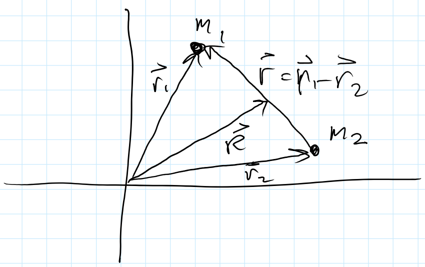
Oblique Collisions

In general, $\vec{p}_1 + \vec{p}_2 = \vec{q}_1 + \vec{q}_2$, is quite complicated so its often easier to work in center-of-mass coordinates



Definitions: Consider 2 particles w/ positions $\vec{r}_1$ & $\vec{r}_2$ & masses m_1 & m_2.

If the internal force is $\vec{F}$ & particles are in a uniform grav. field then the e.o.m. are

$$m_1 \ddot{\vec{r}}_1 = m_1 \vec{g} + \vec{F}$$
$$m_2 \ddot{\vec{r}}_2 = m_2 \vec{g} - \vec{F}$$

Now, define the center-of-mass position vector

$$\vec{R} = \frac{m_1 \vec{r}_1 + m_2 \vec{r}_2}{m_1 + m_2}$$

& the relative position

$$\vec{r} = \vec{r}_1 - \vec{r}_2$$

Adding the 2 equations of motion,

$$m_1 \ddot{\vec{r}}_1 + m_2 \ddot{\vec{r}}_2 = (m_1 + m_2)\vec{g}$$

$$(m_1 + m_2)\ddot{\vec{R}} = (m_1 + m_2)\vec{g}$$

or $M\ddot{\vec{R}} = M\vec{g}$ where $M = m_1 + m_2$

$\hookrightarrow$ e.o.m for the center of mass (its just uniform acc'n)

Now, need e.o.m. for $\vec{r}$:

$$\ddot{\vec{r}}_1 = \vec{g} + \frac{\vec{F}}{m_1}$$

$$\ddot{\vec{r}}_1 = \vec{g} + \frac{\vec{F}}{m_1}$$

$$\ddot{\vec{r}}_2 = \vec{g} - \frac{\vec{F}}{m_2}$$

Subtract: $\ddot{\vec{r}}_1 - \ddot{\vec{r}}_2 = \dfrac{\vec{F}}{m_1} + \dfrac{\vec{F}}{m_2}$

$$\ddot{\vec{r}} = \frac{\vec{F}m_2}{m_1 m_2} + \frac{\vec{F}m_1}{m_1 m_2} = \frac{\vec{F}m_1 + \vec{F}m_2}{m_1 m_2} = \frac{\vec{F}(m_1+m_2)}{m_1 m_2}$$

or $\left(\dfrac{m_1 m_2}{m_1+m_2}\right)\ddot{\vec{r}} = \vec{F}$

or $\mu\ddot{\vec{r}} = \vec{F}$ where $\mu = \dfrac{m_1 m_2}{(m_1+m_2)}$ is called the reduced mass

Now, we have 2 e.o.m., 1 for the pos'n of the c.o.m. & the other for the relative position

For the c.o.m, if $\vec{g} = 0$

$$M\ddot{\vec{R}} = 0$$

$$M\dot{\vec{R}} = \text{constant} = (m_1+m_2)\frac{(m_1\dot{\vec{r}}_1 + m_2\dot{\vec{r}}_2)}{(m_1+m_2)} = \vec{p}_1 + \vec{p}_2$$

cons. of momentum

For the rel. position, the e.o.m. we found ($\mu\ddot{\vec{r}} = F$) is the same e.o.m. as a 1 particle system w/ mass μ, so we can just solve this as before. Then decompose as

$$\vec{r}_1 = \vec{R} + \frac{m_2\vec{r}}{M} \; ; \; \vec{r}_2 = \vec{R} - \frac{m_1\vec{r}}{M} \quad \text{(verify!)}$$

Can also write angular momentum & KE in terms of c.o.m. & relative positions:

$$\vec{J} = m_1(\vec{r}_1 \times \dot{\vec{r}}_1) + m_2(\vec{r}_2 \times \dot{\vec{r}}_2)$$

$$= m_1\left(\vec{R} + \frac{m_2}{M}\vec{r}\right)\times\left(\dot{\vec{R}} + \frac{m_2}{M}\dot{\vec{r}}\right) + m_2\left(\vec{R} - \frac{m_1}{M}\vec{r}\right)\times\left(\dot{\vec{R}} - \frac{m_1}{M}\dot{\vec{r}}\right)$$

