Section 1.1 : Mathematical Models and Solutions

Chapter 1 : Introduction

Math 2552 Differential Equations

Section 1.1

Topics

We will cover these topics in this section.

1. Mathematical Models and Direction Fields
2. Newton's Law of Cooling

Objectives

For the topics covered in this section, students are expected to be able to do the following.

1. Apply an exponential growth/decay model to solve and analyze first order DEs

Example: Newton's Law of Cooling

Suppose a system under observation is an object at temperature $u(t)$, at time t, and is located in an environment with constant ambient temperature T.

Newton's Law of Cooling: the rate of change of the temperature of an object is negatively proportional to the difference between $u(t)$ and T.

$$\frac{d}{dt}u = -k(u - T) \tag{1}$$

Here, u is an **unknown**, and k and T are **parameters** of the system.

Equation (1) is an example of a **differential equation**.

Definition: Differential Equation

A **differential equation** is an equation involving a function and its derivatives.

Solution to a DE

- A **solution** of a DE is a differentiable function that satisfies that DE on some interval.
- To determine whether a function is a solution to a given DE, what can we do?

Example: Verify that $Ce^{-kx} + T$, $C \in \mathbb{R}$, is a solution to the DE

$$\frac{d}{dt}u = -k(u - T)$$

.

Dynamical Systems

Newton's Law is one example of an equation that describes a **dynamical system**.

A dynamical system is composed of:

- A **system**: Which means that we are observing a phenomenon which behaves according to a set of laws.
- The phenomenon may be mechanical, biological, social, etc.
- **Dynamics**: the system evolves in time.

It is our task to **predict** and **characterize** (as much as possible) the long-term behavior of the dynamical system and how it changes. This leads us to the use of derivatives and the methods we explore for the rest of the course.

Section 1.2 : Qualitative Methods

Chapter 1 : Introduction

Math 2552 Differential Equations

Section 1.2

Topics
We will cover these topics in this section.

1. Mathematical Models and Direction Fields
2. Diagrams

Objectives
For the topics covered in this section, students are expected to be able to do the following.

1. Determine and classify equilibrium solutions
2. Sketch direction fields and phase lines
3. Sketch solution curves of autonomous DEs based on a qualitative analysis

Direction Fields

- Sometimes we encounter situations where we wish to describe the solutions to a differential equation that we cannot solve.
- When this happens, we might be able to use a qualitative method to characterize the system.
- In the next slide, we explore one qualitative method: the **direction field**, and how we can construct it.

Direction Field Example

Example: recall Newton's Law of Cooling, $\frac{d}{dt}u = -k(u-T)$.

If $T = 1$ and $k = 2$. Then

$$\frac{d}{dt}u = -k(u-T) = 2 - 2u$$

If $u = 0$, $\dfrac{du}{dt} = \_\_\_\_\_\_$

If $u = 1$, $\dfrac{du}{dt} = \_\_\_\_\_\_$

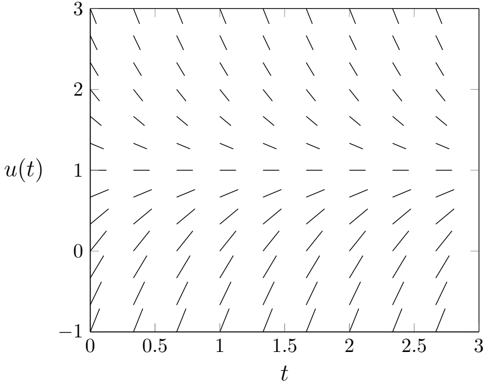
If $u = 2$, $\dfrac{du}{dt} = \_\_\_\_\_\_$

If $u = 3$, $\dfrac{du}{dt} = \_\_\_\_\_\_$

We use this information to sketch a set of line segments that characterize $u'(t)$ for a particular k and T.

Direction Field for $u'(t) = 2 - 2u$

We can plot line segments for a set of points (t, u), whose slope are determined by $u'(t)$.



For what values of u is $\frac{du}{dt} = 0$?

Autonomous DEs and Equilibrium Solutions

- An **autonomous differential equation**: is of the form $\dfrac{dy}{dt} = f(y)$
- An **equilibrium solution** of a DE in $y(t)$ satisfies $y =$ constant.
- Equilibrium points are also referred to as **critical points**, **fixed points**, **stationary points**.

An equilibrium solution must therefore satisfy

$$\frac{dy}{dt} =$$

Phase Portraits and Stability

Example 1: Suppose $\frac{dy}{dt} = f(y) = y(y-1)(y-2)$, $y_0 \geq 0$, $t \geq 0$.

a) Determine the equilibrium points.

b) Sketch $f(y)$ vs y.

c) Use the information in parts (a) and (b) to sketch the phase portrait of the DE.

d) Use the information in parts (a), (b), and (c) to sketch a few solution curves, or **integral curves** for the DE.

Classification of Equilibrium Points of an

Possible equilibrium patterns of an autonomous DE, $\frac{dy}{dt} = f(y)$ near an equilibrium point, $y = y_1$.

classification	**interpretation**
stable equilibrium point	solution curves, close to and on either side of y_1, tend to y_1 as $t \to \infty$
unstable equilibrium point	solution curves, close to and on either side of y_1, tend away from y_1 as $t \to \infty$
semi-stable equilibrium point	solution curves, close to and on one side of y_1, tends to y_1 as $t \to \infty$, the other side tends away

Example 2 (if time permits)

Example: Suppose $\dfrac{dy}{dt} = f(y) = y^2(1-y)^2$, $y_0 \in \mathbb{R}$, $t \geq 0$.

a) Sketch $f(y)$ vs y.

b) Sketch the phase portrait of the DE.

c) Classify the equilibrium points.

d) Use the information in parts (a), (b), and (c) to sketch a few solution curves of the DE.

Section 1.3 : Definitions, Classifications, Terminology

Chapter 1 : Introduction

Math 2552 Differential Equations

Section 1.3

Topics

We will cover these topics in this section.

1. Classification of ODEs
2. Standard Form

Objectives

For the topics covered in this section, students are expected to be able to do the following.

1. Classify differential equations
2. Convert a first order linear ODE into standard form

Classifying Differential Equations

Section 1.3 introduces some of the ways we can classify differential equations.

Classification allows us to determine which methods we can use to solve a DE.

Ordinary and Partial Differential Equations

- **Ordinary Differential Equation (ODE)**: the functions in the DE depend on only one a single independent variable.
- **Partial Differential Equation (PDE)**: the functions in the DE depend on more than one independent variable.

This course focuses on ODEs. PDEs are the focus of more advanced courses that are offered by the School of Math.

Examples

The heat equation, $\dfrac{\partial}{\partial t}u(x,t) = D\dfrac{\partial u^2}{\partial x^2}$ is an example of a PDE.

Newton's Law of Cooling, $\frac{d}{dt}u(t) = -k(u-T)$, is an example of an ODE.

The Order of an ODE

- The **order** of an ODE is the highest degree derivative which appears in the equation.
- This course mostly focuses on first and second order ODEs.

Example: What is the order of the ODE $u^{(3)} + 2e^t u'' + uu' = t^4$?

Linear ODEs

Definition: Linear ODE

An n^{th} order **linear ODE** takes the form

$$\sum_{k=0}^{n} a_k u^{(n-k)} = a_0(t)u^{(n)}(t) + a_1(t)u^{(n-1)}(t) + \ldots + a_n(t)u(t) = g(t)$$

The **coefficients** $a_0(t),\ a_1(t), \ldots,\ a_n(t)$ and $g(t)$ are given, $u(t)$ is unknown.

If $g(t) = 0$, we say that the linear ODE is **homogeneous**.

A DE that is not of this form is **non-linear**.
The coefficients may be non-linear with respect to t.

Standard Form of a First Order DE

The general first order linear equation is of the form

$$a_0(t)\frac{dy}{dt} + a_1(t)y = h(t) \quad [*]$$

Again, a_0, a_1 and h are given, and y is the unknown.

Definition: Standard Form, 1st Order Linear ODE

If $a_0(t) \neq 0$, we can divide by $a_0(t)$ and put [*] in the form

$$\frac{dy}{dt} + p(t)y = g(t)$$

where $p(t) := a_1(t)/a_0(t)$ and $g(t) := h(t)/a_0(t)$. This the **Standard Form**

Examples

Which of the following DEs are linear? What is the order of the DEs?

1. $t^3 \dfrac{d^2y}{dt^2} + \sin(t)\dfrac{dy}{dt} = y$
2. $y\dfrac{dy}{dt} = t$
3. $(1+t)\dfrac{d^3y}{dt^3} = \sin(t+y)$

Summary

Classification allows us to determine which methods we can use to solve the DE.

We can classify an ODE based on

- the order of the equation
- linearity
- whether the DE is homogeneous

We will introduce other ways to classify DEs.

Section 2.1 : Separable Equations

Chapter 2 : First Order Differential Equations

Math 2552 Differential Equations

Section 2.1

Topics

We will cover these topics in this section.

1. Solving first order separable differential equations.

Objectives

For the topics covered in this section, students are expected to be able to do the following.

1. Classify differential equations as separable
2. Solve separable first order linear ODEs

Lecture Organization

- We explore a method to solve **separable differential equations** in this section.
- We will
    1. define what a separable differential question is
    2. then solve an ODE
    3. then discuss the solution method more generally.
- Our approach will make use of the **differential**, a topic often covered in a differential calculus course.

Definition

Definition: Separable ODE

A first order differential equation is **separable** if it can be written in the form

$$\frac{dy}{dx} = f(x, y) = p(x)q(y)$$

Example

Consider the differential equation

$$\frac{dy}{dt} = (1 - 12t)y^2, \ y = y(t), \ y(0) = \frac{1}{8}.$$

a) Is this differential equation linear?
b) Compute the differential $dy = \frac{dy}{dt}dt$.
c) If possible, separate variables on either side of the equation in the previous step.
d) If possible, integrate and solve for y.

General Procedure

This gives us a procedure for solving a separable DE.

1. Compute the differential $dy = \frac{dy}{dt} dt$.
2. Separate variables on either side of the equation in the previous step.
3. Integrate and solve for y.

Note that our differential equation need not be linear for us to use this method.

Section 2.2 : Linear Equations: Integrating Factors

Chapter 2 : First Order Differential Equations

Math 2552 Differential Equations

Section 2.2

Topics

We will cover these topics in this section.

1. Solving a first order linear ordinary differential equations using a procedure that uses an integrating factor

Objectives

For the topics covered in this section, students are expected to be able to do the following.

1. Convert differential equations into standard form
2. Classify differential equations as linear
3. Solve first order linear ODEs using an integrating factor

Lecture Organization

- We explore a method to solve a first order **linear** ODE in this section.
- Recall that an ODE is in **standard form** if it can be written as

$$\frac{dy}{dt} + p(t)y(t) = y(t)$$

- We will solve an equation in standard form on the next slide, and then discuss the solution method more generally.

Example 1

Consider the differential equation $ty' + 2y = 4t$ for $t \geq 0$.

