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Assignment 5 Solutions

1. a) The e.o.m. for this system is

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$$m\ddot{x} + \underbrace{3\beta m}_{\lambda}\dot{x} + \underbrace{\frac{17\beta^2 m}{2}}_{K} x = mA\cos\omega t$$

$$\therefore \gamma = \frac{\lambda}{2m} = \frac{3}{2}\beta \; ; \; \omega_0^2 = \frac{K}{m} = \frac{17\beta^2}{2}$$

The maximum amplitude during steady-state oscillations occurs at the resonance frequency

$$\omega_{res} = \sqrt{\omega_0^2 - 2\gamma^2} = \sqrt{\frac{17\beta^2}{2} - \frac{9}{2}\beta^2} = \sqrt{\frac{8}{2}}\beta = \underline{2\beta}$$ 3

b) Max. amplitude is

$$a_{1,max} = \frac{(F/m)}{2\gamma\sqrt{\omega_0^2 - \gamma^2}} \quad \text{(from notes)}$$

In this case, $F_1 = mA$, so

$$a_{1,max} = \frac{A}{3\beta\sqrt{\frac{17\beta^2}{2} - \frac{9}{4}\beta^2}} = \frac{A}{3\beta\sqrt{\frac{25\beta^2}{4}}} = \frac{A}{3\beta^2\left(\frac{5}{2}\right)} = \underline{\frac{2A}{15\beta^2}}$$ 2

2. a) Critical damping is when $\gamma = \omega_0$, so, in this case, $\gamma = 0.5\omega_0$

The resonant freq. is $\omega_{res} = \sqrt{\omega_0^2 - 2\gamma^2} = \sqrt{\omega_0^2 - 2\left(\frac{1}{4}\right)\omega_0^2} = \sqrt{\frac{1}{2}\omega_0^2} = \underline{\frac{\omega_0}{\sqrt{2}}}$

b) $$Q = \frac{\omega_d}{2\gamma} = \frac{\sqrt{\omega_0^2 - \gamma^2}}{2\gamma} = \frac{\sqrt{\omega_0^2 - \frac{1}{4}\omega_0^2}}{\omega_0} = \frac{\sqrt{\frac{3}{4}}\cancel{\omega_0}}{\cancel{\omega_0}} = \underline{\frac{\sqrt{3}}{2}}$$

c) $$\tan\theta_1 = \frac{2\gamma\omega_1}{\omega_0^2 - \omega_1^2} = \frac{2(0.5\omega_0)(2\omega_0)}{\omega_0^2 - 4\omega_0^2} = \frac{2\cancel{\omega_0^2}}{-3\cancel{\omega_0^2}} = -\frac{2}{3}$$

$$\therefore \theta_1 = \tan^{-1}\left(-\frac{2}{3}\right) = \underline{146.°3} \quad (\text{or } -33.°7)$$

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d) $$a_1 = \frac{(F_1/m)}{\sqrt{(\omega_0^2-\omega_1^2)^2+4\gamma^2\omega_1^2}} = \frac{(F_1/m)}{\sqrt{(\omega_0^2-4\omega_0^2)^2+4\left(\frac{\omega_0}{4}\right)^2 4\omega_0^2}}$$

$$= \frac{(F_1/m)}{\sqrt{9\omega_0^4+4\omega_0^4}} = \frac{(F_1/m)}{\sqrt{13}\,\omega_0^2} = \underline{0.277\left(\frac{F_1}{m\omega_0^2}\right)}$$

3. From the notes we have that for $\gamma \ll \omega_0$ & around resonance

$$a_1 \simeq \frac{a_{1,max}\,\gamma}{\sqrt{(\omega_0-\omega_1)^2+\gamma^2}}$$

In this case $a_1 = \frac{1}{2} a_{1,max}$, so

$$\frac{1}{2} = \frac{\gamma}{\left((\omega_0-\omega_1)^2+\gamma^2\right)^{1/2}} \rightarrow (\omega_0-\omega_1)^2+\gamma^2 = 4\gamma^2$$

$$\downarrow$$

$$(\omega_0-\omega_1) = \pm\sqrt{3}\,\gamma$$

$$\downarrow$$

$$\underline{\omega_1 \simeq \omega_0 \pm \gamma\sqrt{3}}$$

4. The equation of motion is

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$$m\ddot{x} + \lambda\dot{x} + Kx = F_0 e^{-\alpha t}\cos\omega_1 t$$

