MATH 3215 SPRING 2019
MIDTERM 2
MAR. 15, 2019

Instructions.

1. There are four problems with three parts each. Each part is four points. If you do not know how to do a part, check the box in that part and write nothing else to receive one point.
2. If you turn in your paper by the time I call the end of the exam, you get two extra points.
3. Partial credit will be given. Show all your work for the highest chance of getting full credit. If you just write the answer with no explanation, you will get no credit. If you get the correct answer, but by a method that is incorrect, the answer will be considered incorrect.
4. You cannot use calculators. Therefore you do not need to simplify your answers: it is ok to write an answer like $\binom{10}{2}\binom{5}{3}$ or an answer like $\frac{.1+.3+.2}{3}$. You also cannot use any notes, you cannot talk or give signals to anyone.
5. You cannot use your phone. The only things on your desk should be writing tools and the exam.
6. Write legibly. If I cannot read your writing, you answer will be considered incorrect.

DO NOT OPEN THE TEST UNTIL YOU ARE INSTRUCTED TO DO SO.

NAME:

KEY

TO BE USED FOR GRADING:

1 : 12 pts	2 : 12 pts	3 : 12 pts	4 : 12 pts	Total /48

1. 128 (which is 2^7) people play a game using a fair coin. Each player flips their coin 6 times and records their outcomes. A player is a "winner" if their outcomes are all heads and a "super winner" if their outcomes are all tails.

(a) (4 pts) Find an exact expression for the probability that there are no "winners" and no "super winners."

$$\mathbb{P}(\text{one person is not winner or super winner}) = \frac{2^6-2}{2^6} = 1-\frac{1}{2^5}$$

$$\mathbb{P}(\text{no winner or super winner}) = \left(1-\frac{1}{2^5}\right)^{2^7} \boxed{= \left(1-\frac{1}{32}\right)^{128}}$$

(b) (4 pts) If X is the number of "winners," use the Poisson approximation to find an approximate value of $\mathbb{P}(X \leq 2)$.

$$X \sim \text{Bin}\left(2^7, \frac{1}{2^6}\right) = \text{Bin}\left(2^7, \frac{2}{2^7}\right) \approx \text{Poisson}(\underbrace{2}_{\lambda})$$

$$\mathbb{P}(X \leq 2) \approx \sum_{k=0}^{2} e^{-\lambda}\frac{\lambda^k}{k!} = \boxed{\sum_{k=0}^{2} e^{-2}\frac{2^k}{k!}}$$

(c) (4 pts) Let Y be the number of "super winners." Find an exact expression for $\mathbb{P}(X = 2, Y = 3)$. Are X and Y independent?

$(X,Y) \sim$ Trinomial with $n = 2^7$

$p_A = \frac{1}{2^6}$ $\quad p_B = \frac{1}{2^6}$, $\quad p_C = 1-\frac{2}{2^6}$

$$\mathbb{P}(X=2, Y=3) = \boxed{\frac{(2^7)!}{2!\,3!\,(2^7-2-3)!}\left(\frac{1}{2^6}\right)^2\left(\frac{1}{2^6}\right)^3\left(1-\frac{2}{2^6}\right)^{2^7-2-3}}$$

Not indep., since (X,Y) is trinomial (its range is not rectangular)

2. (a) Let f be the real function defined by $f(x) = c/\sqrt{x}$ for $x \in (0,1]$ and $f(x) = 0$ otherwise. (Here, c is some real number.)

i. (4 pts) Find the value of c so that f is a pdf.

$$1 = \int_0^1 \frac{c}{\sqrt{x}}\,dx = c\,\frac{x^{1/2}}{1/2}\Big|_0^1 = 2c$$

$$\Rightarrow \boxed{c = 1/2}$$

ii. (4 pts) Suppose that X is a random variable with f as its pdf. Find $\mathrm{Var}(X^{1/8})$.

$$\mathrm{Var}(X^{1/8}) = \mathbb{E}(X^{1/4}) - (\mathbb{E}(X^{1/8}))^2$$

$$\mathbb{E}(X^{1/4}) = \int_0^1 x^{1/4}\cdot\frac{1}{2\sqrt{x}}\,dx = \frac{1}{2}\int_0^1 x^{-1/4}\,dx = \frac{1}{2}\,\frac{x^{3/4}}{3/4}\Big|_0^1 = \boxed{\frac{2}{3}}$$

