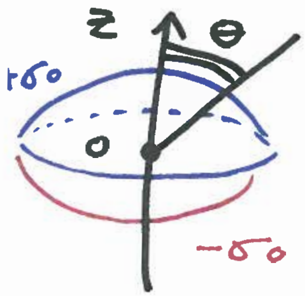
Problem A



We want to calculate the potential $V^{outside}(r,\theta) \equiv V^{o}(r,\theta)$ outside the shell and the potential $V^{inside}(r,\theta) \equiv V^{i}(r,\theta)$ inside the shell. At first glance it looks like two independent problems. We will see that boundary conditions unite these two problems.

$\boxed{\text{inside the shell}}$ (i) $\nabla^2 V = 0$ on domain Ω defined as points $(r<R, \theta)$ $\boxed{A.1}$

(ii) $\dfrac{\partial V}{\partial m}$ is given on boundary $\partial\Omega$ of the domain by surface charge density $\sigma(\theta)$

Given (i) and (ii) we know that there is a unique solution for $V^{i}(r<R,\theta)$

Our guess is
$$V^{i}(r,\theta) = \frac{1}{4\pi\varepsilon_0}\sum_{\ell=0}^{+\infty}\left[A^{i}_{\ell}\, r^{\ell} + \frac{B^{i}_{\ell}}{r^{\ell+1}}\right]P_\ell(\cos\theta)$$

note the indices "i" on A^{i}_{ℓ} and B^{i}_{ℓ} to define it's inside the shell

We don't want the potential to blow up when $r \to 0 \Rightarrow \boxed{B^{i}_{\ell} = 0 \;\forall \ell}$

Therefore
$$\boxed{V^{inside}(r,\theta) = \frac{1}{4\pi\varepsilon_0}\sum_{\ell=0}^{+\infty} A^{i}_{\ell}\, r^{\ell} P_\ell(\cos\theta)}$$

So far, so good.

outside the shell Same reasoning + no blow up for $r \to +\infty$ $\Rightarrow$ $A^o_\ell = 0 \ \forall \ell$

Therefore
$$\boxed{V^{outside}(r,\theta) = \frac{1}{4\pi\varepsilon_0} \sum_{\ell=0}^{+\infty} \frac{B^o_\ell}{r^{\ell+1}} P_\ell(\cos\theta)}$$

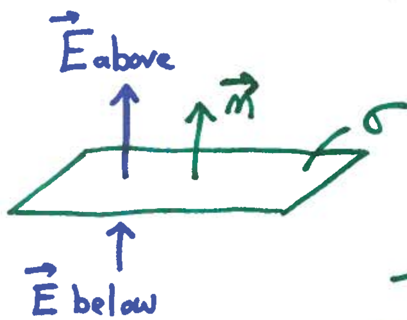
but V is continuous! Therefore $V^{outside}(r=R,\theta) = V^{inside}(r=R,\theta)$

which we can write as
$$\sum_{\ell=0}^{+\infty} \frac{B^o_\ell}{R^{\ell+1}} P_\ell(\cos\theta) = \sum_{\ell=0}^{+\infty} A^i_\ell R^\ell P_\ell(\cos\theta)$$

by orthogonality of P_ℓ's it means $\frac{B^o_\ell}{R^{\ell+1}} = A^i_\ell R^\ell \ \forall \ell$ $\Rightarrow$ $\boxed{A^i_\ell = \frac{B^o_\ell}{R^{2\ell+1}} \ \forall \ell}$

it is now time to use boundary conditions

Reminder



Using Gauss's Law $\vec{E}_{above}\cdot\vec{n} - \vec{E}_{below}\cdot\vec{n} = \sigma/\varepsilon_0$

$$-\frac{\partial V_{above}}{\partial n} + \frac{\partial V_{below}}{\partial n} = \sigma/\varepsilon_0$$

