Final Exam

How to turn your work in: This final finishes by 2:10 pm on 12/02, at which point you must drop your pen and prepare to upload your work to Canvas. You have until 2:20 pm to upload it to Canvas. If you encounter technical difficulties to upload, send a picture of your work to mourigal@gatech.edu or by text message to 404-747-4969.There are three problems. Questions marked with a $*$ are "must-do" questions.

Honor code: This quiz is administered under a strict Honor Code. By signing your name on each page of your uploaded quiz, you certify that you took the test with closed book and notes, closed internet, no calculator, and no communications with anyone.

Useful Formulas:

$$\int \frac{dx}{x} = \ln x$$

$$\int \frac{dx}{x+a} = \ln(a+x)$$

$$\int \frac{xdx}{x+a} = x - a\ln(a+x)$$

$$\int \frac{dx}{\sqrt{x^2 \pm a^2}} = \ln(x + \sqrt{a^2+x^2})$$

$$\int \frac{xdx}{\sqrt{x^2 \pm a^2}} = \sqrt{a^2+x^2}$$

$$\int \frac{dx}{x^2 \pm a^2} = \frac{1}{a}\tan^{-1}(x/a)$$

$$\int \frac{dx}{(x^2 \pm a^2)^{3/2}} = \frac{\pm x}{a^2\sqrt{x^2 \pm a^2}}$$

$$\int \frac{xdx}{(x^2 \pm a^2)^{3/2}} = -\frac{1}{\sqrt{x^2 \pm a^2}}$$

$$\int \cos(ax)dx = \frac{1}{a}\sin(ax)$$

$$\int \sin(ax)dx = -\frac{1}{a}\cos(ax)$$

$c^2 = a^2 + b^2 - 2ab\cos\theta$ Law of cosines

$$(1+\epsilon)^\alpha = 1 + \alpha\epsilon + \frac{\alpha(\alpha-1)}{2!}\epsilon^2$$

$$V = \frac{1}{4\pi\varepsilon_0}\iiint \frac{\rho(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}d\tau'$$

$$V = \frac{1}{4\pi\varepsilon_0}\iint \frac{\sigma(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}da'$$

$$\nabla\cdot\mathbf{E} = \rho/\varepsilon_0 \quad \text{or} \quad \oiint \mathbf{E}\cdot d\mathbf{a} = \frac{1}{\varepsilon_0}\iiint \rho\, d\tau$$

$$\Delta V = -\int \mathbf{E}\cdot d\mathbf{l} \quad \text{and} \quad \mathbf{E} = -\nabla V$$

$$\nabla^2 V = -\rho/\varepsilon_0$$

$$\partial V/\partial n|_{\text{above}} - \partial V/\partial n|_{\text{below}} = -\sigma/\varepsilon_0$$

$$W = \frac{1}{2}\sum_i q_i V(\mathbf{r}_i)$$

$$W = \frac{\varepsilon_0}{2}\iiint |\mathbf{E}|^2 d\tau + \frac{1}{2\mu_0}\iiint |\mathbf{B}|^2 d\tau$$

$$V(r,\theta) = \sum_{\ell=0}^{+\infty}\left(A_\ell r^\ell + \frac{B_\ell}{r^{\ell+1}}\right)P_\ell(\cos\theta)$$

$$P_0(x) = 1, P_1(x) = x, P_2(x) = 3x^2/2 - 1/2,$$

$$P_3(x) = \frac{1}{2}(5x^3 - 3x)$$

$$\int_{-1}^{+1} P_\ell(x)P_{\ell'}(x) = \frac{2}{2\ell+1}\delta_{\ell,\ell'}$$

$$\int_0^a \sin(n\pi x/a)\sin(n'\pi x/a)\, dx = \frac{a}{2}\delta_{n,n'}$$

$$V(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\sum_{\ell=0}^{+\infty}\frac{1}{r^{\ell+1}}\iiint (r')^\ell P_\ell(\cos\theta')\rho(\mathbf{r}')d\tau'$$

$$V(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\left(\frac{Q}{r} + \frac{\mathbf{p}\cdot\hat{\mathbf{r}}}{r^2} + \sum_{\alpha,\beta}\frac{Q_{\alpha\beta}\hat{r}_\alpha\hat{r}_\beta}{r^3}\right)$$

$$\mathbf{p} = \sum_i q_i\mathbf{r}_i \equiv \iiint \rho(\mathbf{r}')\mathbf{r}'d\tau'$$

$$Q_{\alpha\beta} = \sum_i \frac{q_i}{2}\left(3r_{i\alpha}r_{i\beta} - r_i^2\delta_{\alpha\beta}\right) \equiv \iiint \frac{\rho(\mathbf{r}')}{2}\left(3r'_\alpha r'_\beta - r'^2\delta_{\alpha\beta}\right)d\tau'$$

