Ramsey Theory

Graphs, Equations, and Rainbows

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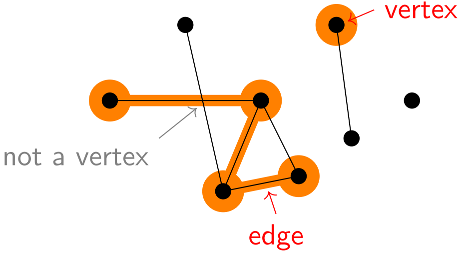
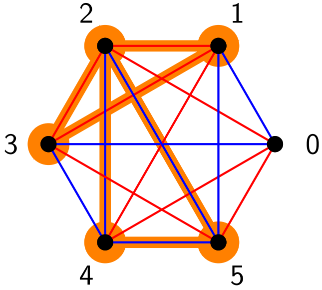
October 29, 2024

What is a Graph?

A bunch of dots and lines connecting some of them:



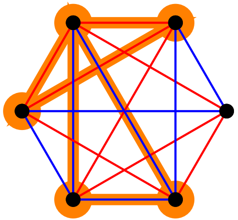
Subgraph: any subset of vertices and edges that is itself a graph



This edge-colored graph contains two monochromatic triangle subgraphs.

Minimizing Monochromatic Subgraphs

Edge colorings of K_n (graph with all possible edges):



This edge-colored graph (K_6) contains two monochromatic triangle (K_3) subgraphs.

Theorem (Ramsey, 1928)

For every graph H, there exists an n so that every possible coloring of the edges of a K_n contains a monochromatic subgraph H.

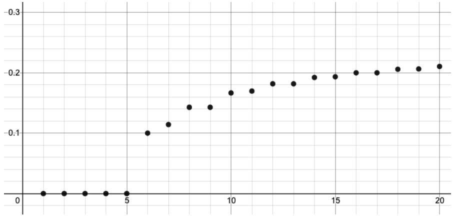
Traditional Ramsey theory: Find the smallest such n.

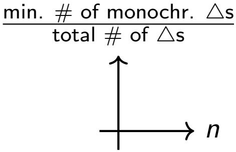
Ramsey multiplicities: How many monochromatic Hs are guaranteed as $n \to \infty$?

A Fundamental Example

Question

What is the minimum number of monochromatic triangles in a coloring of K_n? Stated another way: what *fraction* of triangles are guaranteed to be monochromatic? 2/3? 1/2? 21%?





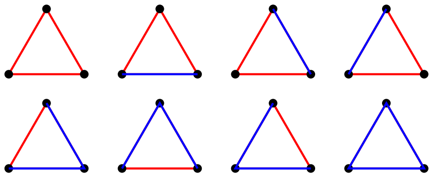
Asymptotic questions: Does this fraction approach something as $n \to \infty$? If so, what? (Less concerned with exact value.)

The Best Way to Color?

What kind of colorings produce these minimum values?

One way to color (not necessarily best): randomly

Probability that some fixed triangle in a randomly colored K_n is monochromatic:



$$\frac{1 \text{ red } \triangle + 1 \text{ blue } \triangle}{8 \text{ total colorings}} = \frac{1}{4}$$

$\Rightarrow$ We *expect* $1/4$ of all $\triangle$s to be monochromatic in a randomly colored K_n

Is random the best way?

Yes

Theorem (Goodman, 1959)

$$\frac{\textit{min. \# of monochr. } \triangle\textit{s in } K_n}{\textit{total \# of } \triangle\textit{s in } K_n} \to \frac{1}{4} \textit{ as } n \to \infty.$$

Takeaway: Every coloring of K_n has at least as many monochromatic triangles as what you would expect from a random coloring, i.e. random colorings achieve the (asymptotic) minimum.

We refer to a graph with this property as **common**. So the graph  is common.

What other graphs are common?

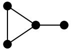
A Classification Question

"Triangle" (K_3) can be replaced with any fixed graph H.

Question

For which graphs H is the minimum number of monochromatic copies of H in K_n (asymptotically) the same as what you would expect from a random coloring, i.e. which graphs are common?

Some results:

- Common: paths, cycles, trees, even-spoked wheels
- Uncommon:  , K_s for $s \geq 4$
- Unknown: bipartite graphs (see Sidorenko's conjecture)

Avoiding Monochromatic Solutions to $x + y = z$

Question

Suppose we have 2 colors and want to color $[n] = \{1, 2, \ldots, n\}$ in order to minimize the number of monochromatic solutions to $x + y = z$.

- What is the minimum fraction of monochromatic solutions?
- Which coloring(s) achieve(s) this minimum?

Notation: A solution is a triple (x, y, z), e.g. $(1, 3, 4)$ is a solution.

Example: $n = 5$ 1 2 3 4 5

$(1, 1, 2)$ $(1, 2, 3)$ $(2, 1, 3)$ $(1, 3, 4)$ $(2, 2, 4)$
$(3, 1, 4)$ $(1, 4, 5)$ $(2, 3, 5)$ $(3, 2, 5)$ $(4, 1, 5)$ mc. frac. $= \frac{2}{10}$.

Question: Does the minimum fraction of monochromatic solutions approach something as $n \to \infty$? If so, what?

Best Way to Color

Question: If we colored each element of $[n]$ or $\mathbb{Z}_n$ uniformly randomly and independently, what would we expect the monochromatic fraction to be?

Answer: $1/4$. An arbitrary* solution (x, y, z) has a $1/4$ probability of being monochromatic.

Question: Is this also the minimum?

