UNIVERSITA DEGLI STUDI DI TORINO

Idempotent Clifford Modules

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ABSTRACT+ Elementary Background

IDEMPOTENT CLIFFORD MODULES AND THOM CLASSES

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ABSTRACT. The aim of this paper is to construct examples orientation classes for oriented vector bundles E in the K - theory of their thom spaces $\tau(E)$. The method of constrction is the use of idempotent Clifford modules.

Definition 1.2. *If X be a Riemannian manifold. A complex vector bundle $S \xrightarrow{\pi} X$ is called a* ***Clifford module*** *if it carries a fibrewise action,called* ***Clifford multiplication***

$$\mu : \mathbb{C}l(X) \to End_{\mathbb{C}}(S).$$

Definition 1.3. *If an Oriented Riemannian vector bundle E has a spin structure it also has an associated* ***real spinor bundle*** *$\mathcal{S}(E) = P_{spin}(E) \times_\rho M$, where M is a left Clifford module and $\rho : Spin(n) \to SO(M)$ is the left module multiplication by elements in the spin group. The complex spinor bundle is the complexification the real spinor bundle, that is $S_{\mathbb{C}}(E) = P_{spin}(E) \times_\rho (M \otimes \mathbb{C})$, where $M \otimes \mathbb{C}$ is a complex left module for $Cl_n \otimes \mathbb{C}$*

Theorem 1.2.

- *Let $E \xrightarrow{\pi} X$ be a real $8m$-dimensional vector bundle over a compact space X, then $\pi^* S(E)$ with its original $\mathbb{Z}_2$ grading inherited from the $\mathbb{Z}_2$ grading of the spinor bundle $S(E)$, along with Clifford mulplication defined as*

$$\mu : \pi^* S^{\pm}(E) \to \pi^* S^{\mp}(E)$$

, with $\mu_e \phi = e \cdot \phi$ is the fibrewise Clifford multiplication, for a given $e \in E_x$. Then the class $\theta(E) := [\pi^* S^+(E), \pi^* S^-(E), \mu] \in KO_{cpt}(E)$ is a KO theory orientation class for E. In particular the map

$$i_! : KO(X) \xrightarrow{\cong} KO_{cpt}(E),$$

$i_!([E]) = [\pi^* E] \cdot \theta(E)$ is the Thom isomorphism in the KO-theory of E.

- *If $E \xrightarrow{\pi} X$ be an oriented vector bundle of dimension $2m$ over a compact space, if E has a spin structure then the class*

$$s(E) = [\pi^* \mathcal{S}^+_{\mathbb{C}}(E), \pi^* \mathcal{S}^-_{\mathbb{C}}(E), \mu] \in K_{cpt}(E)$$

is a K-theory orientation for E. and the map

$$i_! : K(X) \xrightarrow{\cong} K_{cpt}(E)$$

, where $i_!([E]) = (\pi^*[E]) \cdot s(E)$ is a Thom isomorphism for the K theory of E.

Thus according to this theorem the classes $\theta(E)$, $\delta(E)$, $s(E)$ are orientation/ Thom classes for KO ans K theory respectivley and they generate the isomorphisms that make $K_{cpt}(E)$ (resp $KO_{cpt}(E)$) $K(X)$ (resp $KO(X)$) modules, generated by corresponding Thom class.

Definition 2.4. *Minimal left ideals of the form $Cl_{p,q} \cdot F$,where F is the idempotent $F = \frac{1+e_{\alpha_1}}{2} \cdot \ldots \cdot \frac{1+e_{\alpha_k}}{2}$ constructed from the maximal set of coummuting involutions , will be what we call* ***idempotent Clifford modules*** *or* ***idempotent Spinor spaces****.*

Generic construction of the Thom classes

For even dimensions the bundles $\mathcal{S}(E) = P_{spin}(E) \times_\rho Cl_n \cdot F$ and $\mathcal{S}^{\mathbb{C}}{}_n = P_{spin}(E) \times_\rho Cl_n \cdot F \otimes \mathbb{C}$ decompose into Eigen bundles via the left multiplication endomorphism of the volume element, also these bundles inhert the natural $\mathbb{Z}_2$ ove grading over X from the Clifford algebras.