This is linking the two problems rather intimately

Translated to our problem:
$$\boxed{-\frac{\partial V^o(r,\theta)}{\partial r}\bigg|_{r=R} + \frac{\partial V^i(r,\theta)}{\partial r}\bigg|_{r=R} = \sigma(\theta)/\varepsilon_0}$$

③

Therefore as

$$\begin{cases} \left.\dfrac{\partial V^o}{\partial r}\right|_{r=R} = \dfrac{1}{4\pi\varepsilon_0}\displaystyle\sum_{\ell=0}^{+\infty} -(\ell+1)\dfrac{B^o_\ell}{R^{\ell+2}} P_\ell(\cos\theta) \\ \left.\dfrac{\partial V^i}{\partial r}\right|_{r=R} = \dfrac{1}{4\pi\varepsilon_0}\displaystyle\sum_{\ell=0}^{+\infty} \ell A^i_\ell R^{\ell-1} P_\ell(\cos\theta) \end{cases}$$

We get

$$\frac{1}{4\pi\varepsilon_0}\sum_{\ell=0}^{+\infty}\left[\frac{(\ell+1)B^o_\ell}{R^{\ell+2}} + \ell A^i_\ell R^{\ell-1}\right] P_\ell(\cos\theta) = \frac{\sigma(\theta)}{\varepsilon_0}$$

using continuity relationship e.g. $\left(A^i_\ell = \dfrac{B^o_\ell}{R^{2\ell+1}}\right)$

$$\frac{1}{4\pi\varepsilon_0}\sum_{\ell=0}^{+\infty} B^o_\ell\left[\frac{\ell+1}{R^{\ell+2}} + \frac{\ell R^{\ell-1}}{R^{2\ell+1}}\right] P_\ell(\cos\theta) = \frac{\sigma(\theta)}{\varepsilon_0}$$

Same algebra

$$\frac{1}{4\pi\varepsilon_0}\sum_{\ell=0}^{+\infty} B^o_\ell \underbrace{\left[\frac{\ell+1}{R^{\ell+2}} + \frac{\ell}{R^{\ell+2}}\right]}_{\frac{2\ell+1}{R^{\ell+2}}} P_\ell(\cos\theta) = \frac{\sigma(\theta)}{\varepsilon_0}$$

using Fourier's Trick:

$$\frac{1}{4\pi}\sum_{\ell=0}^{+\infty}\int_0^\pi \sin\theta\, P_{\ell'}(\cos\theta)\, P_\ell(\cos\theta)\, B^o_\ell\, \frac{2\ell+1}{R^{\ell+2}}\, d\theta = \int_0^\pi \sin\theta\, P_{\ell'}(\cos\theta)\, \sigma(\theta)\, d\theta \quad \forall \ell'$$

④

and therefore

$$\frac{1}{4\pi}\frac{2}{2\ell'+1}B^{o}_{\ell'}\frac{2\ell'+1}{R^{\ell'+2}} = \int_0^{\pi} \sin\theta\, P_{\ell'}(\cos\theta)\,\sigma(\theta)\,d\theta \qquad \forall \ell'$$

$$\frac{1}{4\pi}\frac{2B^{o}_{\ell'}}{R^{\ell'+2}} = \int_0^{\pi} \sin\theta\, P_{\ell'}(\cos\theta)\,\sigma(\theta)\,d\theta \quad \forall \ell'$$

$\boxed{\ell'=0}$

$$\int_0^{\pi} \sin\theta\,\sigma(\theta)\,d\theta = \int_0^{\pi/2} +\sigma_0 \sin\theta\,d\theta - \int_{\pi/2}^{\pi} \sigma_0 \sin\theta\,d\theta = 0$$

