1 Information Theory

1.1 Information

Highly improbable events bring more information to us, while certain events bring no information.

The information of an event x will therefore depends on the probability distribution $p(x)$ of its random variable X.

1.2 Construct Information Fomula

Let $h(\cdot)$ be a monotonic function of any distribution $p(x)$ that returns the information of $p(x)$. If x and y are unrelated events, we hope the information they take are also unrelated, so

$$h(x, y) = h(x) + h(y) \tag{1}$$
$$p(x, y) = p(x)p(y) \tag{2}$$

Note that we can interpret $h(p(x))$ as $h(x)$ and interpret $h(p(x, y))$ as $h(x, y)$.
$\log_2(x)$ is a monotonic function that satisfied both (1) and (2), hence we can define

$$h(x) = -\log_2 p(x)$$

Then $h(x)$ satisfies $2^{h(x)} = \frac{1}{p(x)}$. We can interpret this as

$h(x)$ is the amount of bits that being enough for representing $\frac{1}{p(x)}$ in binary.

When $p(x)$ is low, the probability is low, we need more bits to represent it.

1.3 Entropy

Let X be the random variable of the state that transmitted from a sender to a receiver. Intuitively, *the average amount of the information* that X carries is obtained by taking the expectation of information $h(x)$ with respect to the p.d.f. $p(x)$

$$\sum_{x \in X} p(x)h(x) = -\sum_{x \in X} p(x) \log_2 p(x)$$

This is called the *entropy* of the random variable X and denote it by $\mathrm{H}(X)$ or $\mathrm{H}(p)$ or $\mathrm{H}(x)$, based on the context of the paragraph.
Since $\lim_{p \to 0} p \ln p = 0$, we just take $p(x) \ln p(x) = 0$ when we encounter $p(x) = 0$ for some x.

1.3.1 Nats

In practice, we use $\ln p(x)$ instead of $\log_2 p(x)$. That is,

$$h(x) = -\ln p(x)$$
$$\mathrm{H}(p) = -\sum_{x \in X} p(x) \ln p(x)$$

In this situation, we said the information is measured in the units of 'nats'.

1.3.2 Entropy as Lower Bound

> *Entropy is a lower bound of the amount of bits that a random variable can transmits.*

by the *Noiseless Coding Theorem.*

1.3.3 Noiseless Coding Theorem

> N i.i.d. random variables each with entropy $\mathrm{H}(X)$ can be compressed into more than $N\,\mathrm{H}(X)$ bits with negligible risk of information loss, as $N \to \infty$; but conversely, if they are compressed into fewer than $N\,\mathrm{H}(X)$ bits it is virtually certain that information will be lost.

(ChatGPT) In essence, the theorem states that for any given data source with a certain probability distribution of symbols, it is possible to encode the source in such a way that the average length of the encoded message per symbol is close to the entropy of the source.

1.4 Maximize Entropy in Discrete Case

TL;DR

> *The distribution that can carry the most average amount of information is the uniform.*

Proof

Let $X = \{x_i\}_{i=1}^M$ be a discrete random variable and let p be the distribution of X. For the optimization problem

$$\max\left(-\sum_{i=1}^{M} p(x_i)\ln p(x_i)\right)$$

with the normalization constraint on the probabilities

$$\sum_{i=1}^{M} p(x_i) = 1$$

The Lagrangian is

$$\mathcal{L} = -\sum_{i=1}^{M} p(x_i)\ln p(x_i) + \lambda\left(\sum_{i=1}^{M} p(x_i) - 1\right)$$

From $\partial\mathcal{L}/\partial p(x_i) = -(\ln p(x_i) + 1) + \lambda = 0$, we have $\lambda = \ln p(x_i) + 1$, $\forall i = 1, \ldots, M$. By $\sum_{i=1}^{M} p(x_i) = 1$ and $\lambda = \ln p(x_i) + 1$, it's easy to get $\lambda = \ln(1/M) + 1$, we then have

$$p(x_i) = \frac{1}{M}, \forall i$$

is the stationary point.

To verify the maximum, first compute the Hessian matrix

$$\frac{\partial\mathcal{L}}{\partial p(x_i)\partial p(x_j)} = -I_{ij}\frac{1}{p(x_i)}$$

It's obvious that all the eigenvalues are negative (negative definite). So $p(x_i) = \frac{1}{M}$ actually attains a maximum, and the maximum entropy is $\mathrm{H}(p) = \ln M$.

