Week 7: Gamma, χ^2 and Gaussian distributions

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Clarification for Problems 1 and 7 in the homework

Theorem

A function $F : \mathbb{R} \to [0,1]$ is a cdf of some random variable, if and only if

1. $\lim\limits_{x \to -\infty} F(x) = 0$,
2. $\lim\limits_{x \to \infty} F(x) = 1$,
3. $x \le y \Rightarrow F(x) \le F(y)$ *(F is non-decreasing)*,
4. $\lim\limits_{y \to x^+} f(y) = f(x)$ *for any $x \in \mathbb{R}$ (F is right continuous).*

Theorem

$f(x)$ is a pdf of a continuous random variable if and only if

1. $f(x) \ge 0$ *for every $x \in \mathbb{R}$.*
2. $\int\limits_{\mathbb{R}} f(x)\,\mathrm{d}x = 1$.

Tuesday class

Poisson process (reminding from last week)

Definition (Poisson process)

Suppose we have an action that happens regularly over time (bus arrival, visits of a website by users, etc) that satisfies the following properties:

1. The occurrence of the action at any time interval in the future is independent of past occurrences.
2. The average number of actions happening in a given time interval is proportional to the interval length.

Denote by λ the average number of times the action happens in the unit time interval $[0, 1]$. Then we call this type of action a **Poisson process with rate** λ.

- Poisson process is a random experiment where the outcomes are the arrival times (the times when the actions happen).
- Poisson process is the continuous version of Bernoulli trials.
- If we denote by Y the number of actions in a Poisson process that happened in a certain fixed interval, $[a, b]$, then Y will have a Poisson distribution with parameter $\lambda \cdot (b - a)$.

Gamma distribution: motivation

Let X_k be the time when the action happens for the k-th time in a Poisson process with rate λ.

- $\text{Range}(X_k) = [0, \infty)$.
- $F(X_k \leq x) = 0$ when $x < 0$.
- For $x \geq 0$,
$$F(x) = P(X_k \leq x) = 1 - P(X_k > x).$$
- $P(X_k > x)$ is the probability that there are $k-1$ or less actions in the interval $[0, x]$.
- Since the number of actions in the interval $[0, x]$ is a Poisson random variable with parameter $\lambda \cdot x$,
$$P(X_k > x) = e^{-\lambda x}\frac{(\lambda x)^0}{0!} + \cdots + e^{-\lambda x}\frac{(\lambda x)^{k-1}}{(k-1)!}.$$
- Hence,
$$F(x) = 1 - e^{-\lambda x}\sum_{i=0}^{k-1}\frac{(\lambda x)^i}{i!}, \quad x \geq 0.$$

For $x \geq 0$,

$$
\begin{aligned}
f(x) &= F'(x) \\
&= \lambda e^{-\lambda x} \sum_{i=0}^{k-1} \frac{(\lambda x)^i}{i!} - e^{-\lambda x} \sum_{i=1}^{k-1} \lambda i \frac{(\lambda x)^{i-1}}{i!} \\
&= \lambda e^{-\lambda x} \sum_{i=0}^{k-1} \frac{(\lambda x)^i}{i!} - \lambda e^{-\lambda x} \sum_{i=1}^{k-1} \frac{(\lambda x)^{i-1}}{(i-1)!} \\
&= \lambda e^{-\lambda x} \sum_{i=0}^{k-1} \frac{(\lambda x)^i}{i!} - \lambda e^{-\lambda x} \sum_{i=0}^{k-2} \frac{(\lambda x)^i}{i!} \\
&\boxed{= \lambda e^{-\lambda x} \frac{(\lambda x)^{k-1}}{(k-1)!}}.
\end{aligned}
$$

k is assumed a positive integer, but turns out we can "extend" this distribution to other positive numbers k.

Gamma function

Gamma function is defined as $\boxed{\Gamma(\alpha) = \int_0^\infty y^{\alpha-1} e^{-y}\, \mathrm{d}y, \quad \alpha > 0.}$

Theorem

$$\Gamma(\alpha) = (\alpha - 1)\Gamma(\alpha - 1) \quad \textit{for } \alpha > 1.$$

Proof.

