Larmor Effect

Switching to a rotating frame can sometimes simplify the analysis of problems of various types. For example, consider a particle w/ charge q orbiting around a fixed point charge $-q'$ in the prescence of a mag. field $\vec{B}$.

Inertial frame e.o.m

$$\underbrace{m\frac{d^2\vec{r}}{dt^2} = -\frac{K}{r^2}\hat{r}} + q\frac{d\vec{r}}{dt}\times\vec{B} \qquad \text{where } K = \frac{qq'}{4\pi\epsilon_0}$$

What would the motion be w/ just these two terms? ellipse

Switch to a rotating frame:

$$m\left(\ddot{\vec{r}} + 2\vec{\omega}\times\dot{\vec{r}} + \vec{\omega}\times(\vec{\omega}\times\vec{r})\right) = -\frac{K}{r^2}\hat{r} + q\left(\dot{\vec{r}} + \vec{\omega}\times\vec{r}\right)\times\vec{B}$$

$$\ddot{\vec{r}} + 2\vec{\omega}\times\dot{\vec{r}} + \vec{\omega}\times(\vec{\omega}\times\vec{r}) = -\frac{K}{mr^2}\hat{r} + \frac{q}{m}\dot{\vec{r}}\times\vec{B} + \frac{q}{m}(\vec{\omega}\times\vec{r})\times\vec{B}$$

Choose $\vec{\omega} = -\frac{q}{2m}\vec{B}$ (ie, half the gyrofreq.)

$$\therefore\ \ddot{\vec{r}} - \cancel{\frac{q}{m}\vec{B}\times\dot{\vec{r}}} + \left(\frac{q^2}{4m^2}\right)\vec{B}\times(\vec{B}\times\vec{r}) = -\frac{K\hat{r}}{mr^2} + \frac{q}{m}\dot{\vec{r}}\times\vec{B} - \frac{q^2}{2m^2}(\vec{B}\times\vec{r})\times\vec{B}$$

$$= -\frac{K\hat{r}}{mr^2} - \cancel{\frac{q}{m}\vec{B}\times\dot{\vec{r}}} + \frac{q^2}{2m^2}\vec{B}\times(\vec{B}\times\vec{r})$$

$$\therefore\ \ddot{\vec{r}} = -\frac{K\hat{r}}{mr^2} + \frac{q^2}{4m^2}\vec{B}\times(\vec{B}\times\vec{r}) = -\frac{K\hat{r}}{mr^2} + \left(\frac{q}{2m}\right)^2\underline{\vec{B}\times(\vec{B}\times\vec{r})}$$

Assume $|\vec{B}|$ is small enough that the 2nd term is negligible compared to the first: ie. $\left(\frac{qB}{2m}\right)^2 \ll \left(\frac{K}{mr^3}\right) = \frac{qq'}{4\pi\epsilon_0 mr^3}$

When this is satisfied, $\ddot{\vec{r}} \simeq \dfrac{-K\hat{r}}{mr^2}$

So, the orbit in the rotating frame is an ellipse.

→ In an inertial frame, the ellipse precesses w/ ang. vel. ω.

The rotation of electron 'orbits' in the prescence of a $\vec{B}$-field is called the Larmor Effect, $\omega = \omega_L = \frac{qB}{2m}$ is the Larmor freq.

Experimentally, one observes spectral lines from an atom splitting in a $\vec{B}$-field (called the Zeeman Effect), as there are diff. ang. mom. states available in the atom. Really a QM effect.

Potential Theory

In astronomy & astrophysics it is often important to add the gravitational contributions from multiple objects in order to compute orbits of particles. Similarly, real objects are not always spherical & their grav. field must be computed through other means. In all these cases, it is often easier to compute the gravitational potential and then find forces, etc. Similar approach is used in electromagnetism. We'll return to convention $\dot{\vec{r}} = \frac{dr}{dt}$.

First, recall the Divergence Theorem

If V is a volume in space bounded by the closed surface S then for any vector field $\vec{A}$

$$\oint_{\substack{\uparrow \\ \text{evaluated at the surface}}} \vec{A}\cdot d\vec{S} = \int_V \vec{\nabla}\cdot\vec{A}\, dV$$

where $dV = dx\,dy\,dz$ & the +ve side of S is taken to be outside

Recall the gravitational potential energy of a mass m moving in the field of a fixed mass m' at $\vec{r}'$ is $-\frac{Gmm'}{|\vec{r}-\vec{r}'|}$. If instead of one mass (m') m is moving in a region with j masses, each w/ a different mass and location. Then the gravitational potential energy at $\vec{r}$ will be

$$V(\vec{r}) = -\sum_j \frac{Gmm_j}{|\vec{r}-\vec{r}_j|}$$

If we divide out the mass of our 'test' particle then we get a quantity that depends only on the mass dist'n of our system.

Call this quantity the gravitational potential (the potential energy per unit mass)

$$\Phi(\vec{r}) = \frac{V(\vec{r})}{m} = -\sum_j \frac{Gm_j}{|\vec{r}-\vec{r}_j|} \qquad (\Phi \text{ is always negative})$$

As the grav. force can be found by $\vec{F} = -\vec{\nabla} V$

$$\rightarrow \vec{F} = -m\vec{\nabla}\Phi$$

$\therefore$ the acc'n at $\vec{r}$ is $\cancel{m}\ddot{\vec{r}} = -\cancel{m}\vec{\nabla}\Phi$

$$\boxed{\ddot{\vec{r}} = -\vec{\nabla}\Phi}$$

Since $-\vec{\nabla}\Phi(\vec{r})$ is the grav. acc'n at $\vec{r}$ it is called $\vec{g} = -\vec{\nabla}\Phi(\vec{r})$ the grav. field

The potential is useful b/c it is a scalar field that is easier to visualize than the vector grav. field. Also, in many situations the easiest way to obtain $\vec{g}$ is first to calculate the potential and then take the gradient.