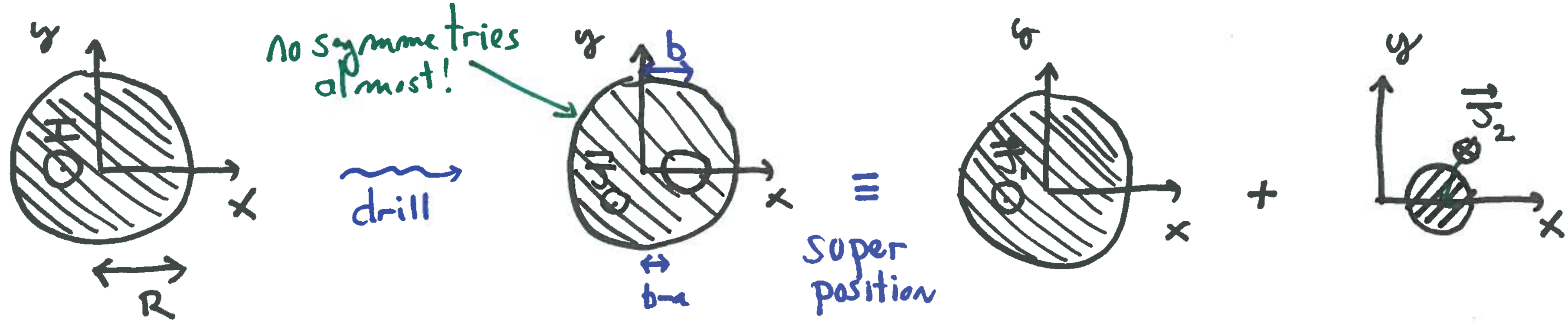
Homework 10

Problem A It's a classic problem. Let's use the superposition principle

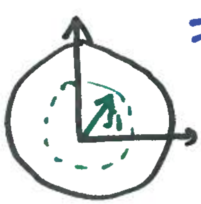


$|\vec J| = I_0/\pi R^2$ $\qquad$ $|\vec J| = I/(\pi R^2 - \pi a^2)$ $\qquad$ $\vec J_1 = \vec J$ $\qquad$ $\vec J_2 = -\vec J$

less material to distribute I

It's ok if you kept $\vec J = \dfrac{I}{\pi R^2}$

Ampère



$\vec B_1 = B_1(s_1)\hat\phi_1$

$\oint \vec B_1 \cdot d\vec l = \mu_0 I_{through}$

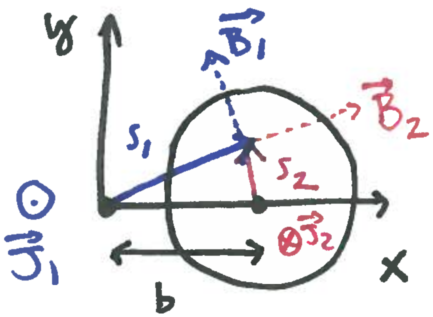
$\vec B_2 = B_2(s_2)\hat\phi_2$

$\oint \vec B_2 \cdot d\vec l = \mu_0 I_{through}$

$B_1(s_1)\cdot 2\pi s_1 = \mu_0 J \cdot \pi s_1^2$

$\to \boxed{\vec B_1(s_1) = \dfrac{\mu_0 J s_1}{2}\hat\phi_1}$ ← different coordinate systems! → $\boxed{\vec B_2(s_2) = -\dfrac{\mu_0 J s_2}{2}\hat\phi_2}$

In the hole see that $\vec{B}_1$ and $\vec{B}_2$ have different directions ??



Look at that $\vec{B}_1 = \frac{\mu_0 J s_1}{2}\hat{\phi}_1 \equiv \frac{\mu_0}{2}\vec{J}_1 \times \vec{s}_1 = \frac{\mu_0}{2}\vec{J}\times\vec{s}_1$

can you see it ?

