Energy Conservation for Central Conservative Forces

For motion in a central, conservative force know $\vec{J} = m\vec{r}\times\dot{\vec{r}}$ is constant, but also know that $V(r)$, the PE, can be defined $(\vec{F} = -\vec{\nabla}V)$, so there is an energy conservation eqn that can be written as $\frac{1}{2}m\dot{\vec{r}}^2 + V(r) = E = \text{const.}$

Rewrite both these conservation laws in polar form:

$$mr^2\dot{\theta} = J$$

$$\frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\theta}^2\right) + V(r) = E$$

Combine: $\frac{1}{2}m\dot{r}^2 + \frac{J^2}{2mr^2} + V(r) = E$, the radial energy eqn.

For a given J, can define an effective potential energy function

$$U(r) = \frac{J^2}{2mr^2} + V(r)$$

$\therefore$ Energy eq'n: $\frac{1}{2}m\dot{r}^2 + U(r) = E$

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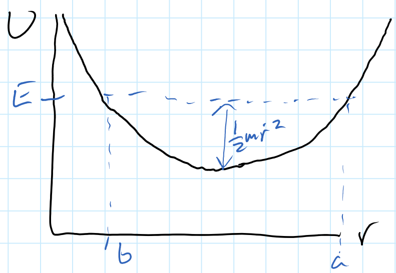
Now, since $\dot{r}^2$ is always positive, any motion is limited to the range of r for which $U(r) = \frac{J^2}{2mr^2} + V(r) \leq E$ where the max & min values correspond to equality w/ E

Ex: $V(r) = \frac{1}{2}Kr^2$ (isotropic harmonic oscillator)



Consider an inverse-square force $\vec{F} = \frac{K}{r^2}\hat{r} \rightarrow V(r) = \frac{K}{r}$

$\therefore$ The radial energy equation is

$$\frac{1}{2}m\dot{r}^2 + \frac{J^2}{2mr^2} + \frac{K}{r} = E$$

The effective p.e. function is

$$U(r) = \frac{J^2}{2mr^2} + \frac{K}{r}$$

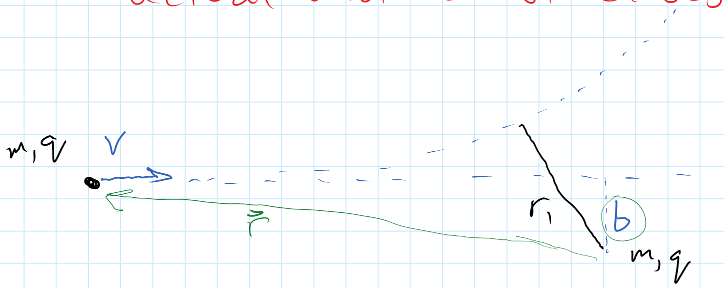
If $K > 0$, this is a repulsive force & $U(r)$ looks like

It has no minimum & circular motion is impossible in this potential.

For any positive E, there is a $r = r_1$ where $U(r_1) = E$ but no maximum position

So, if there is some initial radial motion, will go to r_1 & back out w/ no limit

Ex: A charged particle w/ charge q, moving in the field of another point charge w/ the exact same charge. Initially it is approaching the center w/ velocity v (at a large distance) which would pass the central charge at a distance b. Find the actual distance of closest approach.



The distance b is called the impact parameter

Let the particle start infinitely far away so that $E = \frac{1}{2}mv^2$ (ie, $V(\infty) = 0$)

What about J? $\vec{J} = m\vec{r} \times \vec{v}$

$|\vec{J}| = m|\vec{r}_\perp||\vec{v}|$ where $|\vec{r}_\perp|$ is the comp of $\vec{r} \perp$ to $\vec{v}$

$= mbv$

The distance of closest approach, r_1, is when $\dot{r} = 0$

$$\therefore \quad E = U(r_1)$$

$$\rightarrow \frac{1}{2}mv^2 = \frac{m^2 b^2 v^2}{2\cancel{m} r_1^2} + \frac{K}{r_1}$$

$$r_1^2\left(\frac{1}{2}mv^2\right) = Kr_1 + \frac{mb^2v^2}{2}$$

$$r_1^2 - \frac{2Kr_1}{mv^2} - b^2 = 0$$

$$\therefore r_1 = \frac{K}{mv^2} + \sqrt{\frac{K^2}{m^2v^4} + b^2}$$ where $K = \frac{q^2}{4\pi\epsilon_0}$

If $K < 0$, this is the gravitational case

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Define $\ell = \dfrac{J^2}{m|K|}$

The effective PE function is $U(r) = |K|\left(\dfrac{\ell}{2r^2} - \dfrac{1}{r}\right)$