition 3.1.1. Let $\epsilon\in\mathbb{Q}_+$ be a fixed number. We say a tuple (C,L,q) is *ϵ-stable* if

[3]More precisely, we have (C,L,α) where $\alpha: L|_{p_1}\xrightarrow{\sim}\mathbb{C}$ is the trivialization at the first marking.

(i) Q is locally free at nodes and markings of C,

(ii) for any $x \in C$, the length of torsion subsheaf of Q at x is less than or equal to ϵ^{-1}, and

(iii) $\omega_{\log} \otimes (\wedge^r S^\vee)^{\otimes \epsilon}$ is ample.

When (C, L, q) is ϵ-stable for all $\epsilon \in \mathbb{Q}_+$, it is called *stable.*

By (i), the kernel S is locally free of rank r. For a line bundle L on a prestable curve C/k, it is ample if and only if L restricted to each irreducible component has positive degree. Therefore, when (L, q) is stable, the underlying nodal curve C is semi-stable. Moreover when a rational irreducible component of C has exactly two special points, then $S^\vee$ restricted to that component must have positive degree.

An isomorphism of ϵ-stable quotients is a pair

$$(\phi, \varphi) : (C, L, q) \to (C', L', q')$$

consists of an isomorphism of underlying marked curves $\phi : (C, p_1, \dots, p_m) \xrightarrow{\sim} (C', p_1', \dots, p_m')$ and an isomorphism of line bundles $\varphi : L \xrightarrow{\sim} \phi^* L'$[4] such that q and $\varphi^* q'$ are equal. Family of stable Picard quotients is staightforward to define.

Definition 3.1.2. A family of stable Picard quotients over a scheme B consists of data

$$(\pi : C \to B, p_1, \dots, p_m, L, q)$$

where

- $(C/B, (p_i)_i)$ is a family of nodal curves,
- L is a line bundle on C,
- the coherent sheaf Q is flat over B,
- $q : L^{a_1} \oplus \cdots \oplus L^{a_n} \twoheadrightarrow Q$ is a a surjection of coherent sheaves

such that the restriction of the data to each geometric fiber C_b is a twisted stable quotient.

For given $r, d \in \mathbb{N}$, let $\mathsf{P}_{g,m}(r, A, d)$ denote the prestack of stable Picard quotients with $\mathrm{rk}\,(S) = r, \deg(S) = -d$.

Proposition 3.1.3. For fixed g, m, r, A, d, families of stable Picard quotients are bounded.

[4]When $m \geq 1$, the isomorphism should be compatible with the rigidification.

Proof. Let C be an underlying curve of a stable Picard quotient. Since L^{a_i}s have same slopes, $\oplus_{i=1}^n L^{a_i}$ is a slope semi-stable bundle. Therefore the degree of S on each irreducible component of C is non-positive integer. Now we show that underlying pointed curves are bounded. It suffices to show that the number of unstable rational components are bounded. By the stability condition, such component should be semi-stable and the degree of S on such component should be strictly negative. Therefore the number of unstable rational component is bounded. Now for fixed curve, there are bounded family of multi-degree zero line bundles. By boundedness of Quot functor we have boundedness of stable Picard quotients. □

Lemma 3.1.4. Fix $A \in \mathbb{Z}^n, r, d \in \mathbb{N}$. For $\epsilon \in \mathbb{Q}_+$, consider a $\mathbb{Q}$-line bundle $L_\epsilon := \omega_{\log} \otimes (\wedge^r S^\vee)^\epsilon$. Then there exists sufficiently large number $f \in \mathbb{N}$ (only depending on A, r, d) such that $L_\epsilon^{\otimes f}$ is very ample and $H^1(C, L_\epsilon^{\otimes f}) = 0$.

Proof. In the proof of [38, Lemma 5], the degree argument for $L_\epsilon^{\otimes f}$(or its dual) on each irreducible component was needed. When L is a multi-degree zero line bundle, the same argument applies. □

Proposition 3.1.5. The moduli space of stable Picard quotients $\mathsf{P}_{g,m}(r, A, d)$ parametrizing stable quotients

$$(C, (p_i)_i, 0 \to S \to L^{a_1} \oplus \cdots \oplus L^{a_n} \xrightarrow{q} Q \to 0)$$

with $\mathrm{rk}\,(S) = r$, $\deg(S) = -d$ is a separated Deligne-Mumford stack of finite type over k.

Proof. By Proposition 3.1.3 the family of stable Picard quotients is bounded. Denote

$$\nu : \mathsf{P}_{g,m}(r, A, d) \to \mathsf{Pic}_{g,m} \tag{3.1}$$

the forgetful morphism. Since the family is bounded, there exists a finite type open substack $\mathfrak{P} \subset \mathfrak{Pic}_{g,m}^{\mathrm{rel}}$ where ν factors through. Let $\pi : \mathfrak{C} \to \mathfrak{P}$ be the unversal curve. Following [17] (together with Lemma 3.1.4), $\mathsf{P}_{g,m}(r, A, d)$ can be realized as a closed substack of a smooth algebraic stack. By [17, Lemma 3.1.6] for each geometric point, the automorphism group of q is finite and reduced. Therefore it is a Deligne-Mumford stack.

The separatedness of $\mathsf{P}_{g,m}(r, A, d)$ can be checked by the valuative criterion. Let $(R, \mathfrak{m})$ be a discrete valuation ring with the fraction field K. Consider two prestable curves $C_i \to \mathrm{Spec}(R)$, $p_1^i, \cdots, p_m^i : \mathrm{Spec}(R) \to C_i$ with multi-degree zero line bundles L_i on C_i and Picard stable quotients (L_i, q_i) for $i = 1, 2$. Suppose two Picard stable quotients are isomorphic on the generic fiber. By semi-stable reduction for nodal curves, after finite extension K' of K there exists a marked prestable curve $\widetilde{C}/R'$ together with dominant morphisms $\pi_i : \widetilde{C} \to C_i$ which are isomorphism away from nodes of the special fiber and compatible with sections. Since $\mathsf{Pic}^0_{\widetilde{C}/R'}$ is separated over R', the isomorphism of $L_1|_{C_K} \cong L_2|_{C'_K}$ extends after pulling back to $\widetilde{C}$. By the separatedness of Quot functor [29, Thm. 2.2.4], the isomorphism of two Picard stable quotients at generic fiber extends to $\widetilde{C}$. By stability, central fiber $\widetilde{C}_0$ cannot contain components which are contracted over the nodes of $C_1|_0$ but not contracted over the nodes of $C_2|_0$. □

For studying tautological relations on $\mathsf{Pic}^0_{g,m}$ we only consider rank two Picard stable quotients with $r = 1, A = (0, 1)$. For simplicity, we denote $\mathsf{P}_{g,m}(d) := \mathsf{P}_{g,m}(1, (1, 0), d)$.

Remark 3.1.6. When $A = \vec{0} := (0, \dots, 0)$, our moduli space can be recovered from the moduli space of stable quotients [38]. Let $\overline{Q}_{g,m}(\mathbb{G}(r, n), d) \to \mathfrak{M}_{g,m}$ be the moduli space of stable quotients. Then there exists an isomorphism $\mathsf{P}_{g,m}(r, \vec{0}, d) \cong \overline{Q}_{g,m}(\mathbb{G}(r, n), d) \times_{\mathfrak{M}_{g,m}} \mathsf{Pic}_{g,m}$.

3.2 Boundary strata of stable Picard quotients

The moduli space of stable Picard quotients has universal structure similar to the mapping stack

$$\mathrm{Map}_{\mathfrak{M}_{g,m}}(\mathfrak{C}_{g,m}, [\mathrm{Gr}(r, n)/\mathbb{G}_m] \times \mathfrak{M}_{g,m})$$

where $\mathrm{Gr}(r, n)$ denotes the Grassmannian of r-planes in n-dimensional vector space and $\mathbb{G}_m$ action is determined by A. Consider the universal curve $\pi : \mathfrak{C} \to \mathsf{P}_{g,m}(r, A, d)$ with m-sections $p_1, \cdots, p_m$. Over the universal curve, there exists the universal short exact sequence

$$0 \to \mathcal{S} \to \mathcal{L}^{a_1} \oplus \cdots \oplus \mathcal{L}^{a_n} \to \mathcal{Q} \to 0\,. \tag{3.2}$$

By the stability condition (Definition 3.1.1) $\mathcal{S}$ and $p_i^*\mathcal{Q}$ are locally free. Therefore pulling back (3.2) along the section p_i, we get a short exact sequence of locally free sheaves on $\mathsf{P}_{g,m}(r, A, d)$.

For an n-dimensional vector space V and $r \le n$, let $\mathrm{Gr}(r, n)$ be the Grassmanian of r dimensional subspaces of V.

Definition 3.2.1. For $A = (a_1, \dots, a_n) \in \mathbb{Z}^n$, the *twisted Grassmannian* $\mathrm{Gr}(r, A)$ is defined as the stack quotient $\mathrm{Gr}(r, A) := [\mathrm{Gr}(r, n)/\mathbb{G}_m]$ where $\mathbb{G}_m$ acts on V by weights $a_1, \dots, a_n$.

For $1 \le i \le m$, the evaluation map

$$\mathrm{ev}_i : \mathsf{P}_{g,m}(r, A, d) \to \mathrm{Gr}(r, A) \tag{3.3}$$

is given by the universal property of the twisted Grassmannian. In particular if $A = (a, \cdots, a)$, then $\mathrm{Gr}(r, A) \cong \mathrm{Gr}(r, n) \times B\mathbb{G}_m$ and our evaluation map factors through the usual one.

Boundary strata for $\mathsf{P}_{g,m}(r,A,d)$ can be described using evaluation maps. Let Γ be a $\mathbb{N}$-valued prestable graph of genus g with m markings. We define an evaluation map at each edge of Γ. First we consider a prestable graph Γ with one separating edge e connecting vertices v, v'. Since the universal line bundle is trivialized along the first marking, (3.3) factors through $\mathrm{Gr}(r,n)$. Thus we have

$$\mathrm{ev}_1 \times \mathrm{ev}_1 : \mathsf{P}_{g_1,m_1+1}(r,A,d_1) \times \mathsf{P}_{g_2,m_2+1}(r,A,d_2) \to \mathrm{Gr}(r,n) \times \mathrm{Gr}(r,n)\,.$$

The fiber product along the diagonal $\Delta : \mathrm{Gr}(r,n) \to \mathrm{Gr}(r,n)$ we have $\mathsf{P}_\Gamma(r,A,d)$. By taking the refined Gysin pullback along the diagonal, $\mathsf{P}_\Gamma(r,A,d)$ carries a natural virtual fundamental class. Second, we consider a prestable graph Γ with one non-separating edge e. Then (3.3) factors through

$$\mathrm{ev}_1 \times \mathrm{ev}_2 : \mathsf{P}_{g-1,m+2}(r,A,d) \to \mathrm{Gr}(r,n) \times \mathrm{Gr}(r,A)\,.$$

The fiber product along the morphism $\mathrm{Gr}(r,n) \to \mathrm{Gr}(r,n) \times \mathrm{Gr}(r,A)$ we have $\mathsf{P}_\Gamma(r,A,d)$.

3.3 Properness

The moduli space of stable Picard quotients $\mathsf{P}_{g,m}(r,A,d)$ is ususally not proper over the base field k. In this section we prove that $\mathsf{P}_{g,m}(r,A,d)$ is proper over the relative Picard scheme $\mathsf{Pic}^{\underline{0}}_{g,m}$. This will be enough for later application.

Lemma 3.3.1. When $2g-2+m>0$, the morphism

$$\mu : \mathsf{P}_{g,m}(r,A,d) \to \mathsf{Pic}^{\underline{0}}_{g,m} \tag{3.4}$$

induced by the contraction $f : C \to C'$ and $(C,L,q) \mapsto (C', L' = f_*L)$ is well defined.

Proof. It is enough to check the morphism is well-defined over any base scheme. For a nodal curve C/B, there exists the contraction map [46, Lemma 0E8A]

$$\begin{tikzcd} C \arrow[rr, "f"] \arrow[dr] & & C' \arrow[dl] \\ & B & \end{tikzcd}$$

where C'/B is the associated stable curve. Let L be a multi-degree zero line bundle on C. For each $x \in C'$ the inverse image $f^{-1}(x)$ is a chains of $\mathbb{P}^1$s. In particular it does not form a loop. Therefore $H^i(f^{-1}(x), L|_{f^{-1}(x)}) = 0$ for all $i>0$. By cohomology and base change $L' := f_*L$ is a multi-degree zero line bundle on C'. $\square$

The stabilization morphism does not change the relative Picard groups of multi-degree zero line bundles.

Lemma 3.3.2. Let $f : C \to C'$ be a stabilization morphism of a prestable curve C over a discrete valuation ring $(R, \mathfrak{m})$. Then the pullback morphism $f^* : \mathsf{Pic}^0_{C'/R} \to \mathsf{Pic}^0_{C/R}$ is an isomorphism.

Proof. Since $f_*\mathcal{O}_C \cong \mathcal{O}_{C'}$ the composition $\mathsf{Pic}^0_{C'/R} \xrightarrow{f^*} \mathsf{Pic}^0_{C/R} \xrightarrow{f_*} \mathsf{Pic}^0_{C'/R}$ is an isomorphism by the projection formula. Consider a canonical map $f^*f_*L \to L$ induced by adjunction. Since L restricted to contracted component is trivial, the canonical map is an isomorphism and hence f^* is bijective. Therefore f^* is an isomorphism because both schemes are smooth. $\square$

Lemma 3.3.3. For $2g-2+m>0$, the stabilization morphism $f : \mathfrak{M}_{g,m} \to \overline{\mathcal{M}}_{g,m}$ satisfies the existence part of the valuative criterion for discrete valuation rings.

