Sample Midterm 1B, Math 1554

PLEASE PRINT YOUR NAME CLEARLY IN ALL CAPITAL LETTERS

First Name ____________________ Last Name ____________________

GTID Number: ______________________

Student GT Email Address: _________________________@gatech.edu

Section Number (e.g. A4, QH3, etc.) _______ TA Name _________________________

Student Instructions

- **Show your work** and justify your answers for all questions unless stated otherwise.
- **Organize your work** in a reasonably neat and coherent way.
- Calculators, notes, cell phones, books are not allowed.
- Use dark and clear writing: your exam will be scanned into a digital system.
- Exam pages are double sided. Be sure to complete both sides.
- Leave a 1 inch border around the edges of exams.
- Any work done on scratch paper will not be collected and will not be graded.

Math 1554, Sample Midterm 1B. Your initials: ______________

You do not need to justify your reasoning for questions on this page.

1. (a) (5 points) Suppose A is an $m \times n$ matrix, $\vec{b} \in \mathbb{R}^m$. Select **true** if the statement is true for all choices of A and $\vec{b}$. Otherwise, select **false**.

true	false	
○	○	If $A\vec{x} = \vec{0}$ and A has reduced echelon form E, then $E\vec{x} = \vec{0}$.
○	○	If every column of A is pivotal, then $A\vec{x} = \vec{b}$ is consistent for any $\vec{b}$.
○	○	The echelon form of A is unique.
○	○	If A is 5×7 and has two pivot columns, then $A\vec{x} = \vec{b}$ has 3 free variables.
○	○	If A has linearly dependent columns, then the columns of A cannot span $\mathbb{R}^m$.

(b) (3 points) Fill in the entries of the matrices below so that they are standard matrices for a one-to-one linear transformation. If it is not possible, write "NP" below the matrix.

$$\begin{pmatrix} 1 & 1 \\ 0 & \\ 0 & \end{pmatrix}, \qquad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & \end{pmatrix}, \qquad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & \\ 1 & 0 & 0 \end{pmatrix}$$

(c) (2 points) Suppose T_A maps $\mathbb{R}^4$ to $\mathbb{R}^2$ has standard matrix

$$A = \begin{pmatrix} 5 & -3 & 2 & -3 \\ 1 & 0 & -2 & 1 \end{pmatrix}$$

Fill in the blanks below:

$$T_A \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} \boxed{}\, x_1 + \boxed{}\, x_2 + \boxed{}\, x_3 + \boxed{}\, x_4 \\ \boxed{}\, x_1 + \boxed{}\, x_2 + \boxed{}\, x_3 + \boxed{}\, x_4 \end{pmatrix}$$

(d) (4 points) Let $A \in \mathbb{R}^{3\times 4}$, $B \in \mathbb{R}^{4\times 3}$ and $\vec{x} \in \mathbb{R}^4$. Circle the operations below that are defined.

$$B^T B \qquad A^T B \qquad \vec{x}^T A \vec{x} \qquad \vec{x}^T BA$$

2. (10 points) If possible, write down an example of a matrix with the following properties. If it is not possible to do so, write *not possible*. You do not need to justify your reasoning.

   (a) A 3×3 matrix A, in echelon form, with just the first and third columns pivotal.

   (b) A 3×2 non-zero matrix A, that is in row reduced echelon form, and $A\vec{x} = \vec{0}$ has a non-trivial solution.

   (c) A 2×3 non-zero matrix in echelon form that is the standard matrix for linear transform T. T is not one-to-one, and T is not onto.

   (d) A matrix $A \in \mathbb{R}^{2\times 2}$ such that $T(\vec{x}) = A\vec{x}$, where T is a linear transformation that reflects vectors in $\mathbb{R}^2$ about the line $x_1 = x_2$ and then projects them onto the x_2 axis.

3. (3 points) Suppose $T_A : \mathbb{R}^4 \mapsto \mathbb{R}^3$ is a onto linear transformation. Circle the matrices, if any, that A could be equal to. You do not need to justify your reasoning for this question.

$$\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \qquad \begin{pmatrix} 1 & 2 & 3 & 3 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad \begin{pmatrix} 1 & 2 & -2 & 2 \\ 1 & 2 & 2 & 2 \\ 1 & 2 & 2 & -2 \end{pmatrix}$$

4. (7 points) For what values of $\vec{b} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$ is the system below consistent? Express your answer using parametric vector form.

$$\begin{pmatrix} 0 & 5 \\ 1 & 3 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$$

5. (6 points) For what values of t, if any, are the three vectors below linearly independent?

$$\begin{pmatrix} 1 \\ 0 \\ t \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}, \quad \begin{pmatrix} t \\ 8 \\ 2t \end{pmatrix}$$

6. (10 points) Consider the linear system $A\vec{x} = \vec{b}$, where

$$A = \begin{pmatrix} 1 & -3 & 7 & -5 \\ 0 & 1 & -2 & 3 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad \vec{b} = \begin{pmatrix} 10 \\ 0 \\ 2 \end{pmatrix}$$

(a) Express the augmented matrix $[A\,|\,\vec{b}]$ in row reduced echelon form.

