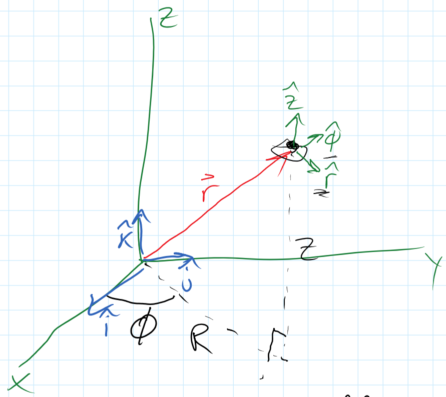
Cylindrical Coordinates - needed for 3D motion

(R, ϕ, z)



$\vec{r} = \underline{R\hat{r}} + z\hat{z}$

From this diagram

$\hat{r} = \cos\phi\,\hat{i} + \sin\phi\,\hat{j}$

$\hat{\phi} = -\sin\phi\,\hat{i} + \cos\phi\,\hat{j}$

$\hat{z} = \hat{k}$

Differentiate these wrt time ($\hat{i}, \hat{j}, \hat{k}$ are fixed)

$$\dot{\hat{r}} = \left(-\sin\phi\,\hat{i} + \cos\phi\,\hat{j}\right)\dot{\phi} = \dot{\phi}\,\hat{\phi}$$

$$\dot{\hat{\phi}} \;\; -\dot{\phi}\,\hat{r}\,, \quad \dot{\hat{z}} = 0$$

if $\vec{r} = R\hat{r} + z\hat{z}$, then $\dot{\vec{r}} = \dot{R}\hat{r} + R\dot{\hat{r}} + \dot{z}\hat{z}$

$$= \dot{R}\hat{r} + R\dot{\phi}\hat{\phi} + \dot{z}\hat{z}$$

and $\ddot{\vec{r}} = \left(\ddot{R} - R\dot{\phi}^2\right)\hat{r} + \left(2\dot{R}\dot{\phi} + R\ddot{\phi}\right)\hat{\phi} + \ddot{z}\hat{z}$ verify!

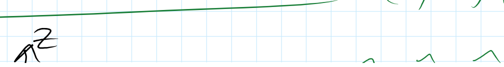
Ex: A butterfly is flying w/ $R = $ constant, $\phi = \omega t$, $z = \alpha t$ where ω & α are constants. Find $\dot{\vec{r}}$ & $\ddot{\vec{r}}$ in cyl. coords.

Take derivatives, $\dot{R} = \ddot{R} = 0$, $\dot{\phi} = \omega$ & $\ddot{\phi} = 0$, $\dot{z} = \alpha$ & $\ddot{z} = 0$

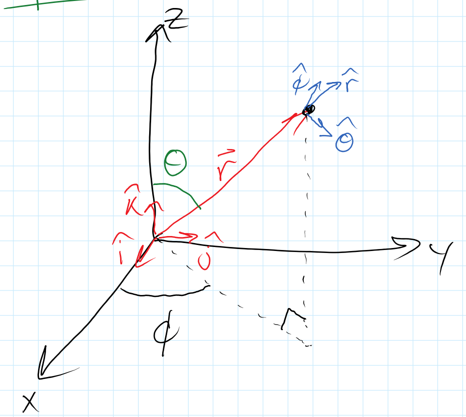
$$\therefore \quad \dot{\vec{r}} = R\omega\hat{\phi} + \alpha\hat{z} \quad \& \quad \ddot{\vec{r}} = -R\omega^2\hat{r}$$

What is trajectory? Helix

Spherical Coordinates (r, θ, ϕ)



Spherical Coordinates (r, θ, ϕ)



$\hat{r}, \hat{\theta}, \hat{\phi}$ are all orthogonal unit vectors

Position vector of particle $\vec{r} = r\hat{r}$

From diagram

$$\hat{r} = \sin\theta\cos\phi\,\hat{i} + \sin\theta\sin\phi\,\hat{j} + \cos\theta\,\hat{k}$$
$$\hat{\theta} = \cos\theta\cos\phi\,\hat{i} + \cos\theta\sin\phi\,\hat{j} - \sin\theta\,\hat{k}$$
$$\hat{\phi} = -\sin\phi\,\hat{i} + \cos\phi\,\hat{j}$$

