# Condensation of SIP particles and sticky Brownian motion

Webinar: Inclusion Process and Sticky Brownian Motions

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# Outline

- Symmetric Inclusion Process.
- Condensation of IPS.
- Condensation of SIP particles (Goal).
- Duality
- Two-particles convergence
- Further consequences

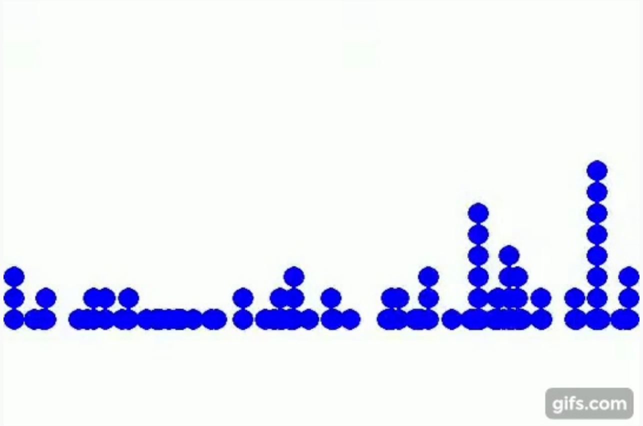
# Symmetric Inclusion Process

# Symmetric Inclusion Process (SIP($\alpha$))

- IPS introduced in [Giardinà, Kurchan, Redig '07].
- State space $\Omega = \mathbb{N}^S$ with $S \subseteq \mathbb{Z}$
- Markov process $\{\eta_t : t \geq 0\}$ where for $i \in S$, $\eta_i$ = number of particles.
- Infinitesimal generator

$$\mathscr{L}f(\eta) = \sum_{i,j \in S} p(j-i)\eta_i(\alpha+\eta_j)(f(\eta^{i,j})-f(\eta))$$

where $n^{i,j} = \eta - \delta_i + \delta_j$



# SIP($\alpha$): Dynamics

$$\mathscr{L}f(\eta) = \alpha\,\mathscr{L}^{\mathrm{IRW}} f(\eta) + \mathscr{L}^{\mathrm{IN}} f(\eta)$$

- Independent part:

$$\mathscr{L}^{\mathrm{IRW}} f(\eta) = \sum_{i,j\in\mathbb{Z}} p(j-i)\eta_i(f(\eta^{ij}) - f(\eta))$$

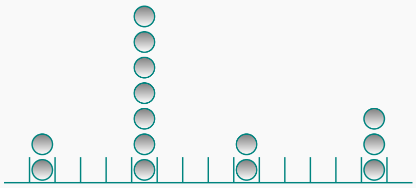
- Purely inclusion part:

$$\mathscr{L}^{\mathrm{IN}} f(\eta) = \sum_{i,j\in\mathbb{Z}} p(j-i)\eta_i\eta_j(f(\eta^{ij}) - f(\eta))$$

# CONDENSATION

# Condensation

Accumulation of particles



# Condensation on IPS

## Probabilistic

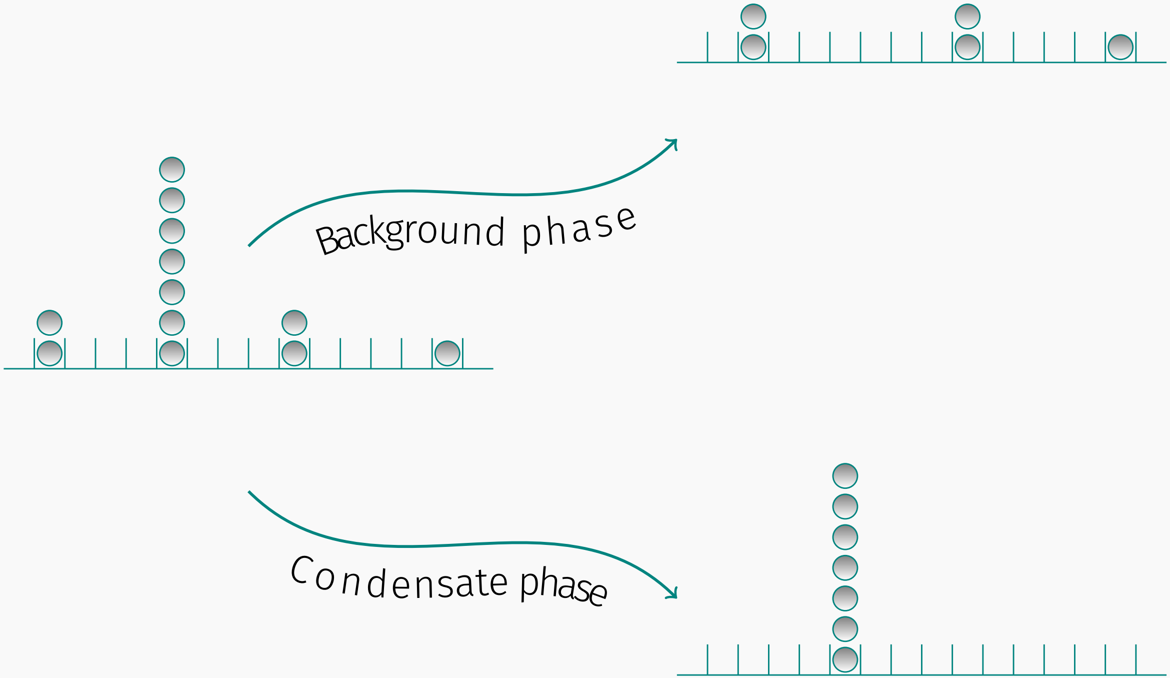
- Phase transition:

$$I_e = \{\mu_\rho : \rho \in [0, \eta_{\max}]\}$$

- Condensation:
    - Unbounded local state.
    - $\exists \rho_c < \infty$ such that

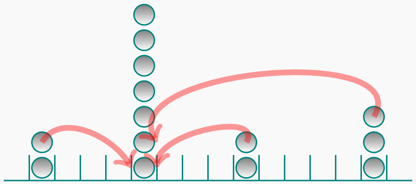
$$\text{No extremal invariant measures for } \rho > \rho_c$$

# Phase separation

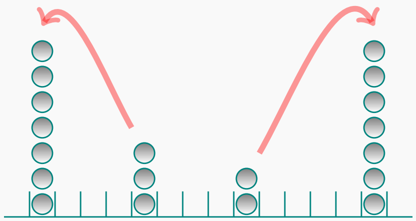


# Some possible reasons

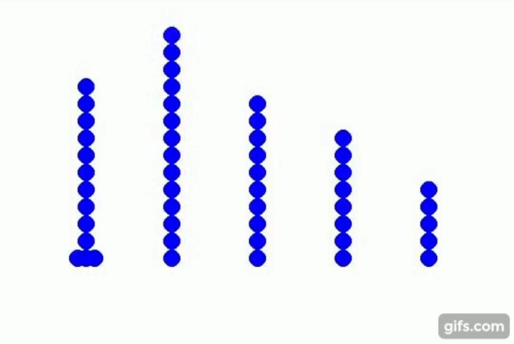
- Spatial inhomogeneities:

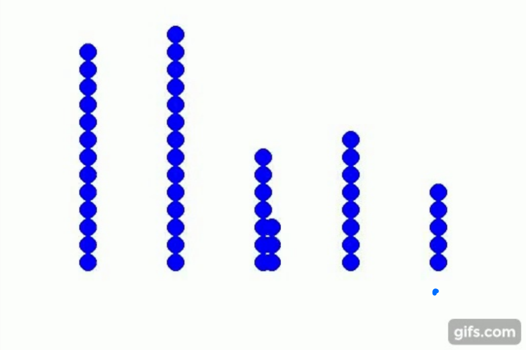


- Particle interactions.



# SIP: small parameter $\alpha$




$$\mathscr{L} f(\eta) = \alpha \mathscr{L}^{\mathrm{IRW}} f(\eta) + \mathscr{L}^{\mathrm{IN}} f(\eta)$$

# SIP: condensation regime

**Rescaling parameter $n \to \infty$:**

- Condensive rescaling:

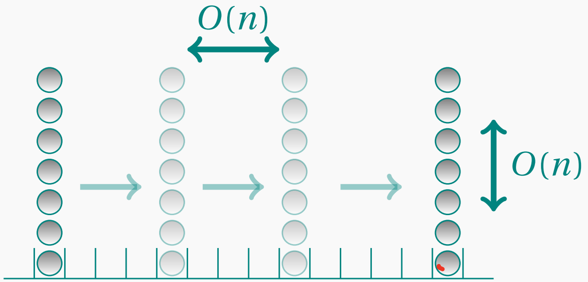
$$\frac{1}{n}\mathbb{Z} \to \mathbb{R}, \quad s(n,t) = \frac{n^3\gamma}{\sqrt{2}}t, \qquad \alpha(n) = \frac{1}{\sqrt{2}\gamma n}$$

