Assignment 4 Solutions

20

1\. Since $F = -\frac{dV}{dx}$, we have $\frac{dV}{dx} = Kx - \frac{c}{x}$, so

5

$$V = \int \left(Kx - \frac{c}{x}\right) dx = \underline{\frac{1}{2}Kx^2 - c\ln x + V_0}$$ where V_0 is a constant

The position of equilibrium is found when $F = 0$

$$\therefore \quad 0 = -Kx + \frac{c}{x}$$

$$0 = -Kx^2 + c \rightarrow x^2 = \frac{c}{K} \rightarrow \underline{x = \sqrt{\frac{c}{K}}}$$ (note: positive root only)

2

The angular frequency of oscillations about the equilibrium is

$$\omega_0 = \sqrt{\frac{k}{m}} \quad \text{where} \quad k = \frac{d^2V}{dx^2}(\text{equil. pt.})$$

We know $\frac{dV}{dx} = Kx - \frac{c}{x}$, so $\frac{d^2V}{dx^2} = K + \frac{c}{x^2}$

At the equil. pt. $\frac{d^2V}{dx^2} = K + \frac{c}{\left(\sqrt{\frac{c}{K}}\right)^2} = K + K = 2K$

$$\therefore \quad \underline{\omega_0 = \sqrt{\frac{2K}{m}}}$$

3

2\. Equate the total energy of the oscillator at the 2 positions

$$\frac{1}{2}m\dot{x}_1^2 + \frac{1}{2}Kx_1^2 = \frac{1}{2}m\dot{x}_2^2 + \frac{1}{2}Kx_2^2$$

$$\rightarrow K\left(x_1^2 - x_2^2\right) = m\left(\dot{x}_2^2 - \dot{x}_1^2\right)$$

$$\therefore\ \omega_0 = \sqrt{\frac{K}{m}} = \left(\frac{\dot{x}_2^2 - \dot{x}_1^2}{x_1^2 - x_2^2}\right)^{1/2}$$

The total energy of the oscillator is simply written as the potential energy at one of the turning points, which is the amplitude of the oscillator. $E = \frac{1}{2}Ka^2$

$$\therefore\ \tfrac{1}{2}Ka^2 = \tfrac{1}{2}m\dot{x}_1^2 + \tfrac{1}{2}Kx_1^2$$

$$a^2 = \frac{m}{K}\dot{x}_1^2 + x_1^2 = \left(\frac{x_1^2 - x_2^2}{\dot{x}_2^2 - \dot{x}_1^2}\right)\dot{x}_1^2 + x_1^2, \text{ from above}$$

Can simplify, $a^2 = \dfrac{x_1^2\dot{x}_1^2 - x_2^2\dot{x}_1^2}{\dot{x}_2^2 - \dot{x}_1^2} + x_1^2 = \dfrac{\cancel{x_1^2\dot{x}_1^2} - x_2^2\dot{x}_1^2 + x_1^2\dot{x}_2^2 \cancel{- x_1^2\dot{x}_1^2}}{\dot{x}_2^2 - \dot{x}_1^2}$

$$a^2 = \frac{x_1^2\dot{x}_2^2 - x_2^2\dot{x}_1^2}{\dot{x}_2^2 - \dot{x}_1^2} \quad \text{or} \quad a = \left(\frac{x_1^2\dot{x}_2^2 - x_2^2\dot{x}_1^2}{\dot{x}_2^2 - \dot{x}_1^2}\right)^{1/2}$$

3. The space averages of the kinetic and potential energies are

7

$$\bar{T} = \frac{1}{a}\int_0^a \frac{1}{2}m\dot{x}^2\,dx \quad \text{and} \quad \bar{V} = \frac{1}{a}\int_0^a \frac{1}{2}Kx^2\,dx$$

$$\bar{V} = \frac{m\omega_0^2}{2a}\int_0^a x^2\,dx$$

$$\bar{V} = \frac{m\omega_0^2 a^2}{6}$$ 2

For $\bar{T}$, need an expression for $\dot{x}$ in terms of x.

