Homework 2

Due Date: Week of Friday **09/04**. All homework submitted by Sunday **09/06** 11:59pm will be graded together. Homework submitted past that time may be graded late. Submit your homework through Canvas as a single pdf file. Do not use solution sets from previous years. You are encouraged to discuss homework assignments with each other, the TAs or myself, but the solutions have to be executed and submitted individually.

Exercise 1 [10%]. Let $\boldsymbol{\mathscr{r}}$ be the usual separation vector from a fixed source point $\mathbf{r}' = (x', y', z')$ to the field point $\mathbf{r} = (x, y, z)$, and let $\mathscr{r}$ be its magnitude. Show that:

(1) $\boldsymbol{\nabla}(\mathscr{r}^2) = 2\boldsymbol{\mathscr{r}}$,
(2) $\boldsymbol{\nabla}(1/\mathscr{r}) = -\hat{\boldsymbol{\mathscr{r}}}/\mathscr{r}^2$.
(3) What is the general formula for $\boldsymbol{\nabla}\,\mathscr{r}^n$ with $n \in \mathbb{N}$?

[*Hint: Realize that spatial derivatives in* $\boldsymbol{\nabla} \equiv \boldsymbol{\nabla}_{\mathbf{r}}$ *are meant with respect to the field point* $\mathbf{r}$ *at fixed* $\mathbf{r}'$.]

Exercise 2 [10%]. Write an expression for the volume charge density $\rho(\mathbf{r})$ of the following charge distributions:

(1) Two point charges: $+3q$ located at $\mathbf{r} = -D\hat{\mathbf{x}}$ and $-q$ located at $\mathbf{r} = +2D\hat{\mathbf{y}}$.
(2) An infinite and uniform linear charge distribution λ running along the y-axis and otherwise located at the origin in the xz-plane.
(3) A spherical shell of radius R, centered at the origin, and carrying a uniform surface charge density σ.

Problem A [30%]. The time-averaged electric potential of a neutral hydrogen atom is given by

$$V(\mathbf{r}) = \frac{q}{4\pi\varepsilon_0}\frac{e^{-\alpha r}}{r}\left(1 + \alpha r/2\right)$$

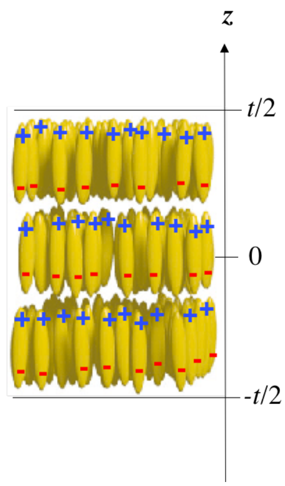
where q is the magnitude of the electronic charge, and $\alpha^{-1} = a_0/2$, a_0 being the Bohr radius.

(1) Determine the electric field $\mathbf{E}(\mathbf{r})$ associated with this potential.

(2) Find the charge distribution $\rho(\mathbf{r})$ that produces this potential. Think carefully about what happens at the origin! Sketch this function $\rho(\mathbf{r})$ in a manner that clearly showcases its characteristics.

(3) Show by explicit integration over $\rho(\mathbf{r})$ that the net charge carried by this distribution is zero. If you don't get zero, think again about what happens at $\mathbf{r} = 0$.

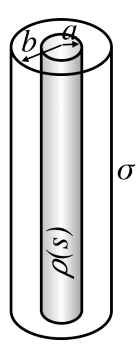
Problem B [20%]. Liquid crystals are made of long rod-like molecules with a positive head and a negative tail that are packed together to form long, thin sheets. As a function of temperature, confinement (for instance squeezing the molecules between plates) or application of an external electric field, liquid crystals organize in different phases that are qualitatively distinct from the usual phases of matter you are used to (gas, liquid and solid). One such phase, called *smectic-A*, is represented in the figure below. The volume charge density in the smectic phase is obviously complicated, but can be quite successfully modeled with the rather simple form $\rho(z) = \rho_0 \sinh(z/z_0)$ where ρ_0 and z_0 are constants, *i.e.* uniform in x and y, but varying in z, with $z = 0$ defined to be the middle of the sheet.



(1) Assuming the sheet is infinitely long in the xy-plane, use Gauss's Law to find the electric field everywhere in space, in terms of the constants ρ_0, z_0, and the sheet thickness t.

(2) Sketch the magnitude of the resulting electric field. Where is it biggest?

Problem C [30%]. Long coaxial cables are used in many applications for instance in dilution refrigerators to excite and measure superconducting qubits at milikevlin temperatures with microwave radiation. The cylindrical cable is made of two parts, the core and the shell, see drawing. Assume that the core is a cylindrical rod of radius a that carries a *non-uniform* <u>volume</u> charge density $\rho(s)$ with $\rho(s) = \rho_0\, s/a$. The shell is an infinitely-thin hollow cylinder of radius b that carries a *uniform* <u>surface</u> charge density σ. The cable is overall electrically neutral. The space between $s \geq a$ and $s = b$ is filled with vaccum.



(1) Find $\mathbf{E}$ everywhere, and sketch it.