- $\alpha(n) \to 0$. (Random Walk << Inclusion Part.)

$$\mathcal{L}_n^{\mathrm{IRW}} f(\eta) = \frac{n^2}{2} \sum_{i,j \in S(n)} p(j-i)\eta_i (f(\eta^{i,j}) - f(\eta))$$

$$\mathcal{L}_n^{\mathrm{IN}} f(\eta) = \frac{n^3\gamma}{\sqrt{2}} \sum_{i,j \in S(n)} p(j-i)\eta_i\eta_j (f(\eta^{i,j}) - f(\eta))$$

# Stationary level:[Grosskinsky-Redig-Vafayi'11]




Approximate infinite volume:

$$\mathcal{X}_{S_n} := \frac{1}{S_n} \sum_{i=1}^{S_n} \eta_i$$

where for a fixed $\rho > 0$, initially started from a stationary spatially homogeneous product measure:

$$\nu_\rho(\eta) = \prod_{x \in S} \mu_\rho(\eta_x)$$

# Goal

# Goal

Understand the coarsening behaviour of the $SIP(\alpha)$ in the condensation regime directly on the infinite lattice $\mathbb{Z}$.

# Density fluctuation field

In the condensation regime we define:

$$\mathscr{Y}_n(\eta,\phi,t) := \frac{1}{n}\sum_{x\in\mathbb{Z}} \phi(x/n)\left(\eta_x(s(n,t)) - \rho\right)$$

Where $\phi \in S(\mathbb{R})$.

**Remark:**

In non-condensive settings the scaling limit of the density fluctuation field is known:

$$\mathscr{X}_n(\eta,\phi,t) = \frac{1}{\sqrt{n}}\sum_{x\in\mathbb{Z}} \phi(x/n)\left(\eta_x(n^2 t) - \rho\right)$$

# Density fluctuation field

## Specific goal

To compute:

$$\lim_{n\to\infty} \mathbb{E}_\nu \left[ \mathscr{Y}_n(\eta,\phi,t)^2 \right]$$

Spoilers [A.,Carinci, Redig '21]:

$$\lim_{n\to\infty} \mathbb{E}_\nu \left[ \mathscr{Y}_n(\eta,\phi,t)^2 \right]$$
$$= \sqrt{2}\gamma\rho^2 \left(1 - p_t^{\mathrm{sbm}}(0,0)\right) \int_{\mathbb{R}} \phi(u)^2\, du - \frac{\rho^2}{2} \int_{\mathbb{R}^2} \phi(\tfrac{u+v}{2})\phi(\tfrac{u-v}{2}) p_t^{\mathrm{sbm}}(v,0)\, dv\, du$$

# Compute:

$$\mathbb{E}_\nu\left[\mathcal{Y}_n(\eta,\phi,t)^2\right]$$
$$= \frac{1}{n^2}\sum_{x,y\in\mathbb{Z}}\phi(\tfrac{x}{n})\phi(\tfrac{y}{n})\int\mathbb{E}_\eta\left(\eta_{s(n,t)}(x)-\rho\right)\left(\eta_{s(n,t)}(y)-\rho\right)\nu(d\eta)$$

# Duality

# Duality

$\xi_t$ is dual to $\eta_t$ with duality function

$$D : \Omega^{\text{dual}} \times \Omega \to \mathbb{R} \tag{1}$$

if for all $t \geq 0$

$$\mathbb{E}_\xi \left[ D(\xi_t, \eta) \right] = \mathbb{E}_\eta \left[ D(\xi, \eta_t) \right] \tag{2}$$

# Self-duality

## Self-duality of the SIP($\alpha$)

Let $\eta(t)$ and $\xi(t)$ be two copies of the SIP($\alpha$). The process is self-dual with duality function:

$$D(\xi,\eta) = \prod_x \frac{\eta_x!}{(\eta_x - \xi_x)!} \frac{\Gamma(\alpha)}{\Gamma(\alpha+\xi_x)}$$

$k$-point correlation functions

$$D(\delta_x,\eta) = \tfrac{1}{\alpha}\eta_x \qquad \text{and} \qquad D(\delta_x+\delta_y,\eta) = \begin{cases} \frac{1}{\alpha^2}\eta_x\eta_y & \text{if } x \neq y \\ \frac{1}{\alpha(\alpha+1)}\eta_x(\eta_x-1) & \text{if } x = y \end{cases}$$

