Answers

Sample Midterm 2B, Math 1554

PLEASE PRINT YOUR NAME CLEARLY IN ALL CAPITAL LETTERS

First Name ____________________ Last Name ____________________

GTID Number: ______________________

Student GT Email Address: __________________________@gatech.edu

Section Number (e.g. A4, QH3, etc.) ________ TA Name _________________________

Student Instructions

- **Show your work** and justify your answers for all questions unless stated otherwise.
- **Organize your work** in a reasonably neat and coherent way.
- Calculators, notes, cell phones, books are not allowed.
- Use dark and clear writing: your exam will be scanned into a digital system.
- Exam pages are double sided. Be sure to complete both sides.
- Leave a 1 inch border around the edges of exams.
- Any work done on scratch paper will not be collected and will not be graded.

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You do not need to justify your reasoning for questions on this page.

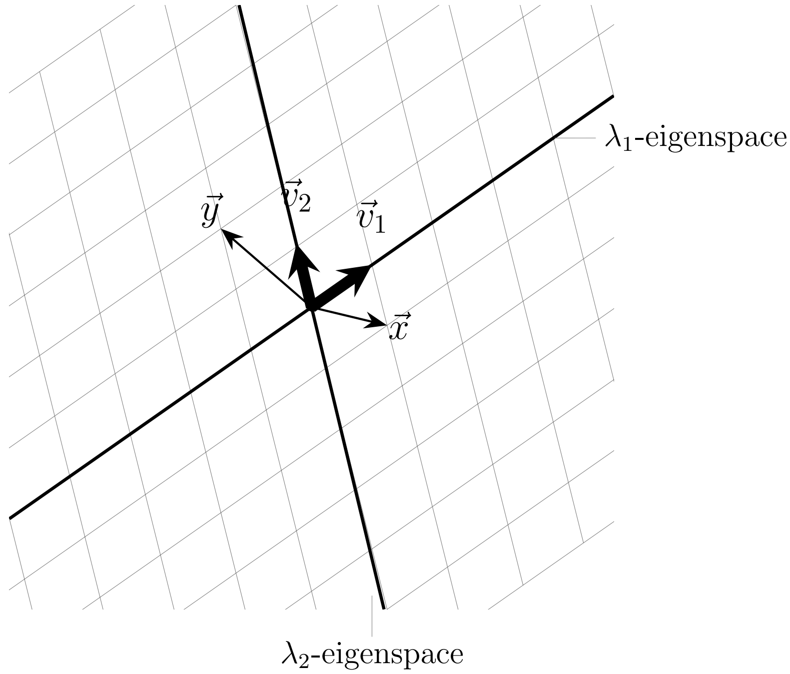
1. (10 points) Indicate **true** if the statement is true, otherwise, indicate **false**.

	true	false
a) If $A\vec{x} = \vec{b}$ has exactly one solution for every $\vec{b}$, A must be singular.	○	○
b) If $\vec{x}, \vec{y} \in \mathbb{R}^3$ are linearly independent, then $\{\vec{x}, \vec{y}, \vec{x} + \vec{y}\}$ is a basis for $\mathbb{R}^3$.	○	○
c) The set of solutions to $A\vec{x} = \vec{b}$, for any $\vec{b} \in \mathbb{R}^n$, is a subspace.	○	○
d) If $A, B \in \mathbb{R}^{n\times n}$ and $AB = I$, then $BA = I$.	○	○
e) Any matrix that is similar to the identity matrix must be equal to the identity matrix.	○	○
f) If $A, B \in \mathbb{R}^{m\times n}$ have the same null space, then they have the same RREF.	○	○
g) If A is $n \times n$, and there exists a $\vec{b} \in \mathbb{R}^n$ such that $A\vec{x} = \vec{b}$ is inconsistent, then $\det(A) = 0$.	○	○
h) If A has an LU factorization, then A is invertible.	○	○
i) If $A \in \mathbb{R}^{n\times n}$ has eigenvector $\vec{x}$ then $2\vec{x}$ is also an eigenvector of A.	○	○
j) Swapping the rows of A does not change the value of $\det(A)$.	○	○

Answers:

a) false
b) false
c) false
d) true
e) true
f) true
g) true
h) false
i) true
j) false

2. (2 points) A 2×2 matrix A has eigenvalues $\lambda_1 = 1$ and $\lambda_2 = 2$, with eigenvectors and eigenspaces indicated in the picture. Draw $A\vec{x}$ and $A\vec{y}$.



