From Newton's 1st & 2nd theorems, it follows that the grav. attraction of a spherical density dist'n $\rho(r')$ on a mass at r is entirely dependent on the mass interior to r:

$$\vec{g}(r) = -\frac{GM(r)}{r^2}\hat{r}$$

where $M(r) = 4\pi \int_0^r r'^2 \rho(r')\,dr'$

$$\left[M(r) = \int_0^r dM = \int_0^r \rho(r')\, 4\pi r'^2 dr' \right]$$

The grav. potential at $\vec{r}$ generated by an arbitrary spherically symmetric density dist'n $\rho(r')$ is calculated by adding the contributions by shells w/ $r' < r$ & $r' > r$

$$\Phi(r) = -\frac{G}{r}\int_0^r dM(r') - G\int_r^\infty \frac{dM(r')}{r'}$$

$$= -4\pi G\left[\frac{1}{r}\int_0^r dr' r'^2 \rho(r') + \int_r^\infty dr' r' \rho(r')\right]$$

Ex: check that $-\vec{\nabla}\Phi$ gives the $g(\vec{r})$ eq'n above

In a spherical matter dist'n, many of the characteristic speeds can be written in terms of the grav. potential

1) Circular speed $V_c(r)$ is when the grav. attraction is balanced by the centripetal acc'n

$$\frac{V_c^2}{r} = |\vec{g}| = \frac{d\Phi}{dr}$$

or $V_c^2 = r\frac{d\Phi}{dr} = \frac{GM(r)}{r}$ from above or $V_c^2 = r\frac{d\Phi}{dr} = \frac{GM(r)}{r}$ from above

Note that $V_c(r)$ measures the mass interior to r.

2) Escape speed. If $\Phi(r) \to 0$ as $r \to \infty$, then

$$V_e(r) = \sqrt{2|\Phi(r)|}$$

$V_e(r)$ depends on mass both inside & outside r

3) Potential energy of a spherical system

$$V = -4\pi G \int_0^\infty r \rho(r) M(r)\, dr$$

Potentials of Some Simple Systems

1) Point mass

$$\Phi(r) = -\frac{GM}{r} \; ; \; V_c(r) = \sqrt{\frac{GM}{r}} \; , \; V_e(r) = \sqrt{\frac{2GM}{r}}$$

Keplerian orbits

2) Homogeneous Sphere

If the density is some constant ρ, then the mass $M(r) = \frac{4}{3}\pi r^3 \rho$

and $V_c = \sqrt{\frac{4\pi G\rho}{3}}\, r$. Orbital speed increases w/ radius.

$\therefore$ Orbital period $T = \frac{2\pi r}{V_c} = \sqrt{\frac{3\pi}{G\rho}} = \text{const.}$

$$\frac{T}{2\pi} = \frac{r}{V_c} = \sqrt{\frac{3}{4\pi G\rho}} = 0.4886\,(G\rho)^{-1/2}$$

$\therefore$ Thus, $(G\rho)^{-1/2}$ is a good estimate of the timescale of the motion independent of the size & shape of the orbit. This result also holds for inhomogenous systems as long as ρ is replaced by $\bar{\rho}$, the avg. density interior to the particle's current radius.

$\therefore$ $t_{dyn} \sim t_{cross} \sim (G\bar{\rho})^{-1/2}$ is used as a characteristic time associated w/ a motion of a star in a galaxy or GC.

The potential energy of a homogeneous sphere of radius a & density ρ

$$V = -4\pi G \int_0^\infty r\rho(r) M(r)\,dr = -4\pi\rho\left(\frac{4}{3}\right)\pi\rho \int_0^a r^4\,dr = -\frac{16}{15}\pi^2 G\rho^2 a^5$$

$$\therefore \; V = -\frac{3}{5}\frac{GM^2}{a}$$

Ex: Show that the grav. pot. of a homogeneous sphere of radius a is

$$\Phi(r) = \begin{cases} -2\pi G\rho\left(a^2 - \frac{1}{3}r^2\right) & r < a \\ -\dfrac{4\pi G\rho a^3}{3r} & r > a \end{cases}$$