Proof. Let $(R, \mathfrak{m})$ be a DVR with a quotient field K. Let C/R be a pointed stable curve and $f : D_K \to C_K$ be a stabilization morphism from a prestable curve. Consider a partial normalization $\widetilde{D}_K = \big(\sqcup D_K^{(i)}\big) \sqcup \big(\sqcup D_K^{(\alpha)}\big) \to D_K$ such that each connected component is either contracted chain of $\mathbb{P}^1$s (denoted $D_K^{(i)}$s) or components of C_K (denoted $D_K^{(\alpha)}$s). We can identify the locus where the partial normalization occurred by additional markings. By the semistable reduction for stable curves, there exist stable curve $D^{(\alpha)}/R$ which extends $D_K^{(\alpha)}$ after passing to a ramification of R. For $D_K^{(i)}$s we take a trivial family $D^{(i)}$ over R. By [12, Proposition 2.4] we can glue $D^{(i)}$s and $D^{(\alpha)}$s along the additional sections and get a prestable curve D over $\mathrm{Spec}(R)$ together with a stabilization morphism $f : D \to C$. By construction the generic fiber of D is isomorphic to D_K. $\square$

The following proof is a modification of [38, Section 6.3].

Proposition 3.3.4. When $2g-2+m>0$, the forgetful morphism (3.4) is proper.

Proof. Proof by valuative criterion. Let $(R, \mathfrak{m})$ be a DVR with a quotient field K. For any given commutative diagram

$$\begin{tikzcd} \mathrm{Spec}(K) \arrow[r] \arrow[d] & \mathsf{P}_{g,m}(r, A, d) \arrow[d, "\mu"] \\ \mathrm{Spec}(R) \arrow[r] & \mathsf{Pic}^0_{g,m} \end{tikzcd}$$

we need to find a diagonal arrow, possibly after ramified cover of $\mathrm{Spec}(R)$, which makes the diagram commutes. The above diagram is equivalent to the data

- a pointed stable curve C/R together with a commutative diagram

$$\begin{tikzcd} D_K \arrow[r, "f"] & C_K \arrow[r] \arrow[d] & C \arrow[d] \\ & \mathrm{Spec}(K) \arrow[r] & \mathrm{Spec}(R) \end{tikzcd}$$

where C_K is the generic fiber, D_K/K is a pointed prestable curve and f is a contraction,

- L a multi-degree zero line bundle on C, (M,q) be a stable Picard quotient on D_K with $f_*M \cong L|_{C_K}$.

By Lemma 3.3.3 there exists a stabilization morphism $f : D \to C$ over $\mathrm{Spec}(R)$ with generic fiber isomorphic to D_K. By Lemma 3.3.2 there exists $\widetilde{M} \in \mathsf{Pic}^{\underline{0}}_{D/R}$ with $f_*\widetilde{M} \cong L$. By taking base change and normaliation, we may assume that C_K is smooth and $f : D_K \to C_K$ is an isomorphism after putting additional sections. Once we prove the existence part for those cases, we can reconstruct the original family by gluing stable Picard quotients described in Section 3.2.

By the properness of Quot scheme [29, Theorem 2.2.4], we extend (M,q) to

$$0 \to S \to \widetilde{M}^{a_1} \oplus \cdots \oplus \widetilde{M}^{a_n} \to Q \to 0$$

over D. This extension may fail to be a stable Picard quotient at the central fiber. Following [38, Section 6.3] one can make it to be stable after further blowup and modification □

3.4 Rank one twisted stable quotients

Let $d > 0$. Consider the twisted stable quotient

$$0 \to S \to L \to Q \to 0.$$

Then $S = L \otimes \mathcal{O}_C(P)$ for some Cartier divisor $P \subset C$. Let $\overline{\mathcal{M}}_{g,m|d}$ be the moduli space of genus g curves with weighted markings. A k-point of $\overline{\mathcal{M}}_{g,m|d}$ consists of nodal curve C with markings $\{p_1,\ldots,p_m\} \cup \{\hat{p}_1,\ldots,\hat{p}_d\} \in C^{\mathrm{sm}}$ where p_is are distinct, $\hat{p}_j$s are distinct from p_is with stability condition:

$$\omega_C(\sum_{i=1}^{m} p_i + \epsilon \sum_{j=1}^{d} \hat{p}_j) \text{ is ample for all } \epsilon \in \mathbb{Q}_+\,.$$

The points $\hat{p}_j$ and $\hat{p}_{j'}$ can collide. By [27] the moduli space $\overline{\mathcal{M}}_{g,m|d}$ is a smooth irreducible Deligne-Mumford stack. Let $\mathfrak{C} \to \overline{\mathcal{M}}_{g,m|d}$ be the universal curve and $\mathsf{Pic}^{\underline{0}}(\overline{\mathcal{M}}_{g,m|d})$ be the associated rigidified relative Picard scheme. For a point $(C, p_i, \hat{p}_j) \in \mathsf{Pic}^{\underline{0}}(\overline{\mathcal{M}}_{g,m|d})$, one can associate a twisted stable quotient

$$0 \to L(-\hat{p}_1 - \cdots - \hat{p}_d) \to L \to Q \to 0\,.$$

This induces a morphism

$$\phi : \mathsf{Pic}^{\underline{0}}(\overline{\mathcal{M}}_{g,m|d}) \to \mathsf{P}_{g,m}((1),1,d). \tag{3.5}$$

Proposition 3.4.1. The morphism (3.5) induces an isomorphism of coarse moduli spaces

$$\mathsf{Pic}^{\underline{0}}(\overline{\mathcal{M}}_{g,m|d})/\Sigma_d \to \mathsf{P}_{g,m}((1),1,d)$$

where the symmetric group Σ_d acts by permuting the markings $\hat{p}_j$.

Similarly we have an isomorphism of coarse moduli spaces

$$\mathsf{Pic}^{\underline{0}}(\overline{\mathcal{M}}_{g,m|d})/\Sigma_d \to \mathsf{P}_{g,m}((0),1,d).$$

4 Virtual fundamental class

4.1 Perfect obstruction theory

We show that the moduli space of twitsted stable quotients carries a Behrend-Fantechi virtual fundamental class [10].

Theorem 4.1.1. Let $m \geq 1$. The moduli space of stable Picard pairs $\mathsf{P}_{g,m}(r,A,d)$ carries a perfect obstruction theory.

Proof. Let $\pi : \mathfrak{C} \to \mathsf{P}_{g,m}(r,A,d)$ be the universal curve with a universal line bundle $\mathcal{L}$ on $\mathfrak{C}$ trivialized along the first marking. Consider the universal quotient (3.2). For $\nu : \mathsf{P}_{g,m}(r,A,d) \to \mathsf{Pic}_{g,m}$, let $\mathbb{L}_\nu$ be the relative cotangent complex and $\tau^{\geq -1}\mathbb{L}_\nu$ be its truncation. By the deformation theory of relative Quot scheme, the relative Atiyah class induces a canonical relative obstruction theory

$$\phi : \mathbb{E}_\nu := R\mathcal{H}om_\pi(\mathcal{S},\mathcal{Q})^\vee \to \tau^{\geq -1}\mathbb{L}_\nu\,. \tag{4.1}$$

Then it is standard to check that ϕ is an obstruction theory and $\mathbb{E}$ can be resolved by a 2-term complex of locally free sheaves. See [38, Theorem 2] and [17, Section 5]. Since $\mathsf{Pic}_{g,m}$ is a smooth algebraic stack of pure dimension $4g-3+m$, (4.1) induces an absolute perfect obstruction theory [26, Appendix B]. □

By Theorem 4.1.1, we get a virtual fundamental class on $\mathsf{P}_{g,m}(A,r,d)$

$$[\mathsf{P}_{g,m}(r,A,d)]^{\mathrm{vir}} \in \mathsf{CH}_{\mathrm{vdim}}(\mathsf{P}_{g,m}(A,r,d))$$

where $\mathrm{vdim} = \chi(S,Q) + 4g - 3 + m = nd + r(n-r)(1-g) + 4g - 3 + m$.

We present some axioms that virtual fundamental classes defined in Theorem 4.1.1 satisfy. This will be a variant of Picard CohFT axioms suggested in [42]. For writing down axioms much symmetric way, we consider Picard stacks and stable Picard quotient before rigidification. The construction of moduli space of stable Picard quotients can be done over the Picard stack $\mathfrak{Pic}^{\underline{0}}(\overline{\mathcal{M}}_{g,m})$ which we denote by $\mathfrak{P}_{g,m}(r,A,d)$. Let $V := H^*(\mathrm{Gr}(r,A),\mathbb{Q})$ be the rational cohomology ring and $\mathbb{1} \in V$

be the fundamental class. The cycle class map $\mathsf{CH}^*(\mathrm{Gr}(r,A)) \to H^{2*}(\mathrm{Gr}(r,A))$ is an isomorphism.

Let q be a formal variable recording the degree. For $2g-2+m>0, m>0$ we consider a system of $\mathbb{Q}$-linear maps

$$\Omega_{g,m} : V^{\otimes m} \to H^*(\mathfrak{Pic}^0_{g,m}, \mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{Q}[[q]] \tag{4.2}$$

defined by

$$\Omega_{g,m}(v_1, \cdots, v_m) := \sum_{d\geq 0} \mu_*\Big(\prod_{i=1}^{m} \mathrm{ev}_i^*(v_i) \cap [\mathfrak{P}_{g,m}(r,A,d)]^{\mathrm{vir}}\Big) q^d \,.$$

Let $\{e_i\}$ be a basis of V over $H^*(B\mathbb{G}_m)$. We choose a pure degree class $\sum_{i,j} g^{i,j} e_i \otimes e_j$ in $H^*(\mathrm{Gr}(r,A)) \otimes_{\mathbb{Q}} H^*(\mathrm{Gr}(r,A))$ such that after pulling back to $H^*(\mathrm{Gr}(r,A) \times_{B\mathbb{G}_m} \mathrm{Gr}(r,A))$, we get the class of the relative diagonal $\mathrm{Gr}(r,A) \to \mathrm{Gr}(r,A) \times_{B\mathbb{G}_m} \mathrm{Gr}(r,A)$.

For a stable graph $\Gamma \in \mathsf{G}_{g,m}$, let $\xi_\Gamma : \mathfrak{Pic}^0_\Gamma \to \mathfrak{Pic}^0_{g,m}$ be the morphism defined in (2.4). By the argument in [11] the system of maps $\{\Omega_{g,m}\}$ satisfies the splitting axiom for Picard CohFTs.

Proposition 4.1.2. Let $\Gamma \in \mathsf{G}_{g,m}$ be a stable graph with one edge where the genus splits by g_1, g_2 and markings split by $I \sqcup J = \{1, \cdots, m\}$. Then we have

$$\xi_\Gamma^* \Omega_{g,m}(v_1, \cdots, v_m) = \sum_{i,j} \rho^*\big(\Omega_{g_1, I+1}(v_I, g^{ij} e_i) \otimes \Omega_{g_2, J+2}(e_j, v_J)\big) \,.$$

The moduli space of stable Picard quotients are not proper over k where the non-properness comes from the non-properness of the relative Picard scheme over $\overline{\mathcal{M}}_{g,m} \setminus \mathcal{M}^{\mathrm{ct}}_{g,m}$. Therefore is not immediately clear how to get numbers out of Theorem 4.1.1. On the other hand, there is a way to get invariants using the λ_g-class. Integrals of tautological classes on compactified relative Picard stack will be addressed in [7].

4.2 Torus fixed loci

Fiberwise GL_n action on $\mathcal{L}^{a_1} \oplus \cdots \oplus \mathcal{L}^{a_n}$ in (3.2) lifts to a GL_n action on $\mathsf{P}_{g,m}(r,A,d)$. Let $T \subset \mathrm{GL}_n$ be the maximal torus with weights $w_1, \cdots, w_n$. T-fixed loci of $\mathsf{P}_{g,m}(r,A,d)$ can be described by decorated graphs. The following description is standard [38, Section 7.3.1]. Let $I(r,A)$ be the set consisting of size r subsets of $\{a_1, \cdots, a_n\}$. Hence $|I(r,A)| = \binom{n}{r}$. Let $J(r,A)$ be the set consisting of size 2 subsets of $I(r,A)$.

Definition 4.2.1. A decorated graph is a tuple

$$(\Gamma, \nu, \mathsf{g}, \mathsf{e}, s, \mathsf{d}, \ell)$$

where

(i) $\Gamma = (V(\Gamma), E(\Gamma))$ is a connected graph[5],

(ii) $\nu : V(\Gamma) \to I(r, A)$ is a map to each vertex specifying splitting type of the subsheaf,

(iii) $\mathsf{g} : V(\Gamma) \to \mathbb{Z}_{\geq 0}$ is genus assignment,

(iv) $s = (s_1(v), \cdots, s_r(v)) \in \mathbb{Z}_{\geq 0}^r$ with $\mathbf{s}(v) := \sum_{i=1}^r s_i(v)$ together with inclusion

$$\iota_s : \{1, \cdots, r\} \to \{1, \cdots, n\}\,,$$

(v) $\mathsf{e} : E(\Gamma) \to J(r, A)$ with degree assignment $\mathsf{d} : E(\Gamma) \to \mathbb{Z}_{\geq 1}$,

(vi) $\ell : L \to \{1, \cdots, m\}$ is a distribution of markings.

