Modern Particle Physics
Mark Thomson

Youngwan Kim[1], Jin Choi[1]
Email: youngwan.kim@cern.ch, jin.choi@cern.ch

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1: Seoul National University

1 Introduction

Problem 1.1
Feynman diagrams are constructed out of the Standard Model vertices shown in Figure 1.4. Only the weak charged-current interaction can change the flavour of the particle at the interaction vertex. Explaining your reasoning, state whether each of the sixteen diagrams below represents a valid Standard Model vertex.

Solution:

(a) Valid.

(b) Invalid, due to the fact that ν_e has no electric charge.

(c) Valid.

(d) Valid.

(e) Invalid. Flavor shouldn't change.

(f) Valid.

(g) Invalid. Flavor shouldn't change.

(h) Invalid. Flavor shouldn't change.

(i) Invalid, leptons do not carry color charge.

(j) Valid.

(k) Valid.

(l) Invalid.

(m) Invalid.

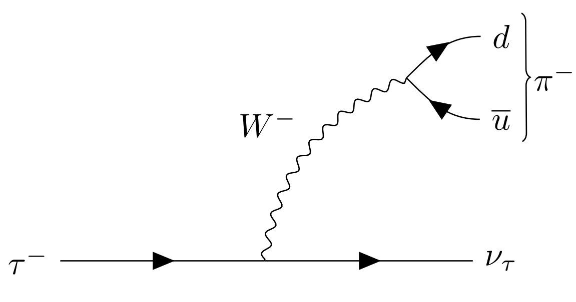
(n) Valid.

(o) Valid.

(p) Invalid. No such 4-point vertex in QED.

Problem 1.2
Draw the Feynman diagram for $\tau^- \to \pi^- \nu_\tau$. (The π^- is the lightest $d\bar{u}$ meson)

Solution:



Problem 1.3
Explain why it is not possible to construct a valid Feynman diagram using the Standard Model vertices for the following processes :

(a) $\mu^- \to e^+e^-e^+$

(b) $\nu_\tau + p \to \mu^- + n$

(c) $\nu_\tau + p \to \tau^+ + n$

(d) $\pi^+(u\bar{d}) + \pi^-(d\bar{u}) \to n(udd) + \pi^0(u\bar{u})$

Solution:

(a) $\mu^- \to e^+e^-e^+$: Charge is not conserved, as well as lepton numbers.

(b) $\nu_\tau + p \to \mu^- + n$: Charge is not conserved, as well as baryon numbers.

(c) $\nu_\tau + p \to \tau^+ + n$: Both baryon and lepton number is not conserved.

(d) $\pi^+(u\bar{d}) + \pi^-(d\bar{u}) \to n(udd) + \pi^0(u\bar{u})$: Baryon number is not conserved.

Problem 1.4
Draw the Feynman diagram for the decays:

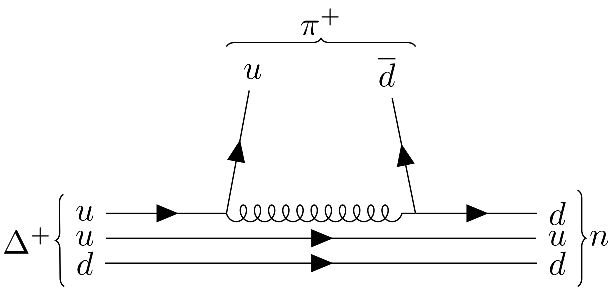
(a) $\Delta^+(uud) \to n(udd)\pi^+(u\bar{d})$

(b) $\Sigma^0(uds) \to \Lambda(uds)\gamma$

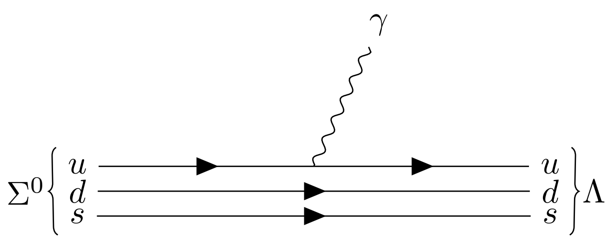
(c) $\pi^+(u\bar{d}) \to \mu^+\nu_\mu$

Solution:

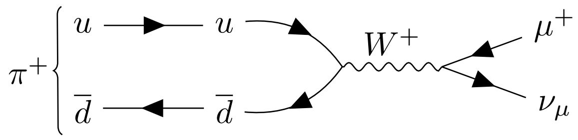
(a) $\Delta^+(uud) \to n(udd)\pi^+(u\bar{d})$



(b) $\Sigma^0(uds) \to \Lambda(uds)\gamma$



(c) $\pi^+(u\bar{d}) \to \mu^+\nu_\mu$



Problem 1.5
Treating the π^0 as a uu bound state, draw the Feynman diagrams for:

(a) $\pi^0 \to \gamma\gamma$

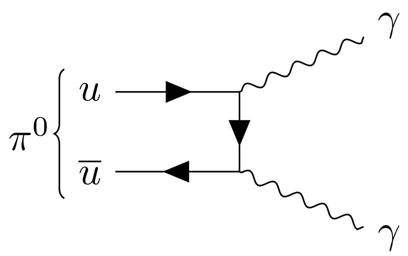
(b) $\pi^0 \to \gamma e^+e^-$

(c) $\pi^0 \to e^+e^-e^+e^-$

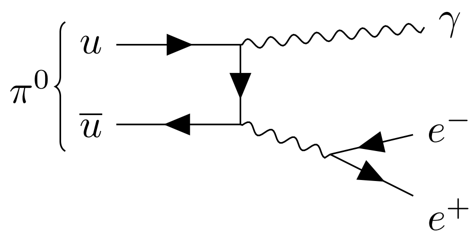
(d) $\pi^0 \to e^+e^-$

Solution:

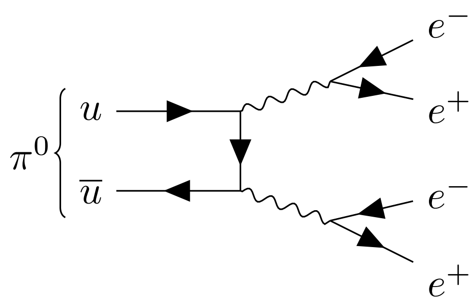
(a) $\pi^0 \to \gamma\gamma$



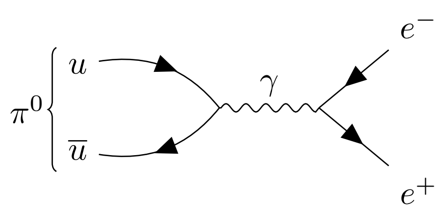
(b) $\pi^0 \to \gamma e^+e^-$



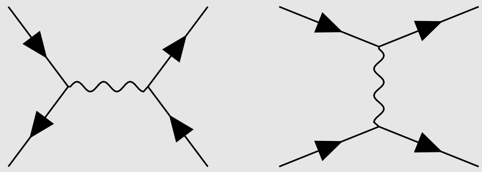
(c) $\pi^0 \to e^+e^-e^+e^-$



(d) $\pi^0 \to e^+e^-$



Problem 1.6
Particle interactions fall into two main categories, scattering processes and annihilation processes, as indicated by the Feynman diagrams below.



Draw the lowest-order Feynman diagrams for the scattering and/or annihilation processes:

(a) $e^-e^- \to e^-e^-$

(b) $e^+e^- \to \mu^+\mu^-$

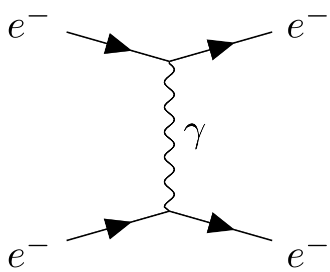
(c) $e^+e^- \to e^+e^-$

(d) $e^-\nu_e \to e^-\nu_e$

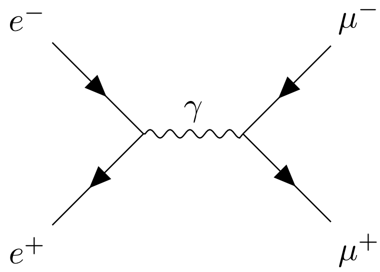
(e) $e^-\overline{\nu_e} \to e^-\overline{\nu_e}$

Solution:

(a) $e^-e^- \to e^-e^-$



(b) $e^+e^- \to \mu^+\mu^-$



(c) $e^+e^- \to e^+e^-$

(d) $e^-\nu_e \to e^-\nu_e$

(e) $e^-\overline{\nu_e} \to e^-\overline{\nu_e}$

Problem 1.7
High-energy muons traversing matter lose energy according to

$$-\frac{1}{\rho}\frac{\mathrm{d}E}{\mathrm{d}x} \approx a + bE$$

where a is due to ionisation energy loss and b is due to the bremsstrahlung and e^+e^- pair-production processes. For standard rock, taken to have $A = 22, Z = 11$ and $\rho = 2.65\mathrm{gcm}^{-3}$, the parameters a and b depend only weakly on the muon energy and have values $a \approx 2.5$ MeVg^{-1}cm^2 and $b \approx 3.5 \times 10^{-6}$g^{-1}cm^2.

(a) At what muon energy are the ionisation and bremsstrahlung/pair production processes equally important?

(b) Approximately how far does a 100 GeV cosmic-ray muon propagate in rock?

Solution:

(a) One could assume that ionisation and bremsstrahlung/pair production processes become equally important for a certain energy scale E^* when $a \simeq bE^*$. Such $E^* \simeq a/b$ can be calculated as ~ 700 GeV.

