Consider $F(t) = \sum_{n=-\infty}^{+\infty} F_n e^{in\omega t}$ where n runs over all integers

In this case, ω = fundamental angular frequency

$n\omega$ = its harmonic frequencies

Each term in the sum is unchanged when $t \to t + \frac{2\pi}{\omega} = t + \tau$

so $F(t)$ is periodic

Fourier analysis says that <u>any</u> periodic function can be expressed as an infinite sum of sines & cosines. To find the solution to the motion of an oscillator driven by $F(t)$ we just need to find the coefficients F_n.

Multiply our $F(t)$ by $e^{-im\omega t}$ & integrate over a period

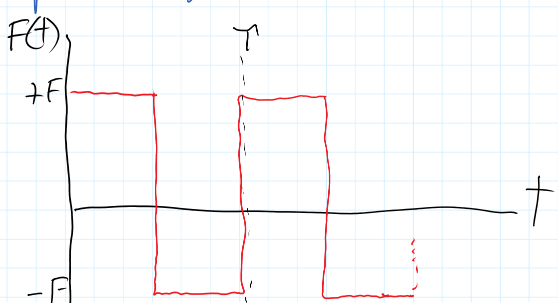
$$\int_0^\tau F(t) e^{-im\omega t} dt = \sum_{n=-\infty}^{+\infty} F_n \int_0^\tau e^{i(n-m)\omega t} dt$$

2 cases, $n \neq m$, the integrals are all $\int_0^\tau \sin x\omega t\,dt \overset{=0}{}$ or $\int_0^\tau \cos x\omega t\,dt \overset{=0}{}$

$$n = m \quad F_m \tau = \int_0^\tau F(t) e^{-im\omega t} dt$$

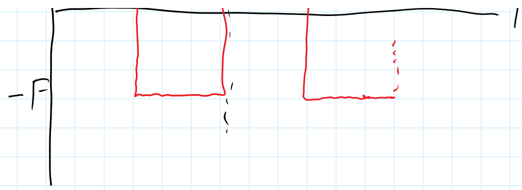
$$\text{or} \quad F_m = \frac{1}{\tau} \int_0^\tau F(t) e^{-im\omega t} dt$$

Example: Square-wave force



Find the position after a long-time of an oscillator subject to a square-wave force.

$$F(t) = \begin{cases} F & r\tau < t < (r+\tfrac{1}{2})\tau \end{cases}$$



$$F(t) = \begin{cases} F & r\tau < t < (r+\tfrac{1}{2})\tau \\ -F & (r-\tfrac{1}{2})\tau < t < r\tau \end{cases}$$

Since $\omega = \dfrac{2\pi}{\tau}$

$$F_n = \frac{F}{\tau}\int_0^{\tau/2} e^{-2\pi i n t/\tau}\,dt - \frac{F}{\tau}\int_{\tau/2}^{\tau} e^{-2\pi i n t/\tau}\,dt$$

$$= \frac{F}{\tau}\left(\frac{-\tau}{2\pi n i} e^{-2\pi i n t/\tau}\right]_0^{\tau/2} - \frac{F}{\tau}\left(\frac{-\tau}{2\pi n i} e^{-2\pi i n t/\tau}\right]_{\tau/2}^{\tau}$$

$$= \frac{F}{\tau}\left(\frac{-\tau e^{-\pi i n}}{2\pi n i} + \frac{\tau}{2\pi n i}\right) - \frac{F}{\tau}\left(\frac{-\tau e^{-2\pi i n}}{2\pi n i} + \frac{\tau e^{-\pi i n}}{2\pi n i}\right)$$

$$= \frac{F}{2\pi n i} - \frac{F e^{-\pi i n}}{2\pi n i} + \frac{F e^{-2\pi i n}}{2\pi n i} - \frac{F e^{-\pi i n}}{2\pi n i}$$

$$= \frac{F}{2\pi n i} - \frac{F e^{-\pi i n}}{\pi n i} + \frac{F e^{-2\pi i n}}{2\pi n i} \qquad e^{-2\pi i n} \rightarrow (\cos(2\pi n) - i\sin(2\pi n)) = 1$$

$$= \frac{F}{\pi n i}\left(1 - e^{-\pi i n}\right) \qquad e^{-\pi i n} \rightarrow (\cos(\pi n) - i\,\cancel{\sin(\pi n)}^{\,0}), \quad \cos(\pi n) \rightarrow \begin{cases} 1, & n \text{ even} \\ -1, & n \text{ odd} \end{cases}$$

$$= \frac{F}{\pi n i}\left(1 - (-1)^n\right)$$

For even values of n, $F_n = 0$, so only odd values of n remain

$$\therefore\ F_n = \frac{2F}{i\pi n} \quad (n \text{ odd})$$

and
$$A_n = \frac{2F}{i\pi n \underbrace{m}_{\text{mass}}(\omega_0^2 - n^2\omega^2 + 2in\gamma\omega)} \quad (n \text{ odd})$$

$$i\pi n m(\omega_0^2 - n^2\omega^2 + i n \gamma \omega)$$

$\hookrightarrow$ mass

so $$x = \sum_{r=-\infty}^{\infty} A_{2r+1} e^{i(2r+1)\omega t} + \text{transient}$$

In general, amplitudes decrease rapidly w/ n, so only the 1st few terms are usually needed