An important consequences of this result:

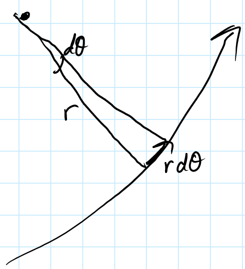
The vectors $\vec{r}$ & $\vec{v}$ define a plane in space. $\vec{J}$ is always $\perp$ to this plane (e.g. let $\vec{r}$ & $\vec{v}$ lie in the xy plane, then $\vec{J}$ is in the $\hat{k}$ direction). So, if $\vec{J}$ is constant, it must be constant in both mag. and direction. $\therefore$ Orientation of the plane of motion is fixed in space. So the motion of any body whose net force is central is simply motion in a plane.

The meaning of the constancy of $|\vec{J}|$ is easiest to see in polar coordinates

$$\vec{v} = \dot{r}\hat{r} + r\dot{\theta}\hat{\theta}$$

$$\text{So, } |\vec{J}| = |\vec{r} \times m\vec{v}| = |r\hat{r} \times m(\dot{r}\hat{r} + r\dot{\theta}\hat{\theta})|$$

$$= mr^2\dot{\theta}\,\underbrace{|\hat{r}\times\hat{\theta}|}_{1} = mr^2\dot{\theta}$$



So, for central forces, $|\vec{J}| = mr^2\dot{\theta} = \text{constant}$

Consider the area swept out by the radius vector when the particle moved by $d\theta$

$$dA = \frac{1}{2}r^2 d\theta$$

So, the rate of area being swept is

$$\frac{dA}{dt} = \frac{1}{2}r^2\dot{\theta} = \frac{J}{2m} = \text{constant}$$

Thus, the rate of area being swept out is constant $\Rightarrow$ Kepler's 2nd Law in the context of bodies moving in a grav. field. Although it applies to any motion subject to a central force (even non-conser. central force)

Orbit of a Particle in Central Force Field

Work in polar coordinates as interested in the trace of (r,θ) of the particle. $\therefore m\ddot{\vec{r}} = f(r)\hat{r}$

where $f(r)$ is the central force acting on the the particle of mass m

From the 1st week: $\ddot{\vec{r}} = (\ddot{r} - r\dot{\theta}^2, 2\dot{r}\dot{\theta} + r\ddot{\theta})$

Thus, the 2 e.o.m are

$$m(\ddot{r} - r\dot{\theta}^2) = f(r)$$

$$m\underbrace{(2\dot{r}\dot{\theta} + r\ddot{\theta})}_{\times \text{ by } r} = 0 \longrightarrow m\frac{d}{dt}(r^2\dot{\theta}) = 0$$

so, $r^2\dot{\theta} = \text{constant} = \frac{J}{m}$ where $J = |\vec{J}|$

specific ang momentum

To find the path in (r,θ) plane (the orbit), re-write these ode's w/o time. Easiest to define a new variable $u = \frac{1}{r} \rightarrow r = \frac{1}{u}$

then $\dot{r} = -\frac{1}{u^2}\dot{u} = -\frac{1}{u^2}\frac{du}{dt} = -\frac{1}{u^2}\frac{du}{d\theta}\frac{d\theta}{dt} = -\frac{1}{\cancel{u^2}}\frac{J}{m\cancel{r^2}}\frac{du}{d\theta} = -\frac{J}{m}\frac{du}{d\theta}$

$$\rightarrow \ddot{r} = -\frac{J}{m}\frac{d}{dt}\left(\frac{du}{d\theta}\right) = -\frac{J}{m}\frac{d\theta}{dt}\left(\frac{d^2u}{d\theta^2}\right) = \frac{-J^2}{m^2r^2}\left(\frac{d^2u}{d\theta^2}\right) = -\frac{J^2u^2}{m^2}\frac{d^2u}{d\theta^2}$$

Sub. this into the r-component of our e.o.m.

$$m\left(-\frac{J^2u^2}{m^2}\frac{d^2u}{d\theta^2} - \frac{1}{u}\frac{J^2}{m^2}u^4\right) = f(u^{-1})$$

$$-\frac{J^2u^2}{m}\frac{d^2u}{d\theta^2} - \frac{J^2u^3}{m} = f(u^{-1})$$

$$\times \text{ by } \left(\frac{-m}{J^2u^2}\right) \rightarrow \boxed{\frac{d^2u}{d\theta^2} + u = -\frac{mf(u^{-1})}{J^2u^2}}$$

Diff. eqn of the orbit of a particle moving under a central force.

Ex: A particle in a central field moves in a spiral orbit

$r = c\theta^2$. Determine the force function.

$$u = \frac{1}{r} = \frac{1}{c\theta^2}, \quad \text{so } \frac{du}{d\theta} = -\frac{2}{c}\theta^{-3}; \quad \frac{d^2u}{d\theta^2} = \frac{6}{c}\theta^{-4} = \frac{6}{c\theta^4} = 6cu^2$$

Sub. into o.d.e:

$$6cu^2 + u = -\frac{mf(u^{-1})}{J^2u^2} \quad \rightarrow \quad f(u^{-1}) = -\frac{J^2u^2 6cu^2}{m} - \frac{J^2u^3}{m}$$

$$= -\frac{J^2}{m}\left(6cu^4 + u^3\right)$$

$$\text{or } f(r) = -\frac{J^2}{m}\left(\frac{6c}{r^4} + \frac{1}{r^3}\right)$$