When Graphs Meet Matrices

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Graphs

Example

A graph consists of

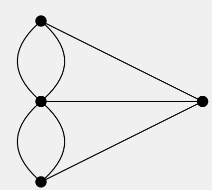
- dots (vertices) and
- line segments (edges) connecting (some of) them.

Definition

> **Definition:** A **graph** G is an ordered pair (V, E) where:
>
> - V is a non-empty finite set of **vertices**.
> - E is a set of **edges**, where each edge connects two vertices.

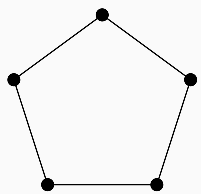
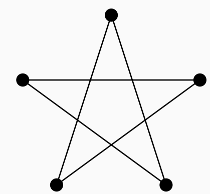
Edges are represented as unordered pairs $\{u, v\}$, where $u, v \in V$.

Non-example



- Multiple edges between vertices

Example

G1	G2
	

Question: Are they the "same"?

- When representing graphs we are free to put vertices wherever we like.
- Roughly speaking, graphs $G1$ and $G2$ are **isomorphic** (the sameness in graphs) if we can get $G1$ by relabeling $G2$.

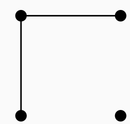
Fact: Isomorphic graphs share the graph-theoretic properties, e.g., number of vertices, edges, etc.

Connected Graph

Definition:

- A graph is **connected** if there is a path between every pair of vertices.
- A **path** is a sequence of edges that joins a sequence of distinct vertices.

Example (Disconnected):

disconnected graph


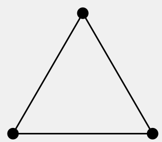
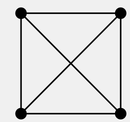
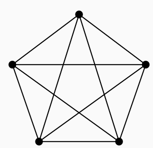
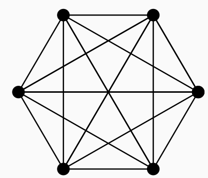
Common Graphs

Path Graph P_n

- Vertices connected in a single line.
- n vertices and $n - 1$ edges.

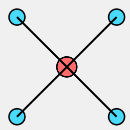
Complete Graph K_n

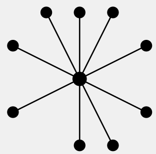
- Every pair of distinct vertices is connected.
- $\frac{n(n-1)}{2}$ edges.

Star Graph $K_{1,n}$

- One central vertex connected to all others.
- n vertices and $n-1$ edges.





Remark:

- Star graphs are special cases of tree and bipartite graphs.

Remark

> - Often we can reduce to simple graphs.
> - Isomorphic ("same") graphs have the same graph-theoretic properties such as connectedness, degree sequences, etc.
>   - Given two graphs, determining if they are isomorphic is a challenging task.

That is, we study simple graphs up to isomorphism. Also, it is difficult to find a new property that holds for all graphs. We study

> - a family of graphs such as path, complete, star graphs to name a few
> - or graphs satisfying certain properties like the planar graphs (e.g., Euler's formula).

Linear Algebra

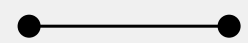
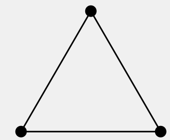
Adjacency Matrix

> **Definition**: For a graph G with n vertices, the **adjacency** matrix $A = (a_{ij})$ is an $n \times n$ matrix where:
>
> $$a_{ij} = \begin{cases} 1 & \text{if there is an edge between vertices } i \text{ and } j, \\ 0 & \text{otherwise.} \end{cases}$$

Properties:

- Entries are either 0 or 1
- Graphs can be reconstructed from adjacency matrices
- Square matrix of size $n \times n$
- Symmetric

Example

	
path,K2	K3

Adjacency matrices of K_2 and K_3:

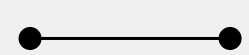
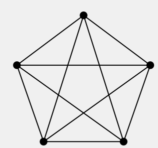
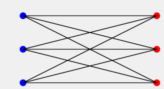
$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$$

Q?

Walk

- A **walk** from vertex u to vertex v (not necessarily distinct) is a sequence of vertices $(w_0, w_1, \ldots, w_k)$, not necessarily distinct, such that
  - w_{i-1} and w_i are connected by an edge, $w_0 = u$, and $w_k = v$.
- Here, k is called the **length** of a walk.
- A **closed walk** is a walk that starts and ends at the same vertex.

Example:

		
path	complete	bipartite

- The number of walks of length k depends on graphs
- The **degree** of a vertex is the number of edges connected to it.
  - The number of length 2 closed walks of a vertex v is the degree.
- Sometimes, we need to count the number of K_3 triangle subgraphs.

Theorems

Theorem: For any graph G with vertex set $V = \{v_1, v_2, ..., v_n\}$, the (i, j) entry of A^k is the number of walks from v_i to v_j of length k.

