Let $\mathbb{C}$ denote the set of complex numbers and let $\mathcal{H} = \mathbb{C}^n$ be the set of all n-tuples $(\xi_1, ..., \xi_n)$ of complex numbers with addition and scalar multiplication defined as follows.

For $x = (\xi_1, ..., \xi_n)$ and $y = (\eta_1, ..., \eta_n)$,

$$x + y = (\xi_1 + \eta_1, ..., \xi_n + \eta_n) \text{ and } \alpha x = (\alpha\xi_1, ..., \alpha\xi_n), \alpha \in \mathbb{C}^n.$$

The complex valued function $\langle , \rangle$, defined on the Cartesian product $\mathcal{H} \times \mathcal{H}$ by

$$\langle x, y \rangle = \sum_{i=1}^{n} \xi_i \overline{\eta}_i$$

is called an *inner product* on $\mathcal{H}$.

Practice

Show

1. $\langle x, x \rangle = 0$ if and only if $x \neq 0$,
2. $\overline{\langle x, y \rangle} = \langle y, x \rangle$,
3. $\langle \alpha x, y \rangle = \alpha \langle x, y \rangle$,
4. $\langle x + y, z \rangle = \langle x, z \rangle + \langle y, z \rangle$.

Sol

1. Let $\xi_i = a_i + b_i i$. Then $\langle x, x \rangle = \sum_{i=1}^n \xi_i \overline{\xi}_i = \sum_{i=1}^n a_i^2 + b_i^2 = 0$ implies that $a_i^2 + b_i^2 = 0, \forall i$, hence $a_i = b_i = 0, \forall i$. Conclude that if $\langle x, x \rangle = 0$, $x = 0$. The other direction is easy to show.
2. $\overline{\langle x, y \rangle} = \overline{\sum_{i=1}^n \xi_i \overline{\eta}_i} = \sum_{i=1}^n \overline{\xi_i \overline{\eta}_i} = \sum_{i=1}^n \overline{\xi}_i \eta_i = \sum_{i=1}^n \eta_i \overline{\xi}_i = \langle y, x \rangle$.
3. $\langle \alpha x, y \rangle = \sum_{i=1}^n \alpha \xi_i \eta_i = \alpha \sum_{i=1}^n \xi_i \eta_i = \alpha \langle x, y \rangle$.
4. Let $z = (\gamma_1, ..., \gamma_n)$.

$$\begin{aligned} \langle x + y, z \rangle &= \sum_{i=1}^{n} (\xi_i + \eta_i) \overline{\gamma}_i = \sum_{i=1}^{n} (\xi_i \overline{\gamma}_i + \eta_i \overline{\gamma}_i) \\ &= \sum_{i=1}^{n} \xi_i \overline{\gamma}_i + \sum_{i=1}^{n} \eta_i \overline{\gamma}_i = \langle x, y \rangle + \langle x, z \rangle \end{aligned}$$

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Practice

Show

1. $\langle x, \alpha y + \beta z \rangle = \overline{\alpha} \langle x, y \rangle + \overline{\beta} \langle x, z \rangle$
2. $\langle x, 0 \rangle = \langle 0, x \rangle = 0$

Sol

1. $$\begin{aligned} \langle x, \alpha y + \beta z \rangle &= \overline{\langle \alpha y + \beta z, x \rangle} = \overline{\alpha \langle y, x \rangle + \beta \langle z, x \rangle} = \overline{\alpha \langle y, x \rangle} + \overline{\beta \langle z, x \rangle} \\ &= \overline{\alpha}\, \overline{\langle y, x \rangle} + \overline{\beta}\, \overline{\langle z, x \rangle} = \overline{\alpha} \langle x, y \rangle + \overline{\beta} \langle x, z \rangle \end{aligned}$$
2. It's simply a direct computation result.

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