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi}\iiint \frac{\mathbf{J}(\mathbf{r}')\times(\mathbf{r}-\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|^3}d\tau' = ... = \frac{\mu_0 I}{4\pi}\int \frac{d\boldsymbol{\ell}'\times(\mathbf{r}-\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|^3}$$

$\mathbf{I} = \lambda\mathbf{v}$ or $\mathbf{K} = \sigma\mathbf{v}$ or $\mathbf{J} = \rho\mathbf{v}$
$\mathbf{F} = I\int d\boldsymbol{\ell}\times\mathbf{B} = ... = \iiint(\mathbf{J}\times\mathbf{B})\,d\tau$

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi}\iiint \frac{\mathbf{J}(\mathbf{r}')\times(\mathbf{r}-\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|^3}d\tau' = ... = \frac{\mu_0 I}{4\pi}\int \frac{d\boldsymbol{\ell}'\times(\mathbf{r}-\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|^3}$$

$$\nabla\times\mathbf{B} = \mu_0\mathbf{J} \quad \text{or} \quad \oint \mathbf{B}\cdot d\mathbf{l} = \mu_0\oiint \mathbf{J}\cdot d\mathbf{a}$$

$$\mathbf{B} = \nabla\times\mathbf{A} \text{ and } \mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi}\iiint \frac{\mathbf{J}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}d\tau' = ... = \frac{\mu_0 I}{4\pi}\int \frac{d\boldsymbol{\ell}'}{|\mathbf{r}-\mathbf{r}'|}$$

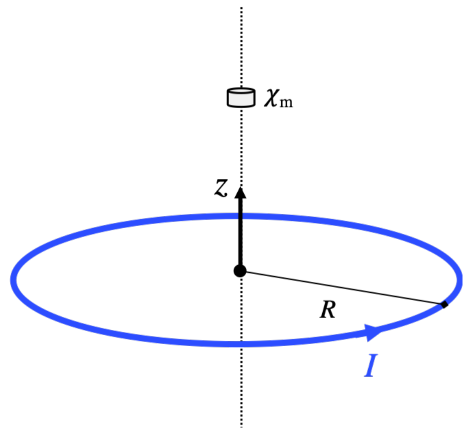
$$\mathbf{A}_{\text{dip}}(\mathbf{r}) = \frac{\mu_0}{4\pi}\frac{\mathbf{m}\times\hat{\mathbf{r}}}{r^2} \quad \text{with} \quad \mathbf{m} = \frac{1}{2}\iiint \mathbf{r}'\times\mathbf{J}(\mathbf{r}')d\tau' = \frac{I}{2}\int \mathbf{r}'\times d\boldsymbol{\ell}'$$

Force on dipoles: $\mathbf{F} = \nabla(\mathbf{p}\cdot\mathbf{E})$ or $\mathbf{F} = \nabla(\mathbf{m}\cdot\mathbf{B})$

Dielectric (General): $\mathbf{D} = \varepsilon_0\mathbf{E} + \mathbf{P}$ and $\nabla\cdot\mathbf{D} = \rho_f$.
Also $\sigma_b = \mathbf{P}\cdot\hat{\mathbf{n}}$, $\rho_b = -\nabla\cdot\mathbf{P}$
Dielectric (Linear): $\mathbf{P} = \varepsilon_0\chi\mathbf{E}$
And if you prefer $\mathbf{D} = \varepsilon_0(1+\chi)\mathbf{E}$ or also $\mathbf{D} \equiv \varepsilon_0\varepsilon_r\mathbf{E} \equiv \varepsilon\mathbf{E}$.

Magnetic Material (General): $\mathbf{H} = \mathbf{B}/\mu_0 - \mathbf{M}$.
Bound currents: $\mathbf{K}_b = \mathbf{M}\times\hat{\mathbf{n}}$ and $\mathbf{J}_b = \nabla\times\mathbf{M}$.
Magnetic Material (Linear): $\mathbf{M} = \chi_m\mathbf{H}$
$\chi_m > 0$ for paramagnets and $\chi_m < 0$ for diamagnets.
Also $\mu = \mu_0\mu_r = \mu_0(1+\chi_m)$.