Now if if we E be an even dimensional oriented vector bundle with a spin structre over X a compact space, and $D(E)$ is unit disk bundle with $S(E)$ the unit sphere bundle. Then under the pull back diagram

$$\begin{tikzcd}
\pi^* \mathcal{S}(E)^{\mathbb{C}}_{2m} \arrow[d] & \mathcal{S}(E)^{\mathbb{C}}_{2m} \arrow[d] \\
D(E) \arrow[r, "\pi"] & X
\end{tikzcd}$$

Under Clifford multiplication have the isomorphism $\mu : (\pi^* \mathcal{S}^{\mathbb{C}}_{2m})^{\pm} \xrightarrow{\cong} (\pi^* \mathcal{S}^{\mathbb{C}}_{2m})^{\mp}$, over $S(E)$

The triple $\eta(E)_{\mathbb{C}} = [(\pi^* \mathcal{S}(E)^{\mathbb{C}}_{2m})^+, (\pi^* \mathcal{S}(E)^{\mathbb{C}}_{2m})^-, \mu] \in \tilde{K}(\tau(E))$ determines a class in the reduced complex K-theory of the Thom space $\tau(E) = D(E)/S(E)$.

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Dimension 2 example

Here we compute our examples in dimension 2. (In dimension 2 we express the representatives in terms of complexified quaternionic bundles.), In the paper we also come up with a dimension 4 example

As is well known $Cl_{0,2} \cong \mathbb{H}$,thus we choose $Cl_{0,2}$ as a $\mathbb{Z}_2$ graded Clifford module over itsef with $Cl^0_{0,2} = span_{\mathbb{R}}\{1, e_{12}\}$, and $Cl^1_{0,2} = span_{\mathbb{R}}\{e_1, e_2\}$.It also worth noting that $Cl^0_{0,2} \cong \mathbb{C}$, with $Spin(2) \cong SO(2)$. When we complexify the Clfford algebra we get isomorphism $\mathbb{C}l_2 = Cl_{0,2} \otimes_{\mathbb{R}} \mathbb{C} \cong \mathbb{C}(2) \cong \mathbb{C} \otimes \mathbb{H}$

Complexifying the Clifford module and choosing the primitive idempotent $F = \frac{1 \otimes 1 + e_{12} \otimes i}{2}$, gives us the complex Clifford module, $\mathbb{C}l_2 \cdot F = span_{\mathbb{R}}\{(1 \otimes 1) \cdot F, (e_1 \otimes 1) \cdot F, (e_2 \otimes 1) \cdot F, (e_{12} \otimes 1) \cdot F\}$, giving us the following equivalences that define Clifford multiplication;

- $(1 \otimes 1) \cdot F = (e_{12} \otimes i) \cdot F$
- $(e_1 \otimes 1) \cdot F = -(e_2 \otimes i) \cdot F$
- $(e_2 \otimes 1) \cdot F = (e_1 \otimes i) \cdot F$
- $(e_{12} \otimes 1) \cdot F = -(1 \otimes i) \cdot F$.

With the obvious grading and Clifford multiplication on the fibres defined by the module equivalences , we obtain for a Riemannian spin vector bundle of rank 2 over a compact Riemannian manifold X, the triple

$$\eta_{\mathbb{C}}(E) = [\pi^*(\mathcal{S}(E))^{\mathbb{C}}_{0,2})^0, \pi^*(\mathcal{S}(E)^{\mathbb{C}}_{0,2})^1, \mu].$$

Giving us a representative of the Thom class in the reduced k-theory of the Thom space of E.

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