Additional Review Set for Midterm 1

This is a set of questions that students can use to help them prepare for an upcoming exam. Note that the topics and sections from the textbook that the midterms covers are listed in the syllabus.

Topics Covered in this Review Set

This review set focuses on following sections from Lay 5^{th} Edition:

- 1.1, 1.2, 1.3, 1.4, 1.5, 1.7, 1.8, 1.9, 2.1

Review Set Solutions

A primary goal of this course is to prepare students for more advanced courses that have this course as a pre-requisite. To help us meet this goal, *solutions are not are provided for the additional review problem sets*. This is intentional: most upper level courses do not have recitations, MML, and solutions for everything. Students must develop and implement their own strategies to check their solutions in those courses.

In this course, students are encouraged to ask questions they may have about the course on Piazza, office hours, by checking their answers with their peers, or by asking their instructor after class. Calculators and software are also great ways to check your work. All of these methods are valuable skills that are transferable to higher level courses, and beyond.

Recommended Strategies for Preparing for Midterms

Students are encouraged to prepare for their exams by:

1. completing all of the problems in the practice review sets,
2. completing additional problems from the textbook,
3. spend most of their preparing by solving problems and solving lots of problems,
4. studying with other people and comparing study strategies with them,
5. asking questions during office hours and/or piazza,
6. start studying as early possible,
7. spread your studying out over many days,
8. identify your goals (i.e. - what you want to accomplish by taking this course), write them down, and align your study strategies with them,
9. review the learning objectives of the course and align your study strategies with them,
10. organize your studying: schedule times you want to study in an online calendar or in your planner.

There are *many* other effective study strategies and resources that you can take advantage of at Georgia Tech. Take time to think about what works for you to reach your goals. And ask other students, your instructors, and TAs what approaches they recommend for preparing for exams.

The MML Study Plan

The MML Study Plan has hundreds of problems you can solve that are automatically graded for you. MML will tell you if your work is correct and offers a few different study aids. To access the study plan for a specific textbook section:

1. navigate to mymathlab.com and log in
2. select your course
3. select Lay Linear Algebra (the online textbook)
4. select a chapter
5. select a section
6. click study plan

1 True/False Exercises

Indicate whether the statement is true or false.

1.1) If a linear system has more equations than unknowns, then the system can have a unique solution.

1.2) If a linear system has more unknowns than equations, then the system can have a unique solution.

1.3) If for every $\vec{b}$, $A\vec{x} = \vec{b}$ has at least one solution, there must be a pivot in every row of A.

1.4) If $A\vec{x} = A\vec{y}$ for some $\vec{x} \neq \vec{y}$, then A could have a pivot in every row, but A cannot have a pivot in every column.

1.5) If $AB = AC$, then $B = C$.

1.6) If matrix B has two columns, the columns of B are $\vec{b}_1$ and $\vec{b}_2$, so that $B = \begin{bmatrix}\vec{b}_1 & \vec{b}_2\end{bmatrix}$, then $AB = \begin{bmatrix}A\vec{b}_1 & A\vec{b}_2\end{bmatrix}$.

1.7) If $A \in \mathbb{R}^{m\times n}$ has linearly dependent columns, then the columns of A cannot span $\mathbb{R}^m$.

1.8) If A and B are 2×2 matrices, the columns of B are $\vec{b}_1$ and $\vec{b}_2$, and $\vec{b}_1 = \vec{b}_2$, then the columns of AB are linearly dependent.

1.9) If $A \in \mathbb{R}^{2\times 2}$ and A has linearly dependent columns, the span of the columns of A is a line that passes through the origin.

1.10) If there are some vectors $\vec{b} \in \mathbb{R}^m$ that are not in the range of $T(\vec{x}) = A\vec{x}$, then there cannot be a pivot in every column of A.

1.11) If transform T is linear, then $T(\vec{u} + \vec{v}) = T(\vec{u}) + T(\vec{v})$ for all $\vec{u}$ and $\vec{v}$ in the domain of T.

1.12) If $\vec{x}_1$ is a solution to the inhomogeneous system $A\vec{x} = \vec{b}$, then any vector in Span$\{\vec{x}_1\}$ is also a solution to $A\vec{x} = \vec{b}$.

1.13) If A is 2×3, then the transformation $\vec{x} \to A\vec{x}$ cannot be one-to-one.

1.14) If a linear system is consistent, then the solution set either contains a unique solution when there are no free variables, or infinitely many solutions when there is at least one free variable.

Note that students are welcome (and encouraged) to post their answers to these questions on Piazza to discuss them. Likewise with the remaining problems on this worksheet.

2 Example Construction Exercises

If possible, give one example of the following.

2.1) A 4×2 non-zero matrix A, that is in RREF, such that $A\vec{x} = \vec{0}$ has a non-trivial solution.