Doing integration by parts

$$\Gamma(\alpha) = \int_0^\infty y^{\alpha-1} e^{-y}\, \mathrm{d}y = -y^{\alpha-1}e^{-y}\Big|_{y=0}^{\infty} + \int_0^\infty (\alpha-1) y^{\alpha-2} e^{-y}\, \mathrm{d}y = 0 + (\alpha-1)\Gamma(\alpha-1).$$

$\square$

For k positive integer

$$\Gamma(k) = (k-1)\Gamma(k-1) = (k-1)(k-2)\Gamma(k-2) = \cdots = (k-1)\cdots 2 \cdot \Gamma(1) = (k-1)! \cdot \Gamma(1).$$

And

$$\Gamma(1) = \int_0^\infty e^{-y}\, \mathrm{d}y = -e^{-y}\Big|_{y=0}^{\infty} = 1 \quad \Rightarrow \quad \boxed{\Gamma(k) = (k-1)!}$$

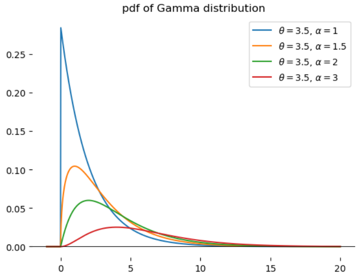
Gamma distribution: definition

Definition (Gamma distribution)

We say that a continuous random variable X has **Gamma distribution** with parameters (θ, α) if its pdf is given by

$$f(x) = \begin{cases} 0 & x < 0 \\ \frac{1}{\theta^\alpha \Gamma(\alpha)} x^{\alpha-1} e^{-\frac{x}{\theta}} & x \geq 0 \end{cases}.$$

- X is the waiting time until the "α-th" occurrence.
- When $\alpha = 1$, Gamma distribution is the same as exponential distribution.
- It is the continuous analogue of the negative binomial distribution.



Mean and variance of Gamma distribution

Theorem (Problem 3.6-7 in the textbook)

The mgf of the Gamma distribution with parameters (θ, α) *is*

$$M(t) = \frac{1}{(1-\theta t)^{\alpha}}, \quad t < \frac{1}{\theta}.$$

- Hence

$$M'(t) = \frac{\alpha\theta}{(1-\theta t)^{\alpha+1}}, \quad M''(t) = \frac{\alpha(\alpha+1)\theta^2}{(1-\theta t)^{\alpha+2}}.$$

- Therefore

$$E[X] = M'(0) = \alpha\theta$$

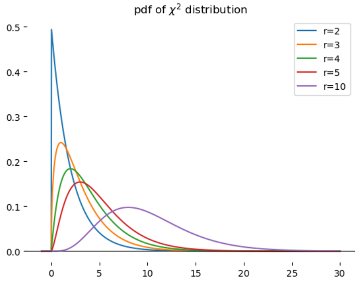
$$\mathrm{Var}(X) = E[X^2] - E[X]^2 = M''(0) - E[X]^2 = \alpha(\alpha+1)\theta^2 - \alpha^2\theta^2 = \alpha\theta^2.$$

Definition (χ^2 distribution)

The Gamma distribution for values $\theta = 2$ and $\alpha = r/2$ ($r = 1, 2, 3, \dots$) is called **χ^2 (chi-squared) distribution with r degrees of freedom:**

$$f(x) = \begin{cases} 0 & x < 0 \\ \frac{1}{2^{\frac{r}{2}}\Gamma(\frac{r}{2})} x^{\frac{r}{2}-1} e^{-\frac{x}{2}} & x \geq 0 \end{cases}.$$

We denote the χ^2 distribution with r degrees of freedom as $\chi^2(r)$.



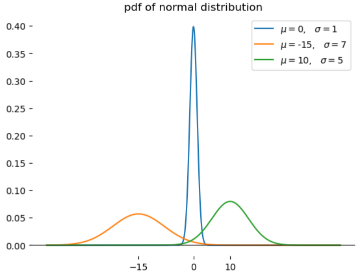
Normal distribution

Definition

A random variable X has **normal** (also called **Gaussian**) distribution with parameters (μ, σ) if its pdf has the form

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}.$$

- We denote the normal distribution with parameters (μ, σ) as $N(\mu, \sigma^2)$.
- Normal distribution shows up a lot due to **Central Limit Theorem** to be discussed later.



