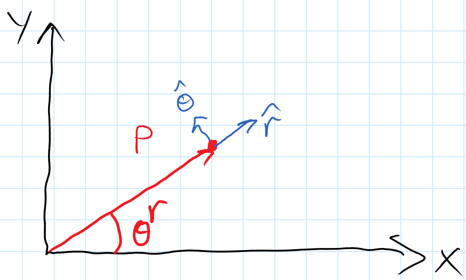
Example: Uniform Circular Motion

A particle moves around a circle of radius R with const. angular speed ω (rad. s^{-1}). Show that the centripetal acc'n is towards the center & has mag. $\omega^2 R$.

Work in polar coordinates.



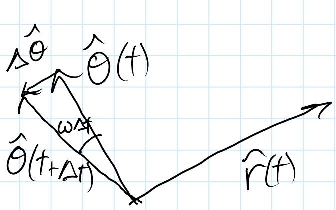
At any point P, there is a unit vector $\hat{r}$ pointing in the direction of increasing r for $\theta = \theta_p$ = constant. A unit vector $\hat{\theta}$ point in direction of increasing θ when r = const.

Now, $\vec{r}(t) = R\,\hat{r}$

so, $\dot{\vec{r}}(t) = \vec{v}(t) = R\,\dot{\hat{r}}$ & $\ddot{\vec{r}}(t) = \vec{a}(t) = R\,\ddot{\hat{r}}$

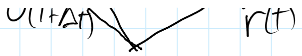
Need to find $\dot{\hat{r}}$ & $\ddot{\hat{r}}$

In the limit $\Delta t \to 0$, $\Delta\hat{r}$ has magnitude $d\theta = \omega dt$ & points along $\hat{\theta}$ at time t. ie $d\hat{r} = \omega\, dt\, \hat{\theta}$ or $\frac{d\hat{r}}{dt} = \dot{\hat{r}} = \omega\hat{\theta} = \dot{\theta}\hat{\theta}$



$\Delta\hat{\theta} = -\omega\Delta t\,\hat{r}$

$\therefore \frac{\Delta\hat{\theta}}{\Delta t} = -\hat{r}\omega \to \dot{\hat{\theta}} = -\hat{r}\omega$



$\therefore \frac{\Delta\hat{\theta}}{\Delta t} = -\hat{r}\omega \rightarrow \dot{\hat{\theta}} = -\hat{r}\omega$

From above, $\ddot{\hat{r}} = \ddot{\theta}\hat{\theta} + \dot{\theta}\dot{\hat{\theta}} = \overset{\text{unif. rot.}}{0} - \dot{\theta}\omega\hat{r} = -\dot{\theta}^2\hat{r}$

Thus, we have for uniform circular motion, $\vec{r} = R\hat{r}(t)$

$$\vec{V} = \dot{\vec{r}} = R\dot{\hat{r}}(t) = R\omega\hat{\theta}$$

$$\vec{a} = \ddot{\vec{r}} = -R\omega\dot{\theta}\hat{r} = \underline{-R\omega^2\hat{r}}$$

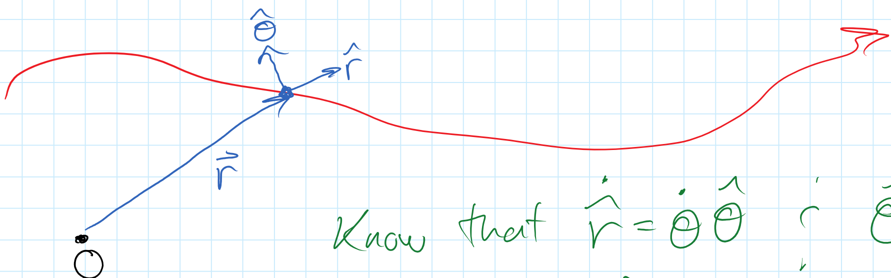
Note, for non-uniform circular motion, $\omega = \dot{\theta} \neq \text{const.}$

$$\dot{\vec{r}} = R\omega\hat{\theta} = R\dot{\theta}\hat{\theta} \quad \text{as before}$$

$$\ddot{\vec{r}} = \underbrace{R\ddot{\theta}\hat{\theta}}_{\vec{a}_{tangential}} - \underbrace{R\dot{\theta}^2\hat{r}}_{\vec{a}_{radial}}$$

$$\therefore |\vec{a}| = a = \sqrt{(R\ddot{\theta})^2 + R^2\dot{\theta}^4} = \sqrt{\dot{V}^2 + \frac{V^4}{R^2}}$$

Now consider velocity & acc'n for arbitrary motion in polar coordinates



Know that $\dot{\hat{r}} = \dot{\theta}\hat{\theta}$ & $\dot{\hat{\theta}} = -\dot{\theta}\hat{r}$

So, since $\vec{r} = r\hat{r}$

$$\vec{V} = \underline{\dot{\vec{r}} = \dot{r}\hat{r} + r\dot{\theta}\hat{\theta}}$$

$$\therefore \vec{a} = \ddot{\vec{r}} = \ddot{r}\hat{r} + \dot{r}\dot{\hat{r}} + (\dot{r}\dot{\theta} + r\ddot{\theta})\hat{\theta} + r\dot{\theta}\dot{\hat{\theta}}$$

$$\rightarrow \underline{\ddot{\vec{r}} = (\ddot{r} - r\dot{\theta}^2)\hat{r} + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\hat{\theta}}$$

Check that these reduce correctly for circular motion & for radial motion.

Example: A turntable is rotating at constant angular speed. There is an ant crawling outward on a radial line such that the ant's distance from the center increases as $r = bt^2$, $\theta = \omega t$ where b & ω are constants. Find the acc'n of the ant.

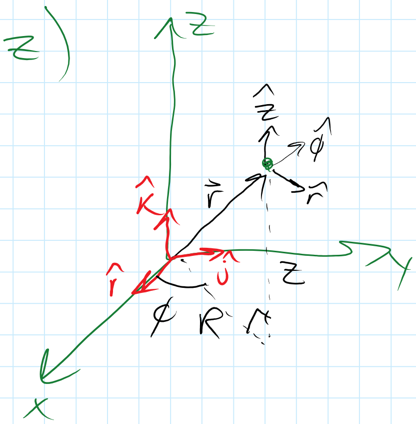
Take all the derivatives: $\dot{r} = 2bt$, $\ddot{r} = 2b$, $\dot{\theta} = \omega$, $\ddot{\theta} = 0$

$$\therefore \ddot{\vec{r}} = (2b - bt^2\omega^2)\hat{r} + (0 + 4bt\omega)\hat{\theta}$$
$$= b(2 - t^2\omega^2)\hat{r} + 4bt\omega\,\hat{\theta}$$

Note the radial comp. of the acc'n becomes negative for large t although r is always increasing. Slowing down...

Cylindrical Cooridnates - needed for 3D motion

(R, ϕ, z)



$\vec{r} = R\hat{r} + z\hat{z}$

From this diagram

$\hat{r} = \cos\phi\,\hat{i} + \sin\phi\,\hat{j}$

$\hat{\phi} = -\sin\phi\,\hat{i} + \cos\phi\,\hat{j}$

$\hat{z} = \hat{k}$