For gravity, $K = GMm$, so $e = \sqrt{1 + \underbrace{\dfrac{2EJ^2}{G^2M^2m^3}}_{\text{all of these are constants throughout the motion}}}$

→ For a given $J$, orbits are classified according to total energy

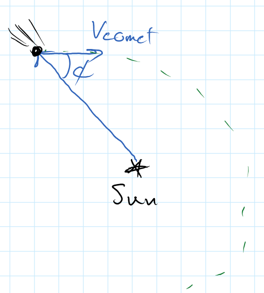
$E < 0$, $e < 1$, closed, elliptical orbit (or, even circular)

$E = 0$, $e = 1$, parabolic orbit

$E > 0$, $e > 1$, hyperbolic orbit

Since $E = T + V$ is constant, closed orbits are those w/ $T < |V|$; open orbits are those w/ $T > |V|$. For gravity, $\dfrac{mv^2}{2} - \dfrac{GMm}{r} = E$, so orbit is elliptical, parabolic or hyperbolic depending on whether $v^2$ is $<, =, > \dfrac{2GM}{r}$

Ex: A comet is observed to have a speed $v_{comet}$ when it is a distance $r_{comet}$ from the Sun. Its direction of motion makes an angle $\phi$ w/ the radius vector from the Sun. Find the eccentricity of the comet's orbit.




$$e = \sqrt{1 + \frac{2EJ^2}{mK^2}}, \quad |\vec{J}| = m|\vec{r} \times \dot{\vec{r}}| = m|\vec{r}||\dot{\vec{r}}|\sin\phi$$
$$= m r_{comet} v_{comet} \sin\phi$$

$$E = \frac{1}{2} m v_{comet}^2 - \frac{GMm}{r_{comet}} \quad ; \quad K = GMm$$

$$\therefore \quad e = \sqrt{1 + \frac{2\left(\dfrac{\cancel{m} v_{comet}^2}{2} - \dfrac{GM\cancel{m}}{r_{comet}}\right)\cancel{m^2} r_{comet}^2 v_{comet}^2 \sin^2\phi}{\cancel{m} G^2 M^2 \cancel{m^2}}}$$

$$e = \sqrt{1 + \frac{\left(r_{comet}^2 V_{comet}^4 \sin^2\phi - 2GM r_{comet} V_{comet}^2 \sin^2\phi\right)}{(GM)^2}}$$

$$\rightarrow \quad e = \sqrt{1 + \frac{r_{comet}\left(V_{comet}\sin\phi\right)^2\left(r_{comet}V_{comet}^2 - 2GM\right)}{(GM)^2}} \quad \boxed{B}$$

NB: Given the same initial position, speed & direction of motion, a grain of sand, a drifting spaceship, or a comet would all have identical orbits, provided no other bodies came near enough. (assuming, also, all body's have small mass compared to Sun).

You can prove for elliptical orbits that the semi-major axis is determined by the total energy $\left(a = \frac{K}{2|E|}\right)$ & that the semi-minor axis depends on the ang. momentum $\left(b = \frac{J}{\sqrt{2m|E|}}\right)$

Ex: What is the semi-major axis of the comet from above?

$$a = \frac{GM\cancel{m}}{\cancel{m}V_{comet}^2 - \frac{2GM\cancel{m}}{r_{comet}}} = \frac{GMr_{comet}}{V_{comet}^2 r_{comet} - 2GM}$$

---

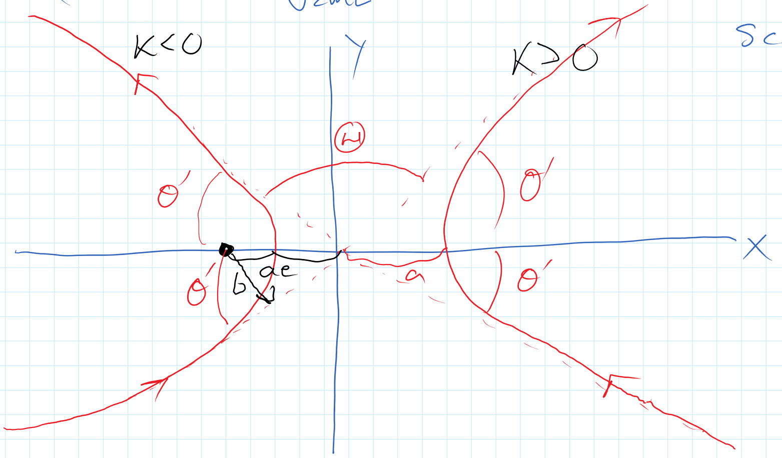
## Hyperbolic Orbits ($E>0$, $e>1$)

For both repulsive ($K>0$) & attractive ($K<0$) cases, the orbit can be written in Cartesian coordinates (chosen so that the $\theta_0 = 0$)

$$\frac{(x-ae)^2}{a^2} - \frac{y^2}{b^2} = 1.$$

where $a = \frac{l}{e^2-1} = \frac{|K|}{2E}$ & $b^2 = al = \frac{J^2}{2mE}$. When $e>1$, this the the eqn of a hyperbola w/ center at $(ae, 0)$ & semi-axes $a$ & $b$.

(Aside: $b = \frac{J}{\sqrt{2mE}}$. If $E = \frac{1}{2}mv^2$, $b = \frac{J}{mv}$ = impact parameter from earlier scattering example.)




From the polar equation, determine directions where $r \to \infty$. $(\theta')$