Week 1: Introduction, probability, combinatorics

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Tuesday class

Syllabus

- Canvas
- GradeScope
- Piazza
- Lecture notes: `https://petrosyan.page/fall2020math3215`

Outcome of an experiment

We conduct an experiment or observe a phenomenon that has a resulting **outcome**; e.g.

- Coin toss (head or tail),
- Follow stock market fluctuations,
- Draw a card from a deck.

Denote $S = \{$the set of all outcomes of a given experiment$\}$.

For a single coin toss experiment,

$$S = \{H, T\}.$$

The set of outcomes for two coin tosses is

$$S = \{HH, HT, TH, TT\}.$$

Events

Often times, we are not interested in a specific outcome but rather if the outcome has certain qualities.

- Did the market go up, or down, i.e. the change is a positive or negative number?
- Is the pulled card a spade?

In other words, we are often interested in sets (collections) of outcomes. Such sets we conventionally call **events**.

- Outcome has both quality A and quality $B \Leftrightarrow A \cap B$.
  (e.g. "all red 4 wheel drive cars" is the intersection of all "red cars" and all "4 wheel drive cars")
- Outcome has quality A and not quality $B \Leftrightarrow A \setminus B$.
- Outcome does not have quality $A \Leftrightarrow A' = S \setminus A$.

Definition

1. Events $A_1, \ldots, A_n$ are called mutually exclusive if $A_i \cap A_j = \emptyset$ for every $i \neq j$ (i.e. no overlaps).
2. Events $A_1, \ldots, A_n$ are called exhaustive if

$$\bigcup_{i=1}^{n} A_i = S.$$

Exhaustive events do not need to be mutually exclusive.

Probability

Random = repetitive phenomenon, each time with a different outcome.

Toss a coin, one time it can be head, some other time can be tail.

Probability of an outcome= how "frequent" a specific outcome comes up.

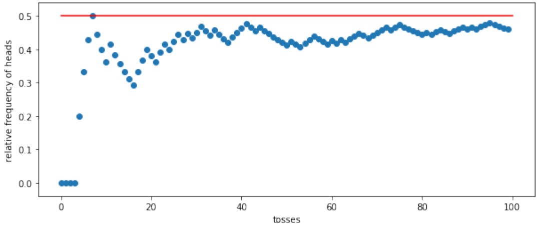
Similarly, we can define **probability of an event** as the proportion of experiment outcomes that terminate in A.

Experiment 1: fair coin toss

I experimented and recorded the outcome of hundred coin tosses (used a quarter):

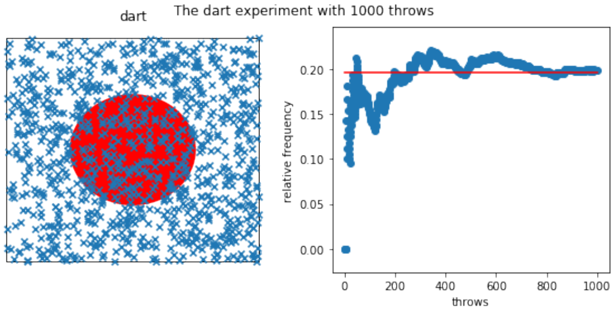
$$[T, T, T, H, H, H, H, T, T, T, H, T, T, T, T, T, H, H, H, T, T, H, H, T, H,$$
$$H, T, H, T, H, H, T, T, H, T, T, T, H, H, H, H, T, T, H, T, T, T, T, T, T,$$
$$H, T, T, H, H, H, T, T, T, H, T, H, T, H, H, H, T, T, H, H, H, H, T, H, H,$$
$$T, T, T, T, T, H, T, H, H, T, T, H, H, H, T, H, T, H, H, H, T, T, T, T, T].$$

$$\text{relative frequency of } H = \frac{\#\text{ of } H \text{ after i tosses}}{i}.$$



Experiment 2: aimless dart throw

I ran the experiment of aimlessly throwing a dart at a square board and computed the frequency at which it lands in the target disk. Here the outcomes are individual points in the square and the event is the disk.



I observed, that frequency approaches to the proportion the area.