# Application of self-duality

Let $\{\eta(t) : t \geq 0\}$ be a process with infinitesimal generator $\mathscr{L}$, then

$$\int \mathbb{E}_\eta \left(\eta_t(x) - \rho\right)\left(\eta_t(y) - \rho\right) \nu(d\eta)$$
$$= \left(1 + \frac{1}{\alpha}\mathbf{1}_{\{x=y\}}\right)\left(\frac{\alpha\sigma}{\alpha+1} - \rho^2\right)\mathbb{E}_{x,y}\left[\mathbf{1}_{\{X_t=Y_t\}}\right] + \mathbf{1}_{\{x=y\}}\left(\frac{\rho^2}{\alpha} + \rho\right)$$

where $\nu$ is a homogeneous product measure with:

$$\rho := \int \eta_x \nu(d\eta) \qquad \text{and} \qquad \sigma := \int \eta_x(\eta_x - 1)\nu(d\eta)$$

# Time variances of the density fluctuation field

$$
\begin{aligned}
&\mathbb{E}_\nu\left[\mathcal{Y}_n(\eta,\phi,t)^2\right] \\
&= \frac{1}{n^2}\sum_{x,y\in\mathbb{Z}}\phi(\tfrac{x}{n})\phi(\tfrac{y}{n})\left(\frac{\sigma}{1+\sqrt{2}\gamma n}-\rho^2\right)p_{s(n,t)}(x-y,0) \\
&+\frac{\sqrt{2}\gamma}{n}\sum_{x\in\mathbb{Z}}\phi(\tfrac{x}{n})^2\left(\frac{\sigma}{1+\sqrt{2}\gamma n}-\rho^2\right)p_{s(n,t)}(0,0) \\
&+\frac{1}{n^2}\sum_{x\in\mathbb{Z}}\phi(\tfrac{x}{n})^2\left(\sqrt{2}\gamma n\rho^2+\rho\right)
\end{aligned}
$$

where:

$$
\mathbb{E}_{x,y}\left[\mathbf{1}_{\{X_{s(n,t)}=Y_{s(n,t)}\}}\right]=p_{s(n,t)}(x-y,0)
$$

# Two SIP particles

# Two-particles scaling limit

The difference process:

$$w_n(t) := \frac{1}{n}\left(x_1(s(n,t)) - x_2(s(n,t))\right)$$

Generator:

$$\begin{aligned}(L_n f)(w) &= n^2\left[f(w+1) - 2f(w) + f(w-1)\right] \\ &+ \frac{2n^3\gamma}{\sqrt{2}}\mathbf{1}_{w=1}\left[f(w+1) - 2f(w) + f(w-1)\right]\end{aligned}$$

for $w \in \frac{1}{n}\mathbb{Z}$

# Mosco convergence for SIP

**Theorem [A.,Carinci, Redig '21]**

Let $(\mathcal{E}_n, D(\mathcal{E}_n))$ the sequence of Dirichlet forms associated to the process given by the difference of two SIP particles, in the condensation regime, on $\mathbb{Z}$ with finite-range interaction. Then, as $n \to \infty$

$$(\mathcal{E}_n, D(\mathcal{E}_n)) \to (\mathcal{E}, D(\mathcal{E}))$$

where $(\mathcal{E}, D(\mathcal{E}))$ denotes the Dirichlet form associated to the two-sided sticky Brownian motion with sticky parameter $\sqrt{2}\gamma$.

# Consequences of Mosco convergence

As a consequence, for $f_N = \mathbf{1}_{\{0\}}$ we have that the sequence

$$T_n(t)f_n(w) = \mathbb{E}_w^n\left[\mathbf{1}_{\{0\}}(w_t)\right] = p_{s(n,t)}(w,0)$$

converges strongly to

$$T_t f(w) = \mathbb{E}_w^{\mathrm{sbm}}\left[\mathbf{1}_{\{0\}}(w_t)\right] = p_t^{\mathrm{sbm}}(w,0)$$

we even get convergence

$$p_{s(n,t)}(0,0) \to p_t^{\mathrm{sbm}}(0,0)$$

# Heuristics (conjecture)

# Consequences

Non-centered field:

$$\mathcal{Z}_n(\eta,\phi,t) = \frac{1}{n}\sum_{x\in\mathbb{Z}}\phi(x/n)\eta_x(s(n,t))$$

