Monday, November 16, 2015

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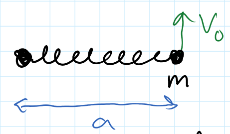
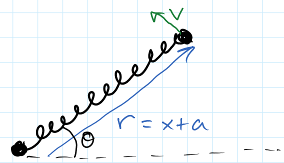
Quiz 2 Solutions

t=0 

t>0 

1. a)

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The restoring force on the mass is $F = -Kx$ where $x = (r-a)$, the extension of the spring from its equilibrium length

In polar coordinates $\ddot{\vec{r}} = (\ddot{r} - r\dot{\theta}^2)\hat{r} + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\hat{\theta}$

So, the 2 equations of motion are

$$m(\ddot{r} - r\dot{\theta}^2) = -Kx = -K(r-a)$$
$$m(2\dot{r}\dot{\theta} + r\ddot{\theta}) = 0$$

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b) From 2nd e.o.m, $m(2\dot{r}\dot{\theta} + r\ddot{\theta}) = 0$

$\times$ by r, $m(2r\dot{r}\dot{\theta} + r^2\ddot{\theta}) = 0$

$$\rightarrow m\frac{d}{dt}(r^2\dot{\theta}) = 0$$

$$r^2\dot{\theta} = \text{constant}$$

but angular momentum, $J = mr^2\dot{\theta}$, so the constant is $\frac{J}{m}$.

At $t=0$, $r=a$; velocity V_0 is in the θ direction, so $r\dot{\theta} = V_0$

$$\therefore \; aV_0 = \frac{J}{m}$$

At max. extent, $r = 2a$ and total speed must again be in the θ direction so

$$2aV = \frac{J}{m} = aV_0$$

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$$\therefore \; V = \frac{V_0}{2}$$

c) Cons. of energy : $\frac{1}{2}mv^2 + V(r) = E$

$$F = -K(r-a) = -Kx$$

$$\therefore F = -\frac{dV}{dx} = -Kx$$

$$\frac{dV}{dx} = Kx$$

$$V = \frac{1}{2}Kx^2 + \cancel{\text{const}} \rightarrow 0$$

$$= \frac{1}{2}K(r-a)^2$$

$$\therefore \frac{1}{2}mv^2 + \frac{1}{2}K(r-a)^2 = E$$

$$\frac{1}{2}m\dot{r}^2 + \frac{1}{2}m(r\dot{\theta})^2 + \frac{1}{2}K(r-a)^2 = E$$

At $t=0$, $r=a$, $\dot{r}=0$; $r\dot{\theta}=v_0$, so

$$\frac{1}{2}mv_0^2 = E$$

$\therefore$ At max extent, $r=2a$, $\dot{r}=0$, $r\dot{\theta}=\frac{v_0}{2}$ (from (b))

$$\cancel{\frac{1}{2}}m\frac{v_0^2}{4} + \cancel{\frac{1}{2}}Ka^2 = \cancel{\frac{1}{2}}mv_0^2$$

$$Ka^2 = \frac{3}{4}mv_0^2$$

$$\therefore v_0^2 = \frac{4Ka^2}{3m} \quad 3$$

d) From part (a)

$$m(\ddot{r} - r\dot{\theta}^2) = -K(r-a)$$

$$\ddot{r} - \frac{r^2\dot{\theta}^2}{r} = -\frac{K}{m}(r-a)$$

When $r=2a$, $r\dot{\theta}=\frac{v_0}{2}$

$$\ddot{r} - \frac{v_0^2}{4(2a)} = -\frac{K}{m}a$$

$$\ddot{r} = -\frac{Ka}{m} + \frac{v_0^2}{8a} = -\frac{Ka}{m} + \frac{1}{\cancel{8a}_2}\frac{\cancel{4}Ka^{\cancel{2}}}{3m} = -\frac{Ka}{m} + \frac{1}{6}\frac{Ka}{m}$$

$$\therefore \ddot{r} = -\frac{5}{6}\frac{Ka}{m} \quad 2$$

2. a) If $u = a(1+b\theta) = a + ab\theta$ and the orbit eq'n is $\frac{d^2u}{d\theta^2} + u = -\frac{f(u^{-1})}{ml^2u^2}$ where $l = r^2\dot{\theta} = \frac{J}{m}$

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$$\frac{du}{d\theta} = ab$$

$$\frac{d^2u}{d\theta^2} = 0 \rightarrow 0 + u = -\frac{f(u^{-1})}{ml^2u^2}$$

$$\rightarrow f(u^{-1}) = -ml^2u^3$$

so $f(r) = -\frac{ml^2}{r^3}$ or $f(r) = -\frac{J^2}{r^3m}$ 5

b) Angular momentum is always conserved for central force motion

$$\therefore \quad r^2\dot{\theta} = \ell = \text{constant}$$

$$\text{so } \dot{\theta} = \frac{\ell}{r^2} = \ell u^2 = \ell a^2(1+b\theta)^2$$

$$\therefore \int \frac{d\theta}{(1+b\theta)^2} = \ell a^2 \int dt$$

$$-\frac{1}{b(1+b\theta)} = \ell a^2 t + \text{const.}$$

At $t=0$, $\theta=0 \rightarrow -\frac{1}{b} = \text{const.}$

$$\therefore \quad -\frac{1}{b(1+b\theta)} = \ell a^2 t - \frac{1}{b}$$

$$\frac{1}{1+b\theta} = -\ell a^2 b t + 1$$

$$1 = (1+b\theta)(1-\ell a^2 b t)$$

$$\frac{1}{(1-\ell a^2 b t)} = 1 + b\theta$$

$$b\theta = \frac{1}{1-\ell a^2 b t} - 1 = \frac{1-1+\ell a^2 b t}{1-\ell a^2 b t} = \frac{\ell a^2 b t}{1-\ell a^2 b t}$$

$$\therefore \theta = \frac{\ell a^2 t}{1-\ell a^2 b t} = \frac{\ell a^2 t}{1-\ell a^2 b t} \quad 5$$

3. A 3D isotropic harmonic oscillator has a potential energy function $V(r) = \frac{1}{2}Kr^2$ where r is the distance from the origin and K is a constant.

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Conservation of energy for this oscillator is $E = \frac{1}{2}mv^2 + \frac{1}{2}Kr^2$

At $t=0$, $r=0$; $v = v_0 \rightarrow E = \frac{1}{2}mv_0^2$

When the particle is at its max. value of r, v will be 0, but total energy will be the same, so

$$\frac{1}{2}mv_0^2 = \frac{1}{2}Kr_{max}^2$$

$$\therefore r_{max}^2 = \frac{mv_0^2}{K} = \frac{v_0^2}{\omega^2}$$

$$\therefore \; r_{max}^2 = \frac{mV_0^2}{K} = \frac{V_0^2}{\omega_0^2}$$

$$\text{or} \quad r_{max} = \frac{V_0}{\omega_0} \qquad 7$$