Week 9: Conditional distributions, continuous bivariate distributions

Armenak Petrosyan

Tuesday class

Conditional pmf

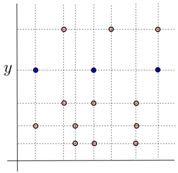
- X, Y be discrete random variables with joint pmf $f(x, y)$.

Definition

Let $f_Y(y) > 0$. The **conditional pmf of** X given $Y = y$ is the following pmf:

$$g(x|y) = \frac{f(x,y)}{f_Y(y)}, \quad x \in \mathrm{Range}(X)$$

- Notice that
$$g(x|y) = P(X = x|Y = y).$$



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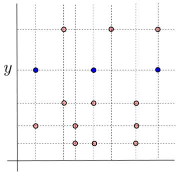

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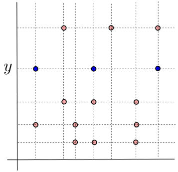

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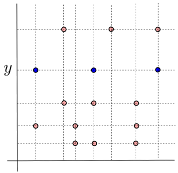

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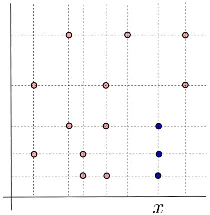

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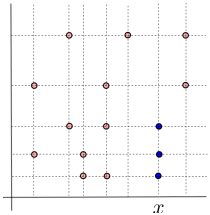

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Exercise 1

Problem (4.3-1 in the textbook)

Let X and Y have the joint pmf

$$f(x,y) = \frac{x+y}{32} \quad x = 1,2 \quad y = 1,2,3,4.$$

(a) *Display the joint pmf and the marginal pmfs on a graph.*
(b) *Find $g(x|y)$ and draw a figure depicting the conditional pmfs for $y = 1,2,3, and 4$.*
(c) *Find $h(y|x)$ and draw a figure depicting the conditional pmfs for $x = 1,2$.*
(d) *Find $P(1 \leq Y \leq 3|X = 1)$, $P(Y \leq 2|X = 2)$, and $P(X = 2|Y = 3)$.*
(e) *Find $E(Y|X = 1)$ and $\mathrm{Var}(Y|X = 1)$.*

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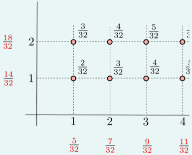

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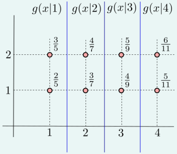

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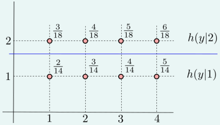

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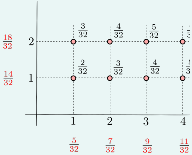

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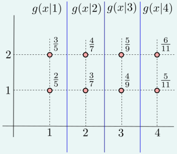

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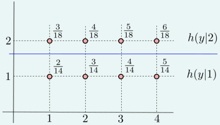

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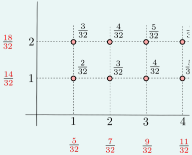

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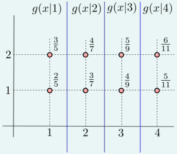

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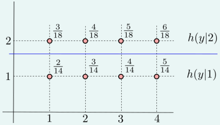

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(d) *From the graph in (c)*

$$P(1 \le Y \le 3|X=1) = \frac{2}{14} + \frac{3}{14} + \frac{4}{14} \boxed{= \frac{9}{14}}.$$

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From the graph in (b)

$$P(X=2|Y=3) \boxed{= \frac{5}{9}}.$$

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$$E(Y|X=1) = 1 \cdot \frac{2}{14} + 2 \cdot \frac{3}{14} + 3 \cdot \frac{4}{14} + 4 \cdot \frac{5}{14} \boxed{= \frac{40}{14}}.$$

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And, finally,

$$\mathrm{Var}(Y|X=1) = \frac{130}{14} - \left(\frac{40}{14}\right)^2 \boxed{= \frac{220}{196}}.$$

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Definition

$$\mu_{Y|x} = E[Y|X = x] = \sum_{y \in \mathrm{Range}(Y)} y h(y|x)$$

is called **conditional mean** of Y given $X = x$.

