MIDTERM 1
MATH 3215-C (PROBABILITY AND STATISTICS)

TUESDAY, SEPTEMBER 22

INSTRUCTOR: ARMENAK PETROSYAN

IMPORTANT: Please read carefully (1pt)

- You have a **12 hour** window to take and submit your exam (**7 am - 7 pm**).
- Be warned: **exam ends at 7 pm** (e.g. if you start at 6 pm, you only have 1 hour).
- After you opened this file you have 100 minutes to finish the work and 20 minutes to submit it **(120 minutes in total)**.
- If you run into difficulties submitting on GradeScope, email the files to the instructor before the 120 minutes expire and before 7 pm. **Late submissions will not be accepted**.
- If you encounter technical problems, email the instructor as soon as possible.
- You **CAN** use the course textbook and the lecture notes/slides for reference.
- You **CAN** use any fact we presented in class without proving them; anything else used must be proved.
- You **CAN** use any calculator you want.
- You **CANNOT** get any help from anyone.
- Posting the problems online to get help or to let others know what the problems are will be a violation; it will be reported and result in a penalty.
- **Write complete answers to get full credit.**
- The total amount of points for this exam is 75. Different problems have different weights.
- Be wise with your time. You can handwrite your answers on a different paper, and submit a photocopy. Make sure it is readable. No need to print the problem sheet or copy the problems.

Problem 1. *(10pt) Post office employs three postmen and they need to deliver post to 120 different households. In how many different ways can they split the job equally?*

Solution. *The job is split equally implies each postmen is going to visit* $120/3 = 40$ *houses. The first postman can visit the 40 houses in* $\binom{120}{40}$ *ways. The remaining 80 houses the second postman can visit in* $\binom{80}{40}$ *ways, and the third postman will visit the remaining houses. Using the multiplication principle, the total number is going to be*

$$\binom{120}{40} \cdot \binom{80}{40} \cdot 1 = \frac{120!}{80!40!} \frac{80!}{40!40!} = \frac{120!}{(40!)^3}.$$

Problem 2 (10pt)**.** *Four people are asked to guess a number between 0 and 9. What is the probability that at least two of them will guess the same number (assuming equiprobability space)?*

Solution. *The total number of all possible number combinations is* 10^4*. The number of all different 4 digit combinations is* $P(10, 6) = \frac{10!}{6!}$*. Therefore, the probability that none of them guesses the same number is* $\frac{10!}{6!10^4}$*. Consequently, the probability of at least tow have them guessing the same number is* $1 - \frac{10!}{6!10^4} = 0.496$*.*

Problem 3 (10pt)**.** *a) Let* A, B *be any two events. Show that*

$$P(A \cap B) \geq P(A) + P(B) - 1.$$

b) For any events $A_1, \dots, A_n$*, use induction to show*

$$P(A_1 \cap \cdots \cap A_n) \geq P(A_1) + \cdots P(A_n) - n + 1.$$

Solution. *a) Using the inclusion-exclusion principle*

$$P(A \cup B) = P(A) + P(B) - P(A \cap B).$$

Probability of an event is always less than or equal than 1, therefore

$$1 \geq P(A) + P(B) - P(A \cap B)$$

which implies

$$P(A \cap B) \geq P(A) + P(B) - 1.$$

b) For any events $A_1, \dots, A_n$*, use induction to show*

$$P(A_1 \cap \cdots \cap A_n) \geq P(A_1) + \cdots P(A_n) - n + 1.$$

The first step of the induction is proved in (a). Assume the inequality is true for $A_1, \dots, A_{n-1}$*:*

$$P(A_1 \cap \cdots \cap A_{n-1}) \geq P(A_1) + \cdots P(A_{n-1}) - n + 2.$$

Put $B = A_1 \cap \cdots \cap A_n$*. From (a) and the inductive assumption,*

$$\begin{aligned} P(A_1 \cap \cdots \cap A_n) = P(B \cap A_n) &\geq p(B) + P(A_n) - 1 \\ &\geq P(A_1) + \cdots P(A_{n-1}) - n + 2 + P(A_n) - 1 \\ &= P(A_1) + \cdots P(A_n) - n + 1. \end{aligned}$$

Problem 4 (10pt). *Are you more likely to win exactly 3 games out of 4 or exactly 5 games out of 8 against an equally strong opponent?*

Solution. *Both opponents are equally strong, implies they each have the same probability* $p = 0.5$ *to win in each round. The number of wins in four games is a binomial distribution with parameters* $(n = 4, p = 0.5)$ *so the probability of winning 3 games is* $\binom{4}{3}0.5^4 = 0.25$.

Similarly, the number of wins in 8 games is a binomial distribution with parameters $(n = 8, p = 0.5)$ *so the probability of winning 5 games is* $\binom{8}{5}0.5^8 = 0.21875$.

The probability of 3 wins in 4 games is larger than the probability of 5 wins in 8 games.

