3) Plummer Model

Consider a spherical system where the density is roughly constant near the center, and falls to zero at large radii.

A simple model w/ these properties is the Plummer model

$$\Phi = -\frac{GM}{\sqrt{r^2+b^2}}$$

where M is the system's total mass & b is the Plummer scale length

$$\nabla^2\Phi = \frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d\Phi}{dr}\right) = \frac{1}{r^2}\frac{d}{dr}\left(r^2\left(\frac{\frac{1}{2}GM2r}{(r^2+b^2)^{3/2}}\right)\right)$$

$$= \frac{1}{r^2}\frac{d}{dr}\left(\frac{GMr^3}{(r^2+b^2)^{3/2}}\right) = \frac{1}{r^2}\left(3GMr^2(r^2+b^2)^{-3/2} - \frac{3}{2}(r^2+b^2)^{-5/2}2rGMr^3\right)$$

$$= \frac{3GMr^2}{r^2}\left((r^2+b^2)^{-3/2}\left(\frac{(r^2+b^2)}{(r^2+b^2)}\right) - (r^2+b^2)^{-5/2}r^2\right)$$

$$= 3GM\left(\frac{r^2+b^2-r^2}{(r^2+b^2)^{5/2}}\right) = \frac{3GMb^2}{(r^2+b^2)^{5/2}}$$

Poisson's eqn says $\nabla^2\Phi = 4\pi G\rho$

$$\therefore \rho(r) = \frac{3Mb^2}{4\pi(r^2+b^2)^{5/2}} = \frac{3Mb^2}{4\pi\left(b^2\left(1+\frac{r^2}{b^2}\right)\right)^{5/2}}$$

$$= \frac{3M}{4\pi b^3}\left(1+\frac{r^2}{b^2}\right)^{-5/2}$$

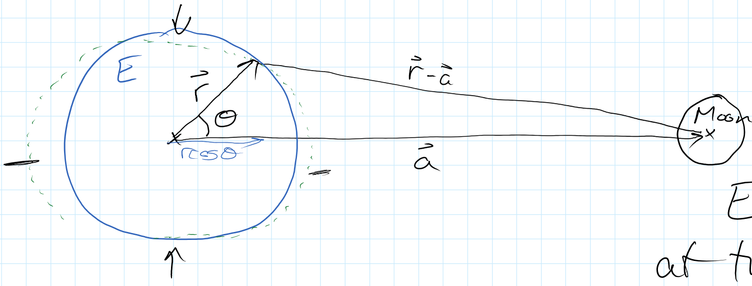
Confirm that the PE of a Plummer model is

$$V = -\frac{3\pi GM^2}{32b}$$

The position of a star orbiting in a Plummer potential cannot be given by ordinary functions.

The Tides

Tidal forces arise b/c there is a difference in the grav. attraction of the Moon b/w the close & far side of the Earth.



Let $\vec{a}$ be the position of the Moon relative to Earth's center and consider a point $\vec{r}$ on the Earth's surface. The potential at this point due to the Moon is

$$\Phi(\vec{r}) = -\frac{Gm}{|\vec{r}-\vec{a}|}$$ where m is the mass of the Moon

Since $r \ll a$, expand this in powers of $\left(\frac{r}{a}\right)$

From above diagram

$$|\vec{r}-\vec{a}|^2 = r^2 - 2ar\cos\theta + a^2$$

$$|\vec{r}-\vec{a}| = \left(r^2 - 2ar\cos\theta + a^2\right)^{1/2} = a\left(1 - \frac{2r}{a}\cos\theta + \frac{r^2}{a^2}\right)^{1/2}$$

so $$\frac{1}{|\vec{r}-\vec{a}|} = \frac{1}{a}\left(1 - \frac{2r\cos\theta}{a} + \frac{r^2}{a^2}\right)^{-1/2}$$

$$\simeq \frac{1}{a}\left(1 - \frac{1}{2}\left(-\frac{2r}{a}\cos\theta + \frac{r^2}{a^2}\right) + \frac{3}{8}\left(-\frac{2r}{a}\cos\theta + \frac{r^2}{a^2}\right)^2 - \cdots\right)$$

$$= \frac{1}{a} + \frac{r}{a^2}\cos\theta + \frac{r^2}{a^3}\left(\frac{3}{2}\cos^2\theta - \frac{1}{2}\right) + \cdots$$

$$\therefore \Phi(\vec{r}) = -Gm\left[\frac{1}{a} + \frac{r}{a^2}\cos\theta + \frac{r^2}{a^3}\left(\frac{3}{2}\cos^2\theta - \frac{1}{2}\right) + \cdots\right]$$

To find the grav. acc'n due to the Moon, we would need $\vec{\nabla}\Phi$

The 1st term of $\Phi(\vec{r})$ is const & won't yield any force.

The 2nd term just points from the Earth's center to the Moon. Can't be responsible for tides

Focus on the quadratic term which gives the grav. field

$$g_r = -\partial\Phi = 2Gmr\left(\frac{3}{2}\cos^2\theta - \frac{1}{2}\right) = Gmr\left(3\cos^2\theta - 1\right)$$

$$\boxed{a^3} \left(\quad \right)$$

note cubic dependence

$$g_\theta = -\frac{1}{r}\frac{\partial \Phi}{\partial \theta} = -\frac{Gmr}{a^3}\left(\frac{3}{2} \cdot 2\cos\theta\sin\theta\right) = -\frac{3Gmr}{a^3}\cos\theta\sin\theta$$

The field is directed outwards along the line of centers, towards & away from the Moon, and inwards in the xy plane.

Note that this is a very weak field, smaller than the change in g due to the oblateness of the Earth.

Consider $\dfrac{g_E}{g_t} = \dfrac{\frac{GM}{r^2}}{\frac{Gmr}{a^3}} = \dfrac{M}{r^2} \cdot \dfrac{a^3}{mr} = \dfrac{M}{m}\dfrac{a^3}{r^3}$

For the Moon $\left(\dfrac{M}{m}\right) = 81.3$ & $\dfrac{a}{r} = 60.3$, so $\dfrac{g_E}{g_t} \simeq 1.79 \times 10^7$

Can do the same for tides raised by the Sun, $\dfrac{g_E}{g_{t,\odot}} \simeq 3.89 \times 10^7$

So, Sun's tidal field is $\sim \frac{1}{2}$ of the Moon's. Need to include both when predicting tides.

If the Sun & Moon are in a line (New or Full Moon) then the tidal forces add & get Spring tides.

At 1st & 3rd quarters, they partially cancel & get Neap tides.

As the Earth rotates, θ changes, but b/c of the $\cos^2\theta$ term, there are 2 peaks to the tides per day.