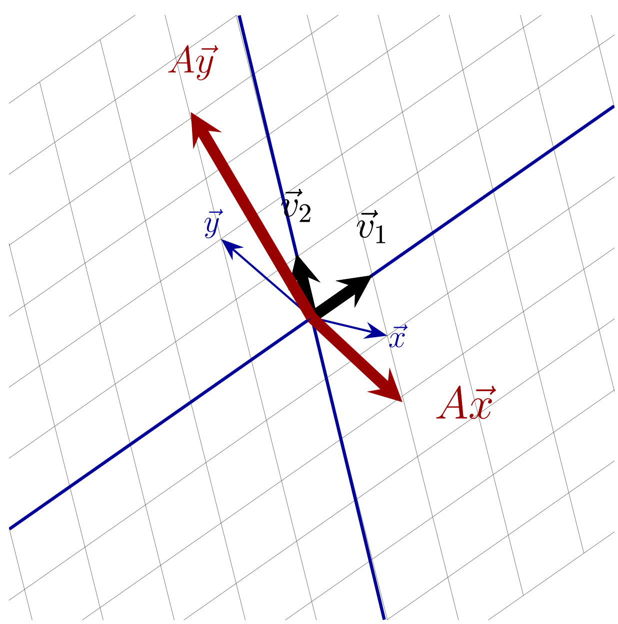
Answer:

We can see from the graph that

$$\vec{x} = \vec{v}_1 - \vec{v}_2, \qquad \vec{y} = 2\vec{v}_2 - \vec{v}_1$$

Using $A\vec{v}_1 = \vec{v}_1$ and $A\vec{v}_2 = 2\vec{v}_2$,

$$A\vec{x} = A(\vec{v}_1 - \vec{v}_2) = A\vec{v}_1 - A\vec{v}_2 = \vec{v}_1 - 2\vec{v}_2$$
$$A\vec{y} = A(2\vec{v}_2 - \vec{v}_1) = 2A\vec{v}_2 - A\vec{v}_1 = 4\vec{v}_2 - \vec{v}_1$$



You do not need to justify your reasoning for questions on this page.

3. (2 points) Fill in the missing entries of the 3×3 matrix A with **non-zero** numbers so that A has null space spanned by $\vec{v}$.

$$\vec{v} = \begin{pmatrix} 2 \\ -1 \\ -1 \end{pmatrix}, \quad A = \begin{pmatrix} 1 & 2 & 0 \\ 1 & 0 & 2 \\ 0 & \_ & \_ \end{pmatrix}$$

An acceptable answer is:

$$\begin{pmatrix} 1 & 2 & 0 \\ 1 & 0 & 2 \\ 0 & 1 & -1 \end{pmatrix}$$

Or, more generally:

$$\begin{pmatrix} 1 & 2 & 0 \\ 1 & 0 & 2 \\ 0 & k & -k \end{pmatrix}, \quad k \in \mathbb{R}$$

4. (6 points) If possible, write down an example of a matrix with the following properties. If it is not possible to do so, write *not possible*. You do not need to justify your reasoning.

   (a) A 4×3 matrix A with $\text{rank}(A) = 3$ and $\text{rank}(A^T) = 4$.

   Answer: not possible (because A^T is 3×4, its rank is at most 3).

   (b) A 2×3 matrix in RREF whose null space is spanned by $\begin{pmatrix} 2 \\ -4 \\ 1 \end{pmatrix}$.

   Answer: $\begin{pmatrix} 1 & 0 & -2 \\ 0 & 1 & 4 \end{pmatrix}$. This is the unique answer.