Following the hint, write this in complex notation

$$m\ddot{z} + \lambda\dot{z} + Kz = F_0 e^{\beta t}$$

Assuming a solution of the form $z = a_1 e^{\beta t - i\theta_1}$

$$(m\beta^2 + \lambda\beta + K)\cancel{z} = \left(\frac{F_0}{a_1}\right)\cancel{z}e^{i\theta_1}$$

Sub in for β:

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$$m\alpha^2 - 2im\alpha\omega_1 - m\omega_1^2 - \lambda\alpha + i\lambda\omega_1 + K = \left(\frac{F_0}{a_1}\right)(\cos\theta_1 + i\sin\theta_1)$$

Equating real & imaginary parts:

(a_1)

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$$m(\alpha^2-\omega_1^2)-\lambda\alpha+k=\frac{F_0}{a_1}\cos\theta_1 \qquad 1$$

$$\omega(-2m\alpha+\lambda)=\frac{F_0}{a_1}\sin\theta_1 \qquad 1$$

$\div$ the 2 equations:

$$\tan\theta_1=\frac{\omega_1(\lambda-2m\alpha)}{m(\alpha^2-\omega_1^2)-\lambda\alpha+k} \qquad 1$$

Squaring & adding the 2 equations gives

$$a_1=\frac{F_0}{\left\{\left[m(\alpha^2-\omega_1^2)-\lambda\alpha+k\right]^2+\omega_1^2(\lambda-2m\alpha)^2\right\}^{1/2}} \qquad 1$$

$\therefore$ the sol'n is $\underline{x(t)=a_1e^{-\alpha t}\cos(\omega_1t+\theta_1)+\text{transient term}}$ 1

5. The equation of motion is

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$$m(\ddot{x}+\omega_0^2x)-F_0\sin\omega_1t=0$$

Sub. the sol'n $x=a_1\sin\omega_1t$ & get

$$m(-a_1\omega_1^2\sin\omega_1t+a_1\omega_0^2\sin\omega_1t)-F_0\sin\omega_1t=0$$

or $$a_1m(\omega_0^2-\omega_1^2)-F_0=0$$

$$\Rightarrow a_1=\frac{F_0}{m(\omega_0^2-\omega_1^2)}$$

$\therefore$ The general solution is

$$x(t)=\frac{F_0}{m(\omega_0^2-\omega_1^2)}\sin\omega_1t+a\cos(\omega_0t-\theta) \qquad 2$$

If $x=0$ @ $t=0$: $\quad 0=a\cos(-\theta)\rightarrow\theta=-\pi/2$

but $a\cos(\omega_0 t + \frac{\pi}{2}) = a\sin(\omega_0 t)$

$$\therefore\ x(t) = \frac{F_0 \sin\omega_1 t}{m(\omega_0^2 - \omega_1^2)} + a\sin\omega_0 t \qquad 2$$

If $V = 0$ @ $t = 0$:

$$V = \dot{x}(t) = \frac{\omega_1 F_0 \cos\omega_1 t}{m(\omega_0^2 - \omega_1^2)} + \omega_0 a\cos\omega_0 t$$

at $t=0$,

$$0 = \frac{\omega_1 F_0}{m(\omega_0^2 - \omega_1^2)} + \omega_0 a$$

$$\therefore\ a = -\frac{F_0\omega_1}{m\omega_0(\omega_0^2 - \omega_1^2)} \qquad 2$$

$\therefore$ Sol'n at $t > 0$ w/ these conditions is

$$x(t) = \frac{F_0 \sin\omega_1 t}{m(\omega_0^2 - \omega_1^2)} - \frac{F_0\omega_1}{m\omega_0(\omega_0^2 - \omega_1^2)}\sin\omega_0 t \qquad 2$$

$$= \underline{\frac{F_0}{m(\omega_0^2 - \omega_1^2)}\left[\sin\omega_1 t - \left(\frac{\omega_1}{\omega_0}\right)\sin\omega_0 t\right]}$$