Additional Review Set for Midterm 3

This is a set of questions that students can use to help them prepare for an upcoming exam. Note that the topics and sections from the textbook that the midterms covers are listed in the syllabus.

Topics Covered in this Review Set

This review set focuses on the following sections from Lay 5^{th} Edition:

- 5.3, 5.5, Google Page Rank, 6.1, 6.2, 6.3, 6.4, 6.5, 6.6

Review Set Solutions

A primary goal of this course is to prepare students for more advanced courses that have this course as a pre-requisite. To help us meet this goal, *solutions are not are provided for the additional review problem sets*. This is intentional: most upper level courses do not have recitations, MML, and solutions for everything. Students must develop and implement their own strategies to check their solutions in those courses.

In this course, students are encouraged to ask questions they may have about the course on Piazza, office hours, by checking their answers with their peers, or by asking their instructor after class. Calculators and software are also great ways to check your work. All of these methods are valuable skills that are transferable to higher level courses, and beyond.

Contents

Recommended Strategies for Preparing for Midterms

Students are encouraged to prepare for their exams by:

1. completing all of the problems in the practice review sets,
2. completing additional problems from the textbook,
3. spend most of their preparing by solving problems and solving lots of problems,
4. studying with other people and comparing study strategies with them,

5. asking questions during office hours and/or piazza,
6. start studying as early possible,
7. spread your studying out over many days,
8. identify your goals (i.e. - what you want to accomplish by taking this course), write them down, and align your study strategies with them,
9. review the learning objectives of the course and align your study strategies with them,
10. organize your studying: schedule times you want to study in an online calendar or in your planner.

There are *many* other effective study strategies and resources that you can take advantage of at Georgia Tech. Take time to think about what works for you to reach your goals. And ask other students, your instructors, and TAs what approaches they recommend for preparing for exams.

The MML Study Plan

The MML Study Plan has hundreds of problems you can solve that are automatically graded for you. MML will tell you if your work is correct and offers a few different study aids. To access the study plan for a specific textbook section:

1. navigate to mymathlab.com and log in
2. select your course
3. select Lay Linear Algebra (the online textbook)
4. select a chapter
5. select a section
6. click study plan

1 True/False Exercises

Indicate whether each statement is true or false.

1.1) The $n \times n$ zero matrix can be diagonalized.

1.2) Matrix A is a 3×3 matrix with two eigenvalues, λ_1 and λ_2. The geometric multiplicity of λ_1 is 1, and the geometric multiplicity of λ_2 is 2. A is diagonalizable.

1.3) If $A \in \mathbb{R}^{2\times 2}$, and $\lambda = -1$, then A reflects eigenvectors that are associated with λ through a line that passes through the origin.

1.4) If $A \in \mathbb{R}^{2\times 2}$ is singular, then at least one of the eigenvalues of A can have a non-zero imaginary component.

1.5) An eigenvector of a matrix could be associated with two distinct eigenvalues.

1.6) If $A \in \mathbb{R}^{8\times 8}$, has 4 eigenvalues, each eigenvalue has algebraic multiplicity 1, and the geometric multiplicity of each eigenvalue is 2, then A can be diagonalized.

1.7) If A has eigenvalue λ, then A^T has eigenvalue λ.

1.8) The 3×3 zero matrix is diagonalizable.

1.9) If A is diagonal, then A can be diagonalized.

1.10) If A is triangular, then A can be diagonalized.

1.11) If A is a 9×9 matrix with 3 distinct eigenvalues, and the eigenspace corresponding to one of the eigenvalues has dimension 7, then A is diagonalizable.

1.12) Any stochastic matrix with a zero entry can not be regular.

1.13) If a stochastic matrix is not regular then it cannot have a steady state.

1.14) A diagonalization of a matrix is unique.

1.15) If $\vec{v} \in \mathbb{R}^n$ is an eigenvector of a square matrix, then all of the elements of $\vec{v}$ cannot be equal to zero.

1.16) The eigenvalues of a triangular matrix are the elements on the diagonal of the matrix.

