

Sample Final B, Math 1554

PLEASE PRINT YOUR NAME CLEARLY IN ALL CAPITAL LETTERS

First Name _____ Last Name _____

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Section Number (e.g. A4, M2, QH3, etc.) _____ TA Name _____

Circle your instructor:

Dr. Barone, Dr. Bloomquist, Dr. Vilaça Da Rocha, Dr. Lacey, Dr. Mayer

Student Instructions

- **Show your work** and justify your answers for all questions unless stated otherwise.
- **Organize your work** in a reasonably neat and coherent way.
- Calculators, notes, cell phones, books are not allowed.
- Use dark and clear writing: your exam will be scanned into a digital system.
- Exam pages are double sided. Be sure to complete both sides.
- Leave a 1 inch border around the edges of exams.
- The last two pages are for scratch work. Please use them if you need extra space.

You do not need to justify your reasoning for questions on this page.

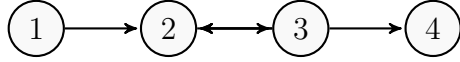
1. (10 points) Determine whether the statements are true or false.

	true	false
a) If $A = A^T$, $\vec{x} \neq \vec{y}$, $A\vec{x} = 3\vec{x}$, and $A\vec{y} = -\vec{y}$, then $\vec{x}^T \vec{y} = 0$.	<input type="radio"/>	<input type="radio"/>
b) If \vec{x}_1 maximizes a quadratic form, $Q(\vec{x})$, subject to the constraint $\ \vec{x}\ = 1$, then so does $-\vec{x}_1$.	<input type="radio"/>	<input type="radio"/>
c) If U is an echelon form of matrix A , then $\text{Col}(U) = \text{Col}(A)$.	<input type="radio"/>	<input type="radio"/>
d) $S = \{\vec{x} \in \mathbb{R}^3 \mid x_1 + x_2 + x_3 = 1\}$ is a subspace of \mathbb{R}^3 .	<input type="radio"/>	<input type="radio"/>
e) An eigenspace is a subspace spanned by a single eigenvector.	<input type="radio"/>	<input type="radio"/>
f) Row operations on a $n \times n$ matrix do not change its eigenvalues.	<input type="radio"/>	<input type="radio"/>
g) If $A \in \mathbb{R}^{n \times n}$ and \vec{x} and \vec{y} are vectors in \mathbb{R}^n , then $A\vec{x} \cdot A\vec{y} = \vec{x}^T A^T A \vec{y}$.	<input type="radio"/>	<input type="radio"/>
h) If the eigenvalues of 3×3 matrix A are 1, 2, 4, and $A = PBP^{-1}$, for 3×3 matrices P and B , then the eigenvalues of B are 1, 2, 4.	<input type="radio"/>	<input type="radio"/>
i) The orthogonal compliment of the subspace $H = \{\vec{x} \in \mathbb{R}^3 \mid x_1 = x_2 - x_3\}$ is a line.	<input type="radio"/>	<input type="radio"/>
j) If A is a $n \times n$ matrix, \vec{x} and \vec{y} are in \mathbb{R}^n , and $A\vec{x} = A\vec{y}$ for a particular $\vec{x} \neq \vec{y}$, then $\dim(\text{Row}(A)^\perp) \neq 0$.	<input type="radio"/>	<input type="radio"/>

You do not need to explain your reasoning for questions on this page.

2. (10 points) If possible, give an example of the following. If it is not possible, write “not possible”.

(a) A Google Matrix, G , for the set of web pages that link to each other according to the diagram below.



(b) A homogeneous linear system with two equations that has no solutions.

(c) A matrix $A \in \mathbb{R}^{2 \times 2}$ that is orthogonally diagonalizable but is not invertible.

(d) A 2×2 matrix whose eigenvalues are $\lambda_1 = 5$ and $\lambda_2 = 10$, and whose corresponding eigenvectors are $\vec{v}_1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$, $\vec{v}_2 = \begin{pmatrix} -1 \\ 2 \end{pmatrix}$.

You do not need to explain your reasoning for questions on this page.

3. (10 points) If possible, give an example of the following. If it is not possible, write “not possible”.

(a) A 2×2 matrix that is stochastic and orthogonal.

(b) A 5×3 matrix, A , in reduced echelon form, such that $\dim(\text{Col}(A)^\perp) = 1$.

