일반적 열 전달 방정식
Heat Transfer In General Formula

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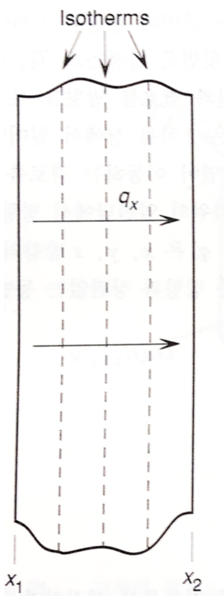
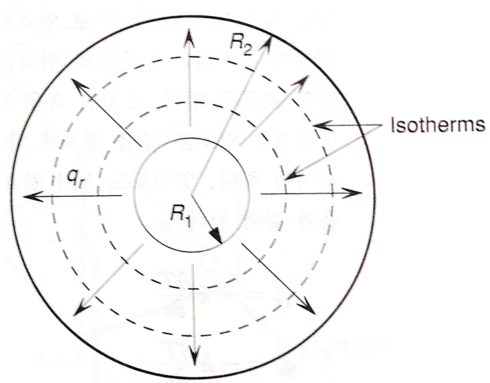
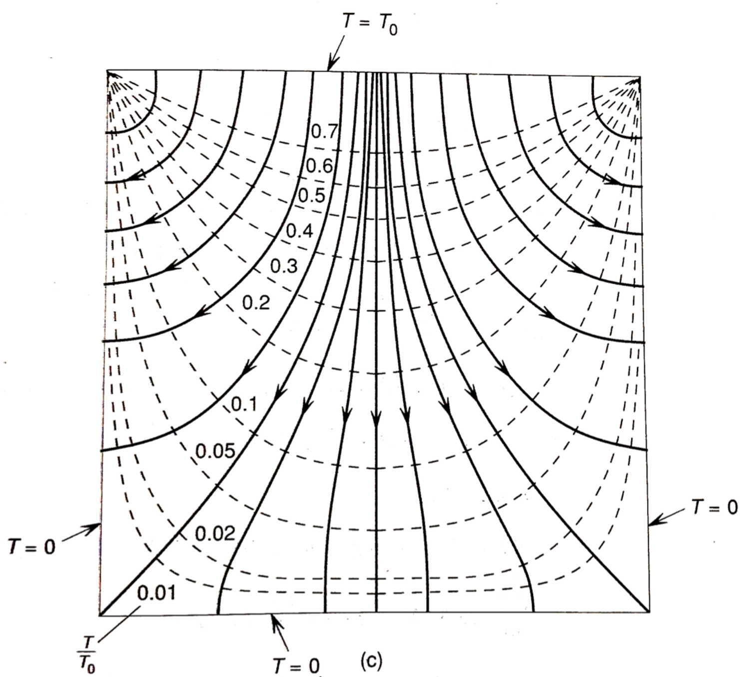
일반(다차원적) 열전도 방정식

- 앞서 우리는 평면벽이나 원통형벽을 통해 1차원적인 열 전도 (즉, 단 한방향으로의 열전도)만을 고려하였다.







- 더욱 복잡한 형상이나, 다차원적 열 경계 조건에서는 일반적 열전도 방정식을 사용하여 열전도 현상을 설명해야 한다.

등방성 재료의 열전도도

- 따라서, 열 흐름 (heat flux) q는 벡터로 표현되며, 3차원 공간에서 벡터 $\vec{\boldsymbol{q}}$는 세 성분으로 이루어진다: $\vec{q}_x, \vec{q}_y, \vec{q}_z$
- 3차원 공간의 일반적 열흐름 문제를 설명하기 위해서는 Fourier 법칙을 다음과 같이 표현하여야 한다.

$$\vec{q}_x = -kA_x \frac{dT}{dx}$$

$$\vec{\boldsymbol{q}} = -\bar{\bar{\boldsymbol{k}}} \cdot \vec{\boldsymbol{\nabla}} T$$

$$\vec{q}_i = -\bar{\bar{k}}_{ij} \vec{\nabla}_j T$$

$\vec{\boldsymbol{\nabla}} T$: Temperature Gradient

$\bar{\bar{\boldsymbol{k}}}$: 열전도도 tensor

$$\vec{q}_x = -\bar{\bar{k}}_{xx}\vec{\nabla}_x T - \bar{\bar{k}}_{xy}\vec{\nabla}_y T - \bar{\bar{k}}_{xz}\vec{\nabla}_z T$$

$$\vec{q}_y = -\bar{\bar{k}}_{yx}\vec{\nabla}_x T - \bar{\bar{k}}_{yy}\vec{\nabla}_y T - \bar{\bar{k}}_{yz}\vec{\nabla}_z T$$

$$\vec{q}_z = -\bar{\bar{k}}_{zx}\vec{\nabla}_x T - \bar{\bar{k}}_{zy}\vec{\nabla}_y T - \bar{\bar{k}}_{zz}\vec{\nabla}_z T$$