1.17) To find the eigenvalues of square matrix A, we can row reduce A to echelon form, and then read off the pivots.

1.18) If A is $n \times n$, and the rank of $A - \lambda I$ is n, then λ is an eigenvalue of A.

1.19) If 2 is an eigenvalue of A, and A is invertible, then $\frac{1}{2}$ is an eigenvalue of A^{-1}.

1.20) All eigenvalues are non-zero scalars.

1.21) All eigenvectors are non-zero vectors.

1.22) An example of a regular stochastic matrix is $P = \begin{pmatrix} 1 & 0.2 \\ 0 & 0.8 \end{pmatrix}$.

1.23) An example of a regular stochastic matrix is $\begin{pmatrix} 1/3 & 1/2 & 1/3 \\ 1/3 & 0 & 1/3 \\ 1/3 & 1/2 & 1/3 \end{pmatrix}$.

1.24) A and B are square $n \times n$ matrices. If 0 is an eigenvalue of the product AB, then 0 is an eigenvalue of BA.

1.25) If A is square $n \times n$ and has n linearly independent eigenvectors, then so does A^T.

1.26) If the determinant of a matrix is zero, then zero is a factor of the characteristic polynomial of the matrix.

1.27) If A is diagonalizable, then so is A^k for $k = 2, 3, 4, \ldots$

1.28) If A is $n \times n$, and A is diagonalizable, then A has n distinct eigenvalues.

1.29) If $\vec{y}$ is in subspace W, and $\vec{y} \in W^{\perp}$, then $\vec{y} = \vec{0}$.

1.30) Any vector $\vec{v} \in \mathbb{R}^n$ is a projection of some other vector, $\vec{u} \in \mathbb{R}^n$, where $\vec{u} \neq \vec{v}$, onto a line.

1.31) The orthogonal compliment of the subspace $V = \{\vec{x} \in \mathbb{R}^3 \mid x_1 = x_3\}$ is a line.

1.32) If W is the line in $\mathbb{R}^3$ spanned by $\vec{v} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$, then $W^{\perp}$ is the plane $ax_1 + bx_2 + cx_3 = 0$.

1.33) Every subspace of $\mathbb{R}^n$, W, contains a vector that is in W and the orthogonal complement $W^{\perp}$.

1.34) If a linear system has infinitely many solutions, then it also has infinitely many least squares solutions.

1.35) If A is 3×4, then Col$(A)^{\perp}$ is a subspace in $\mathbb{R}^3$.

1.36) If A has QR factorization $A = QR$, then the columns of R span Col(A).

1.37) If A has QR factorization $A = QR$, then the columns of Q span Col(A).

1.38) If $\vec{v}$ is a vector in $\mathbb{R}^n$ and W is a subspace, then $\text{proj}_W(\text{proj}_W \vec{v}) = \text{proj}_W \vec{v}$.

1.39) The least squares solution, $\hat{x}$, to a linear system $A\vec{x} = \vec{b}$ is the vector that minimizes the quantity $||A\hat{x} - \vec{b}||^2$.

1.40) If $A \in \mathbb{R}^{m \times n}$ and $m \neq n$, then the vectors in Null(A^T) are in $\mathbb{R}^n$.

1.41) An orthogonal matrix is a standard matrix for a linear transform that preserves lengths.

1.42) If A and B are orthogonal square matrices, then so is $A + B$.

1.43) If A and B are orthogonal square matrices, then so is AB.

1.44) If $A \in \mathbb{R}^{n \times n}$ is an orthogonal matrix, then $A\vec{x} = \vec{b}$ has at least one solution for all $\vec{b} \in \mathbb{R}^n$.

1.45) The dimension of the null space of an orthogonal matrix is always equal to 0.

1.46) Orthogonal $n \times n$ matrices are non-singular.

1.47) An orthogonal matrix cannot be the standard matrix for a transformation that performs a dilation with scale factor k, unless $k = 1$.

