3.1 — a)

Yes. Let V be a subspace of $\mathbb{R}^2$. By theorem 12, $(0,0) \in V$.

If V contains only one element, then $V = \{(0,0)\}$, which is a trivial subspace.

Now, suppose that V contains more than one element, such that there exists $u = (x_0, y_0) \in V$ and $u \neq 0$.
Since V is a subspace, for any $\lambda \in \mathbb{R}$, $\lambda u \in V$.
Therefore, V contains the line connects the origin and u, call it L. As L goes through the origin, it is a subspace of $\mathbb{R}^2$.

If $L \not\subset V$, V contains another point $v = (x_1, y_1)$, which is not a multiple of u.
Assume that there is a point $(a,b) \in \mathbb{R}^2$, $(a,b) \in V$.
Then, if there exists $\lambda \in V$ and $\mu \in \mathbb{R}$ such that $(a,b) = \lambda u + \mu v$. In the linear system, it can be expressed as $\begin{cases} a = \lambda x_0 + \mu x_1 \\ b = \lambda y_0 + \mu y_1 \end{cases}$

Since u and v are linearly independent, the above system has a unique solution. Therefore, (a,b) is a linear combination of u and v, and the corresponding subspace is $\mathbb{R}^2$ itself.

Therefore, there can be a proper nontrivial subspace $V \subset \mathbb{R}^2$.

3.1 – b) Find a basis of the space spanned by $b_0, b_1, b_2, b_3, b_4 = A$

✓ Find a pivot columns → form a basis

$$\begin{bmatrix} 1/\sqrt{2} & 0 & 1/\sqrt{2} & 0 & 0 \\ 0 & 1/\sqrt{2} & 0 & 1/\sqrt{2} & 0 \\ 0 & 1/\sqrt{2} & 0 & 0 & 1/\sqrt{2} \\ 0 & 0 & 1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 1/\sqrt{2} & 0 & 0 & 1/\sqrt{2} & 0 \end{bmatrix} \quad R_5 \to R_5 - R_1$$

$$\Leftrightarrow \begin{bmatrix} 1/\sqrt{2} & 0 & 1/\sqrt{2} & 0 & 0 \\ 0 & 1/\sqrt{2} & 0 & 1/\sqrt{2} & 0 \\ 0 & 1/\sqrt{2} & 0 & 0 & 1/\sqrt{2} \\ 0 & 0 & 1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 0 & 0 & -1/\sqrt{2} & 1/\sqrt{2} & 0 \end{bmatrix} \begin{array}{l} \\ \\ R_3 \to R_3 - R_2 \\ \\ R_5 \to R_5 + R_4 \end{array}$$

$$\Leftrightarrow \begin{bmatrix} 1/\sqrt{2} & 0 & 1/\sqrt{2} & 0 & 0 \\ 0 & 1/\sqrt{2} & 0 & 1/\sqrt{2} & 0 \\ 0 & 0 & 0 & -1/\sqrt{2} & 1/\sqrt{2} \\ 0 & 0 & 1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 0 & 0 & 0 & 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} \begin{array}{l} \\ \\ R_3 \\ R_4 \end{array} \text{exchange}$$

$$\Leftrightarrow \begin{bmatrix} 1/\sqrt{2} & 0 & 1/\sqrt{2} & 0 & 0 \\ 0 & 1/\sqrt{2} & 0 & 1/\sqrt{2} & 0 \\ 0 & 0 & 1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 0 & 0 & 0 & -1/\sqrt{2} & 1/\sqrt{2} \\ 0 & 0 & 0 & 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} \quad R_5 \to R_5 + R_4$$

③

$$\Leftrightarrow \begin{bmatrix} 1/\sqrt{2} & 0 & 1/\sqrt{2} & 0 & 0 \\ 0 & 1/\sqrt{2} & 0 & 1/\sqrt{2} & 0 \\ 0 & 0 & 1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 0 & 0 & 0 & -1/\sqrt{2} & 1/\sqrt{2} \\ 0 & 0 & 0 & 0 & 2/\sqrt{2} \end{bmatrix}$$

pivot columns $\Rightarrow$ all columns form the basis

$$\det(A) = (-1)^1 \cdot \left(\frac{1}{\sqrt{2}} \cdot \frac{1}{\sqrt{2}} \cdot \frac{1}{\sqrt{2}} \cdot \left(-\frac{1}{\sqrt{2}}\right) \cdot \frac{2}{\sqrt{2}}\right)$$

$$= (+1) \cdot \left(\frac{1}{2} \cdot \left(+\frac{1}{\cancel{2}}\right) \cdot \frac{\cancel{2}}{\sqrt{2}}\right) = \frac{1}{2\sqrt{2}} = \frac{\sqrt{2}}{4}$$

$$\therefore A = \begin{bmatrix} 1/\sqrt{2} & 0 & 1/\sqrt{2} & 0 & 0 \\ 0 & 1/\sqrt{2} & 0 & 1/\sqrt{2} & 0 \\ 0 & 1/\sqrt{2} & 0 & 0 & 1/\sqrt{2} \\ 0 & 0 & 1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 1/\sqrt{2} & 0 & 0 & 1/\sqrt{2} & 0 \end{bmatrix}, \quad \det(A) = \sqrt{2}/4$$