Quiz 1

How to turn your work in: This quiz finishes by 11:50 am on 09/18, at which point you must drop your pen and prepare to upload your work to CANVAS. You have until 12:10 am to upload it to CANVAS. If you encounter technical difficulties to upload, send a picture of your work to mourigal@gatech.edu or by text message to 404-747-4969.

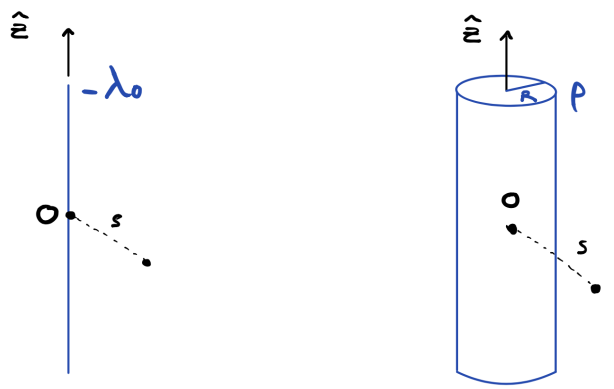
Honor code: This quiz is administered under a strict Honor Code. By signing your name on each page of your uploaded quiz, you certify that you took the test with closed book and notes, closed internet, no calculator, and no communications with anyone – it was just you and the virtual photons ♡.

Grading Rubric: This quiz contains two problems of equal importance. In each problem, questions are related but can be answered independently. If you can't solve one question, move to the next. The points you earned for each question will be decided based on the overall performance of the class. I am expecting that only 0% to 10% of the students in the class will be able to finish all the questions. Questions marked with $*$ are the bare minimum that you should be able to answer. Show your detailed work.

Useful Formulas:

$$\int \frac{dx}{x} = \ln x$$
$$\int \frac{dx}{x+a} = \ln(a+x)$$
$$\int \frac{xdx}{x+a} = x - a\ln(a+x)$$
$$\int \frac{dx}{\sqrt{x^2 \pm a^2}} = \ln(x + \sqrt{a^2+x^2})$$
$$\int \frac{xdx}{\sqrt{x^2 \pm a^2}} = \sqrt{a^2+x^2}$$
$$\int \frac{dx}{x^2 \pm a^2} = \frac{1}{a}\tan^{-1}(x/a)$$
$$\int \frac{dx}{(x^2 \pm a^2)^{3/2}} = \frac{\pm x}{a^2\sqrt{x^2 \pm a^2}}$$
$$\int \frac{xdx}{(x^2 \pm a^2)^{3/2}} = -\frac{1}{\sqrt{x^2 \pm a^2}}$$
$$(1+\epsilon)^\alpha = 1 + \alpha\epsilon + \frac{\alpha(\alpha-1)}{2!}\epsilon^2$$
$$V = \frac{1}{4\pi\varepsilon_0}\iiint \frac{\rho(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}\mathrm{d}\tau'$$
$$V = \frac{1}{4\pi\varepsilon_0}\iint \frac{\sigma(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}\mathrm{d}a'$$
$$\nabla\cdot\mathbf{E} = \rho/\varepsilon_0 \ \text{ or } \ \oiint \mathbf{E}\cdot\mathrm{d}\mathbf{a} = \frac{1}{\varepsilon_0}\iiint \rho\,\mathrm{d}\tau$$
$$\Delta V = -\int \mathbf{E}\cdot\mathrm{d}\mathbf{l} \ \text{ and } \ \mathbf{E} = -\nabla V$$
$$\nabla^2 V = -\rho/\varepsilon_0$$
$$\partial V/\partial n|_{\text{above}} - \partial V/\partial n|_{\text{below}} = -\sigma/\varepsilon_0$$
$$W = \frac{1}{2}\sum_i q_i V(\mathbf{r}_i)$$
$$W = \frac{\varepsilon_0}{2}\iiint |\mathbf{E}|^2\mathrm{d}\tau$$

Problem A. Energy of a negatively charged wire. We first consider an infintely long wire running along the z-axis and carrying a linear charge density $-\lambda_0$ with $\lambda_0 > 0$ (left side of figure below).



$*$(1) Using Gauss's law, show that the electric field anywhere is

$$\mathbf{E} = \frac{-\lambda_0}{2\pi\epsilon_0}\frac{\hat{s}}{s}$$

where s is the distance from the wire.

$*$(2) Calculate the following quantities

$$\nabla \cdot \mathbf{E} \quad \forall s \quad \text{and} \quad \oint_{\mathcal{C}} \mathbf{E} \cdot \mathrm{d}\boldsymbol{\ell}$$

where $\mathcal{C}$ is a closed loop going around the above wire.

(3) Calculate the total electrostatic energy w stored per unit-length of infinitely-thin wire. Weird right?