This is indeed a pdf:

$$\int_{-\infty}^{\infty} f(x)\,\mathrm{d}x = \int_{-\infty}^{\infty} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}\,\mathrm{d}x \quad \left(\text{after changing } z = \frac{x-\mu}{\sigma}\right)$$

$$= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}}\,\mathrm{d}z = I.$$

$$I^2 = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}}\,\mathrm{d}z \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}}\,\mathrm{d}y = \frac{1}{2\pi} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-\frac{z^2+y^2}{2}}\,\mathrm{d}z\,\mathrm{d}y$$

(change to polar coordinates $\quad z = r\cos\phi, \quad y = r\sin\phi$)

$$= \frac{1}{2\pi} \int_0^{2\pi}\int_0^{\infty} r e^{-\frac{r^2}{2}}\,\mathrm{d}r\,\mathrm{d}\phi = \frac{1}{2\pi}\int_0^{2\pi} 1\,\mathrm{d}\phi = 1.$$

Therefore, $I = 1$.

Theorem

If X has normal distribution with parameters (μ, σ) then

$$M(t) = e^{\mu t + \frac{\sigma^2 t^2}{2}}.$$

Proof

$$M(t) = \int_{-\infty}^{\infty} e^{tx} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} \, \mathrm{d}x \left(\text{after changing } z = \frac{x-\mu}{\sigma}\right)$$

$$= \int_{-\infty}^{\infty} e^{t(\sigma z + \mu)} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{z^2}{2}} \, \mathrm{d}z$$

$$= e^{\mu t} \int_{-\infty}^{\infty} e^{t\sigma z} \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} \, \mathrm{d}z$$

Proof (cont.)

$$
\begin{aligned}
&= e^{\mu t} \int_{-\infty}^{\infty} e^{t\sigma z} \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} \, \mathrm{d}z \\
&= e^{\mu t} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2 - 2t\sigma z}{2}} \, \mathrm{d}z \\
&= e^{\mu t} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2 - 2t\sigma z + t^2\sigma^2}{2}} e^{\frac{t^2\sigma^2}{2}} \, \mathrm{d}z \\
&= e^{\mu t} e^{\frac{t^2\sigma^2}{2}} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2 - 2t\sigma z + t^2\sigma^2}{2}} \, \mathrm{d}z \\
&= e^{\mu t} e^{\frac{t^2\sigma^2}{2}} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{(z - t\sigma)^2}{2}} \, \mathrm{d}z \\
&= e^{\mu t} e^{\frac{t^2\sigma^2}{2}} .
\end{aligned}
$$

The last integrant is the pdf of the normal distribution with parameters $(\mu = t\sigma, \sigma = 1)$. □

Mean and variance of normal distribution

- We found
$$M(t) = e^{\mu t + \frac{\sigma^2 t^2}{2}}.$$
- Hence
$$M'(t) = [\mu + t\sigma^2] \cdot e^{\mu t + \frac{\sigma^2 t^2}{2}}, \quad M''(t) = [\sigma^2 + (\mu + t\sigma^2)^2] \cdot e^{\mu t + \frac{\sigma^2 t^2}{2}}.$$
- Therefore
$$E[X] = M'(0) = \mu,$$
$$\mathrm{Var}(X) = E[X^2] - E[X]^2 = M''(0) - E[X]^2 = \sigma^2 + \mu^2 - \mu^2 = \sigma^2.$$

The parameters μ and σ are the mean and standard deviation of normal distribution with parameters (μ, σ).

Thursday class

Standard normal distribution

The normal distribution for parameters $(\mu = 0, \sigma = 1)$ is called the **standard normal distribution**.

Theorem

If X has normal distribution with parameters (μ, σ) then $\frac{X-\mu}{\sigma}$ has standard normal distribution.

