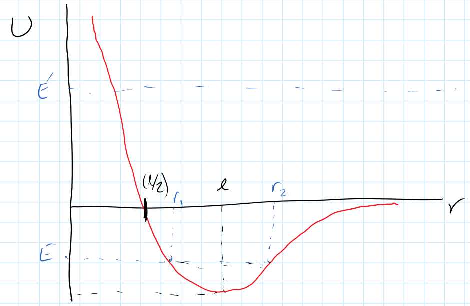
If $K<0$, this is the gravitational case.

Define $\ell = \frac{J^2}{m|K|}$

The effective PE function is $U(r) = |K|\left(\frac{\ell}{2r^2} - \frac{1}{r}\right)$



$U\left(\frac{1}{2}\ell\right) = 0$

$U(r)$ has a min at $r=\ell$

w/ min. value $U(\ell) = \frac{-|K|}{2\ell}$

Several types of motion depending on value of E.

1) $E = -\frac{|K|}{2\ell} = U_{min} \rightarrow \dot{r}=0$; particle moves in a circle w/ radius ℓ

The circular velocity from $\frac{1}{2}mV_c^2 = E - V$

$$= \frac{-|K|}{2\ell} + \frac{|K|}{\ell} = \frac{|K|}{2\ell}$$

$\therefore V_c = \sqrt{\frac{|K|}{m\ell}}$ (agrees w/ earlier result w/ $\ell = r_0$)

NB: For a circular orbit PE = 2 x KE

2) $-\frac{|K|}{2\ell} < E < 0$, the radial distance varies b/w r_1 ; r_2

- Particle 'bound'
- orbit shape? ellipse

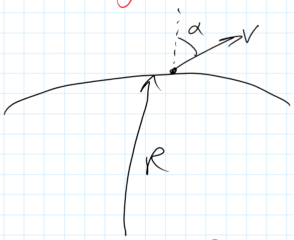
3) $E=0$. There is a minimum distance $r_1 = \frac{1}{2}\ell$ but the max r_2 is unbound. Particle reaches ∞ w/ zero KE

- orbit shape? parabola

4) $E>0$. Particle 'unbound'. There is a min. distance but no max. distance. The particle can escape to ∞ w/ non-zero velocity.

- orbit shape? hyperbola.

Ex: What is the min. velocity w/ which a projectile launched from the surface of the Earth (taken to be a sphere of mass M ; radius R) can escape ; how does it depend on angle of launch?



Suppose mass launched w/ velocity v at an angle α to the vertical. The energy & ang. mom. are $E = \frac{1}{2}mv^2 - \frac{GMm}{R}$, $J = mRv\sin\alpha$

Re-write this in terms of g: $mg = \frac{GMm}{R^2} \rightarrow GM = gR^2$

$\therefore\ E = \frac{1}{2}mv^2 - mgR.$

From above, object will escape to ∞ if $E \geq 0$. ie

$$\frac{1}{2}mv_e^2 - mgR \geq 0 \rightarrow v_e \geq \sqrt{2gR} \geq 11.2\ \text{km s}^{-1}$$

(compare this to v_c for Earth, $v_c = \sqrt{gR} \sim 7\ \text{km s}^{-1}$)

Ex: If the launch speed v is equal v_c, how high will the projectile reach for a given α?

In this case, $E<0$ & orbit is b/w r_1 & r_2. Max distance is r_2 and $E = U(r_2)$ at that point.

$$\frac{1}{2}mv^2 - \frac{GMm}{R} = \frac{J^2}{2mr_2^2} - \frac{GMm}{r_2}$$

$$\frac{1}{2}mv^2 - \frac{R^{\cancel{2}}gm}{\cancel{R}} = \frac{m^2R^2v^2\sin^2\alpha}{2\cancel{m}r_2^2} - \frac{R^2gm}{r_2}$$

$$\rightarrow \cancel{m}v^2r_2^2 - 2Rg\cancel{m}r_2^2 - \cancel{m}R^2v^2\sin^2\alpha + 2R^2g\cancel{m}r_2 = 0$$

$$(v^2 - 2Rg)r_2^2 + 2R^2gr_2 - R^2v^2\sin^2\alpha = 0$$

if $v = v_c = \sqrt{gR}$ then

$$(gR - 2gR)r_2^2 + 2R^2\cancel{g}r_2 - R^2\cancel{g}R\sin^2\alpha = 0$$

$\times(-1)$

$$r_2^2 - 2Rr_2 + R^2\sin^2\alpha = 0$$

$$\rightarrow \boxed{r_2 = R(1+\cos\alpha)}$$

Energy Equation of an Orbit

Return to energy equation: $\frac{1}{2}m(\dot{r}^2+r^2\dot{\theta}^2)+V(r)=E$

Write in terms of $u=\frac{1}{r}$. Verify that

$$\frac{J^2}{2m}\left(\frac{du}{d\theta}\right)^2+\frac{J^2}{2m}u^2+V(u^{-1})=E$$