2.2) A 3×7 matrix A, in RREF, with 2 pivot columns, such that $A\vec{x} = \vec{b}$ has exactly 1 free variable.

2.3) A 3×7 matrix A, in RREF, with 2 pivot columns, such that $A\vec{x} = \vec{b}$ has exactly 5 free variables.

2.4) A homogeneous linear system that has no solutions.

2.5) A homogeneous linear system with two equations and two unknowns that has only one solution.

2.6) A is a 3×3 matrix in RREF, has linearly dependent columns, and exactly two pivot columns.

2.7) A 2×2 non-zero matrix in echelon form that does not commute with $A = \begin{pmatrix} 1 & 0 \\ 0 & 7 \end{pmatrix}$.

2.8) a 2×3 non-zero matrix in echelon form that is the standard matrix for linear transform T. T is not one-to-one and is not onto.

2.9) A 5×2 non-zero matrix in echelon form that is the standard matrix for linear transform T. T is not one-to-one and is not onto.

2.10) A matrix, A, that is the standard matrix for the linear transform $T_A : \mathbb{R}^2 \to \mathbb{R}^2$. T_A first reflects points across the line $x_1 = x_2$, and then projects them onto the x_2-axis.

2.11) For each linear transform T, give one example of a standard matrix A associated to T, that is also in RREF.

(a) $T_A : \mathbb{R}^3 \to \mathbb{R}^4$ is one-to-one.
(b) $T_A : \mathbb{R}^4 \to \mathbb{R}^3$ is onto.
(c) $T_A : \mathbb{R}^4 \to \mathbb{R}^3$ is onto, and the second column of A is not pivotal.
(d) Do (a)-(c), using only 0 and 1 as entries, and as few 1's as possible.
(e) Do (a)-(c), using only 0 and 1 as entries, and as many 1's as possible.

3 Multiple Choice, Short Answer, and Fill in the Blank Exercises

3.1) We often have one of two possible goals for applying row reduction algorithms: express a system in echelon form, or in RREF. And we have so far explored many different kinds of questions that are related the row reduction algorithm. You will need to know whether you need to express a system in echelon, or RREF.

For the questions below, A is a $m \times n$ matrix, and T_A is the linear transformation associated to A. For each question choose whether the echelon form (EF) or the RREF is needed.

EF **RREF** Identify a solution to $A\vec{x} = \vec{b}$.

EF **RREF** Compute the coefficients of the polynomial that passes through a set of points.

EF **RREF** Express the set of solutions to $A\vec{x} = \vec{b}$ in parametric vector form.

EF **RREF** Is the system $A\vec{x} = \vec{b}$ consisistent?

EF **RREF** Which columns in a given matrix A are pivotal/free?

EF **RREF** Determine whether the vectors $\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4$ are linearly independent.

EF **RREF** For what values of h is $\vec{y}$ in the span of $\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4$?

EF **RREF** Determine whether a linear transformation $T_A : \mathbb{R}^n \mapsto \mathbb{R}^m$ is one-to-one.

EF **RREF** Is a linear transformation $T_A : \mathbb{R}^n \mapsto \mathbb{R}^m$ onto?

3.2) Matrix A has dimensions 5×6, and has exactly 2 linearly independent columns.

(a) For the system $A\vec{x} = \vec{b}$, how many free variables could there be?

(b) True or False: $A\vec{x} = \vec{b}$ has a solution for all $\vec{b}$.

3.3) Consider the following linear transformation, and answer the questions below.

$$T(x_1, x_2) = \left(\frac{1}{2}x_1 - \frac{1}{2}x_2, -\frac{1}{2}x_1 + \frac{1}{2}x_2\right)$$

(a) What is the domain of T?

(b) What is the co-domain of T?

(c) What is the standard matrix of T?

(d) What are the images of the standard vectors $\vec{e}_1$ and $\vec{e}_2$?

(e) Draw a sketch of $\vec{e}_1$, $\vec{e}_2$, $T(\vec{e}_1)$ and $T(\vec{e}_2)$ on one picture.

(f) Is T onto?

(g) Is T one-to-one?

3.4) Suppose $A \in \mathbb{R}^{5\times 3}$, $B \in \mathbb{R}^{5\times 5}$ and $\vec{x} \in \mathbb{R}^3$. Circle the operations that are defined. You don't need to justify your reasoning for this question.

$$(BA)^T\vec{x} \qquad A^TA\vec{x} \qquad A^2 \qquad \vec{x}\vec{x}^T \qquad (A^TB)^T\vec{x} \qquad A^T\vec{x}$$

3.5) Circle the matrices that are in RREF. You don't need to justify your reasoning for this question.

$$\begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} \qquad \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \qquad \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

3.6) State whether the following sets of vectors span a point, a line, a plane, or all of $\mathbb{R}^3$. Give your answer without doing any row reductions.