$\vec{B}_2 = -\frac{\mu_0 J s_2}{2}\hat{\phi}_2 \equiv \frac{\mu_0}{2}\vec{J}_2\times\vec{s}_2 = -\frac{\mu_0}{2}\vec{J}\times\vec{s}_2$

So $\vec{B}_1 + \vec{B}_2 = \frac{\mu_0}{2}\left(\vec{J}_1\times\vec{s}_1 + \vec{J}_2\times\vec{s}_2\right) = \frac{\mu_0}{2}\left(\vec{J}\times(\vec{s}_1 - \vec{s}_2)\right)$

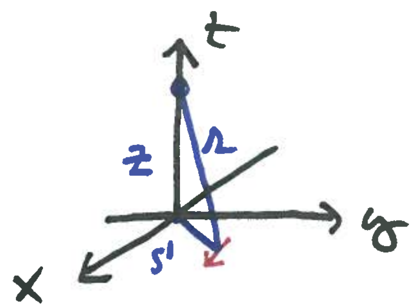
But from drawing: $\vec{b} = \vec{s}_1 - \vec{s}_2$ and therefore $\boxed{\vec{B} = \vec{B}_1 + \vec{B}_2 = \frac{\mu_0}{2}\vec{J}\times\vec{b}}$

The field is uniform in the hole!!!

Problem B

(1) The object has finite extent so $\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi}\iint \frac{\vec{K}(\vec{r}')\,da'}{\mathscr{r}}$ seems appropriate

here $\vec{K} = K_0\hat{x}$



$da' = s'\,d\phi'\,ds'$ cylindrical

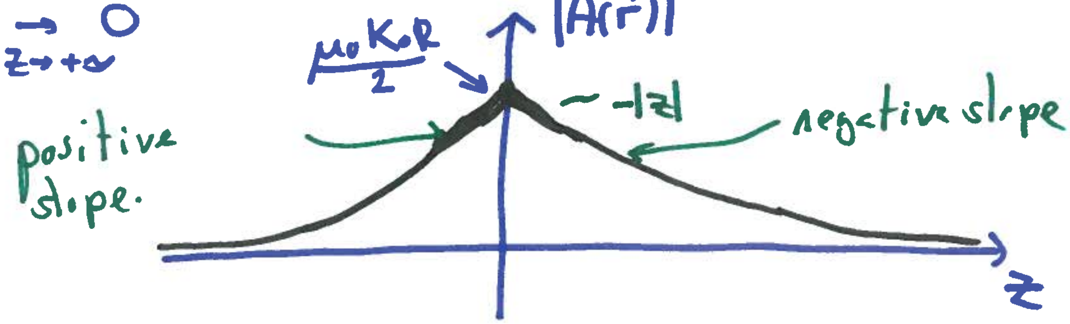
$\mathscr{r} = \sqrt{s'^2 + z^2}$ see drawing then $\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi}\int_0^R\int_0^{2\pi} \frac{K_0 s' ds' d\phi'}{\sqrt{s'^2+z^2}}\hat{x} = \frac{K_0\mu_0}{4\pi}\cdot 2\pi\cdot\hat{x}\cdot\int_0^R \frac{s' ds'}{\sqrt{s'^2+z^2}} = \frac{K_0\mu_0}{2}\left[\sqrt{s'^2+z^2}\right]_0^R\hat{x}$

therefore $\vec{A}(\vec{r}) = \frac{K_0\mu_0}{2}\left(\left[\sqrt{R^2+z^2} - \sqrt{z^2}\right]\right)\hat{x} = \frac{\mu_0 K_0}{2}\hat{x}\cdot\left[\sqrt{R^2+z^2} - |z|\right]$ ③

⚠ $\sqrt{z^2} = |z|$

$\vec{A}(\vec{r}) \xrightarrow[z\to 0]{} \frac{\mu_0 K_0}{2}\hat{x}\cdot\left[R\sqrt{1+\frac{z^2}{R^2}} - |z|\right] \simeq \frac{\mu_0 K_0}{2}\hat{x}\left[R + \frac{z^2}{2R} - |z|\right] \simeq \frac{\mu_0 K_0}{2}\hat{x}\left[R - |z|\right]$

$\vec{A}(\vec{r}) \xrightarrow[z\to+\infty]{} 0$



This tells us there is a break in the slope of $|\vec{A}(\vec{r})|$ close to $z=0$

$\sim R - |z| \ll R \;\forall z!$

② <u>Theorem</u>: $\frac{\partial\vec{A}^{above}}{\partial n} - \frac{\partial\vec{A}^{below}}{\partial n} = -\mu_0\vec{K}$ at a current sheet.

