HOMEWORK 8 - Problem B

B

The bound surface charge density and volume charge density are given by

$$\sigma_b = \vec{P} \cdot \hat{n} \tag{1}$$

$$\rho_b = -\nabla \cdot \vec{P} \tag{2}$$

where $\vec{P}$ is the polarization vector field.

B.1

Therefore for $\vec{P} = Cs\hat{s}$ in cylindrical coordinates (s, θ, z)

$$\sigma_b = Ca$$

$$\rho_b = -\frac{1}{s}\frac{\partial}{\partial s}Cs^2 = -2C.$$

The dimension of C is that of volume charge density, $charge/length^3$.

B.2

$$\rho = \rho_b + \rho_f = \rho_b \tag{3}$$

Using Gauss' law and (3) we get

$$|\vec{E}(s)|2\pi s\epsilon_0 = \int_0^s \int_0^{2\pi} \rho s' ds' d\theta$$

$$|\vec{E}(s)| = \frac{Cs}{\epsilon_0}$$

Therefore

$$\vec{E}(s) = -\frac{Cs}{\epsilon_0}\hat{s} \quad ; \quad s < a.$$

At the surface and outside the rod the electric field is zero because the electric field due to surface bound charge density cancels the effect of the volume bound charge density. Therefore

$$\vec{E}(s \geq a) = 0.$$

B.3

The electric displacement field is given as

$$\vec{D} = \epsilon_0 \vec{E} + \vec{P} \tag{4}$$

Hence

$$\vec{D}(s) = 0 \tag{5}$$

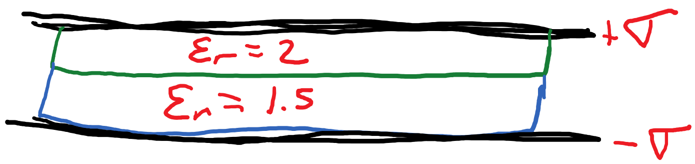
everywhere. This agrees with the result obtained using Gauss' law since $\rho_f = 0$, $\sigma_f = 0$ and

$$\nabla \cdot \vec{D}(s) = \rho_f$$
$$(\vec{D}_{out} - \vec{D}_{in}) \cdot \hat{n} = \sigma_f.$$

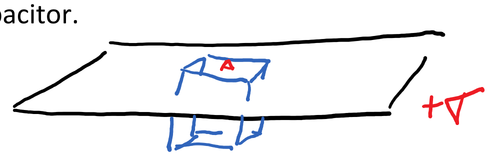
HOMEWORK 8 - Problem A

Done in Class

C



1) The electric displacement will have contributions from each of the two sheets that make up the capacitor.



Apply Gauss's Law.

$$\oint \vec{D}_t \cdot d\vec{A} = Q_{free}$$

$$\left|\vec{D}_t\right| 2A = \sigma A$$

$$\vec{D}_t = \frac{\sigma}{2} \quad \text{Away from the sheet}$$

Note When applying Gauss's Law you must account for ALL of the flux entering/exiting the Gaussian surface from the enclosed charge. In this case, the surface must capture the flux produced on both sides of the sheet from the charge on the sheet.

The bottom sheet is similar, but the displacement field goes toward the sheet due to the opposite charge

$$\vec{D}_b = \frac{\sigma}{2} \quad \text{Towards the sheet}$$

Apply the superposition principle to find the total Displacement Field.

$$\vec{D} = \vec{D}_t + \vec{D}_b$$

Inside:

$$\vec{D} = \frac{\sigma}{2}(-\hat{z}) + \frac{\sigma}{2}(-\hat{z}) = -\sigma\hat{z}$$

Note We can also verify that the field is zero above and below the capacitor:

Above the capacitor:

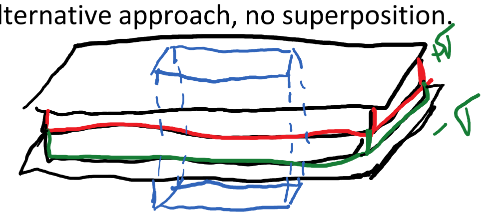
$$\vec{D} = \frac{\sigma}{2}(\hat{z}) + \frac{\sigma}{2}(-\hat{z}) = \vec{0}$$

Below the capacitor:

$$\vec{D} = \frac{\sigma}{2}(-\hat{z}) + \frac{\sigma}{2}(\hat{z}) = \vec{0}$$

If you choose NOT to use superposition, you must first prove the flux outside the entire arrangement is 0 then you can carefully calculate Gauss's Law on either side of the sheet.

