MATH 3215 SPRING 2019
FINAL EXAM
APR. 29, 2019

Instructions.

1. There are six problems with three parts each. Each part is four points. If you do not know how to do a part, check the box in that part and write nothing else to receive one point.

2. If you turn in your paper by the time I call the end of the exam, you get two extra points.

3. Partial credit will be given. Show all your work for the highest chance of getting full credit. If you just write the answer with no explanation, you will get no credit. If you get the correct answer, but by a method that is incorrect, the answer will be considered incorrect.

4. You cannot use calculators. Therefore you do not need to simplify your answers: it is ok to write an answer like $\binom{10}{2}\binom{5}{3}$ or an answer like $\frac{.1+.3+.2}{3}$. You also cannot use any notes, you cannot talk or give signals to anyone.

5. You cannot use your phone. The only things on your desk should be writing tools and the exam.

6. Write legibly. If I cannot read your writing, you answer will be considered incorrect.

DO NOT OPEN THE TEST UNTIL YOU ARE INSTRUCTED TO DO SO.

NAME: KEY

TO BE USED FOR GRADING:

1 : 12 pts	2 : 12 pts	3 : 12 pts	4 : 12 pts	5 : 12 pts	6 : 12 pts	Total /72

1. (a) (4 pts) Let X be a Cauchy random variable: its pdf is $f(x) = \frac{1}{\pi(1+x^2)}$ for $x \in \mathbb{R}$. Let $Y = X^3$ and compute the pdf for Y.

For $y \in \mathbb{R}$, $\mathbb{P}(Y \le y) = \mathbb{P}(X^3 \le y) = \mathbb{P}(X \le \sqrt[3]{y}) = \int_{-\infty}^{\sqrt[3]{y}} \frac{1}{\pi(1+x^2)}\,dx$

$f(y) = \frac{d}{dy}\mathbb{P}(Y \le y) = \boxed{\frac{1}{3} y^{-2/3} \cdot \frac{1}{\pi(1+y^{2/3})},\ y \neq 0}$

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(b) (4 pts) Suppose that X and Y are standard normal random variables (that is, $N(0,1)$) with correlation coefficient $\rho = 0.5$. Compute $\mathbb{E}[(5X+6Y)^2]$.

$= \mathbb{E}[25X^2 + 60XY + 36Y^2] = 25 + 36 + 60\,\mathbb{E}XY$

$\mathbb{E}XY = \mathrm{Cov}(X,Y) + \mathbb{E}X\mathbb{E}Y = \rho(X,Y) = 0.5$

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$= 25 + 36 + 60(0.5) = 25 + 36 + 30 = \boxed{91}$

(c) (4 pts) Let $X_1, \ldots, X_n$ be an i.i.d. sample from a Bernoulli distribution with parameter $p = 1/2$. For sample size $n = 100$, use the central limit theorem to find the approximate probability $\mathbb{P}(\bar{X} \ge 11/20)$, where $\bar{X}$ is the sample mean $(1/n)(X_1 + \cdots + X_n)$.

$\bar{X} \approx N(\mu, \sigma^2/n)$

$= N\left(\frac{1}{2}, \frac{1}{400}\right)$

$\mathbb{P}\left(\bar{X} \ge \frac{11}{20}\right) = \mathbb{P}\left(\frac{\bar{X} - \frac{1}{2}}{1/20} \ge \frac{\frac{11}{20} - \frac{1}{2}}{1/20}\right)$

$\approx 1 - \Phi(1)$

☐

$= 1 - 0.8413$

$= \boxed{0.1587}$

2. (a) Let $X_1, \ldots, X_n$ be an i.i.d. sample from an exponential distribution with parameter θ, there $\theta > 0$.

i. (4 pts) Write an expression for the log-likelihood function for a maximum likelihood estimator for θ.

$$\log L(x_1, \ldots, x_n) = \log\left[\left(\frac{1}{\theta}\right)e^{-x_1/\theta} \cdot \ldots \cdot \left(\frac{1}{\theta}\right)e^{-x_n/\theta}\right]$$

$$= \log\left[\frac{1}{\theta^n} e^{-\frac{1}{\theta}(x_1+\cdots+x_n)}\right]$$

$$= \boxed{-n\log\theta - \frac{1}{\theta}(x_1+\cdots+x_n)}$$

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ii. (4 pts) Using maximum likelihood estimation, compute an estimator $\hat{\theta}$ for θ in terms of the X_i's.

