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Math 2552 Differential Equations

Modeling Project

Ice Melt Problem

Description

I chose the Ice Melt problem (link:https://www.simiode.org/resources/1880/download/1-20-S-IceMelt-StudentVersion.pdf), which required me to set mathematical models for ice balls and ice cubes and decide which one has a bigger melting velocity and can stay for a long time. Since this is a practical problem that need to consider more than solving a math problem, I made two types of models, one for the same diameter (the diameter of the sphere equals to the side length of the cube) and one for the same volume (the volume of the sphere equals to the volume of the cube). I will set 4 differential equations to involve 2 situations and show the decreasing velocity and the total melting time for each model. After setting the general models, I will test and verify them by trying a specific data for the parameters and running the model by the computer to test the result.

Method

1. Same Volume

The volume of sphere $V_{sphere} = \frac{4}{3}\pi r^3$, the exposed surface area of sphere $S_{sphere} = 4\pi r^2$

the volume of cube $V_{cube} = l^3$, the exposed surface area of cube $s_{cube} = 6l^2$

Since the sphere and the cube have the same volume, which means that $V_s = V_c$,

$$\frac{4}{3}\pi r^3 = l^3$$

$$\sqrt[3]{\frac{4\pi}{3}}\ r = l$$

We can assume that for melting snow/ice is the rate at which the object is melting (rate of change of volume) is proportional to the exposed surface area, so the mathematical model for the melting ice is

$$\frac{dV}{dt} \propto -S \Rightarrow \frac{dV}{dt} = -kS$$

Where k can be considered as a melting parameter (a constant positive real number). For ice sphere, the model would be

$$\frac{d(\frac{4}{3}\pi r^3)}{dt} = -k4\pi r^2$$

$$\frac{4}{3}\pi\ \frac{dr^3}{dt} =- k4\pi r^2$$

$$\frac{1}{3}3r^2\frac{dr}{dt} =- kr^2$$

$$\frac{dr}{dt} =- k$$

This means that the ice sphere's radius will continue to decrease at a constant k velocity until the radius becomes 0.

For ice cube, the differential equation model would be

$$\frac{dl^3}{dt} =- k6l^2$$

$$3l^2\frac{dl}{dt} =- k6l^2$$

$$\frac{dl}{dt} =- 2k$$

This means that the side length of the ice cube will continue to decrease at a constant $2k$ velocity until the side length becomes 0. Now I need to convert side length l into r.

$$\frac{dr\sqrt[3]{\frac{4}{3}\pi}}{dt} = -2k$$

$$\frac{dr}{dt} = -\frac{2}{\sqrt[3]{\frac{4}{3}\pi}}k \approx -1.241k$$

$$1.241 > 1$$

When the ice sphere and ice cube have the same volume, the melting velocity of the ice cube is bigger than the ice sphere, which means that the ice sphere outlasts the ice cube in the same condition. The advertisement's claim is true.

2. **Same Diameter**

When the ice sphere's diameter equals the ice cube's side length, they will have different volumes. So even though the melting velocity of the ice cube is bigger, I still need to make sure which one can stay for a longer period.

$$2r = l$$

$$V_{cube} = l^3 = 8r^3$$

$$S_{cube} = 6l^2 = 24r^2$$

$$V_{sphere} = \frac{4}{3}\pi r^3$$

$$S_{sphere} = 4\pi r^2$$

The melting velocity for ice cube

$$\frac{dv}{dt} = -ks$$

$$\frac{d8r^3}{dt} = -k24r^2$$

$$8 * 3r^2\frac{dr}{dt} = -k24r^2$$

$$\frac{dr}{dt} = -k$$

The melting velocity for ice sphere

$$\frac{d\frac{4}{3}\pi r^3}{dt} = -k4\pi r^2$$

$$\frac{dr}{dt} = -k$$

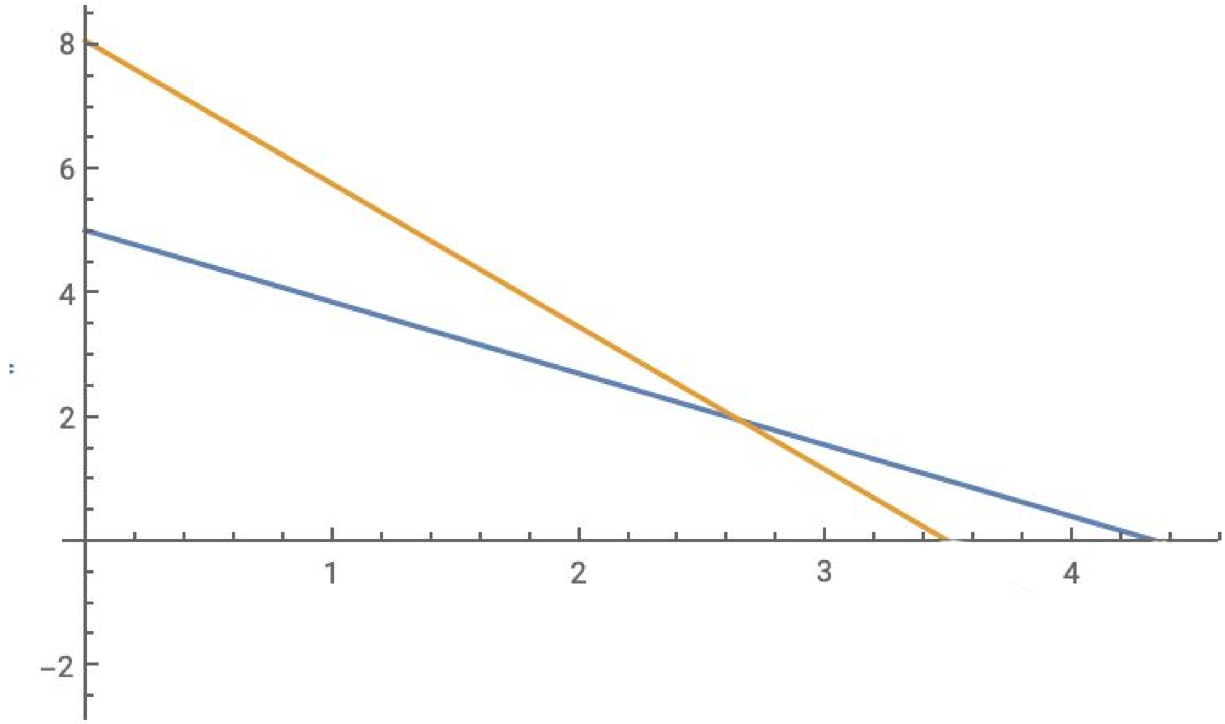
When the ice cube and the ice sphere have the same diameter (side length), they have the same melting velocity. Since the ice cube has a bigger volume, it will have a longer melting period. In this situation, the advertisement's claim is false.

