A Damped, Driven Oscillator

It is common to consider an oscillator that is driven by an external force $F(t)$. In this case, we have an inhomogeneous diff. eqn as the eqn of motion:

$$m\ddot{x} + \lambda\dot{x} + Kx = F(t)$$

If $x_1(t)$ is the solution to this equation, the general sol'n is found by adding $x_0(t)$, the solution to the homogenous equation (ie, $m\ddot{x} + \lambda\dot{x} + Kx = 0$)

Let $F(t)$ be periodic (e.g., an EM wave) where $F(t) = F_1 \cos\omega_1 t$ where F_1 & ω_1 are real constants. It will be easier to solve the e.o.m if we move to complex notation

$$m\ddot{z} + \lambda\dot{z} + Kz = F_1 e^{i\omega_1 t}$$

The real part of this solution will be the solution we want. The force has a freq. ω_1, so a good guess is the solution will also be oscillatory w/ that freq. So, sub in $z = A_1 e^{i\omega_1 t}$ but $A_1 = a_1 e^{-i\theta_1}$ is a complex constant, so $\underline{z = a_1 e^{i(\omega_1 t - \theta_1)}}$

$$\rightarrow \quad (-m\omega_1^2 + i\lambda\omega_1 + K)A_1 = F_1$$

$\div$ both sides by $me^{-i\theta_1}$ & use $\underline{\omega_0^2 = \frac{K}{m}, \ \gamma = \frac{\lambda}{2m}}$

$$\rightarrow \quad (\omega_0^2 + 2i\gamma\omega_1 - \omega_1^2)\, a_1 = \frac{F_1}{m} e^{i\theta_1}$$

Equating real & imaginary parts

$$(\omega_0^2 - \omega_1^2)\, a_1 = \left(\frac{F_1}{m}\right)\cos\theta_1$$

$$(\omega_0^2-\omega_1^2)\,a_1 = \left(\frac{F_1}{m}\right)\cos\theta_1$$

$$2\gamma\omega_1 a_1 = \left(\frac{F_1}{m}\right)\sin\theta_1$$

Square & add these equations:

$$\frac{(\omega_0^2-\omega_1^2)^2 a_1^2}{\left(\frac{F_1}{m}\right)^2} = \cos^2\theta_1$$

$$\frac{4\gamma^2\omega_1^2 a_1^2}{\left(\frac{F_1}{m}\right)^2} = \sin^2\theta_1$$

$$\frac{a_1^2}{\left(\frac{F_1}{m}\right)^2}\left[(\omega_0^2-\omega_1^2)^2+4\gamma^2\omega_1^2\right] = 1$$

$$\therefore \boxed{a_1 = \frac{(F_1/m)}{\sqrt{(\omega_0^2-\omega_1^2)^2+4\gamma^2\omega_1^2}}}$$

$\div$ one equation by the other gives the phase:

$$\boxed{\tan\theta_1 = \frac{2\gamma\omega_1}{\omega_0^2-\omega_1^2}} \qquad (0<\theta_1<\pi \text{ for } F_1>0)$$

$\therefore$ The real part of the solution is $x = a_1\cos(\omega_1 t-\theta_1)$ where a_1 & θ_1 are given by above eqns.

To get the general solution, add the solution to the homo. eqn

$$\rightarrow\ x = a_1\cos(\omega_1 t-\theta_1) + a\,e^{-\gamma t}\cos(\underline{\omega t}-\theta) \text{ where } a, \theta$$

are fixed by initial condions

The 2nd term dies away w/ time and is called the transient term.

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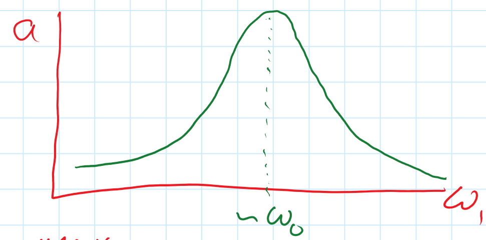
Resonance

Examine the amplitude of the long-term oscillation

$$a_1 = \frac{(F_1/m)}{\sqrt{(\omega_0^2-\omega_1^2)+4\gamma^2\omega_1}}$$



To determine the freq. at which the max amplitude occurs

$$\frac{da_1}{d\omega_1} = \left(\frac{F_1}{m}\right)\left(-\frac{1}{2}\right)\frac{\left(2(\omega_0^2-\omega_1^2)(-2\omega_1)+8\gamma^2\omega_1\right)}{\left((\omega_0^2-\omega_1^2)^2+4\gamma^2\omega_1^2\right)^{3/2}} = 0$$

$$\rightarrow -4\cancel{\omega_1}(\omega_0^2-\omega_1^2)+8\gamma^2\cancel{\omega_1}=0$$

$$-4\omega_0^2+4\omega_1^2+8\gamma^2=0$$

$$\rightarrow \omega_1^2=\omega_{res}^2=\omega_0^2-2\gamma^2$$

For small damping $\omega_{res}\simeq\omega_0$, but in general $\omega_{res}=\sqrt{\omega_0^2-2\gamma^2}$

So, $a_{1,max} = \dfrac{(F_1/m)}{\sqrt{(\omega_0^2-\omega_0^2+2\gamma^2)+4\gamma^2(\omega_0^2-2\gamma^2)}} = \dfrac{(F_1/m)}{\sqrt{4\gamma^2+4\gamma^2\omega_0^2-8\gamma^4}}$

$$= \frac{(F_1/m)}{\sqrt{4\gamma^2\omega_0^2-4\gamma^4}} = \frac{(F_1/m)}{2\gamma\sqrt{\omega_0^2-\gamma^2}}$$

For weak damping, $a_{1,max} \simeq \dfrac{(F_1/m)}{2\gamma\omega_0} = \dfrac{F_1}{2m\gamma\omega_0} = \dfrac{F_1}{\lambda\omega_0}$

So, at resonace, the amplitude becomes v. large when λ is v. small.