   (c) A 3×3 matrix in echelon form, A, such that $\text{Col}(A)$ is spanned by the vectors $\begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}$.

   Answer: $\begin{pmatrix} 2 & 1 & * \\ 0 & 2 & * \\ 0 & 0 & 0 \end{pmatrix}$, where * can be anything.

   (d) A 4×4 stochastic matrix, P, such that the Markov Chain $x_{k+1} = Px_k$ for $k = 0, 1, 2, \ldots$, does not have a unique steady-state.

   Answer: $P = I_4$, the 4×4 identity matrix.

5. (1 point) Suppose $\vec{v}_1$, $\vec{v}_2$ are eigenvectors of an 3×3 matrix A that correspond to eigenvalues λ_1 and λ_2.

$$\vec{v}_1 = \begin{pmatrix} 1 \\ 8 \\ 0 \end{pmatrix}, \quad \vec{v}_2 = \begin{pmatrix} 5 \\ 0 \\ 2 \end{pmatrix}, \quad \lambda_1 = 1, \quad \lambda_2 = \frac{1}{10}$$

Vector $\vec{p}$ is such that $\vec{p} = \vec{v}_1 - 13\vec{v}_2$. What does $A^k\vec{p}$ tend to as $k \to \infty$? $\boxed{}$

Answer: by inspection, $A^k p \to v_1$. To see this, note that $Av_1 = v_1$ and $Av_2 = \frac{1}{10}v_2$.

$$\begin{aligned}
Ap &= A(v_1 - 13v_2) = Av_1 - 13Av_2 = v_1 - \frac{13}{10}v_2 \\
A^2p &= A^2(v_1 - 13v_2) = A^2v_1 - 13A^2v_2 = v_1 - \frac{13}{10^2}v_2 \\
A^3p &= A^3(v_1 - 13v_2) = A^3v_1 - 13A^3v_2 = v_1 - \frac{13}{10^3}v_2 \\
A^kp &= A^k(v_1 - 13v_2) = A^kv_1 - 13A^kv_2 = v_1 - \frac{13}{10^k}v_2
\end{aligned}$$

The second term goes to zero as $k \to \infty$. If you would like to see another example like this, check out Example 5 from Section 5.2 of our textbook. Or solve Problem #25, or #27 from that section.

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You do not need to justify your reasoning for questions on this page.

6. (3 points) If the determinant $\left|\begin{pmatrix} a & b \\ 1 & 0 \end{pmatrix}\right| = 3$, compute the value of $\left|\begin{pmatrix} -1 & 0 & 0 \\ 2a & 2b & 0 \\ 0 & 0 & 5 \end{pmatrix}\right|$.

Answer: $\begin{vmatrix} -1 & 0 & 0 \\ 2a & 2b & 0 \\ 0 & 0 & 5 \end{vmatrix} = 5\begin{vmatrix} -1 & 0 \\ 2a & 2b \end{vmatrix} = -5\begin{vmatrix} 2a & 2b \\ -1 & 0 \end{vmatrix} = 10\begin{vmatrix} a & b \\ 1 & 0 \end{vmatrix} = 30$

7. A is the 3×6 matrix $A = \begin{bmatrix} 1 & 6 & -4 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & 2 & -6 \\ 0 & 0 & 0 & 0 & 1 & 6 \end{bmatrix}$

   (a) (1 point) The rank of A is _____.

   (b) (1 point) The dimension of Null(A) is _____.

   (c) (2 points) Write down a basis for Col(A).

   (d) (3 points) Construct a basis for Null(A).

