Sample Midterm 2A, Math 2552, Summer 2019

Instructor: Dr. Greg Mayer   Date: May 2019   Time: 10:05 am to 11:20 am

Student GT Email Address: ______________________________@gatech.edu

Student Instructions

- **Show your work** and justify your answers for all questions unless stated otherwise.
- You will have 75 minutes to take the exam. There are 50ish total points possible.
- Calculators, notes, cell phones, books are not allowed.
- Please write your answers neatly and show all of your work.
- Use dark and clear writing: your exam will be scanned into a digital system.
- Exam pages are double sided. Be sure to complete both sides.
- Leave a 1 inch border around the edges of exams.
- Check that every page has the same booklet number.

1. (10 points) Solve the differential equation.

$$y'' + y = \sec x$$

You may find the formula

$$\int \tan x \, dx = \ln |\cos x|$$

helpful.

2. (10 points) Solve the initial value problem.

$$\vec{x}\,' = \frac{1}{2}\begin{pmatrix} -2 & -1 \\ 1 & 0 \end{pmatrix}\vec{x}, \quad \vec{x}(0) = \begin{pmatrix} -1 \\ 0 \end{pmatrix}$$

3. (8 points) Solve the differential equation using the method of undetermined coefficients.

$$y'' + 4y = 3\sin(2t)$$

4. (8 points) The position of a moving object, $y(t)$, satisfies the IVP

$$y'' + 2y' + 5y = 0, \quad y(0) = 0, \quad y'(0) = 4$$

The solution is $y(t) = 2e^{-t}\sin(2t)$.

(a) Give a rough sketch of the solution, $y(t)$ for $t > 0$. Label your axes.

(b) For $t \geq 0$, sketch the trajectory of the object in the phase plane. Indicate the location corresponding to $t = 0$, the direction of motion, and label your axes.

5. (4 points) Construct an initial value problem for the following situation.

   A spring is stretched 0.5 m by a force of 0.25 newtons (N). A mass weighing 2 kg is attached to a spring that is also attached to a viscous damper that applies a force of 10 N when the velocity of the mass is 5 m/s. The mass is pulled down 2 m below its equilibrium position and given an initial upward velocity of 4 m/s.

6. (3 points) If $W[f, g]$ is the Wronskian of f and g, and if $u = 2f - g,\ v = f + 2g$, find the Wronskian $W[u, v]$ of u and v in terms of $W[f, g]$.

7. (7 points) Use the method of reduction of order to find a second solution y_2 of the given differential equation such that $\{y_1, y_2\}$ is a fundamental set of solutions on the given interval.

$$t^2y'' - t(t+2)y' + (t+2)y = 0, \quad t > 0, \quad y_1(t) = t$$

SAMPLE MIDTERM 2

1) $y'' + y = \sec x$, $\int \tan x \, dx = \ln|\cos x|$

HOMOG. SOL'N : $y''+y=0 \Rightarrow y = c_1 \cos x + c_2 \sin x$ (2)

PARTICULAR: $y_p = y_1 u_1 + y_2 u_2$ (1)

SOLVE : $\left.\begin{array}{l} y_1 u_1' + y_2 u_2' = 0 \\ y_1' u_1' + y_2' u_2' = \sec x \end{array}\right\}$ (1)

OR: $\left\{\begin{array}{ll} \cos x \, u_1' + \sin x \, u_2' = 0 & (A) \\ -\sin x \, u_1' + \cos x \, u_2' = \sec x & (B) \end{array}\right\}$ (1)

(A) $\cos x$ − (B) $\sin x$ yields : $\left.\begin{array}{l} u_1' = -\sec x \sin x = -\tan x \\ \Rightarrow u_1 = -\int \tan x \, dx = \ln|\cos x| \end{array}\right\}$ (2)

(A) $\sin x$ + B $\cos x$ yields $u_2' = 1 \Rightarrow u_2 = x$ (2)

$$\Rightarrow \boxed{y = c_1 \cos x + c_2 \sin x + \cos x \ln|\cos x| + x \sin x}$$ (1)

$$\vec{x}\,' = \frac{1}{2}\begin{pmatrix} -2 & -1 \\ 1 & 0 \end{pmatrix} \vec{x}, \quad \vec{x}(0) = \begin{pmatrix} -1 \\ 0 \end{pmatrix}$$

EIGENVALUES: $\begin{vmatrix} -1-\lambda & -1/2 \\ 1/2 & -\lambda \end{vmatrix} = \lambda^2 + \lambda + 1/4 = (\lambda + 1/2)^2 = 0$ 1

$\Rightarrow \lambda = -1/2$ 1

EIGENVECTORS: $\begin{pmatrix} -1/2 & -1/2 \\ \cdot & \cdot \end{pmatrix} \vec{v}_1 = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \vec{v}_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ 2