$$\frac{d}{d\theta}\log L(x_1, \ldots, x_n) = \frac{-n}{\theta} + \frac{1}{\theta^2}(x_1+\cdots+x_n) = 0 \text{ when}$$

$$n = \frac{1}{\theta}(x_1+\cdots+x_n)$$

$$\Rightarrow \boxed{\hat{\theta} = \bar{X}}$$

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$$\frac{d^2}{d\theta^2}\log L(x_1, \ldots, x_n) = \frac{n}{\theta^2} - \frac{2}{\theta^3}(x_1+\cdots+x_n) = \frac{n}{(\bar{X})^2} - \frac{2n}{(\bar{X})^2} < 0. \text{ So max.}$$

(b) (4 pts) Let $X_1, \ldots, X_n$ be an i.i.d. sample from a distribution with pdf $f(x) = \theta x^{\theta-1}$, for $x \in (0,1)$ (the parameter θ is assumed to be > 0). Use the method of moments to estimate θ in terms of the X_i's.

$$\text{Set } \bar{X} = \theta\int_0^1 x \cdot x^{\theta-1}\,dx = \theta\int_0^1 x^\theta\,dx = \theta \cdot \left.\frac{x^{\theta+1}}{\theta+1}\right|_0^1$$

$$= \frac{\theta}{\theta+1}$$

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$$\Rightarrow \bar{X}\theta + \bar{X} = \theta$$

$$\Rightarrow \boxed{\hat{\theta} = \frac{\bar{X}}{1-\bar{X}}}$$

3. (a) (4 pts) We sample soda cans from around the GT campus and find the following data for grams of sugar in $n = 9$ cans:

$$21, 22, 20, 23, 18, 20, 9, 50, 1.$$

Draw a boxplot for this data, indicating potential outliers and outliers.

1
9
18
20
20
21
22
23
50

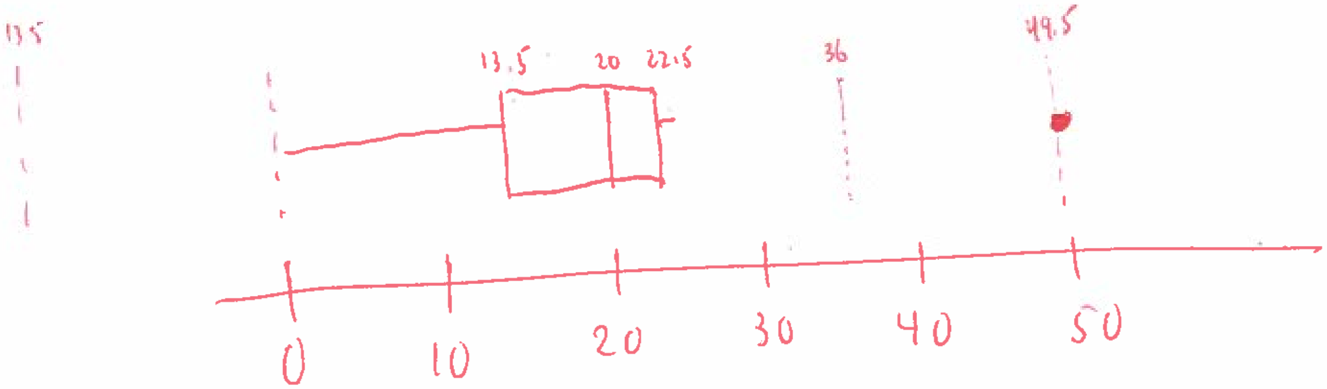
$n = 9$

$\tilde{\pi}_{0.25} = y_2 + 0.5(y_3 - y_2) = 9 + (0.5)(9) = \boxed{13.5}$

$\tilde{\pi}_{0.5} = y_5 = \boxed{20}$

$\tilde{\pi}_{0.75} = y_7 + 0.5(y_8 - y_7)$

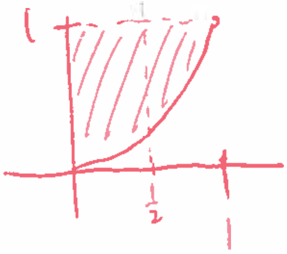
$= 22 + 0.5(23 - 22) = \boxed{22.5}$



IQR = 9

4. Let $f(x,y) = 3/2$, $x^2 \le y \le 1$, $0 \le x \le 1$, be the joint pdf of X and Y.

(a) (4 pts) Find $\mathbb{P}(0 \le X \le 1/2)$.

