Linear Oscillators

Consider a particle in equilibrium. Means total force on the particle is zero. If this a conservative force then $F = -\frac{dV}{dx} = 0$, ie, V is a constant at that position. Lets choose the equilibrium point to be x=0 & the arbitrary constant so that $V(0)=0$. For small displacements around the equilibrium position expand V(x) in a Taylor's Series:

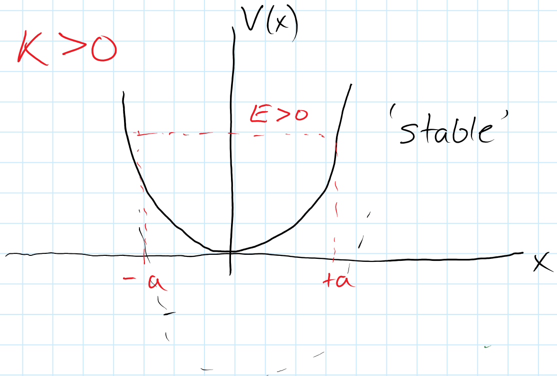
$$V(x) = \cancelto{0}{V(0)} + x\cancelto{0}{\frac{dV}{dx}(0)} + \frac{1}{2}x^2\frac{d^2V}{dx^2}(0) + \cdots$$

near $x=0$, $V(x) \simeq \frac{1}{2}Kx^2$ where $K = \frac{d^2V(0)}{dx^2} \neq 0$

Thus motion near any equilibrium point is described approx. by this potential. → Very common in physics!

Can anticipate motion in this potential by sketching it

$K>0$



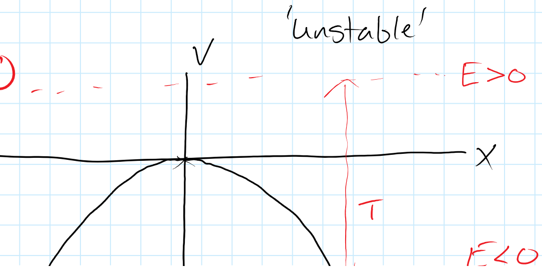
If $E>0$, 2 points where $E=V$

$E = \frac{1}{2}Ka^2$

$\therefore a = \pm\sqrt{\frac{2E}{K}}$

motion will oscillate b/w $\pm a$

$K<0$



If $E<0$, particle will come to rest at potential 'barrier' & reverse

rest at potential barrier & reverse

If $E>0$, particle will surmount the barrier

The force corresponding to the potential energy function $V(x)=\frac{1}{2}Kx^2$

$$F(x) = -\frac{dV}{dx} = -Kx$$ attractive if $K>0$ (Hook's Law) repulsive if $K<0$

Eq'n of motion: $m\ddot{x} + Kx = 0$

This is a linear ODE (ie, linear in $x, \dot{x}, \ddot{x}$, etc.). Satisfy superposition principle: if $x_1(t)$ & $x_2(t)$ are 2 independent solutions than any linear combination

$$x(t) = a_1 x_1(t) + a_2 x_2(t)$$ is the general sol'n

(a_1 & a_2 are constants)

If $K<0$, the general solution is $x = Ae^{pt} + Be^{-pt}$ (check by substition)

- 2nd term decays v. quickly, so displacement increases exponentially. Unstable equil.

If $K>0$, get simple harmonic oscillator eq'n:

$$\ddot{x} + \left(\frac{K}{m}\right)x = 0$$

Define $\omega_0^2 = \frac{K}{m} \rightarrow \ddot{x} + \omega_0^2 x = 0$

Sol'n: $x = c\cos\omega_0 t + d\sin\omega_0 t$ where c & d are set by initial conditions

A more useful way of writing this is

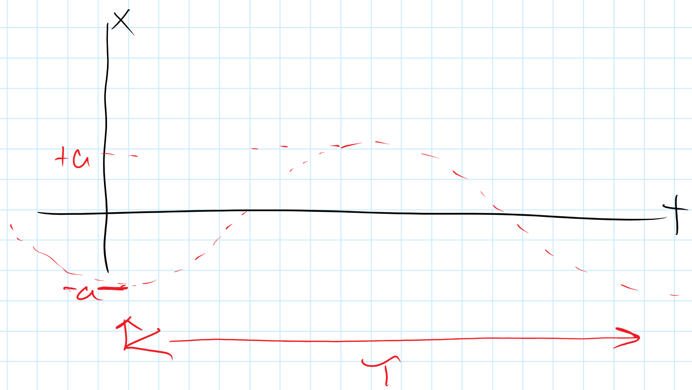
$$x = a\cos(\omega_0 t - \theta)$$ where a & θ are constants

a = amplitude of oscillation (as in prev. sketch)

ω_0 = ang. freq. (rad s^{-1})

$T = \dfrac{2\pi}{\omega_0}$ period (s)

$f = \dfrac{1}{T} = \dfrac{\omega_0}{2\pi}$ (Hz)



For sufficiently small displacements, any system w/ a stable equil. point behaves like a simple harmonic oscillator. In particular, T or f may always be found from

$\dfrac{d^2V}{dx^2}$ at equil. points.