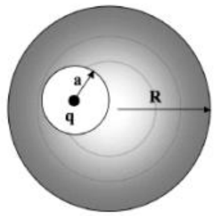
Homework 4 Solutions

Problem A. Polarized Conductor [20%]. A spherical cavity (radius a) is dug out of a larger solid (and overall neutral) conducting sphere of radius R. Note that the cavity is off-centered by a vector $\mathbf{c}$ with respect to the center of the large sphere, which is at the origin. At the center of the small cavity (so at vector $\mathbf{r} = \mathbf{c}$), we put a point charge q. **This problen can be answered qualitatively** with drawings and little to no calculations.



(1) Give the surface charge density σ_a on the cavity wall and σ_R (at $r = R^+$), and sketch the distribution of charges.

(2) Give and then sketch the $\mathbf{E}$ field everywhere: inside the cavity, in the bulk of the sphere, and outside.

(3) Imagine that q is moved a little off to one side, so it is no longer exactly at the center of the cavity, what changes? Please explain qualitatively and/or draw the resulting situation for the surface charge densities.

A1)

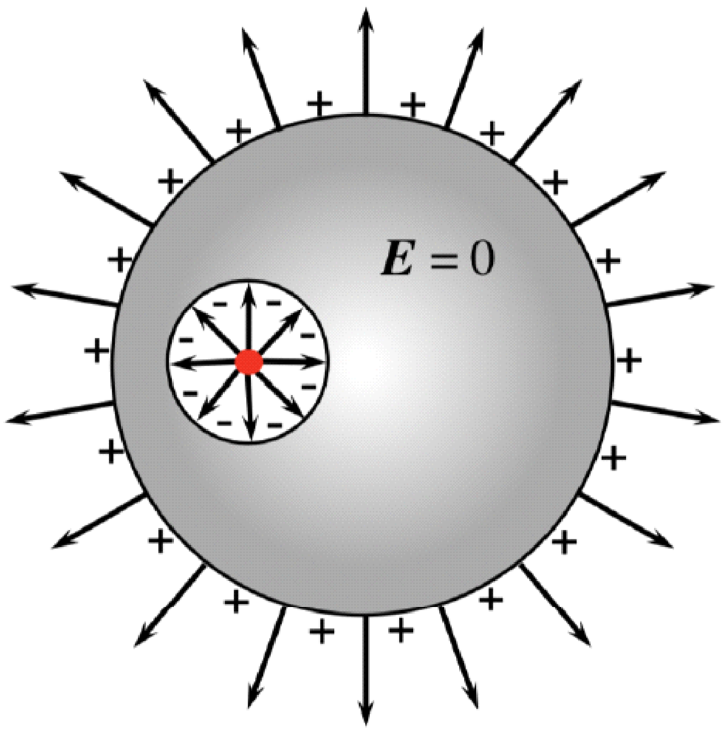
The electric field inside the bulk of the conductor is zero. The spherical symmetry still holds so the surface charge density on both surface of the cavity and the surface of the spherical conductor is uniform. The total charge on the surface of the cavity is $-q$ so that inside the bulk of the conductor the electric field is zero by Gauss law. And since the conductor electrically neutral there is $+q$ charge on the outer surface. Therefore

$$\sigma_a = \frac{-q}{4\pi a^2}$$
$$\sigma_R = \frac{q}{4\pi R^2}$$

A2)

Let the vector pointing from the center of the cavity to the center of the spherical conductor be $\vec{A}$ then

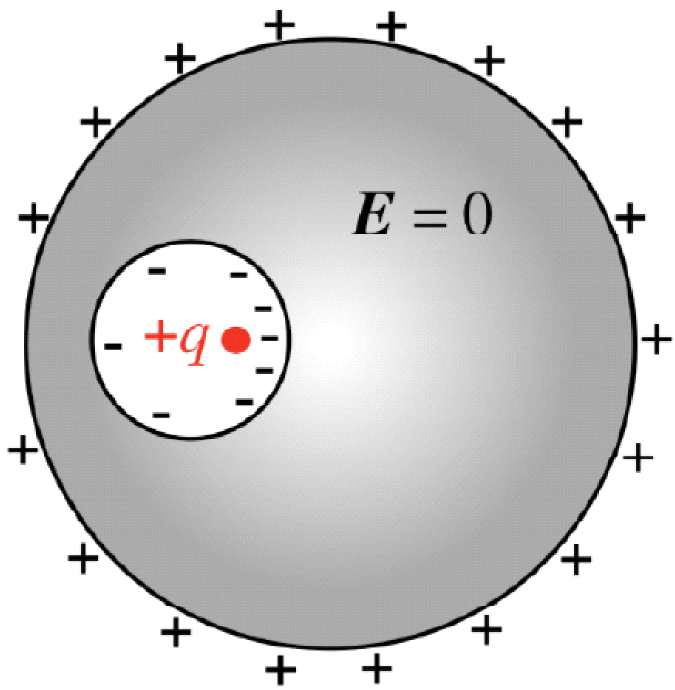
$$\vec{E} = \begin{cases} \frac{q}{4\pi\epsilon_0 r^2}\hat{r} & r < a \\ \frac{q}{4\pi\epsilon_0 |\vec{r}-\vec{A}|^3}(\vec{r}-\vec{A}) & \text{outside the conductor} \\ 0 & \text{inside the conductor and outside the cavity.} \end{cases}$$



