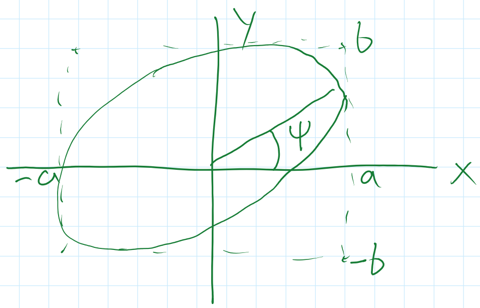
$$\therefore \frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{2xy\cos\Delta}{ab} = \sin^2\Delta$$

Recall the quadratic equation $ax^2 + bxy + cy^2 + dx + ey + f = 0$ represents a conic section. See Appendix B for a review of conics.

If the discriminant $(b^2 - 4ac)$ is negative, then the shape is an ellipse.

In this case $b^2 - 4ac = \frac{4\cos^2\Delta}{a^2b^2} - \frac{4}{a^2b^2} = \frac{-4}{a^2b^2}(1 - \cos^2\Delta) = -\frac{4\sin^2\Delta}{a^2b^2}$

which is always negative, so the path is an ellipse



If $\Delta = \frac{\pi}{2}$, then $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

which is an ellipse coincident w/ the axes semi-major axis a ; semi-minor axis b

If $\Delta = 0$ or π, then $\frac{x^2}{a^2} \pm \frac{2xy}{ab} + \frac{y^2}{b^2} = 0$

$$\left(\frac{x}{a} \pm \frac{y}{b}\right)^2 = 0 \Rightarrow y = \pm\frac{b}{a}x \quad \begin{pmatrix} +\text{ sign if } \Delta = 0 \\ -\text{ sign if } \Delta = \pi \end{pmatrix}$$

The path is a straight line

See §4.1 in text to see that $\tan 2\psi = \frac{2ab\cos\Delta}{a^2 - b^2}$

In 3D, the path is also an ellipse, but in a plane off the xy plane.

plane.

Nonisotropic Oscillator

If the magnitudes of the components of the restoring force depend on the direction of the displacement, this is a nonisotropic oscillator.

For a suitable choice of axes in 2D

$$m\ddot{x} = -K_1 x$$

$$m\ddot{y} = -K_2 y$$

So, $\omega_1 = \sqrt{\frac{K_1}{m}}$ & $\omega_2 = \sqrt{\frac{K_2}{m}}$ and

$$x = a\cos(\omega_1 t + \alpha), \quad y = b\cos(\omega_2 t + \beta)$$

The oscillation lies entirely within a rectangle w/ sides $2a, 2b$ centered on the origin. If ω_1 & ω_2 are commensurate ie if $\frac{\omega_1}{n_1} = \frac{\omega_2}{n_2}$ where n_1 & n_2 are integers then the path is called a Lissajous figure, and will be closed. After a time $\frac{2\pi n_1}{\omega_1} = \frac{2\pi n_2}{\omega_2}$ the particle will return to the initial pos'n.

However, if ω's are not commensurate, the path is not closed & the particle will eventually 'fill the box'

Example: A particle of mass m moves in 2D under the following PE function $V(\vec{r}) = \frac{1}{2}K(x^2+4y^2)$. Find the resulting motion, given the initial conditions at $t=0$: $x=a$, $y=0$, $\dot{x}=0$, $\dot{y}=V_0$

The force is $\vec{F} = -\vec{\nabla}V = -Kx\hat{i} - 4Ky\hat{j} = m\ddot{\vec{r}}$

Separable, but not isotropic

e.o.m's: $m\ddot{x} + Kx = 0$ & $m\ddot{y} + 4Ky = 0$

The x-motion has angular freq. $\omega_x = \sqrt{\frac{K}{m}}$ while the y-motion has freq. $\omega_y = \sqrt{\frac{4K}{m}} = 2\omega_x$. The general solution is then

$$x = a_1 \cos\omega_x t + b_1 \sin\omega_x t \xrightarrow{@t=0} x = a = a_1$$

$$y = a_2 \cos 2\omega_x t + b_2 \sin 2\omega_x t \qquad y = 0 = a_2$$

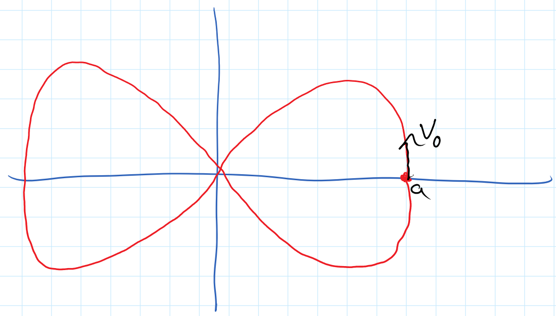
So, the velocities are

$$\dot{x} = -a_1 \omega_x \sin\omega_x t + b_1 \omega_x \cos\omega_x t \xrightarrow{@t=0} \dot{x} = 0 = b_1 \omega_x \rightarrow b_1 = 0$$

$$\dot{y} = -2a_2 \omega_x \sin 2\omega_x t + 2b_2 \omega_x \cos 2\omega_x t \qquad \dot{y} = V_0 = 2b_2\omega_x \rightarrow b_2 = \frac{V_0}{2\omega_x}$$

$\therefore$ Final eqns are $x = a\cos\omega_x t$; $y = \frac{V_0}{2\omega_x}\sin 2\omega_x t$

Path is a Lissajous figures



Central Force ; Gravitation

The harmonic oscillator is one example of a central, conservative force. More generally, in 3D a conservative force has components $F_r = -\frac{\partial V}{\partial r}$, $F_\theta = -\frac{1}{r}\frac{\partial V}{\partial \theta}$, $F_\phi = -\frac{1}{r\sin\theta}\frac{\partial V}{\partial \phi}$

$\therefore$ The force will be central iff V is independent of θ ; ϕ

Then $\vec{F} = -\hat{r}\frac{\partial V}{\partial r}$

Thus, the electric & gravitational forces are central, conservative forces.

While the force is only radial, the motion of a particle may be in any direction. In particular, it is interesting to consider the component of motion $\perp$ to the radial direction

The angular momentum of a particle located a distance $\vec{r}$ from a given origin & moving w/ momentum $\vec{p} = m\vec{v}$ is defined as

$$\vec{J} = \vec{r} \times \vec{p}$$

The time derivative is : $\dot{\vec{J}} = \dot{\vec{r}} \times \vec{p} + \vec{r} \times \dot{\vec{p}}$

But $\dot{\vec{r}} \times \vec{p} = \dot{\vec{r}} \times (m\dot{\vec{r}}) = 0$, $\dot{\vec{J}} = \vec{r} \times \dot{\vec{p}} = \vec{r} \times \vec{F}$ from Newton's 2nd Law

The cross product $\vec{G} = \dot{\vec{J}}$ is the moment of the force, or torque

For central forces, $\vec{r} \parallel \vec{F}$, so $\dot{\vec{J}} = 0$ & $\vec{J}$ is constant w/ time.