Friday, August 28, 2015 1:03 PM

Quadratic resistance: $F=-bv^2$ $(b>0$, constant$)$

$$-bv^2 = m\frac{dv}{dt}$$

$$\therefore -\frac{b}{m}\int_0^t dt = \int_{v_0}^{v}\frac{dv}{v^2} = -\left(\frac{1}{v}-\frac{1}{v_0}\right)$$

$$\rightarrow v(t) = \frac{v_0}{\left(1+\frac{b}{m}v_0 t\right)}$$

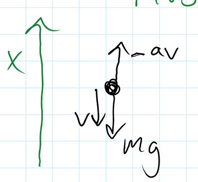
Now, set $K=\frac{bv_0}{m}$, then the 2nd integration gives

$$x(t) = \int_0^t \frac{v_0\,dt}{1+Kt} = \frac{v_0}{K}\ln(1+Kt)+x_0$$

Interestingly, v decreases as $1/t$, but x diverges as $t\to\infty$. As v gets small, F gets small faster so resistance becomes negligible.

Examples: Vertical Fall through a fluid

First, consider linear drag:



E.o.m: $-mg-av = m\frac{dv}{dt}$

Separate variables

$$dt = \frac{m\,dv}{(-mg-av)}$$

$$t = \int_{v_0}^{v}\frac{m\,dv}{(-mg-av)} = -\frac{m}{a}\ln\left(\frac{mg+av}{mg+av_0}\right)$$

$$\therefore -\frac{at}{m} = \ln\left(\frac{mg+av}{mg+av_0}\right)$$, exp. both sides & solve for v

$$v = -\frac{mg}{a}+\left(\frac{mg}{a}+v_0\right)e^{-\left(\frac{a}{m}\right)t}$$

The exp. term becomes negligible after $t \gg \frac{m}{a}$. So, v

approaches a limiting value $= -\frac{mg}{a}$. The terminal velocity where the drag force & gravity balance

of

Define the terminal velocity $V_t = \frac{mg}{a}$ & $\tau = \frac{m}{a}$ (the characteristic time)

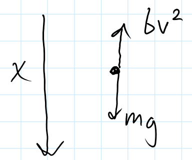
$$V = -V_t\left(1 - e^{-t/\tau}\right) + V_0 e^{-t/\tau}$$

The 1st term exponentially 'fades in' & the initial velocity 'fades out' due to drag. For an object dropped at rest

$$V = -V_t\left(1 - e^{-t/\tau}\right). \text{ After a time } t = 5\tau,$$

$$V = -0.993 V_t$$

Now, consider drag with a body dropped w/ initial speed V_0 w/ quadratic drag force.



$$mg - bv^2 = m\frac{dv}{dt}$$

$$mg\left(1 - \frac{b}{mg}v^2\right) = m\frac{dv}{dt} \quad \Rightarrow \text{define } V_t = \sqrt{\frac{mg}{b}}$$

$$\text{so } V_t^2 = \frac{mg}{b}$$

$$\therefore \frac{dv}{dt} = g\left(1 - \frac{V^2}{V_t^2}\right)$$

Integrating,

$$t - t_0 = \int_{V_0}^{V} \frac{dv}{g\left(1 - \frac{V^2}{V_t^2}\right)} = \tau\left(\tanh^{-1}\left(\frac{V}{V_t}\right) - \tanh^{-1}\left(\frac{V_0}{V_t}\right)\right)$$

$$\text{where } \tau = \frac{V_t}{g} = \sqrt{\frac{m}{bg}}$$

Solving for v:

$$V = V_t \tanh\left(\frac{t - t_0}{\tau} - \tanh^{-1}\frac{V_0}{V_t}\right)$$

If $V_0 = 0$ @ $t_0 = 0$, then $V = V_t \tanh\frac{t}{\tau} = V_t\left(\frac{e^{2t/\tau} - 1}{e^{2t/\tau} + 1}\right)$

Recall: $\cosh x = \frac{1}{2}\left(e^{x} + e^{-x}\right)$

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$$\sinh x = \frac{1}{2}(e^{x} - e^{-x})$$

$$\tanh x = \frac{\sinh x}{\cosh x}$$

After $t = 5\tau$

$V = 0.99991\, V_t$. Faster than the linear case.