(b) Using the values given,

$$-\frac{\mathrm{d}E}{\mathrm{d}x} \approx a\rho + b\rho E \impliedby \left(a\rho \sim 6.6 \text{ MeV/cm} \text{ , } b\rho \sim 9.275 \times 10^{-6}/\text{cm}\right)$$
$$\simeq 7.52 \text{ MeV/cm}$$

which shows that a 100 GeV muon will go through around 132 metres of rock.

Problem 1.8
Tugsten has a radiation length of $X_0 = 0.35$ cm and a critical energy of $E_c = 7.97$ MeV. Roughly what thickness of tungsten is required to fully contain a 500 GeV electromagnetic shower from an electron?

Solution: Getting x_{max} for the given situation, one obtains :

$$x_{\text{max}} = \frac{1}{\ln 2} \ln\left(\frac{E}{E_c}\right) = \frac{1}{\ln 2} \ln\left(\frac{500 \text{ GeV}}{7.97 \text{ MeV}}\right) \sim 16$$

Thus, roughly around $x_{\text{max}}X_0 \simeq 5.6$cm of tungsten would be able to contain a 500 GeV electromagnetic shower from an electron.

Problem 1.9
The CPLEAR detector consisted of: tracking detectors in a magnetic field of 0.44 T; and electromagnetic calorimeter; and Čerenkov detectors with a radiator of refractive index $n = 1.25$ used to distinguish $\pi^\pm$ from $K^\pm$.

A charged particle travelling perpendicular to the direction of the magnetic field leaves a track with a measured radius of curvature of $R = 4$m. If it is observed to give a Čerenkov signal, is it possible to distinguish between the particle being a pion or kaon? Take $m_\pi \approx 140$ MeV/c^2 and $m_K \approx 494$ MeV/c^2

Solution: First, the momentum could be extracted from the fact that the charged particles are travelling perpendicular ($\lambda = 0$) to the 0.44 T magnetic field, which eventually gives $p = 0.3BR = 0.528$ GeV. The threshold mass for Čerenkov radiation in this case would be,

$$\sqrt{n^2 - 1}p = 0.75 \times p = 0.396 \text{ GeV}$$

thus in such situation it would be able to identity whether the track corresponds to a pion or kaon, as only m_π is smaller than the threshold Čerenkov radiation mass.

Problem 1.10
In a fixed-target pp experiment, what proton energy would be required to achieve the same centre-of-mass energy as the LHC, which will ultimately operate at 14 TeV.

Solution: Let the four-momentum of the beam proton and the fixed target proton as $p_1 = (E, 0, 0, p)$ and $p_2 = (m_p, 0, 0, 0)$. Using the following expression of the centre-of-mass energy $\sqrt{s}$, the proton energy E to satisfy the required situation would be :

$$\sqrt{s} = (p_1 + p_2)^2 = 2m_p^2 + 2p_1 \cdot p_2$$
$$= 2m_p\,(m_p + E) = 14 \text{ TeV} \implies \boxed{E \simeq 7.4 \text{ PeV}}$$

Problem 1.11
At the LEP e^+e^- collider, which had a circumference of 27 km, the electron and positron beam currents were both 1.0 mA. Each beam consisted of four equally spaced bunches of electrons/positrons. The bunches had an effective area of $1.8 \times 10^4 \mu m^2$. Calculate the instantaneous luminosity on the assumption that the beams collided head-on.

Solution: The instantaneous luminosity $\mathcal{L}$ could be computed as,

$$\mathcal{L} = f \frac{n_1 n_2}{4\pi\sigma_x\sigma_y}$$

As the problem states, the given effective area $1.8 \times 10^4 \mu m^2 = 1.8 \times 10^{-4} cm^2$ of the bunches will correspond to $\sigma_x\sigma_y$. In this case, the bunches are seperated $27/4 = 6.75$km and as the leptons were accelerated $\sim 0.99c$, the temporal separation between the beams will be $\simeq 5\mu s$ where relativistic effects are folded in, thus the collision frequency will be $f \simeq 200$kHz. Using the beam current $1.0 \times 10^{-3} C \cdot s^{-1}$, the number of electrons can be derived using

$$n_1 = n_2 = \frac{1.0\text{mA}}{ef} = \frac{1.0 \times 10^{-3} \text{C} \cdot \text{s}^{-1}}{1.62 \times 10^{-19}\text{C}} \times 5\mu\text{s} \simeq 3.5 \times 10^{10}$$

Then using all the numbers derived, the instantaneous luminosity can be calculated as,

$$\mathcal{L} = 2.0 \times 10^5 \text{s}^{-1} \times \frac{\left(3.5 \times 10^{10}\right)^2}{1.8 \times 10^{-4}\text{cm}^2} \simeq 1.3 \times 10^{30} \text{cm}^{-2} \cdot \text{s}^{-1}$$

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2 Underlying Concepts

Problem 2.1
When expressed in natural units the lifetime of the W boson is approximately $\tau \approx 0.5$ GeV^{-1}. What is the corresponding value in S.I. units?