Corollary:

- The degree sequence is the list of diagonal entries of the matrix A^2
- 2 |E| = trace of A^2 (The Handshake Lemma)
- 6 (number of K_3 subgraphs) = trace of A^3

Eigenvalues of Square Matrices

> Eigenvalues λ and Eigenvectors v satisfy:
>
> $$Av = \lambda v$$

Eigenvalues of Adjacency Matrices:

- Adjacency matrices are real and symmetric.
  - all eigenvalues are real and it has an orthonormal basis.
- The **Spectrum** of A (or G) is the set of its eigenvalues.
  - Non-isomorphic graphs can have the same spectrum.
- Properties:
  - Largest eigenvalue relates to (algebraic) graph connectivity.
  - Eigenvalues can provide bounds on graph parameters like diameter and chromatic number.
- Another matrix of interest is the Laplacian matrix.

Eigenvalues of Path Graphs

- Path Graph P_n:
- Eigenvalues:

$$\lambda_k = 2\cos\left(\frac{k\pi}{n+1}\right), \quad k = 1, 2, \ldots, n$$

- Eigenvalues are distinct and between -2 and 2.

Eigenvalues of Complete Graphs

- Complete Graph K_n:
- Adjacency Matrix:

$$A = J - I$$

where J is the matrix of all ones and I is the identity matrix.

- Eigenvalues: $n-1, -1, -1, \ldots, -1$.

Questions

- It is often hard to find the eigenvalues of a matrix.
- There are several (upper and lower)-bounds of their eigenvalues.

Y. Hong, *Bounds of eigenvalues of graphs* (1993) Discrete Math. contains results such as

2.(7). If G is a connected graph and λ the largest eigenvalue of G, then

$$\lambda(G) \leq \sqrt{2e - n + 1},$$

where the equality holds iff G is one of the following graphs:

1. the star $K_{1,n-1}$;
2. the complete graph K_n.

Questions in Hong (1993)

Problem 2: Let G be a connected graph with n vertices and chromatic number k. We already know that

$$k - 1 \leq \lambda(G) \leq (k - l)\frac{n}{k}.$$

What is the best possible lower bound?

Problem (Brualdi-Li) Let T_n denote a tourament with n vertices. Is it true that $\lambda(\tilde{T}_n) < \lambda(T_n) < \lambda(\bar{T}_n)$, where

$\tilde{T}_n$	$\bar{T}_n$
$A(\tilde{T}_n)=\begin{pmatrix} 0 & 1 & 0 & & & & & & 0 \\ 0 & 0 & 1 & 0 & & & & & \\ 1 & 0 & 0 & 1 & 0 & & & & \\ & 1 & 0 & 0 & 1 & \cdot & & & \\ & & \cdot & \cdot & \cdot & \cdot & \cdot & & \\ & & & \cdot & \cdot & \cdot & \cdot & \cdot & \\ & & & & \cdot & \cdot & \cdot & & 0 \\ & & & & & \cdot & \cdot & \cdot & 1 \\ 1 & & & & & & 1 & 0 & 0 \end{pmatrix},$	$A(\bar{T}_n)=\left(\begin{array}{ccccccc\|ccccccc} 0 & 1 & & & & & 1 & 0 & 0 & \cdot & \cdot & \cdot & & 0 \\ 0 & 0 & 1 & & & & & 1 & 0 & 0 & \cdot & \cdot & & \\ & 0 & 0 & 1 & & \mathbf{1} & & & 1 & 0 & 0 & \cdot & & \cdot \\ & & \cdot & \cdot & \cdot & & & & & \cdot & \cdot & \cdot & & \cdot \\ & \mathbf{0} & & \cdot & \cdot & \cdot & & & \mathbf{1} & & \cdot & \cdot & \cdot & \cdot \\ & & & & \cdot & \cdot & 1 & & & & & \cdot & \cdot & 0 \\ 0 & & & & & 0 & 0 & 1 & & & & & 1 & 0 \\ \hline 1 & 0 & & & & & 0 & 0 & 1 & & & & & 1 \\ 1 & 1 & 0 & & & & & 0 & 0 & 1 & & & & \\ \cdot & 1 & 1 & 0 & & \mathbf{0} & & & 0 & 0 & 1 & \mathbf{1} & & \\ \cdot & \cdot & \cdot & \cdot & \cdot & & & & & \cdot & \cdot & \cdot & & \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & & & \mathbf{0} & & \cdot & \cdot & \cdot & \\ & & & & \cdot & \cdot & 0 & & & & & \cdot & \cdot & 1 \\ 1 & \cdot & \cdot & \cdot & & 1 & 1 & 0 & & & & & 0 & 0 \end{array}\right).$

Suggestion

Suggestion:

- For background, the book on algebraic graph theory by Aguilar available free at https://www.geneseo.edu/~aguilar/public/notes/Graph-Theory-HTML/index.html
- The book and lecture notes by Spielman, e.g., http://cs-www.cs.yale.edu/homes/spielman/sagt/, target more advanced readers
- For research, read papers, e.g., Hong 1993

Questions?

Image Sources:

- Discrete Mathematics: An Open Introduction, 3rd edition Oscar Levin
- ChatGPT4

Thank you!