They should satisfy the following conditions:

- when $e = (v, v')$, invariant curve $\mathsf{e}(e)$ joins $\nu(v)$ and $\nu(v')$,
- $\sum_{v \in V(\Gamma)} \mathsf{g}(v) + h^1(\Gamma) = g$,
- $\sum_{v \in V(\Gamma)} s(v) + \sum_{e \in E(\Gamma)} \mathsf{d}(e) = d$,
- each $v \in V(\Gamma)$ is either degenerate, i.e. $\mathsf{g}(v) = 0, \mathrm{val}(v) = 2, s(v) = 0$, or it satisfies the stability condition $2\mathsf{g}(v) - 2 + \mathrm{val}(v) + \epsilon\mathbf{s}(v) > 0$ for all $\epsilon \in \mathbb{Q}_+$.

The T-fixed loci of $\mathsf{P}_{g,m}(r, A, d)$ are described by, up to a finite morphism, the relative Picard scheme over Hassett moduli spaces. For a decorated graph Γ, let

$$\overline{\mathcal{M}}_\Gamma := \prod_{v \in V(\Gamma)} \overline{\mathcal{M}}_{\mathsf{g}(v), \mathrm{val}(v)|s(v)}$$

be the product of Hassett spaces. When the vertex v is degenerated we set $\overline{\mathcal{M}}_{\mathsf{g}(v), \mathrm{val}(v)|s(v)}$ to be the point. It has a gluing morphism $\xi_\Gamma : \overline{\mathcal{M}}_\Gamma \to \overline{\mathcal{M}}_{g,m|d}$. Let $\mathsf{Pic}^{\underline{0}}_\Gamma$ be the pullback of $\mathsf{Pic}^{\underline{0}}(\overline{\mathcal{M}}_{g,m|d})$ along ξ_Γ. We describe the T-fixed stable Picard stack as follows.

(i) On the component C_v, the stable Picard quotient is given by

$$0 \to \bigoplus_{j=1}^{r} L^{a_{\iota_s(i)}} \otimes \mathcal{O}_{C_v}(-\hat{p}_1 - \cdots - \hat{p}_{s(v)}) \to \bigoplus_{i=1}^{n} L^{a_i} \to Q \to 0\,, \qquad (4.3)$$

where the injection is determined by i_s.

[5] For the definition of prestable graphs, see [8, Section 2.1].

(ii) For each edge e, consider the degree $\mathsf{d} : E(\Gamma)$ covering of T-invariant curve $\mathbb{P}^1$ in the Grassmannian $\mathrm{Gr}(r, n)$ ramified only over two torus fixed points. The stable Picard quotient is obtained by pulling back the tautological sequence of $\mathrm{Gr}(r, n)$.

(iii) The gluing of stable Picard quotients on different components are given by the gluing map described in Section 3.2.

Then there exists a finite morphism $j_\Gamma : \mathsf{Pic}^0_\Gamma \to \mathsf{P}_{g,m}(r, A, d)$ obtained by the stable Picard quotient over Pic^0_Γ.

4.3 Virtual localization

By [16, Theorem 3.5], the virtual localization formula [26] holds for $\mathsf{P}_{g,m}(r, A, d)$. Consider a connected component of the T-fixed loci of $\mathsf{P}_{g,m}(r, A, d)$ indexed by a decorated graph Γ. Then we have

$$[\mathsf{P}_{g,m}(r, A, d)]^{\mathrm{vir}} = \sum_\Gamma j_{\Gamma *} \frac{[\mathsf{Pic}^0_\Gamma]^{\mathrm{vir}}}{e^T(N^{\mathrm{vir}}_\Gamma)}$$

in $\mathsf{CH}^T_*(\mathsf{P}_{g,m}(r, A, d))_{\mathrm{loc}}$. We compute the contribution of the virtual normal bundle at that fixed locus. Consider a graph Γ corresponding to a fixed locus of the moduli space of stable Picard quotients. Then the contribution of the virtual normal bundle to the localization formula takes the form

$$\prod_v \mathrm{Cont}(v) \cdot \prod_e \mathrm{Cont}(e) \cdot \prod_{(v,e)} \mathrm{Cont}(v, e)$$

for vertices v, edges e and flags (v, e).

Let v be a nondegenerate vertex. Then

$$\mathrm{Cont}(v) = \frac{e(\mathrm{Ext}^1(S, Q)^m)}{e(\mathrm{Ext}^0(S, Q)^m)} \cdot \frac{1}{\prod_e w(e)/\mathsf{d}(e) - \psi_e} \tag{4.4}$$

where the second term appears in the moving part of deforming the underlying curve. We denote $\check{\iota}_s : \{1, \cdots, n-r\} \to \{1, \cdots, n\}$ by the complement of ι_s. Let $\hat{P}_i \subset C$ be the divisor associated to s_i. In (4.3) we have

$$S = \bigoplus_{i=1}^r L^{a_{\iota_s(i)}}(-\hat{P}_i)\,, \text{ and } Q = \bigoplus_{j=1}^{n-r} L^{a_{\check{\iota}_s(j)}} \oplus \bigoplus_{i=1}^r L^{a_{\iota_s(i)}}|_{\hat{P}_i}\,.$$

We can compute the moving part of $R\mathrm{Hom}(S, Q)$ using the short exact sequence

$$0 \to \bigoplus_{i=1}^r \left(L^{a_{\iota_s(i)}}\right)^\vee \to S^\vee \to \bigoplus_{i=1}^r \left(L^{a_{\iota_s(i)}}\right)^\vee(\hat{P}_i)|_{\hat{P}_i} \to 0\,.$$

Therefore (4.4) is equivalent to

$$\begin{aligned}\mathrm{Cont}(v) &= \frac{1}{e(R\pi_*(\bigoplus_{i=1}^{r} L^{-a_{\iota_S(i)}} \otimes \bigoplus_{j=1}^{n-r} L^{a_{\bar\iota(j)}})} \cdot \frac{1}{\prod_e w(e)/\mathsf{d}(e) - \psi_e} \\ &\cdot \frac{1}{\prod_{1\le i\neq j\le r} e(H^0(L^{a_{\iota(j)}-a_{\iota(i)}}(\hat P_i)|_{\hat P_j}) \otimes [w_j - w_i]} \\ &\cdot \frac{1}{\prod_{1\le i\le r, r+1\le j^*\le n} e(H^0(L^{a_{\bar\iota(j^*)}-a_{\iota(i)}}(\hat P_i)|_{\hat P_i}) \otimes [w_j - w_i]}\,,\end{aligned}$$

where $[-]$ denote the trivial line bundle with the specified weight.

The vertex contribution to the fixed part of $R\mathrm{Hom}(S, Q)$ equals

$$\bigoplus_{i=1}^{r} R\mathrm{Hom}(L^{a_{\iota_S(i)}}(-\hat P_i), L^{a_{\iota_S(i)}}|_{\hat P_i}) \cong \bigoplus_{i=1}^{r} \mathrm{Hom}(\mathcal{O}_C(-\hat P_i), \mathcal{O}_{\hat P_i})\,,$$

and thus the virtual fudamental class for the fixed locus is the fundamental class.

5 Tautological relations on the smooth locus

In this section we investigate tautological relations on $\mathsf{Jac}_{g,1}$ using moduli space of stable Picard quotients. This strategy can be extended to the enter space $\mathsf{Pic}^{\underline{0}}_{g,m}$.

5.1 K-theory classes

For $g \ge 2$ and $d \ge 1$, let $\mathfrak{C}_g \to \mathcal{M}_{g,1}$ be the universal curve. Let $\mathfrak{C}_g^d \to \mathcal{M}_{g,1}$ be the d-fold symmetric product of the universal curve. Pulling back it to the relative Jacobian $\mathsf{Jac}_g \to \mathcal{M}_{g,1}$, we get $\pi^d : \mathfrak{C}_g^d \to \mathsf{Jac}_g$. Consider the universal curve U obtained from the pullback from Jac_g

$$\begin{tikzcd} U \arrow[r] \arrow[d, "\mu"] & \mathfrak{C}_g \arrow[d] \\ \mathfrak{C}_g^d \arrow[r, "\pi^d"] & \mathsf{Jac}_g\,. \end{tikzcd}$$

For $1 \le j \le d$, denote $p_j : \mathfrak{C}_g^d \to U$ by the j-th section. For $S_U := \mathcal{O}_U(-\sum_{j=1}^d p_j)$, consider the universal quotient sequence $0 \to S_U \to \mathcal{O}_U \to Q_U \to 0$ on U. Let L be a universal line bundle on U pulled back from $\mathfrak{C}_g$. Consider a K-theory class

$$\mathbb{F}_d = -R\mu_*(S_U^\vee \otimes L^\vee) \in K^0(\mathfrak{C}_g^d)$$

and similarly consider a K-theory class

$$\mathbb{F}'_d = -R\mu_*(S_U^\vee \otimes L)\,.$$

Denote a K-theory class
$$\mathbb{E}_d = -R\mu_* L\,.$$

Let $\mathfrak{C} \subset \mathfrak{C}_g^2$ be the diagonal. For $i \neq j$, consider the projection $\mathfrak{C}_g^d \to \mathfrak{C}_g^2$ for i,j-th factor. Let $D_{i,j} \subset \mathfrak{C}_g^d$ be the pullback of the diagonal $\mathfrak{C}$ via the projection. Set
$$\Delta_i := D_{1,i} + \cdots + D_{i-1,i}\,, i \geq 2$$
and we denote $\Delta_1 := 0$. Taking the cohomology of the sequence
$$0 \to L(\sum_{j=1}^{d-1} p_j) \to L(\sum_{j=1}^{d} p_j) \to L(\sum_{j=1}^{d} p_j)|_{p_d} \to 0$$
we compute the total Chern class of $\mathbb{F}_d$ inductively.

Lemma 5.1.1. Let $s : \mathfrak{C}_g^d \to U$ be a section corresponds to the last marking. Then
$$c_1(\mu_*(L(p_1 + \cdots + p_d)|_{\mathrm{Im}(s)}))) = \Delta_d - \psi_d + \xi_d\,.$$

Proof. By a direct computation, we have
$$\begin{aligned}\mu_*(L(p_1 + \cdots + p_d)|_{\mathrm{Im}(s)}) &= \mu_* s_* s^*(L(p_1 + \cdots + p_d)) \\ &= s^*\mathcal{O}(p_1 + \cdots + p_{d-1}) \otimes s^*\mathcal{O}(s) \otimes s^* L.\end{aligned}$$
Because $c_1(s^*\mathcal{O}(s)) = -\psi_d$, we get the result. $\square$

By Lemma 5.1.1 we can express the total Chern class of $\mathbb{F}_d$ as
$$c(\mathbb{F}_d) = \frac{c(\mathbb{E}_d)}{(1 + \Delta_1 - \psi_1 + \xi_1)\cdots(1 + \Delta_d - \psi_d + \xi_d)}$$
and similarly for $\mathbb{F}'_d$
$$c(\mathbb{F}'_d) = \frac{c(\mathbb{E}_d)}{(1 + \Delta_1 - \psi_1 - \xi_1)\cdots(1 + \Delta_d - \psi_d - \xi_d)}.$$
We express the total Chern class of $\mathbb{E}_d$ as tautological classes.

Lemma 5.1.2. Let $\mu : C \to B$ be a smooth curve. Let L be a muli-degree zero line bundle on C. Let u be a formal variable. The total Chern class of the K-theory class $\mathbb{E} := R\mu_*(L)$ is
$$\begin{aligned}\sum_{j=1}^{\infty}\sum_{w=0}^{j+1} c_j(\mathbb{E})_{(w)} t^j u^w = \exp\Big(&\sum_{w=2}^{\infty} \frac{1}{w(w-1)}\kappa_{-1,w} t^{w-1}(-u)^w + \sum_{w=1}^{\infty} \frac{1}{2w}\kappa_{0,w} t^w (-u)^w \\ &\sum_{w=0}^{\infty}\sum_{s=1}^{\infty} \binom{2s-2+w}{w} \frac{B_{2s}}{2s(2s-1)} \kappa_{2s-1,w} t^{2s+w-1}(-u)^w\Big) - 1\,.\end{aligned}$$
Here $c_j(\mathbb{E})_{(w)}$ is the weight decomposition given by Proposition 2.2.1.

Proof. By the Grothendieck-Riemann-Roch formula we have $\mathrm{ch}(\mathbb{E}) = \mu_*(\mathrm{ch}(L).\mathrm{Td}(T_\mu))$. Since μ is smooth of relative dimension one, we have $\mathrm{Td}(T_\mu) = \frac{\psi}{\exp(\psi)-1}$ where $\psi := c_1(\omega_\mu)$. Using the transformation formula between chern characters and chern classes

$$\sum c_i(\mathbb{E})t^i = \exp(\sum(-1)^{s-1}(s-1)!\,\mathrm{ch}_s(\mathbb{E})t^s)\,,$$

the result follows. □

A parallel computation for family of nodal curves appears in [18].