Problem A [40%]. Consider a large loop of current of radius R carrying a free current I as shown. A small piece of linear magnetic material is held on the loop's axis at a distance $z_0 > 0$ from the center. The magnetic material is a tiny puck of mass m, volume V, and magnetic susceptibility χ_m. The goal of this problem is to calculate the force on the puck.



$*$ (1) Calculate the magnetic field $\mathbf{B}(z)$ generated by the large loop everywhere on the axis of the loop as a function of z using Biot & Savart. Plot the magnitude of the magnetic field as a function of z

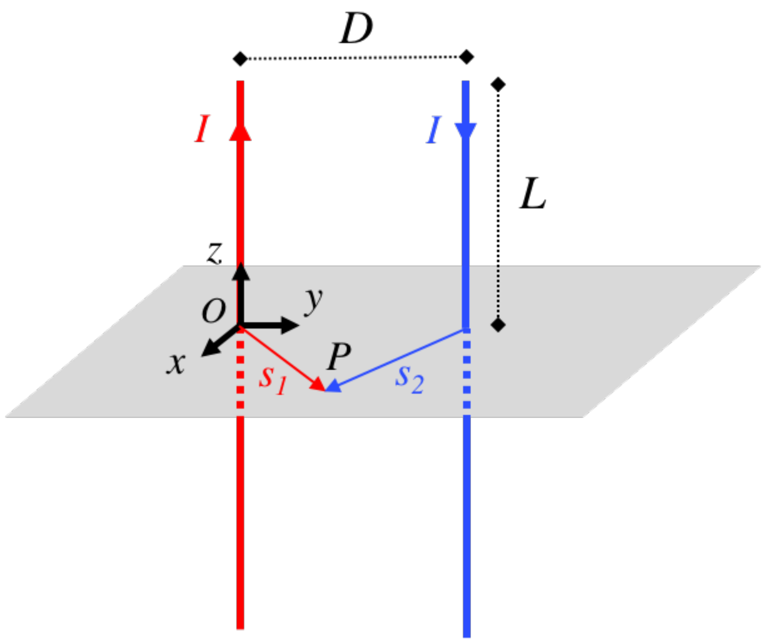
(2) Calculate the magnetization $\mathbf{M}$ and the total magnetic dipole moment $\mathbf{m}$ induced on the puck by the large loop as a function of z. You can assume that the puck is so tiny that all its parts experience the same field $\mathbf{B}(z_0)$ calculated above.

$*$ (3) Assuming that the puck is made of a paramagnetic material ($\chi_m > 0$) calculate the direction and magnitude of the force it experiences. How does the magnitude of the force changes as the position of the puck z_0 is increased?

$*$ (4) Assume now that the puck is made of a superconductor, that is a perfect diamagnetic material ($\chi_m = -1$). What changes compared to the above situation?

(5) Draw the magnetic-field lines around the puck in the case of the paramagnet and the superconductor.

Problem B [30%]. The goal of this problem is to calculate the vector-potential $\mathbf{A}(\mathbf{r})$ and the magnetic-field $\mathbf{B}(\mathbf{r})$ generated by a pair of long parallel wires separated by distance D, each carrying a current I, but in opposite directions ($+\hat{z}$ and $-\hat{z}$ here), see drawing.



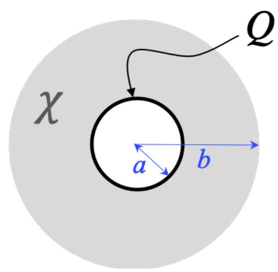
$*$ (1) Let's start with only one wire. Calculate the vector potential $\mathbf{A}(\mathbf{r})$ generated at point P by the wire passing through the origin, which is the red wire with the current going up. Provide direction and magnitude for $\mathbf{A}(\mathbf{r})$. To accomplish this calculation, assume that the total wire length is $2L$ and that the point P is in a plane midway between the ends of the wire, at a distance s_1 from the wire and .