Ex: In an earlier example we had for the spiral orbit $r=c\theta^2$

$$\frac{du}{d\theta}=-\frac{2}{c}\theta^{-3}=-2c^{1/2}u^{3/2}$$

So, the energy eq'n of the orbit is

$$\frac{1}{2}\frac{J^2}{m}(4cu^3)+\frac{J^2}{2m}u^2+V=E$$

$$\Rightarrow V=E-\frac{1}{2}\frac{J^2}{m}(u^2+4cu^3) \Rightarrow V(r)=E-\frac{1}{2}\frac{J^2}{m}\left(\frac{1}{r^2}+\frac{4c}{r^3}\right)$$

$$f(r)=-\frac{\partial V}{\partial r}=+\frac{1}{2}\frac{J^2}{2m}\left(-\frac{2}{r^3}-\frac{12c}{r^4}\right)=-\frac{J^2}{m}\left(\frac{6c}{r^4}+\frac{2}{r^3}\right) \text{ as before.}$$

Apply an Inverse-Square Law, $V(r)=-\frac{K}{r}=-Ku$ Then energy eq'n is

$$\frac{J^2}{2m}\left(\frac{du}{d\theta}\right)^2+\frac{J^2}{2m}u^2-Ku=E\ ;\ \div \text{ by } K,\ \frac{J^2}{2mK}\left(\frac{du}{d\theta}\right)^2+\frac{J^2}{2mK}u^2-u=\frac{E}{K}$$

Let $l=\frac{J^2}{mK}$ $\Rightarrow$ $\frac{l}{2}\left(\frac{du}{d\theta}\right)^2+\frac{l u^2}{2}-u=\frac{E}{K}$

Separate variables, $\left(\frac{du}{d\theta}\right)^2+u^2-\frac{2u}{l}=\frac{2E}{Kl}$

$$\left(\frac{du}{d\theta}\right)^2=\frac{2E}{Kl}-u^2+\frac{2u}{l}$$

$$\frac{du}{d\theta}=\left(\frac{2E}{Kl}-u^2+\frac{2u}{l}\right)^{1/2}$$

$$\rightarrow d\theta=\left(\frac{2E}{Kl}-u^2+\frac{2u}{l}\right)^{-1/2}du$$

Integrate both sides

LHS : $\theta-\theta_0$

RHS : $\displaystyle\int\frac{du}{\sqrt{\frac{2E}{Kl}+\frac{2}{l}u-u^2}}$

My table of integrals says $\int \frac{dx}{\sqrt{a+bx+cx^2}} = -\frac{1}{\sqrt{-c}} \sin^{-1}\left(\frac{2cx+b}{\sqrt{q}}\right) \quad c<0$

$$q = 4ac - b^2$$

So $q = 4\left(\frac{2E}{k\ell}\right)(-1) - \frac{4}{\ell^2} = \frac{-8E}{k\ell} - \frac{4}{\ell^2}$

$$\therefore \quad \theta - \theta_0 = -\sin^{-1}\left(\frac{-2u + 2/\ell}{\left(\frac{8E}{k\ell} + \frac{4}{\ell^2}\right)^{1/2}}\right) = -\sin^{-1}\left(\frac{-\frac{2}{\ell}(u\ell - 1)}{\frac{2}{\ell}\left(1 + \frac{2E\ell}{k}\right)^{1/2}}\right) = \sin^{-1}\left(\frac{u\ell - 1}{\left(1+\frac{2E\ell}{k}\right)^{1/2}}\right)$$

θ_0 is a constant of integration. Let $\theta_0 = -\frac{\pi}{2}$; solve for u

$$\sin\left(\theta + \frac{\pi}{2}\right) = \frac{u\ell - 1}{\left(1 + \left(\frac{2E\ell}{k}\right)\right)^{1/2}} \rightarrow \left[1 + \frac{2E\ell}{k}\right]^{1/2} \cos\theta = u\ell - 1$$

$$\therefore \quad u = \frac{1}{\ell}\left[1 + \left(1 + \frac{2E\ell}{k}\right)^{1/2} \cos\theta\right]$$

or $$\boxed{r = \frac{\ell}{\left(1 + \left(1 + \frac{2E\ell}{k}\right)^{1/2} \cos\theta\right)}}$$

This is the polar eq'n of the orbit w/ eccentricity is

now $e = \sqrt{1 + \frac{2E\ell}{k}} = \sqrt{1 + \frac{2EJ^2}{mk^2}}$