The eccentricity can be written in terms of speed at pericenter.

We had $r_0 = \frac{l}{1+e} \Rightarrow r_0(1+e) = l \Rightarrow r_0 + er_0 = l \Rightarrow r_0 e = l - r_0$

$$e = \frac{l}{r_0} - 1 = \frac{J^2}{mKr_0} - 1$$

Let v_0 = speed of particle at pericenter $(\theta = 0)$

$$V_0 = r_0 \dot{\theta}_0 \Rightarrow r_0 V_0 = r_0^2 \dot{\theta}_0 = \frac{J}{m}$$

$$\Rightarrow J^2 = m^2 r_0^2 V_0^2$$

$$\therefore e = \frac{m^{\cancel{2}} r_0^{\cancel{2}} V_0^2}{\cancel{m} K \cancel{r_0}} = \frac{m r_0 V_0^2}{K} - 1$$

From last time, we saw $V_c = \sqrt{\frac{K}{mr_0}} \Rightarrow V_c^2 = \frac{K}{mr_0}$

$$\therefore e = \left(\frac{V_0}{V_c}\right)^2 - 1$$

So, the eq'n of the orbit can be written

$$r = r_0 \left[\frac{(V_0/V_c)^2}{1 + \left[(V_0/V_c)^2 - 1\right] \cos\theta} \right]$$

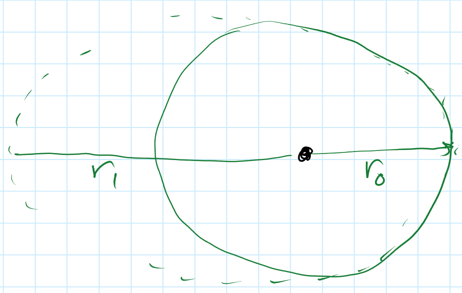
r_1 is found by setting $\theta = \pi$ in this equation

$$r_1 = \frac{r_0 (V_0/V_c)^2}{2 - (V_0/V_c)^2}$$

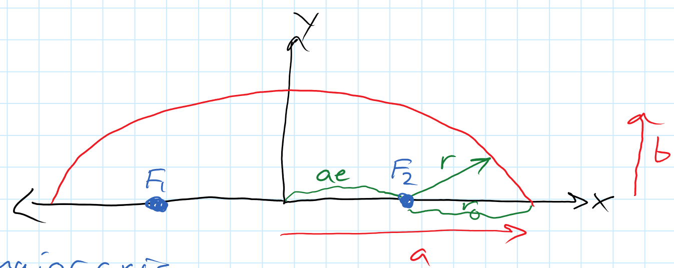
Ex: A rocket satellite is going around the Earth in a circular orbit of radius r_0. A sudden blast of the motor increases the speed by 15%. Find the eq'n of the new orbit & compute the apogee distance.

Let V_c be the circular speed & V_0 be the new initial speed, then $\frac{V_0}{V_c} = 1.15$ & $r = \frac{r_0(1.3225)}{(1+0.3225\cos\theta)}$

Set $\theta = \pi$, gives $r_1 = 1.95\, r_0$



Other Properties of Ellipses



Eqn's: $r(\theta) = \frac{r_0(1+e)}{(1+e\cos\theta)}$

$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, $a > b$, a = semi-major axis
b = semi-minor axis

- F_1, F_2 are the focii of the ellipse
- $r(\theta)$ measured relative to F_2 when $e<1$
- Can show $b = a\sqrt{1-e^2}$. Distance from origin to $F_{1,2} = ae$
- $a = \frac{r_0}{1-e}$
- Area $= \pi ab$

Relationship B/w Orbital Size & Orbital Period

Recall that the orbit sweeps out area at a rate

$$\frac{dA}{dt} = \frac{J}{2m}$$

So, integrate this over a period τ

$$A = \int_0^\tau \frac{dA}{dt}\,dt = \frac{J\tau}{2m}$$

$$\tau = \frac{2mA}{J}$$

The area of an ellipse is $A = \pi ab = \pi a^2\sqrt{1-e^2}$

$$\therefore \quad \tau = \frac{2m}{J}\pi a^2\sqrt{1-e^2}$$

or

$$\tau^2 = \frac{4m^2\pi^2a^4(1-e^2)}{J^2}$$

But $2a = r_0 + r_1 = r_0 + r_0\left(\frac{1+e}{1-e}\right) = r_0\left(1+\frac{1+e}{1-e}\right) = r_0\left(\frac{1-e+1+e}{1-e}\right) = \frac{r_0 2}{1-e}$

$$2a = \frac{J^2}{mK}\frac{1}{(1+e)}\left(\frac{2}{1-e}\right) = \frac{2J^2}{mK}\frac{1}{(1-e)(1+e)} = \frac{2J^2}{mK(1-e^2)}$$

$$\rightarrow (1-e^2) = \frac{J^2}{mKa}$$

Substitute:

$$\tau^2 = \frac{4m^{\cancel{2}}\pi^2a^{\cancel{4}3}}{\cancel{J^2}}\cdot\frac{\cancel{J^2}}{\cancel{m}K\cancel{a}} = \left(\frac{4m\pi^2}{K}\right)a^3$$

i.e. $\tau^2 \propto a^3$

For gravity, $K = GMm$; $\tau^2 = \left(\frac{4\pi^2}{GM}\right)a^3$ Kepler's 3rd Law of Planetary Motion

Ex: Work out constant if a is measured in AU (1.5×10^8 km) & τ is in yrs.

Energy Conservation for Central Conservative Forces

Energy Conservation for Central Conservative Forces

For motion in a central, conservative force know $\vec{J} = m\vec{r} \times \dot{\vec{r}}$ is constant, but also know that $V(r)$, the PE, can be defined $(\vec{F} = -\vec{\nabla} V)$, so there is an energy conservation eqn that can be written as $\frac{1}{2} m \dot{\vec{r}}^2 + V(r) = E = \text{const.}$