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Assignment 3 Solutions

1. Equation of motion

$$F(v) = -cv^{3/2} = m\frac{dv}{dt} = m\frac{dv}{dx}\frac{dx}{dt} = mv\frac{dv}{dx}$$

$$\therefore -\frac{c}{m} v^{1/2} = \frac{dv}{dx}$$

$$\rightarrow -\frac{c}{m}\int_0^x dx = \int_{v_0}^0 \frac{dv}{v^{1/2}}$$

$$-\frac{c}{m}x = 2v^{1/2}\Big]_{v_0}^0 = 2(0 - v_0^{1/2}) = -2v_0^{1/2}$$

$$\therefore \underline{x = \frac{2mv_0^{1/2}}{c}}$$

2. Equation of motion

$$F = -Kx^{-2} = m\frac{dv}{dt} = m\frac{dv}{dx}\frac{dx}{dt} = mv\frac{dv}{dx}$$

$$\therefore -\frac{K}{m}\int_b^x \frac{dx}{x^2} = \int_0^v v\,dv$$

$$-\frac{K}{m}\left(-x^{-1}\right]_b^x = \frac{K}{m}\left(\frac{1}{x} - \frac{1}{b}\right) = \frac{1}{2}v^2$$

$$\therefore v = \left[\frac{2K}{m}\left(\frac{1}{x} - \frac{1}{b}\right)\right]^{1/2}$$

So, $$\frac{dx}{dt} = \left[\frac{2K}{m}\left(\frac{1}{x} - \frac{1}{b}\right)\right]^{1/2} = \left[\frac{2K}{mb}\left(\frac{b-x}{x}\right)\right]^{1/2}$$

$$\frac{dx}{dt} \quad \left[m(x \quad b) \right] \quad \left[mb(\quad x \quad) \right]$$

$$\rightarrow \int_0^t dt = \int_b^0 \left[\frac{mb}{2K}\left(\frac{x}{b-x}\right) \right]^{1/2} dx = \left(\frac{mb^3}{2K}\right)^{1/2} \int_1^0 \left(\frac{\frac{x}{b}}{1-\frac{x}{b}} \right)^{1/2} d\left(\frac{x}{b}\right)$$

Since $x \le b$, let $\frac{x}{b} = \sin^2\theta$, then $d\left(\frac{x}{b}\right) = 2\sin\theta\cos\theta\, d\theta$

$$\therefore \quad t = \left(\frac{mb^3}{2K}\right)^{1/2} \int_{-\pi/2}^{0} \left(\frac{\sin^2\theta}{1-\sin^2\theta} \right)^{1/2} 2\sin\theta\cos\theta\, d\theta$$

$$= \left(\frac{2mb^3}{K}\right)^{1/2} \int_{-\pi/2}^{0} \sin^2\theta\, d\theta = \underline{\left(\frac{mb^3}{8K}\right)^{1/2} \pi}$$

3. Equation of motion:

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$$F = Kxv = m\frac{dv}{dt}$$

$$Kx = \frac{m}{v}\frac{dv}{dt} = \frac{m}{v}\frac{dv}{dx}\frac{dx}{dt} = \frac{m}{\cancel{v}}\frac{dv}{dx}\cancel{v}$$

$$\therefore \quad \int Kx\,dx = m\int dv$$

$\frac{Kx^2}{2} = mv + C$ where C is a constant 2

at $x = 0,\ t = 0,\ v = V_0$ so $C = -mV_0$

$$\therefore \quad \frac{Kx^2}{2} + mV_0 = m\frac{dx}{dt}$$

So, $\int dt = \int \frac{dx}{\left(\frac{K}{2m}\right)x^2 + V_0} = \int \frac{dx}{A^2 + B^2 x^2}$ where $A^2 = V_0$;

So, $\int dt = \int \frac{dx}{\left(\frac{K}{2m}\right)x^2 + V_0} = \int \frac{dx}{A^2 + B^2 x^2}$ where $A^2 = V_0$; $B^2 = \frac{K}{2m}$

Tables of integration tell us that

$$t = \frac{1}{AB}\tan^{-1}\left(\frac{Bx}{A}\right)$$

Solving for x ; subbing back in for A and B,

$$\underline{x = \left(\frac{2mV_0}{K}\right)^{1/2} \tan\left[\left(\frac{KV_0}{2m}\right)^{1/2} t\right]}$$ 4

4. The force is $F = -\frac{dV}{dx} = \begin{cases} +Kx & \text{for } |x| < a \\ 0 & \text{for } |x| \geq a \end{cases}$ 1