Test and Verify

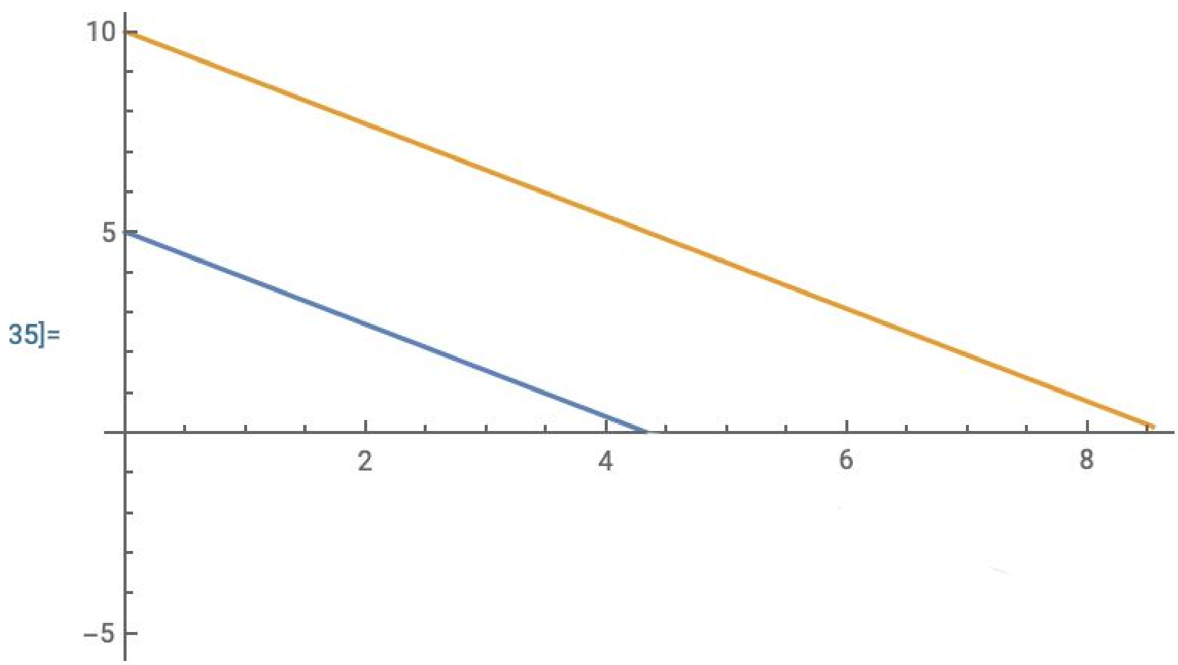
To test and verify whether my models are correct, I need to set specific number to the parameter k and calculate the melting period so that I can compare them to get the result.

To simplify the problem, for both situations, I set the diameter of the ice sphere to be 10cm. So for the first situation, the radius $r = 5cm$, and the side length of the ice cube $l = \sqrt[3]{\frac{4\pi}{3}} * 5cm \approx 8.06cm$. For the second situation, the radius $r = 5cm$, and the side length of the ice cube $l = 10cm$. Also, for a room temperature 25°C, set k for 1.153.

The result is shown below (blue line is for the ice sphere, and yellow line is for the ice cube)(horizontal axis stands for time, and vertical axis stands for radius).



Same Volume



Same Diameter (Side Length)

As two graphs shown above, the rate of decrease of radius (side length) is constant, and results are corresponding to my mathematical models.

Reflection

During the solving process, most of the calculating process focus on solving first order differential equations, and I set my mathematical models for the variation of r, the radius of the ice sphere. Also, in order to compare the melting velocity of the ice cube and the ice sphere, I need to transform the side length of the ice cube into the radius. These are knowledge I learned from the Math 2552 course. To improve my work in a more comprehensive and more realistic way, I set two situations to check whether the claim is correct. However, there are still many improvements I can take. First of all, in the testing part, I still used the change of radius, rather than volume, to check my result. If I can show how the total volume changes over time, it would be more clear for viewers to know the result.

Conclusion

In the introduction of this project, my work is to check whether the melting velocity of the ice sphere is slower than the melting velocity of the ice cube. I assume that the rate at which the object is melting (rate of change of volume) is proportional to the exposed surface area, and I present two situations in order to cover all the possibilities. The result shows that when the ice sphere and the ice cube have the same volume, the ice sphere outlasts the cube; when they have the same diameter (side length), the ice cube outlasts the ice sphere.

Bibliography

Brian Winkel (2015), "1-020-S-IceMelt," https://www.simiode.org/resources/761.

https://en.wikipedia.org/wiki/Ice