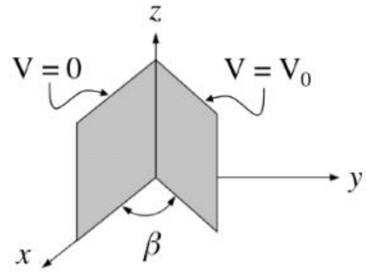
A3)

If the charge q inside the cavity is placed off center then the surface charge density of the inner cavity becomes non uniform. Qualitatively speaking the charge density is highest at the closest point and is lowest at the farthest which are diametrically opposite to each other and in between it varies continuously. The surface charge density on the outer spherical surface of radius R is again uniform with total charge q.

The electric field inside the cavity is different from the previous case and the spherical symmetry no longer holds. The fields in other regions is still the same.



Problem B. The Wedge [20%]. Two very large plates form a wedge of angle β. One plate is held at $V = 0$ and the other one at $V = V_0$ as show below.



(1) Write Laplace's equation for the region between the plates. Use cylindrical coordinates and assume that the electric potential does not depend on z or s. *Hint: You should obtain a 1D differential equation in ϕ – pretty easy to solve no?*

(2) By solving the above Laplace's equation calculate the electric potential and the electric field (magnitude and direction) in the region between the plates assuming the plates have fixed potential as shown.

B1)

In cylindrical coordinates, the Laplacian takes the form

$$\nabla^2 V = \frac{1}{s}\frac{\partial}{\partial s}\left(s\frac{\partial V}{\partial s}\right) + \frac{1}{s^2}\frac{\partial^2 V}{\partial \phi^2} + \frac{\partial^2 V}{\partial z^2}$$

If we assume the potential depends upon only ϕ inside the wedge, then Laplace's equation reduces to

$$\frac{\partial^2 V}{\partial \phi^2} = 0$$

B2)

Equation (1) above admits a general solution of the form

$$V(\phi) = A\phi + B$$

Here, we demand

$$\begin{aligned} V(0) &= 0 \\ V(\beta) &= V_0 \end{aligned}$$

which leads to the system of equations

$$\begin{aligned} B &= 0 \\ \beta A + B &= V_0 \end{aligned}$$

The final solution for V is then

$$V(\phi) = \frac{V_0 \phi}{\beta}$$

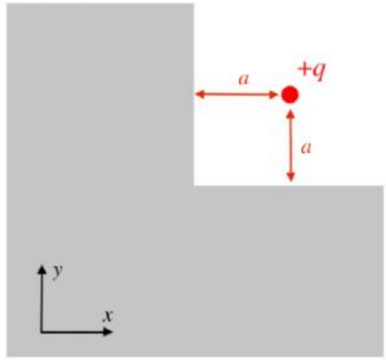
In general, the electric field is related to the potential by

$$\vec{E} = -\nabla V,$$

so

$$\begin{aligned} \vec{E} &= -\frac{\partial V}{\partial s}\hat{s} - \frac{1}{s}\frac{\partial V}{\partial \phi}\hat{\phi} - \frac{\partial V}{\partial z}\hat{z} \\ &= -\frac{V_0}{\beta s}\hat{\phi} \end{aligned}$$

Problem C. The Corner [30%]. A point-charge $+q$ is located next to a very large grounded conductor (distance a). The conductor is bent at a right-angle as shown below. Our goal is to calculate the electric potential generated in the upper-right corner. We assume that the problem is two-dimensional, namely that there is no dependence on the coordinate z. **The use of Mathematica/Wolfram Alpha is allowed.**



(1) Set up a configuration of image charges (what charges do you need, where should they be located, etc) that allows you to solve for the potential in the upper right corner. Calculate the potential in this region.

(2) Find the surface charge density induced on the plane at $x = 0$.