Know, $x = a\cos(\omega_0 t - \theta)$. Set $\theta = 0$, so $x = a\cos\omega_0 t$

$\therefore\ \dot{x} = -a\omega_0 \sin\omega_0 t$

Know, $x = a\cos(\omega_0 t - \theta)$. Set $\theta = 0$,

$$\therefore \quad \dot{x} = -a\omega_0 \sin\omega_0 t$$

$$\text{or } \dot{x}^2 = a^2\omega_0^2\sin^2\omega_0 t = a^2\omega_0^2(1-\cos^2\omega_0 t) = \omega_0^2(a^2 - x^2)$$

Can use this in $\overline{T}$ integral.

4

$$\therefore \quad \overline{T} = \frac{m\omega_0^2}{2a}\int_0^a (a^2 - x^2)\,dx = \frac{m\omega_0^2}{2a}\left(a^3 - \frac{a^3}{3}\right) = \frac{2m\omega_0^2 a^2}{6}$$

We see that $\overline{T} = 2\overline{V}$ 1

4. The displacement of a damped harmonic oscillator is

8

$$x = e^{-\gamma t} a\cos(\omega_d t - \theta)$$

$$\frac{dx}{dt} = -e^{-\gamma t} a\omega_d \sin(\omega_d t - \theta) - \gamma e^{-\gamma t} a\cos(\omega_d t - \theta)$$

maxima occur at $\frac{dx}{dt} = 0 = \omega_d \sin(\omega_d t - \theta) + \gamma\cos(\omega_d t - \theta)$

$$\rightarrow \tan(\omega_d t - \theta) = -\frac{\gamma}{\omega_d} \quad 3$$

Thus the condition of relative maximum occurs every time that t increases by $\frac{2\pi}{\omega_d}$: $\quad t_{i+1} = t_i + \frac{2\pi}{\omega_d}$

For the i^{th} maximum : $x_i = e^{-\gamma t_i} a\cos(\omega_d t_i - \theta)$

$i^{th}+1$ max. : $x_{i+1} = e^{-\gamma t_{i+1}} a\cos(\omega_0 t_{i+1} - \theta) = e^{-\gamma\frac{2\pi}{\omega_d}} x_i$

$$\therefore \frac{x_{i+1}}{x_i} = e^{-\gamma\frac{2\pi}{\omega_d}} = e^{-\gamma\tau_d}$$ where τ_d is the period of the damped oscillator

5

5. From the previous problem we know that the amplitude drops by $e^{-\gamma\tau_d}$ each period. In this case we are told that after n periods, the amplitude has dropped by $1/e$

$$\therefore \left(e^{-\gamma\tau_d}\right)^n = \frac{1}{e} = e^{-1}$$

$$\text{or } \gamma\tau_d n = 1 \rightarrow \gamma = \frac{1}{\tau_d n} = \frac{\omega_d}{2\pi n}$$

Now, $\omega_d = (\omega_0^2 - \gamma^2)^{1/2}$, so $\omega_0 = (\omega_d^2 + \gamma^2)^{1/2} = \omega_d\left(1 + \frac{1}{4\pi^2 n^2}\right)^{1/2}$

$$\therefore \frac{\tau_d}{\tau_0} = \frac{2\pi/\omega_d}{2\pi/\omega_0} = \frac{\omega_0}{\omega_d} = \left(1 + \frac{1}{4\pi^2 n^2}\right)^{1/2}$$

For large n, the binomial theorem says $\left(1 + \frac{1}{4\pi^2 n^2}\right)^{1/2} \simeq 1 + \frac{1}{8\pi^2 n^2}$

$$\text{so } \frac{\tau_d}{\tau_0} \simeq 1 + \frac{1}{8\pi^2 n^2}$$