1.48) If S is a subspace, then $\text{proj}_S \vec{u}$ is a vector in S.

1.49) For any matrix A, $A^T A$ is the matrix of dot products of the columns of A.

1.50) If $A^T A$ is invertible, then the columns of A are linearly independent.

1.51) If V is a subspace spanned by $\vec{v}_1$ and $\vec{v}_2$, and $\vec{x} \cdot \vec{v}_1 = \vec{x} \cdot \vec{v}_2 = 0$, then $\vec{x} \in V^{\perp}$.

1.52) The length of each column of a stochastic matrix is 1.

1.53) If the columns of an $n \times n$ matrix A are orthonormal, then the linear mapping $\vec{x} \mapsto A\vec{x}$ preserves lengths.

1.54) Every linearly independent set of nonzero vectors in $\mathbb{R}^n$ is orthogonal.

1.55) If $\vec{x}_1$ and $\vec{x}_2$ are non-zero linearly independent vectors in $\mathbb{R}^n$ that span subspace W, then an orthogonal basis for W is the set $\{\vec{x}_1, \vec{x}_2 - \text{proj}_{\vec{x}_1}\vec{x}_2\}$.

1.56) If $A \in \mathbb{R}^{n\times n}$ and $\vec{x}$ and $\vec{y}$ are vectors in $\mathbb{R}^n$, then $A\vec{x} \cdot A\vec{y} = \vec{x}^T A^T A\,\vec{y}$.

1.57) If $\vec{u}$ is in subspace S, then the projection of $\vec{u}$ onto $S^\perp$ is $\vec{0}$.

1.58) If S is a two-dimensional subspace of $\mathbb{R}^2$, then the dimension of $S^\perp$ is 2.

1.59) If matrix A has QR factorization $A = QR$, then Col(A) equals Col(R).

1.60) If $\vec{x}$ is not in subspace W, then $\text{proj}_W\vec{x} \neq \vec{0}$.

1.61) If V is a subspace spanned by $\vec{v}_1$ and $\vec{v}_2$, and $\vec{x}\cdot\vec{v}_1 = \vec{x}\cdot\vec{v}_2 = 0$, then $\vec{x} \in V^\perp$.

1.62) For orthonormal $\{\vec{v}_1, \vec{v}_2\}$, the magnitude of $\vec{v}_1 + \vec{v}_2$ is $\sqrt{2}$.

1.63) If the columns of a 5×2 matrix U are orthonormal, then $UU^T\vec{y}$ is the orthogonal projection of $\vec{y}$ onto the column space of U.

1.64) A least-squares line that best fits the data points $(0, y_1), (1, y_2), (2, y_3)$ is unique for any values y_1, y_2, y_3.

1.65) If $\vec{v}_1$ and $\vec{v}_2$ are non-zero orthogonal vectors, then they are linearly independent.

1.66) A least squares solution to a linear system $A\vec{x} = \vec{y}$ is a vector $\hat{x}$ such that $A\hat{x}$ is as close as possible to $\vec{y}$.

1.67) This question explores some of the properties that orthogonal matrices have. Indicate whether the following statements are true or false.

I) The determinant of an orthogonal matrix is always equal to 1.
II) The determinant of an orthogonal matrix is always equal to 1 or -1.
III) If the determinant of a matrix is 1, then the matrix must be orthogonal.

2 Example Construction Exercises

If possible, give an example of the following.

2.1) A matrix whose columns form an orthogonal basis for $\mathbb{R}^4$.

2.2) A matrix, A, that is in echelon form, and

$$\dim\left((\text{Row}(A))^\perp\right) = 2, \qquad \dim\left((\text{Col}(A))^\perp\right) = 3$$

2.3) A vector $\vec{v} \in \mathbb{R}^3$ and a subspace W such that $\text{proj}_W\vec{v} = \vec{v}$, and $\dim(W) = 2$.

2.4) An orthogonal matrix, in echelon form, whose columns span a 2-dimensional subspace of $\mathbb{R}^3$.