C1)

Note first that placing an image charge $-q$ at $(-a, a, 0)$ causes the potential on the $y - z$ plane to be zero everywhere. Now, we need to ensure that the potential everywhere on the $x - z$ plane is zero. We can do this with two image charges: a charge $+q$ at $(-a, -a, 0)$, and a charge $-q$ at $(a, -a, 0)$. Note that the two charges below the $x - z$ plane offset each others' contributions to the overall potential on the $y - z$ plane. The potential is then

$$
\begin{aligned}
V(\vec{r}) &= \sum_i \frac{q_i}{4\pi\epsilon_0 |\vec{r} - \vec{r}_i|} \\
&= \frac{q}{4\pi\epsilon_0}\left(\frac{1}{|\vec{r} - (a, a, 0)|} + \frac{1}{|\vec{r} - (-a, -a, 0)|} - \frac{1}{|\vec{r} - (a, -a, 0)|}\right. \\
&\quad \left. - \frac{1}{|\vec{r} - (a, -a, 0)|}\right)
\end{aligned}
$$

C2)

The surface charge density can be obtained from the discontinuity in the component of the electric field normal to the interface:

$$\sigma = \epsilon_0 \left(\vec{E}_2 - \vec{E}_1\right) \cdot \hat{n}$$

where $\hat{n}$ is a vector pointing from region 1 to region 2.

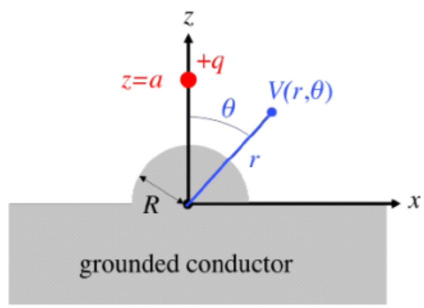
The net electric field outside the conductor will be that due to the one true point charge and the three image charges:

$$\begin{aligned}
\vec{E} &= \frac{q}{4\pi\epsilon_0}\left(\frac{\vec{r}-(a,a,0)}{|\vec{r}-(a,a,0)|^{3/2}} + \frac{\vec{r}-(-a,-a,0)}{|\vec{r}-(-a,-a,0)|^{3/2}}\right. \\
&\quad \left. - \frac{\vec{r}-(-a,a,0)}{|\vec{r}-(-a,a,0)|^{3/2}} - \frac{\vec{r}-(a,-a,0)}{|\vec{r}-(a,-a,0)|^{3}}\right)
\end{aligned}$$

Inside the conductor, the electric field is zero. Therefore, along the $y-z$ plane,

$$\begin{aligned}
\sigma &= \epsilon_0 \vec{E}(0,y,0)\cdot\hat{x} \\
&= \frac{q}{4\pi}\left(\frac{-a}{[a^2+(y-a)^2+z^2]^{3/2}} + \frac{a}{[a^2+(y+a)^2+z^2]^{3/2}}\right. \\
&\quad \left. - \frac{a}{[a^2+(y-a)^2+z^2]^{3/2}} + \frac{a}{[a^2+(y+a)^2+z^2]^{3/2}}\right) \\
&= \frac{qa}{4\pi}\left(\frac{1}{[a^2+(y+a)^2+z^2]^{3/2}} - \frac{1}{[a^2+(y-a)^2+z^2]^{3/2}}\right)
\end{aligned}$$

Problem D. The Bump [30%]. Imagine a *grounded* infinite conducting plane in the xy plane, that has a conducting hemispherical bump (radius R) in it, centered at the origin, as shown. A charge q sits a distance a above the plane, i.e. at the point $(x = 0, y = 0, z = a)$.



(1) I claim that you can find the potential V anywhere in the plane above the conductor using the method of images, with **three image charges**. Where should they be? Explain your reasoning; you need to ensure the boundary condition $V = 0$ on the entire conductor.

(2) What is $V(r, \theta)$?

(3) What fraction of the total induced charge is induced on the hemispherical bump?

Due to the azimuthal symmetry (if we use spherical coordinates (r, θ, ϕ) then the potential does not depend on ϕ) of the system the problem effectively becomes

two dimensional. As shown in the figure we can use plane polar coordinates (r, θ) where θ is the angle made by the position vector with the X axis.

D1)

The underlying principle of solving for a potential using method of images is to find a set of point charges inside the conductor so that the boundary conditions are satisfied at the surface of the conductor. For example, in case of a grounded conductor with planar boundary the image charge $(-q)$ is placed inside the conductor at a distance equal to the separation of the source point charge (q) and the plane. So without doing any computation we see that there must be a image charge $-q$ at $z = -a$.