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$$\sigma^2_{Y|x} = E[(Y - \mu_{Y|x})^2 | X = x] = \sum_{y \in \mathrm{Range}(Y)} [(y - \mu_{Y|x})^2 h(y|x)$$

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- The $\mu_{X|y}, \sigma_{X|y}$ are defined in a similar way.

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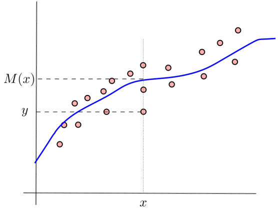
- Typically there is no function $m : \mathrm{Range}(X) \to \mathbb{R}$ so that $y = m(x)$ for any $(x, y) \in \mathrm{Range}(X, Y)$.
- For the same x there can be multiple y-s with $(x, y) \in \mathrm{Range}(X, Y)$.

Example

Represent the weights of students in class as function of their heights.

- Conditional mean
$$m(x) = E[Y|X = x]$$
is the "best" function representing y in terms of x.
- $\sigma_{Y|x}$ measures how much Y varies from $m(x)$ when $X = x$.



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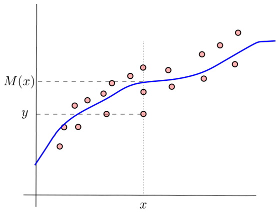

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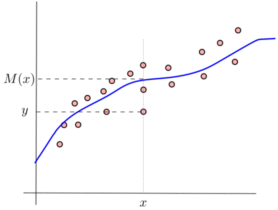

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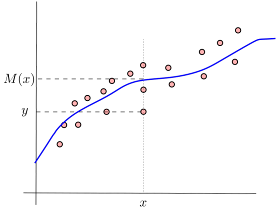

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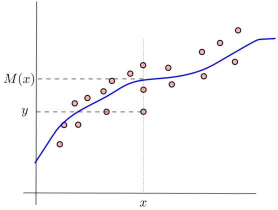

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Trinomial distribution case

- Consider the trinomial distribution with parameters (n, P_S, P_I, p_F)
- $p_S + p_I + p_F = 1.$
- $\mathrm{Range}(X, Y) = \{(x, y) : x + y \leq n\}.$
- $f(x,y) = \binom{n}{x}\binom{n-x}{y} p_S^x p_I^y (1 - p_S - p_I)^{n-x-y}.$
- $f_X(x) = \binom{n}{x} p_S^x (1 - p_S)^{n-x}.$

$$
\begin{aligned}
h(y|x) = \frac{f(x,y)}{f_X(x)} &= \binom{n-x}{y} \frac{p_I^y (1 - p_S - p_I)^{n-x-y}}{(1-p_S)^{n-x}} \\
&= \binom{n-x}{y} \left(\frac{p_I}{1-p_S}\right)^y \left(\frac{1 - p_S - p_I}{1-p_S}\right)^{n-x-y} \\
&= \binom{n-x}{y} \left(\frac{p_I}{1-p_S}\right)^y \left(1 - \frac{p_I}{1-p_S}\right)^{n-x-y}.
\end{aligned}
$$

- Let $p = \frac{p_I}{1-p_S}$, then

$$h(y|x) = \binom{n-x}{y} p^y (1-p)^{n-x-y}.$$

- The conditional pmf is a Binomial distribution with parameters $(n - x, p)$.

