Homework 1

Thursday, August 20, 2020 12:46 AM

Problem A [20%]. **Some classic results on triple products.**

(a) Prove the Grassmann identity for arbitrary vectors $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C} \in \mathbb{R}^3$:

$$\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = \mathbf{B}\,(\mathbf{A} \cdot \mathbf{C}) - \mathbf{C}\,(\mathbf{A} \cdot \mathbf{B})$$

(b) Use the Grassmann identity to derive the Jacobi identity:

$$\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) + \mathbf{B} \times (\mathbf{C} \times \mathbf{A}) + \mathbf{C} \times (\mathbf{A} \times \mathbf{B}) = \mathbf{0}$$

a) $\vec{A} \times \left(\vec{B} \times \vec{C}\right) = \left(A_y B_x C_y + A_z B_x C_z - A_y B_y C_x - A_z B_z Cx\right)\hat{x}$ ...

Add two terms that sum to zero to each dimension

$$(+A_x B_x C_x - A_x B_x C_x)\hat{x}$$

Result in one dimension

$$\left(B_x \left(\vec{A} \cdot \vec{C}\right) - C_x \left(\vec{A} \cdot \vec{B}\right)\right)\hat{x}$$

The same follows for the other two dimensions.

b) Using $\vec{A} \times \left(\vec{B} \times \vec{C}\right) = B\left(\vec{A} \cdot \vec{C}\right) - \vec{C}\left(\vec{A} \cdot \vec{B}\right)$

$$\left[\vec{B}\left(\vec{A} \cdot \vec{C}\right) + \vec{C}\left(\vec{B} \cdot \vec{A}\right) + \vec{A}\left(\vec{C} \cdot \vec{B}\right)\right] - \left[\vec{C}\left(\vec{A} \cdot \vec{B}\right) + \vec{A}\left(\vec{B} \cdot \vec{C}\right) + \vec{B}\left(\vec{C} \cdot \vec{A}\right)\right]$$

Dot product is commutative so all terms cancel out

Problem B [20%]. **Working with non-uniform charge densities.**

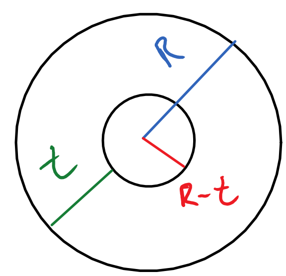
A hollow sphere (outer radius R, thickness $t < R$, inner radius $R - t$) of an unknown material carries a volume charge density $\rho(r, \theta, \phi) = \alpha e^{-\beta r^3} |\sin(\theta)|$ where α and β are constants.

(a) What is the total charge Q on the **sphere** in terms of α and β ?

(b) What are the dimensions (physical units) of α and β ?

(c) Imagine that the thickness t is much smaller than R, *i.e.* $t \ll R$, such that the charge carried by the sphere can be approximated by a surface charge density σ. What is the expression for $\sigma(\theta, \phi)$?

a)



$$Q_T = \iiint \rho \, \mathrm{d}^3 r = \iiint \rho(r, \theta, \phi) r^2 \sin\theta \, \mathrm{d}r \, \mathrm{d}\theta \, \mathrm{d}\phi$$

$$Q_T = \alpha \left(\int_0^{2\pi} \mathrm{d}\phi \right) \left(\int_0^{\pi} |\sin\theta| \sin\theta \, d\theta \right) \left(\int_{R-t}^{R} \mathrm{e}^{-\beta r^3} r^2 \, \mathrm{d}r \right)$$

$$\sin\theta \geq 0 \qquad \text{for} \qquad 0 < \theta < \pi$$

$$Q_T = \alpha \left(\int_0^{2\pi} \mathrm{d}\phi \right) \left(\int_0^{\pi} \sin^2\theta \, d\theta \right) \left(\int_{R-t}^{R} \mathrm{e}^{-\beta r^3} r^2 \, \mathrm{d}r \right)$$

$$Q_T = \left(\frac{\pi^2 \alpha}{3\beta} \right) \left(\mathrm{e}^{-(R-t)^3} - \mathrm{e}^{-\beta R^3} \right)$$

b)

$$\alpha = \frac{C}{m^3} \qquad \beta = \frac{1}{m^3}$$

c)

$$Q_T = \iiint \rho(r, \theta, \phi) r^2 \sin\theta \, \mathrm{d}r \, \mathrm{d}\theta \, \mathrm{d}\phi = \iint \sigma(\theta, \phi) R^2 \sin\theta \, \mathrm{d}\theta \, \mathrm{d}\phi$$

$$\sigma(\theta, \phi) R^2 = \int \rho(r, \theta, \phi) r^2 \, \mathrm{d}r$$

$$\sigma(\theta, \phi) = \left(\frac{\alpha |\sin\theta|}{3\beta R^2} \right) \left(\mathrm{e}^{-\beta(R-t)^3} - \mathrm{e}^{-\beta R^3} \right)$$