(b) Write the set of solutions to $A\vec{x} = \vec{b}$ in parametric vector form.

Solutions

1. (a)
   - true
   - false
   - false
   - false
   - false

   (b) First matrix: Fill in with 1 in the second row, 0 in the third row
   Second matrix: not possible
   Third matrix: not possible

   (c)
$$T_A \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} \boxed{5}\, x_1 + \boxed{-3}\, x_2 + \boxed{2}\, x_3 + \boxed{-3}\, x_4 \\ \boxed{1}\, x_1 + \boxed{0}\, x_2 + \boxed{-2}\, x_3 + \boxed{1}\, x_4 \end{pmatrix}$$

   (d) $B^T B$ $\quad$ $\vec{x}^T BA$

2. (a) $\begin{pmatrix} \bullet & * & * \\ 0 & 0 & \bullet \\ 0 & 0 & 0 \end{pmatrix}$, where $*$ denotes anything, and $\bullet$ denotes anything non-zero

   (b) $\begin{pmatrix} 1 & * \\ 0 & 0 \\ 0 & 0 \end{pmatrix}$ or $\begin{pmatrix} 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}$

   (c) $\begin{pmatrix} * & * & * \\ 0 & 0 & 0 \end{pmatrix}$ or $\begin{pmatrix} 0 & * & * \\ 0 & 0 & 0 \end{pmatrix}$ or $\begin{pmatrix} 0 & 0 & * \\ 0 & 0 & 0 \end{pmatrix}$

   (d) $T = A\vec{x} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \vec{x} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \vec{x}$
   $\implies A = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$

3. $\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$ $\qquad$ $\begin{pmatrix} 1 & 2 & -2 & 2 \\ 1 & 2 & 2 & 2 \\ 1 & 2 & 2 & -2 \end{pmatrix}$

4.

$$\left(\begin{array}{cc|c} 0 & 5 & b_1 \\ 1 & 3 & b_2 \\ 2 & 1 & b_3 \end{array}\right) \sim \left(\begin{array}{cc|c} 1 & 3 & b_2 \\ 0 & 1 & b_1/5 \\ 0 & -5 & b_3 - 2b_2 \end{array}\right)$$

$$\sim \left(\begin{array}{cc|c} 1 & 3 & b_2 \\ 0 & 1 & b_1/5 \\ 0 & 0 & b_3 - 2b_2 + b_1 \end{array}\right)$$

$\implies$ need $b_3 - 2b_2 + b_1 = 0$

$\implies b_1 = 2b_2 - b_3$

$\implies \vec{b} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} = \begin{pmatrix} 2b_2 - b_3 \\ b_2 \\ b_3 \end{pmatrix} = b_2 \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} + b_3 \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$

5.

$$\left(\begin{array}{ccc|c} 1 & 0 & t & 0 \\ 0 & 1 & 8 & 0 \\ t & -1 & 2t & 0 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 0 & t & 0 \\ 0 & 1 & 8 & 0 \\ 0 & -1 & 2t - t^2 & 0 \end{array}\right)$$

$$\sim \left(\begin{array}{ccc|c} 1 & 0 & t & 0 \\ 0 & 1 & 8 & 0 \\ 0 & 0 & 2t - t^2 + 8 & 0 \end{array}\right)$$

$\implies$ for L.I., we need $-t^2 + 2t + 8 \neq 0$, or $-(t+2)(t-4) \neq 0$

$\implies t \neq -2, 4$

6. (a) $\left(\begin{array}{cccc|c} 1 & -3 & 7 & -5 & 10 \\ 0 & 1 & -2 & 3 & 0 \\ 0 & 0 & 1 & 0 & 2 \end{array}\right) \sim \left(\begin{array}{cccc|c} 1 & -3 & 0 & -5 & -4 \\ 0 & 1 & 0 & 3 & 4 \\ 0 & 0 & 1 & 0 & 2 \end{array}\right) \sim \left(\begin{array}{cccc|c} 1 & 0 & 0 & 4 & 8 \\ 0 & 1 & 0 & 3 & 4 \\ 0 & 0 & 1 & 0 & 2 \end{array}\right)$

(b) $\vec{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 8 - 4x_4 \\ 4 - 3x_4 \\ 2 \\ 0 + x_4 \end{pmatrix} = \begin{pmatrix} 8 \\ 4 \\ 2 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} -4 \\ -3 \\ 0 \\ 1 \end{pmatrix}$