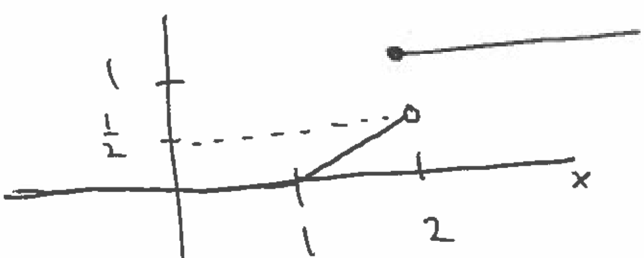
$$\mathbb{E}(X^{1/8}) = \int_0^1 x^{1/8}\cdot\frac{1}{2\sqrt{x}}\,dx = \frac{1}{2}\int_0^1 x^{-3/8}\,dx = \frac{1}{2}\,\frac{x^{5/8}}{5/8}\Big|_0^1 = \boxed{\frac{4}{5}}$$

$$\mathrm{Var}(X^{1/8}) = \boxed{\frac{2}{3} - \left(\frac{4}{5}\right)^2}$$

(b) (4 pts) Consider the cdf

$$F(x) = \begin{cases} 0 & \text{if } x \le 1 \\ \frac{x-1}{2} & \text{if } x \in [1,2) \\ 1 & \text{if } x \ge 2. \end{cases}$$

If X is a random variable with this cdf, find $\mathbb{E}X$.



$\mathbb{P}(X=2) =$ value of jump $= \frac{1}{2}$

pdf in $[1,2) = \frac{d}{dx}\left(\frac{x-1}{2}\right) = \frac{1}{2}$

$$\mathbb{E}X = \int_1^2 x\cdot\frac{1}{2}\,dx + 2\cdot\frac{1}{2}$$

$$= \frac{1}{2}\,x^2\Big|_1^2 + 1$$

$$= \frac{1}{2}(4-1) + 1 = \boxed{\frac{5}{2}}$$

3. Calls arrive at a call center according to a Poisson process with average rate 30/hour.

(a) (4 pts) Find an exact expression for the probability that ≥ 2 calls arrive in the first minute.

P.P. with $\lambda = \frac{1}{2}$ / min.

X = # calls in first minute $\sim$ Poisson$(1/2)$

$$\mathbb{P}(X \geq 2) = \sum_{k=2}^{\infty} e^{-\lambda} \frac{\lambda^k}{k!} = \boxed{e^{-1/2} \sum_{k=2}^{\infty} \frac{\left(\frac{1}{2}\right)^k}{k!}}$$

(b) (4 pts) If T_1 is the time of the first call (in minutes), find the moment generating function for $T_1 + 5$.

$T_1 \sim \exp(\theta)$ with $\theta = \frac{1}{\lambda} = 2$ (pdf is $\frac{1}{\theta} e^{-x/\theta}$ for $x \geq 0$)

$$M(t) = \mathbb{E}\, e^{t(T_1+5)} = e^{5t}\, \mathbb{E} e^{tT_1}$$

$$= e^{5t} \int_0^{\infty} e^{tx} \cdot \frac{1}{2} e^{-x/2}\, dx$$

$$= \frac{1}{2} e^{5t} \cdot \frac{1}{t - \frac{1}{2}} e^{(t-\frac{1}{2})x} \Big|_{x=0}^{x=\infty} = \frac{1}{2} e^{5t} \cdot \frac{1}{\frac{1}{2} - t} \boxed{\text{if } t < \tfrac{1}{2} \;\; = \frac{e^{5t}}{1-2t}}$$

(c) (4 pts) Your friend erroneously believes that the number of calls arriving in the first hour is distributed as a normal random variable with $\mu = 30$ and $\sigma^2 = 25$. Under this assumption, use the attached table to find the approximate probability that the number of calls arriving in the first hour is in the interval $(25, 35]$.