$\boxed{\ell'=1}$

$$\int_0^{\pi} \sin\theta\cos\theta\,\sigma(\theta)\,d\theta = \int_0^{\pi/2} +\sigma_0 \sin\theta\cos\theta\,d\theta - \int_{\pi/2}^{\pi} \sigma_0 \sin\theta\cos\theta\,d\theta$$

$$= \sigma_0 \left[-\frac{\cos^2\theta}{2}\right]_0^{\pi/2} + \sigma_0\left[\frac{\cos^2\theta}{2}\right]_{\pi/2}^{\pi}$$

$$= \sigma_0 \frac{1}{2} + \sigma_0 \frac{1}{2} = \sigma_0 \quad !$$

therefore $\frac{1}{4\pi}\frac{2B^{o}_{1}}{R^3} = \sigma_0 \Rightarrow B^{o}_{1} = 4\pi \cdot \frac{\sigma_0 R^3}{2} \rightarrow A^{i}_{1} = 4\pi\frac{\sigma_0}{2}$

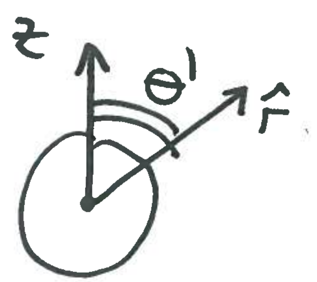
$B^{o}_{0} = 0 \rightarrow A^{i}_{0} = 0$

Therefore, at the end of the day:

$$\boxed{\begin{aligned} V^i(r,\theta) &\simeq \frac{1}{4\pi\varepsilon_0}\cdot\frac{4\pi\sigma_0}{2}\, r\cos\theta = \frac{\sigma_0}{2\varepsilon_0}\, r\cos\theta \\ V^o(r,\theta) &= \frac{1}{4\pi\varepsilon_0}\cdot\frac{4\pi\sigma_0 R^3}{2r^2}\cos\theta = \frac{\sigma_0}{2\varepsilon_0}\,\frac{R^3}{r^2}\cos\theta \end{aligned}}\qquad \begin{aligned} &[O(r^2)] \\ &[O(1/r^3)] \end{aligned}$$

$\boxed{A.2}$

$$V_{dipole}(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\frac{1}{r^2}\,\vec{p}\cdot\hat{r} \quad \text{with } |\vec{p}| = \iiint r'\,P_1(\cos\theta')\,\rho(r')\,d\tau'$$



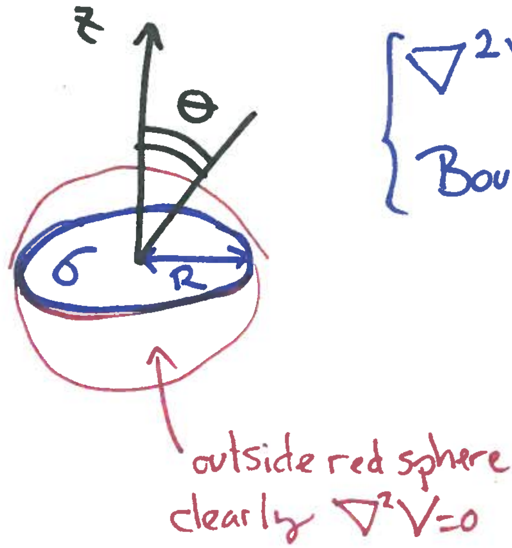
here $|\vec{p}| = \iint \overset{R}{r'}\,P_1(\cos\theta')\,\sigma(\theta')\,da'$

$$|\vec{p}| = \int_0^{2\pi} d\phi' \int_0^{\pi} \sin\theta'\,d\theta'\,R^2\cdot R\cos\theta'\,\sigma(\theta')$$

$$|\vec{p}| = 2\pi R^3 \int_0^{\pi} \sigma(\theta')\sin\theta'\cos\theta'\,d\theta'$$

$$|\vec{p}| = 2\pi R^3\cdot\left[\sigma_0\int_0^{\pi/2}\sin\theta'\cos\theta'\,d\theta' - \sigma_0\int_{\pi/2}^{\pi}\sin\theta'\cos\theta'\,d\theta\right] = 2\pi\sigma_0 R^3\left(\left[\frac{\cos^2\theta'}{2}\right]_{\pi/2}^{\pi} - \left[\frac{\cos^2\theta'}{2}\right]_0^{\pi/2}\right)$$

