Back to our expression for potential w/ a continuous mass distribution

$$\Phi(\vec{r}) = -G\int \frac{\rho(\vec{r}\,')\,dV}{|\vec{r}-\vec{r}\,'|}$$

Take gradient of both sides:

$$\vec{\nabla}\Phi(\vec{r}) = -G\int \rho(\vec{r}\,')\,dV\left[\vec{\nabla}\left(\frac{1}{|\vec{r}-\vec{r}\,'|}\right)\right]$$

This is a gradient w/ respect to the unprimed coordinates

$$\frac{1}{|\vec{r}-\vec{r}\,'|} = \frac{1}{\left((x-x')^2+(y-y')^2+(z-z')^2\right)^{1/2}}$$

So, $$\frac{\partial}{\partial x}\frac{1}{|\vec{r}-\vec{r}\,'|} = \frac{-\frac{1}{2}\,2\,(x-x')}{\left((x-x')^2+(y-y')^2+(z-z')^2\right)^{3/2}}$$

$$\therefore\ \vec{\nabla}\left(\frac{1}{|\vec{r}-\vec{r}\,'|}\right) = \frac{-1}{\left[(x-x')^2+(y-y')^2+(z-z')^2\right]^{3/2}}\left((x-x')\hat{i}+(y-y')\hat{j}+(z-z')\hat{k}\right)$$

$$= \frac{-(\vec{r}-\vec{r}\,')}{|\vec{r}-\vec{r}\,'|^3}$$

$$\therefore\ \vec{\nabla}\Phi(\vec{r}) = G\int \frac{\rho(\vec{r}\,')(\vec{r}-\vec{r}\,')\,dV}{|\vec{r}-\vec{r}\,'|^3}$$

now, $\vec{g}$, the grav. field is

$$\vec{g} = -\vec{\nabla}\Phi(\vec{r}) = -G\int \rho(\vec{r}\,')(\vec{r}-\vec{r}\,')\,dV$$

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Now, take the divergence of this equation

$$\vec{\nabla}\cdot\vec{g}(\vec{r}) = -G\int \rho\, \vec{\nabla}\cdot\left(\frac{(\vec{r}-\vec{r}\,')}{|\vec{r}-\vec{r}\,'|^3}\right)dV$$

Can be shown (e.g. §1.5 in Griffith's Intro. to Electrodynamics) that

$$\vec{\nabla}\cdot\left(\frac{(\vec{r}-\vec{r}\,')}{|\vec{r}-\vec{r}\,'|^3}\right) = 4\pi\,\delta^3(\vec{r}-\vec{r}\,') \quad \text{where} \quad \delta^3(\vec{r}-\vec{r}\,') = \begin{cases} 0 & \vec{r}\neq\vec{r}\,' \\ 1 & \vec{r}=\vec{r}\,' \end{cases}$$

$$\therefore\ \vec{\nabla}\cdot\vec{g} = -4\pi G\rho(\vec{r})$$

but since $\vec{g} = -\vec{\nabla}\Phi$, we get

$$\boxed{\vec{\nabla}^2\Phi = 4\pi G\rho} \quad \text{Poisson's Equation}$$

Given a $\rho(\vec{r})$ & a boundary condition, this eqn can be solved for $\Phi(\vec{r})$. For an isolated system, the boundary condition is $\Phi(\vec{r}) \to 0$ as $|\vec{r}| \to \infty$. Poisson's eqn provides a route to Φ & then to $\vec{g}$ that is more convienent then directly integrating.

(In the special case $\rho = 0$, get Laplace's eqn $\nabla^2\Phi = 0$)

Integrate both sides of Poisson's Eqn over an arbitrary volume:

$$\int \vec{\nabla}^2\Phi\, dV = 4\pi G\int \rho\, dV$$

$$\int \vec{\nabla}\cdot(\vec{\nabla}\Phi)\,dV = 4\pi G M_{total}$$

$$\int \vec{\nabla}\cdot(\vec{\nabla}\Phi)\,dV = 4\pi G M_{total}$$

$$-\int \vec{\nabla}\cdot\vec{g}\,dV = 4\pi G M_{total}$$

div. thm

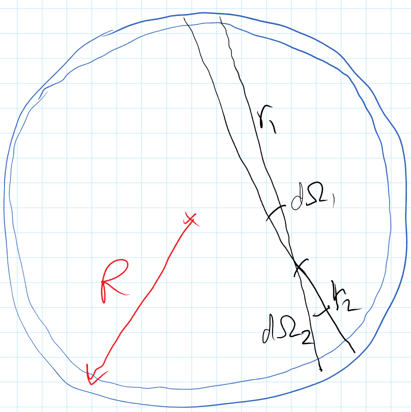
$$-\oint \vec{g}\cdot d\vec{S} = 4\pi G M_{total}$$

$\therefore$ The integral of the grav. field over any closed surface is proportional to the mass enclosed within that surface. (Gauss's Thm.)

Properties of Spherical Systems

Newton proved 2 theorems that enable us to calculate Φ of any spherically symmetric dist'n of matter easily

Newton's 1st Theorem: A body that is inside a spherical shell of matter experiences no net force from that shell.



As the area subtended by a cone grows as r^2

$d\Omega = \dfrac{dA}{r^2}$ is fixed. Then the mass subtended

by cone 1 & 2 is $\delta m_1 = d\Omega\, r_1^2\sigma$; $\delta m_2 = d\Omega\, r_2^2\sigma$

Take the ratio $\delta m_1/\delta m_2 = \left(\dfrac{r_1}{r_2}\right)^2$ or $\left(\dfrac{\delta m_1}{r_1^2}\right) = \left(\dfrac{\delta m_2}{r_2^2}\right)$

So any particle placed at 'x' will feel the same force from both sides of the shell. Summing over all cones centered on 'x' one concludes that the body experiences no net force from the shell.

Corollary: Since $\vec{g}=0$, $\Rightarrow \vec{\nabla}\Phi = 0$ so the grav. potential is a constant. Thus, we can evaluate the potential $\Phi(\vec{r})$ inside the shell by calculating $\Phi(\vec{r}) = -G\int \dfrac{\rho(\vec{r}')\,dV}{|\vec{r}-\vec{r}'|}$ at any point. Pick shell by calculating $\Phi(\vec{r}) = -G\int \frac{\rho(\vec{r}')dV}{|\vec{r}-\vec{r}'|}$ at any point. Pick the center, then $|\vec{r}-\vec{r}'| = R \Rightarrow \Phi(\vec{r}) = -\frac{GM}{R}$.

Newton's 2nd Theorem. The grav. force on a body that lives outside a spherical shell of matter is the same as it would be if all the shell's matter were concentrated into a point at its center.

Proof is above.