1. Is this differential equation separable?
2. Solve the differential equation.

The Integrating Factor

Given a linear first order ODE in standard form,

$$\frac{dy}{dt} + p(t)y = g(t) \tag{2}$$

Multiply by the **integrating factor** $\mu(t) = e^{\int p\,dt}$.

$$\mu(t)\ \frac{dy}{dt} + p(t)\mu(t)y = \mu(t)g(t) \tag{3}$$

μ is constructed so that the left-hand side is a derivative of a product.

$$\frac{d}{dt}\left(\mu(t)y(t)\right) = \mu(t)\ \frac{dy}{dt} + p(t)\mu(t)y \tag{4}$$

We construct μ so that the left-hand side of (2) is the derivative of $\mu(t)\,y$.

Procedure

This gives us a procedure for solving a first order linear DE.

$$\frac{dy}{dt} + p(t)y(t) = y(t)$$

1. Convert given DE to standard form (if necessary).
2. Calculate integrating factor $\mu(t) = e^{\int p(t)\,dt}$
3. Multiply DE by μ, express DE in the form $[\mu(t)y]' = \mu g$.
4. Integrate
5. Solve for y

We will explore more examples of solving linear equations in the next section.

Section 2.3 : Modeling with First Order Equations

Chapter 2 : First Order Differential Equations

Math 2552 Differential Equations

Section 2.3

Topics

We will cover these topics in this section.

1. Solving a first order linear ordinary differential equations using a procedure that uses an integrating factor

Objectives

For the topics covered in this section, students are expected to be able to do the following.

1) Construct a differential equation to model a real-world situation.
2) Solve the differential equation so that we can interpret its solution to characterize a system.
3) Analyze mathematical statements and solutions of differential equations.

Example 1: Water Tank

A tank initially contains 40 pounds of salt dissolved in 600 gallons of water. Starting at time $t = 0$, water that contains $1/2$ pound of salt per gallon is poured into the tank at the rate of 4 gal/min and the mixture is drained from the tank at the same rate.

a) Construct a differential equation for $Q(t)$, which is the number of pounds of salt in the tank at time $t > 0$.

b) Solve the DE to determine an expression for $Q(t)$.

c) After a long period of time, what happens to the concentration of salt in the tank?

Mathematical Modeling

The process we used in the previous example roughly followed this process.

1) Construct a differential equation to model a real-world situation.
2) Solve the differential equation so that we can interpret its solution to characterize a system.
3) Analyze mathematical statements and solutions of differential equations.

The above process is used throughout this course, and so are captured in the course learning objectives that are stated in the syllabus.

Example 2: Population Model

The world population in 2018 is roughly 7.6 billion.

a) The world population is increasing at a rate of 1.2% per year. If the growth rate remains fixed at 1.2%, how long will it take for the population of the world to reach 20 billion people?

b) Assume the earth cannot support a population beyond 20 billion people. If the population growth rate is proportional to the difference between how close the world population is to this limiting value, what is the expression that gives the world population as a function of time?

Section 2.4 : Differences Between Linear and Nonlinear Equations

Chapter 2 : First Order Differential Equations

Math 2552 Differential Equations

Section 2.4

Topics

We will cover these topics in this section.

1. Theorems for first order linear and nonlinear IVPs.

Objectives

For the topics covered in this section, students are expected to be able to do the following.

1) Characterize first order linear and nonlinear IVPs in terms of existence and uniqueness.
2) Determine intervals where a solution to a given first order IVP exists.

Motivation

There are two questions that we exploring in this section.

Given an initial value problem (IVP).

1. **existence**: does the IVP have a solution, and if so, where?
2. **uniqueness**: is the solution unique?

We explore these questions for linear and non-linear cases.

A Motivating Example

Consider the IVP

$$\frac{dy}{dt} = 8ty^{1/5}, \quad y(0) = 0$$

Take a few minutes to answer the following on your own.

a) Is the ODE linear?

b) Solve the IVP to determine an expression for $y(t)$.

c) Is your solution unique? Can you identify another solution to the IVP?

After a few minutes compare your results with someone sitting nearby.

Linear IVP

Theorem: Existence and Uniqueness of 1st Order Linear IVP

If p and g are continuous on (α, β), $t_0 \in (\alpha, \beta)$, then there is a **unique** solution to the IVP

$$y' + p(t)y = g(t), \quad y(t_0) = y_0$$

A proof of this theorem is in the textbook.

Example: Consider the IVP

$$(9 - t^2)y' + 5ty = 3t^2, \ y(-1) = 1$$

Determine an interval where a solution to the IVP exists.

Nonlinear IVP

Theorem: Existence and Uniqueness of 1st Order Nonlinear IVP

If f and $\frac{\partial f}{\partial y}$ are continuous over $\alpha < t < \beta$, and $\gamma < y < \delta$ which contains the point (t_0, y_0), then there is a **unique** solution to the IVP

$$y' = f(t,y), \quad y(t_0) = y_0$$

on an interval contained in $\alpha < t < \beta$.

Note that:

- A proof of this theorem goes beyond the scope of this course.
- These conditions are sufficient, but not necessary.
- The expression $\dfrac{\partial f}{\partial y}$ is a **partial** derivative.
- Does $\frac{dy}{dt} = 8ty^{1/5},\ y(0) = 0$, satisfy the conditions of this theorem?

Other Linear and Nonlinear IVP Comparisons

The 1st order ODE $y' + p(t)y = g(t)$ has the following properties.

1. If the p and g are continuous, there is a general solution containing an arbitrary constant that represents all solutions of the ODE.
2. There is an explicit expression for the ODE.
3. Points where the solution is discontinuous can be found by identified without solving the ODE: they are identified from the coefficients.

A **nonlinear** first order ODE has none of the above properties.

Section 2.5 : Autonomous Equations and Population Dynamics

Chapter 2 : First Order Differential Equations

Math 2552 Differential Equations

Section 2.5

Topics
We will cover these topics in this section.

1. Bifurcation points and bifurcation diagrams
2. Concavity

Objectives
For the topics covered in this section, students are expected to be able to do the following.

1) Use concavity to sketch solution curves of a DE.
2) Sketch a bifurcation diagram for a first order autonomous DE.

Autonomous DEs

Recall the following.

- an **autonomous** DE has the form $\dfrac{dy}{dt} = f(y)$
- **equilibrium solutions** of an autonomous DE can be found by locating roots of $f(y) = 0$
- we can use equilibrium solutions to sketch solution curves

A more accurate sketch of solution curves can be drawn by using **concavity**.

Note that if $\frac{dy}{dt} = f(y)$, then

$$\frac{d^2y}{dt^2} = \frac{d}{dt} f(y) =$$

Thus, if y is concave up when:

Example 1

A population obeys the logistic equation

$$\frac{dy}{dt} = ry\left(1 - \frac{y}{K}\right)$$

a) Sketch f vs y, identify and classify the equilibrium points of y.
b) For $y \in [0, T]$ determine whether y is concave up or concave down.
c) Use the information in parts (a), (b) to sketch a few **integral curves** of the DE.

Example 2 (as time permits)

A population obeys the DE $y' = a - y^2$, for a parameter a. Take a few minutes to solve the following on your own.

a) Determine the equilibrium points for any $a \in \mathbb{R}$. There are three cases.

b) Sketch the phase lines for each case and classify the critical points.

c) For the case when $a > 0$, sketch a few solution curves.

d) Sketch the location of the critical point as a function of a in an ay-plane.

The sketch in part (d) is known as a **bifurcation diagram**.

Section 3.1 : Systems of Two Linear Algebraic Equations

Chapter 3 : Systems of Two First Order Equations

Math 2552 Differential Equations

Section 3.1

Topics
We will cover these topics in this section.

1. Solving first order separable differential equations.

Objectives
For the topics covered in this section, students are expected to be able to do the following.

1. Determine the eigenvalues and eigenvectors of a matrix
2. Solve a system of two linear equations
3. Characterize a linear system in terms of the number of solutions, and whether the system is consistent or inconsistent.
4. Characterize a linear system in terms of its eigenvalues.

Examples

1. Solve the linear system, and determine whether the lines intersect, are parallel, or are coincident.

$$\begin{aligned} x_1 - 2x_2 &= -1 \\ -x_1 + 3x_2 &= 3 \end{aligned}$$

2. Determine the eigenvalues and eigenvectors of the matrices.

$$\text{a)} \quad A = \begin{pmatrix} 5 & 2 \\ 2 & 1 \end{pmatrix}$$

$$\text{b)} \quad B = \begin{pmatrix} -5 & -5 \\ 5 & -5 \end{pmatrix}$$

Section 3.2 : Systems of Two First Order Linear Differential Equations

Chapter 3 : Systems of Two First Order Equations

Math 2552 Differential Equations

Section 3.2

Topics

We will cover these topics in this section.

1. Systems of two first order linear DEs

Objectives

For the topics covered in this section, students are expected to be able to do the following.

1. Express a set of linear ODEs as a matrix equation
2. Characterize a DE by its dimension and whether it is homogeneous
3. Confirm that a given vector function is a solution to a DE system.
4. Identify critical points of a system of DEs.
5. Covert a second order DE into a system of DEs.
6. Sketch component plots of a solution to a DE.

The textbook also introduces phase diagrams in this section, we will introduce them in a later part of this course.

Foxes and Rabbits

Foxes and rabbits live together on an island. The foxes (predators) eat the rabbits, rabbits (prey) eat vegetables.

Photo credits: www.flickr.com/photos/mattyfioner, www.flickr.com/photos/43158397@N02

Suppose

- $x_1(t) =$ number of foxes on the island
- $x_2(t) =$ number of rabbits on the island
- k foxes are removed from the island per day, $k > 0$

For simplicity, assume x_1 and x_2 are continuous and differentiable.

A Linear System of DEs

Populations of the two species can be modeled using a system of linear ODEs.

$$\frac{dx_1}{dt} = ax_1 + bx_2 - k$$
$$\frac{dx_2}{dt} = cx_1 + dx_2$$

We can write this as a matrix equation:

expressing two linear DEs as a matrix equation is our 1st learning objective

Dimension and Homogeneity

$$\frac{d\vec{x}}{dt} = P(t)\vec{x} + \vec{g}(t)$$

Our linear system

- is first order system of **dimension two**, because $\vec{x}$ has two elements.
- is **non-homogeneous** because $\vec{g}(t) \neq \vec{0}$.

If $\vec{g}(t) = \vec{0}$ for all t, the system is **homogeneous**.

dimension and homogeneity covers our 2nd learning objective

Solutions to a System

A **solution** to a two-dimensional system

$$\frac{d\vec{x}}{dt} = P(t)\vec{x} + \vec{g}(t) \tag{5}$$

is a set of two functions, $x_1(t)$ and $x_2(t)$ that satisfy the system for all t on some interval.

Suppose we are given values for $a, b, c, d, \vec{g}$ so that

$$\frac{d\vec{x}}{dt} = \begin{pmatrix} 20 & 0 \\ -10 & 30 \end{pmatrix} \vec{x} \tag{6}$$

It can be shown that there are two **solutions** to this system, which are

$$\vec{u}_1(t) = e^{\lambda_1 t}\,\vec{v}_1(t), \quad \vec{u}_2(t) = e^{\lambda_2 t}\,\vec{v}_2(t)$$

λ_1 and λ_2 are the eigenvalues of P. Identify one solution, and show that it satisfies (2).