Now we concentrate on the class

$$\mathbb{B}_d := \frac{1}{(1+\Delta_1-\psi_1+u\xi_1)\cdots(1+\Delta_d-\psi_d+u\xi_d)} \tag{5.1}$$

where u is a formal variable. The generating series of the classes $\pi^d_*\mathbb{B}_d$ is obtained by the Wick's formula as in [41]. Let $\tilde{M}^d_r$ be the coefficient (5.1)

$$\mathbb{B}_d = \sum_{r=0}^{\infty} \tilde{M}^d_r(\psi_i, u\xi_j, -D_{ij})t^r.$$

Here t keeps track of cohomological degree of each monomial and u keeps track of b-degree of each monomial. Let $\tilde{S}_{r,d,a}$ be the summand of the evaluation $\tilde{M}^d_r(\psi_i = 1, u\cdot\xi_j = 1, -D_{ij} = 1)$ consisting of the contributions of connected monomials in front of u^a.

Lemma 5.1.3. Let $\pi^d : \mathfrak{C}^d_g \to \mathsf{Jac}_g$ be the projection. Let a_j, b_j be non-negative integers and let $c_j \geq 1$. Then

$$\pi^d_*\Big(\prod_{i=1}^{d-1}(-D_{i,i+1})^{c_i}\cdot\prod_{j=1}^{d}\psi_j^{a_j}\cdot\prod_{j=1}^{d}\xi_j^{b_j}\Big) = (-1)^c\kappa_{a+c-d,b}$$

where $a = \sum a_j, b = \sum b_j$ and $c = \sum c_j$.

Proof. By the self-intersection formula $(D_{ij})^2 = \psi_i(-D_{ij}) = \psi_j(-D_{ij})$, the monomial can be written as $\prod_{i=1}^{d-1} D_{i,i+1}\cdot$(monomials in ξ_j and ψ_j). The locus corresponds to $\prod_{i=1}^{d-1} D_{i,i+1}$ is the image of the big diagonal $\mathfrak{C} \to \mathfrak{C}^d_g$. Thus one can pullback ψ,ξ-classes to $\mathfrak{C}$ and get the result. □

Proposition 5.1.4. Let $\pi^d : \mathfrak{C}^d_g \to \mathsf{Jac}_g$ be the projection and $\mathbb{B}_d$ be the class defined in (5.1). We have

$$\sum_{d=1}^{\infty}\sum_{r\geq 0}\pi^d_*([\mathbb{B}_d]^r)t^r\frac{x^d}{d!} = \exp\Big(\sum_{d=1}^{\infty}\sum_{r=0}^{\infty}\sum_{s=0}^{r-d+1}\tilde{S}_{r,d,a}(-1)^{d-1-a}\kappa_{r-d-a,a}t^r u^a\frac{x^d}{d!}\Big)\,.$$

Proof. For connected monomials, the pushforward π^d_* only depends on the degree a and b by Lemma 5.1.3. Therefore the contribution of connected monomials is $\tilde{S}_{r,d,a}\kappa_{r-d-a,a}$. By Wick's formula, disconnected contribution is related to the connected formula by

$$\exp\big(\sum_{d=1}^{\infty}\sum_{r=0}^{\infty}\tilde{S}_{r,d,a}t^r u^a\frac{x^d}{d!}\big) = 1+\sum_{d=1}^{\infty}\prod_{i=1}^{d}\frac{1}{1-(i+u)t}\frac{x^d}{d!}\,.$$

Hence we get the result. □

Corollary 5.1.5. Define $\widetilde{C}_{r,w,d}$ by the coefficients of

$$\sum_{d=1}^{\infty}\sum_{r}\sum_{w}\tilde{C}_{r,w,d}t^r u^w\frac{x^d}{d!} = \log\Big(1+\sum_{d=1}^{\infty}\prod_{i=1}^{d}\frac{1}{1-it-ut}\frac{(-1)^d}{t^d}\frac{x^d}{d!}\Big)\,.$$

Then $\widetilde{C}_{r,w,d}=0$ if $r<w-1$.

Proof. In the proof of Lemma 5.1.3, a connected monomial in the variables ψ_i, ξ_i and $-D_{ij}$ must have at least $d-1$ factors of the variables $-D_{ij}$. Therefore $\widetilde{S}_{r,d,a}=0$ if $r-w<d-1$. This implies the vanishing of $\widetilde{C}_{r,w,d}$ in the prescribed degree. □

5.2 Tautological relations

Let $\mathsf{P}^{\mathrm{sm}}_{g,m}(d)$ be the fiber product $\mathsf{P}_{g,m}(d)\times_{\mathsf{Jac}_{g,m}}\mathfrak{Pic}^{\mathrm{rel}}_{g,m}$. Consider the contraction morphism $\mu:\mathsf{P}^{\mathrm{sm}}_{g,m}(d)\to\mathsf{Jac}_{g,m}$. For $c\geq 1$, we study tautological relations

$$\mu_*(0^c\cap[\mathsf{P}^{\mathrm{sm}}_g(d)]^{\mathrm{vir}})=0$$

using the virtual localization.

Lemma 5.2.1. For $c>0$ and $c+w\equiv 0\mod 2$, we have

$$\pi^d_*(c_{g-d-1+c}(\mathbb{F}_d))=0$$

in $\mathsf{R}^*(\mathsf{Jac}_g)$.

Proof. Let $\mathbb{G}_m$ acts on $L\oplus\mathcal{O}_C$ with weights $[0,1]$. It lifts to the $\mathbb{G}_m$ action on the perfect obstruction theory on $\mathsf{P}_{g,m}(d)$. We lift the $\mathbb{G}_m$ action to a rank one trivial bundle with weight 1. There are two $\mathbb{G}_m$-fixed loci of the moduli of twisted stable quotients. Two fixed loci are both isomorphic to $\mathfrak{C}^d_g/\Sigma_d$.

For each fixed locus, we compute the virtual normal bundle. For simplicity, we denote $P:=\sum_{j=1}^{d}p_j\subset C$ which corresponds to markings. The first fixed locus corresponds to a twisted stable pair

$$0\to S=L(-P)\to L\oplus\mathcal{O}_C\to Q=L|_P\oplus\mathcal{O}_C\to 0.$$

By annalizing the moving part of $R\mathcal{H}om(S,Q)$, we show that the virtual normal bundle corresponds to $\mathbb{F}_d = -R\mu_*(S_U^\vee \otimes L^\vee)$. Therefore the localization contribution of the locus is

$$\frac{1}{d!}\pi_*^d(c_{g-d-1+c}(\mathbb{F}_d)).$$

The second fixed locus corresponds to a twisted stable pair

$$0 \to S = \mathcal{O}_C(-P) \to L \oplus \mathcal{O}_C \to L \oplus \mathcal{O}_C|_P \to 0.$$

The virtual normal bundle corresponds to $\mathbb{F}'_d = -R\mu_*(S_U^\vee \otimes L)$. Let $c_-(\mathbb{F}'_d)$ be the total Chern class of $\mathbb{F}'_d$ evaluated at -1. Then the localization contribution of the locus is

$$\frac{(-1)^{g-d-1}}{d!}\pi_*^d[c_-(\mathbb{F}'_d)]^{g-d-1+c}$$

where $[-]^k$ is the degree k component. Since $c(\mathbb{F}_d)_{(w)} = (-1)^w c(\mathbb{F}'_d)_{(w)}$, we get the vanishing result with the parity condition shifted by w. $\square$

Consider a generating series

$$\Phi(t,u,x) = \sum_{d=0}^{\infty}\prod_{i=1}^{d}\frac{1}{1-(i+u)t}\frac{(-1)^d}{d!}\frac{x^d}{t^d}\,. \tag{5.2}$$

By taking the logarithm, we define coefficients $\tilde{C}_{r,d,a} \in \mathbb{Q}$

$$\log(\Phi) =: \sum_{d=1}^{\infty}\sum_{r=-1}^{\infty}\sum_{a=0}^{\infty}\tilde{C}_{r,d,a}t^r u^a\frac{x^d}{d!}\,.$$

Consider an element $\widetilde{\gamma}$ in $\mathbb{Q}[\{\kappa_{a,b}\}_{a\geq -1.b\geq 0}][[u,t,x]]$

$$\begin{aligned}\widetilde{\gamma} = &\sum_{w=2}^{\infty}\frac{1}{w(w-1)}\kappa_{-1,w}u^w t^{w-1} + \sum_{w=1}^{\infty}\frac{1}{2w}\kappa_{0,w}u^w t^w \\ &+ \sum_{w=0}^{\infty}\sum_{s=1}^{\infty}\binom{2s-2+w}{w}\frac{B_{2s}}{2s(2s-1)}\kappa_{2s-1,w}u^w t^{2s+w-1} \\ &+ \sum_{d=1}^{\infty}\sum_{r=-1}^{\infty}\sum_{w=0}^{\infty}\tilde{C}_{r,d,w}\kappa_{r,w}u^w t^{r+w}\frac{x^d}{d!}\,. \end{aligned} \tag{5.3}$$

Denote the $u^a t^r x^d$-coefficient of $\exp(-\widetilde{\gamma})$ by

$$[\exp(-\widetilde{\gamma})]_{u^a t^r x^d} \in \mathbb{Q}[\{\kappa_{a,b}\}_{a\geq -1.b\geq 0}]\,.$$

Since twisted κ-classes are defined over Jac_g, we have $\kappa_{-1,1} = 0$ and $\kappa_{0,0} = 2g-2$.

Proposition 5.2.2. For $g - 2d - 1 < r$ and $g \equiv r + 1 + w \mod 2$, we have

$$[\exp(-\widetilde{\gamma}))]_{u^w t^r x^d} = 0 \text{ in } \mathsf{R}^r_{(w)}(\mathsf{Jac}_g)\,.$$

Moreover when the parity condition does not hold the expression is trivially zero.

Proof. By Lemma 5.2.1 we have vanishing result on Jac_g. Chern classes of $\mathbb{F}_d$ and $\mathbb{F}'_d$ can be written in terms of tautological classes by Lemma 5.1.2 and Proposition 5.1.4. The u-refinement of relations follows from Proposition 2.2.1. □

5.3 Extending relations

Tautological relations in Section 5.2 can be extended by considering insertions on the stable Picard quotients. Over the universal curve $\pi : \mathfrak{C} \to \mathsf{P}_g(d)$, there are natural bivariant classes

$$s = c_1(S^\vee_{\mathfrak{C}}),\ \omega = c_1(\omega_\pi),\ \xi = c_1(\mathcal{L})$$

in $\mathsf{CH}^1(\mathfrak{C})$. Let $\mu : \mathsf{P}^{\mathrm{sm}}_g(d) \to \mathsf{Jac}_g$ be the forgetful map.

Lemma 5.3.1. Let F_1, F_2 be two $\mathbb{G}_m$-fixed loci of $\mathsf{P}^{\mathrm{sm}}_g(d)$ and U_1, U_2 be the universal curve of F_1 and F_2 respectively. Then

(i) $\pi_*(s\omega^b\xi^c|_{U_1}) = -\kappa_{b-1,c+1} + \psi_1^b\xi_1^c + \cdots + \psi_d^b\xi_d^c$,

(ii) $\pi_*(s\omega^b\xi^c|_{U_2}) = \psi_1^b\xi_1^c + \cdots + \psi_d^b\xi_d^c$.

Proof. Let $\theta_i \in \mathsf{CH}^1(U_1)$ be the class of i-th section. By the proof of Lemma 5.2.1, we have $S^\vee_{\mathfrak{C}}|_{U_1} \cong L^\vee \otimes \mathcal{O}_{U_1}(D)$, so $s|_{U_1} = -\xi + \theta_1 + \cdots + \theta_d$. Therefore we obtain the first identity. For the second identity, we use $S^\vee_{\mathfrak{C}}|_{U_2} \cong \mathcal{O}_{U_2}(D)$. □

We introduce an infinite number of formal variables $\mathbf{p} = \{p_{i,j}\}_{i,j\geq 1}$ indexed by two numbers. For a finite tuple of integers $\sigma = (\sigma_{i,j})$, we denote

$$\ell(\sigma) = \sum \sigma_{ij},\ o_1(\sigma) = \sum i\sigma_{i,j},\ o_2(\sigma) = \sum j\sigma_{i,j},\ |\mathrm{Aut}(\sigma)| = \prod_{ij} \sigma_{ij}!\,.$$

For $\sigma = (\sigma_{ij})$, we denote $\mathbf{p}^\sigma = \prod p_{ij}^{\sigma_{ij}}$.

Lemma 5.3.2. Let $\gamma^* := \log(\Phi)$. Denote

$$\Phi^{\mathbf{p}}(t,u,x) := \sum_\sigma \sum_{d=0}^{\infty} \prod_{i=1}^{d} \frac{1}{1 - it - ut} \frac{(-1)^d}{d!} \frac{x^d}{t^d} \frac{d^{\ell(\sigma)} t^{o_1(\sigma)} u^{o_2(\sigma)} \mathbf{p}^\sigma}{|\mathrm{Aut}(\sigma)|}\,.$$

Then

$$\log(\Phi^{\mathbf{p}}) = \exp\left(\sum_{i=1}^{\infty} p_{ij} t^i u^j x \frac{d}{dx}\right)\gamma^*.$$

Proof. By [41, Proposition 7], we have

$$\exp\Big(\sum_{i,j=1}^{\infty} p_{ij}t^i u^j x\frac{d}{dx}\Big)\,\log(\Phi) = \log\left(\exp\Big(\sum_{i,j=1}^{\infty} p_{ij}t^i u^j x\frac{d}{dx}\Big)\,\Phi\right).$$

The right hand side matches with $\log(\Phi^{\mathbf{p}})$. □

We denote $\widetilde{C}_{r,w,d}$ by coefficient of the expansion of $\log(\Phi^{\mathbf{p}})$

$$\log(\Phi^{\mathbf{p}}) =: \sum_{\sigma}\sum_{d=0}^{\infty}\sum_{r=-1}^{\infty}\sum_{w=0}^{\infty}\widetilde{C}_{r,w,d}(\sigma)t^r u^w \frac{x^d}{d!}\mathbf{p}^{\sigma}\,.$$

By Lemma 5.3.2 the coefficients $\widetilde{C}_{r,w,d}(\sigma)$ only depends on $\widetilde{C}_{r,w,d}$. By Wick's formula, we obtain the following combinatorial expression.

