MATH 3215 SPRING 2019
MIDTERM 2 SOLUTIONS

1. **An airline always overbooks if possible. A particular plane has 95 seats on a flight in which a ticket sells for $300. The airline sells 100 such tickets for this flight.**

   (a) **If the probability of an individual not showing up is 0.05, assuming independence, what is the approximate probability that the airline can accommodate all the passengers who do show up? (Use the Poisson approximation.)**

   Solution. We did this in class.

   (b) **If the airline must return the $300 price plus a penalty of $400 to each passenger that cannot get on the flight, what is the approximate expected payout (penalty plus ticket refund) that the airline will pay?**

   Solution. We did this in class.

2. **Consider the pdf $f(x) = ce^x$ for $x \in [1, 2]$ and zero otherwise.**

   (a) **Find c so that it is a pdf.**

   Solution. We need $c \geq 0$ so that $f(x) \geq 0$ for all x. Also we need $\int_{-\infty}^{\infty} f(x)\,\mathrm{d}x = 1$, so we solve:

$$1 = \int_1^2 ce^x\,\mathrm{d}x = ce^x\Big|_1^2 = c(e^2 - e),$$

   so $c = 1/(e^2 - e)$.

   (b) **Find the moment generating function and use it to compute the expected value.**

   Solution. The moment generating function, $M(t)$, is

$$\begin{aligned}\int_{-\infty}^{\infty} e^{tx} f(x)\,\mathrm{d}x = \int_1^2 e^{tx} ce^x\,\mathrm{d}x = c\int_1^2 e^{x(t+1)}\,\mathrm{d}x &= \frac{c}{t+1} e^{x(t+1)}\Big|_1^2 \\ &= \frac{c}{t+1}(e^{2(t+1)} - e^{t+1}).\end{aligned}$$

   So the expected value is

$$\begin{aligned}M'(0) &= c\frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{t+1}(e^{2(t+1)} - e^{t+1})\Big|_{t=0} \\ &= c\left[-\frac{1}{(t+1)^2}(e^{2(t+1)} - e^{t+1}) + \frac{1}{t+1}(2e^{2(t+1)} - e^{t+1})\right]_{t=0} \\ &= c\left[-(e^2 - e) + (2e^2 - e)\right] \\ &= \frac{e^2}{e^2 - e}.\end{aligned}$$

(c) **Find a median of this distribution and also find $\pi_{1/3}$.**
Solution. The CDF of this distribution is

$$F(x) = \int_1^x ce^t \, \mathrm{d}t = ce^t \Big|_1^x = c(e^x - e) \text{ for } x \in [1, 2],$$

0 for $x \leq 1$ and 1 for $x \geq 2$. To find the median m, we solve

$$\frac{1}{2} = F(m) = c(e^m - e) \Rightarrow m = \log\left(\frac{e^2 - e}{2} + e\right).$$

Similarly to find $\pi_{1/3}$, we solve $1/3 = F(\pi_{1/3})$ to find

$$\pi_{1/3} = \log\left(\frac{e^2 - e}{3} + e\right).$$

3. (a) **Customers call a support line according to a Poisson process with rate 15 per hour.**

i. **Let T_3 be the time (in minutes) that the third customer calls. Find the probability that T_3 is greater than or equal to 8.**
Solution. The Poisson process has $\lambda = 1/4$ (in minutes) and the time T_3 has Gamma distribution with $\alpha = 3$ and $\theta = 1/\lambda = 4$. Therefore if f is the pdf for this Gamma variable, then

$$\mathbb{P}(T_3 \geq 8) = \int_8^\infty f(x) \, \mathrm{d}x = \int_8^\infty \frac{1}{\Gamma(3)4^3} x^2 e^{-x/4} \, \mathrm{d}x.$$

ii. **Find the probability that in the first 4 minutes, exactly 1 customer calls AND in the next 4 minutes (from time 4 to 8), exactly 1 customer calls.**
Solution. The number of arrivals for the Poisson process for disjoint intervals of times are independent. Therefore if X_1 is the number of arrivals in $[0, 4]$ and X_2 is the number of arrivals in $[4, 8]$, then

$$\mathbb{P}(X_1 = 1 \text{ and } X_2 = 1) = \mathbb{P}(X_1 = 1)\mathbb{P}(X_2 = 1).$$

The definition of the Poisson process is exactly the same if we start time at $t = 4$, so both of these probabilities are the same. This gives $\mathbb{P}(X_1 = 1)^2$. The variable X_1 is Poisson distributed with parameter λt, where $t = 4$, so the parameter is 1. Therefore $\mathbb{P}(X_1 = 1) = g(1)$, where $g(k) = e^{-1}/k!$ is the pmf for Poisson with parameter 1. Therefore the final probability is

$$g(1)^2 = e^{-2}.$$

(b) **Let X be a Poisson random variable with $\lambda = 1$ and let Y be a Gamma random variable with $\alpha = 3$ and $\theta = 1$. Compare the two probabilities**

$\mathbb{P}(X \geq 3)$ and $\mathbb{P}(Y \leq 1)$. Are they equal, or is one bigger, or is there not enough information? Explain your answer.