Solution: In natural units, $\hbar = 1.055 \times 10^{-34} \text{J} \cdot \text{s} = 6.582 \times 10^{-25} \text{GeV} \cdot \text{s}$ which is, 1 GeV^{-1} $= 6.582 \times 10^{-25}$s. Thus the lifetime of the W boson in S.I. units can be written as, $\boxed{\tau \simeq 3.291 \times 10^{-25}\text{s}}$.

Problem 2.2
A cross section is measured to be 1 pb; convert this to natural units.

Solution: Taking note that $\hbar c = 0.197$ GeV fm, which is 0.197 GeV $= 1\text{fm}^{-1}$

$$1 \text{ pb} = 10^{-10} \text{ fm}^2 = 10^{-10} \times \left(\frac{1}{0.197}\right)^2 \text{ GeV}^{-2} = \boxed{2.57 \times 10^{-9} \text{ GeV}^{-2}}$$

Problem 2.3
Show that the process $\gamma \to e^+e^-$ can not occur in vacuum.

Solution: If it were so, such process should occur in any frame. Let such frame as the rest frame of

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Problem 2.4
A particle of mass 3 GeV is travelling in the positive z-direction with momentum 4 GeV. What are its energy and velocity?

Solution: Using the relation of $m^2 = E^2 - |\mathbf{p}|^2$, one gets $E^2 = 25$ GeV^2 thus the energy is $\boxed{E = 5 \text{ GeV}}$. Now considering the relation of $|\mathbf{p}| = E\beta$, it is seen that $\beta = |\mathbf{p}|E^{-1} = 0.8$ thus the velocity is $\boxed{0.8c}$.

Problem 2.5
In the laboratory frame, denoted Σ, a particle travelling in the z-direction has momentum $\mathbf{p} = p_z\hat{\mathbf{z}}$ and energy E.

(a) Use the Lorentz transformation to find expressions for the momentum p'_z and energy E' of the particle in a frame Σ' which is moving in a velopcity $\mathbf{v} = +v\hat{\mathbf{z}}$ relative to Σ, and show that $E^2 - p_z^2 = (E')^2 - (p'_z)^2$.

(b) For a system of particles, prove that the total four-momentum squared,

$$p^\mu p_\mu \equiv \left(\sum_i E_i\right)^2 - \left(\sum_i \mathbf{p}_i\right)^2$$

is invariant under Lorentz transformations.

Solution:

(a) Let the four-momentum of the given particle in the frame Σ and Σ' as $p = (E, 0, 0, p_z), p' = (E', \mathbf{p}')$ respectively. Denoting the corresponding matrix representation of the given Lorentz transformation as $\mathbf{\Lambda}$, one could write down the transformation of p as,

$$\begin{aligned} p' = \mathbf{\Lambda} p \implies p'^\mu &= \Lambda^\mu_\nu p^\nu \\ &= \begin{pmatrix} \gamma & 0 & 0 & -\gamma\beta \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\gamma\beta & 0 & 0 & \gamma \end{pmatrix} \begin{pmatrix} E \\ 0 \\ 0 \\ p_z \end{pmatrix} = \gamma \begin{pmatrix} E - \beta p_z \\ 0 \\ 0 \\ -E\beta + p_z \end{pmatrix} \end{aligned}$$

which implies that $E' = \gamma\,(E - \beta p_z)$ and $p'_z = -\gamma(E\beta - p_z)$. Using such expression of p', one could show that :

$$\begin{aligned} (E')^2 - (p'_z)^2 &= \gamma^2(E - \beta p_z)^2 - \gamma^2(E\beta - p_z)^2 \\ &= \gamma^2\left[(E - \beta p_z)^2 - (E\beta - p_z)^2\right] \\ &= \gamma^2\left[(E - \beta p_z + E\beta - p_z)\,(E - \beta p_z - E\beta + p_z)\right] \\ &= \gamma^2(1+\beta)(1-\beta)(E - p_z)(E + p_z) = E^2 - p_z^2 \quad \square \end{aligned}$$

(b)

Problem 2.6
For the decay $a \to 1 + 2$, show that the mass of the particle a can be expressed as

$$m_a^2 = m_1^2 + m_2^2 + 2E_1E_2\,(1 - \beta_1\beta_2\cos\theta)$$

where β_1 and β_2 are the velocities of the daughter particles and θ is the angle between them.