Differtiating wrt time:

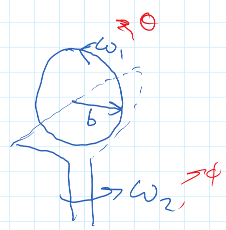
$$\left.\begin{aligned}\dot{\hat{r}} &= \dot{\theta}\,\hat{\theta} + \dot{\phi}\sin\theta\,\hat{\phi}\\ \dot{\hat{\theta}} &= -\dot{\theta}\,\hat{r} + \dot{\phi}\cos\theta\,\hat{\phi}\\ \dot{\hat{\phi}} &= -\dot{\phi}\sin\theta\,\hat{r} - \dot{\phi}\cos\theta\,\hat{\theta}\end{aligned}\right\}\ \text{verify!}$$

So, $\vec{r} = r\hat{r}$

$$\dot{\vec{r}} = \dot{r}\,\hat{r} + r\dot{\theta}\,\hat{\theta} + r\sin\theta\,\dot{\phi}\,\hat{\phi}$$

$$\ddot{\vec{r}} = \left(\ddot{r} - r\dot{\theta}^2 - r(\sin^2\theta)\dot{\phi}^2\right)\hat{r} + \left(r\ddot{\theta} + 2\dot{r}\dot{\theta} - r\sin\theta\cos\theta\,\dot{\phi}^2\right)\hat{\theta}$$
$$+ \left(r\sin\theta\,\ddot{\phi} + 2\dot{r}\dot{\phi}\sin\theta + 2r\cos\theta\,\dot{\theta}\dot{\phi}\right)\hat{\phi}$$

Ex: A wheel of radius b rotates w/ constant angular speed ω_1 about its axis which, in turn, rotates w/ const. ang. speed ω_2 about a vertical axis so that the axis of the wheel stays in the horiz plane ; the center of the wheel is motionless



Use sph. coord's to find the acc'n of any point on the rim of the wheel.

$$\therefore\ r = b,\quad \theta = \omega_1 t,\quad \phi = \omega_2 t$$

$$\dot{r} = \ddot{r} = 0, \quad \dot{\theta} = \omega_1, \quad \ddot{\theta} = 0, \quad \dot{\phi} = \omega_2, \quad \ddot{\phi} = 0$$

$$\therefore \quad \ddot{\vec{r}} = (-b\omega_2^2 \sin^2\theta - b\omega_1^2)\,\hat{r} - (b\omega_2^2 \sin\theta\cos\theta)\,\hat{\theta} + (2b\omega_1\omega_2\cos\theta)\,\hat{\phi}$$

The point at the top of the wheel has $\theta = 0$

$$\ddot{\vec{r}} = \underbrace{-b\omega_1^2\,\hat{r}}_{\text{centripetal acc'n}} + \underbrace{2b\omega_1\omega_2\,\hat{\phi}}_{\text{transv. acc'n perp. to plane of wheel.}}$$

Reminder about vector derivatives

→ the derivative of the mag. of $\vec{r}$, $\frac{d|\vec{r}|}{dt}$, is <u>not</u> the same as the derivative $\left|\frac{d\vec{r}}{dt}\right|$ (e.g. particle moving in a circle)

ex: Take the time derivative of both sides of $\vec{v}\cdot\vec{v} = v^2$

$$\frac{d\vec{v}}{dt}\cdot\vec{v} + \vec{v}\cdot\frac{d\vec{v}}{dt} = 2v\dot{v}, \rightarrow \quad 2\vec{v}\cdot\vec{a} = 2v\dot{v}$$

$$\rightarrow \quad \vec{v}\cdot\vec{a} = \underline{v\dot{v}}$$

For any particle w/ const. speed, $\vec{v}$ & $\vec{a}$ must be $\perp$.

b/c $\dot{v} \neq |\vec{a}|$, $\dot{v}$ is the mag. ~~of~~ acc'n along the direction of motion