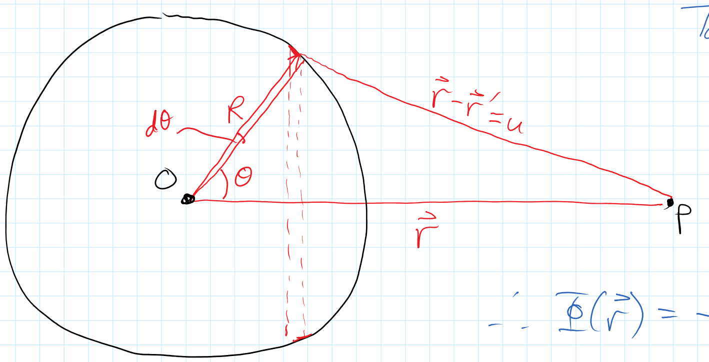
For a continuous mass distribution, the sum becomes an integra[l]

$$\Phi(\vec{r}) = -G\int \frac{dM}{|\vec{r}-\vec{r}\,'|}$$ where $\vec{r}\,'$ varies over the mass dist'n

e.g. $$\Phi(\vec{r}) = -G\int \frac{\rho(\vec{r}\,')\,dV}{|\vec{r}-\vec{r}\,'|}$$ for a mass density $\rho(\vec{r}\,')$

Examples: Find the potential function for a uniform spherical shell.



Take $r \gg R$. Circumference of a ring element $2\pi R\sin\theta$, and its mass

$dM = 2\pi R\sin\theta\, R\,d\theta\,\sigma$ where

σ = mass per unit area

$$\therefore \Phi(\vec{r}) = -G\int \frac{2\pi R^2 \sin\theta\,\sigma\,d\theta}{|\vec{r}-\vec{r}\,'|} = -2\pi R^2 G\sigma \int \frac{\sin\theta\,d\theta}{u}$$

From the triangle: $r^2 + R^2 - 2rR\cos\theta = u^2$

$$\rightarrow \quad rR\sin\theta\,d\theta = u\,du$$

$$\sin\theta\,d\theta = \frac{u\,du}{rR}$$

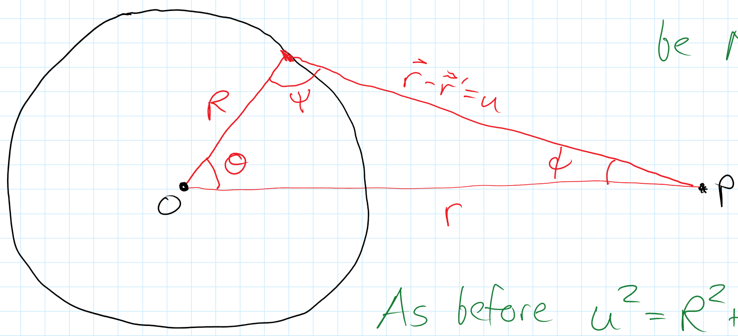
$$\therefore \Phi(\vec{r}) = -2\pi G \cancel{R^2}\sigma \int \frac{\cancel{u}\,du}{r\cancel{R}\cancel{u}} = -\frac{2\pi G R\sigma}{r}\int_{r-R}^{r+R} du$$

$$\Phi(\vec{r}) = -\frac{2\pi G R\sigma}{r}(\cancel{r}+R-\cancel{r}+R) = -\frac{4\pi R^2\sigma G}{r} = -\frac{GM}{r}$$

where M is the mass of the entire shell. This is the same as the potential of a particle of mass M located at the origin. $\therefore$ The grav. field (ie, the grav. force) outside the shell is the same as if the entire mass was concentrated at its center.

Exercise: Show the potential inside the shell is constant.

Example: Find the potential function & the gravitational field in the plane of a thin ring.



Let the radius be R & total mass be M. μ = mass per unit length

$$\therefore \; \Phi(\vec{r}) = -G\int \frac{dM}{u} = -G\int_0^{2\pi} \frac{\mu R\, d\theta}{u}$$

As before $u^2 = R^2 + r^2 - 2Rr\cos\theta$

$$\therefore \; \Phi = -2\mu G R \int_0^{\pi} \frac{d\theta}{(r^2+R^2-2Rr\cos\theta)^{1/2}} = \frac{-2\mu G R}{r}\int_0^{\pi} \frac{d\theta}{\left(1+\frac{R^2}{r^2}-\frac{2R}{r}\cos\theta\right)^{1/2}}$$

Assume $r >> R$ & expand integrand in a power-series of $x = \frac{R}{r}$ making certain to keep terms of x^2 & lower [NB: $(1+x)^{-1/2} \simeq 1 - \frac{1}{2}x + \frac{3}{8}x^2 - \frac{5}{16}x^3 + \cdots, \; x<1$]

$$\Phi = -2x\mu G \int_0^{\pi} \left[\left(1-\tfrac{1}{2}x^2 + x\cos\theta\right) + \tfrac{3}{8}\left(x^2 - 2x\cos\theta\right)^2 + \cdots\right] d\theta$$

$\int$'s to zero

$$= -2x\mu G \int_0^{\pi} \left(1 - \tfrac{1}{2}x^2 + \cancel{x\cos\theta} + \tfrac{3}{2}x^2\cos^2\theta + \cdots\right) d\theta$$

$$= -2x\mu G\left(\pi - \tfrac{1}{2}x^2\pi + \tfrac{3}{2}x^2 \underbrace{\int_0^{\pi}\cos^2\theta\, d\theta}_{\frac{\pi}{2}}\right)$$

$$= -2x\mu G\left(\pi - \frac{\pi}{2}x^2 + \frac{3}{4}\pi x^2\right) = -2x\mu G\left(\pi + \tfrac{1}{4}\pi x^2\right)$$

$$\therefore \; \Phi = -\frac{2\pi R\mu G}{r}\left(1 + \frac{R^2}{4r^2}\right)$$

but $M = 2\pi R\mu$, so $\Phi = -\frac{GM}{r}\left(1+\frac{R^2}{4r^2}\right)$

So, the grav. field is

$$\vec{g} = -\vec{\nabla}\Phi = -\frac{\partial \Phi}{\partial r}\hat{r} - \frac{\partial \Phi}{\partial \theta}\hat{\theta}$$

$$= -\frac{\partial}{\partial r}\left(-\frac{GM}{r} - \frac{GMR^2}{4r^3}\right)\hat{r}$$

$$= -\frac{GM}{r^2} - \frac{3}{4}\frac{GMR^2}{r^4}\hat{r}$$

$$= -\frac{GM}{r^2}\left(1 + \frac{3}{4}\left(\frac{R}{r}\right)^2\right)\hat{r}$$

as $r \to \infty$, this approaches field of a point mass.

For a point near the center of the ring, use $r < R$ & expand in powers of $\frac{r}{R}$. Show that $\Phi = -\frac{GM}{R}\left(1 + \frac{r^2}{4R^2}\right)$ & $\vec{g} = \frac{GM}{2R^3} r\,\hat{r}$ (what does this mean?)