Note will have term

$$\frac{m_1 m_2}{M}\vec{r}\times\dot{\vec{R}}\,,\quad -\frac{m_1 m_2}{M}\vec{r}\times\dot{\vec{R}}$$

$$\frac{m_1 m_2}{M}\vec{R}\times\dot{\vec{r}}\,,\quad -\frac{m_1 m_2}{M}\vec{R}\times\dot{\vec{r}}$$

Only terms that remain will lead to

$$\vec{J} = m_1 \vec{R}\times\dot{\vec{R}} + \frac{m_2^2 m_1}{M^2}\vec{r}\times\dot{\vec{r}} + m_2\vec{R}\times\dot{\vec{R}} + \frac{m_1^2 m_2}{M^2}\vec{r}\times\dot{\vec{r}}$$

$$= (m_1+m_2)\vec{R}\times\dot{\vec{R}} + \frac{\cancel{(m_2+m_1)}m_1 m_2}{\cancel{(m_2+m_1)}(m_1+m_2)}\vec{r}\times\dot{\vec{r}}$$

$$= \underbrace{M\left(\vec{R}\times\dot{\vec{R}}\right)}_{\text{ang. mom. of the c.o.m.}} + \underbrace{\mu\left(\vec{r}\times\dot{\vec{r}}\right)}_{\text{ang. mom. of particles about each other}}$$

Similarly for KE

$$T = \frac{1}{2}m_1\dot{r}_1^2 + \frac{1}{2}m_2\dot{r}_2^2$$

sub in for $\dot{r}_1$ & $\dot{r}_2$, expand & collect

$$T = \frac{1}{2}M\dot{R}^2 + \frac{1}{2}\mu\dot{r}^2$$

Center-of-Mass Reference Frame

When dealing w/ 2 (or even more) bodies it is often easier to work in a reference frame where c.o.m is at rest. Even if there is a $\vec{g}$ & the frame is non-inertial, this will still simplify the problem. This procedure is common in nuclear & particle physics kinematic reactions.

Notation: quantities in c.o.m frame have an asterix

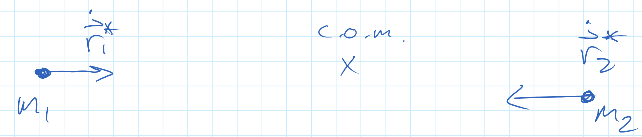
since c.o.m. is stationary in this frame, place c.o.m at the origin $\therefore \vec{R}^* = 0$

; $\vec{r}_1^{\,*} = \frac{m_2}{M}\vec{r}$; $\vec{r}_2^{\,*} = -\frac{m_1}{M}\vec{r}$ (note: $\vec{r}$ does not have an asterix)

In this frame, the momenta of the 2 particles are equal & opposite:
(the total linear mom. is zero)

$$m_1 \dot{\vec{r}}_1^{\,*} = -m_2 \dot{\vec{r}}_2^{\,*} = +\frac{m_1 m_2}{M}\dot{\vec{r}} = \mu \dot{\vec{r}} = \vec{p}^{\,*}$$

Center-of-mass frame



From above $\vec{J}^{\,*} = \mu(\vec{r} \times \dot{\vec{r}}) = \vec{r} \times \vec{p}^{\,*}$

$$T^* = \frac{1}{2}\mu \dot{r}^2 = \frac{\vec{p}^{\,*2}}{2\mu}$$

To convert to frame where c.o.m. is moving w/ $\dot{\vec{R}}$ just add

$$\dot{\vec{r}}_1 = \dot{\vec{R}} + \dot{\vec{r}}_1^{\,*}, \quad \dot{\vec{r}}_2 = \dot{\vec{R}} + \dot{\vec{r}}_2^{\,*}, \quad \vec{p}_1 = m_1 \dot{\vec{R}} + \vec{p}^{\,*}, \quad \vec{p}_2 = m_2 \dot{\vec{R}} - \vec{p}^{\,*}$$

$$\vec{J} = M(\vec{R} \times \dot{\vec{R}}) + \vec{J}^{\,*} \quad ; \quad T = \frac{1}{2}MR^2 + T^*$$