Sample Midterm 3A, Math 2552, Summer 2019

Instructor: Dr. Greg Mayer   Date: 2019   Time: 10:05 am to 11:20 am

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Student Instructions

- **Show your work** and justify your answers for all questions unless stated otherwise.
- You will have 75 minutes to take the exam. There are 50ish total points possible.
- Calculators, notes, cell phones, books are not allowed.
- Please write your answers neatly and show all of your work.
- Use dark and clear writing: your exam will be scanned into a digital system.
- Exam pages are double sided. Be sure to complete both sides.
- Leave a 1 inch border around the edges of exams.
- Check that every page has the same booklet number.

Questions

1. (15 points) Compute the Laplace transform of the following functions.
   (a) $f(t) = 13e^{-3t} - 21\sin 9t$
   (b) $g(t) = \int_0^t \sin(t-\tau)e^{\tau}\, d\tau$
   (c) $h(t) = \begin{cases} t-1, & 2 \le t < 4 \\ 1, & \text{else} \end{cases}$

2. (5 points) Calculate the inverse Laplace transform of the function.

$$F(s) = \frac{e^{7s}}{(s-1)(s-3)}$$

3. (10 points) Solve the initial value problem using the Laplace Transform.

$$y'' - 6y' + 9y = t, \quad y(0) = 0, \quad y'(0) = 1$$

4. (10 points) Solve the initial value problem.

$$y'' + 2y' + 10y = 21\delta(t-\pi), \quad y(0) = 0, \quad y'(0) = -13$$

5. (10 points) Consider the system

$$\frac{dx}{dt} = x(1-x-y), \qquad \frac{dy}{dt} = \frac{y}{4}(3-4y-2x)$$

   Identify all critical points and classify them according to stability and type.

Note: this sample test covers some of the material in chapters 5 and 7. Certainly not all of it. Be sure to review recitation worksheets, lecture slides, assigned homework problems, webwork.

SAMPLE TEST 3 ANSWERS

1) a) $\mathcal{L}\{13e^{-3t} - 21\sin 9t\}$

$$= 13\,\frac{1}{s+3} - 21\,\frac{9}{s^2+9^2}$$

b) $\mathcal{L}\left\{\int_0^t \sin(t-\tau)\, e^{\tau}\, d\tau\right\}$

$$= \mathcal{L}\{\sin t\}\,\mathcal{L}\{e^t\}$$

$$= \frac{1}{s^2+1}\,\frac{1}{s-1}$$

c) $h = (t-1)\,u_{24}$

$$= (t-1)(u_2 - u_4)$$

$$= (t-1)u_2 - (t-1)u_4$$

$$= \big((t-1)+1-1\big)u_2 - \big((t-1)+3-3\big)u_4$$

$$= (t-2)u_2 + u_2 - (t-4)u_4 - 3u_4$$

$$\mathcal{L}\{h\} = \frac{1}{s^2}e^{-2s} + \frac{1}{s}e^{-2s} - \frac{1}{s^2}e^{-4s} - 3\,\frac{1}{s}e^{-4s}$$

2) $\dfrac{1}{(s-1)(s-3)} = \dfrac{A}{s-1} + \dfrac{B}{s-3}$

$1 = A(s-3) + B(s-1)$

if $s=3$, $B = 1/2$

$s=1$, $A = -1/2$

$\Rightarrow F = -1/2 \dfrac{e^{7s}}{s-1} + \dfrac{1}{2} \dfrac{e^{7s}}{s-3}$

$\Rightarrow f(t) = -\frac{1}{2} e^{t+7} u_7 + \frac{1}{2} e^{3(t+7)} u_7$

$$y'' - 6y' + 9y = t, \quad y(0) = 0, \quad y'(0) = 1$$

$$s^2 Y - s y(0) - y'(0) - 6(sY - y(0)) + 9Y = \frac{1}{s^2} \qquad 2$$

$$\Rightarrow (s^2 - 6s + 9)\, Y = \frac{1}{s^2} + 1$$

$$Y = \frac{\frac{1}{s^2} + 1}{s^2 - 6s + 9}$$

$$= \frac{s^2 + 1}{s^2 (s-3)^2} \qquad 2$$

$$= \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s-3} + \frac{D}{(s-3)^2} \qquad 1$$

