Why is Q useful?

Consider the energy of the damped oscillator

$$E = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}kx^2 \text{ which will not be constant}$$

so $$\frac{dE}{dt} = m\dot{x}\ddot{x} + kx\dot{x} = (m\ddot{x} + kx)\dot{x}$$

Recall from the eom $m\ddot{x} + kx = -\lambda\dot{x}$

$$\therefore \frac{dE}{dt} = -\lambda\dot{x}^2,$$ energy always decreases due to damping (goes into heat)

Now, consider the energy lost by an underdamped oscillator. Assume $\theta = 0$ ∴ the mass was released from rest after an initial displacement of a.

$$x = a e^{-\gamma t}\cos\omega t$$

$$\rightarrow \dot{x} = -a e^{-\gamma t}(\gamma\cos\omega t + \omega\sin\omega t)$$

The energy lost by this oscillator over one cycle (time T) is

$$\Delta E = \int_0^T \dot{E}\,dt = \frac{1}{\omega}\int_0^{2\pi} \dot{E}\,d\phi \quad \text{since } \omega t = \phi$$

since $\dot{E} = -\lambda\dot{x}^2$

$$\Delta E = -\frac{\lambda a^2}{\omega}\int_0^{2\pi} e^{-2\gamma t}\left[\gamma^2\cos^2\phi + 2\gamma\omega\cos\phi\sin\phi + \omega^2\sin^2\phi\right]d\phi$$

$$\simeq -\frac{\lambda a^2}{\omega}\underbrace{e^{-2\gamma t}}_{\text{doesn't change much over 1 cycle if } \gamma \text{ is small}}\int_0^{2\pi}\left[\gamma^2\cos^2\phi + 2\gamma\omega\cos\phi\sin\phi + \omega^2\sin^2\phi\right]d\phi$$

$\rightarrow$ doesn't change much over 1 cycle if γ is small

$$= -\lambda a^2 \frac{T}{2\pi} e^{-2\gamma t}\left[\gamma^2+\omega^2\right]\pi = -\frac{\lambda a^2 T e^{-2\gamma t}}{2}\omega_0^2$$

$$= -\left(\frac{\lambda}{2m}\right) m\omega_0^2 a^2 T e^{-2\gamma t}. \quad \Delta E \text{ falls by } e^{-1}, \text{ in } t_0 = \frac{1}{2\gamma} = \frac{1}{2} t_{rel} = \frac{m}{\lambda}$$

$$\text{so,} \quad \Delta E = -\left(\frac{1}{2} m\omega_0^2 a^2 e^{-t/t_0}\right)\frac{T}{t_0}$$

( ) can be identified as the energy stored in the oscillator at time t (see earlier discussion of E in oscillators)

$$\therefore \frac{|\Delta E|}{E} = \frac{T}{t_0} = \frac{\left(\frac{2\pi}{\omega}\right)}{\left(m/\lambda\right)} = 2\pi\left(\frac{\lambda}{m\omega}\right) = \frac{2\pi}{Q} \quad \text{if } \omega \simeq \omega_0$$

$\therefore$ Q measures the fractional energy loss in a period of a weakly damped oscillator. Large Q, small energy loss.

Example Qs: Earthquake driven ground: 250-1400

Earthquake driven ground	250-1400
Piano string	3000
Excited atom	10^7
Neutron Star	10^{12}

Example: A car suspension system is critically damped & its period of free oscillations w/ no damping is 1s. If the system is initially displaced by an amount x_0 & released w/ no velocity, find the displacement at $t=1s$.

For critical damping $\gamma = \omega_0 = \frac{2\pi}{T_0}$

In this case $\tau_0 = 1\text{s}$, so $\gamma = 2\pi$

For critical damping $x = (a + bt)e^{-\gamma t}$

at $t=0$, $x = x_0 \rightarrow a = x_0$

$t=0 \quad \dot{x} = 0 = -\gamma a \cancel{e^{-\gamma t}}^{1} + b\cancel{e^{-\gamma t}}^{1} - \cancel{\gamma b t e^{-\gamma t}}^{0}$

$\therefore b = x_0 \gamma = 2\pi x_0$

$\therefore \; x = (x_0 + 2\pi x_0 t)e^{-2\pi t}$

$= x_0(1 + 2\pi t)e^{-2\pi t}$

$\therefore$ Displacement at 1s is $x = x_0(1+2\pi)e^{-2\pi} = 0.0136\, x_0$

ie, basically returned to equil.

Example: A 5×10^{-4} Kg mass is attached to a spring w/ $K = 0.05\,\text{Nm}^{-1}$ underwater. If $\lambda = 5\times10^{-5}\,\frac{\text{Ns}}{\text{m}^2}$, (a) find the # of oscillations that the mass makes in the time that the amplitude drops by a factor of 2 from its initial value, (b) the Q of the oscillator.

Change in amplitude for an underdamped oscillator

$$a(t) = a_0 e^{-\gamma t} = a_0 e^{-\frac{\lambda}{2m}t}$$

$$\rightarrow \quad \frac{1}{2}\cancel{a_0} = \cancel{a_0}\, e^{-\frac{\lambda}{2m}t_{1/2}}$$

$$e^{\frac{\lambda}{2m}t_{1/2}} = 2$$

$$\rightarrow t_{1/2} = \left(\frac{2m}{\lambda}\right)\ln 2 = (t_{rel})\ln 2$$

So, number of oscillations during this time is

$$\left(\frac{\omega}{2\pi}\right) t_{1/2} = \left(\frac{\omega}{2\pi}\right)\left(\frac{2m}{\lambda}\right) \ln 2 = n$$

$$\omega = \sqrt{\omega_0^2 - \gamma^2} \quad , \quad \omega_0^2 = \frac{k}{m} = 100\ s^{-2} \quad , \quad \gamma = \frac{\lambda}{2m} = 0.05\ s^{-1}$$

$$\therefore \quad \omega \simeq 10\ s^{-1} \simeq \omega_0$$

$$\therefore \quad n = \left(\frac{10\ s^{-1}}{2\pi}\right) 20\ s \cdot \ln 2 \simeq 22$$

$$\rightarrow \quad Q = \frac{\omega_0}{2\gamma} = \frac{(10\ s^{-1})}{(2 \cdot 0.05\ s^{-1})} = 100$$