Let's assume $\bar{\bar{k}}_{ij} = 0 \quad \text{if } i \neq j$

$$\vec{q}_x = -\bar{\bar{k}}_{xx}\vec{\nabla}_x T$$

$$\vec{q}_y = -\bar{\bar{k}}_{yy}\vec{\nabla}_y T$$

$$\vec{q}_z = -\bar{\bar{k}}_{zz}\vec{\nabla}_z T$$

열전도도가 등방성을 가질 때.

$$\vec{q}_x = -k\vec{\nabla}_x T$$

$$\vec{q}_y = -k\vec{\nabla}_y T$$

$$\vec{q}_z = -k\vec{\nabla}_z T$$

열 에너지 수지

$$\vec{q}_x = -k\vec{\nabla}_x T$$

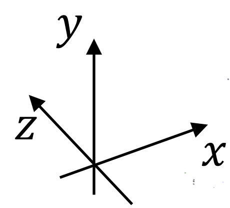
$$q_x = -k\frac{\partial T}{\partial x}$$

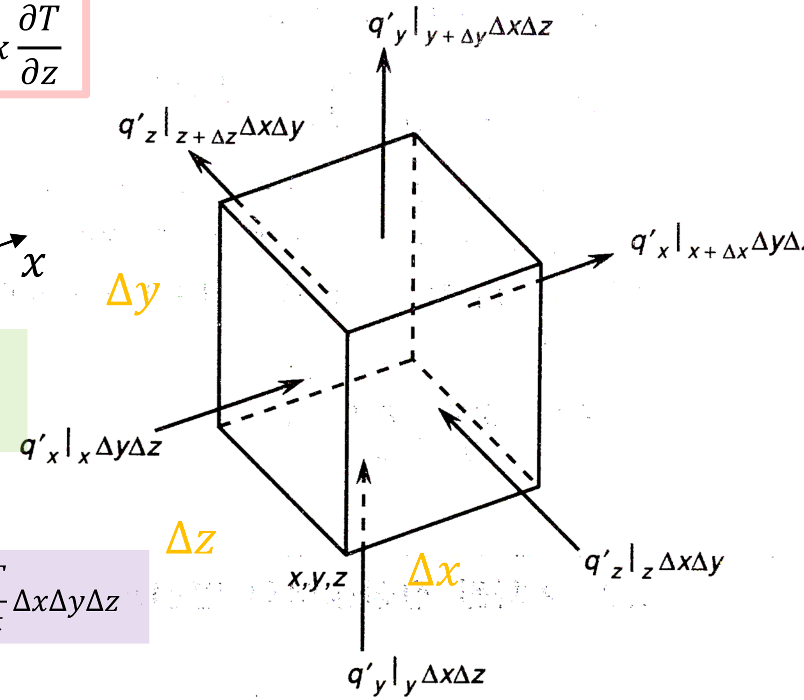
$$q_y = -k\frac{\partial T}{\partial y}$$

$$q_z = -k\frac{\partial T}{\partial z}$$

$$\vec{q}_y = -k\vec{\nabla}_y T$$

$$\vec{q}_z = -k\vec{\nabla}_z T$$





$$\left(\frac{\text{들어온열}}{t}\right) - \left(\frac{\text{나간 열}}{t}\right) + \left(\frac{\text{발생한 열}}{t}\right) = \text{시간당 열에너지 변화}$$

$$(\text{발생한 열}/t) = \dot{q}\Delta V = \dot{q}\cdot\Delta x\Delta y\Delta z$$

$$\text{열에너지 변화}/t = \text{부피내 재료의 엔탈피 변화}/t = \frac{\partial H}{\partial t}\Delta x\Delta y\Delta z = \rho C_{p(m)}\frac{\partial T}{\partial t}\Delta x\Delta y\Delta z$$

