B) Linear Air Resistance - not conservative motion

Write air resistance as $-m\gamma\vec{v}$ where γ is a constant

e.o.m. $m\ddot{\vec{r}} = -m\gamma\dot{\vec{r}} - mg\hat{k}$

Write in components:

$$\ddot{x} = -\gamma\dot{x}$$
$$\ddot{y} = -\gamma\dot{y}$$
$$\ddot{z} = -\gamma\dot{z} - g$$

Equations are separated ; solve them individually

These are v. similar to our earlier case of drag motion w/ $\gamma = \frac{a}{m}$

Therefore, we can write the solutions as

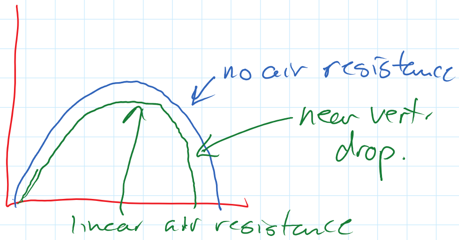
$$\dot{x} = \dot{x}_0 e^{-\gamma t}$$
$$\dot{y} = \dot{y}_0 e^{-\gamma t}$$
$$\dot{z} = \dot{z}_0 e^{-\gamma t} - \frac{g}{\gamma}\left(1 - e^{-\gamma t}\right)$$

and $x = \frac{\dot{x}_0}{\gamma}\left(1 - e^{-\gamma t}\right)$, $y = \frac{\dot{y}_0}{\gamma}\left(1 - e^{-\gamma t}\right)$, $z = \left(\frac{\dot{z}_0}{\gamma} + \frac{g}{\gamma^2}\right)\left(1 - e^{-\gamma t}\right) - \frac{gt}{\gamma}$ ←

where the initial pos'n to be the origin ; $\vec{V}_0 = (\dot{x}_0, \dot{y}_0, \dot{z}_0)$

$$\text{vectorally,} \quad \vec{r}(t) = \left(\frac{\vec{V}_0}{\gamma} + \frac{\hat{k}g}{\gamma^2}\right)\left(1 - e^{-\gamma t}\right) - \hat{k}\frac{gt}{\gamma}$$

Confirm that trajectory lies in the plane $y = \left(\frac{\dot{y}_0}{\dot{x}_0}\right)x$. The path is not a parabola, but lies below it



NB: actual air drag is a much more complicated function of velocity

Horizontal range of the projectile. Set $\dot{y}_0 = 0$ so motion is in (x, z) plane & set $z=0$ & eliminate t w/ the $x(t)$ eq'n

From $x(t)$, $t = -\gamma^{-1} \ln\left(1 - \frac{\gamma x}{\dot{x}_0}\right)$

$$\therefore \left(\frac{\dot{z}_0}{\gamma} + \frac{g}{\gamma^2}\right)\frac{\gamma x_h}{\dot{x}_0} + \frac{g}{\gamma^2}\ln\left(1 - \frac{\gamma x_h}{\dot{x}_0}\right) = 0$$ can't be solved analytically

Can get close to the answer by expanding the $\ln(\,)$ term:

$$\ln(1-u) = -u - \frac{u^2}{2} - \frac{u^3}{3} - \cdots \quad (\text{for } |u| \ll 1)$$

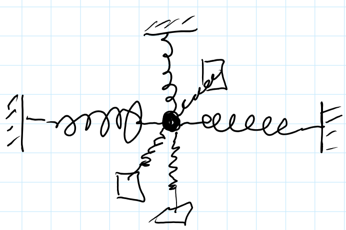
Pluggin this in & ASA

$$x_h = \frac{2\dot{x}_0 \dot{z}_0}{g} - \frac{8\dot{x}_0 \dot{z}_0^2 \gamma}{3g^2} + \cdots$$

If the projectile is fired at an elevation angle α w/ initial speed V_0 then $\dot{x}_0 = V_0 \cos\alpha$, $\dot{z}_0 = V_0 \sin\alpha$, $2\dot{x}_0\dot{z}_0 = 2V_0^2 \sin\alpha\cos\alpha = V_0^2 \sin 2\alpha$

$$\text{so,} \quad x_h = \underbrace{\frac{V_0^2 \sin 2\alpha}{g}}_{\text{no air resistance}} - \underbrace{\frac{4V_0^3 \sin 2\alpha \sin\alpha\, \gamma}{3g^2}}_{\text{reduction from air resistance}} + \cdots$$

Harmonic Oscillator in 2D or 3D



Consider a linear restoring force alway directed to the origin $\vec{F} = -K\vec{r}$

$\therefore m\ddot{\vec{r}} = -K\vec{r} \rightarrow$ a 3D linear isotropic oscillator

Start in 2D & note that this is a separable e.o.m:

$$m\ddot{x} = -Kx$$
$$m\ddot{y} = -Ky$$
conservative? yes

Solution: $x = a\cos(\omega t + \alpha)$, $y = b\cos(\omega t + \beta)$

where $\omega^2 = \frac{K}{m}$; a, b, α, β are const. determined by initial conditions

Want to find path in xy-plane. Need to eliminate t from eqns

Write $y = b\cos(\omega t + \alpha + \Delta)$ where $\Delta = \beta - \alpha$

$\therefore$ $y = b[\cos(\omega t+\alpha)\cos\Delta - \sin(\omega t + \alpha)\sin\Delta]$ from trig-identity

$$\frac{y}{b} = \left(\frac{x}{a}\cos\Delta - \underline{\left(1-\frac{x^2}{a^2}\right)^{1/2}\sin\Delta}\right) \quad *$$

Square both sides

$$\frac{y^2}{b^2} = \frac{x^2}{a^2}\cos^2\Delta - \frac{2x}{a}\cos\Delta\sin\Delta\left(1-\frac{x^2}{a^2}\right)^{1/2} + \left(1-\frac{x^2}{a^2}\right)\sin^2\Delta$$

$$\frac{y^2}{b^2} = \frac{x^2}{a^2}\cos^2\Delta + \sin^2\Delta - \frac{x^2}{a^2}\sin^2\Delta - \frac{2x}{a}\cos\Delta\sin\Delta\left(1-\frac{x^2}{a^2}\right)^{1/2}$$

$$= \sin^2\Delta + \frac{x^2}{a^2}\cos^2\Delta - \frac{x^2}{a^2}\sin^2\Delta - \frac{2x}{a}\cos\Delta\left(\frac{x}{a}\cos\Delta - \frac{y}{b}\right) \quad \text{from } *$$

$$= \sin^2\Delta + \frac{x^2}{a^2}\cos^2\Delta - \frac{x^2}{a^2}\sin^2\Delta - \frac{2x^2}{a^2}\cos^2\Delta + \frac{2xy\cos\Delta}{ab}$$

$$= \sin^2\Delta - \frac{x^2}{a^2}\left(\sin^2\Delta + \cos^2\Delta\right) + \frac{2xy\cos\Delta}{ab}$$

$$\therefore \frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{2xy\cos\Delta}{ab} = \sin^2\Delta$$

Recall the quadratic equation $ax^2 + bxy + cy^2 + dx + ey + f = 0$ represents a conic section. See Appendix B for a review of conics.