Solution: Let the four-momenta of the daughters as $p_i = (E_i, \mathbf{p}_i)$ for $i = 1, 2$. Momentum conservation states that $p_a = p_1 + p_2$ where p_a is the four-momentum of the mother particle. Squaring both sides, one obtains

$$\begin{aligned} p_a \cdot p_a = m_a^2 &= (p_1 + p_2)^2 \\ &= p_1 \cdot p_1 + p_2 \cdot p_2 + 2p_1 \cdot p_2 \\ &= m_1^2 + m_2^2 + 2(E_1E_2 - \mathbf{p}_1 \cdot \mathbf{p}_2) \\ &= m_1^2 + m_2^2 + 2(E_1E_2 - |\mathbf{p}_1||\mathbf{p}_2|\cos\theta) \\ &= m_1^2 + m_2^2 + 2(E_1E_2 - E_1\beta_1E_2\beta_2\cos\theta) \end{aligned}$$

$$= m_1^2 + m_2^2 + 2E_1E_2\left(1 - \beta_1\beta_2\cos\theta\right) \quad \square$$

Problem 2.7
In a collider experiment, Λ baryons can be identified from the decay $\Lambda \to \pi^- p$, which gives rise to a displaced vertex in a tracking detector. In a particular decay, the momenta of the π^+ and p are measured to be 0.75 GeV and 4.25 GeV respectively, and the opening angle between the tracks is 9°. The masses of the pion and proton are 189.6 MeV and 938.3 MeV.

(a) Calculate the mass of the Λ baryon.

(b) On average, Λ baryons of this energy are observed to decay at a distance of 0.35 m from the point of production. Calculate the lifetime of the Λ.

Solution:

(a) Let the four-momenta of π^-, p as $p_\pi = (0.75 \text{ GeV}, \mathbf{p}_\pi)$,$p_p = (4.25 \text{ GeV}, \mathbf{p}_p)$ respectively, which gives $p_\Lambda = p_\pi + p_p = (5 \text{ GeV}, \mathbf{p}_\pi + \mathbf{p}_p)$ as the four-momenta of Λ. Using the mass of the pion and proton, one could obtain

$$|\mathbf{p}_\pi|^2 = (0.75 \text{ GeV})^2 - m_\pi^2 \simeq 0.5265 \text{ GeV}^2 \implies |\mathbf{p}_\pi| \simeq 0.725 \text{ GeV}$$
$$|\mathbf{p}_p|^2 = (4.25 \text{ GeV})^2 - m_p^2 \simeq 17.18 \text{ GeV}^2 \implies |\mathbf{p}_\pi| \simeq 4.144 \text{ GeV}$$

The mass of the Λ baryon m_Λ can be acquired as :

$$\begin{aligned} m_\Lambda^2 = p_\Lambda \cdot p_\Lambda &= (5 \text{ GeV})^2 - |\mathbf{p}_\pi + \mathbf{p}_p|^2 \\ &= (5 \text{ GeV})^2 - \left[|\mathbf{p}_\pi|^2 + |\mathbf{p}_p|^2 + 2\,|\mathbf{p}_\pi|\,|\mathbf{p}_p|\cos 9^\circ\right] \\ &= (5 \text{ GeV})^2 - [0.5265 + 17.18 + 2 \cdot 0.725 \cdot 4.144 \cdot 0.98] \text{ GeV}^2 \\ &\simeq 1.35 \text{ GeV}^2 \implies \boxed{m_\Lambda \simeq 1.16 \text{ GeV}} \end{aligned}$$

which agrees well with experimental values.

(b) Let the lifetime and β of Λ as τ_Λ and β_Λ then one could realize that $c\beta_\Lambda\tau_\Lambda \sim 0.35$m. β_Λ can be simply derived using $\beta_\Lambda = |\mathbf{p}_\Lambda|/E_\Lambda \simeq 0.97$. Thus the lifetime of Λ becomes $\boxed{\tau_\Lambda \simeq 0.12 \times 10^{-8}\text{s}}$

Problem 2.8
In the laboratory frame, a proton with total energy E collides with proton at rest. Find the minimum proton energy such that process

$$p + p \to p + p + \bar{p} + \bar{p}$$

is kinematically allowed.

Solution:

Problem 2.9
Find the maximum opening angle between the photons produced in the decay $\pi^0 \to \gamma\gamma$ if the energy of the neutral pion is 10 GeV, given that $m_{\pi^0} = 135$ MeV.

Solution: Using the results derived in Problem 2 and taking account on the fact that photons are massless, one could write down

$$m_{\pi_0^2} = 2E_1E_2\left(1-\beta_1\beta_2\cos\theta\right) = 2E_1E_2\left(1-\cos\theta\right) \implies \cos\theta = \frac{m_{\pi_0}^2}{2E_1E_2} - 1$$

Taking account that $E_1 + E_2 = 10$ GeV, let $E_1 = E$ and express θ in terms of E as,

$$\cos\theta = \frac{m_{\pi_0}^2}{2E(10-E)} - 1$$

In the range of $E \in [0, 10]$ GeV the RHS of the above identity will take a local minimum when $E = 5$ GeV which will give the maximum value of θ, which will be denoted as θ^*. One could get θ^* as,

$$\cos\theta^* = \frac{\left(1.35\times10^{-1}\ \text{GeV}\right)^2}{100\ \text{GeV}^2} - 1 = -0.99981775 \implies \boxed{\theta^* \simeq 178.906099^\circ}$$

which is nearly back-to-back.