(2) Use the fact that $L \gg s_1$ to simplify the above result using a Taylor expansion in $\epsilon = s_1/L$ (keep the first non-zero term for $\mathbf{A}(\mathbf{r})$). From your result for the vector potential $\mathbf{A}(\mathbf{r})$ in the limit of large $L \gg s_1$, derive the magnetic field $\mathbf{B}(\mathbf{r})$ produced by the red wire in the same limit.

$*$ (3) Still with only one wire and assuming $L \to \infty$. Derive the magnetic field $\mathbf{B}(\mathbf{r})$ produced by the red wire direclty using Ampére's Law.

(4) Using the superposition principle, calculate the approximate vector-potential $\mathbf{A}(\mathbf{r})$ generated by the pair of wires at point P ($L \gg s_1, s_2$). Express your result as a function of the ratio s_2/s_1, I, and μ_0.

Problem C [30%]. A thin metallic shell of radius a is embedded in a plastic sphere of radius b. The metallic shell carries uniformly distributed free charges of total $+Q$ and is otherwise empty. The plastic sphere is overall neutral and modeled as an isotropic linear dielectric of susceptibility χ.



$*$ (1) Solve for the displacement field $\mathbf{D}$, the electric field $\mathbf{E}$ and the polarization $\mathbf{P}$ *everywhere*. Note that there are three distinct regions to consider in this problem: (i) $r \leq a$, (ii) $a < r < b$, and (iii) $r \geq b$.

$*$ (2) Using Gauss' Law for $\mathbf{E}$, determine the total charge within a sphere of radius r with $a < r < b$. This is not $+Q$!

$*$ (3) Find the bound surface charge density on the outside of the plastic sphere.

Cartesian. $d\mathbf{l} = dx\,\hat{\mathbf{x}} + dy\,\hat{\mathbf{y}} + dz\,\hat{\mathbf{z}}$; $\quad d\tau = dx\,dy\,dz$

Gradient: $$\nabla t = \frac{\partial t}{\partial x}\,\hat{\mathbf{x}} + \frac{\partial t}{\partial y}\,\hat{\mathbf{y}} + \frac{\partial t}{\partial z}\,\hat{\mathbf{z}}$$

Divergence: $$\nabla \cdot \mathbf{v} = \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}$$

Curl: $$\nabla \times \mathbf{v} = \left(\frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z}\right)\hat{\mathbf{x}} + \left(\frac{\partial v_x}{\partial z} - \frac{\partial v_z}{\partial x}\right)\hat{\mathbf{y}} + \left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y}\right)\hat{\mathbf{z}}$$

Laplacian: $$\nabla^2 t = \frac{\partial^2 t}{\partial x^2} + \frac{\partial^2 t}{\partial y^2} + \frac{\partial^2 t}{\partial z^2}$$

Spherical. $d\mathbf{l} = dr\,\hat{\mathbf{r}} + r\,d\theta\,\hat{\boldsymbol{\theta}} + r\sin\theta\,d\phi\,\hat{\boldsymbol{\phi}}$; $\quad d\tau = r^2\sin\theta\,dr\,d\theta\,d\phi$

Gradient: $$\nabla t = \frac{\partial t}{\partial r}\,\hat{\mathbf{r}} + \frac{1}{r}\frac{\partial t}{\partial \theta}\,\hat{\boldsymbol{\theta}} + \frac{1}{r\sin\theta}\frac{\partial t}{\partial \phi}\,\hat{\boldsymbol{\phi}}$$

Divergence: $$\nabla \cdot \mathbf{v} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 v_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\, v_\theta) + \frac{1}{r\sin\theta}\frac{\partial v_\phi}{\partial \phi}$$

Curl: $$\nabla \times \mathbf{v} = \frac{1}{r\sin\theta}\left[\frac{\partial}{\partial\theta}(\sin\theta\, v_\phi) - \frac{\partial v_\theta}{\partial\phi}\right]\hat{\mathbf{r}}$$
$$+ \frac{1}{r}\left[\frac{1}{\sin\theta}\frac{\partial v_r}{\partial\phi} - \frac{\partial}{\partial r}(r v_\phi)\right]\hat{\boldsymbol{\theta}} + \frac{1}{r}\left[\frac{\partial}{\partial r}(r v_\theta) - \frac{\partial v_r}{\partial\theta}\right]\hat{\boldsymbol{\phi}}$$

Laplacian: $$\nabla^2 t = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial t}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial t}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 t}{\partial\phi^2}$$