$$\left(q'_x\Big|_x - q'_x\Big|_{x+\Delta x}\right)\Delta y\Delta z + \left(q'_y\Big|_y - q'_y\Big|_{y+\Delta y}\right)\Delta x\Delta z + \left(q'_z\Big|_z - q'_z\Big|_{z+\Delta z}\right)\Delta x\Delta y + \dot{q}\cdot\Delta x\Delta y\Delta z = \rho C_{p(m)}\frac{\partial T}{\partial t}\Delta x\Delta y\Delta z$$

$$\frac{(q'_x|_x - q'_x|_{x+\Delta x})}{\Delta x} + \frac{(q'_y|_y - q'_y|_{y+\Delta y})}{\Delta y} + \frac{(q'_z|_z - q'_z|_{z+\Delta z})}{\Delta z} + \dot{q} = \rho C_{p(m)}\frac{\partial T}{\partial t}$$

$$-\left(\frac{\partial q_x}{\partial x} + \frac{\partial q_y}{\partial y} + \frac{\partial q_z}{\partial z}\right) + \dot{q} = \rho C_{p(m)}\frac{\partial T}{\partial t}$$

$$-k\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2}\right) + \dot{q} = \rho C_{p(m)}\frac{\partial T}{\partial t}$$

열전도 방정식

$$-k\left(\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}+\frac{\partial^2 T}{\partial z^2}\right)+\dot{q}=\rho C_{p(m)}\frac{\partial T}{\partial t}$$

Steady-state

No heat-generation

$$-k\left(\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}+\frac{\partial^2 T}{\partial z^2}\right)+\dot{q}=0 \rightarrow -k\left(\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}+\frac{\partial^2 T}{\partial z^2}\right)=0 \rightarrow \left(\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}+\frac{\partial^2 T}{\partial z^2}\right)=0$$

$$\left(\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}+\frac{\partial^2 T}{\partial z^2}\right)=0$$

Laplacian: Divergence of Gradient: $\sum_i \left(\frac{\partial^2 f}{\partial x_i^2}\right)$

정상상태의 수학적 표현방법에 널리 쓰임

2차원 정상상태의 열전도

(x, y) 2차원 공간

$$\left(\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}+\frac{\partial^2 T}{\partial z^2}\right)=0 \longrightarrow \left(\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}\right)=0$$

온도 분포 $T(x, y)$ 및

온도 구배 $\left(\frac{\partial T}{\partial x}, \frac{\partial T}{\partial y}\right)$?

적절한 열경계 조건 (boundary condition)과 적분을 통해 구할 수 있겠다.

$$T(x, y)=\frac{4T_0}{\pi}\sum_{n}^{\infty}\frac{1}{n}\frac{\sinh(n\pi y/L)}{\sinh(n\pi)}\sin\frac{n\pi x}{L}$$

왼편에 보시다 시피, 이러한 방법은 매우 까다롭거나, 어렵다…

이런경우, 해를 근사적으로 구할 수 있는 유한 차분법 (Finite difference method)이 유용하다.

유한 차분법

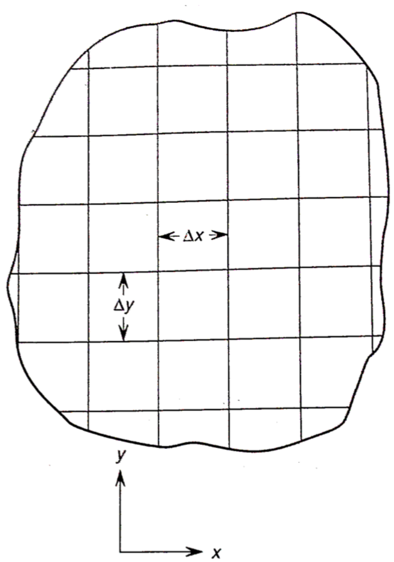
미분

$$\frac{\partial T}{\partial x} = \lim_{\Delta x \to 0} \frac{\Delta T}{\Delta x}$$

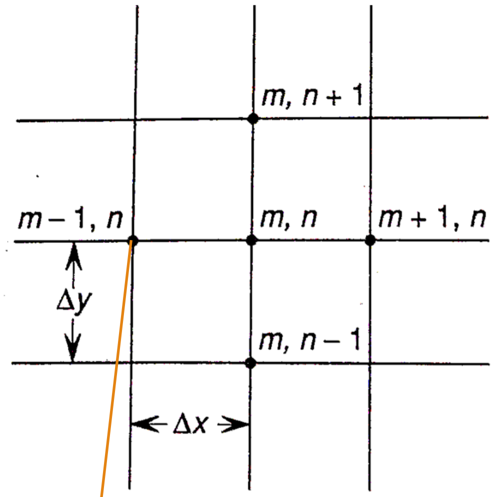
차분

$$\frac{\partial T}{\partial x} \approx \frac{T(x+\Delta x) - T(x)}{\Delta x}$$

$$\frac{\partial^2 T}{\partial x^2} \approx \frac{T(x+\Delta x) - 2T(x) + T(x-\Delta x)}{(\Delta x)^2}$$



열 구조물을 유한한 크기($\Delta x, \Delta y$)의 grid로 세분한다.