Formal definition of probability

Definition

A probability P is any set function (i.e. a function that takes sets as input and outputs numbers) with properties

- for every event A, $P(A) \geq 0$,
- P(S)=1,
- for any (finite or infinite) collection of mutually exclusive events $\{A_1, A_2, \dots\}$,

$$P(A_1 \cup A_2 \cup \cdots) = P(A_1) + P(A_2) + \cdots .$$

The set of events with the associated probabilities is called a **probability space**.

Statistician vs Probability theorist

Statistician

Give me data and I will find the probabilities (e.g. how we did above). The heuristic methods (approach to problem solving that employs a practical method) I use are consistent with theory but I also rely on empirical evidence (verifiable by observation or experience rather than theory).

Probability theorist

Give me probabilities, doesn't matter how you found them, and I will prove mathematical theorems about them. Sometimes I will also prove theorems justifying the heuristic methods employed by statisticians.

Theorem

For any event A

1. $P(A') = 1 - P(A)$.
2. $P(\emptyset) = 0$.
3. $0 \leq P(A) \leq 1$.

Proof.

1. A and A' are mutually exclusive and $A \cup A' = S$ hence $P(A) + P(A') = P(S) = 1$.
2. The compliment of the empty set is S, hence, from 1., $P(\emptyset) = 1 - P(S) = 0$.
3. $P(A) \geq 0$ by definition and $P(A) = 1 - P(A') \leq 1$.

$\square$

Theorem

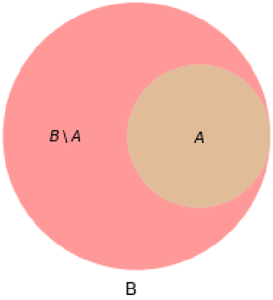
For any events A and B,

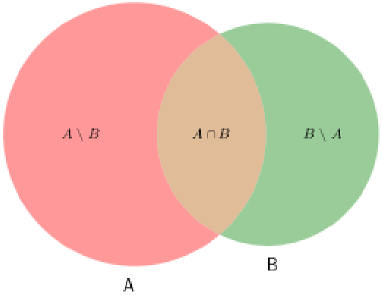
1. *If $A \subseteq B$ then $P(A) \leq P(B)$.*
2. *$P(A \cup B) = P(A) + P(B) - P(A \cap B)$ (**inclusion-exclusion principle**).*

Proof.

1. Let $A \subseteq B$. A and $B \setminus A$ are mutually exclusive and $A \cup (B \setminus A) = B$ (see the Venn diagram). Hence
$$P(B) = P(A) + P(B \setminus A) \geq P(A).$$
2. These sets are mutually exclusive
$$A_1 = A \setminus B, \;\; A_2 = B \setminus A, \;\; A_3 = A \cap B,$$
and $A \cup B = A_1 \cup A_2 \cup A_3$ (see the Venn diagram). Hence
$$\begin{aligned} P(A \cup B) &= P(A \setminus B) + P(B \setminus A) + P(A \cap B) \\ &= [P(A \setminus B) + P(A \cap B)] + [P(B \setminus A) + P(A \cap B)] - P(A \cap B) \\ &= P(A) + P(B) - P(A \cap B). \end{aligned}$$

$\square$


B \ A
A
B


A \ B
A ∩ B
B \ A
A
B

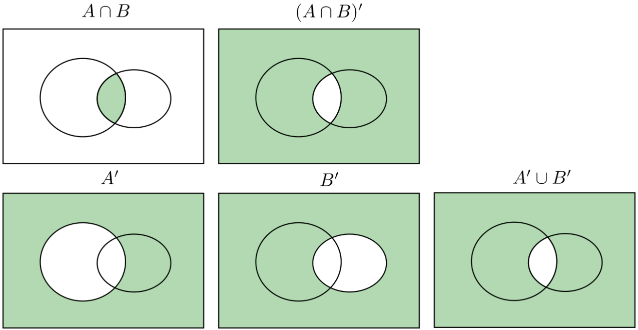
Thursday class

Use the Venn Diagrams to convince yourself that the following is true.

De Morgan's laws

For any two events A and B

1. $(A \cup B)' = A' \cap B'$,
2. $(A \cap B)' = A' \cup B'$.



Exercise 1

Problem (1.1-7 in the book)

Given that $P(A \cup B) = 0.76$ *and* $P(A \cup B') = 0.87$, *find* $P(A)$.