If Y = # calls in first hour is $N(30, 25)$, then

$$\mathbb{P}(Y \in (25, 35]) = \mathbb{P}\left(\frac{Y-30}{5} \in (-1, 1]\right)$$

$$= \Phi(1) - \Phi(-1)$$

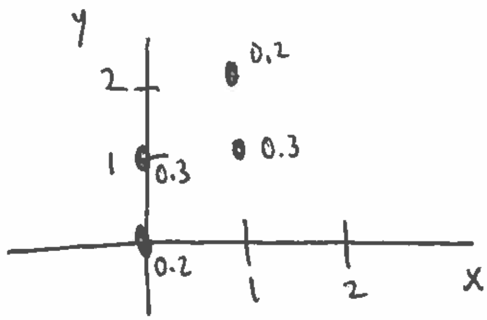
$$= \Phi(1) - (1 - \Phi(1))$$

$$= 2\Phi(1) - 1$$

$$\approx \boxed{2 \cdot (0.8413) - 1}$$

4. Let X and Y have the joint pmf $f(x, y)$ defined by $f(0,0) = f(1,2) = 0.2$, $f(0,1) = f(1,1) = 0.3$.

(a) (4 pts) Depict the points and corresponding probabilities on a graph (as we have done in class).



(b) (4 pts) Compute the mean of X and the mean of Y.

$$f_X(x) = \begin{cases} 0.5 & x = 0 \\ 0.5 & x = 1 \end{cases} \Rightarrow \mathbb{E}X = 0 \cdot 0.5 + 1 \cdot 0.5 = \boxed{0.5}$$

$$f_Y(y) = \begin{cases} 0.2 & y = 0 \\ 0.6 & y = 1 \\ 0.2 & y = 2 \end{cases} \Rightarrow \mathbb{E}Y = 0 \cdot 0.2 + 1 \cdot 0.6 + 2 \cdot 0.2 = \boxed{1}$$

(c) (4 pts) Does the least squares regression line for X and Y have positive, negative, or zero slope? Justify your answer.

slope is $\rho \frac{\sigma_Y}{\sigma_X}$, which is >0, <0, $=0$ if and only if ρ is. (since $\frac{\sigma_Y}{\sigma_X} > 0$)

$\rho = \frac{Cov(X,Y)}{\sigma_X \sigma_Y}$ is >0, <0, or $=0$ if and only if $Cov(X,Y)$ is.

$$\begin{aligned} Cov(X,Y) = \mathbb{E}XY - \mu_X\mu_Y &= \mathbb{E}XY - (0.5)(1) \\ &= (1)(0.3) + (2)(0.2) - 0.5 \\ &= 0.7 - 0.5 \\ &= 0.2 \\ &> 0 \end{aligned}$$

, $\boxed{\text{so slope is } > 0.}$

Formulas

- Bayes' Theorem: If $B_1, \ldots, B_k$ form a partition of the sample space S, and A is any event, then for any $i = 1, \ldots, k$,

$$\mathbb{P}(B_i \mid A) = \frac{\mathbb{P}(A \mid B_i)\mathbb{P}(B_i)}{\sum_{j=1}^{k} \mathbb{P}(A \mid B_j)\mathbb{P}(B_j)}.$$

- The following are probability mass functions for distributions we discussed.

1. Bernoulli with parameter p:

$$f(x) = \begin{cases} p & x = 1 \\ 1-p & x = 0 \\ 0 & \text{otherwise} \end{cases}$$

2. Geometric with parameter p:

$$f(x) = \begin{cases} (1-p)^{x-1}p & x = 1, 2, \ldots \\ 0 & \text{otherwise} \end{cases}$$

3. Binomial with parameters n, p:

$$f(x) = \begin{cases} \binom{n}{x} p^x (1-p)^{n-x} & x = 0, 1, 2, \ldots, n \\ 0 & \text{otherwise} \end{cases}$$

4. Hypergeometric with parameters N_1, N_2, n:

$$f(x) = \begin{cases} \frac{\binom{N_1}{x}\binom{N_2}{n-x}}{\binom{N_1+N_2}{n}} & \text{for } \max\{0, n-N_2\} \leq x \leq \min\{N_1, n\} \\ 0 & \text{otherwise} \end{cases}.$$

5. Negative binomial with parameters r, p:

$$f(x) = \binom{x-1}{r-1} p^r (1-p)^{x-r}, \; x = r, r+1, \ldots$$

6. Poisson with parameter λ:

$$f(x) = e^{-\lambda}\frac{\lambda^x}{x!}, \; x = 0, 1, 2, \ldots$$

7. Trinomial with parameters n, p_A, p_B, p_C:

$$f(x, y) = \frac{n!}{x!y!(n-x-y)!} p_A^x p_B^y p_C^{n-x-y} \text{ for } x, y \geq 0 \text{ and } x + y \leq n.$$