Taylor expansion for when t is close to 0 gives:

$$\sigma(\theta, \phi) = -\alpha |\sin\theta| \mathrm{e}^{-\beta R^3} t$$

Problem C [60%]. Charged cylindric can.

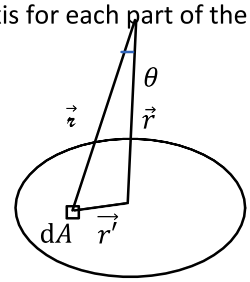
An aluminum can of La Croix (Pamplemousse flavor) has been charged electrically. Consider that the walls, top and bottom of the can are infinitely thin, and that the can is well modeled by a closed cylinder of radius R and height h carrying a uniform <u>surface</u> charge density σ. Take the reference of cylindrical coordinates, point O, at the center of the can.

(a) Using Coulomb's law, calculate the magnitude of the electric field at a point P (coordinate z) located on the axis of the cylinder ($s = 0$) but outside the can ($|z| > h$). Would it be possible to solve this problem using Gauss' law?

(b) By taking the limit $z \to \infty$, give the first two terms in the "far-field" expansion of the can's electric field.

a)

Due to symmetry E-field will point along z-axis for each part of the can
Start with the top of the can:



$$\mathrm{d}E_z = \frac{\sigma\,\mathrm{d}A}{4\pi\varepsilon_0 |\vec{\mathscr{r}}|^2}\cos\theta$$

$$\vec{r} = \left(z - \frac{h}{2}\right)\hat{z}$$

$$\mathrm{d}A = s\mathrm{d}s\,\mathrm{d}\phi$$

$$\vec{r'} = (s)\hat{s}$$

$$\cos\theta = \frac{\left(z - \frac{h}{2}\right)}{\sqrt{s^2 + \left(z - \frac{h}{2}\right)^2}}$$

$$|\vec{\mathscr{r}}| = \sqrt{s^2 + \left(z - \frac{h}{2}\right)^2}$$

$$E_{z,top} = \int_0^{2\pi}\int_0^R \frac{\sigma\left(z-\frac{h}{2}\right)s}{4\pi\varepsilon_0\left(s^2+\left(z-\frac{h}{2}\right)^2\right)^{\frac{3}{2}}}\,\mathrm{d}s\,\mathrm{d}\phi$$

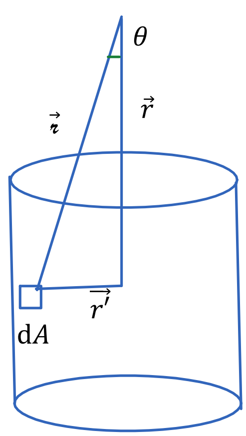
$$E_{z,top} = \left(\frac{\sigma}{2\varepsilon_0}\right)\left(1-\frac{z-\frac{h}{2}}{\sqrt{R^2+\left(z-\frac{h}{2}\right)^2}}\right)$$

The bottom is similar except the sign for h/2 is changed.

$$E_{z,bottom} = \left(\frac{\sigma}{2\varepsilon_0}\right)\left(1-\frac{z+\frac{h}{2}}{\sqrt{R^2+\left(z+\frac{h}{2}\right)^2}}\right)$$

For the sides, the infinitesimal area is different than the top and bottom

$$\mathrm{d}E_z = \frac{\sigma\,\mathrm{d}A}{4\pi\varepsilon_0|\vec{\mathscr{r}}|^2}\cos\theta$$



$$\mathrm{d}A = R\mathrm{d}z\,\mathrm{d}\phi$$

$$\vec{r} = (z-z')\hat{z}$$

$$\overrightarrow{r'} = (R)\hat{s}$$

$$|\vec{\mathscr{r}}| = \sqrt{(z-z')^2+R^2}$$

$$\cos\theta = \frac{z-z'}{\sqrt{(z-z')^2+R^2}}$$

$$E_{z,sides} = \int_0^{2\pi}\int_{-\frac{h}{2}}^{\frac{h}{2}} \frac{\sigma R(z-z')}{4\pi\varepsilon_0[(z-z')^2+R^2]^{\frac{3}{2}}}\,\mathrm{d}\phi\,\mathrm{d}z'$$