Proof.

$$P\left(\frac{X-\mu}{\sigma} \leq x\right) = P(X \leq \sigma x + \mu) = \int_{-\infty}^{\sigma x+\mu} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(y-\mu)^2}{2\sigma^2}} \, \mathrm{d}y = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} \, \mathrm{d}z$$

after changing $z = \frac{y-\mu}{\sigma}$. □

For standard normal distribution, due to symmetry,

$$P(X > \alpha) = 1 - P(X < \alpha).$$

Exercise 1

Problem (3.3-7 from the textbook)

If X is $N(650, 625)$, *find*

(a) $P(600 \leq X < 660)$.

(b) *A constant* $c > 0$ *such that* $P(|X - 650| \leq c) = 0.9544$.

Solution

Solution (cont.)

Exercise 2

Problem (3.3-14 from the textbook)

The strength X of a certain material is such that its distribution is found by $X = e^Y$, where Y is $N(10, 1)$. Find the cdf and pdf of X, and compute $P(10,000 < X < 20,000)$. Note: $F(x) = P(X \leq x) = P(e^Y \leq x) = P(Y \leq \log(x))$ so that the random variable X is said to have a **lognormal distribution**.

Solution

Solution (cont.)

Theorem

If X has standard normal distribution then X^2 has $\chi^2(1)$ distribution.

Proof.

- For $\chi^2(1)$,

$$f(x) = \begin{cases} 0 & x < 0 \\ \frac{1}{2^{\frac{1}{2}}\Gamma(\frac{1}{2})} x^{-\frac{1}{2}} e^{-\frac{x}{2}} & x \geq 0 \end{cases}.$$

- $F(x) = 0$ for $x < 0$.
- For $x \geq 0$,

$$P(X^2 \leq x) = P(-\sqrt{x} \leq X \leq \sqrt{x}) = \int_{-\sqrt{x}}^{\sqrt{x}} \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}}\,\mathrm{d}z = 2\int_0^{\sqrt{x}} \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}}\,\mathrm{d}z$$

$$= 2\int_0^{\sqrt{x}} \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}}\,\mathrm{d}z \quad (\text{take } y = \sqrt{z}) = \frac{1}{\sqrt{2\pi}} \int_0^{x} y^{-\frac{1}{2}} e^{-\frac{y}{2}}\,\mathrm{d}y.$$

- $\Gamma(\frac{1}{2}) = \sqrt{2\pi}$ automatically from here because both are pdf-s.

□

Exercise 3

Problem (3.3-9(b) from the textbook)

Find the distribution of $W = X^2$ *when* X *is* $N(0, 4)$.

Problem (3.1-21 from the textbook)

Let $X_1, X_2, ..., X_k$ be random variables of the continuous type, and let $f_1(x), f_2(x), ..., f_k(x)$ be their corresponding pdfs, each with sample space $S = (-\infty, \infty)$. Also, let $c_1, c_2, ..., c_k$ be non-negative constants such that $\sum_{i=1}^{k} c_i = 1$.

(a) *Show that $f(x) = \sum_{i=1}^{k} c_i f_i(x)$ is a pdf of a continuous-type random variable on S.*

(b) *If X is a continuous-type random variable with pdf $f(x) = \sum_{i=1}^{k} c_i f_i(x)$ on S, $E(X_i) = \mu_i$, and $\mathrm{Var}(Xi) = \sigma_i^2$ for $i = 1, \ldots, k$, find the mean and the variance of X.*

Definition

$f(x)$ is called the **mixture density** of $f_1(x), f_2(x), ..., f_k(x)$.

- The most widely used mixture is the **mixture of Gaussians**: $f(x) = \sum_{i=1}^{k} c_i N(\mu_i, \sigma_i)$.
- Any distribution can be approximated by a Gaussian mixture (in sense of cdf convergence).

Mixed type random variable

Definition

The mixture of a continuous and discrete random variables is called a **mixed random variable**.

Problem (3.4-9 in the textbook)

Consider the following game: A fair die is rolled. If the outcome is even, the player receives a number of dollars equal to the outcome on the die. If the outcome is odd, a number is selected at random from the interval $[0, 1)$ with a balanced spinner, and the player receives that fraction of a dollar associated with the point selected.

1. *Define and sketch the cdf of X, the amount received.*
2. *Find the expected value of X.*

Solution