Problem 2.10
The maximum of the $\pi^- p$ cross section, which occurs at $p_\pi = 300$ MeV, corresponds to the resonant production of the Δ^0 baryon (i.e. $\sqrt{s} = m_\Delta$). What is the mass of the Δ?

Solution:

Problem 2.11
Tau-leptons are produced in the process $e^+e^- \to \tau^+\tau^-$ at a centre-of-mass energy of 91.2 GeV. The angular distribution of the π^- from the decay $\tau^- \to \pi^-\nu_\tau$ is

$$\frac{\mathrm{d}N}{\mathrm{d}\left(\cos\theta^*\right)} \propto 1 + \cos\theta^*$$

where θ^* is the polar angle of the π^- in the tau-lepton rest frame, relative to the direction defined by the τ spin. Determine the laboratory frame energy distribution of the π^- for the cases where the tau lepton spin is (i) *aligned with* or (ii) *opposite* to its direction of flight.

Solution:

Problem 2.12
For the process $1 + 2 \to 3 + 4$, the Mandelstam variables s, t and u are defined as $s = (p_1 + p_2)^2, t = (p_1 - p_3)^2$ and $u = (p_1 - p_4)^2$. Show that

$$s + t + u = m_1^2 + m_2^2 + m_3^2 + m_4^2.$$

Solution: By definition of the Mandelstam variables, one could express $(s + t + u)$ as

$$\begin{aligned}
s + t + u &= (p_1 + p_2)^2 + (p_1 - p_3)^2 + (p_1 - p_4)^2 \\
&= \sum_i p_i \cdot p_i + 2p_1 \cdot p_1 + 2p_1 \cdot p_2 - 2p_1 \cdot p_3 - 2p_1 \cdot p_4 \\
&= \sum_i m_i^2 + 2p_1 \cdot (p_1 + p_2 - p_3 - p_4) \\
&= \sum_i m_i^2 \quad \square
\end{aligned}$$

The fact that in any frame $p^\mu p_\mu = m^2$ for a particle with mass m is used in the third identity, and in the last step the conservation of momentum $p_1 + p_2 = p_3 + p_4$ is used.

Problem 2.13
At the HERA collider, 27.5 GeV electrons were collided head-on with 820 GeV protons. Calculate the centre-of-mass energy.

Solution: Let the four-momentum of the electron and proton as $p_e = (E_e, \mathbf{p}_e)$, $p_p = (E_p, \mathbf{p}_p)$ respectively. The centre-of-mass energy $\sqrt{s}$ can be expressed as,

$$
\begin{aligned}
s &= (p_e + p_p)^2 = p_e \cdot p_e + p_p \cdot p_p + 2p_e \cdot p_p \\
&= m_e^2 + m_p^2 + 2\left(E_e E_p - \mathbf{p}_e \cdot \mathbf{p}_p\right) \\
&= m_e^2 + m_p^2 + 2\left(E_e E_p + |\mathbf{p}_e|\,|\mathbf{p}_p|\right) \simeq 4E_e E_p \qquad \left(|\mathbf{p}_i|^2 = E_i^2 - m_i^2 \sim E_i^2\right)
\end{aligned}
$$

As the collision is occuring head-on, one could say that $\mathbf{p}_e \cdot \mathbf{p}_p = -|\mathbf{p}_e|\,|\mathbf{p}_p|$ which was used in the last identity. Looking upon the order of the variables, $m_e \simeq 0.5$ MeV$, m_p \simeq 93.8$ MeV and $E_e = 27.5$ GeV$, E_p = 820$ GeV for an approximation it is okay to consider $m_e, m_p \sim 0$. Thus the centre-of-mass energy $\boxed{\sqrt{s} \simeq 300 \text{ GeV}}$ when all the needed values are plugged in.

Problem 2.14
Consider the Compton scattering of a photon of momentum $\mathbf{k}$ and energy $E = |\mathbf{k}| = \mathbf{k}$ from an electron at rest. Writing the four-momenta of the scattered photon and electron respectively as k' and p', conservation of four-momentum is expressed as $k + p = k' + p'$. Use the relation $p'^2 = m_e^2$ to show that the energy of the scattered photon is given by

$$E' = \frac{E}{1 + (E/m_e)(1 - \cos\theta)}$$

Solution:

Problem 2.15
Using the commutation relations for position and momentum, prove that

$$\left[\hat{L}_x, \hat{L}_y\right] = i\hat{L}_z$$

Using the commutation relations for the components of angular momenta prove

$$\left[\hat{L}^2, \hat{L}_x\right] = 0$$

and

$$\hat{L}^2 = \hat{L}_-\hat{L}_+ + \hat{L}_z + \hat{L}_z^2$$

Solution:

Problem 2.16
Show that the operators $\hat{S}_i = \frac{1}{2}\sigma_i$, where σ_i are the three Pauli spin-matrices,

$$\hat{S}_x = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \hat{S}_y = \frac{1}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \text{and} \quad \hat{S}_z = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

satisfy the same algebra as the angular momentum operators, namely

$$\left[\hat{S}_x, \hat{S}_y\right] = i\hat{S}_z \quad \left[\hat{S}_y, \hat{S}_z\right] = i\hat{S}_x \quad \text{and} \quad \left[\hat{S}_z, \hat{S}_x\right] = i\hat{S}_y$$

Find the eigenvalue(s) of the operator $\hat{\mathbf{S}}^2 = \frac{1}{4}\left(\hat{S}_x^2 + \hat{S}_y^2 + \hat{S}_z^2\right)$ and deduce that the eigenstates of $\hat{S}_z$ are a suitable representation of a spin-half particle.

Solution:

Problem 2.17
Find the third-order term in the transition matrix element of Fermi's golden rule.

Solution:

---

3 Decay Rates and Cross Sections

Problem 3.1
Calculate the energy of the μ^- produced in the decay at rest $\pi^- \to \mu\bar{\nu}_\mu$. Assume $m_\pi = 140$ MeV, $m_\mu = 106$ MeV and take $m_\nu \sim 0$.

Solution: Let the four-momenta of the muon and the neutrino to be $p_1 = (E_1, 0, 0, E_2)$ and $p_2 = (E_2, 0, 0, -E_2)$. In the pion rest frame, $E_1 + E_2 = m_\pi$ and from the muon mass constraint $m_\mu^2 = E_1^2 - E_2^2$. Solving these equation gives

$$\boxed{E_1 = \frac{m_\pi^2 + m_\mu^2}{2m_\pi} = 110.13 \text{ GeV}}$$

Problem 3.2
For the decay $a \to 1 + 2$, show that the momenta of both daughter particles in the centre-of mass frame p^* are

$$p^* = \frac{1}{2m_a}\sqrt{\left[m_a^2 - (m_1^2 + m_2^2)\right]\left[m_a^2 - (m_1^2 - m_2^2)\right]}$$

.

Solution: Let the four-momenta of the mother particle and the daughter particles to be $p_a = (m_a, 0, 0, 0)$, $p_1 = (E_1, 0, 0, p^*)$, $p_2 = (E_2, 0, 0, -p^*)$ From the mass constraints, we get $E_1 + E_2 = m_a$, $E_1^2 - p^{*2} = m_1^2$, and $E_2^2 - p^{*2} = m_2^2$.

Since we have three unknown variables E_1, E_2, p^* and three equations, it is possible to get p^* in terms of m_a, m_1 and m_2, which gives the desired solution. In detail,

$$p^{*2} = E_1^2 - m_1^2 = E_2^2 - m_2^2 \implies E_1^2 = E_2^2 + m_1^2 - m_2^2$$

$$= (m_a - E_1)^2 + m_1^2 - m_2^2 \implies E_1 = \frac{1}{2m_a}\left(m_a^2 + m_1^2 - m_2^2\right)$$

which leads to a similar expression of E_2 using $E_1 + E_2 = m_a$

$$E_2 = m_a - E_1 = \frac{1}{2m_a}\left(m_a^2 - m_1^2 + m_2^2\right)$$

Then one could finally write down p^* in terms of m_a, m_1, m_2 as, using the fact that $p^{*2} = E_1^2 - m_1^2 = E_2^2 - m_2^2$

$$p^{*2} = \frac{1}{2}\left[E_1^2 + E_2^2 - \left(m_1^2 + m_2^2\right)\right]$$

$$= \frac{1}{2}\left[(E_1 + E_2)^2 - 2E_1E_2 - \left(m_1^2 + m_2^2\right)\right]$$

$$
\begin{aligned}
&= \frac{1}{2}\left[m_a^2 - \frac{1}{2m_a^2}\left[m_a^2-(m_1^2+m_2^2)\right]\left[m_a^2-(m_1^2-m_2^2)\right] - \left(m_1^2+m_2^2\right)\right] \\
&= \frac{1}{2}\left[m_a^2-\left(m_1^2+m_2^2\right)\right]\left[1-\frac{1}{2m_a^2}\left[m_a^2-\left(m_1^2-m_2^2\right)\right]\right] \\
&= \frac{1}{2}\left[m_a^2-\left(m_1^2+m_2^2\right)\right]\left[1-\frac{1}{2m_a^2}\left[m_a^2-\left(m_1^2-m_2^2\right)\right]\right] \\
&= \frac{1}{4m_a^2}\left[m_a^2-(m_1^2+m_2^2)\right]\left[m_a^2-(m_1^2-m_2^2)\right] \quad \square
\end{aligned}
$$

Problem 3.3
Calculate the branching ratio for the decay $K^+ \to \pi^+\pi^0$, given the partial decay width $\Gamma(K^+ \to \pi^+\pi^0) = 1.2 \times 10^{-8}$ eV and the mean kaon lifetime $\tau(K^+) = 1.2 \times 10^{-8}$ s.