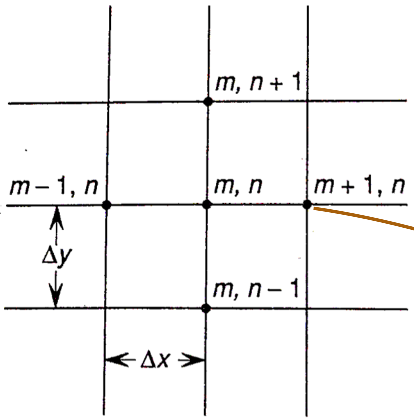


각 교차점에서의 온도T를 구하는 것이 목표.

X축은 m
Y축은 n을 사용하여
$T(x, y)$를 불연속적인 유한한 크기의 요소 (예를 들어 $T_{m-1,n}, T_{m,n}, T_{m+1,n}$ 로 표기된 온도 분포) 에서 찾아보는 방법.

$$\frac{\partial^2 T}{\partial x^2} \approx \frac{T(x+\Delta x) - 2T(x) + T(x-\Delta x)}{(\Delta x)^2}$$

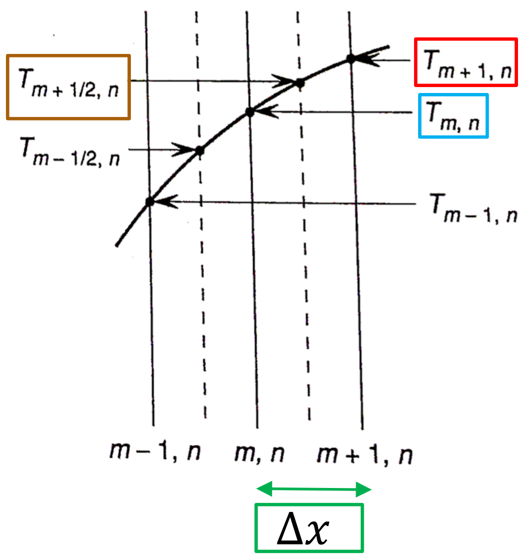
도출해보자



각 교차점에서의 온도T를
구하는 것이 목표.

각 교차점의 온도를 $T_{m,n}$의
$T_{m+1,n}$ 등으로 표기 가능.

$(m+\frac{1}{2}, n)$ 에서 x 방향으로의 온도 구배는?



$$\left.\frac{\partial T}{\partial x}\right|_{(m+\frac{1}{2},n)} \approx \frac{\boxed{T_{m+1,n}} - \boxed{T_{m,n}}}{\boxed{\Delta x}}$$

$$\left.\frac{\partial T}{\partial x}\right|_{(m-\frac{1}{2},n)} \approx \frac{T_{m,n} - T_{m-1,n}}{\Delta x}$$

$$\left.\frac{\partial^2 T}{\partial x^2}\right|_{(m,n)} \approx \frac{\left.\frac{\partial T}{\partial x}\right|_{(m+\frac{1}{2},n)} - \left.\frac{\partial T}{\partial x}\right|_{(m-\frac{1}{2},n)}}{\Delta x}$$

$$\left.\frac{\partial^2 T}{\partial x^2}\right|_{(m,n)} \approx \frac{\frac{T_{m+1,n} - T_{m,n}}{\Delta x} - \frac{T_{m,n} - T_{m-1,n}}{\Delta x}}{\Delta x}$$

$$\rightarrow \frac{T_{m+1,n} - T_{m,n} - (T_{m,n} - T_{m-1,n})}{(\Delta x)^2}$$

$$\left.\frac{\partial^2 T}{\partial x^2}\right|_{(m,n)} \approx \frac{T_{m+1,n} - 2T_{m,n} + T_{m-1,n}}{(\Delta x)^2}$$

$$\left.\frac{\partial^2 T}{\partial y^2}\right|_{(m,n)} \approx \frac{T_{m,n+1} - 2T_{m,n} + T_{m,n-1}}{(\Delta y)^2}$$

$$\frac{\partial^2 T}{\partial x^2} \approx \frac{T(x+\Delta x) - 2T(x) + T(x-\Delta x)}{(\Delta x)^2}$$

