Answers

Sample Midterm 2A, Math 1554

PLEASE PRINT YOUR NAME CLEARLY IN ALL CAPITAL LETTERS

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Student Instructions

- **Show your work** and justify your answers for all questions unless stated otherwise.
- **Organize your work** in a reasonably neat and coherent way.
- Calculators, notes, cell phones, books are not allowed.
- Use dark and clear writing: your exam will be scanned into a digital system.
- Exam pages are double sided. Be sure to complete both sides.
- Leave a 1 inch border around the edges of exams.
- Any work done on scratch paper will not be collected and will not be graded.

Math 1554, Sample Midterm 2A. Your initials: ______________

You do not need to justify your reasoning for questions on this page.

1. (10 points) Indicate **true** if the statement is true, otherwise, indicate **false**.

	true	false
a) If a square matrix is not invertible, then it does not have an LU factorization.		
b) $H = \{\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathbb{R}^2 \mid x_1 + x_2 = 1\}$ is a subspace.		
c) If $A\vec{x} = A\vec{y}$ for some $\vec{x} \neq \vec{y} \in \mathbb{R}^n$, then A cannot be invertible.		
d) If A is non-singular, then A^n must be non-singular, for any integer $n > 1$.		
e) If A and B are square $n \times n$ matrices and $AB = I$, then A is invertible.		
f) All elementary matrices are triangluar.		
g) If $T_A : \mathbb{R}^3 \mapsto \mathbb{R}^4$ is one-to-one, then $\text{rank}(A) = 3$ and $\dim(\text{Null}(A))$ is 0.		
h) A 5×5 matrix A with rank 3 has an eigenvalue $\lambda = 0$.		
i) The algebraic multiplicity of an eigenvalue λ can be zero.		
j) If A is square and row equivalent to an identity matrix, then $\det(A) \neq 0$.		

Answers:

a) false: any matrix that can be reduced to echelon form without row swaps has an LU

b) false: the subset does not include the zero vector

c) true: $A\vec{x} = A\vec{y}$ for some $\vec{x} \neq \vec{y} \in \mathbb{R}^n$ implies that $T = A\vec{x}$ is not one-to-one.

d) true: think about how determinants can be used

e) true: B is the inverse of A and vice versa

f) false: think about the elementary matrices that perform a row swap

g) true: every column must be pivotal

h) true: having a zero eigenvalue implies that the matrix does not have a pivot in very column

i) false: algebraic multiplicity is always at least 1

j) true: A must be invertible, which implies that its determinant is non-zero.

2. (2 points) Fill in the blanks.

(a) $T_A = A\vec{x}$, where $A \in \mathbb{R}^{2\times 2}$, is a linear transform that rotates vectors in $\mathbb{R}^2$ clockwise by θ radians about the origin, then reflects them through the line $x_1 = 0$, then projects them onto the x_1-axis. Compute $\det(A)$. $\boxed{}$

(b) An eigenvector of $A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ -1 & 1 & -1 & 1 \\ 1 & -1 & -1 & 1 \end{pmatrix}$ is $\vec{v}_1 = \begin{pmatrix} 0 \\ -1 \\ 1 \\ 0 \end{pmatrix}$. What is the eigenvalue associated with $\vec{v}_1$? $\boxed{}$ *Hint: do not compute the characteristic polynomial!*

Answers:

a) Not necessary to show work, but here is one way at arriving at the answer:

$$\begin{aligned} \det A &= \det(A_{\text{Projection}} A_{\text{Reflection}} A_{\text{Rotation}}) \\ &= \det A_{\text{Projection}} \det A_{\text{Reflection}} \det A_{\text{Rotation}} \\ &= 0 \cdot (-1) \cdot (1) \\ &= 0 \end{aligned}$$

b) Multiplying Av_1 we obtain $Av_1 = -2v_1$, so the eigenvalue is -2.

You do not need to justify your reasoning for questions on this page.

3. (10 points) If possible, write down an example of a matrix with the following properties. If it is not possible to do so, write *not possible*. You do not need to justify your reasoning.

(a) If possible, give an example of a 2×2 matrix whose column space is the line $x_1 = -2x_2$, and whose null space is the line $x_1 = 4x_2$.

A possible answer:

$$A = \begin{pmatrix} 2 & -8 \\ -1 & 4 \end{pmatrix}$$

One way to contruct this matrix: a vector in Col$A$ is $\begin{pmatrix} -2 \\ 1 \end{pmatrix}$. Or any non-zero multiple of that vector can be used, such as $\begin{pmatrix} 2 \\ -1 \end{pmatrix}$. A vector in Null$A$ is $\begin{pmatrix} 4 \\ 1 \end{pmatrix}$. $\implies A = \begin{pmatrix} 2 & -8 \\ -1 & 4 \end{pmatrix}$. We can check to make sure $\begin{pmatrix} 4 \\ 1 \end{pmatrix}$ is in the null space by multiplying A with that vector. There are many other acceptable answers.

