General Motion of a Particle in 3D

Not all motion is in 1D, so need to generalize concepts & treatment to 3D systems. In general, solutions to 3D motions are often not analytical. However, when forces are conservative, simplifications are possible. Need to determine properties of a 3D conservative force

The KE of a particle moving in 3D is

$$T = \frac{1}{2}m\dot{\vec{r}}^2 = \frac{1}{2}m\left(\dot{x}^2+\dot{y}^2+\dot{z}^2\right)$$

Its rate of change is

$$\dot{T} = m\left(\dot{x}\ddot{x}+\dot{y}\ddot{y}+\dot{z}\ddot{z}\right) = m\dot{\vec{r}}\cdot\ddot{\vec{r}} = \dot{\vec{r}}\cdot\left(m\ddot{\vec{r}}\right) = \dot{\vec{r}}\cdot\vec{F}$$

or, in a time interval dt,

$$dT = d\vec{r}\cdot\vec{F}$$

or $\vec{F}\cdot d\vec{r} = F_x\,dx + F_y\,dy + F_z\,dz$ is the work done by the force in the displacement $d\vec{r}$

$\rightarrow$ $\vec{F}\cdot\dot{\vec{r}} = \vec{F}\cdot\vec{v} =$ rate of working = rate of changing KE

Recall that a conservative force leads to an energy conservation equation $T+V=$ constant

If $V(\vec{r})$, then $\dot{V}(\vec{r}) = \frac{\partial V}{\partial x}\dot{x} + \frac{\partial V}{\partial y}\dot{y} + \frac{\partial V}{\partial z}\dot{z}$

or since $\vec{\nabla}V = \frac{\partial V}{\partial x}\hat{i} + \frac{\partial V}{\partial y}\hat{j} + \frac{\partial V}{\partial z}\hat{k}$ where $\vec{\nabla} = \left(\frac{\partial}{\partial x},\frac{\partial}{\partial y},\frac{\partial}{\partial z}\right)$

$$\dot{V}(\vec{r}) = \dot{\vec{r}} \cdot \vec{\nabla} V$$

So, energy conservation

$$\dot{T} + \dot{V} = 0$$

$$\dot{\vec{r}} \cdot \vec{F} + \dot{\vec{r}} \cdot \vec{\nabla} V = 0$$

$$\dot{\vec{r}} \cdot (\vec{F} + \vec{\nabla} V) = 0$$

Since this must be true for any velocity

$$\boxed{\vec{F} = -\vec{\nabla} V} \text{ for a conservative force}$$

In cartesian coord's, $F_x = -\frac{\partial V}{\partial x}$, $F_y = -\frac{\partial V}{\partial y}$, $F_z = -\frac{\partial V}{\partial z}$

(diff. in cyl. or sph. coords)

The $\vec{F} = -\vec{\nabla} V$ relation imposes a condition on the $\vec{F}$-fields for it to be conservative. It must be curless:

$$\vec{\nabla} \times \vec{F} = \left(\frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z}\right)\hat{i} + \left(\frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x}\right)\hat{j} + \left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y}\right)\hat{k} \text{ in cart. coords}$$

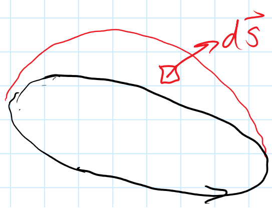
$$= \underbrace{\cancel{\left(-\frac{\partial^2 V}{\partial z \partial y} + \frac{\partial^2 V}{\partial y \partial z}\right)}}_{0}\hat{i} + 0 + 0$$

In fact, $\vec{\nabla} \times (\vec{\nabla} f) = 0$ where f is any scaler field.

This fact has another consequence.

Recall Stoke's Thm:

Let C be a closed contour. This contour bounds an open surface S. If $\vec{F}(\vec{r},t)$ is a vector function then



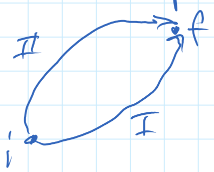
$$\oint_C \vec{F}(\vec{r},t) \cdot d\vec{r} = \int_S (\vec{\nabla} \times \vec{F}) \cdot d\vec{s}$$

line integral around

$\vec{C}$ around path

So, if $\vec{F}$ is conservative, $\vec{\nabla}\times\vec{F}=0$, and $\oint \vec{F}\cdot d\vec{r}$ for all contours is zero. But $\vec{F}\cdot d\vec{r}$ is the work done along $dr = \Delta T$

$\therefore$ There is no net work done by a conservative force along a closed path.



$$\rightarrow \int_{I} \vec{F}\cdot d\vec{r} - \int_{II} \vec{F}\cdot d\vec{r} = 0 \rightarrow \int_{I} \vec{F}\cdot d\vec{r} = \int_{II} \vec{F}\cdot d\vec{r}$$

$\therefore$ The work done by a conservative force in a displacement from $\vec{r}_0$ to $\vec{r}$ is independent of the path b/w these 2 points.

$\vec{F}$ is a conservative force $\Longleftrightarrow$ $\vec{F} = -\vec{\nabla}V$

$\vec{F}$ is a conservative force $\Longleftrightarrow$ $\vec{\nabla}\times\vec{F}=0$

$\vec{F}$ is a conservative force $\Longleftrightarrow$ Work done by $\vec{F}$ is path independent