검증

$$\left.\frac{\partial^2 T}{\partial x^2}\right|_{(m,n)} \approx \frac{T_{m+1,n} - 2T_{m,n} + T_{m-1,n}}{(\Delta x)^2}$$

$$\left.\frac{\partial^2 T}{\partial y^2}\right|_{(m,n)} \approx \frac{T_{m,n+1} - 2T_{m,n} + T_{m,n-1}}{(\Delta y)^2}$$

(x, y) 2차원 공간에서의 정상상태 열전도 방정식에 대입

$$\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2}\right) = 0$$

$$\left(\frac{T_{m+1,n} - 2T_{m,n} + T_{m-1,n}}{(\Delta x)^2} + \frac{T_{m,n+1} - 2T_{m,n} + T_{m,n-1}}{(\Delta y)^2}\right) = 0$$

$\Delta x = \Delta y$ 의 정사각형 요소를 사용한다면

$$T_{m+1,n} - 2T_{m,n} + T_{m-1,n} + T_{m,n+1} - 2T_{m,n} + T_{m,n-1} = 0$$

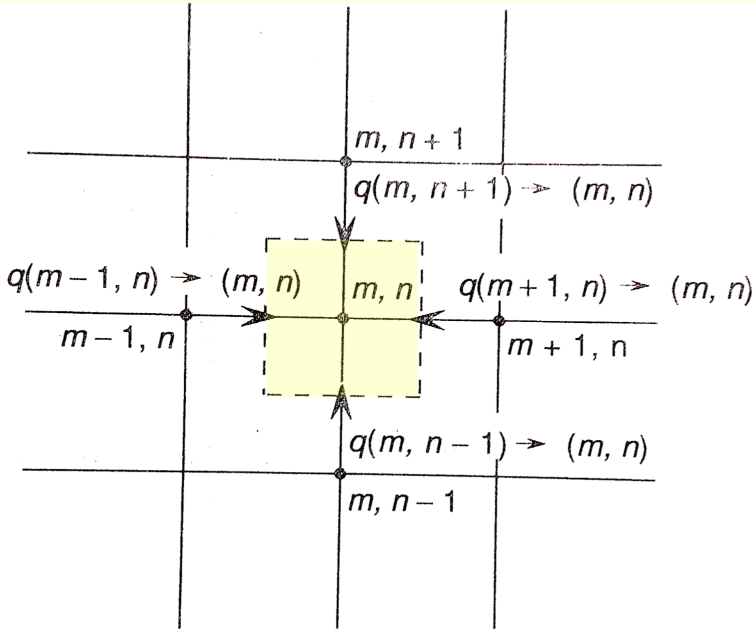
$$T_{m+1,n} - 4T_{m,n} + T_{m-1,n} + T_{m,n+1} + T_{m,n-1} = 0$$

$$T_{m+1,n} + T_{m-1,n} + T_{m,n+1} + T_{m,n-1} = 4T_{m,n}$$

$$T_{m,n} = \frac{T_{m+1,n} + T_{m-1,n} + T_{m,n+1} + T_{m,n-1}}{4}$$

유한 차분법

$$T_{m,n} = \frac{T_{m+1,n} + T_{m-1,n} + T_{m,n+1} + T_{m,n-1}}{4}$$



(m,n)을 중심으로 하는 노란 요소 안으로 주위에서 유입되는 열량과 요소 내 발생하는 열량의 합이 zero ∵ Steady-state!

$$q_{(m-1,n)\to(m,n)} = -k \cdot A \cdot \frac{\partial T}{\partial x} \approx -k \cdot \Delta y \Delta z \cdot \frac{T_{(m,n)} - T_{(m-1,n)}}{\Delta x}$$

$$q_{(m,n)\to(m+1,n)} \approx -k \cdot \Delta y \Delta z \cdot \frac{T_{(m+1,n)} - T_{(m,n)}}{\Delta x}$$

$$q_{(m,n-1)\to(m,n)} \approx -k \cdot \Delta x \Delta z \cdot \frac{T_{(m,n)} - T_{(m,n-1)}}{\Delta y}$$

$$q_{(m,n)\to(m,n+1)} \approx -k \cdot \Delta x \Delta z \cdot \frac{T_{(m,n+1)} - T_{(m,n)}}{\Delta y}$$