1) Alternative approach, no superposition.



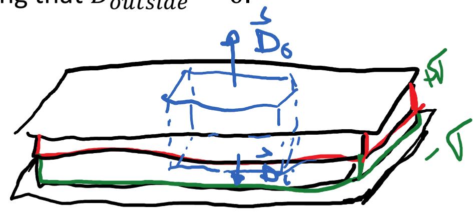
Apply Gauss's Law.

$$\oint \vec{D}_{outside} \cdot d\vec{A} = Q_{free}$$

$\left|\vec{D}_{outside}\right| 2A = (\sigma - \sigma)A = 0$

$\vec{D}_{outside} = \vec{0}$

Pick either side and calculate Gauss's Law remembering that $\vec{D}_{outside} = \vec{0}$.



Looking at the top:

$\left|\vec{D}_{outside}\right| \mathrm{A} + \left|\vec{D}_{inside}\right| \mathrm{A} = \sigma A$

$\left|\vec{D}_{inside}\right| = \sigma$

But the displacement goes away from the sheet, so:

$\vec{D}_{inside} = -\sigma\hat{z}$

The bottom gives the same result. In this case, we are looking at the entire configuration so either the top OR the bottom give the total displacement inside.

2) To find the E-field:

$\vec{D} = \varepsilon\vec{E} = \varepsilon_r\varepsilon_0\vec{E}$

For slab 1:

$$\vec{E}_1 = \frac{\vec{D}}{\varepsilon_1\varepsilon_0} = \frac{-\sigma}{2\varepsilon_0}\hat{z}$$

For slab 2:

$$\vec{E}_2 = \frac{\vec{D}}{\varepsilon_2\varepsilon_0} = \frac{-\sigma}{1.5\varepsilon_0}\hat{z} = \frac{-2\sigma}{3\varepsilon_0}\hat{z}$$

3) To find the polarization:

$\vec{D} = \varepsilon_0\vec{E} + \vec{P}$

$\vec{P} = \vec{D} - \varepsilon_0\vec{E}$

For slab 1:

$$\vec{P}_1 = \vec{D} - \varepsilon_0\vec{E}_1 = -\sigma\hat{z} - \frac{-\sigma}{2}\hat{z} = \frac{-\sigma}{2}\hat{z}$$

For slab 2:

$$\vec{P}_2 = \vec{D} - \varepsilon_0\vec{E}_2 = -\sigma\hat{z} - \frac{-2\sigma}{3\varepsilon_0}\hat{z} = \frac{-\sigma}{3}\hat{z}$$

4) Recall that the potential difference of a capacitor can be found using:

$$\mathrm{V} = -\int_{-}^{+} \vec{E} \cdot \mathrm{d}\vec{L}$$

Where + corresponds to the positive plate and - corresponds to the negative plate.
In this case, the E-field has a different (constant) value in each slab, so we separate the line integral into two parts.

$$\mathrm{V} = -\int_{middle}^{+} \vec{E}_1 \cdot \mathrm{d}\vec{L} + -\int_{-}^{\mathrm{middle}} \vec{E}_2 \cdot \mathrm{d}\vec{L} = \frac{\sigma a}{2\varepsilon_0} + \frac{2\sigma a}{3\varepsilon_0} = \frac{7\sigma a}{6\varepsilon_0}$$

5) To find the bound charges:

$\rho_b = -\vec{\nabla} \cdot \vec{P}$

Since $\vec{P}$ is constant in each region,

$\rho_b = 0$ everywhere.

$\sigma_b = \vec{P} \cdot \hat{n}$ at the surface of interest

	$\vec{P}$	$\hat{n}$	σ_b
Top of slab 1	$\vec{P}_1$	$+\hat{z}$	$\frac{-\sigma}{2}$

Bottom of slab 1	$\vec{P}_1$	$-\hat{z}$	$\frac{\sigma}{2}$
Top of slab 2	$\vec{P}_2$	$+\hat{z}$	$\frac{-\sigma}{3}$
Bottom of slab 2	$\vec{P}_2$	$-\hat{z}$	$\frac{\sigma}{3}$