Exercise 1

Because only one coordinate $\vec{r}_0$ is varied at a time, we can use a simple generalization of the one-dimensional Taylor Theorem:

$$\begin{aligned}
\phi\left(\vec{r}_0+\delta \hat{x}_i\right) &= \sum_{k=0}^{\infty} \frac{1}{k!} \frac{\partial^k \phi}{\partial x_i^k}\bigg|_{\vec{r}=\vec{r}_0} \delta^k \\
&= \phi\left(\vec{r}_0\right)+\frac{\partial \phi}{\partial x_i}\bigg|_{\vec{r}=\vec{r}_0} \delta+\frac{1}{2} \frac{\partial^2 \phi}{\partial x_i^2}\bigg|_{\vec{r}=\vec{r}_0} \delta^2+\frac{1}{6} \frac{\partial^3}{\partial x_i^3}\bigg|_{\vec{r}=\vec{r}_0} \delta^3+\mathcal{O}\left(\delta^4\right)
\end{aligned}$$

Therefore

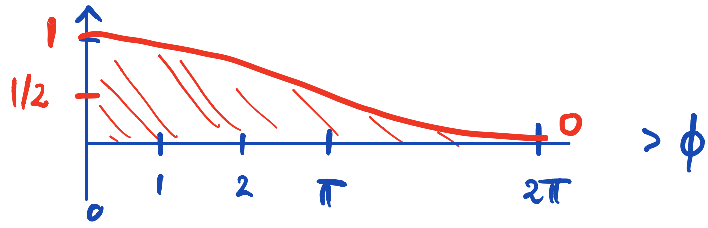
$$\begin{aligned}
\phi(\vec{r}_1) &= \phi\left(\vec{r}_0\right)+\frac{\partial \phi}{\partial x}\bigg|_{\vec{r}=\vec{r}_0} \delta+\frac{1}{2} \frac{\partial^2 \phi}{\partial x^2}\bigg|_{\vec{r}=\vec{r}_0} \delta^2+\frac{1}{6} \frac{\partial^3}{\partial x^3}\bigg|_{\vec{r}=\vec{r}_0} \delta^3+\mathcal{O}\left(\delta^4\right) \\
\phi(\vec{r}_2) &= \phi\left(\vec{r}_0\right)-\frac{\partial \phi}{\partial x}\bigg|_{\vec{r}=\vec{r}_0} \delta+\frac{1}{2} \frac{\partial^2 \phi}{\partial x^2}\bigg|_{\vec{r}=\vec{r}_0} \delta^2-\frac{1}{6} \frac{\partial^3}{\partial x^3}\bigg|_{\vec{r}=\vec{r}_0} \delta^3+\mathcal{O}\left(\delta^4\right) \\
\phi(\vec{r}_3) &= \phi\left(\vec{r}_0\right)+\frac{\partial \phi}{\partial y}\bigg|_{\vec{r}=\vec{r}_0} \delta+\frac{1}{2} \frac{\partial^2 \phi}{\partial y^2}\bigg|_{\vec{r}=\vec{r}_0} \delta^2+\frac{1}{6} \frac{\partial^3}{\partial y^3}\bigg|_{\vec{r}=\vec{r}_0} \delta^3+\mathcal{O}\left(\delta^4\right) \\
\phi(\vec{r}_4) &= \phi\left(\vec{r}_0\right)-\frac{\partial \phi}{\partial y}\bigg|_{\vec{r}=\vec{r}_0} \delta+\frac{1}{2} \frac{\partial^2 \phi}{\partial y^2}\bigg|_{\vec{r}=\vec{r}_0} \delta^2-\frac{1}{6} \frac{\partial^3}{\partial y^3}\bigg|_{\vec{r}=\vec{r}_0} \delta^3+\mathcal{O}\left(\delta^4\right) \\
\phi(\vec{r}_5) &= \phi\left(\vec{r}_0\right)+\frac{\partial \phi}{\partial z}\bigg|_{\vec{r}=\vec{r}_0} \delta+\frac{1}{2} \frac{\partial^2 \phi}{\partial z^2}\bigg|_{\vec{r}=\vec{r}_0} \delta^2+\frac{1}{6} \frac{\partial^3}{\partial z^3}\bigg|_{\vec{r}=\vec{r}_0} \delta^3+\mathcal{O}\left(\delta^4\right) \\
\phi(\vec{r}_6) &= \phi\left(\vec{r}_0\right)-\frac{\partial \phi}{\partial z}\bigg|_{\vec{r}=\vec{r}_0} \delta+\frac{1}{2} \frac{\partial^2 \phi}{\partial z^2}\bigg|_{\vec{r}=\vec{r}_0} \delta^2-\frac{1}{6} \frac{\partial^3}{\partial z^3}\bigg|_{\vec{r}=\vec{r}_0} \delta^3+\mathcal{O}\left(\delta^4\right)
\end{aligned}$$

Summing these terms yields

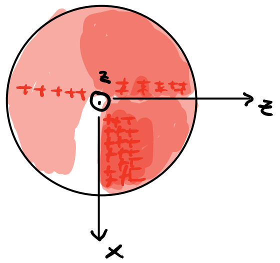
$$\begin{aligned}
\frac{1}{6} \sum_{i=1}^{6} \phi(\vec{r}_i) = \phi(\vec{r}_0) + \frac{1}{6} \nabla^2 \phi(\vec{r})\Big|_{\vec{r}=\vec{r}_0} + \mathcal{O}(\delta^4) \\
\approx 0
\end{aligned}$$

where the final approximation holds if terms of order δ^4 may be neglected.

