Damped Oscillator

If there is energy loss during the oscillation, there will be a frictional term in the e.o.m. Again, consider small displacements from equilibrium, so x & $\dot{x}$ will be small and therefore any non-linear terms (x^2, $x\dot{x}$ or $\dot{x}^2$) can be neglected. Thus, a damped harmonic oscillator has a $F = -Kx - \lambda\dot{x}$ where λ is a +ve constant

$\therefore$ e.o.m is $m\ddot{x} + \lambda\dot{x} + Kx = 0$ (LRC circuits, atomic physics)

Consider solutions of the form $\underline{x = e^{pt}}$ & substitute into e.o.m.

$$\rightarrow mp^2 + \lambda p + K = 0$$

solving this quadratic equation

$$p = -\gamma \pm \sqrt{\gamma^2 - \omega_0^2}$$

where $\gamma = \frac{\lambda}{2m}$ & $\omega_0 = \sqrt{\frac{K}{m}}$ (the freq. of the undamped oscillator)

3 Possible Cases

1) Overdamping ($\gamma > \omega_0$)

If λ is large enough then both roots are real & negative

$$p = -\gamma_\pm, \quad \gamma_\pm = \gamma \pm \sqrt{\gamma^2 - \omega_0^2}$$

and $x = \frac{1}{2}Ae^{-\gamma_+ t} + \frac{1}{2}Be^{-\gamma_- t}$ where A & B are constants & the $\frac{1}{2}$'s are going to be useful later

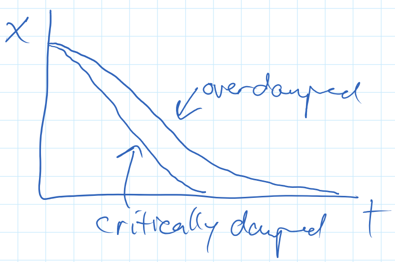
Displacement exponentially drops to zero, dominated by the γ_- term

2) Critical Damping ($\gamma = \omega_0$)

In this case $p = -\gamma$ & we only have $x = Ae^{-\gamma t}$ so we need another solution in order to have 2 arbitrary constants.

Verify that $x = te^{-pt}$ is also a solution, so the general solution is $\underline{x = (a+bt)e^{-\gamma t}}$

Motion is again non-oscillatory & displacement goes to zero faster than the overdamped case.



Critical damping is often ideal in measurement devices & in suspension systems.

3) Underdamped. If γ is small, so that $\gamma < \omega_0$, the roots (for p) are complex conjugates.

(See review of complex numbers)

so, $p = -\gamma \pm \sqrt{\gamma^2 - \omega_0^2} \rightarrow p = -\gamma \pm i\omega$ where

$$\omega = \sqrt{\omega_0^2 - \gamma^2}$$

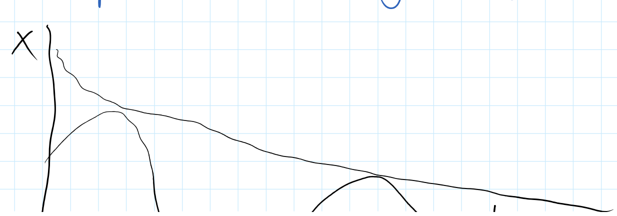
so, the general solution is

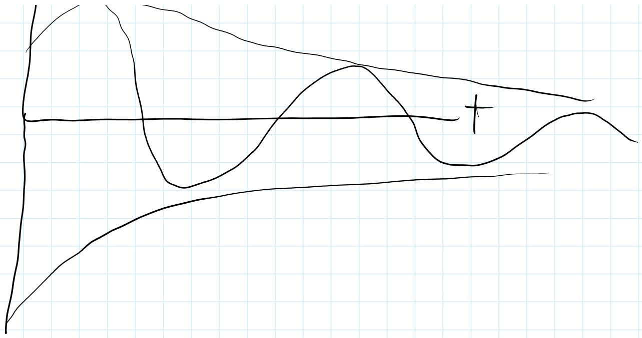
$$x = \tfrac{1}{2}Ae^{i\omega t - \gamma t} + \tfrac{1}{2}Be^{-i\omega t - \gamma t} \quad (A = ae^{-i\theta} \;\&\; B = ae^{i\theta})$$

Since these are complex conjugates

$$x = \mathrm{Re}\left(Ae^{i\omega t - \gamma t}\right) = ae^{-\gamma t}\cos(\omega t - \theta)$$

Thes solution are oscillatory w/ an exponentially decreasing amplitude & ang. freq. $\omega < \omega_0$





The time in which the amplitude is reduced by a factor of $1/e$ is called the relation time. $\gamma t_{rel} = 1 \Rightarrow t_{rel} = \frac{1}{\gamma}$

or $t_{rel} = \frac{2m}{\lambda}$

Another useful quantity is the quality factor Q of the oscillator. $Q = \frac{m\omega_0}{\lambda} = \frac{\omega_0}{2\gamma}$. If the damping is small then Q is large.

Why is Q useful?