Solution. Y has the same distribution as the third arrival for a Poisson process with $\lambda = 1$. X has the same distribution as the number of arrivals until $t = 1$ in a Poisson process with $\lambda = 1$. Note that the number of arrivals is ≥ 3 exactly when the third arrival occurs at time ≤ 1. Therefore

$$\mathbb{P}(X \geq 3) = \mathbb{P}(Y \leq 1).$$

4. (a) **We flip a fair coin. If it comes up heads, we wait 1 minute and call Robbie Hootman. If it comes up tails, we spin a spinner with scale $[0, 2)$ and wait the amount of time (in minutes) on which the spinner lands before making the call. Let X be the amount of time we wait, and find the CDF of X.**

Solution. To compute the CDF $F(x)$, we consider different cases. First, X cannot be negative, so if $x < 0$, we have $F(x) = 0$. If $x \in [0, 1)$, then $X \leq x$ exactly when the coin comes up tails, and the spinner is $\leq x$. This has probability $(1/2)(x/2) = x/4$. If $x \in [1, 2)$, then $X \leq x$ exactly when the coin comes up heads, or when the coin comes up tails but the spinner is $\leq x$. This has probability $(1/2) + (1/2)(x/2) = (1/2) + x/4$. Last, if $x \geq 2$, then $X \leq x$ always, so $F(x) = 1$. In total,

$$F(x) = \begin{cases} 0 & \text{if } x < 0 \\ \frac{x}{4} & \text{if } x \in [0, 1) \\ \frac{1}{2} + \frac{x}{4} & \text{if } x \in [1, 2) \\ 1 & \text{if } x \geq 2. \end{cases}$$

(b) **Now we forget about the call and just flip the coin.**

i. **Let Y be the number of flips needed to get 2 heads in total. Find the probability that $Y \leq 4$.**
Solution. Y is negative binomial and has range $\{2, 3, 4, \dots\}$.

$$\begin{aligned} \mathbb{P}(Y = 2) &= \mathbb{P}(\{HH\}) = \frac{1}{4} \\ \mathbb{P}(Y = 3) &= \mathbb{P}(\{HTH, THH\}) = \frac{1}{4} \\ \mathbb{P}(Y = 4) &= \mathbb{P}(\{HTTH, THTH, TTHH\}) = \frac{3}{16}. \end{aligned}$$

Therefore $\mathbb{P}(Y \leq 4) = 1/4 + 1/4 + 3/16 = 11/16$.

ii. **Let Z be the number of flips needed to get 2 heads in a row. Find the probability that $Z \leq 4$. Compare this answer to the answer in the preceding part and explain the discrepancy.**

Solution. Again Z has range $\{2, 3, 4, \dots\}$. Now we compute

$$\begin{aligned}
\mathbb{P}(Z = 2) &= \mathbb{P}(\{HH\}) = \frac{1}{4} \\
\mathbb{P}(Z = 3) &= \mathbb{P}(\{THH\}) = \frac{1}{8} \\
\mathbb{P}(Z = 4) &= \mathbb{P}(\{HTHH, TTHH\}) = \frac{1}{8}.
\end{aligned}$$

Therefore $\mathbb{P}(Z \le 4) = 1/4 + 1/8 + 1/8 = 1/2$. This is smaller than the answer for the preceding part because it is more difficult to get two heads in a row than to get two heads in total.

5. (a) **While at work, Robbie Hootman theorizes that the quarterly earnings X of the airline company at which he works approximately follow a normal distribution with mean (in billions of dollars) 12 and variance 4. Find the probability that X is at least 8.**

Solution. X is approximately $N(12, 4)$. To compute the probability, we standardize:

$$\mathbb{P}(X \ge 8) = \mathbb{P}\left(\frac{X - 12}{2} \ge \frac{8 - 12}{2}\right) = \mathbb{P}\left(Z \ge -2\right),$$

where Z is a standard normal. This is the area to the right of -2 under the standard normal curve, and by symmetry it is equal to one minus the area to the right of 2. From the table, we obtain $1 - 0.0228 = 0.9772$.

(b) **Find the probability that $|X - 10| \le 2$.**

Solution. Again, we standardize. The probability is

$$\mathbb{P}\left(|X - 10| \le 2\right) = \mathbb{P}(8 \le X \le 12) = \frac{1}{2} - \mathbb{P}(X \le 8).$$

From the last part, we have $\mathbb{P}(X \le 8) = 1 - \mathbb{P}(X \ge 8) \sim 0.0228$. Therefore we obtain $1/2 - 0.0228 = 0.4772$.

(c) **What is the distribution of $(X - 12)^2$?**

Solution. We write

$$(X - 12)^2 = \left(2 \cdot \frac{X - 12}{2}\right)^2 = 4 \cdot \left(\frac{X - 12}{2}\right)^2.$$

Since $(X - 12)/2$ is a standard normal (we standardized X), and the square of a standard normal is a $\chi^2(1)$ random variable, we find that the distribution of $(X - 12)^2$ is that of 4 times a $\chi^2(1)$ random variable.