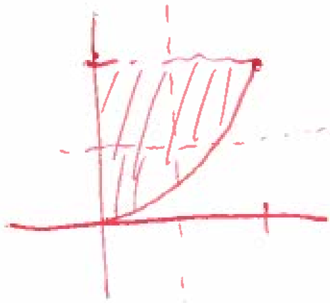


$$= \int_0^{\frac{1}{2}} \int_{x^2}^{1} \frac{3}{2}\, dy\, dx$$

$$= \frac{3}{2}\int_0^{\frac{1}{2}} (1-x^2)\,dx = \frac{3}{2}\left[x - \frac{x^3}{3}\right]_0^{\frac{1}{2}}$$

$$= \frac{3}{2}\left[\frac{1}{2} - \frac{1}{24}\right] = \frac{3}{2}\left[\frac{11}{24}\right] = \boxed{\frac{11}{16}}$$

(b) (4 pts) Find $\mathbb{P}(X \ge 1/2, Y \ge 1/2)$.



$$= \int_{\frac{1}{2}}^{1} \int_{\frac{1}{2}}^{\sqrt{y}} \frac{3}{2}\, dx\, dy = \frac{3}{2}\int_{\frac{1}{2}}^{1} \left(\sqrt{y} - \frac{1}{2}\right) dy$$

$$= \frac{3}{2}\left[\frac{y^{3/2}}{3/2} - \frac{y}{2}\right]_{\frac{1}{2}}^{1}$$

$$= \frac{3}{2}\left[\left(\frac{1}{3/2} - \frac{1}{2}\right) - \left(\frac{(\frac{1}{2})^{3/2}}{3/2} - \frac{1/2}{2}\right)\right]$$

$$= \frac{3}{2}\left[\frac{2}{3} - \frac{1}{2} - \frac{2}{3}\left(\frac{1}{2}\right)^{3/2} + \frac{1}{4}\right] = 1 - \frac{3}{4} - \left(\frac{1}{2}\right)^{3/2} + \frac{3}{8}$$

$$= \boxed{\frac{5}{8} - \left(\frac{1}{2}\right)^{3/2}}$$

$$\frac{5}{8} - \frac{1}{\sqrt{8}}$$

$$= \frac{5}{8} - \frac{1}{2\sqrt{2}}$$

(c) (4 pts) Are X and Y independent? Why or why not?

$\boxed{\text{No. Range not rectangular}}$

(b) (4 pts) For a larger sample of size $n = 100$, we find a sample mean of 21 g. Assuming that the number grams of sugar in soda cans is normally distributed with mean μ and standard deviation $\sigma^2 = 16$, construct a one-sided confidence interval of the type $[a, \infty)$ for the μ with significance level $\alpha = 0.05$. (Use the attached tables for tail probabilities for the normal distribution.)

$n = 100$; $\bar{X} = 21$, $\sigma^2 = 16$; $\alpha = 0.05$

C.I.: $\left[\bar{X} - z_\alpha \frac{\sigma}{\sqrt{n}}, \infty\right)$

$= \boxed{\left[21 - (1.645) \cdot \frac{4}{10}, \infty\right)} = [20.342, \infty)$

□

(c) (4 pts) We now test the null hypothesis $H_0 : \mu = 20$ against the alternative hypothesis $H_1 : \mu \neq 20$ with significance level $\alpha = 0.10$. For $n = 100$ samples, find the p-value associated with the sample mean $\bar{x} = 21$ and known variance $\sigma^2 = 16$. Do we reject or accept H_0?

H_0: $\mu = 20$ $\qquad \alpha = 0.10$, $n = 100$, $\sigma^2 = 16$

H_1: $\mu \neq 20$

□

p-value for $\bar{X} = 21$ $= \mathbb{P}(|\bar{X} - 20| \geq 1 ; \mu = 20)$

$= \mathbb{P}\left(\left|\frac{\bar{X} - 20}{4/10} - \frac{20-20}{4/10}\right| \geq \frac{1}{4/10} ; \mu = 20\right)$

$= \mathbb{P}(|Z| \geq 2.5)$

$= 2(1 - \Phi(2.5))$

$= 2(1 - .9938)$

$= 2(.0062) = \boxed{.0124} \leq 0.10$

$\boxed{\text{Reject}}$

5. Let $f(x) = \frac{c}{x^4}$ for $x \geq 1$.

(a) (4 pts) Find c so that f is a pdf. If X is a random variable with pdf equal to f, find $\mathbb{E}X$.

