# Sharpness of the Resonance

Consider the case $\gamma << \omega_0$ and close to resonance.

Then $\omega_0^2 - \omega_1^2 = (\omega_0 + \omega_1)(\omega_0 - \omega_1)$

$$\simeq 2\omega_0(\omega_0 - \omega_1)$$

and $\gamma\omega_1 \simeq \gamma\omega_0$

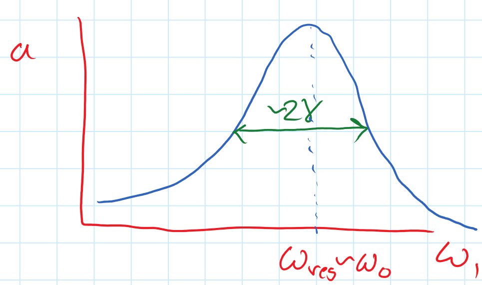
So, the amplitude $a_1(\text{around resonance}) \simeq \dfrac{(F_1/m)}{\sqrt{4\omega_0^2(\omega_0-\omega_1)^2 + 4\gamma^2\omega_0^2}}$

$$= \frac{(F_1/m)}{2\omega_0\sqrt{(\omega_0-\omega_1)^2+\gamma^2}} = \frac{(F_1/m)\gamma}{2\gamma\omega_0\sqrt{(\omega_0-\omega_1)^2+\gamma^2}} = \frac{a_{1,max}\,\gamma}{\sqrt{(\omega_0-\omega_1)^2+\gamma^2}}$$

So, when $|\omega_0 - \omega_1| = \gamma$, $a_1^2 = \frac{1}{2}a_{max}^2$

That is, $\gamma$ is a measure of the width of the resonance curve.

$2\gamma$ is the freq. diff b/w the pts. for which the energy is down by a factor of $1/2$ $(E \propto a^2)$




Can also relate these concepts to the quality factor Q.

Recall $Q = \dfrac{\omega_0}{2\gamma} \simeq \dfrac{\omega_{res}}{2\gamma}$ for weak damping

At the half-energy points, $\Delta\omega = 2\gamma = \dfrac{\omega_{res}}{Q}$

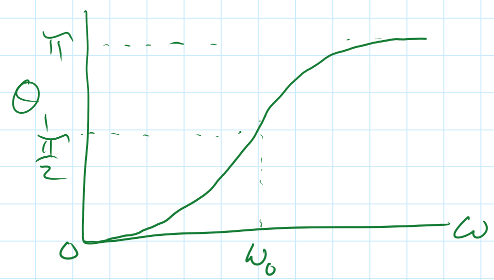
or, since $\omega = 2\pi f$, $\dfrac{\Delta\omega}{\phantom{x}} = \dfrac{\Delta f}{\phantom{x}} = \dfrac{1}{\phantom{x}}$, the fractional

$\omega_{res}$  $f_{res}$  $Q$  width of the resonance peak

So, smaller damping, higher Q, narrower resonance peak.

## Phase Difference, $\theta_1$

$$\tan\theta_1 = \frac{2\gamma\omega_1}{\omega_0^2 - \omega_1^2}$$




At low driving frequencies, $\omega_1 \ll \omega_0$, $\tan\theta_1 \to 0$, $\theta_1 \to 0$ ; response is in phase w/ driving force.

In this case $a_1(\omega_1 \to 0) \simeq \frac{F_1/m}{\omega_0^2} = \frac{F_1/m}{\left(\frac{K}{m}\right)} = \frac{F_1}{K}$ (The spring is all that matters)

At high driving frequencies $\omega_1 \gg \omega_0$, $\tan\theta_1 \to 0$ but from -ve side so $\theta_1 \to \pi$ ; displacement is 180° out of phase w/ driving force.

$a_1(\omega_1 \gg \omega_0) \simeq \frac{F_1/m}{\omega_1^2}$ which gets small as $\omega_1$ increases

Example: The damping factor $\gamma$ of a spring suspension system is 0.10 the critical value. If the undamped freq. is $\omega_0$ find (a) the resonance freq. (b) the quality factor, (c) the phase angle $\theta_1$ when the system is driven at a freq $\omega_1 = \omega_0/2$ ; (d) the steady-state ampl. at this freq.

a) critical damping is when $\gamma = \omega_0$, so in this case $\gamma = 0.10\omega_0$

The resonance freq. is $\omega_{res} = \sqrt{\omega_0^2 - 2\gamma^2} = \sqrt{\omega_0^2 - 2(0.1\omega_0)^2}$

$= \sqrt{0.98\omega_0^2} = 0.99\omega_0$

b) $Q = \frac{\omega_0}{2\gamma} = \frac{\omega_0}{0.2\omega_0} = 5$

c) $\tan\theta_1 = \dfrac{2\gamma\omega_1}{\omega_0^2-\omega_1^2} = \dfrac{2(0.10\omega_0)(0.50\omega_0)}{\omega_0^2-\frac{1}{4}\omega_0^2} = \dfrac{0.1\omega_0^2}{\frac{3}{4}\omega_0^2} = 0.13$

$$\therefore \theta_1 = 7.6°$$

d) $a_1 = \dfrac{(F_1/m)}{\sqrt{(\omega_0^2-\omega_1^2)^2+4\gamma^2\omega_1^2}} = \dfrac{(F_1/m)}{\sqrt{(\omega_0^2-\frac{1}{4}\omega_0^2)^2+\cancel{4}(0.01\omega_0^2)\frac{\omega_0^2}{\cancel{4}}}} = \dfrac{(F_1/m)}{\sqrt{\frac{9}{16}\omega_0^4+0.01\omega_0^4}}$

$$= 1.32\left(\frac{F_1}{m\omega_0^2}\right)$$

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## General Periodic Force

Now, that we understand the behavior for one sinusoidal force, we can generalize to the case where the applied force is a sum of periodic terms,

$$F(t) = F_1 e^{i\omega_1 t} + F_2 e^{i\omega_2 t} + \dots = \sum_r F_r e^{i\omega_r t}$$

where the 'real' force is just the real part of each term.

Because the e.o.m. is linear, the solution in the general case is

$$x = \sum_r A_r e^{i\omega_r t} + \text{transient}$$

where each $A_r$ is related to $F_r$ by

$$(-m\omega_r^2 + i\lambda\omega_r + K)A_r = F_r$$