Georgia Institute of Technology

Physics 3201 – Classical Mechanics I

Final Exam

December 7, 2020; 8:00am – 10:50am

Instructor:	Prof. D. Ballantyne
Time Allowed:	2 hours 50 minutes + 10 minutes to submit to the Instructor

To Be Answered On Paper, Scanned and Uploaded to Canvas

This exam consists of 8 questions. 2 questions have multiple parts, and 6 questions have just a single part. Answer all questions. Make sure to include justification for your answers. *Correct answers without appropriate justification may not be awarded full (or any) marks.*

This exam consists of 4 pages, including the cover page (this one). In addition, there is 1 page of potentially useful mathematical formulas and identities. Students should count the pages on their exam prior to beginning and report any omissions to their instructor.

This is an open-notes/open-book exam. Students may consult any class materials posted to Canvas, the assigned textbook, or the other books listed under "Other Resources" in the class syllabus. No other resources, either online (i.e., Chegg.com) or physical (i.e., your roommate's notes), can be used while taking this test. Communication between students is also not allowed during the exam window. However, students are encouraged to contact the instructor if they need clarification on any of the test questions.

Students who make use of unauthorized materials, communicate with each other during the exam, or appear to engage in similar dishonest practices may be dismissed from the exam and subject to further academic discipline.

1. (5 pts) A particle of constant mass m accelerates along the x-axis under the influence of a force $F(x) = F_0 e^{-ax}$ where F_0 and a are constants. If the particle's speed vanishes at $x = 0$, find its speed as a function of x.

2. (10 pts) A particle of constant mass m oscillates between x_1 and x_2 due to a conservative force. Show that the period of oscillation is

$$\tau = 2\int_{x_1}^{x_2} \sqrt{\frac{m}{2[V(x_2) - V(x)]}}\,dx.$$

   In particular, if $V = \frac{1}{2}m\omega_0^2(x^2 - bx^4)$, show that the period for oscillations of amplitude a is

$$\tau = \frac{2}{\omega_0}\int_{-a}^{a} \frac{dx}{\sqrt{a^2 - x^2}\sqrt{1 - b(a^2 + x^2)}}.$$

   Using the binomial theorem to expand in powers of b, and the substitution $x = a\sin\theta$, show that for small amplitude the period is approximately $\tau \approx 2\pi(1 + \frac{3}{4}ba^2)/\omega_0$.

3. (7 pts) A particle with fixed mass m experiences a conservative force towards the origin according to $F = -mk^2/x^3$, where k is a constant. If the particle starts from rest at a position d from the origin, show that the time required for the particle to reach the origin is d^2/k.

4. (14 pts)

   (a) (3 pts) Give three different mathematical statements that must be true if the three-dimensional force, $\vec{F}(\vec{r})$, is a conservative force.

   (b) (3 pts) Using words and not equations, give two examples of conservative forces found in the real world, and one example of a non-conservative force.

   (c) (2 pts) What conservation law can be precisely defined for a conservative force? Why can you not write down a similar conservation law for a non-conservative force?

   (d) (1 pt) What conservation law can be defined for a *central*, conservative force?

   (e) (5 pts) Find which of the following forces are conservative, and for those that are find the corresponding potential energy function (a and b are constants).

      i. $F_x = ax + by^2, \quad F_y = az + 2bxy, \quad F_z = ay + bz^2$
      ii. $F_x = ay, \quad F_y = az, \quad F_z = ax$
      iii. $F_r = 2ar\sin\theta\ \sin\phi, \quad F_\theta = ar\cos\theta\sin\phi, \quad F_\phi = ar\cos\phi$

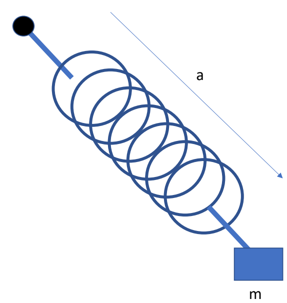

Figure 1: Diagram for Problem #4. The system lies in a horizontal plane.

5. (15 pts) A particle of mass m is attached to the end of a light spring of equilibrium length a, whose other end is fixed, so that the spring is free to rotate in a horizontal plane (Fig. 1). The force exerted by the spring is k times its extension.