(b) A 3×7 matrix, A, in RREF, such that $\dim(\mathrm{Col}(A)) = 4$, and $\dim(\mathrm{Null}(A)) = 3$.

Answer: not possible (cannot have more than 3 pivots if there are 3 rows).

(c) A 3×5 matrix, A, in RREF, such that $\dim(\mathrm{Col}(A)) = 3$.

Answer: $A = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$, or $\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$, etc., many answers here.

(d) A 4×4 lower triangular matrix A, such that $\det(A) = -1$ and $\mathrm{rank}(A) = 4$.

Answer: $\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$ (many other acceptable answers here)

(e) A 3×3 matrix whose determinant is equal to zero, and whose null space is the plane $x_1 + 2x_2 + 3x_3 = 0$.

Answer: The conditions imply we need two free variables (x_2 and x_3). We need a 3x3 matrix that has one pivot, because

$$\begin{aligned} \mathrm{rank}A + \dim \mathrm{Null}A &= \#\text{ columns} \\ &= (\#\text{ basic vars}) + (\#\text{ free vars}) \\ &= (\#\text{ pivots}) + (\#\text{ non-pivots}) \end{aligned}$$

Our matrix can be: $A = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$. Any vector $\vec{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$ will give us the required relation $x_1 + 2x_2 + 3x_3 = 0$ with $A\vec{x} = \vec{0}$.

4. (3 points) Let A and B be $n \times n$ matrices such that $A^2 = B^2 = I_n$, $BA = I_n$, and

$$C = \begin{bmatrix} A & B \\ B & -A \end{bmatrix}$$

Express C^2 in terms of A and B. Simplify as much as possible. Show your work.

Answer:

$$C^2 = \begin{pmatrix} A & B \\ B & -A \end{pmatrix}\begin{pmatrix} A & B \\ B & -A \end{pmatrix} = \begin{pmatrix} A^2+B^2 & AB-BA \\ BA-AB & B^2+A^2 \end{pmatrix}$$

But $BA = I$, so $B = A^{-1}$. And $A^2 = B^2 = I_n$. We can simplify further:

$$\implies \begin{pmatrix} A^2+B^2 & AB-BA \\ BA-AB & B^2+A^2 \end{pmatrix} = \begin{pmatrix} I+I & I-I \\ I-I & I+I \end{pmatrix} = \begin{pmatrix} 2I & 0 \\ 0 & 2I \end{pmatrix} = 2I_{2n}$$

5. (5 points) If a square matrix A has eigenvalue 2 with eigenvector $\vec{x}$ and eigenvalue $-1/2$ with eigenvector $\vec{y}$, express the the following in terms of $\vec{x}$ and $\vec{y}$. It is not necessary to show your work here.

   (a) $A^3\vec{x} =$ $\boxed{}$

   (b) $A^3\vec{y} =$ $\boxed{}$

   (c) $A(\vec{x}+\vec{y}) =$ $\boxed{}$

   (d) $A^2(\vec{x}+\vec{y}) =$ $\boxed{}$

   (e) $A^{-1}(\vec{x}+\vec{y}) =$ $\boxed{}$

Answer:

1. $2^3\vec{x}$. *Note: it is perfectly acceptable to leave your answer as $2^3\vec{x}$, but you can write your answer as $8\vec{x}$ if you prefer. Likewise with the remaining questions.*
2. $(\frac{-1}{2})^3 y$
3. $2x - \frac{1}{2}y$
4. $4x + \frac{1}{4}y$
5. $\frac{1}{2}x - 2y$

6. (3 points) The vector $\vec{x} = (1, 12, 3)^T$ is in the span of $\mathcal{B} = \{\vec{v}_1, \vec{v}_2\}$, where $\vec{v}_1 = (1, 0, -1)^T$ and $\vec{v}_2 = (-1, 3, 2)^T$. Compute $[\vec{x}]_{\mathcal{B}}$.

Answer:

$$\left(\begin{array}{rr|r} 1 & -1 & 1 \\ 0 & 3 & 12 \\ -1 & 2 & 3 \end{array}\right) \sim \left(\begin{array}{rr|r} 1 & -1 & 1 \\ 0 & 1 & 4 \\ 0 & 1 & 4 \end{array}\right) \sim \left(\begin{array}{rr|r} 1 & 0 & 5 \\ 0 & 1 & 4 \\ 0 & 0 & 0 \end{array}\right)$$

$$\implies [\vec{x}_{\mathcal{B}}] = \begin{pmatrix} 5 \\ 4 \end{pmatrix}$$

7. (3 points) Compute the inverse of the matrix $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ -3 & 0 & 3 \end{pmatrix}$.