Trinomial distribution case

- Consider the trinomial distribution with parameters (n, P_S, P_I, p_F)
- $p_S + p_I + p_F = 1$.
- $\text{Range}(X, Y) = \{(x, y) : x + y \leq n\}$.
- $f(x, y) = \binom{n}{x}\binom{n-x}{y} p_S^x p_I^y (1 - p_S - p_I)^{n-x-y}$.
- $f_X(x) = \binom{n}{x} p_S^x (1 - p_S)^{n-x}$.

$$
\begin{aligned}
h(y|x) = \frac{f(x, y)}{f_X(x)} &= \binom{n-x}{y} \frac{p_I^y (1 - p_S - p_I)^{n-x-y}}{(1 - p_S)^{n-x}} \\
&= \binom{n-x}{y} \left(\frac{p_I}{1 - p_S}\right)^y \left(\frac{1 - p_S - p_I}{1 - p_S}\right)^{n-x-y} \\
&= \binom{n-x}{y} \left(\frac{p_I}{1 - p_S}\right)^y \left(1 - \frac{p_I}{1 - p_S}\right)^{n-x-y}.
\end{aligned}
$$

- Let $p = \frac{p_I}{1-p_S}$, then

$$
h(y|x) = \binom{n-x}{y} p^y (1 - p)^{n-x-y}.
$$

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Trinomial distribution case

- Consider the trinomial distribution with parameters (n, P_S, P_I, p_F)
- $p_S + p_I + p_F = 1$.
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- $f_X(x) = \binom{n}{x} p_S^x (1 - p_S)^{n-x}$.

$$
\begin{aligned}
h(y|x) = \frac{f(x,y)}{f_X(x)} &= \binom{n-x}{y} \frac{p_I^y (1 - p_S - p_I)^{n-x-y}}{(1 - p_S)^{n-x}} \\
&= \binom{n-x}{y} \left(\frac{p_I}{1 - p_S}\right)^y \left(\frac{1 - p_S - p_I}{1 - p_S}\right)^{n-x-y} \\
&= \binom{n-x}{y} \left(\frac{p_I}{1 - p_S}\right)^y \left(1 - \frac{p_I}{1 - p_S}\right)^{n-x-y}.
\end{aligned}
$$

- Let $p = \frac{p_I}{1 - p_S}$, then

$$h(y|x) = \binom{n-x}{y} p^y (1 - p)^{n-x-y}.$$

- The conditional pmf is a Binomial distribution with parameters $(n - x, p)$.

Trinomial distribution case

- Consider the trinomial distribution with parameters (n, P_S, P_I, p_F)
- $p_S + p_I + p_F = 1$.
- $\text{Range}(X, Y) = \{(x, y) : x + y \leq n\}$.
- $f(x, y) = \binom{n}{x}\binom{n-x}{y} p_S^x p_I^y (1 - p_S - p_I)^{n-x-y}$.
- $f_X(x) = \binom{n}{x} p_S^x (1 - p_S)^{n-x}$.

$$\begin{aligned} h(y|x) = \frac{f(x,y)}{f_X(x)} &= \binom{n-x}{y} \frac{p_I^y (1 - p_S - p_I)^{n-x-y}}{(1 - p_S)^{n-x}} \\ &= \binom{n-x}{y} \left(\frac{p_I}{1 - p_S}\right)^y \left(\frac{1 - p_S - p_I}{1 - p_S}\right)^{n-x-y} \\ &= \binom{n-x}{y} \left(\frac{p_I}{1 - p_S}\right)^y \left(1 - \frac{p_I}{1 - p_S}\right)^{n-x-y}. \end{aligned}$$

- Let $p = \frac{p_I}{1 - p_S}$, then

$$h(y|x) = \binom{n-x}{y} p^y (1 - p)^{n-x-y}.$$

- The conditional pmf is a Binomial distribution with parameters $(n - x, p)$.