2.5) A matrix C, in echelon form, such that the linear system $C\vec{x} = \vec{b}$ has a unique least-squares solution, where $\vec{x} \in \mathbb{R}^2$ and

$$\vec{b} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

2.6) A non-zero matrix D, in echelon form, such that the linear system $D\vec{x} = \vec{b}$ does not have a unique least-squares solution, where $\vec{x} \in \mathbb{R}^2$ and

$$\vec{b} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

2.7) A subspace S, of $\mathbb{R}^4$, that satisfies $\dim(S) = \dim(S^{\perp}) = 2$.

2.8) A 2×3 matrix, A, that is in reduced echelon form. $\text{Row}(A)^{\perp}$ is spanned by $\begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix}$.

2.9) A 4×3 lower triangular matrix, C, and $\text{Col}(C)^{\perp}$ is spanned by $\vec{v} = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$.

2.10) A 4×3 lower triangular matrix, C, and $\text{Col}(C)^{\perp}$ is spanned by $\vec{v} = \begin{pmatrix} 1 \\ 2 \\ 1 \\ 2 \end{pmatrix}$.

2.11) Two linearly independent vectors that are orthogonal to $\vec{a} = \begin{pmatrix} 2 \\ 0 \\ -1 \end{pmatrix}$.

2.12) A subspace, S, of $\mathbb{R}^3$ such that $\dim(S^{\perp}) = 2$.

2.13) A symmetric orthogonal matrix in $\mathbb{R}^{2\times 2}$ that is not the identity matrix.

2.14) A 3×3 upper triangular matrix whose entries are all real, but whose eigenvalues are complex.

2.15) A matrix,A, that is similar to $B = \begin{pmatrix} 1 & 2 \\ 0 & 5 \end{pmatrix}$, but $A \neq B$.

2.16) A 2×2 singular matrix, whose entries are all real, and has an eigenvalue equal to $3i$.

2.17) A 3×3 matrix whose entries are all real, but whose eigenvalues are complex.

2.18) A diagonalizable 3×3 matrix that has exactly two distinct eigenvalues.

2.19) A 2×2 non-zero matrix that has 0 as an eigenvalue twice.

2.20) A square matrix that is in RREF, diagonalizable, and singular.

2.21) A vector in $\mathbb{R}^4$ that spans the subspace $H = \{\vec{x} \in \mathbb{R}^4 \,|\, x_1 - x_2 = 0\}$ and is a probability vector.

2.22) A 2×2 elementary matrix that cannot be diagonalized.

2.23) A 2×2 stochastic matrix that is singular and not regular.

2.24) A 2×2 matrix that does not have real eigenvalues.

2.25) A 2×2 matrix that is singular and is diagonalizable.

2.26) A 2×2 matrix, A, such that $A\vec{x} = \vec{b}$ has at least one solution for every $\vec{b}$, and A has only one distinct eigenvalue.

2.27) A singular 2×2 matrix whose eigenspace corresponding to eigenvalue $\lambda = 2$ is the line $x_1 = 2x_2$. The other eigenspace of the matrix is the x_2 axis.

2.28) A 2×2 matrix whose null space is the line $x_2 = x_1$ and whose eigenspace corresponding to eigenvalue $\lambda = 1$ is the line $x_1 = -2x_2$.

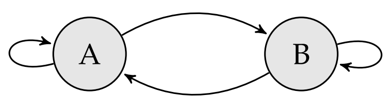
2.29) A 5×5 matrix that cannot be diagonalized.

2.30) A 5×5 matrix that has one eigenvalue that has algebraic multiplicity 5 and geometric multiplicity 2.

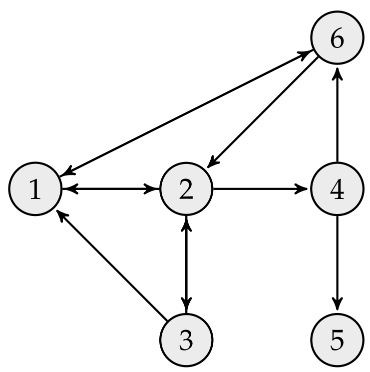
2.31) Matrices A and B that are similar but have different characteristic equations.