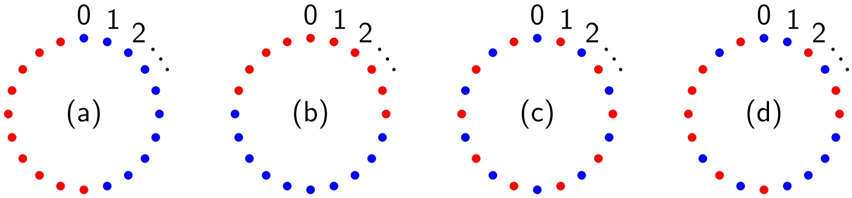
Theorem (Robertson-Zeilberger, 1998)

The minimum fraction of monochromatic solutions to $x + y = z$ over $[n]$ approaches $2/11$ as $n \to \infty$.

This coloring achieves the minimum:

Pop Quiz

Which coloring of $\mathbb{Z}_n$ do you think minimizes the fraction of monochromatic solutions to $x + y = z$? (e.g. $n = 12$: $8 + 6 = 2$)



Answer: All achieve the minimum!

Theorem (Datskovsky, 2003)

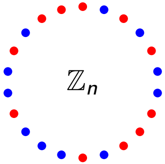
The minimum fraction of monochromatic solutions is $1/4$, *which is achieved whenever there are an equal number of blues and reds.*

Comparing to Random Colorings

Recall: Over $[n]$ or $\mathbb{Z}_n$, the expected proportion of monochromatic solutions to $x + y = z$ approaches $1/4$ as $n \to \infty$.

$[n]$

min. monochr. $\sim 2/11 < 1/4$



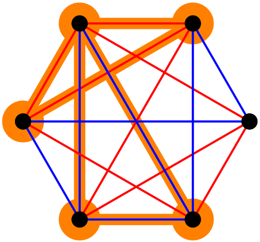
min. monochr. $\sim 1/4$

Definition: Uncommon: random is not best, **Common**: random is best.

Flipping the Script

Recall goal:

- Minimize the number of monochr. H inside a large K_n, or
- Minimize the number of monochr. solns. to an equation over $[n]$ or $\mathbb{Z}_n$.



1 2 3 4 5 6
(2,2,4) (1,5,6)

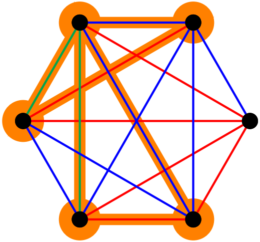
"Flipped" goal:

- *Maximize* the number of *rainbow* H inside a large K_n, or
- *Maximize* the number of *rainbow* solns. to an equation over $[n]$ or $\mathbb{Z}_n$.

Examples

"Flipped" goal:

- *Maximize* the number of *rainbow* H inside a large K_n, or
- *Maximize* the number of *rainbow* solns. to an equation over $[n]$ or $\mathbb{Z}_n$.



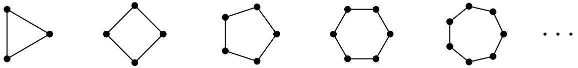
1 2 3 4 5 6
(1,2,3) (1,5,6)

Definition: A graph/equation is **rainbow-common** if the maximum number of rainbow subgraphs/solutions is maximized by uniformly random colorings.

Rainbow Classification of Graphs

Conjecture

A graph is rainbow-common if and only if it is a forest (contains no cycles).

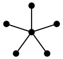
Cycles: 

Theorem (Sun 2024)

If a graph contains a cycle, then it is rainbow-uncommon.

What's left to prove

If a graph is a forest (contains no cycles), then it is rainbow-common.

Known rainbow-common: Stars  (DeSilva et al 2019)

Unknown: Everything else (without a cycle), e.g. P_4: 

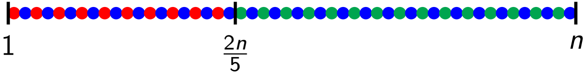
Rainbow Solutions to $x + y = z$

Question

What is the maximum number (or proportion) of rainbow solutions to $x + y = z$?

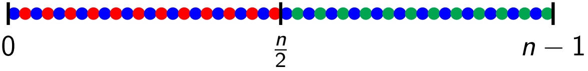
Answer:

- Over $[n]$: nobody knows



rb. prop. $\to \frac{2}{5}$ as $n \to \infty$

- Over $\mathbb{Z}_n$: nobody knows



rb. prop. $\to \frac{3}{8}$ as $n \to \infty$

Everything is Uncommon?

Recall: Every graph that contains a cycle is rainbow-uncommon.

Conjecture

Every equation is rainbow-uncommon.

Questions:

- What is the analogy of "containing a cycle" for an equation?
- Do all equations "contain a cycle"?

List of Open Questions

- Prove (or disprove) P_4 is rainbow-common when using 3 colors,
- Show certain equations are rainbow-uncommon, i.e. given an equation, find a coloring that asymptotically beats uniformly random colorings,
- Prove (or disprove) the max. prop. of rainbow solutions to $x + y = z$
  - $\to 2/5$ as $n \to \infty$, over $[n]$,
  - $\to 3/8$ as $n \to \infty$, over $\mathbb{Z}_n$,
- Prove (or disprove) that for every equation over $[n]$,

$$\lim_{n\to\infty} \frac{\text{min. \# monochr. solns.}}{\text{\# solns.}} \quad \text{exists,}$$

- Determine whether or not there is a useful correspondence between graphs and equations when using more than 2 colors (minimizing monochr. or maximizing rainbow),
- Prove (or disprove) every equation is rainbow-uncommon (over $[n]$ or $\mathbb{Z}_n$).

Thank You!

Questions?

For references, contact `Gabriel.Elvin@csusb.edu`