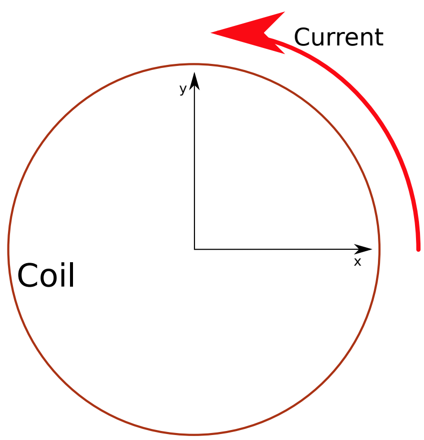
A

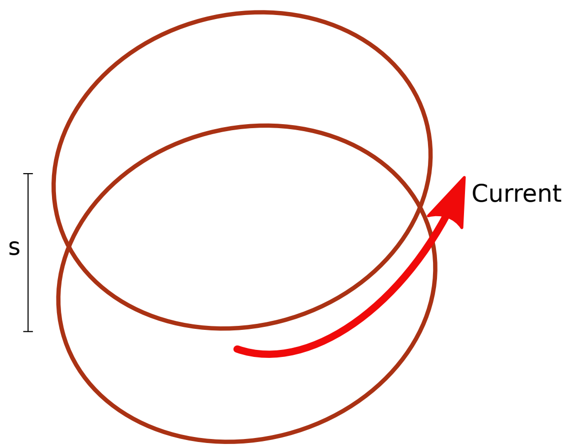
A.1



In this solution, cylindrical coordinates will be used, with the convention that conventional current travels in the positive $\hat{\phi}$ direction. The field due to one coil is

$$
\begin{aligned}
\frac{\mu_0 N}{4\pi}\int \frac{Id\vec{l}\times(\vec{r}-\vec{r}')}{|\vec{r}-\vec{r}'|^3} &= \frac{\mu_0 NIa}{4\pi}\int_0^{2\pi}\frac{\hat{\phi}\times[-a\hat{s}+(z-s/2)\hat{z}]d\phi}{[a^2+(z-s/2)^2]^{3/2}} \\
&= \frac{\mu_0 NIa}{4\pi}\int_0^{2\pi}\frac{(z\hat{s}+a\hat{z})d\phi}{[a^2+(z-s/2)^2]^{3/2}} \\
&= \frac{\mu_0 NIa}{4\pi}\int_0^{2\pi}\frac{[(z-s/2)(-\cos\phi\hat{x}+\sin\phi\hat{y})+a\hat{z}]d\phi}{[a^2+(z-s/2)^2]^{3/2}} \\
&= \frac{\mu_0 NIa^2\hat{z}}{2}\frac{1}{[a^2+(z-s/2)^2]^{3/2}}
\end{aligned}
$$

A.2



For the lower coil, the only change in the displacement vector from source to observer is that $(z - s/2)$ becomes $(z + s/2)$. With this change, the combined field is

$$
\frac{\mu_0 NIa^2\hat{z}}{4\pi}\left[\frac{1}{[a^2+(z-s/2)^2]^{3/2}}+\frac{1}{[a^2+(z+s/2)^2]^{3/2}}\right]
$$

We find

$$
\begin{aligned}
\frac{\partial^2}{\partial z^2}\frac{1}{[a^2+(z\pm s/2)^2]^{3/2}} &= \frac{\partial}{\partial z}\frac{-3(z\pm s/2)}{[a^2+(z\pm s/2)^2]^{5/2}} \\
&= \frac{-3}{[a^2+(z\pm s/2)^2]^{5/2}}+\frac{15(z\pm s/2)^2}{[a^2+(z\pm s/2)^2]^{7/2}}
\end{aligned}
$$

Then

$$
\begin{aligned}
\frac{15(z \pm s/2)^2}{[a^2 + (z \pm s/2)^2]^{7/2}} &= \frac{3}{[a^2 + (z \pm s/2)^2]^{5/2}} \\
5(z \pm s/2)^2 &= a^2 + (z \pm s/2)^2 \\
\left| z \pm \frac{s}{2} \right| &= \frac{a}{2}
\end{aligned}
$$

