Midterm 3, Math 2552

PLEASE PRINT YOUR NAME CLEARLY IN ALL CAPITAL LETTERS

First Name ____________________ Last Name ____________________

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Section Number (A1, A2 or A3) ________ TA Name _________________________

INSTRUCTIONS (PLEASE READ)

Formatting and Timing

- You should only need 75 min to take the exam, but students will have 3 hours to submit the exam, from the time that it is released.
- **Show your work** and justify your answers for all questions unless stated otherwise.
- Please write neatly, and use dark and clear writing so that the scan is easy to read.
- Please write your name or initials at the top of every page.
- Please solve the questions in the exam in the order they are given.
- You do not need to print the exam. As long as you solve problems in the order they are given (just like the written homework sets), you can write your answers on your own paper.But students can print the exam and write their answers on the printed copy if they prefer.

Submission

- Students should scan their work and submit it through Gradescope. There should be an **assignment** in Gradescope for this exam. The process for submitting your work will be similar to what you have used for homework.
- Work must be submitted today by 12:30 PM ET.
- Please upload your work as a single PDF file. If this is not possible you can email your work to your instructor.
- During the upload process in Gradescope, please indicate which page of your work correspondsto each question in the exam.

Questions

- If there are questions during the exam, students can ask them on BlueJeans, email their instructor or message them through Canvas.
- Our course Piazza forum will be temporarily inactive for 24 hrs on the day of the exam.
- If you run into any technical issues or any unanticipated emergencies, please email your instructor as soon as you can.

Integrity

- Students can use any resources while taking these tests including online calculators and Mathematica.
- Students cannot communicate with anyone during these tests including using Reddit or online message boards
- Students cannot use solutions provided from another student or third party.
- In other words: do your own work but you can use technology to solve problems.

1. (8 points) Compute the Laplace Transform of the following functions:

   (a) (4 points)

$$f(t) = e^{2t} t \sin(3t).$$

   We have a multiplication by an exponential (a shift in the s-domain) and a multiplication by t (-1 times a derivative in the s-domain). We can deal with the order we want between these two operations. Here is an example:

$$\begin{aligned}
\mathcal{L}[\sin(3t)](s) &= \frac{3}{t^2+9}, \\
\mathcal{L}[t\sin(3t)](s) &= -\frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{3}{t^2+9}\right) = \frac{6t}{(t^2+9)^2}, \\
F(s) = \mathcal{L}[e^{2t} t \sin(3t)](s) &= \frac{6(t-2)}{((t-2)^2+9)^2}.
\end{aligned}$$

   (b) (4 points)

$$g(t) = \begin{cases} 2 & \text{if} \quad 0 \le t < 2, \\ t & \text{if} \quad 2 \le t < 3, \\ t^2 - 5t + 9 & \text{if } t \ge 3. \end{cases}$$

   Let us first express g in terms of unit step functions:

$$g = 2u_{02} + tu_{23} + (t^2 - 5t + 9)u_3 = 2 + (t-2)u_2 + (t^2 - 6t + 9)u_3 = 2 + (t-2)u_2 + (t-3)^2 u_3.$$

   Using the formula $\mathcal{L}[f(t-c)u_c(t)](s) = e^{-cs}F(s)$, we have

$$\begin{aligned}
G(s) &= \frac{2}{s} + e^{-2s}\mathcal{L}[t](s) + e^{-3s}\mathcal{L}[t^2](s) \\
&= \frac{2}{s} + \frac{e^{-2s}}{s^2} + \frac{2e^{-3s}}{s^3}.
\end{aligned}$$

2. (8 points) Compute the Inverse Laplace Transform of the following functions:

(a) (4 points)

$$F(s) = \frac{e^{-2\pi s}}{s^2 + 2s + 10}.$$

First, let us complete the square in the fraction:

$$F(s) = \frac{e^{-2\pi s}}{s^2 + 2s + 10} = \frac{e^{-2\pi s}}{(s+1)^2 + 9}.$$

Now, we want to use the formula $\mathcal{L}[g(t-c)u_c(t)](s) = e^{-cs}G(s)$, with

$$G(s) = \frac{e^{-2\pi s}}{(s+1)^2 + 9}.$$

We recognize the formula of the Laplace transform of $\frac{1}{3}\sin(3t)$ with a shift, thus

$$g(t) = \frac{1}{3}e^{-t}\sin(3t).$$

Finally, we have

$$f(t) = g(t-2\pi)u_{2\pi} = \frac{1}{3}e^{-(t-2\pi)}\sin(3(t-2\pi)) = \frac{1}{3}e^{-t+2\pi}\sin(3t).$$

(b) (4 points)

$$G(s) = \frac{9s - 16}{(s^2+4)(s-5)}.$$

Using partial fractions, we have

$$G(s) = \frac{9s - 16}{(s^2+4)(s-5)} = \frac{As + B}{s^2 + 4} + \frac{C}{s-5}.$$

To determine the constants A, B and C, we have then the equation:

$$9s - 16 = (As + B)(s - 5) + C(s^2 + 4).$$

Taking $s = 5$, we have $29 = 29C$, thus $C = 1$.
Taking $s = 0$, we have $-16 = -5B + 4C$, thus $B = \frac{4C+16}{5} = \frac{20}{5} = 4$.
Looking at the terms in s^2, we have $0 = A + C$, thus $A = -C = -1$.
Finally we have

$$G(s) = \frac{4 - s}{s^2 + 4} + \frac{1}{s-5} = 2\frac{2}{s^2 + 4} - \frac{s}{s^2 + 4} + \frac{1}{s-5}.$$

Taking the inverse Laplace Transform, we obtain

$$g(t) = 2\sin(2t) - \cos(2t) + e^{5t}.$$

3. (6 points) Using the Laplace Transform, solve the following IVP:

$$\begin{cases} y''(t) - y'(t) - 6y(t) = -36t, \\ y(0) = -1, \quad y'(0) = 16. \end{cases}$$

Using the Laplace Transform on the equation, we have

$$s^2Y(s) - sy(0) - y'(0) - (sY(s) - y(0)) - 6Y(s) = -\frac{36}{s^2}.$$

Using the initial data, we have then:

$$(s^2 - s - 6)Y(s) = 17 - s - \frac{36}{s^2} = \frac{-s^3 + 17s^2 - 36}{s^2}.$$

Thus, we have

$$Y(s) = \frac{-s^3 + 17s^2 - 36}{s^2(s^2 - s - 6)} = \frac{-s^3 + 17s^2 - 36}{s^2(s-3)(s+2)} = \frac{As + B}{s^2} + \frac{C}{s-3} + \frac{D}{s+2}.$$

The constants A, B, C and D satisfy the equation

$$-s^3 + 17s^2 - 36 = (As + B)(s-3)(s+2) + Cs^2(s+2) + Ds^2(s-3).$$

Taking $s = 0$, we have $-36 = -6B$, thus $B = 6$.

Taking $s = -2$, we have $8 + 68 - 36 = 40 = -20D$, thus $D = -2$.

Taking $s = 3$, we have $-27 + 153 - 36 = 90 = 45C$, thus $C = 2$.

Looking at the terms in s^3, we have $-1 = A + C + D$, thus $A = -1 - C - D = -1$.

Finally, we have

$$Y(s) = \frac{-s + 6}{s^2} + \frac{2}{s-3} - \frac{2}{s+2},$$

which gives us

$$y(t) = 6t - 1 + 2e^{3t} - 2e^{-2t}.$$

Remark: you can check that:

- $y_h(t) = C_1e^{3t} + C_2e^{-2t}$ is the general form for the solution of the associated homogeneous equation.
- $y_p(t) = 6t - 1$ is a particular solution.
- The choices $C_1 = 2$ and $C_2 = -2$ allow us to solve the IVP.

4. (9 points) The goal of this exercise is to compute the Laplace Transforms of the function:

$$y(t) = \cos(\alpha t)^2,$$

where α is a real number We want to use three different methods.