Critical Points

Critical points, or **equilibrium solutions** of a system of DEs correspond to points where

$$\frac{d\vec{x}}{dt} =$$

For example, the critical points of

$$\frac{d\vec{x}}{dt} = \begin{pmatrix} 20 & 0 \\ -10 & 30 \end{pmatrix} \vec{x} \tag{7}$$

are:

Second Order Equations

We can always convert a second order linear DEs to a system of first order linear DEs.

Example: convert the second order equation to a system of first order differential equations.

$$\frac{d^2y}{dt^2} - \sin(t)\frac{dy}{dt} + 7y = e^t \cos t + 1$$

Note: Component Plots

Plots of solutions u_1 and u_2 vs t are **component plots** of a two-dimensional system of ODEs.

Example: a solution to

$$\frac{d\vec{x}}{dt} = \begin{pmatrix} 20 & 0 \\ -10 & 30 \end{pmatrix} \vec{x}$$

is:

$$\vec{u}(t) = c_1\vec{u}_1 + c_2\vec{u}_2 =$$

Section 3.3 : Homogeneous Linear Systems with Constant Coefficients

Chapter 3 : Systems of Two First Order Equations

Math 2552 Differential Equations

Section 3.3

Topics

We will cover these topics in this section.

1. Systems of two first order linear DEs

Objectives

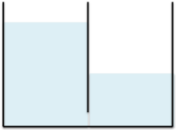
For the topics covered in this section, students are expected to be able to do the following.

1. Solve first order homogeneous linear systems.
2. Sketch component plots and phase portraits of linear systems of differential equations.

The textbook also introduces the Wronskian and discusses uniqueness. We'll briefly explore these topics and go through them in more detail in Section 6.2.

Compartment Model

A tank is divided into two cells. Each cell is filled with a fluid, and small opening allows the fluid to flow freely between the cells.



Assume: height of fluid in a cell changes at a rate proportional to the difference between fluid height in that cell and the fluid height in the other cell.

Questions:

1. What happens to the system after a long period of time?
2. Construct a linear system for the fluid level heights.
3. Solve the linear system.
4. Determine whether the solution is unique.
5. Sketch component plots and a phase portrait of the system.

Section 3.4 : Complex Eigenvalues

Chapter 3 : Systems of Two First Order Equations

Math 2552 Differential Equations

Section 3.4

Topics
We will cover these topics in this section.

1. Systems of two first order linear DEs for the complex eigenvalue case

Objectives
For the topics covered in this section, students are expected to be able to do the following.

1. Solve first order homogeneous linear systems that have complex eigenvalues
2. Sketch component plots and phase portraits of linear systems of differential equations for complex eigenvalues

Example

Determine the general solution to

$$\vec{x}\,' = A\vec{x} = \begin{pmatrix} -1 & 2 \\ -1 & -3 \end{pmatrix} \vec{x}$$

Converting Solution to Real Valued Vector Functions

Suppose we have

$$\vec{x}(t) = c_1 e^{\lambda_1 t} \vec{v}_1 + c_1 e^{\lambda_2 t} \vec{v}_2$$

where λ_1, λ_2 are complex, and $\vec{v}_1$ and $\vec{v}_2$ are complex. We wish to express $\vec{x}$ as a real-valued vector function.

Section 3.5 : Repeated Eigenvalues

Chapter 3 : Systems of Two First Order Equations

Math 2552 Differential Equations

Section 3.5

Topics
We will cover these topics in this section.

1. Systems of two first order linear DEs for the repeated eigenvalue case

Objectives
For the topics covered in this section, students are expected to be able to do the following.

1. Solve first order homogeneous linear systems that have repeated eigenvalues
2. Sketch component plots and phase portraits of linear systems of differential equations for repeated eigenvalues

Participation Activity: Worksheet

Depending on how much time we have, we will run examples in this lecture as a participation activity.

- Please work in groups of ___________
- Each group submits **one** sheet of paper
- Instructor has paper you can use
- Print full names and email addresses on cover
- Every student in a group gets the same grade
- Grading scheme per question:
  - 0 marks for no work or working alone
  - 1 mark for starting the problem or for a final answer with insufficient justification
  - 2 marks for a complete solution

Object Motion

The motion of an object moving in the xy-plane is given by $\vec{r}(t)$, where

$$\vec{r}(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$$

The velocity of the object is constrained by

$$\frac{dx}{dt} = -x + ky \tag{8}$$

$$\frac{dy}{dt} = -y \tag{9}$$

Assume $k \in \mathbb{R}$. At time $t = 0$, our object is located at $\vec{r}(0) = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$.

Take a few minutes on your own to solve this initial value problem. Create a rough sketch of the phase portrait. How your answer depend on the value of k?

Compare your answers with someone sitting nearby.

Example

Construct the general solution to the system.

$$\vec{x}\,' = \begin{pmatrix} 1 & -1 \\ 1 & 3 \end{pmatrix} \vec{x}$$

Section 4.1 : Definitions and Examples

Chapter 4 : Second Order Equations

Math 2552 Differential Equations

Section 4.1

Topics
We will cover these topics in this section.

1. Second order linear constant coefficient differential equations
2. Spring-mass systems

Objectives
For the topics covered in this section, students are expected to be able to do the following.

1. Classify second order differential equations as linear or non-linear, homogeneous or non-homogeneous.
2. Construct second order differential equations, and initial value problems, that represent spring-mass systems.
3. Interpret component-plots of solutions to spring-mass system problems.

The textbook explores the linearized pendulum as well as LRC circuits. We will not have time to explore these concepts at this point. Students aren't expected to be familiar with this material.

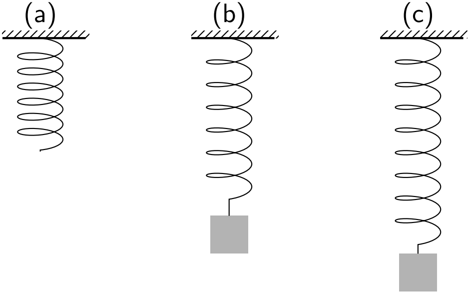
Spring-Mass System

Three cases:

(a) Spring of natural length l_0 attached to horizontal surface

(b) Mass m, attached to spring, spring length in equilibrium position is $L + l_0$

(c) Same as (b) except an external force has extended length of spring



Forces

Our mathematical models include four different forces.

force	**equation**
gravitational	
spring	
external	
damping	

Terminology

classification	**differential equation**
unforced undamped oscillator	$my'' + ky = 0$
forced undamped oscillator	$my'' + ky = F(t)$
unforced damped oscillator	$my'' + \gamma y' + ky = 0$
forced damped oscillator	$my'' + \gamma y' + ky = F(t)$

Participation Activity: Index Card

For each case, construct a differential equation.

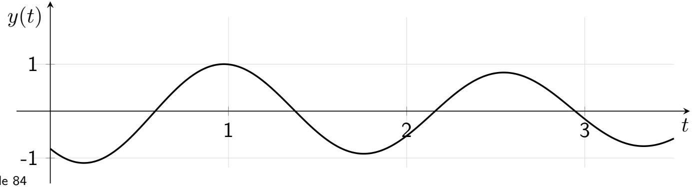
- Please work in groups of ___________
- Each group submits **one** card
- Instructor has paper you can use
- Print full names and email addresses on cover
- Every student in a group gets the same grade
- Grading scheme per question:
  - 0 marks for no work or working alone
  - 1 mark for starting the problem or for a final answer with insufficient justification
  - 2 marks for a complete solution

Examples

Construct an initial value problem for the following situations.

1. A 2 pound mass is attached to a spring and stretches it 4 inches. The spring is released from rest from a point that is 3 inches above the equilibrium point.
2. If a 0.25 kg mass is attached to a spring, it stretches the spring by 0.5 m. The spring is then attached to a viscous damper with a damping constant of 0.01 N·s/m, and the mass is released with a downward velocity of 4 m/s from a point 0.01 m below equilibrium.

The figure below represents the graph of an equation of motion for a damped spring/mass system. Determine whether the object was released above or below equilibrium position, and whether the mass was released with an upward, downward, or no velocity. Assume y is measured downward.



Section 4.2 : Theory of Second Order Homogeneous Equations

Chapter 4 : Second Order Equations

Math 2552 Differential Equations

"At the end of eleventh grade, I took the measure of the situation, and came to the conclusion that rapidity doesn't have a precise relation to intelligence. What is important is to deeply understand things and their relations to each other ... the fact of being quick or slow isn't really relevant"

- Laurent Schwartz, Field's Medal Winning Mathematician

Section 4.2

Topics
We will cover these topics in this section.

1. Existence and uniqueness of second order linear DEs
2. Reduction of order

Objectives
For the topics covered in this section, students are expected to be able to do the following.

1. Determine intervals where a second order IVP will have a unique solution.
2. Apply the method of reduction of order to identify solutions of differential equations.

The textbook introduces the differential operator notation, and describes how we can apply Abel's theorem to evaluate the Wronskian and use it to determine whether a set of functions are a fundamental set.

Motivation

This section reviews the uniqueness questions that we explored for 1^{st} order DEs.

Given an initial value problem (IVP),

1. does the IVP have a **unique** solution, and if so, where?
2. do a set of solutions form a **fundamental** set?

We explore these questions for first order systems, and for second order linear differential equations.

Note: 2^{nd} order DEs are a special case 1^{st} order linear systems.

Recall: Existence and Uniqueness

Theorem

If p, q, and g are continuous on open interval I, $t_0 \in I$, then there is a **unique** solution to the IVP

$$y'' + p(t)\,y' + q(t) = g(t), \quad y(t_0) = y_0, \quad y'(t_0) = y_1$$

Example: determine the largest interval in which the solution to the IVP is certain to exist.

$$(t-2)^2 y'' + ty' + \sqrt{t}\,y = 0, \quad y(1) = 1, \quad y'(1) = 2$$

Recall: Fundamental Set of Solutions

Theorem

Let y_1, y_2 be solutions of

$$y'' + py' + qy = 0$$

on an interval, I, in which each p is continuous.

If for some $t_0 \in I$ their Wronskian is non-zero, then every solution of the DE, $\phi(t)$, can be written as

$$\phi(t) = c_1 y_1 + c_2 y_2$$

Reduction of Order

Given one solution, y_1, to a 2nd order linear homogeneous differential equation, the **reduction of order** method allows to derive y_2 such that $\{y_1, y_2\}$ are a fundamental set.

Example: $y_1(t) = t$ is a solution to

$$y'' - \frac{1}{t}y' + \frac{1}{t^2}y = 0$$

Use the reduction of order procedure to determine another linearly independent solution for $t > 0$.

Section 4.3 : Linear Homogeneous Equations with Constant Coefficients

Chapter 4 : Second Order Equations

Math 2552 Differential Equations

Section 4.3

Topics

We will cover these topics in this section.

1. 2nd order linear homogeneous constant coefficient equations
2. Cauchy-Euler equations

Objectives

For the topics covered in this section, students are expected to be able to do the following.

1. Solve, sketch solutions, and characterize solutions, of 2nd order linear homogeneous constant coefficient equations and Cauchy-Euler equations

Example

Consider the constant coefficient second order homogeneous DE:

$$ay'' + by' + cy = 0 \tag{10}$$

By yourself or working with someone nearby:

1. Express (1) as an equivalent linear system, of the form

$$\vec{x}\,' = A\vec{x}$$

2. Determine the eigenvalues and eigenvectors of A
3. Determine the solution of (1)

General Solution

Theorem

The constant coefficient second order homogeneous DE:

$$ay'' + by' + cy = 0 \tag{11}$$

has solutions of the form $y = e^{\lambda t}$.