Solution

1. *Using De Morgan's law*

$$(A \cup B')' = A' \cap (B')' = A' \cap B = B \setminus A.$$

2. *Hence* $P(B \setminus A) = 1 - P(A \cup B') = 0.13$.
3. *On the other hand,* $B \setminus A$ *and* A *are mutually exclusive and* $(B \setminus A) \cup A = A \cup B$ *hence*

$$P(A \cup B) = P(A) + P(B \setminus A).$$

4. *Therefore,* $P(A) = 0.76 - 0.13 = 0.63$. **The answer is 0.63.**

Exercise 2

Problem

In a class of 100 students, 30 have been to France, 15 have been to Germany, and 7 have been to both countries. Assuming everyone in class is equally probable to be selected, what is the probability that a randomly selected student hasn't been to any of these countries.

Solution

1. $A =$ the set of students that have been to France: $P(A) = \frac{30}{100}$.
2. $B =$ the set of students that have been to Germany: $P(B) = \frac{15}{100}$.
3. $A \cap B =$ the set of students that have been to both countries: $P(A \cap B) = \frac{7}{100}$.
4. $(A \cup B)' =$ the set of students that have been to neither country.

$$P((A \cup B)') = 1 - P(A \cup B).$$

5. On the other hand

$$P(A \cup B) = P(A) + P(B) - P(A \cap B) = \frac{30}{100} + \frac{15}{100} - \frac{7}{100} = \frac{38}{100}.$$

6. **Answer:** $\frac{62}{100}$.

Equiprobability space

Let $S = \{s_1, \ldots, s_n\}$ and let $p_i = P(s_i)$. Then, for any $A = \{s_{i_1}, \ldots, s_{i_k}\}$,

$$P(A) = p_{i_1} + \cdots + p_{i_k}.$$

Definition

The probability space is called equiprobability space if $p_1 = p_2 = \cdots = p_n$.

- In equiprobability space $p_i = \frac{1}{n}$ for every $i = 1 \ldots, n$.
- If event A is of size m, then $P(A) = \frac{m}{n}$.

Example

1. The coin toss experiment with a fair coin is an equiprobability space with $p_1 = p_2 = 1/2$.
2. In 2 coin toss experiment $S = \{HH, HT, TH, TT\}$ therefor in the equiprobability space on S, each outcome has probability.

Question

1. What is the probability of each outcome in the equiprobability space on the set of m coin tosses?
2. What is the probability of the event that the number of heads in m consecutive tosses is k for some given k?

We need *Combinatorics!* .

Combinatorics – a branch of mathematics primarily concerned with counting combinations and arrangements.

Idea: use simpler observations to make complicated computations.

Multiplication principle

If experiment E_1 has n_1 outcomes and experiment E_2 has n_2 outcomes, then the number of joint outcomes will be $n_1 n_2$.

Easily extended to m experiments with n_i outcomes $(i = 1 \ldots, m)$: the joint experiment has $n_1 \ldots n_m$ outcomes.

Example

Number of outcomes for m coin tosses is 2^m.

Definition

A *permutation* of a set is any ordered arrangement of its members.

Example

The sequences $(1, 2, 3)$ and $(2, 1, 3)$ each are permutations of the set $\{1, 2, 3\}$.

Theorem

The number of permutations of a set with n elements is $n! = n(n-1)\cdots 1$.

Proof.

We will use the multiplication principle.

1. The first element of the permutation can be selected in n different ways.
2. After the first element is fixed, the second element can be chosen in $n-1$ ways (since we have to exclude the first element) so the first two can be selected in $n(n-1)$ ways.
3. After the first two are fixed, the third one can be selected in $n-2$ ways, and so on until the last element which is now unique as there is only one left to choose from.

$\square$

$P(n, r)$ – number of permutations of n objects taken r at a time.
Similar to number of permutations, it can be seen that

$$P(n, r) = n(n-1)\cdots(n-r+1) = \frac{n!}{(n-r)!}.$$

Example

Line of succession for US presidency consists of 18 people. What is the number of all such combinations out of 100 mln (made up number) eligible citizens? Answer is

$$\frac{100,000,000!}{99,999,982!}$$

which is a large number.

Samples

Definition

A **sample of size** r from a given set S is any selection of elements from the set that has r elements. We distinguish between

1. unique sample and sample with repetitions (the same element is present more than once),
2. ordered sample and sample without order.