(a) (4 points) Compute the first and second derivatives y' and y''. Using them, prove that y is a solution of the IVP:

$$\begin{cases} y'' + 4\alpha^2 y = 2\alpha^2, \\ y(0) = 1, \qquad y'(0) = 0. \end{cases}$$

Use this IVP to compute $Y(s)$.

We have

$$\begin{aligned} y(t) &= \cos(\alpha t)^2, \\ y'(t) &= -2\alpha \sin(\alpha t)\cos(\alpha t), \\ y''(t) &= -2\alpha^2(\cos(\alpha t)^2 - \sin(\alpha t)^2) = -2\alpha^2(2\cos(\alpha t)^2 - 1) = 2\alpha^2 - 4\alpha^2\cos(\alpha t)^2, \end{aligned}$$

where we used in the last line that $\cos(\alpha t)^2 + \sin(\alpha t)^2 = 1$. Therefore, we have

$$y'' + 4\alpha^2 y = 2\alpha^2 - 4\alpha^2\cos(\alpha t)^2 + 4\alpha^2\cos(\alpha t)^2 = 2\alpha^2.$$

For the initial data, using the formulas above, we have

$$\begin{aligned} y(0) &= \cos(0)^2 = 1, \\ y'(0) &= -2\alpha \sin(0)\cos(0) = 0. \end{aligned}$$

Thus $y(t) = \cos(\alpha t)^2$ is the solution of the IVP.

Taking the Laplace Transform of the IVP, we have

$$s^2 Y(s) - sy(0) - y'(0) + 4\alpha^2 Y(s) = \frac{2\alpha^2}{s}.$$

Using the initial values, we have

$$(s^2 + 4\alpha^2)Y(s) = s + \frac{2\alpha^2}{s} = \frac{s^2 + 2\alpha^2}{s}.$$

Thus, we have

$$Y(s) = \frac{s^2 + 2\alpha^2}{s(s^2 + 4\alpha^2)}.$$

(b) (2 points) Using the trigonometry formula $\cos(\alpha t)^2 = \dfrac{1+\cos(2\alpha t)}{2}$ and the Laplace Transform of the cosine, compute $Y(s)$.

Using the formula above, we have $y(t) = \cos(\alpha t)^2 = \dfrac{1+\cos(2\alpha t)}{2}$. Thus, taking the Laplace Transform, we obtain:

$$Y(s) = \frac{1}{2s} + \frac{1}{2} \times \frac{s}{s^2+4\alpha^2} = \frac{1}{2} \times \frac{s^2+4\alpha^2+s^2}{s(s^2+4\alpha^2)} = \frac{s^2+2\alpha^2}{s(s^2+4\alpha^2)}.$$

We recover the result of part (a).

(c) (2 points) Using Euler's Formula $\cos(\alpha t) = \dfrac{e^{i\alpha t}+e^{-i\alpha t}}{2}$, express $y(t)$ in functions of exponential and constant terms. Use this expression to compute $Y(s)$.

We have $\cos(\alpha t) = \dfrac{e^{i\alpha t}+e^{-i\alpha t}}{2}$, thus

$$y(t) = \cos(\alpha t)^2 = \frac{(e^{i\alpha t}+e^{-i\alpha t})^2}{4} = \frac{e^{2i\alpha t}+e^{-2i\alpha t}+2}{4}.$$

Taking the Laplace Transform, we obtain

$$\begin{aligned} Y(s) &= \frac{1}{4}\left(\frac{1}{s-2i\alpha} + \frac{1}{s+2i\alpha} + \frac{2}{s}\right) = \frac{1}{4}\left(\frac{(s+2i\alpha)+(s-2i\alpha)}{(s-2i\alpha)(s+2i\alpha)} + \frac{2}{s}\right) \\ &= \frac{1}{4}\left(\frac{2s}{s^2+4\alpha^2} + \frac{2}{s}\right) = \frac{s^2+2\alpha^2}{s(s^2+4\alpha^2)}. \end{aligned}$$

We recover again results from part (a) and (b).