More directly, we may substitute

$$y = e^{\lambda t}$$

into (2) to obtain the **characteristic equation** as follows:

General Solution: Three Cases

If λ_1 and λ_2 are the roots of the characteristic equation for

$$ay'' + by' + cy = 0 \tag{12}$$

Then there are three cases for the general solution of (3).

λ_1, λ_2	general solution
real distinct	$c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t}$
real repeated, $\lambda_1 = \lambda_2$	$c_1 e^{\lambda_1 t} + c_2 t e^{\lambda_1 t}$
complex, $\lambda_1 = \alpha + i\beta$	$e^{\alpha t}\left(c_1 \cos \beta t + c_2 \sin \beta t\right)$

Example: Undamped and Unforced Oscillator

Consider the spring-mass system with no forcing, no damping, spring constant $k = 4$:

$$y'' + 4y = 0$$

Initial conditions are $y(0) = 1$, $y'(0) = 0$.

1. Express (1) as an equivalent linear system, of the form

$$\vec{x}\,' = A\vec{x}$$

2. Without solving the DE, sketch the phase portrait
3. Solve the IVP

If Time Permits: Cauchy-Euler

A DE of the form

$$ax^2\frac{d^2y}{dx^2} + bx\frac{dy}{dx} + cy = 0 \tag{13}$$

is a 2nd order **Cauchy-Euler** equation.

To solve we try solutions of the form

$$y = x^m$$

Substitution into (4) yields

$$am^2 + (b-a)m + c = 0$$

There are three cases:

- distinct real roots
- repeated real roots
- complex roots

If Time Permits: Cauchy-Euler Example

Solve

$$x^2\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} - 4y = 0$$

Section 4.4 : Mechanical Vibrations

Chapter 4 : Second Order Equations

Math 2552 Differential Equations

Section 4.4

Topics
We will cover these topics in this section.

1. homogeneous spring-mass problems
2. undamped, critically damped, and damped motion

Objectives
For the topics covered in this section, students are expected to be able to do the following.

1. Solve, sketch solutions, and characterize solutions, of spring-mass systems.
2. Characterize spring-mass systems using component plots and phase diagrams, as well as concepts such as damping, amplitude, period, frequency, phase, when oscillations are below a particular threshold.

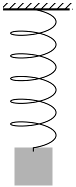
In the interest of time, students are not expected to be familiar with quasi-frequency, quasi-period, electrical vibrations, and circuit problems.

Undamped Motion

Consider the unforced un-damped spring-mass system:

$$my'' + ky = 0$$

Solve this equation, draw a component plot, and characterize the motion in terms of amplitude, frequency, and phase.



Damped Motion

Consider the unforced damped spring-mass system:

$$my'' + \gamma y' + ky = 0$$

Determine an expression for the roots of the characteristic equation as a function of k, m, γ.

Damped Motion Cases

motion	occurs when	motion
underdamped	$\gamma^2 - 4km < 0$	$e^{\alpha t}(A\cos\beta t + B\sin\beta t)$
critically damped	$\gamma^2 - 4km = 0$	$(A + Bt)e^{\lambda_1 t}$
overdamped	$\gamma^2 - 4km > 0$	$Ae^{\lambda_1 t} + Be^{\lambda_2 t}$

As before, $\lambda = \alpha + i\beta$.

We can also describe each case using component plots, direction fields, and phase portraits.

Direction Fields and Phase Portraits

Express the equation

$$my'' + \gamma y' + ky = 0$$

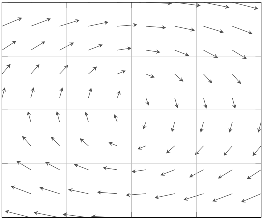
as a first order system using matrix notation, $\vec{x}\,' = A\vec{x}$. What do the elements of $\vec{x}$ represent?

Direction Fields and Phase Portraits

Add solution curves to the direction fields below to create phase portraits.

no damping, $\gamma = 0$

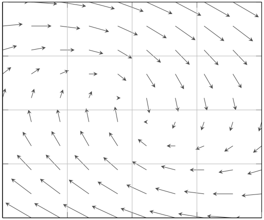
$\frac{dy}{dt}$



position, y

damping, $0 < \gamma^2 < 4km$

$\frac{dy}{dt}$



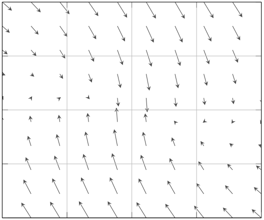
position, y

Direction Fields and Phase Portraits

Add solution curves to the direction fields below to create phase portraits.

critically damped, $\gamma^2 = 4km$

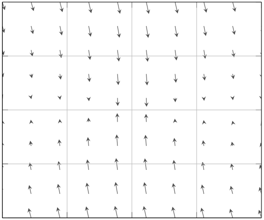
$\dfrac{dy}{dt}$



position, y

overdamped, $\gamma^2 > 4km$

$\dfrac{dy}{dt}$



position, y

In both cases, the origin is a nodal sink.

Example

The motion of a spring-mass system with damping is determined by

$$y'' + 10y' + 24y = 0, \quad y(0) = 1, \quad y'(0) = 0$$

1. Solve the IVP.
2. Plot the solution to the IVP, $y(t)$.
3. Sketch the phase portrait for the differential equation. Include the solution curve that corresponds to the initial conditions.

Section 4.5 : Undetermined Coefficients

Chapter 4 : Second Order Equations

Math 2552 Differential Equations

"I have not failed. I've just found 10,000 ways that won't work."

\- Thomas Edison

Applying the method of undetermined coefficients can involve trying different solutions and testing them until we find one that works.

Section 4.5

Topics
We will cover these topics in this section.

1. The method of undetermined coefficients

Objectives
For the topics covered in this section, students are expected to be able to do the following.

1. Solve differential equations and initial value problems using the method of undetermined coefficients

Theorem

In this section, we seek solutions to the **nonhomogenous** problem

$$y'' + p(t)y' + q(t)y = g(t) \tag{14}$$

The corresponding **homogeneous** problem is

$$y'' + p(t)y' + q(t)y = 0 \tag{15}$$

Note that if Y_1 and Y_2 are solutions of (1), then their difference is:

This implies that the general solution of (1) is:

Solving Nonhomogeneous Differential Equations

To solve a nonhomogeneous DE:

1. Construct solution to homogeneous problem
2. Determine a particular solution to the nonhomogeneous problem using either
   - method of undetermined coefficients
   - variation of parameters
3. Add the functions found in the two previous steps to form the general solution

Participation Activity: Worksheet

Depending on how much time we have, we will run examples in this lecture as a participation activity.

- Please work in groups of ________________
- Each group submits **one** sheet of paper
- Instructor has paper you can use
- Print full names and email addresses on cover
- Every student in a group gets the same grade
- Grading scheme per question:
  - 0 marks for no work or working alone
  - 1 mark for starting the problem or for a final answer with insufficient justification
  - 2 marks for a complete solution

Examples

Determine a particular solution of the following.

1. $y'' + 3y' + 2y = 10e^{3t}$
2. $y'' + 3y' + 2y = \sin t$
3. $y'' - 6y' + 9y = e^{3t}$
4. $y'' + 4y = 5t^2 e^t$ (if time permits)

A Solution Strategy for Undetermined Coefficients

To solve an IVP containing an equation of the form

$$ay'' + by' + cy = g(t)$$

we can use the following steps.

1. Obtain general solution of homogeneous equation
2. Determine if undetermined coefficients can be used
3. If $g(t) = \sum_i^n g_i$, then consider each of the n sub-problems separately
4. Solve first sub-problem: assume particular solution form, determine coefficients
5. Repeat previous step for each sub-problem
6. Form general solution to differential equation
7. Solve IVP

Particular Solutions

In the table below, s is the smallest non-negative integer so that Y is a solution of the homogeneous equation, and

$$P_n(t) = a_0 t^n + a_1 t^{n-1} \ldots + a_n \tag{16}$$

$$Q_n(t) = A_0 t^n + A_1 t^{n-1} \ldots + A_n \tag{17}$$

$$R_n(t) = B_0 t^n + B_1 t^{n-1} \ldots + B_n \tag{18}$$

$g(t)$	particular solution $Y(t)$
$P_n(t)$	$t^s Q_n$
$P_n(t)e^{\alpha t}$	$t^s e^{\alpha t} Q_n$
$P_n(t)e^{\alpha t}\sin(\beta t)$	$t^s e^{\alpha t}\left(\cos(\beta t)Q_n + \sin(\beta t)R_n\right)$
$P_n(t)e^{\alpha t}\cos(\beta t)$	$t^s e^{\alpha t}\left(\cos(\beta t)Q_n + \sin(\beta t)R_n\right)$

Section 4.6 : Forced Vibrations, Frequency Response, Resonance

Chapter 4 : Second Order Equations

Math 2552 Differential Equations

Section 4.6

Topics
We will cover these topics in this section.

1. Forced spring-mass systems

Objectives
For the topics covered in this section, students are expected to be able to do the following.

1. Characterize a spring-mass system using the concepts of beats, resonance, transient solution, and steady-state solution

Students are not expected to solve questions regarding frequency response and gain.

Transient and Steady-State

From our undetermined coefficients lecture, we saw the damped, forced system

$$y'' + 3y' + 2y = \sin t$$

It has the solution

$$y = c_1 e^{-2t} + c_2 e^{-t} + 0.1 \sin t - 0.3 \cos t$$

The homogeneous solution corresponds to the **transient solution** and the particular solution corresponds to the **steady-state** solution.

Participation Activity: Worksheet

Depending on how much time we have, we will run examples in this lecture as a participation activity.

- Please work in groups of ________________
- Each group submits **one** sheet of paper
- Instructor has paper you can use
- Print full names and email addresses on cover
- Every student in a group gets the same grade
- Grading scheme per question:
  - 0 marks for no work or working alone
  - 1 mark for starting the problem or for a final answer with insufficient justification
  - 2 marks for a complete solution

Undamped Oscillator

Consider the spring-mass system

$$y'' + \omega_0^2 y = F_0 \cos(\omega t), \quad y(0) = 0, \quad y'(0) = 0, \quad F_0 > 0$$

Constants ω_0 and ω are positive. Solve the DE for the cases:

1. $\omega \neq \omega_0$
2. $\omega = \omega_0$

Section 4.7 : Variation of Parameters

Chapter 4 : Second Order Equations

Math 2552 Differential Equations

Section 4.7

Topics
We will cover these topics in this section.

1. Variation of parameters for 2nd order differential equations
2. Variation of parameters for first order systems
3. Fundamental matrix

Objectives
For the topics covered in this section, students are expected to be able to do the following.

1. Solve non-homogeneous first order systems and second order differential equations using the method of variation of parameters.

Motivation

Undetermined coefficients:

- does not give us an explicit expression for the particular solution,
- can only be applied when $g(t)$ is a combination of sine, cosine, exponential, and polynomials.