요소 내 발생하는 열량의 합: $\dot{q}\Delta x \Delta y \Delta z$

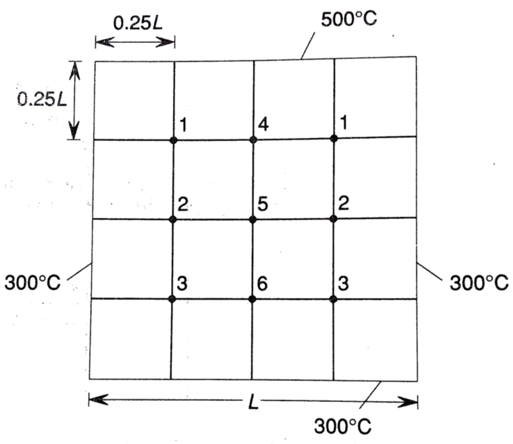
$$-k \cdot \Delta y \Delta z \cdot \frac{T_{(m,n)} - T_{(m-1,n)}}{\Delta x} - k \cdot \Delta x \Delta z \cdot \frac{T_{(m,n)} - T_{(m,n-1)}}{\Delta y}$$
$$-\left[-k \cdot \Delta y \Delta z \cdot \frac{T_{(m+1,n)} - T_{(m,n)}}{\Delta x} - k \cdot \Delta x \Delta z \cdot \frac{T_{(m,n+1)} - T_{(m,n)}}{\Delta y}\right]$$
$$+\dot{q}\Delta x \Delta y \Delta z = 0$$

With $\Delta x = \Delta y$

$$T_{(m-1,n)} + T_{(m,n-1)} + T_{(m+1,n)} + T_{(m,n+1)} + \frac{\dot{q}\Delta x \Delta y}{k} - 4T_{(m,n)} = 0$$

예제 6.10

- 변의 길이가 L인 정사각형 판에서 장상상태하의 온도 분포를 계산하여라.



$at\ Pt.1$ $300 + 500 + T_2 + T_4 = 4T_1$	$at\ Pt.1$ $-4T_1 + T_2 + T_4 = -800$
$at\ Pt.2$ $300 + \mathrm{T}_1 + T_3 + T_5 = 4T_2$	$at\ Pt.2$ $\mathrm{T}_1 - 4T_2 + T_3 + T_5 = -300$
$at\ Pt.3$ $300 + 500 + T_2 + T_6 = 4T_3$	$at\ Pt.3$ $T_2 - 4T_3 + T_6 = -800$
$at\ Pt.4$ $500 + T_1 + T_1 + T_5 = 4T_4$	$at\ Pt.4$ $2T_1 - 4T_4 - T_5 = -500$
$at\ Pt.5$ $T_2 + T_2 + T_4 + T_6 = 4T_5$	$at\ Pt.5$ $2T_2 + T_4 - 4T_5 + T_6 = 0$
$at\ Pt.6$ $300 + T_3 + T_3 + T_5 = 4T_6$	$at\ Pt.6$ $2T_3 + T_5 - 4T_6 = -300$

$aT_1 + bT_2 + cT_3 + dT_4 + eT_5 + fT_6 = g$
형태로 통합하면…

$$\begin{bmatrix} -4 & 1 & 0 & 1 & 0 & 0 \\ 1 & -4 & 1 & 0 & 1 & 0 \\ 0 & 1 & -4 & 0 & 0 & 1 \\ 2 & 0 & 0 & -4 & -1 & 0 \\ 0 & 2 & 0 & 1 & -4 & 1 \\ 0 & 0 & 2 & 0 & 1 & -4 \end{bmatrix} = \begin{bmatrix} -800 \\ -300 \\ -800 \\ -500 \\ 0 \\ -300 \end{bmatrix}$$

$$\begin{bmatrix} -4 & 1 & 0 & 1 & 0 & 0 \\ 1 & -4 & 1 & 0 & 1 & 0 \\ 0 & 1 & -4 & 0 & 0 & 1 \\ 2 & 0 & 0 & -4 & -1 & 0 \\ 0 & 2 & 0 & 1 & -4 & 1 \\ 0 & 0 & 2 & 0 & 1 & -4 \end{bmatrix} \begin{bmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \\ T_5 \\ T_6 \end{bmatrix} = \begin{bmatrix} -800 \\ -300 \\ -800 \\ -500 \\ 0 \\ -300 \end{bmatrix}$$