**Theorem (A.Carinci,Redig'21)**

*Let $\nu_\rho$ be as before, then:*

$$\lim_{n\to\infty}\mathbb{E}_\nu\left[\mathcal{Z}_n(\eta,\phi,t)\right] = \rho\int_{\mathbb{R}}\phi(x)\,dx$$

*and*

$$\lim_{n\to\infty}\mathbb{E}_\nu\left[\mathcal{Z}_n(\eta,\phi,t)\right] = \rho^2\int_{\mathbb{R}^2}\phi(y)\,\phi(x)\,dx\,dy\int_{\mathbb{R}}p_t^{sbm}(z,x-y)\,dz$$

# Heuristics

Previous result is compatible with macroscopic field:

$$\mathcal{Z}^{(n)}(\phi, t) = \frac{\rho}{n} \sum_{i=1}^{n} \int_{\mathbb{R}} \phi(X_i^x(t))\, dx$$

where $(X_1^{x_1}(t), \ldots, X_n^{x_n}(t))$ is an $n$-dimensional diffusion process with:

1. $X_i^{x_i}(t) \sim B^{x_i}(t)$.
2. $(X_i^{x_i}(t) - X_j^{x_j}(t), X_i^{x_i}(t) + X_j^{x_j}(t)) \sim (B^{\mathrm{sbm}, x_i - x_j}(t), \bar{B}^{x_i + x_j}(2t - T(t))$.

Not enough! Need higher-order moments!

Thanks

# Extra slides

# Mosco convergence: varying spaces case

Let $\{(E_n, D(E_n))\}_n$ be a a sequence of Dirichlet forms on a sequence of Hilbert spaces $H_n$, and $(E, D(E))$ be a Dirichlet form on $H$.

[Mosco '94, Kuwae-Shioya '03]

We say that $\{(E_n, D(E_n))\}_n$ Mosco-converges to $(E, D(E))$ if:

- Convergence of Hilbert spaces: $H_n \xrightarrow[n\to\infty]{} H$
- Mosco I. $E(f) \leq \liminf_{n\to\infty} E_n(f_n) \qquad \forall f_n \xrightarrow[n\to\infty]{\text{weakly}} f \in H$
- Mosco II. $\forall f \in H, \exists f_n \in H_n$ s.t. $\quad f_n \xrightarrow[n\to\infty]{\text{strongly}} f$ and

$$E(f) = \lim_{n\to\infty} E_n(f_n)$$

# Convergence of Semigroups

**Strong convergence of semigroups**

The following statements are equivalent:

- $\{(\mathscr{E}_n, D(\mathscr{E}_n))\}_n$ Mosco-converges to $\{(\mathscr{E}, D(\mathscr{E}))\}$.
- The associated sequence of semigroups $\{T_n(t)\}_n$ strongly-converges to the semigroup $T(t)$ for every $t > 0$.

# Setting: Discrete vs Continuous

Difference process:

- Hilbert space:

$$H_n = L^2(\tfrac{1}{n}\mathbb{Z}, \nu_{\gamma,n})$$

- Measure:

$$\nu_{\gamma,n} = \tfrac{1}{n} + \gamma\delta_0$$

- Inner product:

$$\langle f, g\rangle_{H_n} = \sum_{w\in\frac{1}{n}\mathbb{Z}} f(w)g(w)\,\nu_{\gamma,n}(w)$$

- Dirichlet form:

$$\mathcal{E}_n(f) = -\langle f, L_n f\rangle_{H_n}$$

Sticky Brownian motion:

- Hilbert space:

$$H = L^2(\mathbb{R}, \nu_\gamma)$$

- Measure:

$$\nu_\gamma = dx + \gamma\delta_0$$

- Inner product:

$$\langle f, g\rangle_H = \int_{\mathbb{R}} f(x)g(x)\,\nu_\gamma(dx)$$

- Dirichlet form:

$$\mathcal{E}(f) = \int_{\mathbb{R}\setminus\{0\}} f'(x)^2\,\nu_\gamma(dx)$$

# Stationary level:[Grosskinsky-Redig-Vafayi'11]

Approximate infinite volume:

$$\mathscr{X}_{S_n} := \frac{1}{S_n}\sum_{i=1}^{S_n}\eta_i \to \begin{cases} 0 & \text{if } S_n << 1/\alpha(n) \\ X_\rho & \text{if } S_n \sim 1/\alpha(n) \\ \rho & \text{if } S_n >> 1/\alpha(n) \end{cases}$$

where $X_\rho$ is Gamma random variable with mean $\rho$.