Foucoult's Pendulum

An ordinary pendulum free to swing in any direction. Symmetric so period of oscillation is equal. Famous system for demonstrating Earth's rotation. Typically very long. (Tellus Science Museum in Cartersville) If amplitude is small compared to the length then this system is just a 2D pendulum w. e.o.m.

$$\ddot{x} = -\frac{g}{\ell}x + 2\omega\dot{y}\cos\theta \quad ; \quad \ddot{y} = -\frac{g}{\ell}y - 2\omega\dot{x}\cos\theta$$

since $\dot{z} \simeq 0$, and ignore the small vertical component of the Coriolis force

In vector notation $\ddot{\vec{r}} = -\frac{g}{\ell}\vec{r} - 2\omega\cos\theta(\hat{k}\times\dot{\vec{r}})$

Easiest to picuture at pole ($\theta = 0$). An inertial observer sees the pendulum oscillating in a fixed plane while Earth rotates underneath it. An observer on the Earth sees the oscillation plane rotate w/ ang. velocity $-\omega$. At any other latitude, ω is replaced w/ its vertical component $\omega\cos\theta\hat{k}$. Thus, the oscillation plane will rotate with freq. $-\Omega = -\omega\cos\theta$ around the vertical.

To solve the eqns of motion write $\xi = x + iy$, so $-i\dot{\xi} = \dot{y} - i\dot{x}$

$$\therefore \quad \ddot{x} + i\ddot{y} = -\frac{g}{\ell}x + 2\omega\dot{y}\cos\theta - i\frac{g}{\ell}y - 2\omega\dot{x}\cos\theta\, i$$

$$\ddot{\xi} = -\frac{g}{\ell}(x+iy) + 2\omega\cos\theta(\dot{y} - i\dot{x})$$

let $\omega_0^2 = \frac{g}{\ell}$; recall $\Omega = \omega\cos\theta$

$$\therefore \quad \ddot{\xi} = -\omega_0^2\xi - i\Omega 2\dot{\xi}$$

or $\ddot{\xi} + 2\Omega\dot{\xi} i + \omega_0^2 \xi = 0$

Consider solutions $\xi = Ae^{pt}$

$$\cancel{A} p^2 \cancel{e^{pt}} + 2\Omega \cancel{A} p \cancel{e^{pt}} i + \omega_0^2 \cancel{A} \cancel{e^{pt}} = 0$$

$$p^2 + 2i\Omega p + \omega_0^2 = 0$$

$$\rightarrow p = -i\Omega \pm i\omega_1 \quad \text{where } \omega_1^2 = \omega_0^2 + \Omega^2$$

$$\therefore \xi = Ae^{-i(\Omega-\omega_1)t} + Be^{-i(\Omega+\omega_1)t}$$

Set A & B from initial conditions

If pendulum is started at $t=0$ from $(x,y) = (a,0)$ with speed $(0, -a\Omega)$ $[\xi = x+iy]$

$$\xi_0 = A + B = a$$

$$\dot{\xi}_0 = -\cancel{i}a\Omega = -\cancel{i}(\Omega-\omega_1)A - \cancel{i}(\Omega+\omega_1)B$$

$$-a\Omega = -(A+B)\Omega + (A-B)\omega_1$$

$$-a\Omega = -a\Omega + (A-B)\omega_1$$

$$\therefore A - B = 0$$

$$A = B$$

$$\rightarrow A = \frac{a}{2}$$

$$\therefore \xi = \frac{a}{2}e^{-i(\Omega-\omega_1)t} + \frac{a}{2}e^{-i(\Omega+\omega_1t}$$

$$= \frac{a}{2}e^{-i\Omega t}e^{i\omega_1 t} + \frac{a}{2}e^{-i\Omega t}e^{-i\omega_1 t} = \frac{a}{2}e^{-i\Omega t}\underbrace{\left(e^{i\omega_1 t} + e^{-i\omega_1 t}\right)}_{2\cos\omega_1 t}$$

$$\therefore \xi = ae^{-i\Omega t}\cos\omega_1 t$$

In terms of x & y

$$x = a\cos\Omega t\cos\omega_1 t \quad , \quad y = -a\sin\Omega t\cos\omega_1 t$$

Since $\Omega \ll \omega_0$, $\omega_0 \simeq \omega_1$ & the solution represents an oscillation w/ amplitude a in a plane rotating w/ ang. vel. $-\Omega$. The period of rotation $\frac{2\pi}{\Omega} = \frac{2\pi}{\omega\cos\theta}$. 24 hrs @ $\theta = 0°$, 34 hrs @ $\theta = 45°$

See the text for a discussion on Coriolis forces affect rotation of air currents, trade winds & storms.

Larmor Effect

Switching to a rotating frame can sometimes simplify the analysis of problems of various types for example, consider a particle w/ charge q, orbiting around a fixed point charge $-q'$ in the prescence of a mag. field $\vec{B}$.

Inertial frame e.o.m

$$\underbrace{m\frac{d^2\vec{r}}{dt^2} = -\frac{K}{r^2}\hat{r}}_{\text{What would the motion be w/ just these two terms? ellipse}} + q\frac{d\vec{r}}{dt}\times\vec{B} \qquad \text{where } K = \frac{qq'}{4\pi\epsilon_0}$$