$$E_{z,sides} = \frac{\sigma R}{2\varepsilon_0}\left(\frac{1}{\sqrt{\left(z-\frac{h}{2}\right)^2+R^2}} - \frac{1}{\sqrt{\left(z+\frac{h}{2}\right)^2+R^2}}\right)$$

$$\vec{E} = \left(E_{z,top} + E_{z,bottom} + E_{z,sides}\right)\hat{z}$$

$$\vec{E} = \left(\frac{\sigma}{2\varepsilon_0}\right)\left(2 - \frac{z-\frac{h}{2}-R}{\sqrt{R^2+\left(z-\frac{h}{2}\right)^2}} - \frac{z+\frac{h}{2}+R}{\sqrt{R^2+\left(z+\frac{h}{2}\right)^2}}\right)\hat{z}$$

Although Gauss's Law still holds true, it is not useable due to the finite length cylinder causing a lack of simple symmetry in the produced E-field.

b)

For the "far field" approximation, we can rearrange the answer into terms of factors of small parameters and Taylor expand the small parameters around zero. Specifically, if z is large, then $\frac{R}{\mathrm{z}}$ and $\frac{h}{\mathrm{z}}$ (or $\frac{h}{2\mathrm{z}}$)will be small parameters.
Rearranging $\vec{E}$ gives:

$$\vec{E} = \left(\frac{\sigma}{2\varepsilon_0}\right)\left(2 - \frac{1-\frac{h}{2\mathrm{z}}-\frac{R}{\mathrm{z}}}{\sqrt{\left(\frac{R}{\mathrm{z}}\right)^2+\left(1-\frac{h}{2\mathrm{z}}\right)^2}} - \frac{1+\frac{h}{2\mathrm{z}}+\frac{R}{\mathrm{z}}}{\sqrt{\left(\frac{R}{\mathrm{z}}\right)^2+\left(1+\frac{h}{2\mathrm{z}}\right)^2}}\right)\hat{z}$$

First, expanding each fraction when $\frac{R}{\mathrm{z}}$ is near zero and keeping zeroth and first order terms gives:

$$\vec{E} = \left(\frac{\sigma}{2\varepsilon_0}\right)\left(\frac{R}{z}\right)\left(\frac{1}{\left(1-\frac{h}{2\mathrm{z}}\right)} - \frac{1}{\left(1+\frac{h}{2\mathrm{z}}\right)}\right)\hat{z}$$

Next, expanding each fraction when $\frac{h}{2\mathrm{z}}$ is near zero and keeping the zeroth, first, second, and third order terms gives:

$$\vec{E} = \left(\frac{\sigma}{\varepsilon_0}\right)\left(\frac{R}{z}\right)\left(\frac{h}{2z}+\frac{h^3}{8z^3}\right)\hat{z} = \left(\frac{\sigma R h}{2\varepsilon_0 z^2}\right)\left(1+\left(\frac{h}{2z}\right)^2\right)\hat{z}$$

The approximation will vary depending on the terms kept for each expansion, but a more intuitive approach may be more instructive.

When physically far away from any charge distribution, the first approximation (from the multipole expansion you will learn later in the course) is the monopole approximation. This just means that the total charge looks like a point charge from far away. Using this idea, we can find the total charge and plug it into the point charge formula.

$$Q_T = \sigma\left(A_{top} + A_{bottom} + A_{sides}\right) = 2\pi R\sigma(R+h)$$

$$\vec{E} = \left(\frac{Q_T}{4\pi\varepsilon_0 z^2}\right)\hat{z} = \left(\frac{\sigma R^2}{2\varepsilon_0 z^2} + \frac{\sigma R h}{2\varepsilon_0 z^2}\right)\hat{z}$$

Rearranging the terms/factors and you can actually see the terms from the expansions.

$$\vec{E} = \left(\left(\frac{\sigma}{2\varepsilon_0}\right)\left(\frac{R}{z}\right)^2 + \left(\frac{\sigma}{\varepsilon_0}\right)\left(\frac{R}{z}\right)\left(\frac{h}{2z}\right)\right)\hat{z}$$

The first term (from the top and bottom) is related to the second order term from the expansion of $\frac{R}{\mathrm{z}}$ near 0, and the second term (from the sides) is related to the first order terms from both expansions.