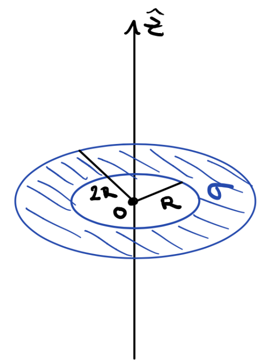
Now let's consider the same infinitely long wire running along the z axis but let's assume it has a finite radius R and carries a uniform volume charge density $\rho = -\rho_0$ (right side of above figure).

$*$(4) Using Gauss's law, calculate the electric field inside ($s < R$) and outside the wire ($s > R$). Plot the magnitude of $|\mathbf{E}|$ as a function of s.

(5) Calculate the total electrostatic energy w stored per unit-length of this finite-radius wire. Comment on this result in light of (3). What is going on now?

(6) Using the expression for the electric field in (1) calculate the difference of potential $\Delta V = V_b - V_a$ between two points a and b located outside the wire at a distance s_a and s_b, respectively. What is the sign of ΔV if $s_a > s_b$? Why is it not possible to send $s_a \to \infty$ in this problem?

Problem B. A disk with a hole. We consider a disk of radius $2R$ in which we punched a hole or radius R. The disk, which is centered on 0, lies in the xy-plane and carries a uniform surface charge density σ. We take the reference of potential at infinity in this problem.



$*$(1) Show that the electric potential $V(z)$ at a point P on the z-axis reads:

$$V(z) = \frac{\sigma}{2\varepsilon_0}\left[\sqrt{z^2+4R^2} - \sqrt{z^2+R^2}\right]$$

(2) Using wathever approach you like, show that the far-field potential is $V(z\to\infty) \approx \alpha/z^2$ and give α.

(3) Furthermore, show that the potential near 0 is $V(z\to 0) \approx V_0 + \beta z^2$ and give V_0 and β.

$*$(4) Using the results above, plot the potential $V(z)$ for any value of z. Explain your reasoning.

$*$(5) Plot the $\hat{z}$-component of the electric-field for any value of z. Explain your reasoning.

We now assume that the inner radius of the disk is sent to zero. We thus have a full disk of radius $2R$.

(6) Without any formal calculations, plot the potential $V(z)$ and the $\hat{z}$-component of the electric-field in this new situation. Explain your reasoning.

Cartesian. $d\mathbf{l} = dx\,\hat{\mathbf{x}} + dy\,\hat{\mathbf{y}} + dz\,\hat{\mathbf{z}}$; $\quad d\tau = dx\,dy\,dz$

Gradient: $$\nabla t = \frac{\partial t}{\partial x}\,\hat{\mathbf{x}} + \frac{\partial t}{\partial y}\,\hat{\mathbf{y}} + \frac{\partial t}{\partial z}\,\hat{\mathbf{z}}$$

Divergence: $$\nabla \cdot \mathbf{v} = \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}$$

Curl: $$\nabla \times \mathbf{v} = \left(\frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z}\right)\hat{\mathbf{x}} + \left(\frac{\partial v_x}{\partial z} - \frac{\partial v_z}{\partial x}\right)\hat{\mathbf{y}} + \left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y}\right)\hat{\mathbf{z}}$$

Laplacian: $$\nabla^2 t = \frac{\partial^2 t}{\partial x^2} + \frac{\partial^2 t}{\partial y^2} + \frac{\partial^2 t}{\partial z^2}$$

Spherical. $d\mathbf{l} = dr\,\hat{\mathbf{r}} + r\,d\theta\,\hat{\boldsymbol{\theta}} + r\sin\theta\,d\phi\,\hat{\boldsymbol{\phi}}$; $\quad d\tau = r^2\sin\theta\,dr\,d\theta\,d\phi$

Gradient: $$\nabla t = \frac{\partial t}{\partial r}\,\hat{\mathbf{r}} + \frac{1}{r}\frac{\partial t}{\partial \theta}\,\hat{\boldsymbol{\theta}} + \frac{1}{r\sin\theta}\frac{\partial t}{\partial \phi}\,\hat{\boldsymbol{\phi}}$$

Divergence: $$\nabla \cdot \mathbf{v} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 v_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}(\sin\theta\, v_\theta) + \frac{1}{r\sin\theta}\frac{\partial v_\phi}{\partial \phi}$$

Curl: $$\nabla \times \mathbf{v} = \frac{1}{r\sin\theta}\left[\frac{\partial}{\partial \theta}(\sin\theta\, v_\phi) - \frac{\partial v_\theta}{\partial \phi}\right]\hat{\mathbf{r}}$$
$$+ \frac{1}{r}\left[\frac{1}{\sin\theta}\frac{\partial v_r}{\partial \phi} - \frac{\partial}{\partial r}(r v_\phi)\right]\hat{\boldsymbol{\theta}} + \frac{1}{r}\left[\frac{\partial}{\partial r}(r v_\theta) - \frac{\partial v_r}{\partial \theta}\right]\hat{\boldsymbol{\phi}}$$