(a)

$$\begin{bmatrix} 1 \\ -2 \\ 5 \end{bmatrix}, \begin{bmatrix} 4 \\ -8 \\ 20 \end{bmatrix}$$

(b)

$$\begin{bmatrix} 1 \\ -2 \\ 5 \end{bmatrix}, \begin{bmatrix} 4 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ \frac{1}{2} \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

3.7) Suppose $A \in \mathbb{R}^{m\times n}$, $m < n$, and $A\vec{x} = \vec{b}_1$ has no solutions for some $\vec{b}_1 \in \mathbb{R}^m$. How many solutions could $A\vec{x} = \vec{b}_2$ have, if $\vec{b}_1 \neq \vec{b}_2$?

4 Computation Exercises

4.1) If possible, express all solutions to the linear system in parametric vector form.

$$\begin{aligned} x_1 \quad +3x_2 \qquad\qquad &= 7 \\ 3x_1 \quad +9x_2 \quad -5x_3 &= 6 \end{aligned}$$

4.2) Let $\vec{v}_1 = \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}, \vec{v}_2 = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}, \vec{v}_3 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$.

(a) Is the set $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ linearly dependent?

(b) Give a geometric description of Span$\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$.

(c) Write $\vec{u} = \begin{pmatrix} 2 \\ 6 \\ -2 \end{pmatrix}$ as a linear combination of $\vec{v}_1$, $\vec{v}_2$, and $\vec{v}_3$.

4.3) Let

$$\vec{e}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad \vec{e}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad \vec{e}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

Also let $T : \mathbb{R}^a \to \mathbb{R}^b$ be the linear transformation that satisfies

$$T(\vec{e}_1) = \begin{pmatrix} 1 \\ 2 \\ 2 \\ 0 \end{pmatrix}, \quad T(\vec{e}_2) = \begin{pmatrix} 0 \\ 0 \\ 2 \\ -1 \end{pmatrix}, \quad T(\vec{e}_3) = \begin{pmatrix} -3 \\ -6 \\ 2 \\ 4 \end{pmatrix}.$$

(a) What are a and b?

(b) What is $T(\vec{u})$, where $\vec{u} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$?

(c) Is T onto?

(d) Is T one to one?

4.4) For each of the following situations, determine (a) whether $A\vec{x} = \vec{0}$ has a nontrivial solution and (b) whether $A\vec{x} = \vec{b}$ has at least one solution for every possible $\vec{b}$ in $\mathbb{R}^m$. Some situations may not be possible.

(i) A is a 3×4 matrix with 4 pivots

(ii) A is a 3×3 matrix with 3 pivots.

(iii) A is a 3×3 matrix with 2 pivots.

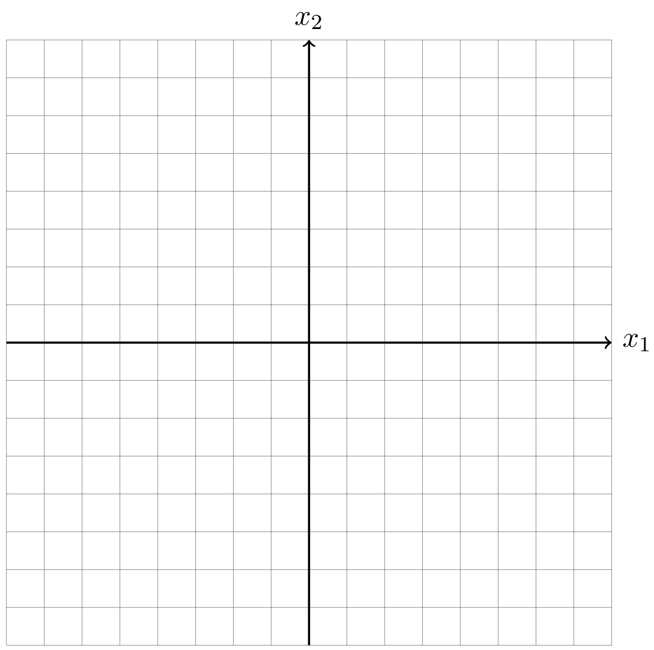
(iv) A is a 3×2 matrix with 2 pivots.

(v) A^T is a 4×2 matrix with 2 pivots.

(vi) A^T is a 4×2 matrix with 3 pivots.

5 Sketching Questions

5.1) Sketch the span of the columns of $A = \begin{pmatrix} 1 & -2 \\ 4 & -8 \end{pmatrix}$. On the same grid sketch a non-zero solution to $A\vec{x} = \vec{0}$ and its span.



5.2) T is the linear transform $x \to Ax$ that projects points in $\mathbb{R}^2$ onto the x_2-axis. Sketch the range of the transform. On the same grid sketch any non-zero vector, $\vec{y}$, whose image under T is the zero vector.