Remark:

A fourth option to compute this Laplace Transform could be to use just Euler's formula for one of the cosines. Indeed, if you write

$$y(t) = \cos(\alpha t)^2 = \left(\frac{e^{i\alpha t}+e^{-i\alpha t}}{2}\right)\cos(\alpha t) = \frac{e^{i\alpha t}}{2}\cos(\alpha t) + \frac{e^{-i\alpha t}}{2}\cos(\alpha t).$$

Then, we can use the fact that a multiplication by an exponential in the $t-$Domain is a shift in the s-Domain to compute $Y(s)$.

(d) (1 point) Compare your results with the $s-$Domain function

$$\frac{s^2}{(s^2+\alpha^2)^2} = \left(\mathcal{L}[\cos(\alpha t)](s)\right)^2.$$

Is it an expected comparison?

$$Y(s) = \frac{s^2+2\alpha^2}{s(s^2+4\alpha^2)} \neq \frac{s^2}{(s^2+\alpha^2)^2}.$$

In other words, we have

$$\mathcal{L}[\cos(\alpha t)^2](s) \neq \left(\mathcal{L}[\cos(\alpha t)](s)\right)^2$$

This is not a surprise, because in general, we have

$$\mathcal{L}[fg] \neq \mathcal{L}[f] \times \mathcal{L}[g].$$

5. (9 points) For the four followings systems, compute the critical points.

(a) (2 points)

$$\begin{cases} x' = (x - y + 2)(y - x + 3), \\ y' = (x - y)(y - x + 1). \end{cases}$$

We have:

$$\begin{aligned} x' = 0 &\Rightarrow x - y = -2 \quad \text{or} \quad x - y = 3, \\ y' = 0 &\Rightarrow x - y = 0 \quad \text{or} \quad x - y = 1. \end{aligned}$$

Therefore, we have to study the four systems:

$$\begin{cases} x - y = -2, \\ x - y = 0, \end{cases} \quad \begin{cases} x - y = -2, \\ x - y = 1, \end{cases} \quad \begin{cases} x - y = 3, \\ x - y = 0, \end{cases} \quad \text{and} \quad \begin{cases} x - y = 3, \\ x - y = 1. \end{cases}$$

None of these systems has a solution, there is no critical point

The nullclines are four parallel lines.

(b) (2 points)

$$\begin{cases} x' = x^2 - 2x + y^2 - 4, \\ y' = x^2 + 2x + y^2 - 4. \end{cases}$$

We have:

$$\begin{aligned} x' = 0 &\Rightarrow x^2 - 2x + y^2 = 4, \\ y' = 0 &\Rightarrow x^2 + 2x + y^2 = 4. \end{aligned}$$

Taking the second row minus the first one, you obtain $4x = 0$ which leads to $x = 0$. Then, the equation for y becomes $y^2 = 4$ which gives $y = \pm 2$.

Therefore, we have two critical points: (0,-2) and (0,2).

The nullclines are two circles, we do have two intersections points between these two circles.

(c) (2 points)

$$\begin{cases} x' = xy, \\ y' = y - x^2 + 1. \end{cases}$$

We have:

$$\begin{aligned} x' &= 0 \Rightarrow x = 0 \quad \text{or} \quad y = 0, \\ y' &= 0 \Rightarrow x^2 - y = 1. \end{aligned}$$

Therefore, we have to study the two systems:

$$\begin{cases} x = 0, \\ x^2 - y = 1, \end{cases} \quad \text{and} \quad \begin{cases} y = 0, \\ x^2 - y = 1. \end{cases}$$

The first one give the critical points (0,1). The second one gives us $x^2 = 1$ which gives $x = \pm 1$.

Therefore, we have three critical points: (0,1), (-1,0) and (1,0).

The $x-$nullclines are the x and y axes, while the $y-$nullcline is a parabola.

