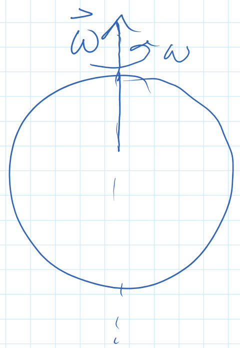
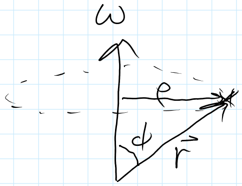
Consider a solid body rotating w/ constant ang. velocity ω. Define $\vec{\omega} = \omega \hat{n}$ where $\hat{n}$ is a unit vector pointing along the rotation axis and its direction is set by the right-hand rule



For Earth, $\vec{\omega}$ points north (Earth rotates from W to E) $|\vec{\omega}| = \frac{2\pi}{86164\,s} = 7.292 \times 10^{-5}\ s^{-1}$



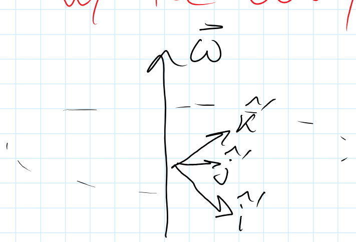
The velocity of a point at position $\vec{r}$

Clearly, $v = \omega\rho = \omega r \sin\phi = |\vec{\omega} \times \vec{r}|$

Direction follows the RHR, so $\underline{\vec{v} = \vec{\omega} \times \vec{r}}$

Actually, any vector $\vec{c}$ that is fixed on the rotating body will change as $\underline{\frac{d\vec{c}}{dt} = \vec{\omega} \times \vec{c}}$

So, let's consider this effect on fixed coordinate axes rotating w/ the body



$$\frac{d\hat{i}'}{dt} = \vec{\omega} \times \hat{i}', \quad \frac{d\hat{j}'}{dt} = \vec{\omega} \times \hat{j}', \quad \frac{d\hat{k}'}{dt} = \vec{\omega} \times \hat{k}'$$

(e.g., if $\hat{k}' \parallel \vec{\omega}$, $\frac{d\hat{i}'}{dt} = \omega \hat{j}'$ $\frac{d\hat{j}'}{dt} = -\omega \hat{i}'$, $\frac{d\hat{k}'}{dt} = 0$)

So, our vector $\vec{c}$ on the rotating body can be written in this rotating frame as $\vec{c} = c_x \hat{i}' + c_y \hat{j}' + c_z \hat{k}'$ (e.g. position of something wrt Earth's surface)

Tricky part: if $\vec{c}$ is changing then its rate of change will be different wrt the rotating frame and the inertial frame

Notation for this chapter only:

$\frac{d\vec{c}}{dt}$ is the rate of change measured by an inertial observer

$\dot{\vec{c}}$ is the rate of change measured by an observer rotating w/ the body.

Both observers will agree on the rates of change of the components c_x, c_y, c_z. That is, $\frac{dc_x}{dt} = \dot{c}_x$, $\frac{dc_y}{dt} = \dot{c}_y$, $\frac{dc_z}{dt} = \dot{c}_z$

Note: the inertial observer will agree on these rates but not on the coordinate system. Think of a ship moving on ocean.

In the rotating frame $\dot{\vec{c}} = \dot{c}_x \hat{i}' + \dot{c}_y \hat{j}' + \dot{c}_z \hat{k}'$

In the stationary frame, the coordinate system is also ~~a~~ varying

$$\frac{d\vec{c}}{dt} = c_x \frac{d\hat{i}'}{dt} + \frac{dc_x}{dt}\hat{i}' + c_y \frac{d\hat{j}'}{dt} + \frac{dc_y}{dt}\hat{j}' + c_z \frac{d\hat{k}'}{dt} + \frac{dc_z}{dt}\hat{k}'$$

$$= \left(\dot{c}_x \hat{i}' + \dot{c}_y \hat{j}' + \dot{c}_z \hat{k}'\right) + \left(c_x \frac{d\hat{i}'}{dt} + c_y \frac{d\hat{j}'}{dt} + c_z \frac{d\hat{k}'}{dt}\right)$$

$$= \dot{\vec{c}} + \vec{\omega} \times \left(c_x \hat{i}' + c_y \hat{j}' + c_z \hat{k}'\right) = \dot{\vec{c}} + \vec{\omega} \times \vec{c}$$

rotation vector

$$\therefore \boxed{\frac{d\vec{c}}{dt} = \dot{\vec{c}} + \vec{\omega} \times \vec{c}}$$

vector in rotating frame

rotating frame

rotating frame

stationary frame

e.g. consider the position vector in the rotating frame:

$$\frac{d\vec{r}}{dt} = \dot{\vec{r}} + \vec{\omega} \times \vec{r}$$

[Note: that if a vector $\vec{c}$ satisfies $\frac{d\vec{c}}{dt} = \vec{\omega} \times \vec{c}$ then it must be a fixed vector rotating w/ speed ω]

Ex: Particle in a Uniform Magnetic Field

A particle w/ charge q moves w/ velocity $\vec{v}$ in a magnetic field $\vec{B}$. $\quad \vec{F} = q(\vec{v} \times \vec{B})$

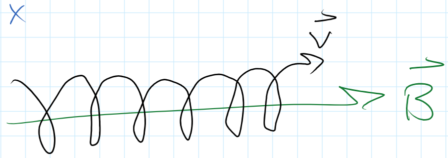
$$\therefore \; m\frac{d\vec{v}}{dt} = q(\vec{v} \times B)$$

$$\frac{d\vec{v}}{dt} = \frac{q}{m}(\vec{v} \times \vec{B}) = -\frac{q}{m}(\vec{B} \times \vec{v})$$

If $\vec{B}$ is uniform & constant than this has the same form as

$$\frac{d\vec{v}}{dt} = \vec{\omega} \times \vec{v} \quad \text{where} \quad \vec{\omega} = -\frac{q}{m}\vec{B}, \quad \text{cyclotron or gyrofreq.}$$

Thus, $\vec{v}$ rotates around $\vec{B}$ w/ constant angular velocity & motion will be a helix



Motion $\parallel$ to $\vec{B}$ is constant.

If $\vec{v}$ is entirely $\perp$ to $\vec{B}$ the motion is circular w/ radius

$$r = \frac{v}{\omega} = \frac{mv}{qB}$$