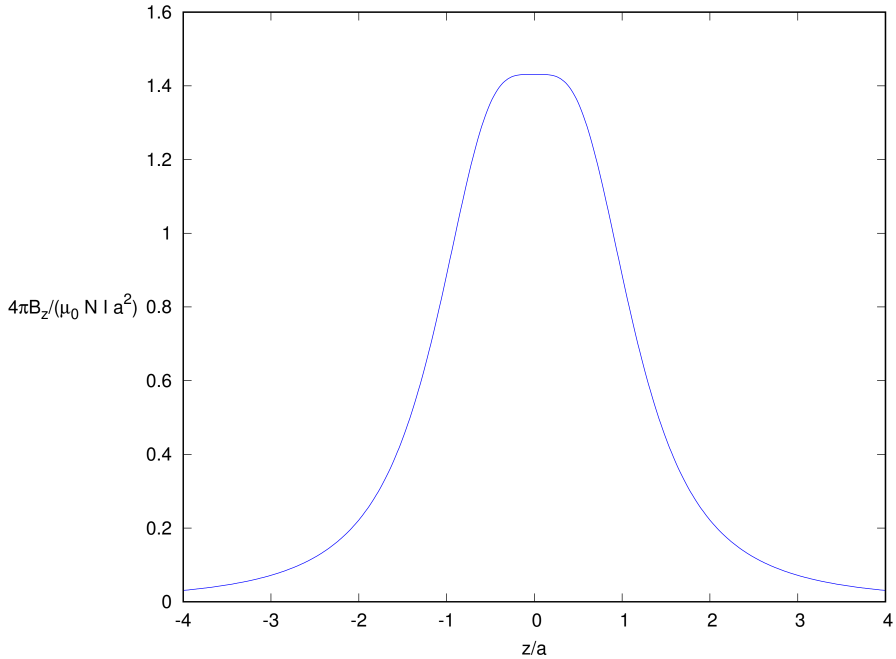
At $z = 0$, this demands $|s/2| = a/2$, so we conclude $s = a$.

Now observe

$$
\begin{aligned}
\frac{\partial^3}{\partial z^3} \frac{1}{[a^2 + (z \pm s/2)^2]^{3/2}} &= \frac{\partial}{\partial z} \left[\frac{-3}{[a^2 + (z \pm s/2)^2]^{5/2}} + \frac{15(z \pm s/2)^2}{[a^2 + (z \pm s/2)^2]^{7/2}} \right] \\
&= \frac{45(z \pm a/2)}{[a^2 + (z \pm a/2)^2]^{7/2}} - \frac{105(z \pm a/2)^3}{[a^2 + (z + a/2)^2]^{9/2}} \\
\left.\frac{\partial^3}{\partial z^3}\right|_{z=0} \frac{1}{[a^2 + (z \pm s/2)^2]^{3/2}} &= \frac{\pm 45\, a/2}{(5a^2/4)^{7/2}} \mp \frac{105\, a^3/8}{(5a^2/4)^{9/2}} \\
&= \frac{\pm 1}{2a^6 (5/4)^{7/2}} \left[45 \mp \frac{105}{5} \right] \\
&= \frac{\pm(45 \pm 21)}{2a^6 (5/4)^{7/2}}
\end{aligned}
$$

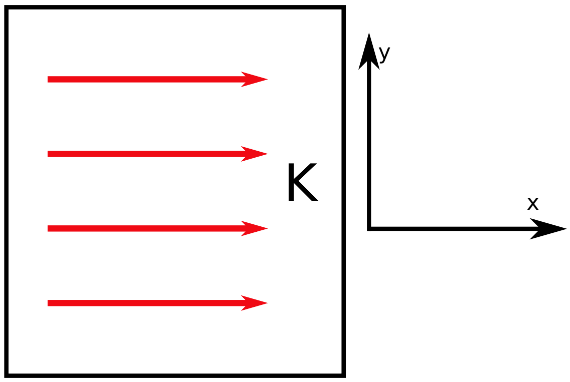
Thus, the contributions to the third partial derivative with respect to z of the magnetic field from the upper and lower coils have equal magnitude and opposite sign, and the third derivative vanishes as well.

