Example: Give the PE function

$$V(\vec{r}) = \alpha x^2 + \beta xy + \gamma z + C$$ where α, β, γ & C are constants.

Find the force function for this $V(\vec{r})$.

$$\vec{F} = -\vec{\nabla}V = -\frac{\partial V}{\partial x}\hat{i} - \frac{\partial V}{\partial y}\hat{j} - \frac{\partial V}{\partial z}\hat{k} = -(2\alpha x + \beta y)\hat{i} - \beta x\hat{j} - \gamma\hat{k}$$

Suppose a particle of mass m is moving in the above force field and at time $t=0$ the particle passes through the origin w/ speed v_0. What will the speed of the particle be if & when it passes through $\vec{r} = \hat{i} + 2\hat{j} + \hat{k}$?

Since this is a consv. force (V exists)

$$T + V = \text{constant}$$

$\therefore$ at $t=0$, $\frac{1}{2}mv_0^2 + C = \text{const.}$

Lets pick $C=0$, so const. $= \frac{1}{2}mv_0^2$

At some later time when particle is at $\vec{r}$

$$\frac{1}{2}mv_0^2 = \frac{1}{2}mv^2 + \alpha + 2\beta + \gamma$$

solve for $v = \sqrt{v_0^2 - \frac{2\alpha}{m} - \frac{4\beta}{m} - \frac{2\gamma}{m}}$

Example: Show that the inverse-square law of force in 3D

$$\vec{F} = \left(-\frac{K}{r^2}\right)\hat{r} \quad \text{is conservative.}$$

Take curl of $\vec{F}$. Use curl in sph. coord's

$$\vec{\nabla}\times\vec{F} = \frac{1}{r^2\sin\theta}\begin{vmatrix} \hat{r} & r\hat{\theta} & r\sin\theta\hat{\phi} \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial\theta} & \frac{\partial}{\partial\phi} \\ F_r & rF_\theta & rF_\phi\sin\theta \end{vmatrix} = 0$$

Projectile Motion

If a Cartesian coordinate system can be chosen such that the Forces involved in a motion involve only the respective coordinates

$$\text{e.g.,}\quad \vec{F}(\vec{r}) = F_x(x)\hat{i} + F_y(y)\hat{j} + F_z(z)\hat{k}$$

the the force is called separable and it will be conservative ($\vec{\nabla}\times\vec{F}=0$)

(note that if $\vec{F}$ is a function of time, or $\dot{\vec{r}}$ then will not be conservative)

In this case, the e.o.m. can be treated as 3 separate equ. that can be treated w/ already studied techniques

$$m\ddot{x} = F_x$$

$$m\ddot{y} = F_y$$

$$m\ddot{z} = F_z$$

Motion of a Projectile in a Uniform Gravitational Field

a) No air resistance so only force is gravity. Choose coord's

So the +ve z-axis is pointing up.

Equ of motion is $m\ddot{\vec{r}} = -mg\hat{k}$ where g is assumed constant

$\therefore$ Force is separable and is conservative

$$\frac{d}{dt}\left(\frac{d\dot{\vec{r}}}{dt}\right) = -g\hat{k}$$

$$\frac{d\vec{r}}{dt} = -gt\hat{k} + \vec{v}_0 \quad \text{where } v_0 \text{ is the initial vel.}$$

$$\vec{r} = -\frac{1}{2}gt^2\hat{k} + \vec{v}_0 t + \cancel{\vec{r}_0} \rightarrow 0, \text{ say initial pos'n is the origin}$$

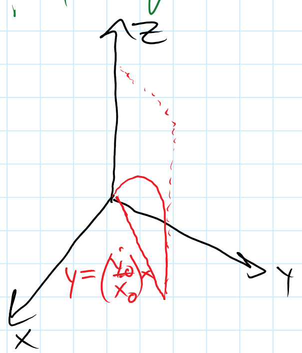
In components

$$x = \dot{x}_0 t$$

$$y = \dot{y}_0 t$$

$$z = \dot{z}_0 t - \frac{1}{2}gt^2$$

To determine the path of the projectile, eliminate t from the x ; y equs. $\quad y = \left(\frac{\dot{y}_0}{\dot{x}_0}\right)x \rightarrow$ path lies entirely in a plane



Eliminate t b/w the x ; z equ's, get

$$z = \left(\frac{\dot{z}_0}{\dot{x}_0}\right)x - \left(\frac{g}{2\dot{x}_0^2}\right)x^2 \quad \text{which is a parabola}$$

To find the range set $z=0$, $\quad 0 = x\left(\frac{\dot{z}_0}{\dot{x}_0} - \frac{gx}{2\dot{x}_0^2}\right)$

$x=0$, obviously, or $\frac{\dot{z}_0}{\dot{x}_0} = \frac{gx}{2\dot{x}_0^2} \rightarrow x = \frac{2\dot{z}_0\dot{x}_0}{g}$

If launched @ angle α, $\dot{x}_0 = V_0\cos\alpha$; $\dot{z}_0 = V_0\sin\alpha$

so $\quad x = \dfrac{2v_0^2 \sin\alpha \cos\alpha}{g} = \dfrac{v_0^2 \sin 2\alpha}{g}$