Proposition 5.3.3. Let s,ω,ξ be classes on $U/\mathfrak{C}_g^d$ as defined above. Then we have

$$\sum_{d=0}^{\infty}\sum_{r\geq 0}\sum_{a\geq -1}\pi^d_*\Big(\exp\Big(\sum_{i,j=1}^{\infty} p_{ij}t^{i+j}u^j\mu_*(s\omega^i\xi^j)\Big)\cdot c_r(\mathbb{F}_d)t^r\Big)\frac{1}{t^d}\frac{x^d}{d!} = \exp(-\gamma^{\mathsf{PQ}})$$

in $\mathsf{CH}_*(\mathfrak{C}_g)$ where

$$\begin{aligned}\gamma^{\mathsf{PQ}} :=& \sum_{w=2}^{\infty}\frac{1}{w(w-1)}\kappa_{-1,w}t^{w-1}(-u)^w + \sum_{w=1}^{\infty}\frac{1}{2w}\kappa_{0,w}t^w(-u)^w \\ &+\sum_{w=0}^{\infty}\sum_{s=1}^{\infty}\binom{2s+w-2}{w}\frac{B_{2s}}{2s(2s-1)}\kappa_{2s-1,w}\,t^{2s+w-1}(-u)^w \\ &+\sum_{\sigma}\sum_{d=1}^{\infty}\sum_{r=-1}^{\infty}\sum_{w=0}^{\infty}\widetilde{C}_{r,w,d}\,\kappa_{r+o_1(\sigma)+o_2(\sigma),w+o_2(\sigma)}t^r(-u)^w\frac{x^d}{d!}\frac{d^{\ell(\sigma)}t^{o_1(\sigma)+o_2(\sigma)}u^{o_2(\sigma)}\mathbf{p}^{\sigma}}{|\mathrm{Aut}(\sigma)|}\,.\end{aligned} \tag{5.4}$$

To get extended tautological relations on Jac_g, we consider the generating series

$$\begin{aligned}\bar{\gamma}^{\mathsf{PQ}} :=& \sum_{w=2}^{\infty}\frac{1}{w(w-1)}\kappa_{-1,w}(-t)^{w-1}u^w + \sum_{w=1}^{\infty}\frac{1}{2w}\kappa_{0,w}(-t)^w u^w \\ &+\sum_{w=0}^{\infty}\sum_{s=1}^{\infty}\binom{2s+w-2}{w}\frac{B_{2s}}{2s(2s-1)}\kappa_{2s+w-1,w}\,(-t)^{2s+w-1}u^w \\ &+\sum_{\sigma}\sum_{d=1}^{\infty}\sum_{r=-1}^{\infty}\sum_{w=0}^{\infty}\widetilde{C}_{r,w,d}\kappa_{r+o_1(\sigma)+o_2(\sigma),w+o_2(\sigma)}(-t)^r u^w\frac{x^d}{d!}\frac{d^{\ell(\sigma)}t^{o_1(\sigma)+o_2(\sigma)}u^{o_2(\sigma)}\mathbf{p}^{\sigma}}{|\mathrm{Aut}(\sigma)|}\,.\end{aligned}$$

The following relations will be the main input for the Faber-Zagier type relation on Jac_g.

Theorem 5.3.4. In $\mathsf{CH}^r(\mathsf{Jac}_g)$, we have relations

$$\begin{aligned}&\Big[\exp\big(-\sum_{r=-1}^{\infty}\sum_{w=1}^{\infty}\kappa_{r,w}t^{r+w}u^w p_{r+1,w-1}\big)\cdot\exp(-\gamma^{\mathsf{PQ}})\Big]_{t^r u^w x^d \mathbf{p}^\sigma}\\ &=(-1)^g\Big[\exp\big(-\sum_{r=-1}^{\infty}\sum_{w=0}^{\infty}\kappa_{r,w}t^{r+w}u^w p_{r+1,w}\big)\cdot\exp(-\bar{\gamma}^{\mathsf{PQ}})\Big]_{t^r u^w x^d \mathbf{p}^\sigma}\end{aligned} \tag{5.5}$$

when $g-2d-1+o_1(\sigma)+o_2(\sigma)<r$.

Proof. Let $\pi:\mathfrak{C}\to\mathsf{P}_g^{\mathrm{sm}}(d)$ be the universal curve and $\mu:\mathsf{P}_g^{\mathrm{sm}}(d)\to\mathsf{Jac}_g$ be the contraction map. Consider a $\mathbb{G}_m$-action on $\mathsf{P}_g^{\mathrm{sm}}(d)$ as in Section 5.2. This action canonically lifts to classes s,ω,ξ. For each $a_i,b_i\in\mathbb{Z}_{\geq 0}$ and $c>0$ we analyze the following system of relations

$$\mu_*\Big(\prod_{i=1}^{\ell}\pi_*(s\omega^{a_i}\xi^{b_i})\cdot 0^c\cap[\mathsf{P}_g^{\mathrm{sm}}(d)]^{\mathrm{vir}}\Big)=0 \tag{5.6}$$

in $\mathsf{CH}_*(\mathsf{Jac}_g)$. By $\mathbb{G}_m$ -localization, and following the proof of Lemma 5.2.1, the contribution of the first fixed locus is

$$\mu_*\Big(\prod_{i=1}^{\ell}\pi_*((-\xi+s)\omega^{a_i}\xi^{b_i})\cdot c_{g-d-1+c}(\mathbb{F}_d)\Big)$$

and the contribution of the second fixed locus is

$$(-1)^{g-d-1}\Big[\mu_*\Big(\prod_{i=1}^{\ell}\pi_*((-1+s)\omega^{a_i}\xi^{b_i})\cdot c_{g-d-1+c}(\mathbb{F}'_d)\Big)\Big]_{g-d-1+\sum_i a_i+b_i+c}.$$

The factor $-\xi$ of $-\xi+s$ in the first contribution induces the overall factor

$$\exp\big(-\sum_{r=-1}^{\infty}\sum_{w=1}^{\infty}\kappa_{r,w}t^{r+w}u^w p_{r+1,w-1}\big).$$

Similarly, the factor -1 of $-1+s$ induces the overall factor $\exp\big(-\sum_{r=-1}^{\infty}\sum_{w=0}^{\infty}\kappa_{r,w}t^{r+w}u^w p_{r+1,w}\big)$. Taking the generating series and applying Proposition 5.3.3 we get the result. $\square$

5.4 Examples

We give some examples from Proposition 5.2.2. Our computation is done by Mathematica.

Example 5.4.1. By [23] the rational cohomology group $H^{2r}(\mathsf{Pic}^0(\mathcal{M}_g))$ should be freely generated by twisted κ-classes when

$$r \leq \frac{2g-3}{6}\,. \tag{5.7}$$

Therefore Proposition 5.2.2 should give trivial relations when r lies in (5.7). We checked that our result matches with (5.7) up to $g \leq 19$ and $d \leq 6$.

Example 5.4.2. By Proposition 2.2.2 we have $(\kappa_{-1,2})^{g+1} = 0$. We checked this result up to $g \leq 6$.

Example 5.4.3. Let $g = 3, r = 2, w = 2$. Then we get the following relation

$$6\kappa_{0,1}^2 - 24\kappa_{0,2} + 5\kappa_{-1,2}\kappa_{1,0} = 0\,.$$

This relation indeed follows from a tautological relations on $\mathcal{M}^{\mathrm{rt}}_{3,2}$

$$\psi_1\psi_2 - \frac{5}{6}(\psi_1^2 + \psi_2^2) - \frac{25}{6}[D,\psi] + \frac{5}{6}[D,\kappa_{1,0}] = 0\,, \tag{5.8}$$

where D is the boundary divisor of $\mathcal{M}^{\mathrm{rt}}_{3,2}$. The relation (5.8) can be easily obtained by [14]. We pullback (5.8) to $\mathfrak{Pic}^{\underline{0}}(\mathcal{M}^{\mathrm{rt}}_{3,2})$, multiply with $\xi_1\xi_2$ and pushforward to $\mathfrak{Pic}^0(\mathcal{M}_{3,0})$. Then we get the result.

6 Rewriting the relations

6.1 Properties of the power series Φ

We study properties of the generating series

$$\Phi(t,u,x) := \sum_{d=0}^{\infty}\prod_{i=1}^{d}\frac{1}{(1-it-u)}\frac{(-1)^d}{d!}\frac{x^d}{t^d}\,.$$

We consider the normalized power series

$$\begin{aligned}\Gamma = -t\Big(&\sum_{w=2}^{\infty}\frac{1}{w(w-1)}u^w t^{-1} + \sum_{w=1}^{\infty}\frac{1}{2w}u^w \\ &+ \sum_{w=0}^{\infty}\sum_{s=1}^{\infty}\binom{2s-2+w}{w}\frac{B_{2s}}{2s(2s-1)}u^w t^{2s-1} + \gamma^*\Big)\,.\end{aligned} \tag{6.1}$$

The first three terms of (6.1) are chosen to match the initial condition. Let $\gamma^* := \log(\Phi)$.

Lemma 6.1.1. The formal power series (6.1) satisfies the following differential equation

$$tx\Gamma_{xx} = (1-u-t)\Gamma_x + x(\Gamma_x)^2 - 1\,. \tag{6.2}$$

Proof. Direct computation shows that Φ satisfies the following differential equation

$$\frac{d}{dx}\big((1-u)\Phi - tx\frac{d}{dx}\Phi\big) = -\frac{1}{t}\Phi.$$

Applying $\gamma^*_x = \frac{d}{dx}\gamma^* = \frac{\Phi}{\Phi_x}$, we get the result. □

Lemma 6.1.2. The function Γ_x is determined recursively as a power series in x. Namely, there exist integers $q_{k,j}$ which can be specified recursively such that

$$\Gamma_x = \frac{-(1-u)+\sqrt{(1-u)^2+4x}}{2x} + \frac{t}{(1-u)^2+4x} + \sum_{k=1}^{\infty}\sum_{j=0}^{k} t^{k+1} q_{k,j}(-x)^j\big((1-u)^2+4x\big)^{-k/2-j-1}.$$

Proof. We expand the function Γ_x in powers of t

$$\Gamma_x = \sum_{k=0}^{\infty} t^k A_k(x), \text{ where } A_k(x) \in \mathbb{Q}(u)[[x]].$$

Taking $t=0$, A_0 should satisfy $xA_0^2 + (1-u)A_0 - 1 = 0$ and hence we can rewrite

$$\Gamma_x = A_0(x) + tZ(t,x), \text{ where } A_0 = \frac{-(1-u)+\sqrt{(1-u)^2+4x}}{2x}.$$

By Lemma 6.1.1 the function Z should satisfy the following differential equation

$$Z = \big((1-u)^2+4x\big)^{-1} + t\big((1-u)^2+4x\big)^{-1/2}\cdot\big(xZ_x - xZ^2 + Z\big)\,.$$

Now we set $Q := \big((1-u)^2+4x\big)\cdot Z$ and take the change of variable

$$y = \frac{-x}{(1-u)^2+4x}\,. \tag{6.3}$$

Since $1+4y = \frac{(1-u)^2}{(1+u)^2+4x}$, the power series Q satisfies the equation

$$Q = 1 + t\frac{(1+4y)^{1/2}}{1-u}\cdot\big((1+4y)(yQ)_y + yQ^2\big)\,. \tag{6.4}$$

Replacing $\tilde{t} := t\cdot\frac{(1+4y)^{\frac{1}{2}}}{1-u}$ and ny the induction on powers of $\tilde{t}$, the series (6.4) has a formal power series expansion

$$Q = \sum_{k=0}^{\infty}\sum_{j=0}^{k}\tilde{t}^k q_{k,j} y^j,\ q_{k,j} \in \mathbb{Z}_{\geq 0}. \tag{6.5}$$

The proof of [30, Lemma 3.1] shows that $q_{k,j}$ are determined by the recursion

$$q_{k,j} = (2k+4j-2)q_{k-1,j-1} + (j+1)q_{k-1,j} + \sum_{m=0}^{k-1}\sum_{\ell=0}^{j-1} q_{m,\ell}q_{k-1-m,j-1-\ell}.$$

Therefore we get the result. $\square$

Integrating Γ_x in x, we get the formula in generating series.