(d) (3 points)

$$\begin{cases} x' = xy - 1, \\ y' = (x - y)(x + y). \end{cases}$$

We have:

$$\begin{aligned} x' &= 0 \Rightarrow xy = 1, \\ y' &= 0 \Rightarrow x = y \quad \text{or} \quad x = -y. \end{aligned}$$

Therefore, we have to study the two systems:

$$\begin{cases} xy = 1, \\ x = y, \end{cases} \quad \text{and} \quad \begin{cases} xy = 1, \\ x = -y. \end{cases}$$

The first one give the equations $x^2 = 1$ and $x = y$, which lead to the critical points (1,1) and (-1,-1). The second one the equations $x^2 = -1$ and $x = -y$, which have no solution.

Therefore, we have two critical points: (-1,-1) and (1,1).

The $x-$nullcline is a hyperbola, while the $y-$nullclines are two lines, one that intersects the hyperbola in two points, and one that never crosses the hyperbola.

6. (8 points) Consider the following system representing two species x and y competing for the same food supply:

$$\begin{cases} x' = x(4 - 2x - y), \\ y' = y(3 - x - y). \end{cases}$$

(a) (2 points) Compute the critical points.

We have:

$$\begin{aligned} x' = 0 &\Rightarrow x = 0 \quad \text{or} \quad 2x + y = 4, \\ y' = 0 &\Rightarrow y = 0 \quad \text{or} \quad x + y = 3. \end{aligned}$$

Therefore, we have to study the four systems:

$$\begin{cases} x = 0, \\ y = 0, \end{cases} \quad \begin{cases} x = 0, \\ x + y = 3, \end{cases} \quad \begin{cases} 2x + y = 4, \\ y = 0, \end{cases} \quad \text{and} \quad \begin{cases} 2x + y = 4, \\ x + y = 3. \end{cases}$$

The three first system lead immediately to the three critical points (0,0), (0,3) and (2,0). For the last system, taking the first row minus the second one, we obtain $x = 1$ which leads (using the first or the second row) to $y = 2$.

We have thus four critical points: (0,0), (0,3), (2,0) and (1,2).

(b) (4 points) For each critical point, construct the linear system near the critical point and classify the stability of the critical point for the linear system.

The system of equations is

$$\begin{cases} x' = x(4 - 2x - y) = -2x^2 - xy + 4x, \\ y' = y(3 - x - y) = -y^2 - xy + 3y. \end{cases}$$

Thus the Jacobian is

$$J(x, y) = \begin{pmatrix} -4x - y + 4 & -x \\ -y & -2y - x + 3 \end{pmatrix}$$

Near each critical point, we study the linear system $\vec{x}' = J\vec{x}$.

At (0,0) we have $J = \begin{pmatrix} 4 & 0 \\ 0 & 3 \end{pmatrix}$. The eigenvalues are 4 and 3, this is an unstable node.

At (0,3) we have $J = \begin{pmatrix} 1 & 0 \\ -3 & -3 \end{pmatrix}$. The eigenvalues are 1 and -3, this is an unstable saddle.

At (2,0) we have $J = \begin{pmatrix} -4 & -2 \\ 0 & 1 \end{pmatrix}$. The eigenvalues are -4 and 1, this is an unstable saddle.

At (1,2) we have $J = \begin{pmatrix} -2 & -1 \\ -2 & -2 \end{pmatrix}$. The eigenvalues satisfy the characteristic equation $\lambda^2 + 4\lambda + 2 = 0$, thus we have $\lambda = \frac{-4 \pm \sqrt{8}}{2} = -2 \pm \sqrt{2}$. Both eigenvalues are negative, we have a stable node.

(c) (2 points) If you start with a positive population of both species x and y, what could you expect for the ratio of both species to be as t goes to infinity.

If we start at $t = 0$ with both x and y positive, the solution will tend to the stable equilibrium point (1,2). In means that for long times, the population of the species y will be around twice the population of the species x.

7. (2 points) A small number of points will be allocated for presentation, neatness, and organization. Please ensure that

- Your work is legible in the scan.
- Your name or initials are at the top of every page.
- Questions are answered in the order in which they were given.
- During the upload process you have indicated which pages correspond to which question, and made sure that none of your pages are upside down or sideways (you can also change the orientation of the pages when you upload in Gradescope). Ensuring that these criteria are met helps ensure that your exam is graded efficiently and accurately.