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- $f_X(x) = \binom{n}{x} p_S^x (1 - p_S)^{n-x}.$

$$
\begin{aligned}
h(y|x) = \frac{f(x,y)}{f_X(x)} &= \binom{n-x}{y} \frac{p_I^y (1 - p_S - p_I)^{n-x-y}}{(1-p_S)^{n-x}} \\
&= \binom{n-x}{y} \left(\frac{p_I}{1-p_S}\right)^y \left(\frac{1-p_S-p_I}{1-p_S}\right)^{n-x-y} \\
&= \binom{n-x}{y} \left(\frac{p_I}{1-p_S}\right)^y \left(1 - \frac{p_I}{1-p_S}\right)^{n-x-y}.
\end{aligned}
$$

- Let $p = \frac{p_I}{1-p_S}$, then

$$h(y|x) = \binom{n-x}{y} p^y (1-p)^{n-x-y}.$$

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Trinomial distribution case

- Consider the trinomial distribution with parameters (n, P_S, P_I, p_F)
- $p_S + p_I + p_F = 1$.
- $\text{Range}(X, Y) = \{(x, y) : x + y \leq n\}$.
- $f(x, y) = \binom{n}{x}\binom{n-x}{y} p_S^x p_I^y (1 - p_S - p_I)^{n-x-y}$.
- $f_X(x) = \binom{n}{x} p_S^x (1 - p_S)^{n-x}$.

$$h(y|x) = \frac{f(x, y)}{f_X(x)} = \binom{n-x}{y} \frac{p_I^y (1 - p_S - p_I)^{n-x-y}}{(1 - p_S)^{n-x}}$$

$$= \binom{n-x}{y} \left(\frac{p_I}{1 - p_S}\right)^y \left(\frac{1 - p_S - p_I}{1 - p_S}\right)^{n-x-y}$$

$$= \binom{n-x}{y} \left(\frac{p_I}{1 - p_S}\right)^y \left(1 - \frac{p_I}{1 - p_S}\right)^{n-x-y}.$$

- Let $p = \frac{p_I}{1-p_S}$, then

$$h(y|x) = \binom{n-x}{y} p^y (1 - p)^{n-x-y}.$$

- The conditional pmf is a Binomial distribution with parameters $(n - x, p)$.

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- Consider the trinomial distribution with parameters (n, P_S, P_I, p_F)
- $p_S + p_I + p_F = 1$.
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- $f(x, y) = \binom{n}{x}\binom{n-x}{y} p_S^x p_I^y (1 - p_S - p_I)^{n-x-y}$.
- $f_X(x) = \binom{n}{x} p_S^x (1 - p_S)^{n-x}$.

$$h(y|x) = \frac{f(x,y)}{f_X(x)} = \binom{n-x}{y} \frac{p_I^y (1 - p_S - p_I)^{n-x-y}}{(1 - p_S)^{n-x}}$$
$$= \binom{n-x}{y} \left(\frac{p_I}{1 - p_S}\right)^y \left(\frac{1 - p_S - p_I}{1 - p_S}\right)^{n-x-y}$$
$$= \binom{n-x}{y} \left(\frac{p_I}{1 - p_S}\right)^y \left(1 - \frac{p_I}{1 - p_S}\right)^{n-x-y}.$$

- Let $p = \frac{p_I}{1-p_S}$, then

$$h(y|x) = \binom{n-x}{y} p^y (1-p)^{n-x-y}.$$

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- $f_X(x) = \binom{n}{x} p_S^x (1 - p_S)^{n-x}.$

$$
\begin{aligned}
h(y|x) = \frac{f(x, y)}{f_X(x)} &= \binom{n-x}{y} \frac{p_I^y (1 - p_S - p_I)^{n-x-y}}{(1 - p_S)^{n-x}} \\
&= \binom{n-x}{y} \left(\frac{p_I}{1 - p_S}\right)^y \left(\frac{1 - p_S - p_I}{1 - p_S}\right)^{n-x-y} \\
&= \binom{n-x}{y} \left(\frac{p_I}{1 - p_S}\right)^y \left(1 - \frac{p_I}{1 - p_S}\right)^{n-x-y}.
\end{aligned}
$$

- Let $p = \frac{p_I}{1 - p_S}$, then

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&= \binom{n-x}{y} \left(\frac{p_I}{1-p_S}\right)^y \left(\frac{1 - p_S - p_I}{1 - p_S}\right)^{n-x-y} \\
&= \binom{n-x}{y} \left(\frac{p_I}{1-p_S}\right)^y \left(1 - \frac{p_I}{1-p_S}\right)^{n-x-y}.
\end{aligned}$$

- Let $p = \frac{p_I}{1-p_S}$, then

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\end{aligned}
$$

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$$
h(y|x) = \binom{n-x}{y} p^y (1-p)^{n-x-y}.
$$

- The conditional pmf is a Binomial distribution with parameters $(n - x, p)$.

