Acceleration in Rotating Frames

From before we had that the velocity of a particle relative to an inertial frame is $\vec{V} = \frac{d\vec{r}}{dt} = \dot{\vec{r}} + \vec{\omega} \times \vec{r}$

Apply this type of expression to acceleration

$$\vec{a} = \frac{d^2\vec{r}}{dt^2} = \underline{\dot{\vec{V}}} + \underline{\vec{\omega} \times \vec{V}}$$

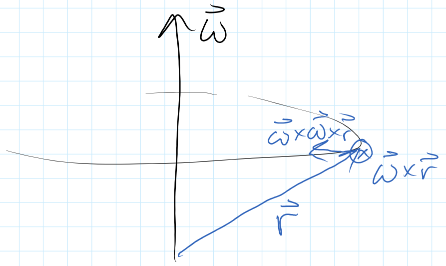
dot both sides of the $\vec{V}$ eq'n: assume $\vec{\omega}$ is constant

$$\dot{\vec{V}} = \ddot{\vec{r}} + \cancel{\dot{\vec{\omega}} \times \vec{r}} + \vec{\omega} \times \dot{\vec{r}}$$

$\times\vec{\omega}$: $\vec{\omega} \times \vec{V} = \vec{\omega} \times \dot{\vec{r}} + \vec{\omega} \times (\vec{\omega} \times \vec{r})$

$$\boxed{\therefore \ \vec{a} = \frac{d^2\vec{r}}{dt^2} = \ddot{\vec{r}} + 2\vec{\omega} \times \dot{\vec{r}} + \vec{\omega} \times (\vec{\omega} \times \vec{r})}$$

The $2\vec{\omega} \times \dot{\vec{r}}$ term is the Coriolos acc'n ; the $\vec{\omega} \times (\vec{\omega} \times \vec{r})$ term is the centripetal acceleration, directed towards the axis of rotation and is $\perp$ to it. Note the Coriolis acc'n appears only when a particle moves in a rotating coordinate system.



In an inertial frame $\vec{F} = m\vec{a}$ where $\vec{F}$ is the vector sum of all real forces acting on a particle (e.g. gravity)

$$\therefore \ m\ddot{\vec{r}} + 2m\vec{\omega} \times \dot{\vec{r}} + m\vec{\omega} \times (\vec{\omega} \times \vec{r}) = \vec{F}$$

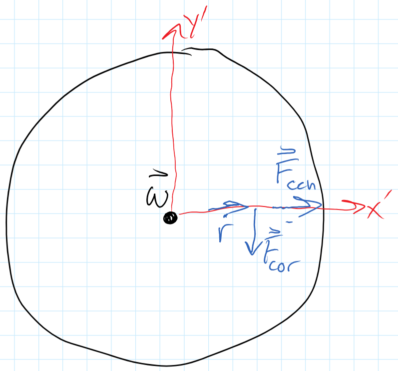
or $m\ddot{\vec{r}} = \vec{F} - 2m\vec{\omega}\times\dot{\vec{r}} - m\vec{\omega}\times(\vec{\omega}\times\vec{r})$

Therefore, an observer in the rotating frame would need to account for 2 apparent (or, fictious) forces to describe the motion of a particle.

- The centrifugal force $-m\vec{\omega}\times(\vec{\omega}\times\vec{r})$ points away from the axis of rotation. Very familar (taking a turn in a car)

- The Coriolis force $-2m\vec{\omega}\times\dot{\vec{r}}$. The 'merry-go-round' force. Always $\perp$ to velocity vector in moving coord system. Appears to deflect a particle at right-angles to its direction of motion.
  - important force for computing trajectories & weather systems
  - responsible for circulation of storms, etc.

e.g. A bug crawling outward w/ a constant speed $\dot{r}$ along the spoke of a wheel that is rotating w/ const. vel. $\vec{\omega}$ about a vertical axis. Find all apparent forces acting on the bug.



Consider the $x'y'$ plane on the wheel rotating at ω. Choose the axes so the bug is on the x' axis & moving outwards.

$$\vec{r} = x\,\hat{i}'$$

$$\dot{\vec{r}} = \dot{r}\,\hat{i}'$$

$$\ddot{\vec{r}} = 0$$

$$\vec{\omega} = \omega\hat{K}'$$

So, the Coriolis force is $-2m\vec{\omega}\times\dot{\vec{r}} = -2m\omega\dot{r}(\hat{K}'\times\hat{i}') = -2m\omega\dot{r}\hat{j}'$

The centrifugal force is $-m\omega\times(\bar{\omega}\times\bar{r}) = -m\omega^2\left[\hat{k}'\times(\hat{k}'\times x\hat{i}')\right]$

$$= -m\omega^2\left(\hat{k}'\times(x\hat{j}')\right)$$

$$= m\omega^2 x\hat{i}'$$

So, e.o.m. in the rotating frame

$$\vec{F} - 2m\omega\dot{r}\hat{j}' + m\omega^2 x\hat{i}' = 0$$ why?

Ex: How far can the bug crawl before it starts to slip, given the coefficient of static friction μ_s b/w the bug & the spoke?

The force of friction has a max value $\mu_s mg$.

So slipping will start when $|\vec{F}| = \mu_s mg$

or $$\sqrt{(2m\omega\dot{r})^2 + (m\omega^2 x)^2} = \mu_s mg$$

solve for x:

$$x = \frac{\sqrt{\mu_s^2 g^2 - 4\omega^2(\dot{r})^2}}{\omega^2}$$