Sample Midterm 1A, Math 1554

PLEASE PRINT YOUR NAME CLEARLY IN ALL CAPITAL LETTERS

First Name ____________________ Last Name ____________________

GTID Number: ______________________

Student GT Email Address: _________________________@gatech.edu

Section Number (e.g. A4, QH3, etc.) _______ TA Name _________________________

Student Instructions

- **Show your work** and justify your answers for all questions unless stated otherwise.
- **Organize your work** in a reasonably neat and coherent way.
- Calculators, notes, cell phones, books are not allowed.
- Use dark and clear writing: your exam will be scanned into a digital system.
- Exam pages are double sided. Be sure to complete both sides.
- Leave a 1 inch border around the edges of exams.
- Any work done on scratch paper will not be collected and will not be graded.

Sample Midterm 1A
You do not need to justify your reasoning for questions on this page.

1. (a) (5 points) Suppose A is an $m \times n$ matrix, $\vec{b} \in \mathbb{R}^m$, and $T_A : \mathbb{R}^m \to \mathbb{R}^n$ is the associated linear transformation. Select **true** if the statement is true for all choices of A and $\vec{b}$. Otherwise, select **false**.

true	false	
◯	◯	If A has dimensions 5×6, and its columns are spanned by two linearly independent vectors, then the system $A\vec{x} = \vec{b}$ has 4 free variables.
◯	◯	The linear transformation T_A is onto if every column of A is pivotal.
◯	◯	If A is a 2×2 matrix with identical columns, then $A\vec{x} = \vec{0}$ is inconsistent.
◯	◯	If the only solution to $T_A(\vec{x}) = \vec{0}$ is $\vec{x} = \vec{0}$, then the columns of A are linearly independent.
◯	◯	If $A\vec{x} = A\vec{y}$ for some $\vec{x} \neq \vec{y}$, then A could have a pivot in every row, but A cannot have a pivot in every column.

(b) (2 points) Fill in the missing entries of each matrix with numbers, so that the columns of the matrices are linearly independent. If it is not possible to do so for a matrix, write *not possible* below the matrix.

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & \end{pmatrix} \qquad \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ & 0 & 0 \end{pmatrix}$$

(c) (3 points) Let $\vec{v}_1, \vec{v}_2$ be linearly independent vectors in $\mathbb{R}^5$. Fill in the circles that are next to expressions equal to $S = \text{Span}(\vec{v}_1, \vec{v}_2)$.

◯ $\text{Span}(\vec{v}_1 + \vec{v}_2,\ -\vec{v}_1 - \vec{v}_2)$
◯ $\text{Span}(\vec{0}, \vec{v}_1, \vec{v}_2)$
◯ $\text{Span}(\vec{v}_1 + \vec{v}_2,\ \vec{v}_1 - \vec{v}_2)$

You do not need to justify your reasoning for questions on this page.

2. (10 points) If possible, write down an example of a matrix with the following properties. If it is not possible to do so, write *not possible*. You do not need to justify your reasoning.

   (a) An augmented matrix that can be used to compute the coefficients a_0, a_1, and a_2 of the polynomial $y(x) = a_0 + a_1 x^2 + a_2 x^4$ that takes the values $y(-1) = 3$, $y(0) = 4$ and $y(2) = 0$.

   (b) A matrix $A \in \mathbb{R}^{2\times 3}$ that is in row reduced echelon form (RREF), such that A^T has two pivot columns.

   (c) A matrix $A \in \mathbb{R}^{2\times 2}$ whose columns span the line $x_1 + 2x_2 = 0$, and whose solutions to $A\vec{x} = \vec{0}$ span the same line.

   (d) A matrix $A \in \mathbb{R}^{2\times 2}$, such that $T(\vec{x}) = A\vec{x}$, where T is a linear transformation that rotates vectors in $\mathbb{R}^2$ counterclockwise by $\pi/2$ radians about the origin, then reflects them about the line $x_1 = x_2$.

3. (4 points) What value(s) of h, if any, is $AB = BA$?

$$A = \begin{pmatrix} 0 & 4 \\ -1 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 2 & 0 \\ 0 & h \end{pmatrix}$$

4. (6 points) For which values of k, if any, do the vectors span $\mathbb{R}^3$?

$$\begin{pmatrix} 1 \\ 0 \\ -k \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}, \begin{pmatrix} k \\ -1 \\ 3k \end{pmatrix}$$

5. (3 points) Suppose $A \in \mathbb{R}^{m\times n}$, $m < n$, and $A\vec{x} = \vec{b}_1$ has no solutions for some $\vec{b}_1 \in \mathbb{R}^m$. How many solutions could $A\vec{x} = \vec{b}_2$ have, if $\vec{b}_1 \neq \vec{b}_2$? Circle all situations that are possible. You do not need to justify your reasoning.

   (a) no solutions

   (b) exactly one solution

   (c) infinitely many solutions

6. (3 points) Suppose $A \in \mathbb{R}^{2\times 9}$, $B \in \mathbb{R}^{9\times 9}$ and $v \in \mathbb{R}^2$. Fill in the circles next to expressions that are **defined**. Leave the other circles empty.

   ○ $v^T(AB)^2$

   ○ $v\,v^T AA^T$

   ○ $(AB)^T v$

7. (6 points) For what vectors $\vec{b} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$ does the equation have a solution? Express your answer in parametric vector form.

$$A\vec{x} = \begin{pmatrix} 0 & 5 \\ 1 & 3 \\ 2 & 1 \end{pmatrix} \vec{x} = \vec{b}$$

8. (8 points) Consider the following linear transformation, and answer the questions below. You do not need to justify your reasoning.

$$T(x_1, x_2) = \left(\frac{1}{2}x_1 - \frac{1}{2}x_2, -\frac{1}{2}x_1 + \frac{1}{2}x_2\right)$$

(a) What is the domain of T? _____

(b) What is the co-domain of T? _____

(c) What is the standard matrix of T?