Laplacian: $$\nabla^2 t = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial t}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial t}{\partial \theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 t}{\partial \phi^2}$$

Cylindrical. $d\mathbf{l} = ds\,\hat{\mathbf{s}} + s\,d\phi\,\hat{\boldsymbol{\phi}} + dz\,\hat{\mathbf{z}}$; $\quad d\tau = s\,ds\,d\phi\,dz$

Gradient: $$\nabla t = \frac{\partial t}{\partial s}\,\hat{\mathbf{s}} + \frac{1}{s}\frac{\partial t}{\partial \phi}\,\hat{\boldsymbol{\phi}} + \frac{\partial t}{\partial z}\,\hat{\mathbf{z}}$$

Divergence: $$\nabla \cdot \mathbf{v} = \frac{1}{s}\frac{\partial}{\partial s}(s v_s) + \frac{1}{s}\frac{\partial v_\phi}{\partial \phi} + \frac{\partial v_z}{\partial z}$$

Curl: $$\nabla \times \mathbf{v} = \left[\frac{1}{s}\frac{\partial v_z}{\partial \phi} - \frac{\partial v_\phi}{\partial z}\right]\hat{\mathbf{s}} + \left[\frac{\partial v_s}{\partial z} - \frac{\partial v_z}{\partial s}\right]\hat{\boldsymbol{\phi}} + \frac{1}{s}\left[\frac{\partial}{\partial s}(s v_\phi) - \frac{\partial v_s}{\partial \phi}\right]\hat{\mathbf{z}}$$

Laplacian: $$\nabla^2 t = \frac{1}{s}\frac{\partial}{\partial s}\left(s\frac{\partial t}{\partial s}\right) + \frac{1}{s^2}\frac{\partial^2 t}{\partial \phi^2} + \frac{\partial^2 t}{\partial z^2}$$

VECTOR IDENTITIES

Triple Products

(1) $\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C}) = \mathbf{B} \cdot (\mathbf{C} \times \mathbf{A}) = \mathbf{C} \cdot (\mathbf{A} \times \mathbf{B})$

(2) $\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = \mathbf{B}(\mathbf{A} \cdot \mathbf{C}) - \mathbf{C}(\mathbf{A} \cdot \mathbf{B})$

Product Rules

(3) $\nabla(fg) = f(\nabla g) + g(\nabla f)$

(4) $\nabla(\mathbf{A} \cdot \mathbf{B}) = \mathbf{A} \times (\nabla \times \mathbf{B}) + \mathbf{B} \times (\nabla \times \mathbf{A}) + (\mathbf{A} \cdot \nabla)\mathbf{B} + (\mathbf{B} \cdot \nabla)\mathbf{A}$

(5) $\nabla \cdot (f\mathbf{A}) = f(\nabla \cdot \mathbf{A}) + \mathbf{A} \cdot (\nabla f)$

(6) $\nabla \cdot (\mathbf{A} \times \mathbf{B}) = \mathbf{B} \cdot (\nabla \times \mathbf{A}) - \mathbf{A} \cdot (\nabla \times \mathbf{B})$

(7) $\nabla \times (f\mathbf{A}) = f(\nabla \times \mathbf{A}) - \mathbf{A} \times (\nabla f)$

(8) $\nabla \times (\mathbf{A} \times \mathbf{B}) = (\mathbf{B} \cdot \nabla)\mathbf{A} - (\mathbf{A} \cdot \nabla)\mathbf{B} + \mathbf{A}(\nabla \cdot \mathbf{B}) - \mathbf{B}(\nabla \cdot \mathbf{A})$

Second Derivatives

(9) $\nabla \cdot (\nabla \times \mathbf{A}) = 0$

(10) $\nabla \times (\nabla f) = 0$

(11) $\nabla \times (\nabla \times \mathbf{A}) = \nabla(\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A}$

FUNDAMENTAL THEOREMS

Gradient Theorem: $\int_{\mathbf{a}}^{\mathbf{b}} (\nabla f) \cdot d\mathbf{l} = f(\mathbf{b}) - f(\mathbf{a})$

Divergence Theorem: $\int (\nabla \cdot \mathbf{A})\, d\tau = \oint \mathbf{A} \cdot d\mathbf{a}$

Curl Theorem: $\int (\nabla \times \mathbf{A}) \cdot d\mathbf{a} = \oint \mathbf{A} \cdot d\mathbf{l}$