2.32) Matrices A and B that are not similar but have the same eigenvalues.

2.33) A two state Markov chain that is not regular (in other words, that does not have a unique steady state).



2.34) A set of 6 webpages link to each other according to the diagram below (a) Construct the transition matrix for this web. (b) Construct the Google Matrix for this web. Use a damping factor of $p = 0.85$.



2.35) The transition matrix, P, for an Internet with 5 web pages is

$$P = \begin{pmatrix} 0 & 1/2 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 1/2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

(a) Sketch the web whose corresponding transition matrix is P.

(b) Construct the Google Matrix, G, for this web.

3 Multiple Choice, Short Answer, and Fill in the Blank Exercises

3.1) Fill in the blanks to determine the diagonalization. *Hint: knowing an eigenvector of a given matrix allows you to quickly determine the corresponding eigenvalue.*

$$A = \begin{bmatrix} 5 & -2 & 3 \\ 0 & 1 & 0 \\ 6 & 7 & -2 \end{bmatrix} = \begin{bmatrix} 3 & \quad & \quad \\ 0 & & \\ 2 & & \end{bmatrix} \begin{bmatrix} \quad & 0 & 0 \\ 0 & \quad & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \quad & \quad & -3 \\ & 0 & 6 \\ & & 8 \end{bmatrix}^{-1}.$$

3.2) Fill in the blanks.

(a) If $\vec{b}$ is in column space of A, then $A\widehat{x} =$ ________.

(b) If U is a square orthogonal matrix, then the inverse of U is $U^{-1} =$ ____

(c) If the columns of A are linearly independent, then the square matrix $A^T A$ is ________________.

(d) If $A = QR$ then R equals ________________.

(e) The least squares solution is unique if the ________ of A are linearly independent.

(f) The Normal Equations for the Least Squares Solution is ________.

(g) If A has QR factorization, the normal equations become ________.

(h) In the QR factorization, Q is a ________ matrix, and R is a ________ matrix.

(i) In most circumstances, solving $R\hat{x} = Q^T\vec{b}$ is faster/slower than solving $\hat{x} = R^{-1}Q^T\vec{b}$.

(j) If $A = Q$, then the solution $\widehat{x} =$ ________.

(k) If W is a subspace of $\mathbb{R}^n$, and $\vec{y} \in \mathbb{R}^n$ has orthogonal projection $\widehat{y}$, then the distance from $\vec{y}$ to W is ________

(l) If the columns of A are linearly independent, then the square matrix $A^T A$ is ________________.

(m) If U is a square orthogonal matrix, then the inverse of U is $U^{-1} =$ ____.

3.3) Circle **possible** if the set of conditions are create a situation that is possible, otherwise, circle **impossible**. You don't need to explain your reasoning. If the situation is possible, give an example.

(a) A is $n \times n$, $A\vec{x} = A\vec{y}$ for a particular $\vec{x} \neq \vec{y}$, $\vec{x}$ and $\vec{y}$ are in $\mathbb{R}^n$, and $\dim(\text{Row}(A)^\perp) \neq 0$.

possible **impossible**

(b) A is $n \times n$, $\lambda \in \mathbb{R}$ is an eigenvalue of A, and $\dim(\text{Col}(A - \lambda I)^\perp) = 0$.

possible **impossible**

(c) A is $n \times n$, $\lambda \in \mathbb{R}$ is an eigenvalue of A, and $\dim(\text{Row}(A - \lambda I)^\perp) = 2$.

possible **impossible**

(d) $\text{proj}_{\vec{v}}\vec{u} = \text{proj}_{\vec{u}}\vec{v}$, $\vec{v} \neq \vec{u}$, and $\vec{u} \neq \vec{0}$, $\vec{v} \neq \vec{0}$.

possible **impossible**

(e) Matrix A is $n \times n$ and invertible, and $A^T A$ is singular.

