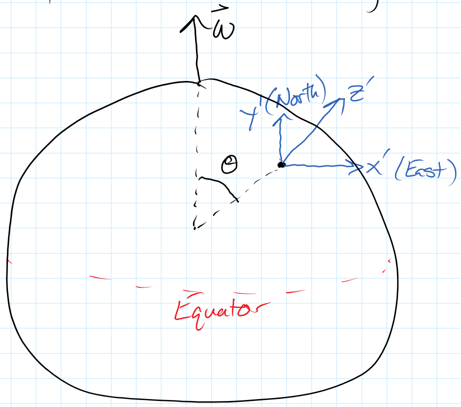
Coroilis Effects on the Motion of a Projectile

To illustrate the effects of the Coriolis force in describing a projectile fired from the Earth, focus on a particle that moves through a distance small enough that both the grav. & centrif. forces are effectively constant. Call it $\vec{g}^*$. Neglect air resistance.

Then e.o.m. would be $m\ddot{\vec{r}} = m\vec{g}^* - 2m\vec{\omega}\times\dot{\vec{r}}$

To solve this, use the following rotating coordinate system:



z' is vertical, or, really, in the direction as $\vec{g}^*$

x' points East

y' points North

$\therefore \quad \vec{g}^* = -g^*\hat{k}'$

The components of $\vec{\omega}$ in the primed system are

$$\omega_{x'} = 0, \quad \omega_{y'} = \omega\sin\theta, \quad \omega_{z'} = \omega\cos\theta$$

The cross-product is $\vec{\omega}\times\dot{\vec{r}} = (\omega\sin\theta\,\dot{z}' - \omega\cos\theta\,\dot{y}')\hat{i}' + (\omega\cos\theta\,\dot{x}')\hat{j}' - \omega\sin\theta\,\dot{x}'\hat{k}'$

$$\therefore \quad \cancel{m}\ddot{\vec{r}} = -\cancel{m}g^*\hat{k}' - 2\cancel{m}\omega\left[(\sin\theta\,\dot{z}' - \cos\theta\,\dot{y}')\hat{i}' + \cos\theta\,\dot{x}'\hat{j}' - \sin\theta\,\dot{x}'\hat{k}'\right]$$

In each coord′

$$\ddot{x}' = -2\omega(\sin\theta\,\dot{z}' - \cos\theta\,\dot{y}')$$

$$\ddot{y}' = -2\omega\cos\theta\,\dot{x}'$$

$$\ddot{z}' = -g^* + 2\omega\sin\theta\,\dot{x}'$$

We can't separate, but we can integrate once wrt to time

$$\dot{x}' = -2\omega(\sin\theta\,z' - \cos\theta\,y') + \dot{x}_0'$$

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$$\left.\begin{aligned} \dot{y}' &= -2\omega\cos\theta\, x' + \dot{y}_0' \\ \dot{z}' &= -g^* t + 2\omega\sin\theta\, x' + \dot{z}_0' \end{aligned}\right\} \text{ sub. them into the } \ddot{x}' \text{ equ}$$

get

$$\ddot{x}' = 2\omega g^* t\sin\theta - 2\omega(\dot{z}_0'\sin\theta - \dot{y}_0'\cos\theta)$$ where we have dropped terms w/ ω^2

Integrate again:

$$\dot{x}' = \omega g^* t^2\sin\theta - 2\omega t(\dot{z}_0'\sin\theta - \dot{y}_0'\cos\theta) + \dot{x}_0'$$

And integrate again:

$$x'(t) = \frac{1}{3}\omega g^* t^3\sin\theta - \omega t^2(\dot{z}_0'\sin\theta - \dot{y}_0'\cos\theta) + \dot{x}_0' t + x_0$$

Sub. this into equ's for $\dot{y}'$ & $\dot{z}'$ above, drop terms w/ ω^2 & integrate

$$y'(t) = \dot{y}_0' t - \omega\dot{x}_0' t^2\cos\theta + y_0'$$

$$z'(t) = -\frac{1}{2}g^* t^2 + \dot{z}_0' t + \omega\dot{x}_0' t^2\sin\theta + z_0'$$

The terms w/ ω express the impact of Earth's rotation on the motion of the projectile in a coor'd system fixed to the Earth.

Ex: A body is dropped from rest at height h above the ground. How far from the point directly below its starting point does it land?

Initial conditions: $t=0$ $(x_0', y_0', z_0') = (0,0,h)$

$$(\dot{x}_0', \dot{y}_0', \dot{z}_0') = (0,0,0)$$

$$\therefore\ x'(t) = \frac{1}{3}\omega g^* t^3\sin\theta$$

$$y'(t) = 0$$

$$z'(t) = -\frac{1}{2} g^* t^2 + h$$

So, body hits ground when $z' = 0$, $h = \frac{1}{2} g^* t^2$, $t = \sqrt{\frac{2h}{g^*}}$

The x' coord. is then $x = \frac{1}{3} \omega g^* \left(\frac{2h}{g^*}\right)^{3/2} \sin\theta$

$$= \frac{\omega}{3} \left(\frac{8h^3}{g^*}\right)^{1/2} \sin\theta$$

Body is deflected East.

e.g. if $h = 100\,\text{m}$ & $\theta = 45°$, $x \simeq 16\,\text{mm}$

This is ahead of the Earth's rotation! B/c of cons. of ang. momentum.