Problem 5 (12pt). *Urn 1 contains one red ball and three blue balls. Urn 2 contains one red ball. A random ball is drawn from urn 1 and placed into urn 2. Then one ball is drawn at random from urn 2 and it turns out red. What is the conditional probability that the ball remaining in the urn 2 is also red?*

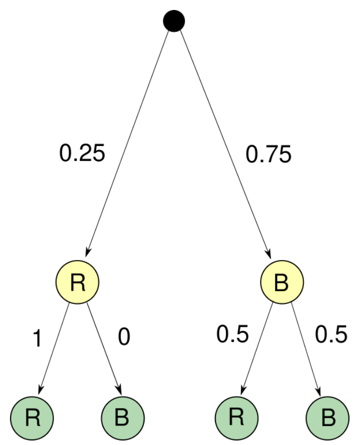
Solution. *Let* R_1 *denote the event if red ball was drawn from urn 1 and* R_2 *if red ball was drawn from urn 2. Same for blue balls. We want to compute* $P(R_1|R_2)$.

Then $P(R_1) = \frac{1}{4} = 0.15$ *and* $P(B_1) = \frac{3}{4} = 0.75$. *Then*

$$P(R_2|R_1) = 1, \quad P(R_2|B_1) = 0.5.$$

Then (easier to see if you draw the probability tree)

$$P(R_1|R_2) = \frac{P(R_2R_1)}{P(R_2)} = \frac{P(R_2R_1)}{P(R_2R_1) + P(R_2B_1)}$$
$$= \frac{P(R_2|R_1)P(R_1)}{P(R_2|R_1)P(R_1) + P(R_2|B_1)P(B_1)} = \frac{0.25}{1 \cdot 0.25 + 0.5 \cdot 0.75} = 0.4.$$



Problem 6 (12pt)**.** *Suppose the cdf of the random variable X is given by*

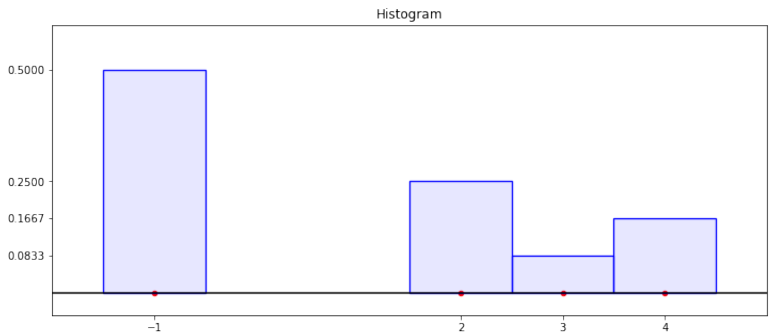
$$F(x) = \begin{cases} 0 & x < -1 \\ \frac{1}{2} & -1 \leq x < 2 \\ \frac{3}{4} & 2 \leq x < 3 \\ \frac{5}{6} & 3 \leq x < 4 \\ 1 & x \geq 4 \end{cases}.$$

(a) Draw the histogram of X.
(b) Find the mean and variance of X.

Solution. *(a) Note that* $\mathrm{Range}(X) = \{-1, 2, 3, 4\}$ *and the pmf is*

$$f(-1) = \frac{1}{2} - 0 = \frac{1}{2}, \quad f(2) = \frac{3}{4} - \frac{1}{2} = \frac{1}{4},$$

$$f(3) = \frac{5}{6} - \frac{3}{4} = \frac{1}{12}, \quad f(4) = 1 - \frac{5}{6} = \frac{1}{6}.$$



(b)

$$E[X] = \sum_{x \in \mathrm{Range}(X)} x f(x) = -1\frac{1}{2} + 2\frac{1}{4} + 3\frac{1}{12} + 4\frac{1}{6} = \frac{11}{12}.$$

$$E[X^2] = \sum_{x \in \mathrm{Range}(X)} x^2 f(x) = 1\frac{1}{2} + 4\frac{1}{4} + 9\frac{1}{12} + 16\frac{1}{6} = \frac{11}{12} = \frac{59}{12}.$$

$$\mathrm{Var} X = E[X^2] - E[X]^2 = \frac{587}{144} \approx 4.076.$$

Problem 7 (10pt)**.** *A fair coin is flipped* 5 *times and the number of heads is recorded as* X_1*. The same coin is then flipped 6 times and the number of heads is recorder as* X_2*. Find the conditional probability*

$$P(X_1 = 2|X_1 + X_2 = 4).$$

Solution. *Notice that*

$$P(X_1 = 2|X_1 + X_2 = 4) = \frac{P(X_1 = 2)P(X_2 = 2)}{P(X_1 + X_2 = 4)}.$$

X_1 *has a binomial distribution with parameters* $(n = 5, p = 0.5)$, X_2 *has a binomial distribution with parameters* $(n = 6, p = 0.5)$ *and* $X_1 + X_2$ *has a binomial distribution with parameters* $(n = 11, p = 0.5)$*. Therefore*

$$P(X_1 = 2|X_1 + X_2 = 4) = \frac{\binom{5}{2}0.5^5\binom{6}{2}0.5^6}{\binom{11}{4}0.5^{11}} = \frac{150}{330} = \frac{5}{11} \approx 0.455.$$