possible **impossible**

3.4) If possible, complete the matrices in such a way so that all columns are orthogonal.

$$A = \begin{bmatrix} -1 & 1 & -1 & 1 \\ 1 & 1 & -1 & \\ 0 & 1 & & \\ 0 & 0 & 1 & \end{bmatrix}, \qquad B = \begin{bmatrix} & 1 & -1 \\ 1 & 1 & \\ 0 & 1 & \\ 0 & 1 & \end{bmatrix}$$

4 Computation Exercises

4.1) W is the subspace spanned by

$$\vec{u} = \begin{pmatrix} 2 \\ 1 \\ -6 \\ 18 \end{pmatrix}$$

Construct a basis for the orthogonal compliment $W^\perp$. What is the dimension of $W^\perp$?

4.2) W is the set of all vectors of the form $\begin{pmatrix} x \\ y \\ x+y \end{pmatrix}$. Which of the vectors are in $W^\perp$?

$$\vec{u} = \begin{pmatrix} 8 \\ -5 \\ 8 \end{pmatrix}, \quad \vec{v} = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}, \quad \vec{w} = \begin{pmatrix} 3 \\ 3 \\ -3 \end{pmatrix}$$

4.3) Determine the coefficients c_1 and c_2 that describe the plane in $\mathbb{R}^3$, $z = c_1 x + c_2 y$, that best fits the points $(1,1,3), (-1,0,-6), (0,-1,6), (1,-1,0)$.

4.4) Construct an orthogonal basis for $D^\perp$.

(a) $D = \{\vec{x} \in \mathbb{R}^3 \ : \ x_1 - 2x_2 + 4x_3 = 0\}$.

(b) $D = \{\vec{x} \in \mathbb{R}^4 \ : \ 2x_1 - x_2 = 0\}$.

4.5) Consider the matrix A.

$$A = \begin{pmatrix} 1 & -3 & 0 & 2 \\ 0 & 0 & 1 & -3 \end{pmatrix}$$

If possible, construct a basis for the following subspaces. Compute the dimension of each space.

(a) $\text{Row}(A)$

(b) $(\text{Row}(A))^\perp$

(c) $\text{Col}(A)$

(d) $(\text{Col}(A))^\perp$

4.6) Let $\vec{u}_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$, $\vec{u}_2 = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}$, and $W = \text{span}\{\vec{u}_1,\ \vec{u}_2\}$.

(a) Find the projection of $\vec{y} = \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix}$ onto $W^\perp$, then use your answer to construct an orthonormal basis for $W^\perp$.

(b) How far is $\vec{y}$ from W? From $W^\perp$?

4.7) Let

$$A = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 0 & 1 \end{pmatrix}, \ \vec{z} = \begin{pmatrix} 1 \\ 3 \\ 8 \\ 2 \end{pmatrix}.$$

Calculate the distance between Col$A$ and $\vec{z}$ without finding an orthogonal basis for col$A$.

4.8) (a) Calculate $A^T A$ and $(A^T A)^{-1}$ for

$$A = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & -1 \end{pmatrix},$$

(b) Use your calculation above solve the least square problem

$$A\hat{x} = \begin{pmatrix} 1 \\ 3 \\ 4 \\ 0 \end{pmatrix}$$

4.9) Let $A = QR$ be as below.

$$A = QR = \frac{1}{2}\begin{bmatrix} 1 & -\sqrt{2} \\ -1 & 0 \\ -1 & 0 \\ 1 & \sqrt{2} \end{bmatrix}\begin{bmatrix} 6 & 2 \\ 0 & 2\sqrt{2} \end{bmatrix}$$

(a) Give an orthogonal basis for $\mathrm{Col}(A)$, and $\mathrm{Col}(A)^{\perp}$

(b) Determine the least-squares solution to $A\hat{x} = \begin{pmatrix} 0 \\ 2 \\ 0 \\ 0 \end{pmatrix}$.