Problem A

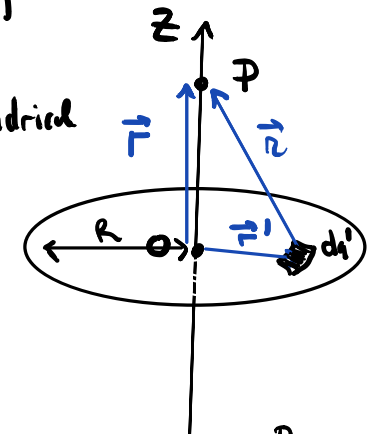


(1)



(2)

cylindrical



$$V(\vec{r}) = \frac{1}{4\pi\varepsilon_0} \iint \frac{dq'}{\imath}$$

- $\left.\begin{array}{l} \vec{r} = z\,\hat{z} \\ \vec{r}\,' = s'\,\hat{s} \end{array}\right\} \imath = \sqrt{z^2 + s'^2}$

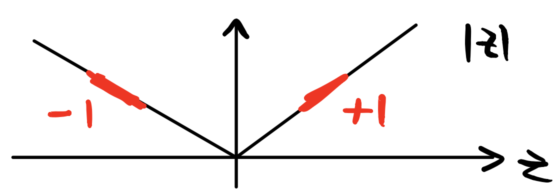
- $dq' = \sigma(\vec{r}\,')\, da' = \sigma_0 \cos(\phi'/4) \cdot s'\, d\phi'\, ds'$ $\quad \phi': 0 \to 2\pi, \quad s': 0 \to R$

$$V(\vec{r}) = \frac{\sigma_0}{4\pi\varepsilon_0} \underbrace{\int_0^R \frac{s'}{\sqrt{z^2 + s'^2}}\, ds'}_{\left[\sqrt{s'^2 + z^2}\right]_0^R} \underbrace{\int_0^{2\pi} \cos(\phi'/4)\, d\phi'}_{4} = \frac{\sigma_0}{\pi\varepsilon_0}\left[\sqrt{z^2 + R^2} - |z|\right]$$

$$V(z) = \frac{\sigma_0}{\pi\varepsilon_0}\left[\sqrt{z^2 + R^2} - |z|\right] = V(z)$$

(3) $\vec{E} = -\vec{\nabla} V = -\dfrac{\partial V}{\partial z}\,\hat{z}$

$$= -\frac{\sigma_0}{\pi\varepsilon_0}\left[\text{blabla} - \overbrace{\frac{\partial}{\partial z}|z|}^{\text{sign}(z)}\right]$$



Problem B

Method 1:

$$
\begin{aligned}
\mathrm{E} &= \frac{\epsilon_0}{2}\int_{allspace} \vec{E}\cdot\vec{E}d\tau \\
&= \frac{\epsilon_0}{2}\int_0^{2\pi} d\phi \int_0^{\pi} d\theta \left(\int_R^{\infty} \frac{Q^2}{16\pi^2\epsilon_0^2 r^2}dr + \int_0^R \frac{Q^2 r^4}{16\pi^2\epsilon_0^2 R^6}dr\right) \\
&= \frac{Q^2}{8\epsilon_0 R}\left(1+\frac{1}{5}\right) \\
&= \frac{3Q^2}{20\epsilon_0 R}
\end{aligned}
$$

Method 2: The charge density ρ is given by

$$\rho = \frac{3Q}{4\pi R^3}.$$

The work done in building up the uniformly charged sphere

$$
\begin{aligned}
\mathrm{E} &= \int_{sphere} V(r)dq \\
&= \int_0^R \frac{q(r)}{4\pi\epsilon_0 r}4\pi r^2 \rho dr \\
&= \frac{\rho^2}{\epsilon_0}\int_0^R \frac{4}{3}\pi r^4 dr \\
&= \frac{4\pi\rho^2 R^5}{15\epsilon_0} \\
&= \frac{3Q^2}{20\epsilon_0 R}
\end{aligned}
$$

Problem C

Since we have to calculate the electrostatic energy per particle we can compute the electrostatic potential energy between one particle and the rest of the system. Without loss of generality consider one of the particles to be at the center i.e., the origin then the potential energy per particle is given by

$$\begin{aligned}
\mathrm{E} = & \frac{1}{2}\left(\int_{a}^{\infty} \frac{-q^2}{4\pi\epsilon_0 x^2} dx + \int_{-\infty}^{-a} \frac{-q^2}{4\pi\epsilon_0 x^2} dx\right) \\
& + \left(\int_{2a}^{\infty} \frac{q^2}{4\pi\epsilon_0 x^2} dx + \int_{-\infty}^{-2a} \frac{q^2}{4\pi\epsilon_0 x^2} dx\right) \\
& + \left(\int_{3a}^{\infty} \frac{-q^2}{4\pi\epsilon_0 x^2} dx + \int_{-\infty}^{-3a} \frac{-q^2}{4\pi\epsilon_0 x^2} dx\right) \\
& + \ldots \\
= & \frac{1}{4\pi\epsilon_0} \Sigma_{j=1}^{\infty} (-1)^j \frac{q^2}{|\vec{r}_j|}
\end{aligned}$$

where j is an integer.

$$\begin{aligned}
\mathrm{E} &= \left(-\frac{q^2}{4\pi\epsilon_0 a} + \frac{q^2}{4\pi\epsilon_0 (2a)} - \frac{q^2}{4\pi\epsilon_0 (3a)} + \ldots\right) \\
&= -\frac{q^2}{\pi\epsilon_0 a}\left(1 - \frac{1}{2} + \frac{1}{3} - \ldots\right) \\
&= -\frac{q^2}{\pi\epsilon_0 a} \Sigma_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} \\
&= -\frac{q^2 \log_e 2}{4\pi\epsilon_0 a}.
\end{aligned}$$

The above series is called the alternating harmonic series and it is the Dirichlet eta function, $\eta(z)$ evaluated at $z = 1$.