Compared to the planar case it is not at all obvious if we can find an image charge to satisfy the boundary condition on the semi circle (hemispherical boundary). But let us assume that it is possible and then due to azimuthal symmetry it must lie on the Z axis. So let the image charge, q' be at $(0, 0, z')$ inside the conductor.

$$V(R,\theta) = \frac{1}{4\pi\epsilon_0}\left(\frac{q}{|\vec{R} - a\hat{z}|} + \frac{q'}{|\vec{R} - z'\hat{z}|}\right)$$

$$0 = \frac{1}{4\pi\epsilon_0}\left(\frac{q}{\sqrt{R^2 + a^2 - 2aR\cos\left(\frac{\pi}{2} - \theta\right)}} + \frac{q'}{\sqrt{R^2 + z'^2 - 2z'R\cos\left(\frac{\pi}{2} - \theta\right)}}\right).$$

By moving one of the terms to the lhs and squaring we get

$$q^2(R^2 + z'^2 - 2z'R\sin\theta) = q'^2(R^2 + a^2 - 2aR\sin\theta).$$

We need two equations for two unknowns, so

$$2q^2 z' R\sin\theta = 2q'^2 aR\sin\theta$$
$$q^2(R^2 + z'^2) = q'^2(R^2 + a^2)$$

Equation (2) must be satisfied for all values of θ implying

$$\frac{q'}{q} = -\sqrt{\frac{z'}{a}}$$

We choose the negative root since it is obvious that the image charge must be of opposite polarity.

$$\frac{R^2 + z'^2}{R^2 + a^2} = \frac{z'}{a}$$
$$z'^2 - \frac{z'}{a}(R^2 + a^2) + R^2 = 0$$

Therefore

$$z' = \frac{R^2}{a} \ ; \ q' = -q\frac{R}{a}.$$

To ensure $V = 0$ on the planar boundary we need to place an image charge $-q'$ at $z = -\frac{R^2}{a}$.

D2)

To calculate the potential outside the conductor use superposition principle.

$$V_1(r,\theta) = \frac{q}{4\pi\epsilon_0}\left(\frac{1}{\sqrt{r^2+a^2-2ar\sin\theta}} - \frac{1}{\sqrt{r^2+a^2+2ar\sin\theta}}\right)$$

$$V_2(r,\theta) = \frac{q}{4\pi\epsilon_0}\left(\frac{\frac{R}{a}}{\sqrt{r^2+\frac{R^4}{a^2}+2\frac{R^2}{a}r\sin\theta}} - \frac{\frac{R}{a}}{\sqrt{r^2+\frac{R^4}{a^2}-2\frac{R^2}{a}r\sin\theta}}\right)$$

So the total potential $V(r,\theta)$ is equal to $V_1(r,\theta) + V_2(r,\theta)$.

D3)

The electric field outside the conductor is given by

$$\begin{aligned}\vec{E}(x,y,z) &= -\nabla V(x,y,z)\\ &= \frac{q}{4\pi\epsilon_0}\left(\frac{\vec{r}-a\hat{z}}{(x^2+y^2+(z-a)^2)^{3/2}} - \frac{\vec{r}+a\hat{z}}{(x^2+y^2+(z+a)^2)^{3/2}}\right)\\ &+ \frac{qR}{4a\pi\epsilon_0}\left(\frac{\vec{r}+(R^2/a)\hat{z}}{(x^2+y^2+(z+(R^2/a))^2)^{3/2}} - \frac{\vec{r}-(R^2/a)\hat{z}}{(x^2+y^2+(z-(R^2/a))^2)^{3/2}}\right)\end{aligned}$$

On the planar surface of the conductor given by $z = 0$ the electric field is perpendicular to the XY plane. Therefore the surface charge density on the planar part of the boundary is given by

$$\sigma(x,y) = \frac{q}{2\pi}\left(-\frac{a}{(x^2+y^2+a^2)^{3/2}} + \frac{(R^3/a^2)}{(x^2+y^2+(R^2/a)^2)^{3/2}}\right)$$

The total induced charge on the plane is given by

$$\begin{aligned}\int_R^\infty drr\sigma(r) &= \int_R^\infty drr\left(-\frac{qa}{(r^2+a^2)^{3/2}} + \frac{(R^3/a^2)}{(r^2+(R^2/a)^2)^{3/2}}\right)\\ &= -q\frac{(a-(R^2/a))}{\sqrt{R^2+a^2}}.\end{aligned}$$

$$= -q\frac{(a - (R^2/a))}{\sqrt{R^2 + a^2}}.$$

The fraction of induced charge on the hemispherical part of the boundary is

$$1 - \frac{(a - (R^2/a))}{\sqrt{R^2 + a^2}}.$$