$$1 = \int_1^\infty \frac{c}{x^4}\,dx = c\int_1^\infty \frac{dx}{x^4} = c \cdot \frac{x^{-3}}{-3}\Big|_1^\infty = \frac{c}{3} \Rightarrow \boxed{c = 3}$$

$$\mathbb{E}X = \int_1^\infty x \cdot \frac{3}{x^4}\,dx = \int_1^\infty \frac{3}{x^3}\,dx = 3 \cdot \frac{x^{-2}}{-2}\Big|_1^\infty = \boxed{\frac{3}{2}}$$

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(b) (4 pts) Find the cdf and a third quartile of the distribution whose pdf is f. Is this quartile unique?

$$F(x) = \mathbb{P}(X \leq x) = \int_1^x \frac{3}{y^4}\,dy = 3 \cdot \frac{y^{-3}}{-3}\Big|_1^x = 3\left[\frac{x^{-3}}{-3} - \frac{1^{-3}}{-3}\right]$$

$$= 3\left[\frac{1}{3} - \frac{1}{3x^3}\right]$$

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$$\boxed{= \begin{cases} 1 - \frac{1}{x^3}, & x \geq 1 \\ 0 & \text{o.w.} \end{cases}}$$

$$1 - \frac{1}{x^3} = 0.75 \Rightarrow \frac{1}{x^3} = \frac{1}{4}$$

$$\Rightarrow \boxed{x = \sqrt[3]{4} \text{ unique}}$$

(c) (4 pts) Let's change f so that it takes two variables: set $f(x, y) = cy/x^4$ for $y \in [1, 2]$ and $x \geq 1$. You do not have to find c, but find the conditional density of x given y.

$$\frac{f(x,y)}{\int_{-\infty}^\infty f(x,y)\,dx} = \frac{cy/x^4}{\int_1^\infty cy/x^4\,dx} = \frac{\frac{1}{x^4}}{\frac{x^{-3}}{-3}\Big|_1^\infty} = \frac{\frac{1}{x^4}}{\frac{1}{3}} = \boxed{\frac{3}{x^4}, \; x \geq 1, \; y \in [1,2]}$$

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6. You roll a fair 6-sided die whose faces display the letters A, A, B, B, C, C. If you get an A, then you pick a number uniformly from $1-10$. If you get B or C, then you pick two numbers uniformly from $1-5$ (not allowing repeats in the number — for example, you cannot select $4,4$).

(a) (4 pts) Find $\mathbb{P}$(you select at least one odd number | you rolled B or C).

$$= \frac{\#\text{ choices with } \geq 1 \text{ odd}}{\#\text{ total choices}} = \frac{\binom{5}{2}-1}{\binom{5}{2}} = \boxed{1-\frac{1}{\binom{5}{2}}}$$

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$$= 1 - \frac{1}{\frac{5\cdot 4}{2}} = 1 - \frac{1}{10} = \boxed{0.9}$$

(b) (4 pts) Find $\mathbb{P}$(you select at least one odd number).

$$= \mathbb{P}(\geq 1 \text{ odd} \mid A)\,\mathbb{P}(A) + \mathbb{P}(\geq 1 \text{ odd} \mid B \text{ or } C)\,\mathbb{P}(B \text{ or } C)$$

$$= \frac{1}{2}\cdot\frac{1}{3} + \frac{\cancel{9}^{3}}{\cancel{10}_{5}}\cdot\frac{\cancel{2}^{1}}{\cancel{3}_{1}}$$

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$$= \frac{1}{6} + \frac{3}{5} = \boxed{\frac{23}{30}}$$

(c) (4 pts) Find $\mathbb{P}$(you rolled B or C | you selected at least one odd number).

$$= \frac{\mathbb{P}(\geq 1 \text{ odd} \mid B \text{ or } C)\,\mathbb{P}(B \text{ or } C)}{\mathbb{P}(\geq 1 \text{ odd})} = \frac{\frac{\cancel{9}^{3}}{\cancel{10}_{5}}\cdot\frac{\cancel{2}^{1}}{\cancel{3}_{1}}}{\frac{23}{30}}$$

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$$= \frac{\frac{3}{5}}{\frac{23}{30}} = \frac{3}{\cancel{5}_{1}}\cdot\frac{\cancel{30}^{6}}{23} = \boxed{\frac{18}{23}}$$