Variation of parameters addresses these points.

Variation of Parameters

We seek solutions to the **nonhomogenous** problem

$$y'' + p(t)y' + q(t)y = g(t) \tag{19}$$

The solution to the corresponding **homogeneous** problem is

$$y_h = c_1 y_1(t) + c_2 y_2(t) \tag{20}$$

To construct a particular solution, we replace c_1 and c_2 with functions:

$$y_p = v_1(t)y_1(t) + v_2(t)y_2(t) \tag{21}$$

Our goal is to determine functions v_1 and v_2.

Procedure for Variation of Parameters

To obtain a particular solution of (1) using variation of parameters:

1. Construct solution to homogeneous problem to obtain y_1, y_2
2. Solve the system of nonlinear equations:

$$y_1 v_1' + y_2 v_2' = 0 \tag{22}$$
$$y_1' v_1' + y_2' v_2' = g \tag{23}$$

3. Integrate v_1' and v_2' to obtain v_1 and v_2.
4. Set $y_p = v_1(t)y_1(t) + v_2(t)y_2(t)$

Note:

- The textbook has an explicit formula for v_1 and v_2 that you don't have to memorize, but you can if you prefer.
- If time permits, we will derive the above procedure in lecture.

Example

Determine a particular solution to

$$t^2y'' - 4ty' + 6y = 4t^3, \quad t > 0$$

given that $y_1 = t^2$ and $y_2 = t^3$ are solutions to the homogeneous equation.

Nonhomogeneous Systems

We seek solutions to the linear **nonhomogenous** system

$$\vec{x}\,' = P\vec{x} + \vec{g}(t) \tag{24}$$

$$P = \begin{pmatrix} p_{11}(t) & p_{12}(t) \\ p_{21}(t) & p_{22}(t) \end{pmatrix}, \quad \vec{g}(t) = \begin{pmatrix} g_1(t) \\ g_2(t) \end{pmatrix} \tag{25}$$

The solution to the corresponding **homogeneous** problem is

$$\vec{x}_h = c_1\vec{x}_1(t) + c_2\vec{x}_2(t) \tag{26}$$

To obtain a particular solution, replace c_1 and c_2 with functions:

$$\vec{x}_p = v_1(t)\vec{x}_1(t) + v_2(t)\vec{x}_2(t) \tag{27}$$

Goal: determine scalar functions $v_1(t)$ and $v_2(t)$.

Fundamental Matrix

Suppose the solutions to the homogeneous problem are:

$$\vec{x}_1 = \begin{pmatrix} x_{11} \\ x_{21} \end{pmatrix}, \qquad \vec{x}_2 = \begin{pmatrix} x_{12} \\ x_{22} \end{pmatrix}$$

Then introduce the **fundamental matrix**, $X(t)$,

$$X(t) = [\vec{x}_1 \ \vec{x}_2] = \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix} \tag{28}$$

If time permits, we will use this matrix to re-write our differential equation and derive formulas for the particular solution during lecture.

Particular Solution Formula

Theorem

Assume entries of matrix P and g are continuous on open interval I, X is the fundamental matrix, then

$$\vec{x}\,' = P\vec{x} + \vec{g}(t)$$

has a particular solution

$$\vec{x}_p = X(t) \int X^{-1}(t)\vec{g}(t)\, dt$$

Example

Determine a particular solution to

$$\vec{x}\,' = \begin{pmatrix} 2 & -3 \\ 1 & -2 \end{pmatrix} \vec{x}(t) + \begin{pmatrix} e^{2t} \\ 1 \end{pmatrix}$$

given that the solutions to the homogeneous equation are

$$\vec{x}_1 = e^{t} \begin{pmatrix} 3 \\ 1 \end{pmatrix}, \quad \vec{x}_2 = e^{-t} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

Section 5.1 : Definition of the Laplace Transform

Chapter 5 : The Laplace Transform

Math 2552 Differential Equations

Section 5.1

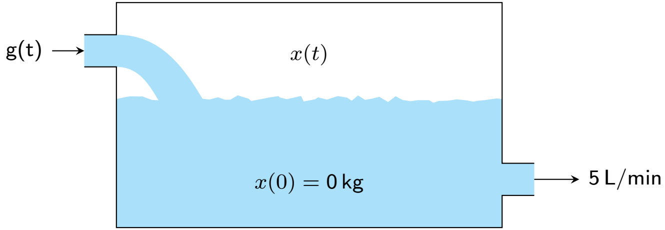
Topics
We will cover these topics in this section.

1. Laplace Transform
2. The Laplace Transform of elementary functions

Objectives
For the topics covered in this section, students are expected to be able to do the following.

1. Determine whether a given function has a Laplace Transform
2. Compute the Laplace Transform of elementary and piecewise functions
3. Sketch piecewise functions
4. Determine whether a function is continuous or piecewise continuous

Motivation



Mass of salt, in kg, in tank at time t is $x(t)$. Tank holds 1000 L, and $x(0) = 0$. The concentration of salt flowing into tank given by:

$$g(t) = \begin{cases} 0.04 \text{ kg/L} \times 6 \text{ L/min}, & 0 < t < 10 \\ 0.02 \text{ kg/L} \times 6 \text{ L/min}, & 10 < t \end{cases}$$

How can we solve this problem?

The Laplace Transform

Definition

Suppose $f(t)$ is defined on $[0, \infty)$. Its Laplace Transform is given by

$$\mathcal{L}[f](s) = \int_0^\infty e^{-st}\, f(t)\, dt$$

The domain of $\mathcal{L}[f]$ is the set of values where the integral exists.

Examples: Compute $\mathcal{L}[1]$, $\mathcal{L}[e^{at}]$, $\mathcal{L}[e^{(a+ib)t}]$.

Linearity

The Laplace transform is a **Linear operator** because:

Example Use the linearity of $\mathcal{L}$ to compute $\mathcal{L}[11 + 5e^{4t} + \sin at]$.

Piecewise Functions

Recall the definition of **piecewise continuous** function. One can also define the Laplace transform for piecewise continuous functions.

Example: Let

$$f(t) = \begin{cases} e^{2t} & 0 \le t < 1 \\ 4 & 1 \le t \end{cases}$$

Compute $\mathcal{L}[f]$.

Exponential Order

What conditions must be satisfied for the Laplace transform of f to exist?

Definition

f is **of exponential order** if there exists constants $K,\ a,\ M$ such that

$$|f(t)| \leq Ke^{at}$$

for all $t > M$.

To check whether a function f is of exponential order, we can show that

$$\frac{f(t)}{e^{at}}$$

is bounded for sufficiently large t.

Example: Determine whether $\cos at$, t^2, e^{t^2} are of exponential order.

Section 5.2 : Properties of the Laplace Transform

Chapter 5 : The Laplace Transform

Math 2552 Differential Equations

Section 5.2

Topics
We will cover these topics in this section.

1. Laplace Transform Theorems
2. Laplace Transforms of IVPs
3. The Laplace Transform of (more) elementary functions

Objectives
For the topics covered in this section, students are expected to be able to do the following.

1. Compute the Laplace Transform of initial value problems

Motivation

Our goal is to use the Laplace Transform to solve differential equations.

Suppose

$$y'' + 9y = e^{-t}\sin(4t), \quad y(0) = 0, \quad y'(0) = 1$$

To apply the Laplace Transform to solve this differential equation, what theorems do we need?

Laplace Transform Theorems

Suppose $F(s) = \mathcal{L}[f(t)]$.

Theorem: Translation in s-domain

$$\mathcal{L}[e^{ct} f(t)] = F(s-c)$$

Theorem: First Derivative

$$\mathcal{L}[f'(t)] = sF(s) - f(0)$$

Example

Compute the Laplace Transform of the initial value problem

$$y' = e^{-5t}\sin(t), \quad y(0) = 2$$

Higher Order Derivatives

Suppose $F(s) = \mathcal{L}[f(t)]$.

Theorem: Second Order Derivative

$$\mathcal{L}[f''(t)] = s^n F(s) - s^{n-1} f(0) - s^{n-2} f'(0)$$

Theorem: Higher Order Derivatives

$$\mathcal{L}[f^{(n)}(t)] = s^n F(s) - s^{n-1} f(0) - s^{n-2} f'(0) - \ldots - f^{(n-1}(0)$$

Example

Compute the Laplace Transform of the initial value problem

$$y'' + 9y = e^{-t}\sin(4t), \quad y(0) = 0, \quad y'(0) = 1$$

Derivatives in the s-Domain

Suppose $F(s) = \mathcal{L}[f(t)]$.

Theorem: Second Order Derivative

$$\mathcal{L}[t^n f(t)] = (-1)^n F^{(n)}(s)$$

It follows from this theorem that

$$\mathcal{L}[t^n] = \frac{n!}{s^{n+1}}$$

Examples (as time permits)

Compute the Laplace Transform of the initial value problems.

1. $y^{(4)} - y = 0, \quad y(0) = y''(0) = y'''(0) = 0, \quad y'(0) = 4/$
2. $y' + ty = t^2, \quad y(0) = 1$

Section 5.3 : The Inverse Laplace Transform

Chapter 5 : The Laplace Transform

Math 2552 Differential Equations

Section 5.3

Topics
We will cover these topics in this section.

1. Inverse Laplace Transform
2. Partial fractions

Objectives
For the topics covered in this section, students are expected to be able to do the following.

1. Compute the Inverse Laplace Transform of rational functions

Motivation

Our goal is to use the Laplace Transform (LT) to solve IVPs.

Our process will be:

1. compute the LT of our IVP
2. solve for $Y(s) = \mathcal{L}(y(t))$
3. compute the inverse LT of Y to determine $y(t)$

The Inverse Laplace Transform

Theorem (definition of $\mathcal{L}^{-1}$)

If $f,\ g$ are piece-wise continuous and of exponential order, and

$$\mathcal{L}[f] = \mathcal{L}[g]$$

then $f = g$.

Thus, under the conditions of this theorem we have an inverse operation $\mathcal{L}^{-1}$, such that

$$\mathcal{L}^{-1}\{\,\mathcal{L}[f]\,\} = f$$

we say that $\mathcal{L}^{-1}$ is the **inverse Laplace operator**.

Examples

Determine $\mathcal{L}^{-1}\{Y\}$, where

$$Y_1(s) = \frac{4}{s^3}, \quad Y_2(s) = \frac{3}{s^2+9}, \quad Y_3(s) = \frac{s-1}{s^2-2s+5}$$

Examples (if time permits)

Note that the Inverse Laplace transform is **linear**.

Examples: Calculate $\mathcal{L}^{-1}[Y]$ for the following.

$$Y_1(s) = \frac{5}{(s+2)^4}$$

$$Y_2(s) = \frac{7s-1}{(s+1)(s+2)(s-3)}$$

$$Y_3(s) = \frac{s^2+9s+2}{(s-1)^2(s+3)}$$

Section 5.4 : Solving Differential Equations with Laplace Transforms

Chapter 5 : The Laplace Transform

Math 2552 Differential Equations

Section 5.4

Topics
We will cover these topics in this section.

1. Using the Laplace Transform to solve IVPs

Objectives
For the topics covered in this section, students are expected to be able to do the following.

1. Solve IVPs using the Laplace Transform

The Laplace Transform easily extends to linear systems, but in the interest of time, students are not expected to be familiar with this for exams.