A.3



Problem B

B.1



$$
\begin{aligned}
\vec{B} &= \frac{\mu_0}{4\pi}\int\int \frac{\vec{K}\times(\vec{r}-\vec{r}\,')dA}{|\vec{r}-\vec{r}\,'|^3} \\
&= \frac{\mu_0}{4\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{K_0\hat{x}\times(x\hat{x}+y\hat{y}+z\hat{z}-x'\hat{x}-y'\hat{y})dx'dy'}{[(x-x')^2+(y-y')^2+z^2]^{3/2}} \\
&= \frac{\mu_0 K_0}{4\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{[(y-y')\hat{z}-z\hat{y}]dx'dy'}{[(x-x')^2+(y-y')^2+z^2]^{3/2}}
\end{aligned}
$$

Now make the subsitution

$$
\begin{aligned}
u &= \frac{x'-x}{\sqrt{(y-y')^2+z^2}} \\
&= \frac{x'-x}{a}
\end{aligned}
$$

where

$$
a \equiv \sqrt{(y-y')^2+z^2}
$$

The limits for u are still $-\infty$ and ∞, and so

$$
\int_{-\infty}^{\infty}\frac{dx'}{[(x-x')^2+(y-y')^2+z^2]^{3/2}} = \int_{-\infty}^{\infty}\frac{du}{a^2[u^2+1]^{3/2}}
$$

Now let $u=\tan(\theta)$, then

$$\int_{-\infty}^{\infty} \frac{du}{a^2[u^2+1]^{3/2}} = \int_{\tan^{-1}(-\infty)}^{\tan^{-1}(\infty)} \frac{\sec^2(\theta)d\theta}{a^2[\tan^2(\theta)+1]^3}$$
$$= \frac{1}{a^2}\int_{-\pi/2}^{\pi/2} \cos(\theta)d\theta$$
$$= \frac{1}{a^2}[\sin(\theta)]_{-\pi/2}^{\pi/2}$$
$$= \frac{2}{a^2}$$

Recalling the definition for a, we find

$$\vec{B} = \frac{\mu_0 K_0}{2\pi}\int_{-\infty}^{\infty} \frac{[(y-y')\hat{z} - z\hat{y}]dy'}{(y-y')^2+z^2}$$

Now define

$$v = \frac{y-y'}{z}$$

If z is positive, then the limits of v are ∞ and ∞, respectively, and they are reversed in z is negative. First we find

$$B_z = \mp\frac{\mu_0 K_0}{2\pi z^2}\int_{-\infty}^{\infty} \frac{v\,z^2 dv}{v^2+1}$$
$$= \mp\frac{\mu_0}{2\pi}\int_{-\infty}^{\infty} \frac{v\,dv}{v^2+1}$$
$$= 0$$

We can see B_z must be zero because it involves the integral of an odd quantity over the range $(-\infty, \infty)$.

Next

$$B_y = \mp\frac{\mu_0 K_0}{2\pi z^2}\int_{-\infty}^{\infty} \frac{z^2\,dv}{v^2+1}$$
$$= \mp\frac{\mu_0 K_0}{2\pi}[\tan^{-1}(v)]_{-\infty}^{\infty}$$
$$= \mp\frac{\mu_0 K_0}{2}$$

Finally

$$\vec{B} = \begin{cases} -\frac{\mu_0 K_0}{2}\hat{y},\, z>0 \\ \frac{\mu_0 K_0}{2}\hat{y},\, z<0 \end{cases}$$

Recall that, in cylindrical coordinates, the magnetic field produced by a line current running along the positive z axis is

$$\vec{B} = \frac{\mu_0 I}{2\pi s}\hat{\phi}$$

Now let the current run along the x axis. The magnetic field should curl counter-clockwise around the x axis, so that, for observation location (x, y, z) and a wire with displaced by y' along the y axis azimuthal direction is given by

$$\begin{aligned}\hat{\phi} &= -\sin(\phi)\hat{x} + \cos(\phi)\hat{z} \\ &= -\frac{z}{\sqrt{(y-y')^2+z^2}}\hat{y} + \frac{y-y'}{\sqrt{(y-y')^2+z^2}}\hat{z}\end{aligned}$$

Here, $\sqrt{(y-y')^2+z^2}$ gives the displacement from the wire to the observer along a path perpendicular to the direction of current.