4.10) Consider the data in the table below.

x	-1	0	2
y	2	0	-1

(a) Construct the normal equations that can be solved to find the values of β_1 and β_0 so that the equation of the least-squares line $y = \beta_1 x + \beta_0$ best fits the data in the table.

(b) Compute the values of β_0 and β_1.

4.11) Let $\vec{v}, \vec{w} \in \mathbb{R}^7$ be orthogonal vectors with $\|v\| = 5$ and $\|w\| = \sqrt{2}$. Determine the values of the following.

$$\|\vec{x}\| = \boxed{}, \qquad \vec{x}\cdot\vec{y} = \boxed{}$$

where $\vec{x} = -\vec{v} + 3\vec{w}$ and $\vec{y} = 3\vec{v} - \vec{w}$.

4.12) Determine all values of a, b, and c, so that the matrix has orthogonal columns.

$$A = \begin{pmatrix} 1 & -7 & a \\ 0 & 1 & b \\ 7 & 1 & c \end{pmatrix}$$

4.13) Let

$$A = \begin{bmatrix} 1 & -2 & 2 \\ 1 & -1 & 5 \\ 1 & 1 & 3 \\ 1 & 2 & 5 \end{bmatrix} \qquad \vec{b} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 5 \end{bmatrix}$$

(a) Find the least squares solutions $\hat{x}$ to $A\vec{x} = \vec{b}$ where

(b) Compute the QR decomposition of A.

(c) Use the QR decomposition to compute the least squares solution $\hat{x}$ above.

4.14) The matrix below is an orthogonal matrix. Fill in the missing entries.

$$A = \begin{bmatrix} 1/\sqrt{10} & \_\_\_ & 1/\sqrt{2} & \_\_\_ \\ 2/\sqrt{10} & -1/\sqrt{2} & 0 & \_\_\_ \\ \_\_\_ & 1/\sqrt{2} & \_\_\_ & 1/\sqrt{10} \\ \_\_\_ & \_\_\_ & \_\_\_ & -2/\sqrt{10} \end{bmatrix}$$

(Bonus question: Could you erase any other entries above, and still determine the matrix?)

4.15) Construct the rotation dilation matrix C for $A = \begin{pmatrix} 1 & 2 \\ -1 & 1 \end{pmatrix}$ and the matrix P so that $AP = PC$.

4.16) Diagonalize the matrix $A = \begin{pmatrix} 3 & 1 \\ 0 & 4 \end{pmatrix}$ and determine the value of A^k for any integer $k > 1$. You should provide a 2×2 matrix that is expressed in terms of k.

4.17) Let $A = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$.

(a) Determine the eigenvalues of A. What is the algebraic multiplicity of each eigenvalue?

(b) Determine an invertible matrix P and a diagonal matrix D such that $A = PDP^{-1}$.

(c) Compute A^{114} and A^{115}.

4.18) (a) B is a 2×2 matrix with two identical rows. What is one of the eigenvalues of B equal to?

(b) Suppose again that B is a 2×2 matrix with two identical rows. Identify a relationship between the two entries in each row such that the second eigenvalue of B is 1.

4.19) Consider the square matrices below.

$$A = I_3 \qquad B = \begin{pmatrix} 1 & 0 & 0 \\ -1 & -1 & 1 \\ 1 & -1 & 1 \end{pmatrix} \qquad C = \begin{pmatrix} 1 & 0 & 0 \\ -1 & -1 & -1 \\ 1 & -1 & 1 \end{pmatrix}$$

For each matrix,

(a) Calculate all the eigenvalues.

(b) Calculate the dimension and basis for all eigenspaces.

(c) Is the matrix diagonalizable? If so, construct invertible matrix P and diagonal matrix D such that the matrix can be diagonalized.

4.20) Let $A = \begin{bmatrix} -2 & -2 \\ 2 & -2 \end{bmatrix}$.

(a) Calculate the eigenvalues and eigenvectors of A.

(b) Write A as a product two matrices so that one of them describes a scaling and the other a rotation.

(c) By how much does A scale?

(d) By how much does A rotate?

