Final Exam, Math 2552

PLEASE PRINT YOUR NAME CLEARLY IN ALL CAPITAL LETTERS

First Name ____________________ Last Name ____________________

GTID Number: ______________________

Student GT Email Address: _________________________@gatech.edu

Section Number (A1, A2 or A3) ________ TA Name _________________________

INSTRUCTIONS (PLEASE READ)

Formatting and Timing

- You should only need 170 min to take the exam, but students will have 4 hours and 30 minutes to submit the exam, from the time that it is released.
- **Show your work** and justify your answers for all questions unless stated otherwise.
- Please write neatly, and use dark and clear writing so that the scan is easy to read.
- Please write your name or initials at the top of every page.
- Please solve the questions in the exam in the order they are given.
- You do not need to print the exam. As long as you solve problems in the order they are given (just like the written homework sets), you can write your answers on your own paper.But students can print the exam and write their answers on the printed copy if they prefer.

Submission

- Students should scan their work and submit it through Gradescope. There should be an **assignment** in Gradescope for this exam. The process for submitting your work will be similar to what you have used for homework.
- Work must be submitted today by 12:30 PM ET.
- Please upload your work as a single PDF file. If this is not possible you can email your work to your instructor.
- During the upload process in Gradescope, please indicate which page of your work correspondsto each question in the exam.

Questions

- If there are questions during the exam, students can ask them on BlueJeans, email their instructor or message them through Canvas.
- Our course Piazza forum will be temporarily inactive for 24 hrs on the day of the exam.
- If you run into any technical issues or any unanticipated emergencies, please email your instructor as soon as you can.

Integrity

- Students can use any resources while taking these tests including online calculators and Mathematica.
- Students cannot communicate with anyone during these tests including using Reddit or online message boards
- Students cannot use solutions provided from another student or third party.
- In other words: do your own work but you can use technology to solve problems.

1. (2 points) A small number of points will be allocated for presentation, neatness, and organization. Please ensure that

    - Your work is legible in the scan.
    - Your name or initials are at the top of every page.
    - Questions are answered in the order in which they were given.
    - During the upload process you have indicated which pages correspond to which question, and made sure that none of your pages are upside down or sideways (you can also change the orientation of the pages when you upload in Gradescope). Ensuring that these criteria are met helps ensure that your exam is graded efficiently and accurately.

2. (8 points) Solve the Initial Value Problem:

$$\frac{\mathrm{d}y}{\mathrm{d}t} = -2y + 4, \qquad y(0) = 3.$$

3. (10 points) (a) (5 points) A population $p(t)$ of bacteria has a rate of growth k proportional to the size of the population. After 5 days, the population of bacteria is the double of the initial population. After 10 days, the population is 100 000.

Write down the IVP satisfied by p (including the value of the population at time 0).

(b) (5 points) Three water tanks A, B and C are placed as follow:

- Tank A has a capacity of 10L. A solution with 2g of salt per liter is pumped in at a rate of 2L/min. Water is pumped from Tank A to Tank B at a rate of 2L/min. Denote by Q_A the quantity of salt in this water tank.
- Tank B has a capacity of 5L. Water is pumped from Tank B to Tank C at a rate of 2L/min. Denote by Q_B the quantity of salt in this water tank.
- Tank C has a capacity of 8L. Water is drained from Tank C at a rate of 2L/min. Denote by Q_C the quantity of salt in this water tank.

Assume the tank are well mixed and filled initially with pure water.

Write down the IVP for the quantity of salt $\vec{Q}(t) = \begin{pmatrix} Q_A(t) \\ Q_B(t) \\ Q_C(t) \end{pmatrix}$ at time t. You don't have to solve it.

4. (10 points) Find the general solution of the system

$$\frac{\mathrm{d}}{\mathrm{d}t}\vec{x} = \begin{pmatrix} -2 & 0 & 0 & 0 \\ 0 & -2 & -3 & 0 \\ 0 & 3 & -2 & 0 \\ 0 & 0 & 0 & -2 \end{pmatrix} \vec{x}.$$

5. (10 points) Find the general solution of the equation

$$y''(t) - y'(t) - 12y(t) = 260\sin(2t) + 14te^{4t}.$$

6. (10 points) Let us draw some phase portraits.

(a) (4 points) Find all equilibria and sketch the phase line of $y' = (y^2 - 4)(y-4)^2(y+1)$. Classify the equilibria.

(b) (3 points) Sketch the phase portrait of $\frac{\mathrm{d}\vec{x}}{\mathrm{d}t} = \begin{pmatrix} -2 & 1 \\ 0 & -2 \end{pmatrix} \vec{x}$ on the plane (x_1, x_2).

(c) (3 points) Sketch the phase portrait of $y'' - 4y' + 8y = 0$ on the plane (y, y').

7. (10 points) Consider the differential equation:

$$y''(t) - 10y'(t) + 25y(t) = \frac{-e^{5t}}{t^2}, \qquad t > 0.$$

Find a particular solution to this equation.

8. (10 points) Let us play with the Laplace Transform.

(a) (5 points) Compute the Laplace transform of the function f defined by

$$f(t) = \begin{cases} \cos(2t) - t & \text{if} \quad 0 < t < 1, \\ \cos(2t) - 10 & \text{if} \quad 1 \leq t < 5, \\ \cos(2t) + t^2 - 10t + 15 & \text{if} \quad t > 5. \end{cases}$$

(b) (2 points) Compute the Inverse Laplace Transform of the function F defined by

$$F(s) = \frac{5(s^2 - 4s + 6)}{(s^2 - 4s + 8)(s - 3)}.$$

9. (10 points) A mass of 1 kg ($m = 1$) is attached to the end of a spring that has spring constant $k = 8$ N/m. The damping constant is $\gamma = 6kg/s$. Assume that an external force of $-6e^{-2t} - e^{-3t}$ Newtons acts on the mass.

   Denote by y(t) the displacement of mass from the equilibrium position.

   (a) (2 points) Write down a differential equation for y(t).

   (b) (8 points) Assume that $y(0) = 3$ and $y'(0) = -14$. Using the Laplace Transform, solve the IVP.

10. (10 points) Consider the following system:

$$\begin{cases} x' = x^2 y + x, \\ y' = (x+y)(x+2). \end{cases}$$

First, find all equilibria for this system. Then, for each equilibrium, give the linear approximating system near the equilibrium and study the stability of each **linear system** obtained.

11. (10 points) Consider the Initial Value Problem:

$$y' = 2t^2 + y - ty, \quad y(0) = 1.$$

Using the Runge-Kutta 4 Method with $h = 2$, find an approximate value of the solution at time $t = 2$.