(c) A quadratic form $Q : \mathbb{R}^3 \mapsto \mathbb{R}$, with no cross-terms, that has maximum value 5, subject to the constraint that $\|\vec{x}\| = 1$.

(d) A matrix that is the standard matrix for the linear transform $T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$. T first rotates points counterclockwise by π radians, and then reflects them through the line $x_1 = x_2$.

(e) A 3×2 matrix whose column space is spanned by the vector $\begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}$ and whose null space is the line $x_1 = 4x_2$.

You do not need to explain your reasoning for questions on this page.

4. Below is a SVD factorization for a matrix $A = U\Sigma V^T$, where

$$U = [\vec{u}_1 \quad \vec{u}_2 \quad \vec{u}_3 \quad \vec{u}_4], \quad \Sigma = \begin{bmatrix} 4 & 0 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad V = [\vec{v}_1 \quad \vec{v}_2 \quad \vec{v}_3 \quad \vec{v}_4 \quad \vec{v}_5]$$

Fill in the blanks.

- (a) (1 point) What is the rank of A ? _____
- (b) (1 point) What is the largest value of $\|A\vec{x}\|$, subject to $\|\vec{x}\| = 1$? _____
- (c) (1 point) List an orthonormal basis for $\text{Row}A$. _____
- (d) (1 point) List an orthonormal basis for $\text{Col}A$. _____
5. (2 points) If possible, fill in the missing entries of the 3×3 matrices with numbers so that the matrices are singular. If it is not possible to do so, write “not possible” below the matrix.

$$B = \begin{bmatrix} 0 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \qquad C = \begin{bmatrix} 3 & 3 \\ 2 & 3 & 2 \\ 1 & 0 & 1 \end{bmatrix}$$

6. (4 points) Fill in the blanks. You do not need to explain your reasoning.

- (a) Suppose $\vec{u} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$, $\vec{v} = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$. The area of the parallelogram determined by $\vec{0}$, \vec{u} , \vec{v} , and $\vec{u} + \vec{v}$ is _____.
- (b) If Q is $n \times n$ and orthogonal, then $Q^T Q$ equals _____.
- (c) If A is square and singular, at least one eigenvalue of A is _____.
- (d) If A is 3×3 , $\dim(\text{Col}(A)) = 1$, then $\dim(\text{Row}(A^T)) =$ _____.

7. (5 points) If possible, compute the LU factorization of A . Clearly indicate matrices L and U .

$$A = \begin{bmatrix} 2 & 4 & 9 & 1 \\ 2 & 1 & 0 & 0 \\ 4 & 5 & 9 & 0 \end{bmatrix}$$

8. $H = \{\vec{x} \in \mathbb{R}^3 : x_1 - x_2 - 9x_3 = 0\}$.

(a) (3 points) Construct a basis for H .

(b) (2 points) Write down a basis for H^\perp .

9. (6 points) Suppose there are two cities, A and B . Every year,
- 60% of the people from A move to B , and 40% stay in A .
 - 10% of the people from B move to A , and 90% stay in B .

The initial populations of A and B are a_0 and b_0 , respectively.

(a) What is the stochastic matrix for this situation?

(b) After a long period of time, what is the population in city A ?

10. (3 points) Identify a non-zero value of k so that P has one real eigenvalue of algebraic multiplicity 2, and another real eigenvalue with algebraic multiplicity 1. Show your work.

$$P = \begin{pmatrix} 3 & 1 & 0 \\ k & 3 & 0 \\ 0 & 0 & 5 \end{pmatrix}$$

11. (10 points) Construct an SVD for $A = \begin{pmatrix} 1 & 0 \\ 2 & 1 \\ 0 & 1 \end{pmatrix}$.

12. (10 points) Find the least squares solution, \hat{x} , to the equation below. Don't forget to show your work.

$$\begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 4 \\ 1 \\ 0 \end{pmatrix}$$

$$\hat{x} = \begin{pmatrix} \\ \\ \\ \end{pmatrix}$$

13. (10 points) Let $A = \begin{pmatrix} 1 & 5 \\ -2 & 3 \end{pmatrix}$.

This question is from sample midterm 3A, its solution is on Canvas and Piazza.

(a) Calculate the eigenvalues of A .

(b) If possible, construct matrices P and C such that $A = PCP^{-1}$.