1.5 Differential Entropy

Let $X \subseteq \mathbb{R}$ be a continuous random variable and $p(x)$ be the distribution of X. By M.V.T, we know there exists some x_i such that

$$\int_{i\Delta}^{(i+1)\Delta} p(x)dx = p(x_i)\Delta$$

where Δ is the length of one partition of X. Now for any $x \in [i\Delta, (i+1)\Delta]$, we can use $p(x_i)\Delta$ to estimate its probability as long as Δ small enough. Here comes an entropy estimation

$$\begin{aligned}
\mathrm{H}_\Delta &= -\sum_i p(x_i)\Delta \ln(p(x_i)\Delta) \\
&= -\sum_i p(x_i)\Delta \ln(p(x_i)\Delta) + \sum_i p(x_i)\Delta \ln \Delta - \sum_i p(x_i)\Delta \ln \Delta \\
&= -\sum_i p(x_i)\Delta \ln\left(\frac{p(x_i)\Delta}{\Delta}\right) - \left(\sum_i p(x_i)\Delta\right) \ln \Delta \\
&= -\sum_i p(x_i)\Delta \ln p(x_i) - \ln \Delta
\end{aligned}$$

Note that $\sum_i p(x_i)\Delta = \int_{x\in X} p(x)dx = 1$.

The limit of the first term of right hand side is

$$\lim_{\Delta\to 0} -\sum_i p(x_i)\Delta \ln p(x_i) = -\int p(x) \ln p(x) dx$$

This integral is called the *differential entropy.*

The difference term $\ln \Delta$ shows the fact that we need lots of bits to describe a continuous variable.

The differential entropy can have negative values when $\sigma^2 < (1/2\pi e)$.

For multi dimension random variable, the differential entropy is similar

$$\mathrm{H}(p) = -\int p(\mathbf{x}) \ln p(\mathbf{x}) d\mathbf{x}$$

1.6 Conditional Entropy

Given a joint probability $p(\mathbf{x}, \mathbf{y})$. When $\mathbf{x}$ is known, the additional information needed to specify the corresponding value of $\mathbf{y}$ is given by $-\ln p(\mathbf{y}|\mathbf{x})$. We can compute the *conditional entropy*

$$\mathrm{H}(\mathbf{y}|\mathbf{x}) = -\iint p(\mathbf{x}, \mathbf{y}) \ln p(\mathbf{y}|\mathbf{x}) d\mathbf{y} d\mathbf{x}$$

Note that

$$\mathrm{H}(\mathbf{x}, \mathbf{y}) = -\iint p(\mathbf{x}, \mathbf{y}) \ln p(\mathbf{x}, \mathbf{y}) d\mathbf{x} d\mathbf{y}$$

$$
\begin{aligned}
&= -\iint p(\mathbf{x},\mathbf{y}) \ln(p(\mathbf{y}|\mathbf{x})p(\mathbf{x})) d\mathbf{x}d\mathbf{y} \\
&= -\iint p(\mathbf{x},\mathbf{y}) \ln p(\mathbf{y}|\mathbf{x}) d\mathbf{x}d\mathbf{y} - \iint p(\mathbf{x},\mathbf{y}) \ln p(\mathbf{x}) d\mathbf{x}d\mathbf{y} \\
&= -\iint p(\mathbf{x},\mathbf{y}) \ln p(\mathbf{y}|\mathbf{x}) d\mathbf{x}d\mathbf{y} - \int \left(\int p(\mathbf{x},\mathbf{y}) d\mathbf{y} \right) \ln p(\mathbf{x}) d\mathbf{x} \\
&= \mathrm{H}(\mathbf{y}|\mathbf{x}) + \mathrm{H}(\mathbf{x})
\end{aligned}
$$

1.7 Cross Entropy

The cross entropy of the distribution $q(x)$ relative to a distribution $p(x)$ is

$$
H(p,q) = -\mathrm{E}_p[\ln q] = -\sum_x p(x) \ln q(x)
$$

In deep learning, $p(x)$ often refers to the ground truth label, and $q(x)$ refers to the output from a deep neural network model.

In information theory, minimize cross entropy means

> *Minimizes the amount of information required to specify the value of x as a result of using $q(x)$.*

1.8 Kullback-Leibler Divergence

Let $p(x)$ be an unknown distribution and we use $q(x)$ to approximate it. This will cause additional amount of information

$$
\begin{aligned}
\mathrm{KL}(p\|q) &= \left(-\int p(\mathbf{x}) \ln q(\mathbf{x}) d\mathbf{x} \right) - \left(-\int p(\mathbf{x}) \ln p(\mathbf{x}) d\mathbf{x} \right) \\
&= -\int p(\mathbf{x}) \ln \left\{ \frac{q(\mathbf{x})}{p(\mathbf{x})} \right\} d\mathbf{x}
\end{aligned}
$$

This is known as the *KL divergence* from q to p. *KL divergence* is also called the *relative entropy.*

Note that $\mathrm{KL}(p\|q) \neq \mathrm{KL}(q\|p)$.

TL;DR

$\mathrm{KL}(p\|q) \geq 0$, $\mathrm{KL}(p\|q) = 0 \iff p = q$.

Convex function

For $0 \leq \lambda \leq 1$, a convex function satisfies

$$
f(\lambda a + (1-\lambda)b) \leq \lambda f(a) + (1-\lambda) f(b)
$$

A convex function is called strictly convex if the equality holds only when $\lambda = 0$ or $\lambda = 1$.