   (a) (5 pts) Write down the effective potential function $U(r)$ for the system, where r is the distance of the particle from the fixed end of the spring.

   (b) (4 pts) The particle is moving in a circular orbit of radius $2a$. Find the orbital angular velocity ω in terms of the natural angular frequency ω_0 of the oscillator when not rotating.

   (c) (6 pts) If the motion is lightly disturbed, the particle will execute small oscillations about the circular orbit. By considering the effective potential energy function $U(r)$ near this equilibrium point, find the angular frequency ω' of small oscillations in terms of ω_0.

6. (10 pts) Consider a spherical star moving through the Galaxy with velocity v. It passes through a uniform cloud of particles with density ρ. The cloud is much larger than the radius of the star, R. If all the particles that collide with the star are trapped by it show that the mass of the star, M, will increase at a rate

$$\frac{dM}{dt} = \pi \rho v \left(R^2 + \frac{2GMR}{v^2} \right)$$

7. (10 pts) If a projectile is fired due east from a point on the surface of the Earth at a northern latitude of λ with a velocity of magnitude v_0 and at an angle of inclination to the horizontal of α, show that, assuming the vertical Coriolis force is negligible, the lateral deflection when the projectile strikes Earth is

$$d = -\frac{4v_0^3}{g^2}\omega \sin\lambda \sin^2\alpha \cos\alpha,$$

where ω is the rotation frequency of the Earth. Is the deflection to the North or the South?

8. (10 pts) Consider a low density spherical cloud with density ρ. It is initially at rest, and then begins to collapse under its own gravitational attraction. Find the radial velocity of a particle which starts at a distance a from the center when it reaches the distance r. Hence, neglecting other forces, show that every particle will reach the center at the same instant, and that the time taken is

$$t = \sqrt{\frac{3\pi}{32\rho G}}.$$

Hint: Assume that particles do not overtake those that start nearer the center. The substitution $r = a\sin^2\theta$ may be used to perform the integration.

Useful Equations and Relationships

- For any scalar function $t(x, y, z)$ and vector function $\vec{v}(x, y, z)$ defined in Cartesian coordinates,

$$\vec{\nabla} t = \frac{\partial t}{\partial x}\hat{x} + \frac{\partial t}{\partial y}\hat{y} + \frac{\partial t}{\partial z}\hat{z}$$

  and

$$\vec{\nabla} \times \vec{v} = \left(\frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z}\right)\hat{x} + \left(\frac{\partial v_x}{\partial z} - \frac{\partial v_z}{\partial x}\right)\hat{y} + \left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y}\right)\hat{z}$$

- For any scalar function $t(r, \theta, \phi)$ and vector function $\vec{v}(r, \theta, \phi)$ defined in spherical coordinates,

$$\vec{\nabla} t = \frac{\partial t}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial t}{\partial \theta}\hat{\theta} + \frac{1}{r\sin\theta}\frac{\partial t}{\partial \phi}\hat{\phi}$$

  and

$$\vec{\nabla} \times \vec{v} = \frac{1}{r\sin\theta}\left[\frac{\partial}{\partial \theta}(\sin\theta v_\phi) - \frac{\partial v_\theta}{\partial \phi}\right]\hat{r} + \frac{1}{r}\left[\frac{1}{\sin\theta}\frac{\partial v_r}{\partial \phi} - \frac{\partial}{\partial r}(r v_\phi)\right]\hat{\theta} + \frac{1}{r}\left[\frac{\partial}{\partial r}(r v_\theta) - \frac{\partial v_r}{\partial \theta}\right]\hat{\phi}$$

- $\int \frac{x}{\sqrt{a^2 - x^2}} dx = -\sqrt{a^2 - x^2}$
- $\int \sin^2 x dx = \frac{x}{2} - \frac{1}{4}\sin 2x$
- Binomial Expansion:
  $(1 - x)^n = 1 - nx + \frac{n(n-1)}{2!}x^2 - \frac{n(n-1)(n-2)}{3!}x^3 + \cdots, \quad |x| < 1$