Consider a thin strip of the current-carrying sheet parallel to the x axis with thickness dy, displaced by y' from the origin along the y axis. In light of the above results, this strip makes a wire-like contribution to the net magnetic field

$$\frac{\mu_0 K_0 dy}{2\pi[(y-y')^2+z^2]}\left[-z\hat{y} + (y-y')\hat{z}\right]$$

Note that, for a given y', there is a strip with coordinate y'' such that $y - y' = y'' - y$. The contributions from these to strips to the z component of $\vec{B}$ will cancel, while the contributions from these two strips to the y component of $\vec{B}$ will be the same. Moreover, the sign of the y component is negative whenever z is positive, and positive whenever z is negative. The direction of the field is therefore sensible.

In assessing the appropriate dependence of the field on x an y, note that on observer with arbitrary x and y coordinates will observe the same current density, moving in the same direction, over an infinitely large area. Thus, the system is translationally invariant with respect to x and y, and the field should not change with observation location.

B.2

From arguments given in B.**1**, we conclude $\vec{B}$ will only have a y component. We choose as an Amperian loop a rectangle of height h and width w, aligned with the y axis and half above and half below the y axis. That path of integration winds counter-clockwise around the positive x axis.

$$\oint \vec{B}\cdot\vec{dl} = \mu_0 \int\int \vec{K}\cdot\hat{n}dA$$

$$\int_0^w |B_y|dy + \int_{h/2}^{-h/2} 0\,dz + \int_0^w |B_y|dy + \int_{-h/2}^{h/2} 0\,dz = \mu_0 \int_0^w \int_{-h/2}^{h/2} K_0\delta(z)dydz$$

$$2|B_y|w = \mu_0 K_0 w$$

$$|B_y| = \frac{\mu_0 K_0}{2}$$

Once again we find

$$B_y = \begin{cases} \frac{-\mu_0 K_0}{2}\hat{y}, z > 0 \\ \frac{\mu_0 K_0}{2}\hat{y}, z < 0 \end{cases}$$

B.3

We already know the magnetic field for a planar sheet carrying a steady surface current K_0 along $\hat{x}$. Hence we can use the superposition principle and add the fields due to two planar sheets to get the net field. Therefore

$$\mathbf{B} = \begin{cases} 0,\ z > a \\ -\mu_0 K_0 \hat{y}\ ,\ a > z > 0 \\ 0\ ,\ z < 0 \end{cases}$$

B.4

We have replaced a planar sheet of current with a slab but in both cases the symmetries are the same i.e. the current density is homogenous in the XY plane and it is pointing along $\hat{x}$. Hence the field is strictly along $\hat{y}$ similar to the previous two situations. Therefore we use Ampere's law to compute the field. To calculate the field at $|z| < h$ consider a rectangular amperian loop parallel to YZ plane with its center concurrent with the origin, length $2z$ and width w, then we get

$$\oint \vec{B}\cdot\vec{dl} = \mu_0 \int\int \vec{J}\cdot\hat{n}dA$$

$$\int_0^w |B_y|dy + \int_z^{-z} 0\,dz' + \int_0^w |B_y|dy + \int_{-z}^{z} 0\,dz' = \mu_0 \int_0^w \int_{-z}^{z} J_0|z|dydz$$

$$2|B_y|w = 2\mu_0 J_0 w\frac{z^2}{2}$$

$$|B_y| = \frac{\mu_0 J_0 z^2}{2}.$$

Similarly for $|z| > a$ we get

$$\begin{aligned}
\oint \vec{B} \cdot \vec{dl} &= \mu_0 \int \int \vec{J} \cdot \hat{n} dA \\
\int_0^w |B_y| dy + \int_z^{-z} 0\, dz' + \int_0^w |B_y| dy + \int_{-z}^{z} 0\, dz' &= \mu_0 \int_0^w \int_{-h}^{h} J_0 |z| dy dz \\
2|B_y| w &= 2\mu_0 J_0 w \frac{h^2}{2} \\
|B_y| &= \frac{\mu_0 J_0 h^2}{2}.
\end{aligned}$$

The direction of the field is given by

$$\frac{\mathbf{B}(z)}{|\mathbf{B}(z)|} = -\frac{z}{|z|} \hat{y}.$$