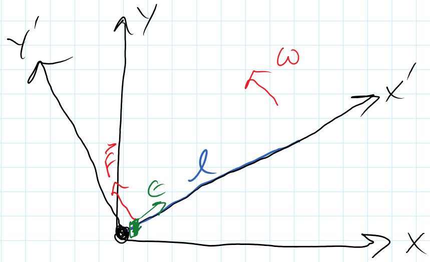
<u>Example:</u> A smooth rod of length l rotates in a plane w/ a constant ang. velocity $\vec{\omega}$ about an axis fixed at the end of the rod and $\perp$ to the plane of rotation. A bead of mass m is initially positioned at the stationary end of the rod & given a slight push such that its initial speed directed down the rod is $\epsilon = \omega l$. How long does it take for the bead to reach the other end of the rod?



Want to analyze this in rotating frame (simpler 1D problem).

Let x' lie along the rod.

There is a force $\vec{F}$ acting on the bead turning it in a circle, $\vec{F} = F\hat{j}'$

So, e.o.m. in rotating frame

$$F\hat{j}' - 2m\omega\hat{k}' \times \dot{x}\hat{i}' - m\omega\hat{k}' \times (\omega\hat{k}' \times x\hat{i}') = m\ddot{x}\hat{i}'$$

$$F\hat{j}' - 2m\omega\dot{x}\hat{j}' + m\omega^2 x\hat{i}' = m\ddot{x}\hat{i}'$$

Component-wise:

$F = 2m\omega\dot{x}$ Corolis force balances reaction force

$\cancel{m}\omega^2 x = \cancel{m}\ddot{x}$

The solution of the 2^{nd} equation is

$$x(t) = Ae^{\omega t} + Be^{-\omega t}$$

$$\dot{x}(t) = \omega A e^{\omega t} - \omega B e^{-\omega t}$$

At $t=0$, $x=0$; $\dot{x} = \epsilon$ $\rightarrow$ $0 = A + B$ $\rightarrow A = -B \rightarrow \frac{\epsilon}{\omega} = -2B$

$\epsilon = \omega(A-B)$

$\therefore B = -\frac{\epsilon}{2\omega}$, $A = \frac{\epsilon}{2\omega}$

$$\therefore x(t) = \frac{\epsilon}{2\omega}\left(e^{\omega t} - e^{-\omega t}\right) = \frac{\epsilon}{\omega}\sinh \omega t$$

The bead flies off when $x = \ell = \frac{\epsilon}{\omega}\sinh \omega T \rightarrow T = \frac{1}{\omega}\sinh^{-1}\left(\frac{\omega \ell}{\epsilon}\right)$

but $\epsilon = \omega \ell$, so $T = \frac{1}{\omega}\sinh^{-1}(1) = \underline{\frac{0.88}{\omega}}$

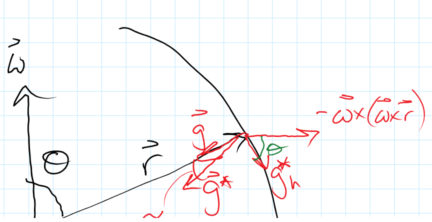
Effects of the Earth's Rotation

For a particle moving under gravity in addition to $\vec{F}$,

$$m\ddot{\vec{r}} = m\vec{g} + \vec{F} - 2m\vec{\omega}\times\dot{\vec{r}} - m\vec{\omega}\times(\vec{\omega}\times\vec{r})$$

where $\vec{g}$ is a vector of magnitude g pointing to the center of the Earth.

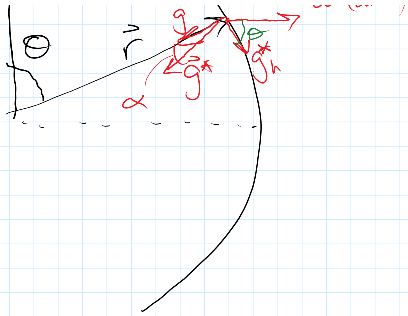
The centrifugal force is everywhere on Earth's surface & proportional to mass of object. When make an experiment to measure $\vec{g}$, actually measure $\vec{g}^* = \vec{g} - \vec{\omega}\times(\vec{\omega}\times\vec{r})$. So, a plumb line does not point directly to Earth's center, but is swung outwards by Earth's rotation.



Consider a point at co-latitude θ (ie., $90°$ - latitude)

Then $|\vec{\omega}\times(\vec{\omega}\times\vec{r})| = \omega|\vec{\omega}\times\vec{r}| = \omega^2 r \sin\theta$

$$g^*_h = \omega^2 r \sin\theta\cos\theta$$



$$g^*_h = \omega^2 r \sin\theta \cos\theta$$

$$g^*_v = g - \omega^2 r \sin^2\theta$$ (Assuming Earth is spherical – its not, b/c of centrifugal forces)

Using $\omega = 7.292\times 10^{-5}\ s^{-1}$ & $r = 6371$ Km (mean radius)

$$\omega^2 r = 34\ \text{mm}\ s^{-2}$$

As this is much smaller than g, can estimate α as

$$\sin\alpha \simeq \alpha = \frac{g_h}{g_v} = \frac{\omega^2 r \sin\theta\cos\theta}{g} = \frac{\omega^2 r}{2g}\sin 2\theta$$

Thus, there is no deflection at poles & equator. Max value is at $\theta = 45^\circ$ & is 1.7×10^{-3} radians or 6 arcminutes

Note gravity is stronger at poles over the equator by 52 mm s^{-2}
Bigger than predicted above b/c of oblateness.