(d) What are the images of the standard vectors $\vec{e}_1$ and $\vec{e}_2$?

(e) Is T onto (yes or no)? __________

(f) Is T one-to-one (yes or no)? __________

(g) Draw a sketch of $\vec{e}_1$, $\vec{e}_2$, $T(\vec{e}_1)$ and $T(\vec{e}_2)$ on one picture. Clearly label your vectors.

This page may be used for scratch work. Please indicate clearly if you would like your work on this page to be graded.

1. (a)
   - true
   - false
   - false
   - true
   - true

   (b) First matrix: not possible
   Second matrix: fill missing entry with any nonzero number

   (c) Span $(\vec{0}, \vec{v}_1, \vec{v}_2)$ Span $(\vec{v}_1 + \vec{v}_2, \vec{v}_1 - \vec{v}_2)$

2. (a) Examples: $\left(\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 1 & 0 & 0 & 4 \\ 1 & 4 & 16 & 0 \end{array}\right)$, $\left(\begin{array}{ccc|c} 1 & 4 & 16 & 0 \\ 1 & 0 & 0 & 4 \\ 1 & 1 & 1 & 3 \end{array}\right)$

   (b) Examples: $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$, $\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

   (c) The columns are to span the line $x_1 + 2x_2 = 0$ and a point on that line is $(2, -1)$. So a vector that spans the line is the vector $\begin{pmatrix} 2 \\ -1 \end{pmatrix}$. We can use this vector as the first column of our 2×2 matrix, giving us $\begin{pmatrix} 2 & a \\ -1 & b \end{pmatrix}$. We need the values of a and b. But the solutions to $A\vec{x} = 0$ is the line $x_1 + 2x_2 = 0$, so we obtain

   $$A\vec{x} = \begin{pmatrix} 2 & a \\ -1 & b \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \Rightarrow 2x_1 + ax_2 = 0, \text{ and } -x_1 + bx_2 = 0$$

   By comparison to $x_1 + 2x_2 = 0$, $a = 4$ and $b = -2$. Our matrix can be: $\begin{pmatrix} 2 & 4 \\ -1 & -2 \end{pmatrix}$

   (Think: Does $\begin{pmatrix} 2 & -4 \\ -1 & 2 \end{pmatrix}$ also satisfy the given criteria? Yes or no?)

   (d) $A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

   It is ok to leave the answer as $\begin{pmatrix} \sin \pi/2 & 0 \\ 0 & -\sin \pi/2 \end{pmatrix}$

3.
$$AB = \begin{pmatrix} 0 & 4 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 2 & 0 \\ 0 & h \end{pmatrix} = \begin{pmatrix} 0 & 4h \\ -2 & h \end{pmatrix}$$
$$BA = \begin{pmatrix} 2 & 0 \\ 0 & h \end{pmatrix} \begin{pmatrix} 0 & 4 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 8 \\ -h & h \end{pmatrix}$$
$$\Longrightarrow h = 2$$

4.
$$\left(\begin{array}{ccc|c} 1 & 0 & k & 0 \\ 0 & 1 & -1 & 0 \\ -k & 2 & 3k & 0 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 0 & k & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 2 & k^2 + 3k & 0 \end{array}\right)$$
$$\sim \left(\begin{array}{ccc|c} 1 & 0 & k & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & k^2 + 3k + 2 & 0 \end{array}\right)$$

$$\implies k^2 + 3k + 2 \neq 0$$
$$\implies (k+2)(k+1 \neq 0$$
$$\implies k \neq -2, k \neq -1$$

5. (a) and (c) only

6. The first expression is undefined, the second and third expressions are defined.

7.

$$\left(\begin{array}{cc|c} 0 & 5 & b_1 \\ 1 & 3 & b_2 \\ 2 & 1 & b_3 \end{array}\right) \sim \left(\begin{array}{cc|c} 1 & 3 & b_2 \\ 0 & 1 & b_1/5 \\ 2 & 1 & b_3 \end{array}\right)$$

$$\sim \left(\begin{array}{cc|c} 1 & 3 & b_2 \\ 0 & 1 & b_1/5 \\ 0 & -5 & b_3 - 2b_2 \end{array}\right)$$

$$\sim \left(\begin{array}{cc|c} 1 & 3 & b_2 \\ 0 & 1 & b_1/5 \\ 0 & 0 & b_3 - 2b_2 + b_1 \end{array}\right)$$

Need $b_3 - 2b_2 + b_1 = 0$, or in parametric vector form:

$$\vec{b} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} = \begin{pmatrix} 2b_2 - b_3 \\ b_2 \\ b_3 \end{pmatrix} = b_2 \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} + b_3 \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$$

8. (a) $\mathbb{R}^2$
   (b) $\mathbb{R}^2$
   (c) $\frac{1}{2}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$
   (d) $\vec{e}_1 \to \begin{pmatrix} 1/2 \\ -1/2 \end{pmatrix}$, $\vec{e}_2 \to \begin{pmatrix} -1/2 \\ 1/2 \end{pmatrix}$
   (e) No
   (f) No
   (g) A rough sketch of what is shown below is sufficient. It is good practice to label your axes. For this class please always place x_1 on the horizontal axis.