Answer:

$$\left(\begin{array}{rrr|c} 1 & 0 & 0 & \\ 0 & 2 & 0 & I_3 \\ -3 & 0 & 3 & \end{array}\right) \sim \left(\begin{array}{rrr|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1/2 & 0 \\ 0 & 0 & 3 & 3 & 0 & 1 \end{array}\right) \sim \left(\begin{array}{c|ccc} & 1 & 0 & 0 \\ I_3 & 0 & 1/2 & 0 \\ & 1 & 0 & 1/3 \end{array}\right)$$

$$\implies \text{inverse is} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1/2 & 0 \\ 1 & 0 & 1/3 \end{pmatrix}$$

8. (4 points) Suppose the determinant $\begin{vmatrix} a & b & c \\ 1 & 2 & 3 \\ 4 & 5 & 6 \end{vmatrix} = 5$, where a, b and c are real numbers. What is the determinant below equal to?

$$\begin{vmatrix} 1 & 2 & 3 \\ 2a & 2b & 2c \\ 5 & 7 & 9 \end{vmatrix}$$

Answer:

$$5 = \begin{vmatrix} a & b & c \\ 1 & 2 & 3 \\ 4 & 5 & 6 \end{vmatrix} = -\begin{vmatrix} 1 & 2 & 3 \\ a & b & c \\ 4 & 5 & 6 \end{vmatrix} = -\begin{vmatrix} 1 & 2 & 3 \\ a & b & c \\ 5 & 7 & 9 \end{vmatrix} = -\frac{1}{2}\begin{vmatrix} 1 & 2 & 3 \\ 2a & 2b & 2c \\ 5 & 7 & 9 \end{vmatrix}$$

$$\implies \begin{vmatrix} 1 & 2 & 3 \\ 2a & 2b & 2c \\ 5 & 7 & 9 \end{vmatrix} = -10$$

9. (4 points) Consider the sequence of row operations that reduce matrix A to echelon form, U.

$$A = \underbrace{\begin{pmatrix} 1 & 2 & 2 \\ -2 & 2 & 0 \\ 1 & 14 & 3 \end{pmatrix}}_{A} \sim \underbrace{\begin{pmatrix} 1 & 2 & 2 \\ 0 & 6 & 4 \\ 1 & 14 & 3 \end{pmatrix}}_{E_1A} \sim \underbrace{\begin{pmatrix} 1 & 2 & 2 \\ 0 & 6 & 4 \\ 0 & 12 & 1 \end{pmatrix}}_{E_2E_1A} \sim \underbrace{\begin{pmatrix} 1 & 2 & 2 \\ 0 & 6 & 4 \\ 0 & 0 & -7 \end{pmatrix}}_{E_3E_2E_1A} = U$$

(a) Construct the elementary matrices E_1, E_2, E_3.

$E_1 =$ $\qquad\qquad$ $E_2 =$ $\qquad\qquad$ $E_3 =$

(b) Consider the matrix products listed below. Which (if any) represents A?

I) $E_3E_2E_1U$
II) $E_1E_2E_3U$
III) $E_1^{-1}E_2^{-1}E_3^{-1}U$
IV) $E_3^{-1}E_2^{-1}E_1^{-1}U$

Answers: Part a) the three matrices are

$$E_1 = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad E_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix} \qquad E_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{pmatrix}$$

If you are not sure how one would construct these elementary matrices, try multiplying E_1A to verify that the product is what it should be. Likewise for the other products.

Part b) The third option is correct.

10. (4 points) Construct a basis for the eigenspace of A associated with the eigenvalue $\lambda = 3$.

$$A = \begin{pmatrix} 5 & -1 & 2 \\ 2 & 2 & 2 \\ 2 & -1 & 5 \end{pmatrix}$$

Answers:

$A - 3I = \begin{pmatrix} 2 & -1 & 2 \\ 2 & -1 & 2 \\ 2 & -1 & 2 \end{pmatrix} \sim \begin{pmatrix} 2 & -1 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \implies 2x_1 - x_2 + 2x_3 = 0$, so x_2, x_3 are free. We may choose any values for the free variables to construct two linearly independent vectors. Choosing 1's and 0's is an acceptable choice for the free variables, but we can choose anything we want for x_2 and x_3 as long as we create two indepdendent vectors.

$\implies \vec{v_1} = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}, \vec{v_2} = \begin{pmatrix} 1/2 \\ 1 \\ 0 \end{pmatrix}$ form an acceptable basis. But another acceptable answer is $\vec{v_1} = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \vec{v_2} = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}$. There are many acceptable answers here.