$$s^2 + 1 = A(s-3)^2 s + B(s-3)^2 + C(s-3)s^2 + Ds^2$$

If $s = 0$, $\quad 0 + 1 = 0 + 9B + 0 + 0 \Rightarrow B = 1/9 \qquad 1$

If $s = 3$, $\quad 10 = 0 + 0 + 0 + 9D \Rightarrow D = 10/9 \qquad 1$

Cubic terms: $\quad 0 = As^3 + Cs^3 = (A + C)\, s^3 \Rightarrow A = -C$

Quadratic terms: $\quad s^2 = -6As^2 + Bs^2 + 3Cs^2 + Ds^2 \qquad 2$

$$1 = -6A + 1/9 + 3A + 10/9$$

$$3A = \frac{11}{9} - 1 = 2/9 \Rightarrow A = 2/27 = -C$$

$$\Rightarrow Y = \frac{2}{27}\frac{1}{s} + \frac{1}{9}\frac{1}{s^2} - \frac{2}{27}\frac{1}{s-3} + \frac{10}{9}\frac{1}{(s-3)^2}$$

$$\Rightarrow \boxed{y = \frac{2}{27} + \frac{1}{9}t - \frac{2}{27}e^{3t} + \frac{10}{9}te^{3t}} \qquad 1$$

MODIFIED VERSION OF 5.7 #1

1. (10 points) Solve the initial value problem.

$$y'' + 2y' + 10y = 4\delta(t-\pi), \quad y(0) = 0, \quad y'(0) = 2$$

$$\mathcal{L}\{y'' + 2y' + 10y\} = \mathcal{L}\{4\delta(t-\pi)\}$$

$$\left(s^2 Y - s\,y(0) - y'(0)\right) + 2\left(sY - y(0)\right) + 10Y = 4e^{-\pi s} \quad ② \mathcal{L}$$

$$\left(s^2 + 2s + 10\right) Y = 4e^{-\pi s} + 2$$

① complete square

③ algebra

$$Y = \frac{4e^{-\pi s}}{s^2+2s+10} + \frac{2}{s^2+2s+10}$$

$$= \frac{4}{3}\,\frac{3}{(s+1)^2+9}\,e^{-\pi s} + \frac{2}{3}\,\frac{3}{(s+1)^2+9}$$

$$\boxed{y(t) = \frac{4}{3}\,e^{-(t-\pi)}\sin 3(t-\pi)u_\pi + \frac{2}{3}\,e^{-t}\sin 3t}$$

Solution to the last question
Critical points:

$$0 = x(1 - x - y) \tag{1}$$
$$0 = \frac{y}{4}(3 - 4y - 2x) \tag{2}$$

Choosing $x = 0$ leads to the points $(0, 0)$ and $(0, \frac{3}{4})$. Choosing $y = 0$ leads to another point, $(1, 0)$. If $x \neq 0$ and $y \neq 0$ then

$$0 = x(1 - x - y) \Rightarrow y = 1 - x$$
$$0 = \frac{y}{4}(3 - 4y - 2x) \Rightarrow 2x + 4y = 3 \Rightarrow 2x + 4(1 - x) = 3 \Rightarrow x = \frac{1}{2}$$

Thus the last critical point is $(\frac{1}{2}, \frac{1}{2})$. Jacobian:

$$J = \begin{pmatrix} 1 - 2x - y & -x \\ -y/2 & 0.75 - 2y - x/2 \end{pmatrix}$$

Classify:

- At $(0, 0)$, $J = \begin{pmatrix} 1 & 0 \\ 0 & 0.75 \end{pmatrix}$, unstable node
- At $(1, 0)$. $J = \begin{pmatrix} -1 & -1 \\ 0 & 0.25 \end{pmatrix}$, unstable saddle
- At $(0, 3/4)$, $J = \begin{pmatrix} 0.25 & 0 \\ -3/8 & -0.75 \end{pmatrix}$, unstable saddle
- At $(\frac{1}{2}, \frac{1}{2})$, $J = \begin{pmatrix} -0.5 & -0.5 \\ -0.25 & -0.5 \end{pmatrix}$. Eigenvalues:

$$0 = (-0.5 - \lambda)^2 - \frac{1}{8} = \lambda^2 + \lambda + \frac{1}{8} \Rightarrow \lambda = -\frac{1}{2} \pm \frac{1}{2}\sqrt{1 - \frac{1}{2}} = -\frac{1}{2} \pm \frac{1}{2\sqrt{2}}$$

Both eigenvalues real and negative, so critical point is a stable node.