$\Rightarrow$ DEFECTIVE MATRIX

$\Rightarrow$ SOLVE $\begin{pmatrix} -1/2 & -1/2 \\ 1/2 & 1/2 \end{pmatrix} \cdot \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ 1

$$\left.\begin{array}{l} \Rightarrow -\frac{1}{2}(w_1 + w_2) = 1 \Rightarrow w_1 = -w_2 - 2 \\ \Rightarrow w_2 \text{ free, choose } w_2 = 0,\ \vec{w} = \begin{pmatrix} -2 \\ 0 \end{pmatrix} \end{array}\right\}$$ 2

$\Rightarrow \vec{x}(t) = c_1 e^{-t/2} \begin{pmatrix} 1 \\ -1 \end{pmatrix} + c_2 e^{-t/2} \left(t \begin{pmatrix} 1 \\ -1 \end{pmatrix} + \begin{pmatrix} -2 \\ 0 \end{pmatrix} \right)$ 1

$$\left.\begin{array}{l} \vec{x}(0) = \begin{pmatrix} -1 \\ 0 \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ -1 \end{pmatrix} + c_2 \begin{pmatrix} -2 \\ 0 \end{pmatrix} \\ \text{By inspection, } c_1 = 0,\ c_2 = 1/2 \end{array}\right\}$$ 1

$\Rightarrow \vec{x} = \frac{1}{2} e^{-t/2} \left(t \begin{pmatrix} 1 \\ -1 \end{pmatrix} + \begin{pmatrix} -2 \\ 0 \end{pmatrix} \right)$ 1

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3. (8 points) Solve the differential equation using the method of undetermined coefficients.

$$y'' + 4y = 3\sin(2t)$$

By inspection, $y_h = c_1 \cos 2t + c_2 \sin 2t$. (2)

$y_p = Atc + Bts$, $c = \cos 2t$, $s = \sin 2t$ (2)

$y_p' = Ac - 2Ats + Bs + 2Btc$

$y_p'' = (-2As) + (-2As - 4Atc) + (2Bc) + (2Bc - 4Bts)$

$y_p'' + 4y_p = t\cos 2t(-4A + 4A)$

$+ t\sin 2t(-4B + 4B)$

$+ \cos 2t(-4B)$

$+ \sin 2t(-2A - 2A)$ (2)

$\Rightarrow B = 0$, $A = -3/4$ (1)

$\Rightarrow y = c_1 \cos 2t + c_2 \sin 2t - \frac{3}{4} t \cos 2t$ (1)

(8 points) The position of a moving object, $y(t)$, satisfies the IVP

$$y'' + 2y' + 5y = 0, \quad y(0) = 0, \quad y'(0) = 4$$

(1) axes labelled in both (a), (b)

The solution is $y(t) = 2e^{-t}\sin(2t)$.

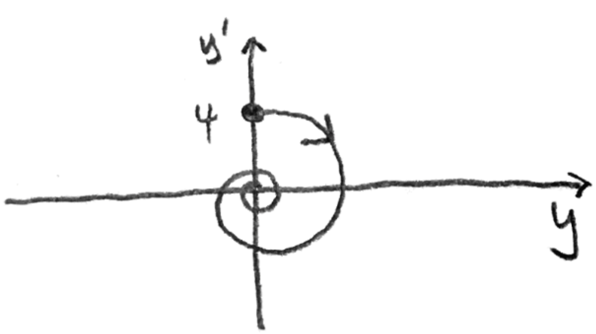
(a) Give a rough sketch of the solution, $y(t)$ for $t > 0$. Label your axes.

(1) $y(0) = 0$

(1) $y'(0) > 0$

(2) general shape: (decaying amplitude sinusoidal)

(b) For $t \geq 0$, sketch the trajectory of the object in the phase plane. Indicate the location corresponding to $t = 0$, the direction of motion, and label your axes.



(1) direction

(1) spiral

(1) curve starts at (0,4)

Answers

5) Using $mg = kL$, $0.25 = \frac{1}{2}k$, so $k = 1/2$. Damping:

$$\begin{aligned} F_d &= \gamma v \\ 10 &= 5\gamma \\ \gamma &= 2 \end{aligned}$$

DE and conditions give the IVP:

$$2y'' + 2y' + \frac{1}{2}y = 0, \quad y(0) = 2, \quad y'(0) = -4$$

6) This is 4.2 # 20. $W(f, g) = fg' - f'g$. Also, $W(u, v) = W(2f - g, f + 2g)$. Then, $W(u, v) = (2f - g)(f + 2g)' - (2f - g)'(f + 2g) = 5fg' - 5f'g = 5W(f, g)$.

7) This is 4.2 # 32. Let $y_2(t) = tv(t)$. Substituting y into the differential equation, we obtain $v'' - v' = 0$. Thus $v'(t) = ce^t$ and, therefore, $v(t) = c_1e^t + c_2$. Therefore, $y_2(t) = c_1te^t + c_2t$. Since we already have $y_1(t) = t$, we set $c_1 = 1$ and $c_2 = 0$. Therefore, we get the solution $y_2(t) = te^t$.