Sample Final B, Answers

1. True/false.

- (a) True
- (b) True
- (c) False
- (d) False
- (e) False
- (f) False
- (g) True
- (h) True
- (i) True
- (j) True

2. Example Construction.

(a) $G = \frac{0.85}{4} \begin{pmatrix} 0 & 0 & 0 & 1 \\ 4 & 0 & 2 & 1 \\ 0 & 4 & 0 & 1 \\ 0 & 0 & 2 & 1 \end{pmatrix} + \frac{0.15}{4} K$, where K is a 4×4 matrix of ones.

(b) Not possible (every homogeneous system has a trivial solution)

(c) $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$

(d) $A = PDP^{-1} = \begin{pmatrix} 2 & -1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} 5 & \\ & 10 \end{pmatrix} \begin{pmatrix} 2/5 & 1/5 \\ -1/5 & 2/5 \end{pmatrix} = \begin{pmatrix} 6 & -2 \\ -2 & 9 \end{pmatrix}$

3. Example Construction.

(a) $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

(b) Not possible

(c) $Q = 5x_1^2$

(d) $A = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$

(e) $\begin{pmatrix} 2 & -8 \\ 0 & 0 \\ 1 & -4 \end{pmatrix}$

4. Fill in the blanks:

(a) 3

- (b) 4
- (c) v_1, v_2, v_3
- (d) u_1, u_2, u_3

5. Matrix completion.

$$B = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \qquad C = \begin{bmatrix} 3 & 3 & 3 \\ 2 & 3 & 2 \\ 1 & 0 & 1 \end{bmatrix}$$

6. Fill in the blanks.

- (a) 2
- (b) I_n
- (c) 0
- (d) 1 since it has one pivot.

7. LU factorization.

$$\begin{aligned} A &= \begin{pmatrix} 2 & 4 & 9 & 1 \\ 2 & 1 & 0 & 0 \\ 4 & 5 & 9 & 0 \end{pmatrix} \\ &\sim \begin{pmatrix} 2 & 4 & 9 & 1 \\ 0 & -3 & -9 & -1 \\ 0 & -3 & -9 & -2 \end{pmatrix} \\ &\sim \begin{pmatrix} 2 & 4 & 9 & 1 \\ 0 & -3 & -9 & -1 \\ 0 & 0 & 0 & -1 \end{pmatrix} \\ &= U \end{aligned}$$

By inspection, based on the row operations we used to produce U from A ,

$$L = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 2 & 1 & 1 \end{pmatrix}$$

8. (a) One equation in three variables: we seek two basis vectors. Choose $x_2 = 1$ and $x_3 = 0$ to obtain $v_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$. Choose $x_2 = 0$ and $x_3 = 1$ to obtain

$$v_2 = \begin{pmatrix} 9 \\ 0 \\ 1 \end{pmatrix}$$

A basis for H is $\{v_1, v_2\}$.

- (b) The basis for H^\perp has only one non-zero vector, v_3 . Any non-zero vector that is perpendicular to both v_1 and v_2 will suffice. By inspection,

$$v_3 = \begin{pmatrix} 1 \\ -1 \\ -9 \end{pmatrix}$$

9. Markov Chain.

(a) $P = \frac{1}{10} \begin{pmatrix} 4 & 1 \\ 6 & 9 \end{pmatrix}$

(b) Solve

$$\begin{aligned} (P - I)\vec{x} &= \vec{0} \\ 0.10 \begin{pmatrix} -6 & 1 \\ 6 & -1 \end{pmatrix} \vec{x} &= \vec{0} \end{aligned}$$

Thus, $\vec{x} = \frac{1}{7} \begin{pmatrix} 1 \\ 6 \end{pmatrix}$. After a long period of time, the population in city A is

$$\frac{a_0 + b_0}{7}$$

10. Set up the usual determinant:

$$\begin{aligned} 0 &= |P - \lambda I| \\ &= (\lambda - 5)((3 - \lambda)(3 - \lambda) - k) \\ &= (\lambda - 5)(9 - 6\lambda + \lambda^2 - k) \end{aligned}$$

If $k = 4$, then $\lambda = 5$ has algebraic multiplicity two.