Lemma 6.1.3. The formal power series (6.1) has the form

$$\Gamma(t,x,u) = \Gamma(0,x,u) + \frac{t}{4}\log(\frac{(1-u)^2+4x}{(1-u)^2}) - \sum_{k=1}^{\infty}\sum_{j=0}^{k}\frac{t^{k+1}}{(1-u)^k}c_{k,j}(-x)^j((1-u)^2+4x)^{-j-\frac{k}{2}}$$

where the coefficients $c_{k,j}$ are determined by

$$q_{k,j} = (2k+4j)c_{k,j} + (j+1)c_{k,j+1}, k \geq 1, k \geq j \geq 0\,.$$

Proof. Following notations in the proof of Lemma 6.1.2, we have

$$\Gamma_x = \Gamma_x(0,x,u) + t((1-u)^2+4x)Q\,.$$

Take the change of variables (6.3). Since $\frac{dy}{dx} = \frac{-(1-u)^2}{((1-u)^2+4x)^2} = -(1-u)^2(1+4y)^2$ and $Z = ((1-u)^2+4x)^{-1}Q = (1-u)^{-2}(1+4y)\cdot Q$, we get

$$\Gamma_y = \Gamma_y(0,y,u) - t(1+4y)^{-1}Q. \tag{6.6}$$

After applying (6.5)

$$(1+4y)^{-1}Q(t,y,u) = \frac{1}{1+4y} + \sum_{k=1}^{\infty} t^k\frac{(1+4y)^{k/2-1}}{(1-u)^k}\sum_{j=0}^{k} q_{k,j}y^j,$$

to (6.6) and integrating, we get

$$\Gamma = \Gamma(0,y,u) - \frac{t}{4}\log(1+4y) - \sum_{k=1}^{\infty}\frac{t^{k+1}}{(1-u)^k}\big(\sum_{j=0}^{k}(1+4y)^{k/2}\tilde{c}_{k,j}y^j + \alpha_k\big)$$

for integration constants α_k. Taking $y=0$, we get

$$\Gamma(t,0,u) = \sum_{k=1}^{\infty}\frac{t^{k+1}}{(1-u)^k}\tilde{c}_{k,0} + \alpha_k.$$

After substituting $u=0$, the function Γ specializes to the function G in [30] and we have $\tilde{c}_{k,0} = c_{k,0} = \frac{B_{k+1}}{k(k+1)}$ by [30, Lemma 3.2, Remark. 3.5]. Therefore the normalization (6.1) shows that $\alpha_k = 0$ for all k and $\tilde{c}_{k,j} = c_{k,j}$. Replacing the variable y back to x, we get the result. $\square$

Consider the change of variables

$$y = \frac{-x}{(1-u)^2+4x}, \; v = \frac{t(1+4y)^{1/2}}{(1-u)} = \frac{t}{((1-u)^2+4x)^{1/2}}\,. \tag{6.7}$$

Under this change of variables the function Γ transfers to $\hat{\Gamma}(v,u,y) := \Gamma(t,u,x)$. By Lemma 6.1.3 $\hat{\Gamma}$ has a simple form

$$\frac{1}{t}\hat{\Gamma}(v,u,y) = \frac{1}{t}\hat{\Gamma}(0,u,y) - \frac{1}{4}\log(1+4y) - \sum_{k=1}^{\infty}\sum_{j=0}^{k}\frac{1}{(1-u)^k}c_{k,j}v^k y^j\,. \tag{6.8}$$

6.2 Rewriting tautological relations

Proposition 6.2.1. The relation for r, n,

$$\Big[\sum_{l\geq 1, m\geq 0, l+m-1\leq \lfloor\frac{n}{2}\rfloor}(-1)^{r+n-l+1}4^{n-m-l+1}\mathsf{P}_l\mathrm{Dec}_{(n,m)}\Big[\exp\Big(-\sum_{k=1}^{\infty}a[k,k]x^k\Big)\Big]_{x^{r+m}}\Big]_{a[k,k]=c_{k,k}} = 0$$

holds when $g-1 < 3r+2n$ and $g \equiv r+1 \pmod 2$.

For the proof of above proposition, we need some explanations. A (k,n,m)-decoration is the ordered set of non-negative integers

$$\sigma = (n_1, n_2, \ldots, n_k, a_1, a_2, \ldots, a_j, b_1, b_2, \ldots, b_m)$$

such that

- $a_i \neq 0$ and $b_i \neq 0$.
- $a_i \leq a_{i+1}$ and $b_i \leq b_{i+1}$.
- $\sum_{i=0}^{k} n_i + \sum_{i=1}^{j} a_j + \sum_{i=1}^{m} b_i = n$.

For $\sigma = (n_1, n_2, \ldots, n_k, a_1, a_2, \ldots, a_j, b_1, b_2, \ldots, b_m)$, we define

$$|\mathrm{Aut}(\sigma)| = j!m!\,.$$

Let $\mathsf{Dec}(k,n,m)$ be the set of all (k,n,m)-decorations.

For $i,j \geq 0$, consider formal variables $a[i,j]$. For $\prod_{i=1}^{k} a[\alpha_i,\beta_i]$, a (n,m)-decoration of $\prod_{i=1}^{k} a[\alpha_i,\beta_i]$ is $\prod_{i=1}^{k} a[\alpha_i,\beta_i]$ with a choice of (k,n,m)-decoration. For $\sigma \in \mathsf{Dec}(k,n,m)$, we denote the (n,m)-decoration of $\prod_{i=1}^{k} a[\alpha_i,\beta_i]$ associated to σ by $\sigma(\prod_{i=1}^{k} a[\alpha_i,\beta_i])$. We define

$$\mathsf{Dec}_{(n,m)}(\prod_{i=1}^{k} a[\alpha_i,\beta_i])$$

to be the formal sum

$$\sum_{\sigma \in \mathsf{Dec}(k,n,m)} \sigma(\prod_{i=1}^{k} a[\alpha_i, \beta_i])$$

of all (n,m)-decoration of $\prod_{i=1}^{k} a[\alpha_i, \beta_i]$.

Let Θ_l be functions on $\mathbb{Z}^2_{\geq 0}$ characterized by the following equations,

$$\begin{aligned}
\Theta_l(1,0) =& l\Big(\prod_{i=2}^{l} \frac{4i-6}{i}\Big) = \prod_{i=1}^{l-1} \frac{4i-2}{i} \\
\Theta_l(2,0) =& 2^{2(l-1)} \\
\Theta_l(2k+1,0) =& \Theta_l(2k-1,0)\frac{2l+2k-3}{2k-1} \\
\Theta_l(2k+2,0) =& \Theta_l(2k,0)\frac{l+k-1}{k} \\
\Theta_l(\alpha,\beta) =& \Theta_l(\alpha+2\beta,0)\,.
\end{aligned}$$

For a decoration of single variable $a[\overset{n}{\alpha}, \beta]$ or empty variables $\emptyset_0$, $\emptyset_{-1}$, we define the function P_l by (if $n=0$,$l=1$ then $\mathsf{P}_1(a[\alpha, \beta]) = 1$)

$$\begin{aligned}
\mathsf{P}_l\Big(a[\overset{n}{\alpha}, \beta]\Big) =& \Theta_l(\alpha,\beta) \prod_{i=1}^{n-(2l-2)} \frac{\alpha+2\beta+i+2l-3}{i} \cdot \kappa_{\alpha,n} \\
\mathsf{P}_l\Big(\overset{n}{\emptyset_0}\Big) =& \frac{2^{2l-4}}{l-1} \prod_{i=1}^{n-(2(l-1))} \frac{2(l-1)-1+i}{i} \cdot \kappa_{0,n} \\
\mathsf{P}_l\Big(\overset{n}{\emptyset_{-1}}\Big) =& \frac{\prod_{i=2}^{2l-1}(n-i)}{((l-1)!)^2\, n\,(n-1)} \cdot \kappa_{-1,n}.
\end{aligned}$$

For a decoration σ of $\prod_i a[\alpha_i, \beta_i]$, let $\sigma = \prod_{i=1}^{k} \sigma_i$ be the decomposition into the decorations of single variables. Then we define the function P_l by

$$\mathsf{P}_l(\sigma) = \frac{1}{\mathrm{Aut}(\sigma)}\Big[\prod_{i}^{k} \sum_{j\geq 0} x^j \mathsf{P}_{j+1}(\sigma_i)\Big]_{x^{l-1}} \tag{6.9}$$

where x is a formal variable.

Define the series in s, y

$$\mathsf{E}[r,d,c] := \Big[(-1)^d (1+4y)^{d-1+r/2-\kappa[0,0]/4+c/4}\, \mathrm{Exp}(\widetilde{\Gamma})\Big]_{s^r y^d}$$

where

$$\widetilde{\Gamma} = \sum_{i,j\geq 0} a[\alpha_i, \beta_j]\, s^i y^j \,.$$

Consider the Γ obtained in Lemma 6.1.3. Let Γ^{formal} to be

$$\Gamma(0,u,x) + t\int \frac{1}{(1-u)^2+4x}dx + \sum_{k=0}^{\infty}\sum_{a=0}^{k+1}\sum_{j=0}^{\infty} t^{k+1}(1+4x)^{-j-k/2}(-x)^j a[k,j]\,.$$

Let $\mathsf{P}_{-1n}(x)$ be the polynomial of degree $\lfloor\frac{n-2}{2}\rfloor$ in x such that

$$\begin{aligned}
[\mathsf{P}_{-1n}(x)]_{x^0} &= \frac{1}{n(n-1)}\,,\\
[\mathsf{P}_{-1n}(x)]_{x^l} &= \frac{\prod_{i=2}^{2l+1}(n-i)}{(l!)^2 n(n-1)}\,.
\end{aligned}$$

Let $\mathsf{P}_{0n}(x)$ be the polynomial of degree $\lfloor\frac{n}{2}\rfloor$ in x such that

$$[\mathsf{P}_{0n}(x)]_{x^0} = \frac{1}{2n}\,, \tag{6.10}$$

$$[\mathsf{P}_{0n}(x)]_{x^l} = \frac{2^{2l-2}}{l}\prod_{i=1}^{n-2l}\frac{2l-1+i}{i}\,. \tag{6.11}$$

Let P_{kn} be the polynomial of degree $\lfloor\frac{n}{2}\rfloor$ in x such that

$$[\mathsf{P}_{kn}(x)]_{x^l} = \Theta_{l+1}(k,0)\prod_{i=1}^{n-2l}\frac{k+i+2l-1}{i}\,. \tag{6.12}$$

We will use the following identities whose proofs are elementary,

Lemma 6.2.2. For $a\in\mathbb{Q}$, we have

$$\begin{aligned}
[(1+4x)^{\frac{a}{4}}]_{x^k} =& [(-1)^k(1+4x)^{\frac{4(k-1)-a}{4}}]_{x^k} = \frac{1}{k!}\prod_{i=0}^{k-1}(a-4i)\,,\\
(1+4x)^{-a-1} =& \sum_{k=0}^{\infty} x^k[(-1)^k(1+4x)^{a+k}]_{x^k}\,.
\end{aligned}$$

Lemma 6.2.3. We have

$$\begin{aligned}
\frac{1}{t}\Gamma^{\text{formal}} =& \frac{1}{t}\Big(f_0(x)+f_1(x)u+\sum_{n=2}^{\infty}\frac{\mathsf{P}_{-1n}(x)u^n}{(1+4x)^{n-\frac{3}{2}}}\Big) + \frac{1}{4}\mathrm{Log}(1+4x) + \sum_{n=1}^{\infty}\frac{\mathsf{P}_{0n}(x)u^n}{(1+4x)^n}\\
&-\sum_{k=1}^{\infty}\sum_{j=0}^{k}\frac{a[k,j]t^k(-x)^j}{((1-u)^2+4x)^{k/2+j}}\,.
\end{aligned}$$

Proof. First, we show

$$\int \frac{1}{(1-u)^2+4x}dx = \frac{1}{4}\mathrm{Log}(1+4x) + \sum_{n=1}^{\infty} \frac{\mathsf{P}_{0n}(x)u^n}{(1+4x)^n}\,. \tag{6.13}$$

Since

$$\frac{1}{(1-u)^2+4x} = \sum_{n=0}^{\infty} \frac{\mathsf{P}_{2n}(x)u^n}{(1+4x)^{1+n}}\,,$$

from Lemma 6.2.4, we need to prove the following equation

$$\frac{d}{dx}\Big(\frac{\mathsf{P}_{0n}(x)}{(1+4x)^n}\Big) = \frac{\mathsf{P}_{2n}(x)}{(1+4x)^{n+1}}$$

which is easy to check from the definitions of P_{0n} and P_{2n} in (6.10) and (6.12). Now observing constant terms with respect to x in the both sides of equation (6.13) are $\sum_{n=1}^{\infty}\frac{1}{2n}u^n$, we obtain equation (6.13).

