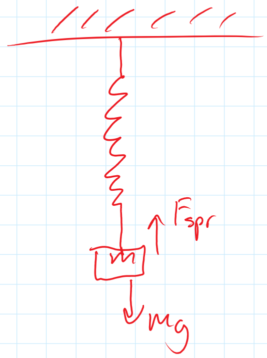
Effect of Constant External Force on a Harmonic Oscillator

Consider a mass on a spring in the vertical position



Force on mass: $F = -K(X - X_e) + mg$

where X_e is the equil. pos'n of the unstreched spring, but gravity will make a new equil. position.

$0 = -K(X_e' - X_e) + mg$ where X_e' is the new equil. position

$$\rightarrow X_e' = X_e + \frac{mg}{K}$$

Now define displacement from this new equil. pos'n as $x = X - X_e' = \cancel{X} - \cancel{X_e} - \frac{mg}{K}$

$$\rightarrow x + \frac{mg}{K} = (\cancel{X} - \cancel{X_e})$$

$$\therefore F = -K\left(x + \frac{mg}{K}\right) + mg = -Kx - mg + mg = -Kx$$

$\therefore$ $m\ddot{x} = -Kx$ will still be e.o.m.

In general, any constant external force applied to a harmonic oscillator just shifts the equilibrium position

Ex: A light spring supports a block of mass m in a vertical position. When in equilibrium, the spring is streched by an amount D_1 over its unstreched length. If the block is streched a distance D_2 from the equilibrium position and released at $t=0$, find (a) the resulting motion $x(t)$, (b) the velocity of the block when it passes upwards through the equil. pos'n; (c) the acc'n of the block at the top of its oscillatory motion.

First, consider the equil. pos'n: $F = 0 = -KD_1 + mg$ $(x>0 \text{ down})$

$$\therefore K = \frac{mg}{D_1}$$

$\therefore$ Ang. freq. of oscillation $\omega_0 = \sqrt{\frac{K}{m}} = \sqrt{\frac{g}{D_1}}$

a) Motion is $x = a\cos(\omega_0 t - \theta)$

@ $t=0$, $x = D_2 = a\cos(-\theta)$, $\dot{x} = 0 = -a\sin(-\theta)$ $\rightarrow \theta = 0$

$$\therefore a = D_2$$

so, $x(t) = D_2 \cos\omega_0 t = D_2\cos\left(\sqrt{\frac{g}{D_1}}\,t\right)$ indep. of mass

b) The velocity is $\dot{x}(t) = -D_2\sqrt{\frac{g}{D_1}}\sin\left(\sqrt{\frac{g}{D_1}}\,t\right)$

When block is back in equil. position, $x = 0 = D_2\cos\sqrt{\frac{g}{D_1}}\,t$

so $\sqrt{\frac{g}{D_1}}\,t = \frac{\pi}{2}$, so $\dot{x}(\text{equil. pos'n}) = -D_2\sqrt{\frac{g}{D_1}}$

c) The acceleration is $\ddot{x}(t) = -D_2\left(\frac{g}{D_1}\right)\cos\left(\sqrt{\frac{g}{D_1}}\,t\right)$

At the top of the motion, $x = -D_2 = D_2\cos\left(\sqrt{\frac{g}{D_1}}\,t\right)$

so $\sqrt{\frac{g}{D_1}}\,t = \pi$

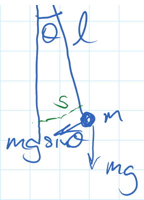
$$\therefore \ddot{x}(\text{top of oscillation}) = \frac{D_2 g}{D_1}$$

If $D_1 = D_2$, $\ddot{x} = g$; block is momentarily in free-fall w/ the spring exerting no force.

Ex: Simple Pendulum Revisited

We've seen the restoring force $F = -mg\sin\theta$

$$\therefore m\ddot{s} = -mg\sin\theta$$



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For small θ, $s \simeq l\theta$ & $\sin\theta \simeq \theta$

so e.o.m. is $\cancel{m}l\ddot{\theta} = -\cancel{m}g\theta$

or $\ddot{\theta} + \frac{g}{l}\theta = 0$, same sho ode w/ $\omega_0 = \sqrt{\frac{g}{l}}$

as long as θ is relatively small, the period of oscillation is $\tau = \frac{2\pi}{\omega_0} = 2\pi\sqrt{\frac{l}{g}}$

Calculate The Time Averaged Kinetic, Potential & Total Energies of a Harmonic Oscillator

$$\langle T \rangle = \frac{1}{\tau}\int_0^\tau T(t)\,dt = \frac{1}{\tau}\int_0^\tau \frac{1}{2}m\dot{x}^2\,dt$$

$$x = A\cos(\omega_0 t - \theta)$$
$$\dot{x} = -A\omega_0\sin(\omega_0 t - \theta)$$

set $\theta = 0$, $\langle T \rangle = \frac{1}{\tau}\left[A^2\frac{1}{2}m\omega_0^2\int_0^\tau \sin^2\omega_0 t\,dt\right]$

let $u = \omega_0 t = \left(\frac{2\pi}{\tau}\right)t \Rightarrow du = \left(\frac{2\pi}{\tau}\right)dt$

$$\Rightarrow \langle T \rangle = \frac{1}{2\pi}\left[\frac{1}{2}m\omega_0^2A^2\int_0^{2\pi}\sin^2u\,du\right]$$

Note that $\frac{1}{2\pi}\int_0^{2\pi}(\sin^2u + \cos^2u)\,du = \frac{1}{2\pi}\int_0^{2\pi}du = 1$

so $\frac{1}{2\pi}\int\sin^2u\,du$ & $\frac{1}{2\pi}\int\cos^2u\,du$ are both $\frac{1}{2}$

$$\therefore \boxed{\langle T \rangle = \frac{1}{4}m\omega_0^2A^2}$$

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$V = \frac{1}{2}Kx^2$ so please show that $\langle V\rangle = \frac{1}{4}KA^2 = \frac{1}{4}m\omega_0^2A^2 = \langle T\rangle$

$$\therefore \langle E\rangle = \langle T\rangle + \langle V\rangle = \frac{1}{2}m\omega_0^2A^2 = \frac{1}{2}KA^2 = E$$

Time avg. T & V are equal. Avg. E is equal to instantaneous E