11. SVD:

$$A^T A = \begin{pmatrix} 1 & 2 & 0 \\ 2 & 0 & 2 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 2 & 0 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} 5 & 2 \\ 2 & 8 \end{pmatrix}$$

By solving characteristic equation, $\lambda = 4, 9$.

$$\Sigma = \begin{pmatrix} 3 & 0 \\ 0 & 2 \\ 0 & 0 \end{pmatrix}$$

V matrix:

$$A^T A - 9I = \begin{pmatrix} -4 & 2 \\ * & * \end{pmatrix}, \quad * = \text{not needed}$$

Thus, an eigenvector is $\vec{v}_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$. And $A^T A$ is symmetric, so $\vec{v}_1 \cdot \vec{v}_2 = 0$. So $\vec{v}_2 = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$. Scale these vectors so they have unit length, place them into a matrix,

$$V = \frac{1}{\sqrt{5}} \begin{pmatrix} 1 & 2 \\ 2 & -1 \end{pmatrix}$$

U matrix:

$$\vec{u}_1 = \frac{1}{\sigma_1} A \vec{v}_1 = \frac{1}{3\sqrt{5}} \begin{pmatrix} 1 & 2 \\ 2 & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \frac{1}{3\sqrt{5}} \begin{pmatrix} 5 \\ 2 \\ 4 \end{pmatrix}$$

$$\vec{u}_2 = \frac{1}{\sigma_2} A \vec{v}_2 = \frac{1}{2\sqrt{5}} \begin{pmatrix} 1 & 2 \\ 2 & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} 2 \\ -1 \end{pmatrix} = \frac{1}{2\sqrt{5}} \begin{pmatrix} 0 \\ 4 \\ -2 \end{pmatrix}$$

$\vec{u}_3 \in \text{Col}A^\perp$, so set

$$\vec{u}_3 = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

and solve equations

$$\vec{u}_1 \cdot \vec{u}_3 = \vec{u}_2 \cdot \vec{u}_3 = 0$$

This yields

$$0 = 5a + 2b + 4c$$

$$0 = 4b - 2c$$

One variable is free: set $c = 2$, then $b = 1$ and $a = 2$.

$$\vec{u}_3 = \frac{1}{3} \begin{pmatrix} -2 \\ 1 \\ 2 \end{pmatrix}$$

The U matrix is

$$U = \begin{pmatrix} \frac{5}{3\sqrt{5}} & \frac{0}{2\sqrt{5}} & \frac{-2}{3} \\ \frac{2}{3\sqrt{5}} & \frac{4}{2\sqrt{5}} & \frac{1}{3} \\ \frac{4}{3\sqrt{5}} & \frac{-2}{2\sqrt{5}} & \frac{2}{3} \end{pmatrix}$$

The SVD is

$$A = U \Sigma V^T$$

with U , Σ , and V as above. There is no need to re-write each element out again, but students should indicate that $A = U \Sigma V^T$ somewhere for full points.

12. $A^T A = \begin{pmatrix} 3 & 2 & 0 \\ 2 & 3 & 0 \\ 0 & 0 & 2 \end{pmatrix}$, $A^T b = \begin{pmatrix} 5 \\ 5 \\ 4 \end{pmatrix}$. Solving the normal equation we find $\hat{x} = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$

13. $0 = |A - \lambda I| = (1 - \lambda)(3 - \lambda) + 10$

$$0 = \lambda^2 - 4\lambda + 13$$

$$\lambda = 2 \pm \frac{1}{2}\sqrt{16 - 52} = 2 \pm \frac{1}{2}\sqrt{-36} = 2 \pm 3i$$

Solve $(A - \lambda I) \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ to find eigenvector.

Choose $\lambda = 2 - 3i$ (you can choose $\lambda = 2 + 3i$ if you prefer)

$$A - \lambda I = \begin{pmatrix} -1 + 3i & 5 \\ -2 & 1 + 3i \end{pmatrix}$$

Thus, using 2nd row, $-2x_1 + (1 + 3i)x_2 = 0$. We only need one of the two rows because the two are multiples of each other. A possible solution is $\vec{v} = \begin{pmatrix} 1 + 3i \\ 2 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} + i \begin{pmatrix} 3 \\ 0 \end{pmatrix}$

$$\implies P = \begin{pmatrix} 1 & 3 \\ 2 & 0 \end{pmatrix}, C = \begin{pmatrix} 2 & -3 \\ 3 & 2 \end{pmatrix}$$