here $z \equiv n$

here $\sqrt{z^2} = |z| = z$ as $z \geq 0$, $\frac{z}{|z|} = 1$

$\left.\frac{\partial\vec{A}^{above}}{\partial z}\right|_{z\geq 0} = \frac{1}{2}\mu_0 K_0\frac{\partial}{\partial z}\left(\sqrt{R^2+z^2} - \sqrt{z^2}\right)\hat{x} = \frac{1}{2}\mu_0 K_0\left(\frac{z}{\sqrt{R^2+z^2}} - \frac{z}{\sqrt{z^2}}\right)\hat{x} = \frac{1}{2}\mu_0 K_0\left(\frac{z}{\sqrt{R^2+z^2}} - 1\right)\hat{x}$

$$\frac{\partial \vec{A}^{below}}{\partial z}\bigg|_{z\le 0} = \frac{1}{2}\mu_0 K_0 \hat{x}\left(\frac{z}{\sqrt{R^2+z^2}} - \frac{z}{\sqrt{z^2}}\right) = \frac{1}{2}\mu_0 K_0\left(\frac{z}{\sqrt{R^2+z^2}} + 1\right)\hat{x}$$

here $\frac{z}{|z|} = -1$

Therefore $$\frac{\partial \vec{A}}{\partial z}\bigg|_{z\ge 0} - \frac{\partial \vec{A}}{\partial z}\bigg|_{z\le 0} = -\frac{1}{2}\mu_0 K_0 \hat{x} - \frac{1}{2}\mu_0 K_0 \hat{x} = -\mu_0 K_0 \hat{x}\,!$$

theorem is right

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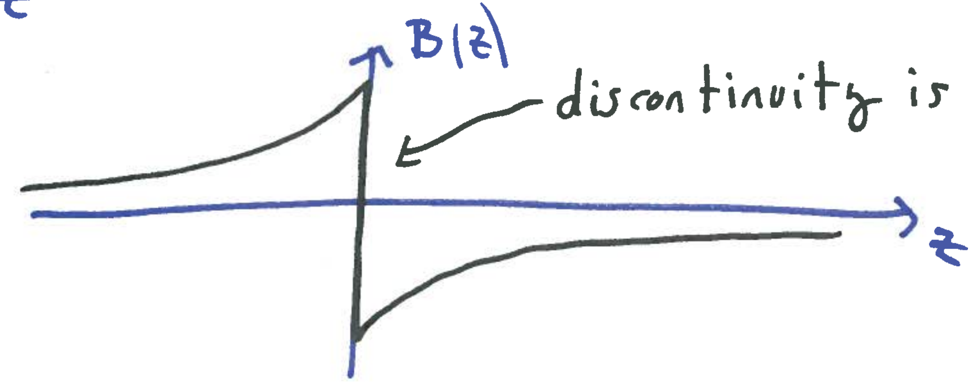
By the way $$\vec{B}(\vec{r}) = \vec{\nabla}\times\vec{A}(\vec{r}) = \vec{\nabla}\times\left(A_x(z)\hat{x}\right) = \frac{\partial A_x(z)}{\partial z}\hat{y}$$

So $$\vec{B}^{above} = \frac{\partial A_x^{above}(z\ge 0)}{\partial z}\hat{y} = \frac{1}{2}\mu_0 K_0\left(\frac{z}{\sqrt{R^2+z^2}} - 1\right)\hat{y}$$

$$\vec{B}^{below} = \frac{\partial A_x^{below}(z\le 0)}{\partial z}\hat{y} = \frac{1}{2}\mu_0 K_0\left(\frac{z}{\sqrt{R^2+z^2}} + 1\right)\hat{y}$$



discontinuity is $-\mu_0 K_0$!

4. $$\lim_{R\to+\infty} \vec{A}(z) = \frac{1}{2}\mu_0 K_0\left(\sqrt{\infty^2+z^2} - \sqrt{z^2}\right)\hat{x} \to +\infty$$

i.e. $\vec{A}$ diverges in the $R\to+\infty$ limit.

But $$\left.\frac{\partial \vec{A}^{above}}{\partial z}\right|_{z=0} - \left.\frac{\partial \vec{A}^{below}}{\partial t}\right|_{z=0} = \left(-\frac{1}{2}\mu_0 K_0 - \frac{1}{2}\mu_0 K_0\right)\hat{x} = -\mu_0 K_0 \hat{x}$$

discontinuity remains finite and so does $\vec{B}$.

$$\vec{B} \xrightarrow[R\to+\infty]{} \mp \frac{1}{2}\mu_0 K_0 \hat{y}$$

$-$: above

$+$: below