Jensen's Inequality

Recall the *Jensen's inequality*, for a convex function f,

$$f\left(\sum_i \lambda_i x_i\right) \leq \sum_i \lambda_i f(x_i)$$

where $\lambda_i \geq 0$ and $\sum_i \lambda_i = 1$.
When each λ_i becomes the probability $p(x_i)$, we have

$$f\left(\mathrm{E}[x]\right) \leq \mathrm{E}[f(x)].$$

For continuous random variable, we have

$$f\left(\int \mathbf{x} p(\mathbf{x}) d\mathbf{x}\right) \leq \int f(\mathbf{x}) p(\mathbf{x}) d\mathbf{x}$$

Proof

Observe that $-\ln x$ is a strictly convex function, so by Jensen's inequality

$$\begin{aligned}
\mathrm{KL}(p\|q) &= -\int p(\mathbf{x}) \ln \frac{q(\mathbf{x})}{p(\mathbf{x})} d\mathbf{x} \\
&\geq -\ln\left(\int p(\mathbf{x}) \frac{q(\mathbf{x})}{p(\mathbf{x})} d\mathbf{x}\right) \\
&= -\ln\left(\int q(\mathbf{x}) d\mathbf{x}\right) = 0
\end{aligned}$$

When the equality holds,

$$\begin{aligned}
&p(\mathbf{x}) \ln \frac{q(\mathbf{x})}{p(\mathbf{x})} = 0 \\
\Rightarrow &\ln \frac{q(\mathbf{x})}{p(\mathbf{x})} = 0 \\
\Rightarrow &\frac{q(\mathbf{x})}{p(\mathbf{x})} = 1 \\
\Rightarrow &q(\mathbf{x}) = p(\mathbf{x})
\end{aligned}$$

1.9 Gaussian Maximizes Differential Entropy

We want to maximize

$$\mathrm{H}(p) = -\int p(x) \ln p(x) dx$$

with three natural constraints

$$\int_{-\infty}^{\infty} p(x) dx = 1$$

$$\int_{-\infty}^{\infty} xp(x)dx = \mu$$
$$\int_{-\infty}^{\infty} (x-\mu)^2 p(x)dx = \sigma^2$$

The Lagrangian is

$$\mathcal{L}[p] = -\int_{\mathbb{R}} p(x)\ln p(x)dx + \lambda_1\left(\int_{\mathbb{R}} p(x)dx - 1\right) + \lambda_2\left(\int_{\mathbb{R}} xp(x)dx - \mu\right) + \lambda_3\left(\int_{\mathbb{R}} (x-\mu)^2 p(x)dx - \sigma^2\right)$$

This is a functional. The derivative of a functional is denoted by $\frac{\delta\mathcal{L}}{\delta p}$ and is defined to satisfy

$$\int \frac{\delta\mathcal{L}[p]}{\delta p}\phi(x)dx = \lim_{\epsilon\to 0}\frac{\mathcal{L}[p+\epsilon\phi]-\mathcal{L}[p]}{\epsilon} = \left[\frac{d}{d\epsilon}\mathcal{L}[p+\epsilon\phi]\right]_{\epsilon=0}$$

where $\phi(x)$ is a variation term.
Deriving from the definition

$$\int \frac{\delta\mathcal{L}[p]}{\delta p}\phi(x)dx = \int \left(-(\ln p(x)+1) + \lambda_1 + \lambda_2 x + \lambda_3(x-\mu)^2\right)\phi(x)dx$$

We have the actual form of $\frac{\delta\mathcal{L}[p]}{\delta p}$ and can let it be zero then get

$$p(x) = e^{-1+\lambda_1+\lambda_2 x+\lambda(x-\mu)^2}$$

Substitute this result back to three constraints leading to

$$p(x) = \frac{1}{(2\pi\sigma^2)^{\frac{1}{2}}}e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

which is the Gaussian.
To verify the maximum, let $f(x)$ be any distribution has the variance σ^2. Since differential entropy is translation invariant, we can also assume $f(x)$ has the same mean μ. Now consider the KL divergence

$$\begin{aligned}
\mathrm{KL}(f\|p) &= -\int f(x)\ln\left(\frac{p(x)}{f(x)}\right)dx \\
&= -\mathrm{H}(f) - \int f(x)\ln\left(\frac{1}{(2\pi\sigma^2)^{\frac{1}{2}}}e^{-\frac{(x-\mu)^2}{2\sigma^2}}\right)dx \\
&= -\mathrm{H}(f) + \frac{1}{2}\ln(2\pi\sigma^2) + \frac{1}{2\sigma^2}\int f(x)(x-\mu)^2 dx \\
&= -\mathrm{H}(f) + \frac{1}{2}\ln(2\pi\sigma^2) + \frac{\sigma^2}{2\sigma^2} \\
&= -\mathrm{H}(f) + \frac{1}{2}\ln(2\pi\sigma^2) + \frac{1}{2} \\
&= -\mathrm{H}(f) + \mathrm{H}(p) \geq 0
\end{aligned}$$

Hence

$$H(p) \geq H(f), \forall f$$

The corresponding maximum entropy is,

$$\mathrm{H}(p) = \frac{1}{2}(1 + \ln(2\pi\sigma^2))$$