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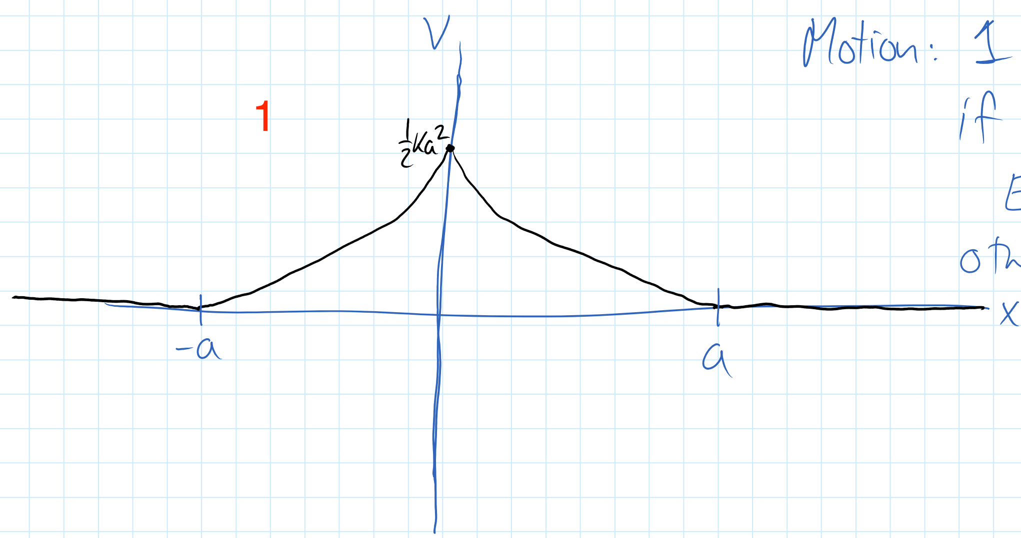
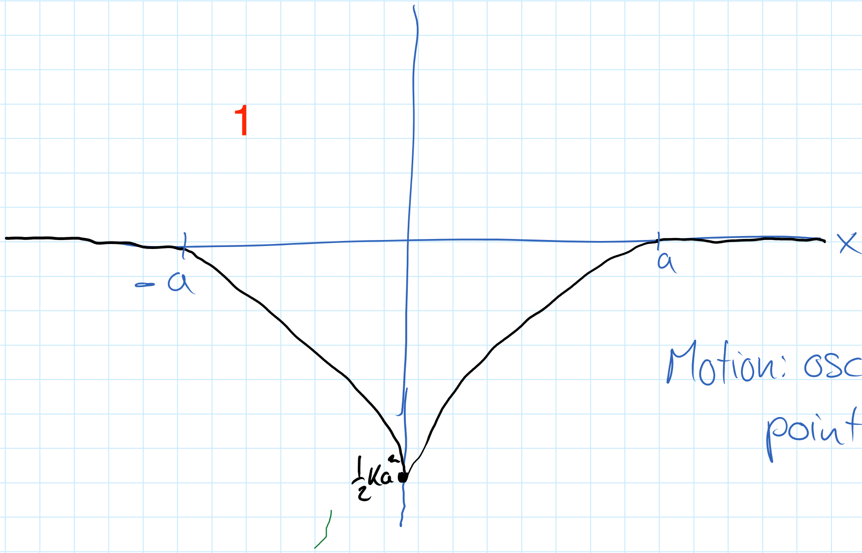
Sketch of V for K>0



Motion: 1 turning point if $K>0$; $E < \frac{1}{2}Ka^2$ 2

otherwise, no turning points

Sketch for $K<0$



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Motion: oscillation b/w 2 turning points if $\kappa<0$ & $E<0$

5. a) Equation of motion $F = -\alpha e^{\beta v} = m\frac{dv}{dt}$

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$$\to \int e^{-\beta v}\,dv = -\frac{\alpha}{m}\int dt$$

$$-\frac{1}{\beta}e^{-\beta v} = -\frac{\alpha}{m}t + C$$

We know $v = v_0$ at $t=0$, so

$$-\frac{1}{\beta}e^{-\beta v_0} = C$$

$$\to -\frac{1}{\beta}\left(e^{-\beta v} - e^{-\beta v_0}\right) = -\frac{\alpha}{m}t$$

Solving for v gives

$$\underline{v(t) = -\frac{1}{\beta}\ln\left[\frac{\alpha\beta t}{m} + e^{-\beta v_0}\right]}$$ 2

b) Solve for t when $v=0$

$$\frac{\alpha\beta t}{m} + e^{-\beta v_0} = 1$$

$$\to \quad t = \frac{m}{\alpha}\left[1 - e^{-\beta v_0}\right]$$

$$\Rightarrow \quad t = \frac{m}{\alpha\beta}\left[1 - e^{-\beta v_0}\right]$$

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c) From (a) we have

$$dx = -\frac{1}{\beta}\ln\left[\frac{\alpha\beta t}{m} + e^{-\beta v_0}\right]dt$$

Using $\int \ln(ax+b)\,dx = \frac{ax+b}{a}\ln(ax+b) - x$ we obtain

$$x + C = -\frac{1}{\beta}\left[\frac{\left[\frac{\alpha\beta t}{m} + e^{-\beta v_0}\right]\ln\left[\frac{\alpha\beta t}{m} + e^{-\beta v_0}\right]}{\alpha\beta/m} - t\right]$$

Evaluating C using $x=0$ at $t=0$ gives

$$C = \frac{v_0 m}{\alpha\beta}e^{-\beta v_0}$$

So, $$x = -\frac{m v_0}{\alpha\beta}e^{-\beta v_0} + \frac{t}{\beta} - \frac{m}{\alpha\beta^2}\left[\frac{\alpha\beta t}{m} + e^{-\beta v_0}\right]\ln\left[\frac{\alpha\beta t}{m} + e^{-\beta v_0}\right]$$

Substituting the time required to stop from (b) gives the distance required to stop

$$x = \frac{m}{\alpha\beta}\left[\frac{1}{\beta} - e^{-\beta v_0}\left[v_0 + \frac{1}{\beta}\right]\right]$$

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