For $E_k \in \mathbb{R}^n$ and $\mathcal{F} = \{E_k : k = 1, 2, ...\}$, we define

$$\bigcup_{E \in \mathcal{F}} E = \{x : x \in E \text{ for some } E \in \mathcal{F}\}, \bigcap_{E \in \mathcal{F}} E = \{x : x \in E \text{ for all } E \in \mathcal{F}\}$$

$$\limsup E_k = \bigcap_{j=1}^{\infty} \left(\bigcup_{k=j}^{\infty} E_k \right), \liminf E_k = \bigcup_{j=1}^{\infty} \left(\bigcap_{k=j}^{\infty} E_k \right).$$

Practice

$$\text{For } \mathcal{U}_j = \cup_{k=j}^{\infty} E_k, \mathcal{U}_j \searrow \limsup E_k.$$

Sol

$\mathcal{U}_j = \bigcup_{k=j}^{\infty} E_k = E_j \bigcup \left(\cup_{k=j+1}^{\infty} E_k \right) = E_j \bigcup \mathcal{U}_{j+1}$, so $\mathcal{U}_{j+1} \subset \mathcal{U}_j$. Hence, $\mathcal{U}_j \searrow \bigcap_j \mathcal{U}_j = \limsup E_k$.

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Practice

$$\text{For } \mathcal{V}_j = \bigcap_{k=j}^{\infty} E_k, \mathcal{V}_j \nearrow \liminf E_k.$$

Sol

$\mathcal{V}_j = \bigcap_{k=j}^{\infty} E_k = E_j \bigcap \left(\cap_{k=j+1}^{\infty} E_k \right) = E_j \bigcap \mathcal{V}_{j+1}$, so $\mathcal{V}_j \subset \mathcal{V}_{j+1}$. Hence $\mathcal{V}_j \nearrow \bigcup_j \mathcal{V}_j = \liminf E_k$.

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Practice

$$\limsup E_k = \{x \in \mathbb{R}^n : x \text{ belongs to infinitely many } E_k\},$$
$$\liminf E_k = \{x \in \mathbb{R}^n : x \in E_k, \forall k \geq k_0(x), k_0(x) \in \mathbb{N}\}.$$

Sol

$$\begin{aligned}
\limsup E_k &= \bigcap_{j=1}^{\infty} \left(\cup_{k=j}^{\infty} E_k \right) = \left\{ x : x \in \cup_{k=j}^{\infty} E_k \text{ for all } j \in \{1, 2, ...\} \right\} \\
&= \{x : x \in \{x : x \in E_k \text{ for some } k \in \{j, j+1, ...\}\} \text{ for all } j \in \{1, 2, ...\}\} \\
&= \{x \in \mathbb{R}^n : x \text{ belongs to infinitely many } E_k\} \\
\liminf E_k &= \bigcup_{j=1}^{\infty} \left(\cap_{k=j}^{\infty} E_k \right) = \left\{ x : x \in \cap_{k=j}^{\infty} E_k \text{ for some } j \in \{1, 2, ...\} \right\} \\
&= \{x : x \in \{x : x \in E_k \text{ for all } k \in \{j, j+1, ...\}\} \text{ for some } j \in \{1, 2, ...\}\} \\
&= \{x \in \mathbb{R}^n : x \in E_k, \forall k \geq k_0(x), k_x \in \mathbb{N}\}
\end{aligned}$$

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