Now, determine how the angle θ varies w/ time.

Use $r^2\dot{\theta} = \text{constant} = \frac{J}{m}$

$$\dot{\theta} = \frac{J}{mr^2} = \frac{J}{m}u^2 = \frac{J}{mc^2\theta^4}$$

$$\rightarrow \theta^4 d\theta = \frac{J}{mc^2}dt$$

$$\rightarrow \frac{\theta^5}{5} = \frac{Jt}{mc^2} + \cancel{\text{const.}} \rightarrow 0$$

$$\rightarrow \theta = \left(\frac{5Jt}{mc^2}\right)^{1/5}$$

Attractive Inverse Square Law

Find the orbit of a particle w/ $f(r) = -\frac{K}{r^2}$ ($K = GMm$ for gravity)

So, $f(r^{-1}) = -Ku^2$

And orbit eqn is:

$$\frac{d^2u}{d\theta^2} + u = \frac{K\cancel{u^2}m}{J^2\cancel{u^2}} = \frac{Km}{J^2} \rightarrow \frac{d^2u}{d\theta^2} + u = \frac{1}{\ell} \quad \text{where } \ell = \frac{J^2}{mK}$$

Compare this w/ the eqn. for the sho: $\frac{d^2x}{dt^2} + \left(\frac{K}{m}\right)x = 0$

$\therefore$ Solution will be $u = A\cos(\theta - \theta_0) + \frac{1}{\ell}$

or $r = \frac{1}{A\cos(\theta-\theta_0) + \frac{1}{\ell}}$ where A & θ_0 are determined from initial conditions

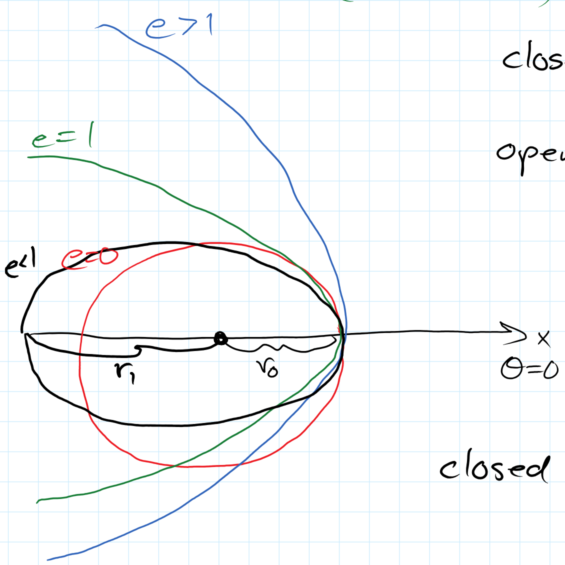
θ_0 sets the orientation wrt the axes, so set $\theta=0$ for this discussion

$$\therefore \quad r = \frac{1}{A\cos\theta + 1/\ell} = \frac{\ell}{A\ell\cos\theta + 1}$$

App. B tells us that this is an equation for a conic section in polar form.

$$r = r(\theta) = \frac{r_0(1+e)}{(1+e\cos\theta)}$$

where e, r_0 are real & positive & represent diff. types of conic sections



closed If $e=0$, a circle of radius r_0

open If $e>1$, a hyperbola where r_0 is the distance of closest approach to the origin

open If $e=1$, a parabola where r_0 ....

closed If $e<1$, an ellipse where r_0 is the distance of closest approach to origin

For orbits, $e=$ 'eccentricity' & $r_0=$ distance of closest approach

$$\therefore \quad e = A\ell = \frac{AJ^2}{mK} \; ; \quad \ell = r_0(1+e)$$

$$\Rightarrow r_0 = \frac{\ell}{1+e} = \frac{J^2}{mK(1+e)}$$ (the value of r when $\theta=0$)

For elliptical orbits at $\theta=\pi$, $r_1 = \dfrac{r_0(1+e)}{(1-e)}$

For orbits around the Sun, r_0 & r_1 are called the perihelion &

aphelion distances.

For orbits around the Earth, r_0 & r_1 are called the perigee & apogee distances

In general, they are the pericenter & apocenter distances

We will see that the energy of the object determines e

and therefore its orbit. The most general bound orbit will be an ellipse. Therefore, planetary orbits should be elliptical (Kepler's 1st Law)

Example: circular orbit, $e=0$

$$r_0 = \frac{J^2}{Km} = \frac{m^{\cancel{2}}(r_0^2\dot{\theta})^2}{K\cancel{m}} = \frac{m(r_0^2\dot{\theta})^2}{K} = \frac{m}{K} r_0^2 V_c^2$$ where V_c = circular velocity

$\Rightarrow V_c = \sqrt{\frac{K}{mr_0}}$. For gravity, $K = GMm$ & $V_c = \sqrt{\frac{GM}{r_0}}$

For Earth, GM_E can be re-written as $mg = \frac{GM_E m}{R_E^2}$ or $GM_E = g R_E^2$

$$\therefore V_c = \sqrt{\frac{g R_E^2}{r_0}}$$

For a satellite close to Earth's surface $V_c = \sqrt{g R_E} = 7.92 \text{ km s}^{-1}$