Trinomial distribution case (cont)

- Using the formula for the mean and variance of the Binomial distribution with parameters $(n-x, p)$.

$$\mu_{Y|x} = (n-x)p = (n-x)\frac{p_I}{1-p_S}$$

$$\sigma^2_{Y|x} = (n-x)p(1-p) = (n-x)\frac{p_I p_F}{(1-p_S)^2}.$$

- In conclusion, $m(x) = \mu_{Y|x}$ is a linear function of x.
- Therefore the least squares line of the trinomial distribution with parameters (n, p_S, p_I) is

$$\boxed{y = (n-x)\frac{p_I}{1-p_S}.}$$

- The distribution is closest to linear when either $p_I \approx 0$ or $p_F \approx 0$ but not both.

Trinomial distribution case (cont)

- Using the formula for the mean and variance of the Binomial distribution with parameters $(n-x, p)$.

$$\mu_{Y|x} = (n-x)p = (n-x)\frac{p_I}{1-p_S}$$

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Trinomial distribution case (cont)

- Using the formula for the mean and variance of the Binomial distribution with parameters $(n - x, p)$.

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Trinomial distribution case (cont)

- Using the formula for the mean and variance of the Binomial distribution with parameters $(n-x,p)$.

$$\mu_{Y|x} = (n-x)p = (n-x)\frac{p_I}{1-p_S}$$
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- In conclusion, $m(x) = \mu_{Y|x}$ is a linear function of x.
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Trinomial distribution case (cont)

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$$\mu_{Y|x} = (n-x)p = (n-x)\frac{p_I}{1-p_S}$$
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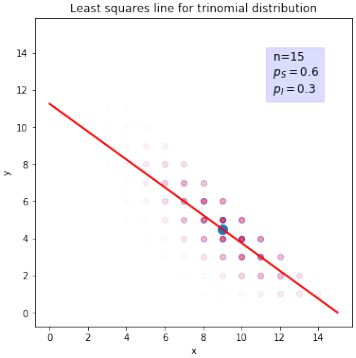
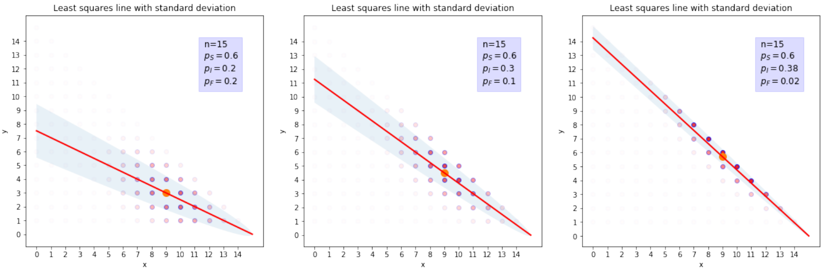
- In conclusion, $m(x) = \mu_{Y|x}$ is a linear function of x.
- Therefore the least squares line of the trinomial distribution with parameters (n, p_S, p_I) is

$$\boxed{y = (n-x)\frac{p_I}{1-p_S}.}$$

- The distribution is closest to linear when either $p_I \approx 0$ or $p_F \approx 0$ but not both.


Least squares line for trinomial distribution
n=15
$p_S = 0.6$
$p_I = 0.3$
y
x
0
2
4
6
8
10
12
14

Affect of conditional variance



- The shaded area is the region between

$$[m(x) - \sigma_{Y|x}, m(x) + \sigma_{Y|x}.]$$