Cylindrical. $d\mathbf{l} = ds\,\hat{\mathbf{s}} + s\,d\phi\,\hat{\boldsymbol{\phi}} + dz\,\hat{\mathbf{z}}$; $\quad d\tau = s\,ds\,d\phi\,dz$

Gradient: $$\nabla t = \frac{\partial t}{\partial s}\,\hat{\mathbf{s}} + \frac{1}{s}\frac{\partial t}{\partial\phi}\,\hat{\boldsymbol{\phi}} + \frac{\partial t}{\partial z}\,\hat{\mathbf{z}}$$

Divergence: $$\nabla \cdot \mathbf{v} = \frac{1}{s}\frac{\partial}{\partial s}(s v_s) + \frac{1}{s}\frac{\partial v_\phi}{\partial\phi} + \frac{\partial v_z}{\partial z}$$

Curl: $$\nabla \times \mathbf{v} = \left[\frac{1}{s}\frac{\partial v_z}{\partial\phi} - \frac{\partial v_\phi}{\partial z}\right]\hat{\mathbf{s}} + \left[\frac{\partial v_s}{\partial z} - \frac{\partial v_z}{\partial s}\right]\hat{\boldsymbol{\phi}} + \frac{1}{s}\left[\frac{\partial}{\partial s}(s v_\phi) - \frac{\partial v_s}{\partial\phi}\right]\hat{\mathbf{z}}$$

Laplacian: $$\nabla^2 t = \frac{1}{s}\frac{\partial}{\partial s}\left(s\frac{\partial t}{\partial s}\right) + \frac{1}{s^2}\frac{\partial^2 t}{\partial\phi^2} + \frac{\partial^2 t}{\partial z^2}$$

VECTOR IDENTITIES

Triple Products

(1) $\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C}) = \mathbf{B} \cdot (\mathbf{C} \times \mathbf{A}) = \mathbf{C} \cdot (\mathbf{A} \times \mathbf{B})$

(2) $\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = \mathbf{B}(\mathbf{A} \cdot \mathbf{C}) - \mathbf{C}(\mathbf{A} \cdot \mathbf{B})$

Product Rules

(3) $\nabla(fg) = f(\nabla g) + g(\nabla f)$

(4) $\nabla(\mathbf{A} \cdot \mathbf{B}) = \mathbf{A} \times (\nabla \times \mathbf{B}) + \mathbf{B} \times (\nabla \times \mathbf{A}) + (\mathbf{A} \cdot \nabla)\mathbf{B} + (\mathbf{B} \cdot \nabla)\mathbf{A}$

(5) $\nabla \cdot (f\mathbf{A}) = f(\nabla \cdot \mathbf{A}) + \mathbf{A} \cdot (\nabla f)$

(6) $\nabla \cdot (\mathbf{A} \times \mathbf{B}) = \mathbf{B} \cdot (\nabla \times \mathbf{A}) - \mathbf{A} \cdot (\nabla \times \mathbf{B})$

(7) $\nabla \times (f\mathbf{A}) = f(\nabla \times \mathbf{A}) - \mathbf{A} \times (\nabla f)$

(8) $\nabla \times (\mathbf{A} \times \mathbf{B}) = (\mathbf{B} \cdot \nabla)\mathbf{A} - (\mathbf{A} \cdot \nabla)\mathbf{B} + \mathbf{A}(\nabla \cdot \mathbf{B}) - \mathbf{B}(\nabla \cdot \mathbf{A})$

Second Derivatives

(9) $\nabla \cdot (\nabla \times \mathbf{A}) = 0$

(10) $\nabla \times (\nabla f) = 0$

(11) $\nabla \times (\nabla \times \mathbf{A}) = \nabla(\nabla \cdot \mathbf{A}) - \nabla^2\mathbf{A}$

FUNDAMENTAL THEOREMS

Gradient Theorem: $\int_{\mathbf{a}}^{\mathbf{b}} (\nabla f) \cdot d\mathbf{l} = f(\mathbf{b}) - f(\mathbf{a})$

Divergence Theorem: $\int (\nabla \cdot \mathbf{A})\, d\tau = \oint \mathbf{A} \cdot d\mathbf{a}$

Curl Theorem: $\int (\nabla \times \mathbf{A}) \cdot d\mathbf{a} = \oint \mathbf{A} \cdot d\mathbf{l}$