Problem B



$$\begin{cases} \nabla^2 V = 0 \quad \text{on domain } \Omega = \{r > R, \theta\} \\ \text{Boundary Condition given along } \theta = 0 \text{ axis only} \end{cases}$$

Our guess $\quad V(r,\theta) = \dfrac{1}{4\pi\varepsilon_0} \displaystyle\sum_{\ell=0}^{+\infty} \left[A_\ell r^\ell + B_\ell / r^{\ell+1}\right] P_\ell(\cos\theta)$

Boundary conditions $\bullet$ $\lim\limits_{r\to+\infty} V(r,\theta) = 0$ therefore $A_\ell = 0 \ \forall \ell$

$\bullet$ $V(r,\theta=0) = \dfrac{1}{4\pi\varepsilon_0} \displaystyle\sum_{\ell=0}^{+\infty} \frac{B_\ell}{r^{\ell+1}} P_\ell(\cos 0) = \frac{1}{4\pi\varepsilon_0} \sum_{\ell=0}^{+\infty} \frac{B_\ell}{r^{\ell+1}} \underbrace{P_\ell(1)}_{=1 \forall \ell!} = \frac{1}{4\pi\varepsilon_0} \sum_{\ell=0}^{+\infty} \frac{B_\ell}{r^{\ell+1}}$

$V(r,\theta=0) = V(z)$ therefore $\dfrac{1}{4\pi\varepsilon_0} \displaystyle\sum_{\ell=0}^{+\infty} \frac{B_\ell}{r^{\ell+1}} = \frac{\sigma_0}{2\varepsilon_0}\left(\sqrt{r^2+R^2} - r\right)$ looks weird but can be done by Taylor expansion

$\dfrac{\sigma_0}{2\varepsilon_0}\left(\sqrt{r^2+R^2} - r\right) = \dfrac{\sigma_0}{2\varepsilon_0} r \left(\sqrt{1+\dfrac{R^2}{r^2}} - 1\right) \xrightarrow[R/r\to 0]{} \ ?$

$$(1+\varepsilon)^{1/2} = 1 + \frac{1}{2}\varepsilon - \frac{1}{8}\varepsilon^2 + \frac{1}{16}\varepsilon^3$$

therefore
$$\frac{\sigma_0}{2\varepsilon_0} r \left(\sqrt{1 + R^2/r^2} - 1 \right) = \frac{\sigma_0}{2\varepsilon_0} r \left(1 + \frac{R^2}{2}\frac{1}{r^2} - \frac{R^4}{8}\frac{1}{r^4} + \frac{1}{16}\frac{R^6}{r^6} - 1 + O(1/r^8) \right)$$

$$= \frac{\sigma_0}{2\varepsilon_0} \left(\frac{R^2}{2}\frac{1}{r} - \frac{R^4}{8}\frac{1}{r^3} + \frac{R^6}{16}\frac{1}{r^5} + O(1/r^7) \right)$$

and subsequently:

$$\frac{1}{4\pi\varepsilon_0} \left[\frac{B_0}{r} + \frac{B_1}{r^2} + \frac{B_2}{r^3} + \frac{B_3}{r^4} + \frac{B_4}{r^5} + \dots \right] = \frac{\sigma_0}{2\varepsilon_0} \left[\frac{R^2}{2}\frac{1}{r} - \frac{R^4}{8}\frac{1}{r^3} + \frac{R^6}{16}\frac{1}{r^5} + \dots \right]$$

therefore:
$$\begin{cases} B_0 = \pi\sigma_0 R^2 \\ B_1 = 0 \\ B_2 = \dfrac{\pi\sigma_0 R^4}{4} \\ B_3 = 0 \\ B_4 = \dfrac{\pi\sigma_0 R^6}{16} \end{cases}$$