Secondly, we show

$$\int \frac{-(1-u)+\sqrt{(1-u)^2+4x}}{2x}dx = f_0(x) + f_1(x)u + \sum_{n=2}^{\infty} \frac{\mathsf{P}_{-1n}(x)u^n}{(1+4x)^{n-\frac{3}{2}}}\,. \tag{6.14}$$

To show above equation, let us write

$$\sqrt{(1-u)^2+4x} = \sum_{n=0}^{\infty} g_n(x)u^n\,.$$

From Lemma 6.2.4 and the following differential equation

$$\frac{d\sqrt{(1-u)^2+4x}}{du} = \frac{1-u}{\sqrt{(1-u)^2+4x}}\,,$$

we obtain

$$f_n = \frac{1}{n}\frac{\mathsf{P}_{1\,n-1} - (1+4x)\mathsf{P}_{1\,n-2}}{(1+4x)^{-1/2+n}}\,.$$

Observing that the constant terms with respect to x in the both side of equation (6.14) are $\sum_{n=2}^{\infty}\frac{1}{n(n-1)}$,the proof of (6.14) follows from the following differential equation,

$$\frac{d}{dx}\Big(\frac{\mathsf{P}_{-1n}(x)}{(1+4x)^{n-3/2}}\Big) = \frac{\mathsf{P}_{1\,n-1}(x) - (1+4x)\mathsf{P}_{1\,n-2}(x)}{2nx(1+4x)^{-1/2+n}}\,.$$

The above equation is easy to check from the definition of P_{-1n} and $\mathsf{P}_{1,n}$. We leave the details to the readers. $\square$

Lemma 6.2.4. For $k \geq 1$, we have

$$\Big(\frac{1}{(1-u)^2+4x}\Big)^{\frac{k}{2}} = \sum_{n=0}^{\infty} \frac{\mathsf{P}_{kn}(x)u^n}{(1+4x)^{\frac{k}{2}+n}}\,.$$

Proof. We first prove the statement for $k = 1$. Let

$$\mathsf{P}_1 := \Big(\frac{1}{(1-u)^2+4x}\Big)^{\frac{1}{2}} = f_0 + f_1 u + f_2 u^2 + f_3 u^3 + \cdots$$

be the series expansion in u. Since P_1 satisfy the following differential equation

$$((1-u)^2+4x)\frac{d\mathsf{P}_1}{du} - (1-u)\mathsf{P}_1 = 0,$$

we have

$$(n+1)f_n - (2n+3)f_{n+1} + (n+2)(1+4x)f_{n+2} = 0\,.$$

If we write $f_n = \frac{\mathsf{P}_{1n}}{(1+4x)^{1/2+n}}$, we obtain

$$(1+4x)(n+1)\mathsf{P}_{1n} - (2n+3)\mathsf{P}_{1\,n+1} + (n+2)\mathsf{P}_{1\,n+2} = 0\,.$$

We can easily check the above equation from the definition of P_{1n} in (6.12). We can similarly prove the statement for $k \geq 2$.

$\square$

For a formal power series in t, u,

$$F = \sum_{r \geq -1, k \geq 0} a_{r,k} t^r u^k$$

with coefficients in a ring, let

$$\{F\}_\kappa = \sum_{r \geq -1, k \geq 0} a_{r,k} \kappa_{r,k} t^r u^k$$

be the series with κ-classes inserted. We set

$$\kappa_{-1,0} = \kappa_{-1,1} = 0\,.$$

Denote by $\mathsf{D}_n[r,d]$ the relation given by

$$[\exp(\{\frac{1}{t}\Gamma^{\mathrm{formal}}\}_\kappa)]_{t^r x^d u^n}\,.$$

Lemma 6.2.5. For $r, d, n \in \mathbb{Z}$, we have

$$\mathsf{D}_n[r,d] = \sum_{l\geq 1, m\geq 0} (-1)^{d-l+1} \mathsf{P}_l\Big(\mathrm{Dec}_{(n,m)}\, \mathsf{E}[r+m, d-(l-1), 4n-6m]\Big)\,.$$

Proof. If we replace c_{kj} with $a[k,j]$ in the equation in Lemma 6.2.3, we can write

$$[\exp(\{\frac{1}{t}\Gamma^{\mathrm{formal}}\}_\kappa)]_{t^r u^n}$$
$$= \Big[(1+4x)^{\frac{1}{4}\kappa_{0,0}} \cdot \exp\Big(\{\frac{1}{t}\Big(\sum_{n=2}^{\infty} \frac{\mathsf{P}_{-1n}(x)u^n}{(1+4x)^{n-\frac{3}{2}}}\Big) + \sum_{n=1}^{\infty} \frac{\mathsf{P}_{0n}(x)u^n}{(1+4x)^n} - \sum_{k=1}^{\infty}\sum_{j=0}^{k} \frac{a[k,j]t^k(-x)^j}{((1-u)^2+4x)^{k/2+j}}\}_\kappa\Big)\Big]_{t^r u^n}$$

as the sum of the following form

$$f_{\alpha_i,\beta_i}(x) \cdot \sigma\Big(\prod_{i=1}^{k} a[\alpha_i,\beta_i] \cdot \kappa_{\alpha_i,n_i}\Big) \prod_{i=1}^{j} \kappa_{0,a_i} \prod_{i=1}^{m} \kappa_{-1,b_i}$$

where $f_{\alpha_i,\beta_i}(x)$ is a rational function in x and

$$\sigma = (n_1, n_2, \ldots, n_k, a_1, a_2, \ldots, a_j, b_1, b_2, \ldots, b_m)$$

is a (k,n,m)-decoration with $\sum_{i=1}^{k} \alpha_i = r$. Moreover, using Lemma 6.2.4, we can uniquely write $f_{\alpha_i,\beta_i}(x)$ as linear combination of the forms

$$\frac{x^q}{(1+4x)^p}\,.$$

We can also write the right-hand side of the equation of Lemma 6.2.5 as a linear sum of the forms

$$\sigma\Big(\prod_{i=1}^{k} a[\alpha_i,\beta_i] \cdot \kappa_{\alpha_i,n_i}\Big) \prod_{i=1}^{j} \kappa_{0,a_i} \prod_{i=1}^{m} \kappa_{-1,b_i}\,.$$

Now by comparing the polynomial $\mathsf{P}_{ij}(x)$ and the values of $\mathsf{P}_l(\sigma)$ defined in (6.9), we can easily check that the coefficients of $\sigma\Big(\prod_{i=1}^{k} a[\alpha_i,\beta_i]\cdot\kappa_{\alpha_i,n_i}\Big)\prod_{i=1}^{j}\kappa_{0,a_i}\prod_{i=1}^{m}\kappa_{-1,b_i}$ in the both side of the equation in Lemma 6.2.5 match using the identity in Lemma 6.2.2,

$$(1+4x)^{-a-1} = \sum_{k=0}^{\infty} x^k[(-1)^k(1+4x)^{a+k}]_{x^k}\,.$$

□

For $r, d, n \in \mathbb{Z}$, we define

$$\mathsf{R}(r,d,n) = \mathsf{D}_n[r,d] + \sum_{i=1}^{r} \frac{(-1)^i}{i!} \prod_{k=1}^{i} \Big(\kappa_{0,0} - 6r - 4n + 4 + 4(k-1)\Big) \mathsf{D}_n[r, d-i]\,. \tag{6.15}$$

Denote by $E\Gamma[r,k]$

$$\Big[\mathrm{Exp}(\sum_{i,j} a[\alpha_i, \beta_j]\, s^i y^j)\Big]_{s^r y^k}$$

Then we have

$$\mathsf{E}[r,j,c] = \Big[(-1)^j (1+4y)^{r/2+j-1-\kappa_{0,0}/4+c/4} \sum_{i=0}^{j} E\Gamma[r,i]\, y^i\Big]_{r,j}. \tag{6.16}$$

Proof of 6.2.1. By applying (6.16) to Lemma 6.2.5, we obtain

$$\mathsf{D}_n[r,d] = (-1)^{d-l+1} \sum_{l,m} \Big[(1+4y)^{r/2+d-1-k_{0,0}/4+n-m-l+1} \sum_{i}^{d-l+1} \mathsf{P}_l \mathrm{Dec}_{(n,m)} E\Gamma[r+m,i] y^i\Big]_{r+m,d-l+1}.$$

Therefore, we get

$$\begin{aligned}
&\mathsf{R}(r,d,n) = \Big[(1+4y)^{(-\kappa_{0,0}+6r+4n-4)/4} \sum_{j=0}^{d} \mathsf{D}_n[r,j] y^j\Big]_d \\
&= \Big[(1+4y)^{(-\kappa_{0,0}+6r+4n-4)/4} \sum_{j=0}^{d} (-1)^{j-l+1} \sum_{l,m} [(1+4y)^{r/2+j-1-\kappa_{0,0}/4+n-m-l+1} \\
&\sum_{i=0}^{j-l+1} \mathsf{P}_l \mathrm{Dec}_{(n,m)} E\Gamma[r+m,i] y^i]_{r+m,j-l+1} y^j\Big]_d \\
&= \Big[(1+4y)^{(-\kappa_{0,0}+6r+4n-4)/4} \sum_{l,m} [\sum_{j=0}^{d} (-1)^{j-l+1} (1+4y)^{j-1+r/2-\kappa_{0,0}/4+n-m-l+1} \\
&\sum_{i=0}^{j-l+1} \mathsf{P}_l \mathrm{Dec}_{(n,m)} E\Gamma[r+m,i]]_{r+m,j-l+1-i} y^{j-l+1-i} y^{l+i-1}\Big]_d \\
&= \sum_{l,m} \Big[\sum_{i} (-1)^i (1+4y)^{(-\kappa_{0,0}+6r+4n-4)/4} (1+4y)^{-i+m-n-r/2+\kappa_{0,0}/4} \mathsf{P}_l \mathrm{Dec}_{(n,m)} E\Gamma[r+m,i] y^{l+i-1}\Big]_d \\
&= \sum_{l,m} \Big[\sum_{i} (-1)^i (1+4y)^{m+r-i-1} \mathsf{P}_l \mathrm{Dec}_{(n,m)} E\Gamma[r+m,i]\Big]_{d-l-i+1}.
\end{aligned}$$

We used Lemma 6.2.2 in the second to last equation in above equations. It is easy to check that we have the following conditions

(i) $m \leq \lfloor \frac{n}{2} \rfloor$,

(ii) $l \leq \lfloor \frac{n}{2} \rfloor - m + 1$.

Recall that $E\Gamma[r+m, i] = 0$ for $i > r+m$. Therefore, if $d+2 > m+r+l$, we have

$$\begin{aligned}\mathsf{R}(r,d,n) &= \sum_{l+m \leq \lfloor \frac{n}{2} \rfloor + 1} \Big[\sum_i (-1)^i (1+4y)^{m+r-i-1} \mathsf{P}_l \mathrm{Dec}_{(n,m)} E\Gamma[r+m, i] \Big]_{d-l-i+1} \\ &= \sum_{l+m \leq \lfloor \frac{n}{2} \rfloor + 1} (-1)^{d-l+1} 4^{d-l-m-r+1} \mathsf{P}_l \mathrm{Dec}_{(n,m)} E\Gamma[r+m, r+m] \,.\end{aligned}$$

Now the proof of Proposition 6.2.1 follows since the relation in Proposition 6.2.1 is $\mathsf{R}(r, r+n, n)$. It is easy to check from Proposition 5.2.2 and (6.15) that $\mathsf{R}(r, r+n, n)$ is relation when $g - 1 < 3r + 2n$ and $g \equiv r + 1 (\mathrm{mod}\ 2)$. □

7 Comparison with other results

7.1 Universal double ramification formula

In [5], a system of tautological relations on the universal Picard stack is obtained along the study of Pixton's formula for the universal double ramification cycle. The η-class corresponds to $\kappa_{-1,2}$ in our convention. We denote $\mathsf{P}^c_{g,B}$ by the codimension c part of Pixton's formula [5, Section 0.3].

Theorem 7.1.1 ([5])**.** Let $B = (b_1, \cdots, b_m) \in \mathbb{Z}^m$ with $\sum_{i=1}^m b_i = 0$. If $c > g$, then $\mathsf{P}^c_{g,B} = 0$ in $\mathsf{CH}^c_{\mathrm{op}}(\mathfrak{Pic}^0_{g,m})$.

The proof involves the double ramification relations with target variety [4] and invariance properties of the Pixton's formula.

Proposition 7.1.2. After restricting to the relative Picard scheme $\mathsf{Pic}^0(\mathcal{C}_{g,m}/\mathcal{M}^{\mathrm{ct}}_{g,m})$ of total degree zero line bundles over $\mathcal{M}^{\mathrm{ct}}_{g,m}$, Theorem 7.1.1 follows from Proposition 2.2.2.

Proof. The statement for $m = 0$ follows from the case for $m = 1$ so we may assume that $m \geq 1$. By [5, Invariance III] it is enough to prove the relation for $B = \vec{0}$. Over $\mathcal{M}^{\mathrm{ct}}_{g,m}$ there exists a morphism

$$\beta : \mathsf{Pic}^0(\mathcal{M}^{\mathrm{ct}}_{g,m}) \to \mathsf{Pic}^{\underline{0}}(\mathcal{M}^{\mathrm{ct}}_{g,m}) \tag{7.1}$$

induced by twisting the universal line bundle by vertical divisors. Using [5, Section 7.6] we have

$$\beta^* \kappa_{-1,2} = \kappa_{-1,2} + \sum_{\Gamma \in \mathsf{G}^{\mathrm{se}}_{g,m}} d^2 [\Gamma_d] \,,$$

where $\mathsf{G}^{\mathrm{se}}_{g,m}$ is the set of graphs in $\mathsf{G}_{g,m}$ having exactly two vertices connected by a single edge described by a partition $(g_1, d|g_2, -d)$ with $g_1 + g_2 = g$ and $[\Gamma_d]$ is the class of boundary divisor Γ_d. By Proposition 2.2.2, we have $(\kappa_{-1,2})^c = 0$ on $\mathsf{CH}^c(\mathsf{Pic}^{\underline{0}}(\mathcal{M}^{\mathrm{ct}}_{g,m}))$ for all $c > g$. By pulling back these relations along (7.1) we get the conclusion by [5, Proposition 36]. □

Remark 7.1.3. The exact sequence (2.2) splits over $\mathcal{M}^{\mathrm{ct}}_{g,m}$ via (7.1). In particular, we can pullback tautological relations on $\mathsf{Pic}^{\underline{0}}(\mathcal{M}^{\mathrm{ct}}_{g,m})$ to $\mathsf{Pic}^0(\mathcal{M}^{\mathrm{ct}}_{g,m})$ and get tautological relations. On the other hand, (2.2) does not split over $\overline{\mathcal{M}}_{g,m}$ in general and it is an interesting question to compare the tautological ring of $\mathsf{Pic}^0(\overline{\mathcal{M}}_{g,m})$ and the tautological ring of $\mathsf{Pic}^{\underline{0}}(\overline{\mathcal{M}}_{g,m})$.