- Unique sample is also called **sample without replacement**.
- Sample with repetitions is called **sample with replacement.**

- Use parentheses for ordered samples: $(1, 2, 3)$ and $(1, 3, 2)$ are different as ordered samples.
- Use curly brackets to denote samples without order: $\{1, 2, 3\}$ and $\{1, 3, 2\}$ are the same as samples without order.

The number of all ordered samples of size r sampled with replacement is equal to n^r.

The number of all ordered samples of size r sampled without replacement is equal to $P(n, r)$.

Theorem

The number of all samples of size r without order and without replacement is denoted by $C(n,r)$ or $\binom{n}{r}$ ("n choose r"), and is equal to the following number

$$\binom{n}{r} = \frac{n!}{r!(n-r)!}.$$

Proof.

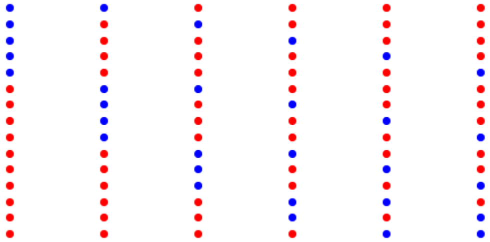
1. Each sample without order can be ordered in $r!$ ways according to the Theorem about permutations.
2. From multiplication principle, $P(n,r) = \binom{n}{r} r!$.
3. Hence $\binom{n}{r} = \frac{P(n,r)}{r!} = \frac{n!}{r!(n-r)!}$.

$\square$

Arrangement of different colored balls

In how many different ways can we arrange r red balls and $n-r$ blue balls? **Answer is:** $\binom{n}{r}$.

All arrangements of 4 red and 2 blue balls



There is one-to-one correspondence between the set of all arrangement of r red and $n-r$ blue balls and the set of all samples of size from $\{1,\ldots,n\}$ without order. To each arrangement put into correspondence the indices where a red ball is placed.

For example, the arrangement (R, B, R) will correspond to $\{1,3\}$.

Question

How many words of length n can we write with r letters a and $n-r$ letters b? **Answer is:** $\binom{n}{r}$**.**

Theorem (Binomial theorem)

For any two numbers a and b,

$$(a+b)^n = \sum_{r=0}^{n} \binom{n}{r} a^r b^{n-r}.$$

Proof.

Notice that in $(a+b)^n$, if we maintain the order of terms after expanding the product, then we will have all the words of length n written in a and b. For example,

$$(a+b)(a+b) = aa + ab + ba + bb.$$

If we group them according to the number of a-s and b-s, we prove the desired result. □

Exercise 3

Problem

What is the probability of getting at least one 6 with 4 throws of a dice?
(it is a fair dice so we are looking at the equiprobability space)

Solution

1. Let A be the event that there is at least one 6.
2. Notice, A' will be the event that there is no 6 in 4 dices (A' is easier to compute).
3. Size of A' = number of ordered samples with replacement of size 4 from $\{1, \dots, 5\} = 5^4$.
4. Size of outcome set S = number of ordered samples with replacement of size 4 from $\{1, \dots, 6\}= 6^4$.
5. Therefor,
$$P(A') = \left(\frac{5}{6}\right)^4 \approx 0.48.$$
6. Finally,
$$P(A) = 1 - P(A') \approx 0.52.$$

Exercise 4

Problem (I.2-6 in the book)

Suppose that Novak Djokovic and Roger Federer are playing a tennis match in which the first player to win three sets wins the match. Using **D** *and* **F** *for the winning player of a set, in how many ways could this tennis match end?*

Solution

1. Let as compute the number of games where **D** won. Then we will multiply by 2.
2. They need to play only 5 games for one of them to win!
3. Case 1: **D** won after 3 sets. That means he won all first 3 sets and the game stopped after the 3rd set.
4. Case 2: **D** won after 4 sets. That means he won only two out of the first 3 sets and he won the 4th set, and the game stopped. The number of such games is $\binom{3}{2}$
5. Case 3: **D** won after 5 games. That means he won only two out of the first 4 sets and he won the 5th set. The number of such games is $\binom{4}{2}$.
6. Thus, the total number of winning sets for D is

$$1 + \binom{3}{2} + \binom{4}{2} = 1 + 3 + 6 = 10.$$

7. The total number of sets for both players will be $10 \times 2 = 20$.