Motivation

Our goal is to use the Laplace Transform (LT) to solve IVPs.

Our process will be:

1. compute the LT of our IVP
2. solve for $Y(s) = \mathcal{L}(y(t))$
3. compute the inverse LT of Y to determine $y(t)$

Participation Activity: Worksheet

Depending on how much time we have, we will run examples in this lecture as a participation activity.

- Please work in groups of ________________
- Each group submits **one** sheet of paper
- Instructor has paper you can use
- Print full names and email addresses on cover
- Every student in a group gets the same grade
- Grading scheme per question:
  - 0 marks for no work or working alone
  - 1 mark for starting the problem or for a final answer with insufficient justification
  - 2 marks for a complete solution

Examples

Solve the following IVPs using the Laplace Transform.

$$\begin{aligned} &1) \quad y' + 3y = 13\sin 2t, \quad y(0) = 6 \\ &2) \quad y'' - 3y' + 2y = e^{-4t}, \quad y(0) = 1, \quad y'(0) = 5 \end{aligned}$$

Helpful formulas:

$$\mathcal{L}[e^{at}] = \frac{1}{s-a}, \quad \mathcal{L}[\sin kt] = \frac{k}{s^2+k^2}, \quad \mathcal{L}[\cos kt] = \frac{s}{s^2+k^2}$$

Section 5.5 : Discontinuous and Periodic Functions

Chapter 5 : The Laplace Transform

Math 2552 Differential Equations

Section 5.5

Topics

We will cover these topics in this section.

1. Step and Indicator functions

Objectives

For the topics covered in this section, students are expected to be able to do the following.

1. Represent functions using the step function.
2. Solve IVPs, with piecewise continuous functions, using the Laplace Transform.

Motivation

Suppose that at time $t = 0$ we place a pie into an oven whose temperature, $y(t)$, is 20°C.

- When $t = 0$, the pie and the oven are both 20°C.
- $y(t)$ increases linearly at rate 50°C per min, until $t = 4$.
- For $t \geq 4$ min, $y = 220$.

Construct an IVP that represents this situation. How can we solve it?

Step and Indicator Functions

The **unit step function** is defined as:

$$u_c(t) = \begin{cases} 0 & 0 \leq t < c \\ 1 & c \leq t \end{cases}$$

The **indicator function** is defined as:

$$u_{bc}(t) = \begin{cases} 0 & 0 \leq t < b \\ 1 & b \leq t \leq c \\ 0 & c \leq t \end{cases}$$

Question

How can we express the indicator function in terms of step functions?

Example

Express the following functions in terms of step functions.

$$a) \qquad f(t) = \begin{cases} 2 & 0 \le t < 3 \\ -2 & 3 \le t \end{cases}$$

$$b) \qquad g(t) = \begin{cases} t & 0 \le t < 2 \\ t^2 & 2 \le t < 4 \\ t^3 & 4 \le t \end{cases}$$

Transform of a Step Function

Theorem

$$\mathcal{L}\{u_c(t)\} = \frac{e^{-cs}}{s}$$

Proof:

Example

Solve the IVPs.

$$y' + y = f, \quad y(0) = 0, \quad f = \begin{cases} 1 & 0 \leq t < 1 \\ -1 & 1 \leq t \end{cases}$$

Shift in the t-Domain

Theorem

$$\mathcal{L}\{u_c\, f(t-c)\} = e^{-cs} F(s)$$

Proof:

Examples

Recall that $\mathcal{L}\{u_c\, f(t-c)\} = e^{-cs}F(s)$.

Compute the following.

1. $\mathcal{L}^{-1}\left\{\dfrac{1}{s-4}e^{-2s}\right\}$
2. $\mathcal{L}^{-1}\left\{\dfrac{s}{s^2+9}e^{-\pi s/2}\right\}$

Transform of a Periodic Function

Theorem

Suppose $f(t)$ is periodic with period T. Then

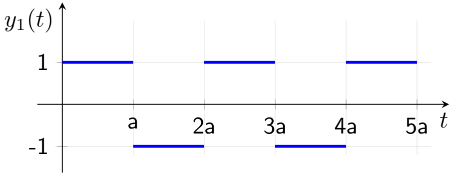
$$\mathcal{L}\{f(t)\} = \frac{1}{1-e^{-Ts}} \int_0^T e^{-st} f(t) dt$$

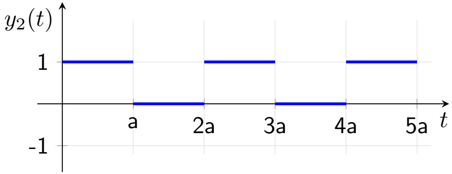
Proof:

Examples

Recall that $\mathcal{L}\{f(t)\} = \frac{1}{1-e^{-Ts}} \int_0^T e^{-st} f(t)dt$.

Compute the Laplace Transform of the following periodic functions.





Additional Example (if time permits)

Evaluate $\mathcal{L}\{\cos(t)\, u_\pi\}$.

Section 5.6 : Discontinuous Forcing Functions

Chapter 5 : The Laplace Transform

Math 2552 Differential Equations

Section 5.6

Topics
We will cover these topics in this section.

1. Constant coefficient second order differential equations with discontinuous forcing functions.

Objectives
For the topics covered in this section, students are expected to be able to do the following.

1. Solve IVPs with discontinuous forcing functions using the Laplace Transform

Examples

Solve the following IVPs using the Laplace Transform.

1. $y'' + 4y = g, \quad y(0) = y'(0) = 0, \quad g(t) = \begin{cases} 1 & 0 \le t < 1 \\ -1 & 1 \le t < 2 \\ 0 & \text{else} \end{cases}$

2. $y'' + 4y' + 4y = g, \quad y(0) = y'(0) = 0, \quad g(t) = u_\pi - u_{2\pi}$

Section 5.7 : Impulse Functions

Chapter 5 : The Laplace Transform

Math 2552 Differential Equations

Section 5.7

Topics

We will cover these topics in this section.

1. Constant coefficient second order differential equations with Dirac delta functions.

Objectives

For the topics covered in this section, students are expected to be able to do the following.

1. Solve IVPs with that involve the Dirac Delta function using the Laplace Transform.

Example

Consider the spring-mass system

$$my'' + \gamma y' + ky = g(t - t_0), \quad y(0) = 1, \quad y'(0) = 0$$

$g(t)$ represents a large force over a short duration at time t_0. What can we use to model this force?

Dirac Delta Function

- Used to model functions that have a large value over a short interval.
- Is not a function.

Dirac Delta Function

The **Dirac delta function** is characterized by the two properties. For any function $f(t)$ that is continuous over an interval containing t_0:

$$\delta(t - t_0) = \begin{cases} 0, & t \neq t_0 \\ \text{undefined}, & t = t_0 \end{cases}$$

$$\int_a^b f(t)\,\delta(t - t_0)\,dt = f(t_0)$$

The Laplace Transform of the Dirac Delta function is:

Example

A mass attached to a spring is released from rest 1 m below equilibrium. After π seconds the mass is struck by a hammer exerting an impulse. The system is governed by

$$y'' + 9y = 3\delta(t - \pi), \quad y(0) = 1, \quad y'(0) = 0$$

Solve this IVP.

Examples

Solve the following IVPs using the Laplace Transform.

1. $y'' + y = -\delta(t-\pi) + \delta(t-2\pi), \quad y(0) = 0, \quad y'(0) = 1$
2. $y'' + 2y' + 2y = \delta(t-\pi), \quad y(0) = y'(0) = 1$

Sketch the solution for the first problem.

Section 5.8 : Convolution

Chapter 5 : The Laplace Transform

Math 2552 Differential Equations

Topics

We will cover these topics in this section.

1. The convolution theorem for the Laplace Transform.

Objectives

For the topics covered in this section, students are expected to be able to do the following.

1. Compute Laplace, and Inverse Laplace Transforms using the convolution theorem.

Note: the textbook goes into more depth than we have time for here. The learning objective states what you are expected to be able to do.

Example

Consider the spring-mass system

$$y'' + \omega^2 y = g(t), \quad y(0) = 0, \quad y'(0) = 0$$

Recall that $\omega^2 = k/m$.

If $g(t)$ is unknown, can we express $y(t)$ in terms of $g(t)$? How?

Convolution

Definition

The **convolution** between piecewise continuous functions f and g is denoted $f * g$, and is

$$f * g = \int_0^t f(t-\tau)g(\tau)\,d\tau$$

Convolution has a number of useful properties:

1. $f * g = g * f$
2. $f * (g_1 + g_2) =$

Convolution Theorem

Definition

Suppose f and g have Laplace Transforms $F(s)$ and $G(s)$ respectively, then

$$\mathcal{L}\left\{f * g\right\} = F(s)G(s)$$

Proof:

Examples

1. Compute the inverse Laplace Transform of the functions.

   a) $\dfrac{1}{(s^2+1)^2}$ *(you may leave your answer in terms of an integral)*

   b) $\dfrac{14}{(s+2)(s-6)}$

2. Compute the Laplace Transform of the following.

   a) $\displaystyle\int_0^t (t-\tau)e^{3\tau}\,d\tau$

   b) $\displaystyle\int_0^t \sin(t-\tau)e^{\tau}\,d\tau$

Section 6.1 : Definitions and Examples

Chapter 6 : Systems of First Order Linear Equations

Math 2552 Differential Equations

Section 6.1

Topics

We will cover these topics in this section.

1. Differentiation and integration of matrix functions
2. Representing an n^{th} order differential equation as an equivalent first order linear system

Objectives

For the topics covered in this section, students are expected to be able to do the following.

1. **Differentiate** and **integrate** $n \times n$ matrix functions
2. **Express** an n^{th} order differential equation into an equivalent first order linear system

Matrix Functions

Our study of systems of DEs involves **matrix functions**, which are matrices whose elements are functions.

$$P(t) = \begin{pmatrix} p_{11}(t) & \cdots & p_{1n}(t) \\ \vdots & & \vdots \\ p_{n1}(t) & & p_{nn}(t) \end{pmatrix}$$

Differentiates and integrals of matrix functions:

- **Differentiation**: element ij of $\dfrac{dP}{dt}$ is equal to $\frac{d}{dt}\, p_{ij}$.
- **Integration**: element ij of $\displaystyle\int_a^b P(t)\, dt$ is equal to $\displaystyle\int_a^b p_{ij}\, dt$.

Example: If $P = \begin{pmatrix} 0 & 1 \\ t & t^2 \end{pmatrix}$, then:

Expressing an n^{th} Order DE as a Linear System

$$y^{(4)} + y = \sin(t)$$

Participation Activity: Worksheet

Depending on how much time we have, we will run examples in this lecture as a participation activity.

