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Assignment 8 Solutions

1. From an example in class we have the eccentricity of a comet at distance r_{com} & moving w/ speed V_{com}, at an angle ϕ wrt the radius vector

is $$e = \sqrt{1 + \frac{r_{com}(V_{com}\sin\phi)^2 (r_{com}V_{com}^2 - 2GM)}{(GM)^2}}$$

Whether or not the orbit is hyperbolic, parabolic or elliptical depends on if $e > 1, 1, < 1$, respectively

As $\left[\frac{r_{com}(V_{com}\sin\phi)^2}{(GM)^2}\right]$ is always positive, this condition

is really just if $(r_{com}V_{com}^2 - 2GM)$ is $>0, 0, <0$, respectively

Convert to Earth units, $d = \frac{r_{com}}{a_E}$ & $q = \frac{V_{com}}{V_E} = \frac{V_{com}\sqrt{a_E}}{\sqrt{GM}}$

where $V_E = \sqrt{\frac{GM}{a_E}}$ is Earth's circular speed. So, $V_{com} = \sqrt{\frac{GM}{a_E}}\, q$

$$\therefore \quad (r_{com}V_{com}^2 - 2GM) >, =, < 0$$

$$\left(\cancel{a_E}\, d\, q^2 \frac{GM}{\cancel{a_E}} - 2GM\right) >, =, < 0$$

$$GM\left(dq^2 - 2\right) >, =, < 0$$

$$\text{or } dq^2 >, =, < 2$$

2. The angular velocity at pericenter is $\omega_{max} = \dot{\theta}_{max} = \frac{J}{mr_0^2}$

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The angular velocity at apcenter is $\omega_{min} = \dot{\theta}_{min} = \frac{J}{mr_1^2}$

Let $$n = \frac{\omega_{max}}{\omega_{min}} = \frac{\cancel{\left(\frac{J}{m}\right)}\frac{1}{r_0^2}}{\cancel{\left(\frac{J}{m}\right)}\frac{1}{r_1^2}} = \frac{r_1^2}{r_0^2}$$

For elliptical orbits at $\theta = \pi$,

$$r_1 = \frac{r_0(1+e)}{(1-e)} \rightarrow \frac{r_1}{r_0} = \frac{(1+e)}{(1-e)}$$

$$\therefore \sqrt{n} = \frac{(1+e)}{(1-e)}$$

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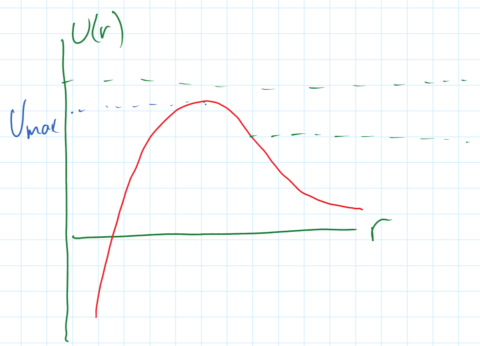
$$\sqrt{n} - \sqrt{n}e = 1+e$$

$$\sqrt{n} - 1 = e(1+\sqrt{n})$$

$$\therefore e = \frac{\sqrt{n}-1}{\sqrt{n}+1}$$

3.a) The effective potential is

8 $$U(r) = \frac{J^2}{2mr^2} - \frac{C}{3r^3}$$ 1



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b) $$\frac{dU}{dr} = -\frac{2J^2}{2mr^3} + \frac{3C}{3r^4} = 0$$

$$\frac{C}{r^4} = \frac{J^2}{mr^3}$$

$$C = \frac{J^2}{m} r \rightarrow r = \frac{mC}{J^2}$$

Substitute to find max. value of $U(r)$

$$U_{max} = \frac{J^2}{2m\left(\frac{mC}{J^2}\right)^2} - \frac{C}{3\left(\frac{mC}{J^2}\right)^3} = \frac{J^6}{2m^3C^2} - \frac{J^6}{3m^3C^2} = \frac{J^6}{6m^3C^2}$$ 3

c) From the sketch we see that if the energy of the particle is less than U_{max}, then the particle will reach a min. value of r and then head back out to infinity. If E is greater than U_{max} then the particle will head all the way in to $r=0$, never to return (since $U \rightarrow -\infty$ as $r \rightarrow 0$, E will become infinitely negative) never to return (since $U \to -\infty$ as $r \to 0$, E will become infinitely negative!)

The condition for capture is thus $U_{max} < E$

Since $E = E_\infty = \frac{1}{2} m v_0^2$,

$$\frac{J^6}{6m^3C^2} < \frac{1}{2} m v_0^2$$

$J = mvr = mvb$ since we are told particle has impact parameter b

$$\therefore \quad b < \left(\frac{3C^2}{m^2 v_0^4}\right)^{1/6} = b_{max}$$

The cross-section for capture is therefore 3

$$\sigma = \pi b_{max}^2 = \pi \left(\frac{3C^2}{m^4 v_0^4}\right)^{1/3}$$

4. Consider a velocity kick Δv applied along the direction of travel at an arbitrary place in the orbit. We seek the optimum location to apply the kick

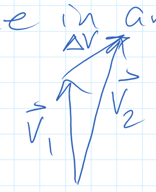
6 $$E_1 = \text{initial energy} = \frac{1}{2} m v^2 - \frac{GMm}{r}$$

$$E_2 = \text{final energy} = \frac{1}{2} m (v + \Delta v)^2 - \frac{GMm}{r}$$

We seek to maximize the energy gain $E_2 - E_1$:

$$E_2 - E_1 = \frac{1}{2} m \left(2v\Delta v + \Delta v^2\right)$$

For a given Δv, this quantity is clearly a maximum when v is a maximum; ie, at perigee. 3 Now consider a velocity kick Δv applied at perigee in an arbitrary direction:



The final energy is $\frac{1}{2}mV_2^2 = \frac{GMm}{r_p}$ where r_p = perigee distance

This will be a maximum for a maximum $|\vec{V_2}|$, which clearly occurs when $\vec{V_1}$ & $\vec{V_2}$ are along the same direction. 3

Thus, the most efficient way to change the energy of an elliptical orbit (for a single engine thrust) is by firing along the direction of travel at perigee.

5. α particle has charge of $2e$

Al nucleus has charge of $13e$

so $b = \frac{qQ}{4\pi\epsilon_0 mv^2}\cot\left(\frac{\Theta}{2}\right) = \frac{26e^2}{4\pi\epsilon_0 2E}\cot(45°)$

$\frac{1}{4\pi\epsilon_0} = 9\times10^9 \frac{Nm^2}{C^2}$, $e = 1.60\times10^{-19}C$

$E = 4000eV\left(\frac{1.60\times10^{-19}J}{1eV}\right) = 6.4\times10^{-16}J$

$\therefore b = \frac{26(1.6\times10^{-19}C)^2(9\times10^9\frac{Nm^2}{C^2})}{2(6.4\times10^{-16}J)}\frac{1}{\tan(45°)}$

$b = 4.68\times10^{-12}m$