Answer:

(a) 3

(b) 3

(c) $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix}$

(d) $\sim \begin{pmatrix} 1 & 6 & -4 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & 0 & -18 \\ 0 & 0 & 0 & 0 & 1 & 6 \end{pmatrix}$

$$\implies \vec{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{pmatrix} = \begin{pmatrix} -6x_2 + 4x_3 + x_6 \\ x_2 \\ x_3 \\ 18x_6 \\ -6x_6 \\ x_6 \end{pmatrix} = x_2 \begin{pmatrix} -6 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} + x_3 \begin{pmatrix} 4 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + x_6 \begin{pmatrix} 1 \\ 0 \\ 0 \\ 18 \\ -6 \\ 1 \end{pmatrix}$$

$$\implies \text{basis is the set } \left\{ \begin{pmatrix} -6 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 4 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \\ 18 \\ -6 \\ 1 \end{pmatrix} \right\}$$

8. (4 points) S is the parallelogram determined by $\vec{v}_1 = \begin{pmatrix} 4 \\ -2 \end{pmatrix}$, and $\vec{v}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$. If $A = \begin{pmatrix} 2 & 3 \\ 2 & 2 \end{pmatrix}$, what is the area of the image of S under the map $\vec{x} \mapsto A\vec{x}$?

Answer:

$$\text{area} = \left| \begin{vmatrix} 4 & 0 \\ -2 & 1 \end{vmatrix} \cdot \begin{vmatrix} 2 & 3 \\ 2 & 2 \end{vmatrix} \right| = |4 \cdot (-2)| = 8$$

9. (4 points) If possible, compute the LU factorization of $A = \begin{pmatrix} 5 & 4 \\ 10 & 6 \\ 0 & 2 \\ -5 & 1 \end{pmatrix}$

Answer:

$$\begin{pmatrix} 5 & 4 \\ 10 & 6 \\ 0 & 2 \\ -5 & 1 \end{pmatrix} \sim \begin{pmatrix} 5 & 4 \\ 0 & -2 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} = U$$

$$L = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ -1 & \frac{-5}{2} & 0 & 1 \end{pmatrix}$$

10. (4 points) List all possible values of k, if any, so that A has a real eigenvalue with geometric multiplicity 2. Show your work.

$$A = \begin{pmatrix} 3 & 2 & 0 \\ 2 & 3 & 0 \\ 0 & 0 & k \end{pmatrix}$$

Answer:

$$\begin{aligned} 0 &= \det(A - \lambda I) && (1) \\ &= (k - \lambda)\ ((3-\lambda)(3-\lambda) - 4) && (2) \\ &= (k-\lambda)(\lambda^2 - 6\lambda + 5) && (3) \\ &= (k-\lambda)(\lambda - 1)(\lambda - 5) && (4) \end{aligned}$$

$\implies k = 1$ or 5

11. (4 points) Construct a basis for the subspace

$$H = \{\vec{x} \in \mathbb{R}^3\ :\ 5x_1 + 4x_2 - 7x_3 = 0\}.$$

Answer:

$$\vec{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} \frac{-4}{5}x_2 + \frac{7}{5}x_3 \\ x_2 \\ x_3 \end{pmatrix} = x_2 \begin{pmatrix} -4/5 \\ 1 \\ 0 \end{pmatrix} + x_3 \begin{pmatrix} 7/5 \\ 0 \\ 1 \end{pmatrix} \implies \text{basis is } \left\{ \begin{pmatrix} -4/5 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 7/5 \\ 0 \\ 1 \end{pmatrix} \right\}$$

12. (5 points) A has only two distinct eigenvalues, 0 and 1. $A = \begin{pmatrix} 0 & 1 & -1 \\ 1 & 0 & 1 \\ 1 & -1 & 2 \end{pmatrix}$.

(a) Construct the eigenbasis for eigenvalue $\lambda = 0$.

(b) Construct the eigenbasis for eigenvalue $\lambda = 1$.

Answer:

a) $A = \begin{pmatrix} 0 & 1 & -1 \\ 1 & 0 & 1 \\ 0 & -1 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix} \implies \vec{v}_1 = \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix}$

$\implies$ basis is $\{\vec{v}_1\}$

b) $A - I = \begin{pmatrix} -1 & 1 & -1 \\ 1 & -1 & 1 \\ 1 & -1 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & -1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

$\implies \vec{v}_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \vec{v}_2 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$

$\implies$ eigenbasis is $\{\vec{v}_1, \vec{v}_2\}$