(e) Find A^{2017} without diagonalizing A. (Hint: use the fact that you know how much A scales and rotates.)

4.21) One of the eigenvalues of A is $\lambda_1 = 1$. One of the eigenvectors of A is $\vec{v}_2$. If possible, construct real matrices P and diagonal D such that $A = PDP^{-1}$. You do not need to construct P^{-1}. But please show your work. *Hint: You do not need to compute the roots of a third order polynomial to obtain the remaining eigenvalues of this matrix.*

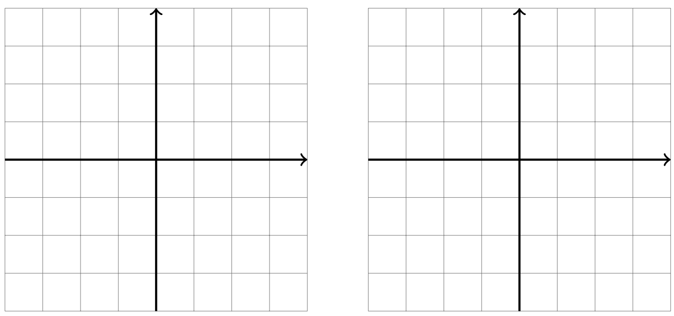
$$A = \begin{pmatrix} 2 & 2 & -1 \\ 1 & 3 & -1 \\ -1 & -2 & 2 \end{pmatrix}, \quad \vec{v}_2 = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$$

5 Sketching Exercises

1. In the grids below, sketch a) Col$A$ and Col$A^\perp$, and b) Null$A$ and Null$A^\perp$.

$$A = \begin{pmatrix} 2 & 1 \\ -4 & -2 \end{pmatrix}$$

(a) Col$A$ and Col$A^\perp$ (b) Null$A$ and Null$A^\perp$

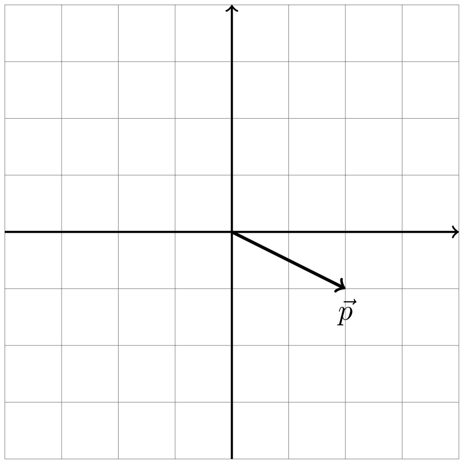


2. Vector $\vec{p}$ is sketched below. Add to the following to the grid.

   (a) The subspace $W = \text{Span}\{\vec{p}\}$, and $W^\perp$

   (b) Any $\vec{v}$, such that $\text{proj}_W\vec{v} = \vec{v}$, $\vec{v} \neq \vec{0}$, and $\vec{v} \neq p$.

   (c) Any $\vec{x}$, such that $\text{proj}_W\vec{x} = -2\vec{p}$, $\vec{x}$ is not in W.



6 Challenge Problems

The following problems go a bit beyond the scope of our course, and are based on questions and discussion from past Math 1554 students.

1. *This question is from a North Fulton High School Mathematics Competition.*

   (a) Reflect A' through the plane spanned by Q' and R', where

   $$A' = \begin{pmatrix} 0 \\ 2 \\ 2 \end{pmatrix}, \quad Q' = \begin{pmatrix} -1 \\ 3 \\ 1 \end{pmatrix}, \quad R' = \begin{pmatrix} 0 \\ 0 \\ 3 \end{pmatrix}$$

   (b) Reflect A through the plane passing through the points P, Q and R, where

   $$A = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \quad P = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \quad Q = \begin{pmatrix} 0 \\ 3 \\ 2 \end{pmatrix}, \quad R = \begin{pmatrix} 1 \\ 0 \\ 4 \end{pmatrix}$$

   What is the product of the entries in the reflected vector?