7.2 Extended Faber's construction

In [24] Faber obtained tautological relations on $\mathcal{M}_g$ using a simple geometric construction. This construction is further studied in [41]. The argument can be lifted over the relative Picard stack over $\mathcal{M}_g$. We follow most of conventions in Section 5.1. For $g \geq 2$, let $\pi : \mathfrak{C}_g \to \mathfrak{Pic}^0_g(\mathcal{M}_g)$ be the universal curve and $\pi^d : \mathfrak{C}^d_g \to \mathfrak{Pic}^0(\mathcal{M}_g)$ be the relative d-th symmetric power of π. Let $\mu : U \to \mathfrak{C}^d_g$ be the universal curve and $\mathcal{L} \to \mathfrak{C}^d_g$ be the universal line bundle. Let $D := \mathcal{O}_U(p_1 + \cdots + p_d)$ be the Cartier divisor corresponding to the sum of d-sections. By cohomology and base change we get the following vanishing result.

Proposition 7.2.1. For $k > d - g$ and $d > 2g - 2$, we have $c_k(R\mu_*(\mathcal{L}(-D)) = 0$ in $\mathsf{CH}^k(\mathfrak{C}^d_g)$.

Further pushing forward relations in Proposition 7.2.1 to $\mathfrak{Pic}^0(\mathcal{M}_g)$, we get tautological relations. Similar to Section 6.1, we introduce a formal power series

$$\Theta(t,u,x) := \sum_{d=0}^{\infty} \prod_{i=1}^{d} (1 + it + ut) \frac{(-1)^d}{d!} \frac{x^d}{t^d} \,. \tag{7.2}$$

Lemma 7.2.2. The formal power series (7.2) can be written as $\Theta = (1+x)^{-\frac{t+tu+1}{t}}$.

Proof. By simple computation, we have

$$t(x+1)\frac{d}{dx}\Theta + (1+t+tu)\Theta = 0 \,,$$

and hence we get the result. □

We consider the logarithm of the power series associated to (7.2)

$$\sum_{d=1}^{\infty} \sum_{w=0}^{\infty} \sum_{r=0}^{d} C_{r,w,d} t^r u^w \frac{x^d}{d!} := \log\big(1 + \sum_{d=1}^{\infty} \prod_{i=1}^{d} (1 + it + ut) \frac{(-1)^d}{t^d} \frac{x^d}{d!}\big) \,. \tag{7.3}$$

Let $Q_{d-g} := R\mu_*(\mathcal{L}(-D))$ be the K-theory class in $K^0(\mathfrak{C}^d_g)$.

Proposition 7.2.3. In $\mathsf{R}^*(\mathfrak{Pic}^0(\mathcal{M}_g))$, we have

$$\sum_{d=1}^{\infty}\sum_{r=0}^{\infty}\sum_{w=0}^{\infty}\pi_*^d(c_r(Q_{d-g}))_{(w)}u^w t^{r-d}\frac{x^d}{d!} = \exp\Big(-\sum_{w=2}^{\infty}\frac{1}{w(w-1)}\kappa_{-1,w}u^w t^{w-1} - \sum_{w=1}^{\infty}\frac{1}{2w}\kappa_{0,w}u^w t^w$$
$$-\sum_{w=0}^{\infty}\sum_{s=1}^{\infty}\binom{2s-2+w}{w}\frac{B_{2s}}{2s(2s-1)}\kappa_{2s-1,w}u^w t^{2s+w-1}$$
$$-\sum_{d=1}^{\infty}\sum_{r=-1}^{\infty}\sum_{w=0}^{\infty}C_{r,d,w}\kappa_{r,w}u^w t^{r+w}\frac{x^d}{d!}\Big).$$

Proof. Consider the short exact sequence

$$0\to\omega_\mu\otimes\mathcal{L}(-D)\to\omega_\mu\otimes\mathcal{L}\to\omega_\mu\otimes\mathcal{L}|_D\to 0$$

on the universal curve U. Taking the derived pushforward, we have

$$Q_{d-g} = R\mu_*(\omega_\mu\otimes\mathcal{L}|_D) - R\mu_*(\omega_\mu\otimes\mathcal{L})$$

in $K^0(\mathfrak{C}_g^d)$. The contribution of $R\mu_*(\omega_\mu\otimes\mathcal{L})$ is computed in Lemma 5.1.2. The contribution of $R\mu_*(\omega_\mu\otimes\mathcal{L}|_D)$ can be computed by the similar method in Proposition 5.1.4. □

By taking products with tautological classes with tautological relations Proposition 7.2.1, we get refined tautological relations on $\mathfrak{Pic}^0(\mathcal{M}_g)$. We introduce a set of variables

$$\mathsf{z} := \{z_{ijk} : i\geq 1, j\geq i-1, k\geq 0\}.$$

For monomials $\mathsf{z}^\sigma = \prod_{i,j,k} z_{i,j,k}^{\sigma_{i,j,k}}$, we denote

$$\ell_1(\sigma) := \sum_{i,j,k} i\sigma_{i,j,k},\ ,\ell_2(\sigma) := \sum_{i,j,k} j\sigma_{i,j,k},\ ,\ell_3(\sigma) := \sum_{i,j,k} k\sigma_{i,j,k}$$

and $|\mathrm{Aut}(\sigma)| := \prod_{i,j,k}\sigma_{i,j,k}!$. Define a differential operator

$$\mathcal{D} := \sum_{i,j,k} z_{i,j,k}t^{j+k}u^k\big(x\frac{d}{dx}\big)^i$$

acting on Θ. Denote

$$\Theta^{\mathcal{D}} := \exp(\mathcal{D})\,\Theta$$

and

$$\log(\Theta^{\mathcal{D}}) =: \sum_\sigma\sum_{d=1}^{\infty}\sum_{r=-1}^{\infty}\sum_{w=0}^{\infty}C_{r,w,d}(\sigma)t^{r+w}u^w\frac{x^d}{d!}\mathsf{z}^\sigma.$$

Consider the following formal power series

$$\begin{aligned}\gamma^{\mathsf{EF}} := & \sum_{w=2}^{\infty} \frac{1}{w(w-1)} \kappa_{-1,w} u^w t^{w-1} + \sum_{w=1}^{\infty} \frac{1}{2w} \kappa_{0,w} u^w t^w \\ & + \sum_{w=0}^{\infty} \sum_{s=1}^{\infty} \binom{2s-2+w}{w} \frac{B_{2s}}{2s(2s-1)} \kappa_{2s-1,w} u^w t^{2s+w-1} \\ & + \sum_{d=1}^{\infty} \sum_{r=-1}^{\infty} \sum_{w=0}^{\infty} C_{r,d,w}(\sigma) \kappa_{r,w} u^w t^{r+w} \frac{x^d}{d!} \,. \end{aligned} \tag{7.4}$$

Theorem 7.2.4. If $g \geq 2, r > -g + \ell_2(\sigma) + \ell_3(\sigma)$ and $d > 2g-2$, then we have

$$\left[\exp(-\gamma^{\mathsf{EF}})\right]_{t^r u^w x^d \mathbf{z}^\sigma} = 0$$

in $\mathsf{R}^r(\mathfrak{Pic}^0(\mathcal{M}_g))$.

Proof. We extend relations from Proposition 7.2.1 using Wick's formula. For a tuple of integers (a, b, c) with $a \geq 0, b \geq 1, c \geq 0$, consider a class

$$\phi[a,b,c] := (-1)^{b-1} \sum_{|I|=b} \psi_I^a \xi_I^c u^c D_I \in \mathsf{CH}^{a+b+c-1}(\mathfrak{C}_g^d)[u]$$

where $I \subset \{1, \cdots, d\}$ and D_I is the class of the associated small diagonal. For a tuple of integers (i, j, k) with $i \geq 1, j \geq i-1, k \geq 0$, there exists a unique $\lambda_{a,b,c} \in \mathbb{Q}$ (depending on i) where the linear combination

$$\Phi[i,j,k] := \sum_{k=c, a+b-1=j} \lambda_{a,b,c} \, \phi[a,b,c]$$

evaluated at $\psi_I = \xi_I = -D_I = 1$ is equal to $d^i u^c$. Using the Wick's formula, we can expand Proposition 7.2.3 as

$$\sum_{d=1}^{\infty} \sum_{r=0}^{\infty} \sum_{w=0}^{\infty} \pi_*^d \Big(\exp\big(\sum_{i \geq 1} \sum_{j \geq i-1} \sum_{k \geq 0} t^{j+k} u^k \Phi[i,j,k] \big) \cdot \sum_{w=0}^{r+1} c_r(Q_{d-g})_{(w)} u^w t^r \Big) \frac{1}{t^d} \frac{x^d}{d!} = \exp(-\gamma^{\mathsf{EF}}) \,. \tag{7.5}$$

By Proposition 7.2.1, we have $c_s(Q_{d-g})$ for $d > 2g-2, s > d-g$. Therefore $t^r u^w x^d \mathsf{z}^\sigma$ coefficient of (7.5) vanishes if $r + d - \ell_2(\sigma) - \ell_3(\sigma) > d - g$. □

It is an interesting question whether Theorem 7.2.4 gives equivalent relations from Theorem 5.3.4. Even after restricting to $\mathsf{R}^*(\mathcal{M}_g)$ this question is still open [41, Conjecture 3].

8 Further directions

In this paper, relative Quot schemes are crucial to study tautological ring of the moduli space of curves with line bundles. This idea can be generalized to study intersection theory of various related moduli spaces.

The moduli space of stable Picard quotients produces tautological relations on $\mathsf{Pic}^0_{g,m}$. The virtual localization contribution gets more complicated over $\mathsf{Pic}^0_{g,m}$. To systematically organize virtual localization formula, it is desirable to develop *cohomological field theory* over the universal Picard stack (Picard CohFT). This formalism and the Givental-Teleman type classification was anticipated in [42]. Since the virual fundamental class of $\mathsf{P}_{g,m}(r, A, d)$ satisfies the splitting axiom (Proposition 4.1.2), we expect the Picard CohFT will be a valuable tool to extend our result over $\mathsf{Pic}^0_{g,m}$.

The idea of constructing stable quotient over $\mathsf{Pic}^0_{g,m}$ can be generalized to the stable Picard quotients over compactified Jacobians. Modular compactifications of $\mathsf{Pic}^0_{g,m}$ has been studied extensively in the literature, including [15, 35]. For a nondegenerate stability θ, we expect moduli spaces of stable Picard quotients can be constructed over the moduli space of compactified Jacobians $\mathsf{Pic}^\theta_{g,m}$. Study of compactified Jacobians gains new interest related to logarithmic intersection theory on logarithmic Picard stack [40, 28]. For a choice of tuples $B = (b_1, \cdots, b_m)$ the Abel-Jacobi map extends to the logarithmic Abel-Jacobi map

$$\mathsf{aj}_B : \overline{\mathcal{M}}^\theta_{g,B} \to \mathsf{Pic}^\theta_{g,m}$$

[28, Definition 25]. The pullback of tautological relations on $\mathsf{Pic}^\theta_{g,m}$ along aj_B defines relations on logarithmic tautological ring $\mathsf{logR}^*(\overline{\mathcal{M}}_{g,m})$.

We can also study the tautological ring of the relative moduli space of semi-stable bundles over the moduli space of smooth curves. For $r, d \geq 0$ and $g \geq 2$, let

$$\mathfrak{Bun}^{\mathrm{ss}}_{r,d}(\mathcal{C}_g/\mathcal{M}_g) \to \mathcal{M}_g$$

be the moduli stack of semi-stable vector bundle of rank r degree d [45]. For a fixed smooth curve, the cohomology ring of the moduli space of stable vector bundles has been an important subject [3, 34], but we do not know systematic studies for the relative version. There exists a relative Quot scheme over $\mathfrak{Bun}^{\mathrm{ss}}_{r,d}(\mathcal{C}_g/\mathcal{M}_g)$ generalizing Section 3 which has potential to study tautological relations on $\mathfrak{Bun}_{r,d}(\mathcal{C}_g/\mathcal{M}_g)$. We note that the method used in Section 7.2 has a direct generalization. Using the semi-stability, we can apply parallel argument in Proposition 7.2.1 and produce tautological relations on $\mathfrak{Bun}^{\mathrm{ss}}_{r,d}(\mathcal{C}_g/\mathcal{M}_g)$. On the other hand, there exists a determinant morphism $\det : \mathfrak{Bun}^{\mathrm{ss}}_{r,d}(\mathcal{C}_g/\mathcal{M}_g) \to \mathfrak{Pic}^d(\mathcal{C}_g/\mathcal{M}_g)$ and hence we can pullback tautological relations on $\mathfrak{Pic}$ to $\mathfrak{Bun}$. It is an interesting question whether it is possible to carry out Quot scheme construction on $\mathfrak{Bun}^{ss}_{r,d}(\mathcal{C}_g/\mathcal{M}_g)$ and find tautological classes. Already in [42] it is speculated that the CohFT formalism can be generalized to $\mathfrak{Bun}_{r,d}(\mathfrak{C}/\mathfrak{M}_{g,m})$. We hope to revisit this direction in future.

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