- Please work in groups of ___________ people.
- Each group submits **one** sheet of paper
- Instructor has paper you can use
- Print full names and email addresses on cover
- Every student in a group gets the same grade
- Grading scheme **per question**:
  - 0 marks for no work or working alone
  - 1 mark for starting the problem or for a final answer with insufficient justification
  - 2 marks for complete solution, justified reasoning

Examples

1. $A(t) = \begin{pmatrix} e^{2t} & \pi \sin(\pi t) \\ 0 & 4t \end{pmatrix}$, compute $\displaystyle\int_0^1 A(t)\,dt$.
2. Express the equation as an equivalent first order linear system.

$$my'' + ky = \sin(t),\ \ m > 0, k > 0$$

3. (Section 3.1) Determine the eigenvalues and eigenvectors of X.

$$X = \begin{pmatrix} -1 & -1 \\ 1 & -1 \end{pmatrix}$$

Section 6.2 : Basic Theory of First Order Linear Systems

Chapter 6 : Systems of First Order Linear Equations

Math 2552 Differential Equations

Section 6.2

Topics

We will cover these topics in this section.

1. Existence and uniqueness of solutions to first order systems
2. Fundamental set of solutions to a DE

Objectives

For the topics covered in this section, students are expected to be able to do the following.

1. Characterize first order linear in terms of its fundamental set of solutions.
2. Determine intervals where a solution to a given first order system exists.
3. Compute the Wronskian for a set of solutions to a differential equation, or system of DEs.

Motivation and Review

There are key questions that we explore in this section.

1. **existence and uniqueness**: where does a linear system of DEs have a solution, and if so, where?
2. **general solution**: what is the set of all possible solutions to a system of DEs? To an n^{th} order DE?

Our questions require the use of concepts from linear algebra.

a) What does it mean for a set of vectors to be linearly independent?

b) What is the span of a set of vectors?

Uniqueness

Theorem: Uniqueness of Solution to Linear System

If P and $\vec{g}$ are continuous on (α, β), $t_0 \in (\alpha, \beta)$, then there is a **unique** solution to the IVP

$$\vec{x}\,' = P(t)\vec{x} + \vec{g}(t), \quad \vec{x}(t_0) = \vec{x}_0$$

- Proof goes beyond the scope of this course.
- This theorem is similar to one we saw in Section 2.4.
- This theorem can be applied to higher order DEs.

Example: Identify an interval on which a unique solution will exist.

$$(t-2)y'' + 3y = t$$

Linear Independence of Functions and the Wronskian

Example: determine whether the functions are linearly independent.

a) $y_1 = e^t, \quad y_2 = e^{-2t}, \quad y_3 = 3e^t - 2e^{-2t}$

b) $y_1 = t, \quad y_2 = t^2, \quad y_3 = 1 - 2t^2$

Fundamental Set of Solutions: n^{th} Order DE

Theorem: Uniqueness of Solution to Linear System

Let $y_1, y_2, \ldots, y_n$ be solutions of

$$y^{(n)} + p_1 y^{(n-1)} + \ldots + p_n y = 0$$

on an interval, I,in which each p is continuous.

If for some $t_0 \in I$ their Wronskian is non-zero, then every solution of the DE,$\phi(t)$, can be written as

$$\phi(t) = \sum_{i=1}^{n} c_i y_i(t)$$

We say that $y_1, y_2, \ldots, y_n$ form a **fundamental set of solutions** for the differential equation.

Linear Independence of Vector Functions, their Wronskian

Example: determine whether the functions are linearly independent.

$$y_1(t) = \begin{pmatrix} t \\ 1-t \\ 0 \end{pmatrix}, \quad y_2(t) = \begin{pmatrix} 0 \\ 1 \\ t \end{pmatrix}, \quad y_3(t) = \begin{pmatrix} t \\ 3-t \\ 2t \end{pmatrix}$$

Fundamental Set of Solutions: Vector Functions

Theorem: Uniqueness of Solution to Linear System

Let $\vec{x}_1, \vec{x}_2, \ldots, \vec{x}_n$ be solutions of

$$\vec{x}' = P(t)\vec{x} + \vec{g}(t)$$

on an interval, I,in which each p is continuous.

If for some $t_0 \in I$ their Wronskian is non-zero, then every solution of the DE,$\phi(t)$, can be written as

$$\phi(t) = \sum_{i=1}^{n} c_i y_i(t)$$

We say that $y_1, y_2, \ldots, y_n$ form a **fundamental set of solutions** for the differential equation.

Example

Determine whether the the vector functions

$$\vec{x}_1(t) = e^{5t}\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad \vec{x}_2(t) = e^{-t}\begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

form a fundamental set of solutions for the system

$$\vec{x}\,' = \begin{pmatrix} 2 & 3 \\ 3 & 2 \end{pmatrix}\vec{x}$$

You may assume that $\vec{x}_1(t)$ and $\vec{x}_2(t)$ are solutions.

Section 6.3 : Homogeneous Linear Systems with Constant Coefficients

Chapter 6 : Systems of First Order Linear Equations

Math 2552 Differential Equations

Section 6.3

Topics

We will cover these topics in this section.

1. Possible forms of the solution to a Homogeneous Linear Systems with Constant Coefficients

Objectives

For the topics covered in this section, students are expected to be able to do the following.

1. Express the fundamental set of solutions, and the general solution to $\vec{x}\,' = A\vec{x}$ for nondefective A.

Motivation and Review

Key question in this section: what is the **general form of the solution**, and **the fundamental set of solutions** to a linear, first order, constant coefficient, system of DEs?

Our questions require the use of concepts from linear algebra.

a) What is the geometric multiplicity of an eigenvalue?

b) What is the algebraic multiplicity of an eigenvalue?

Example: $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$

Three Cases

Consider the linear, first order, constant coefficient system,

$$\vec{x}\,' = P\vec{x}, \quad \vec{x} = \vec{x}(t), \quad P \in \mathbb{R}^{n\times n}.$$

We assume solutions of the form

$$\vec{x}(t) = e^{\lambda t}\vec{v}.$$

Our eigenvalue problem has three cases:

A Nondefective, Real Eigenvalues

Theorem

If $(\lambda_1, \vec{v}_1), \dots (\lambda_n, \vec{v}_n)$ are eigenpairs for $n \times n$ matrix A, and A has n linearly independent eigenvectors $\vec{v}_1, \dots, \vec{v}_n$, then $\vec{x}\,' = A\vec{x}$ has:

- a general solution:
- a fundamental set of solutions:

Note: the eigenvalues need not be distinct.

Example

Determine the general solution to

$$\vec{x}\,' = P\vec{x}, \quad P = \begin{pmatrix} 0 & 1 & -1 \\ 1 & 0 & 1 \\ 1 & -1 & 2 \end{pmatrix}$$

The eigenvalues of P are 0 and 1.

Section 6.4 : Nondefective Matrices with Complex Eigenvalues

Chapter 6 : Systems of First Order Linear Equations

Math 2552 Differential Equations

Section 6.4

Topics

We will cover these topics in this section.

1. First order linear homogeneous systems with complex eigenvalues.

Objectives

For the topics covered in this section, students are expected to be able to do the following.

1. Solve systems of constant coefficient, homogeneous, first order, linear differential equations with complex eigenvalues.

General Solution, Complex Eigenvalues

As we saw in a previous section, we have a general procedure for real-valued solution in the complex case.

1. Compute eigenvalues $\lambda = \alpha \pm i\beta$
2. Compute eigenvector, $\vec{v}$, for $\lambda = \alpha + i\beta$
3. Set $\vec{v} = \vec{a} + i\vec{b}$
4. General solution is $\vec{x}(t) = c_1\vec{u} + c_2\vec{w}$, where

$$\vec{u} = e^{\alpha t}(\vec{a}\cos\beta t - \vec{b}\sin\beta t)$$
$$\vec{w} = e^{\alpha t}(\vec{a}\sin\beta t + \vec{b}\cos\beta t)$$

You may want to memorize the above process and equations.

Particle Motion

The motion of a moving object is described by the system below.

$$\frac{dx}{dt} = -kx \tag{29}$$

$$\frac{dy}{dt} = -z \tag{30}$$

$$\frac{dz}{dt} = y \tag{31}$$

Assume $k > 0$. At time $t = 0$, our particle is located at the point $(x, y, z) = (1, 1, 0)$.

Take a few minutes on your own to solve this initial value problem. Express your solution in terms of real valued functions, and create a rough sketch of the particle motion.

Compare your answers with someone sitting nearby.

Expectations for Quizzes and Exams

- In the previous example, we sketched the solution curve passing through $(1, 1, 0)$ by hand. In general, we need to graph the curve using a computer. On quizzes/midterms, students are expected to sketch solution curves and phase portraits for two dimensional systems, but not for three.
- For three dimensional systems, students will be given the eigenvalues, or the coefficient matrix will have a lot of zeros so that the eigenvalues can be determined quickly.

Section 7.1 : Autonomous Systems and Stability

Chapter 7 : Systems of First Order Linear Equations

Math 2552 Differential Equations

Section 7.1

Topics

We will cover these topics in this section.

1. Critical points of non-linear two dimensional autonomous systems.

Objectives

For the topics covered in this section, students are expected to be able to do the following.

1. Identify and classify critical points of non-linear systems of autonomous differential equations.

Recall: Autonomous Systems

In this section we take a closer look at non-linear autonomous two-dimensional systems:

$$\frac{dy}{dt} = f(x,y), \quad \frac{dx}{dt} = g(x,y)$$

For example,

$$\frac{dx}{dt} = 4 - 2y, \quad \frac{dy}{dt} = y^2 - xy$$

We explored such systems in Section 3.6.

Recall: Critical Point Classification

Critical points of a system correspond to points where

$$\frac{dx}{dt} = 0 \quad \text{and} \quad \frac{dy}{dt} = 0$$

If a trajectory $\vec{x}(t)$ that starts **sufficiently close** to a critical point, $\vec{x}_0$, then the critical point is:

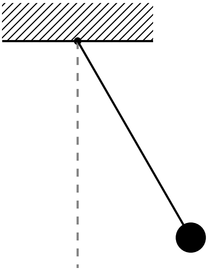
1. stable if:

2. asymptotically stable if:

3. unstable if:

Pendulum

A mass is attached to a inflexible rod that pivots about a point.



Under a gravitational force, the angle $\theta(t)$ the rod makes with the vertical axis satisfies

$$\frac{d^2\theta}{dt^2} + \gamma \frac{d\theta}{dt} + \omega^2 \sin\theta = 0$$

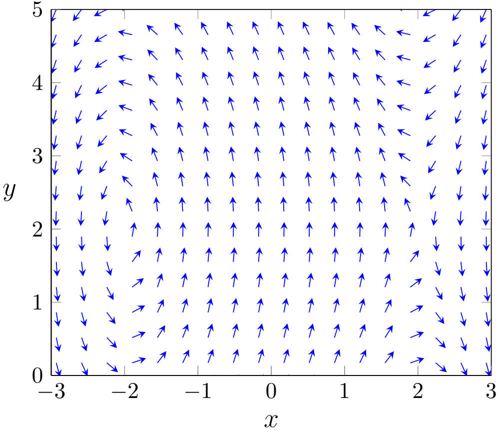
Sketch the phase portrait, θ' vs. θ, for a few different initial conditions. Where are the critical points located?

Example

Determine the critical points of the system:

$$\frac{dx}{dt} = 4 - 2y, \quad \frac{dy}{dt} = 12 - 3x^2$$

A phase portrait of the system is shown below.



Slope Fields

You will need to plot direction fields to complete the homework.

A Google search will point to sites that can plot slope fields.

