I

From the polar eq'n $[r(\theta)]$, determine directions where $r \to \infty$

i.e. when $1 + e\cos\theta' = 0$

$$e\cos\theta' = -1$$

$$\cos\theta' = -\frac{1}{e}$$

Attractive case [$K<0$; LHS of prev. fig.]

$$\theta' = \cos^{-1}\left(-\frac{1}{e}\right) = \pm\left[\pi - \cos^{-1}\left(\frac{1}{e}\right)\right]$$

Repulsive case [$K>0$; RHS of fig.]

$$\theta' = \pm\cos^{-1}\left(\frac{1}{e}\right)$$

$\therefore$ The scattering angle Θ, through which it has been deflected from its original motion is $\Theta = \pi - 2\theta' = \pi - 2\pi + 2\cos^{-1}\left(\frac{1}{e}\right)$

$$= -\left(\pi - 2\cos^{-1}\left(\frac{1}{e}\right)\right)$$

Take abs. value $\to$ $\Theta = \pi - 2\cos^{-1}\left(\frac{1}{e}\right)$

Can relate Θ, b & v: $\Theta - \pi = -2\cos^{-1}\left(\frac{1}{e}\right)$

$$-\frac{1}{2}(\Theta - \pi) = \cos^{-1}\left(\frac{1}{e}\right)$$

$$\cos\left(-\frac{1}{2}(\Theta - \pi)\right) = \cos\left(\frac{1}{2}(\pi - \Theta)\right) = \frac{1}{e}$$

$$e = \sec\left(\frac{1}{2}(\pi - \Theta)\right)$$

trig. identity

Use the identity $b^2 = a^2(e^2 - 1) = a^2\left(\sec^2\left(\frac{1}{2}(\pi - \Theta)\right) - 1\right) = a^2\cot^2\frac{1}{2}\Theta$

but $a = \frac{|K|}{2E} = \frac{|K|}{mv^2}$ so $\boxed{b = \frac{|K|}{mv^2}\cot^2\left(\frac{\Theta}{2}\right)}$

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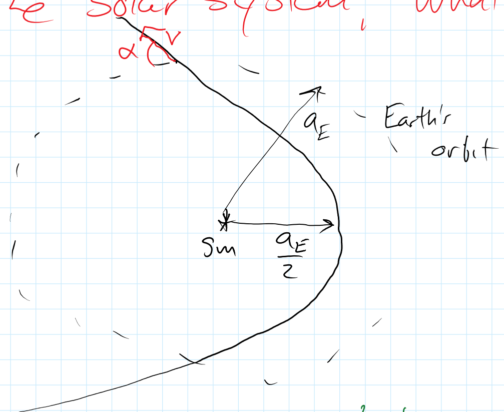
e.g. Orbit of an unbound comet

The min. distance of a comet from the Sun (perihelion distance) is $\frac{1}{2}$ the radius of the Earth's orbit (assumed circular) & its velocity at that point is twice the orbital velocity of the Earth. Find the velocity when the comet crosses Earth's orbit & the angle at which the orbits cross. Will the comet escape from the solar system? What kind of orbit does it follow?



At perihelion, $r = \frac{a_E}{2}$ & $v = 2v_{c,E}$ & $\dot{r} = 0$

PE of the comet at that location $= \frac{2GMm}{a_E}$

KE of the comet at that location $= \frac{1}{2}m(2v_{c,E})^2 = 2mv_{c,E}^2$

but, by definition, $v_{c,E}^2 = \frac{GM}{a_E}$, so $KE = 2m\frac{GM}{a_E}$

$\therefore E = \frac{2GMm}{a_E} - \frac{2GMm}{a_E} = 0$ So, parabolic orbit & comet will escape.

This E is the same at all locations, so at Earth's radius a_E

$$\frac{GM\cancel{m}}{a_E} = \frac{1}{2}\cancel{m}v^2 \rightarrow v^2 = \frac{2GM}{a_E} = 2v_{c,E}^2 \Rightarrow v = \sqrt{2}\,v_{c,E}$$

To find α, consider ang. mom at Earth's orbit:

$$J_E = m(v\cos\alpha)a_E = m\sqrt{2}\,v_{c,E}\cos\alpha\, a_E$$

At perihelion, $J_p = mv\frac{a_E}{2} = m\,2v_{c,E}\frac{a_E}{2} = m\,v_{c,E}\,a_E$

Equate these two expressions

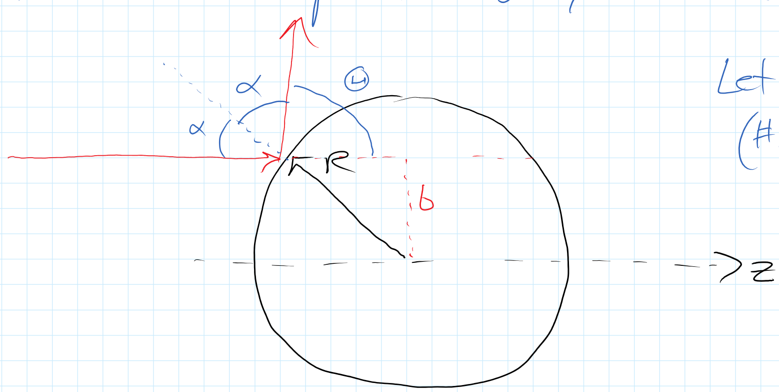
$$\cancel{m}\sqrt{2}\,\cancel{v_{c,E}}\cos\alpha\,\cancel{a_E} = \cancel{m}\cancel{v_{c,E}}\cancel{a_E}$$

$\cos\alpha = \frac{1}{\sqrt{2}} \rightarrow \alpha = 45^\circ$

Scattering Cross-Sections

Repulsive forces allow one to probe the structures of tiny objects by 'bouncing' other things off of them. Measure the dist'n of the scattered particles it tells you about the nature of the force & particles. This is a key concept in experimental physics.

Consider a simple, pefectly hard, fixed sphere of radius R and fire a beam of particles w/ a fixed rate at it



Let the particle flux in the beam (#particles/area/time) be f, then the rate of particles striking our sphere is

$$W = f\sigma$$

where σ = cross-sectional area presented by the sphere $= \pi R^2$

Consider a particle hitting the sphere at impact parameter $b = R\sin\alpha$

If the collision is completely elastic, then the KE & ang. mom. will be the same before & after the collision. ∴ Motion will be in a plane & 'reflection' angle will be equal to the incident angle.

∴ Particle will be deflectied through angle $\Theta = \pi - 2\alpha$

$$\alpha = \frac{(\pi - \Theta)}{2} \rightarrow b = R\sin\left(\frac{\pi}{2} - \frac{\Theta}{2}\right) = R\cos\left(+\frac{\Theta}{2}\right)$$