Here are a few that I've found are ok:

- www.bluffton.edu/homepages/facstaff/nesterd/java/slopefields.html
- www.directionfield.com

So far, I've found wolframalpha to be most useful:

$$\text{streamplot}[\{4-2y, 12-3x^2\}, \{x, -4, 4\}, \{y, -1, 5\}]$$

On an exam I would give you the slope field to interpret.

Additional Examples (as time permits)

For each system, a) identify all of the critical points, and b) classify the critical points.

1. $\dfrac{dx}{dt} = 2x - x^2 - xy, \quad \dfrac{dy}{dt} = 3y - 2y^2 - 3xy$

2. $\dfrac{dx}{dt} = x(6 - x - y), \quad \dfrac{dy}{dt} = -x + 7y - 2xy$

Section 7.2 : Almost Linear Systems

Chapter 7 : Systems of First Order Linear Equations

Math 2552 Differential Equations

Section 7.2

Topics

We will cover these topics in this section.

1. Critical points of non-linear two dimensional autonomous systems.

Objectives

For the topics covered in this section, students are expected to be able to do the following.

1. Identify critical points of non-linear two-dimensional systems of autonomous differential equations.
2. Determine the corresponding linear system near an isolated critical point.
3. Classify the associated linear system according to stability (stable, asymptotically stable, or unstable) and type (center, spiral, proper node, improper node, saddle).

Goal

Suppose

$$\vec{x}\,' = A\vec{x} + \vec{g} \tag{32}$$

and $\vec{x} = \vec{x}_0$ is a critical point of (32).

If $\vec{g} \approx \vec{0}$ near $\vec{x}_0$, then

$$\vec{x}\,' \approx A\vec{x} \quad \text{near} \quad \vec{x}_0$$

We can use this idea to approximate the solution to (32) in a region around $\vec{x} = \vec{0}$.

Isolated Critical Points (ICP)

Definition

Suppose

$$\vec{x}\,' = A\vec{x} + \vec{g} \tag{33}$$

and $\vec{x} = \vec{0}$ is a critical point of (33), and det$A \neq 0$. Then $\vec{x} = \vec{0}$ is an isolated critical point of (33).

Almost Linear System

Definition

If $\vec{x} = \vec{0}$ is an ICP of

$$\vec{x} = A\vec{x} + \vec{g} \tag{34}$$

and

$$\frac{||\vec{g}||}{||\vec{x}||} \to \vec{0}, \quad \text{as} \quad \vec{x} \to \vec{0} \tag{35}$$

then (1) is an **almost linear system** near $\vec{x} = \vec{0}$.

Note: in two dimensions, $\vec{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \quad \vec{g} = \begin{pmatrix} g_1 \\ g_2 \end{pmatrix}$, so (35) can be written as:

Taylor Expansions

We now want to

- consider isolated critical points (ICPs) that are not at the origin, and
- use another test to determine whether a system is almost linear near an ICP.

Consider the autonomous system

$$\vec{x}\,' = F(x,y), \quad \vec{y}\,' = G(x,y) \tag{36}$$

Assume system (36) has an ICP at (x_0, y_0).

Taylor expansions of (36) about (x_0, y_0) is (using multivariable calculus):

2^{nd} Test for Almost Linear

Theorem

Suppose system

$$\vec{x}\,' = F(x,y), \quad \vec{y}\,' = G(x,y)$$

has an ICP at (x_0, y_0). If F and G are twice differentiable in a region around (x_0, y_0), then the system is almost linear.

Note: The corresponding **Jacobian Matrix** of the system is

$$J = \begin{pmatrix} F_x(x_0,\, y_0) & F_y(x_0,\, y_0) \\ G_x(x_0,\, y_0) & G_y(x_0,\, y_0) \end{pmatrix}$$

It can be used to construct the almost linear system near an ICP, as we will see in the next example.

Examples (as time permits)

For each system, a) identify all critical points, b) construct the linear system for each critical point, and c) classify the critical points according to stability.

1. $\dfrac{dx}{dt} = x + x^2 + y^2, \quad \dfrac{dy}{dt} = y - xy$

2. $\dfrac{dx}{dt} = (2 + x)(y - x), \quad \dfrac{dy}{dt} = (4 - x)(y + x)$

Affect of Perturbations

Conisder the two-dimensional non-linear autonomous system

$$\vec{x}\,' = A\vec{x} + \vec{g} \tag{37}$$

and $\lambda = 2i$ is an eigenvalue of A near ICP (x_0, y_0).

1. How does the **linear system** $\vec{x}\,' = A\vec{x}$ behave near the ICP?

2. How does the **almost linear system** (37) behave near the ICP?

Theorem

According to our textbook: the type of critical point and its stability of the **linear system** and the **almost linear system** are the same, except when the eigenvalues of A are pure imaginary or real repeated.

Proof goes beyond scope of course.

You will encounter these changes when completing the homework.

Students will not be asked to classify the almost linear system. Students could be asked to classify the linear systems associated with isolated critical points.

Stability of $\vec{x}' = A\vec{x}$

Please (if you haven't already) memorize Table 7.2.1.

Eigenvalues	Type of Critical Point	Stability
$\lambda_1 > \lambda_2 > 0$	Node	Unstable
$\lambda_1 < \lambda_2 < 0$	Node	Asymptotically stable
$\lambda_2 < 0 < \lambda_1$	Saddle point	Unstable
$\lambda_1 = \lambda_2 > 0$	Proper or improper node	Unstable
$\lambda_1 = \lambda_2 < 0$	Proper or improper node	Asymptotically stable
$\lambda_1, \lambda_2 = \mu \pm i\nu$		
$\mu > 0$	Spiral point	Unstable
$\mu < 0$	Spiral point	Asymptotically stable
$\mu = 0$	Center	Stable

Section 7.3 : Competing Species

Chapter 7 : Systems of First Order Linear Equations

Math 2552 Differential Equations

Section 7.3

Topics
We will cover these topics in this section.

1. Competing species

Objectives
The objectives for this section are the same as those for 7.2. This section applies what we learned in 7.2 to specific cases.

Population Modelling

Recall the **logistic equation**:

$$\frac{dx}{dt} = x(\epsilon - \sigma x) \tag{38}$$

where

- ϵ is a growth rate,
- $\dfrac{\sigma}{\epsilon}$ is a saturation level,
- the quantity $\epsilon - \sigma x)$ represents an environmental capacity for a species (eg - food supply),
- $x(t)$ represents the population of **one species** at time t.

We want to extend theses concepts to **two species**.

Two Species

Now suppose that we have two species that do not interact directly with other, but they do compete for the same food supply.

We could use two separate **logistic equations**:

$$\frac{dx}{dt} = x(\epsilon_1 - \sigma_1 x), \quad \frac{dy}{dt} = y(\epsilon_2 - \sigma_2 y) \tag{39}$$

But the two species rely on the same, limited food source. They impact each others' environmental capacity.

We can instead use

$$\frac{dx}{dt} = x(\epsilon_1 - \sigma_1 x - \alpha_1 y), \quad \frac{dy}{dt} = y(\epsilon_2 - \sigma_2 y - \alpha_2 x) \tag{40}$$

We are interested in cases where $\epsilon_1, \sigma_1, \alpha_1, \epsilon_2, \sigma_2, \alpha_2$ are nonnegative.

Examples (as time permits)

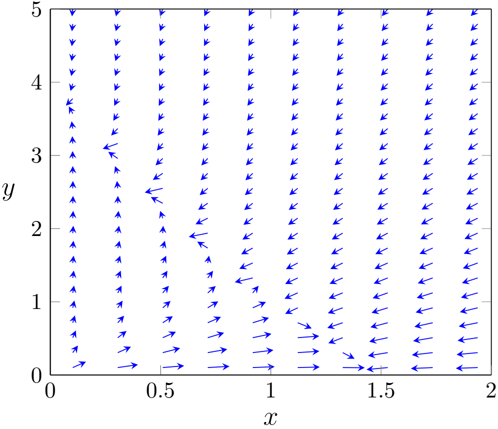
For each system below a) identify all critical points, b) construct the linear system for each critical point, and c) classify the critical points of the linear system according to stability (stable, unstable, asymptotically stable).

$$1) \quad \frac{dx}{dt} = x(1.5 - x - 0.5y), \quad \frac{dy}{dt} = y(2 - 0.5y - 1.5x)$$

$$2) \quad \frac{dx}{dt} = x(1.5 - x/2 - y), \quad \frac{dy}{dt} = y(3/4 - y - 0.125x)$$

Slope Field

A phase portrait of the first system in the previous example is below.



Alternatively, we can obtain the portrait using wolframalpha:

streamplot$[\{x(1.5 - x - y/2), y * (2 - y/2 - 1.5x)\}, \{x, 0, 2\}, \{y, 0, 5\}]$

Section 7.4 : Predator-Prey Equations

Chapter 7 : Systems of First Order Linear Equations

Math 2552 Differential Equations

Section 7.4

Topics
We will cover these topics in this section.

1. Predator-prey equations

Objectives
The objectives for this section are the same as those for 7.2. This section applies what we learned in 7.2 to specific cases.

Predator-Prey Model

Now suppose that we have two species that interact directly with other so that

- $x(t)$ and $y(t)$ are their populations
- y preys on x
- in absence of prey, y dies out, so $y' = -Cy$, $C > 0$
- in absence of predator, $x' = Ax$, $A > 0$
- the number of encounters is proportional to xy

These constraints give us

$$\frac{dx}{dt} = Ax - \alpha xy \tag{41}$$

$$\frac{dy}{dt} = -Cy + \gamma xy \tag{42}$$

These are the **Lotka - Volterra** equations. Students are not expected to memorize their general form.

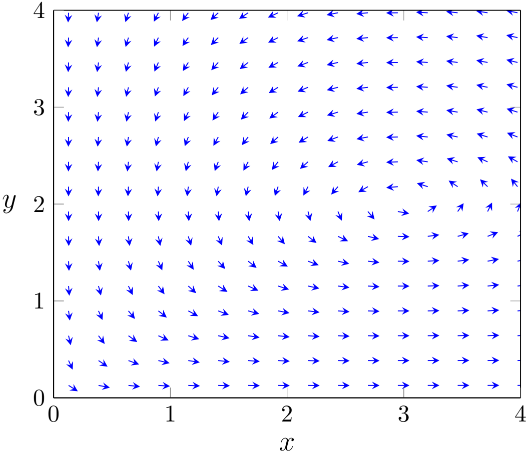
Example

For each system below a) identify all critical points, b) construct the linear system for each critical point, and c) classify the critical points of the linear system according to stability (stable, unstable, asymptotically stable) and type (spiral, proper node, etc).

$$\frac{dx}{dt} = x - 0.5xy, \qquad \frac{dy}{dt} = -0.75y + 0.25xy$$

Slope Field

A phase portrait of the system in the previous example is below.



We can also obtain the portrait using wolframalpha:

$$\text{streamplot}[\{x - 0.5xy, -0.75y + 0.25xy\}, \{x, 0, 4\}, \{y, 0, 4\}]$$

Remaining Questions

From here, there are many questions that we could explore.

- How can we define and improve the accuracy of our model?
- How can we refine our model so that the prey obeys logistic growth in the absence of a predator?
- How can we determine if the solutions are periodic?

Many more details about non-linear systems in remaining sections of Chapter 7, other math courses like Math 4541