Solution: Using the given information,

$$
\begin{aligned}
\mathrm{BR}\left(K^+ \to \pi^+\pi^0\right) &= \frac{1}{\Gamma_{K^+}} \times \Gamma\left(K^+ \to \pi^+\pi^0\right) \\
&= \tau(K^+) \times \Gamma\left(K^+ \to \pi^+\pi^0\right) \\
&= \left(1.2 \times 10^{-8}\mathrm{s}\right) \times \left(1.2 \times 10^{-8}\mathrm{eV}\right) \\
&= \left(1.2 \times 10^{-8}\right) \times \left(\frac{1}{6.58} \times 10^{16}\ \mathrm{eV}^{-1}\right) \times \left(1.2 \times 10^{-8}\mathrm{eV}\right) \\
&= \frac{1.2^2}{6.58} \simeq \boxed{21\%}
\end{aligned}
$$

which is as much as expected from the known branching rate.

Problem 3.4
At a future e^+e^- linear collider operating as a Higgs factory at a centre-of-mass energy of $\sqrt{s} = 250$ GeV, the cross section for the process $e^+e^- \to HZ$ is 250 fb. If the collider has an instantaneous luminosity of $2 \times 10^{34}\mathrm{cm}^{-2}\mathrm{s}^{-1}$ and is operational for 50% of the time, how many Higgs bosons will be produced in five years of running?

Solution: Let the total number of Higgs bosons that will be produced in 5 years of running in such condition as N, then one could calculate N as,

$$
\begin{aligned}
N &= \left(2 \times 10^{34}\mathrm{cm}^{-2}\mathrm{s}^{-1}\right) \times (5\ \mathrm{yrs}) \times (250\ \mathrm{fb}) \times 0.5 \\
&= \left(2 \times 10^{34}\mathrm{cm}^{-2}\mathrm{s}^{-1}\right)\left(1.5768 \times 10^{8}\mathrm{s}\right) \times \left(2.5 \times 10^{-37}\mathrm{cm}^2\right) \times 0.5 \\
&= 3.942 \times 10^5
\end{aligned}
$$

Problem 3.5
The total $e^+e^- \to \gamma \to \mu^+\mu^-$ annihilation cross section is $\sigma = 4\pi\alpha^2/3s$, where $\alpha \simeq 1/137$. Calculate the cross section at $\sqrt{s} = 50$ GeV, expressing your answer in both natural units and in barns. Compare this to the total pp cross section at $\sqrt{s} = 50$ GeV which is approximately 40 mb and comment on the result.

Solution: Plugging in all the values we know in natural units,

$$\sigma = \frac{4\pi}{3 \cdot \left(2.5 \times 10^3 \text{ GeV}^2\right) \cdot 137^2} = 8.9 \times 10^{-8} \text{ GeV}^{-2}$$

which could be converted into barns using 1 $\text{GeV}^{-2} = 0.3894$ mb, gives $\boxed{\sigma = 346.5 \text{ pb}}$.

Problem 3.6
A 1 GeV muon neutrino is fired at a 1m thick block of iron with density $\rho = 7.874 \times 10^3 \text{kg} \cdot \text{m}^{-3}$. If the average neutrino-nucleon interaction cross section is $\sigma = 8 \times 10^{-39} \text{m}^2$, calculate the (small) probability that the neutrino interacts in the block.

Solution: The muon neutrino will pass through $\sim 7.874 \times 10^3 \text{kg} \cdot \text{m}^{-2}$ of iron. As iron has atomic mass of 56, around 56 g of iron will contain 6.022×10^{23} number of nucleons, which is nearly 8.43×10^{28} nucleons for 7.874×10^3 kg of iron. This could be considered as a flux of nucleons per area $\sim 8.43 \times 10^{28} \text{m}^{-2}$. Thus the probability could be derived as the neutrino-nucleon interaction cross section multiplied with such flux of nuclei, which gives $\boxed{\sim 6.74 \times 10^{-14}}$.

Problem 3.7
For the process $a + b \to 1 + 2$ the Lorents-invariant flux term installed

$$F = 4\left[(p_a \cdot p_b)^2 - m_a^2 m_b^2\right]^{\frac{1}{2}}$$

In the non-relativistic limit, $\beta_a, \beta_b \ll 1$, show that

$$F \approx 4 m_a m_b \left|\mathbf{v}_a - \mathbf{v}_b\right|$$

where $\mathbf{v}_a, \mathbf{v}_b$ are the (non-relativistic) velocities of the two particles.

Solution: Let the four-momenta of a,b as $p_a = (E_a, \mathbf{p}_a)$ and $p_b = (E_b, \mathbf{p}_b)$. Under the non-relativistic limit which implies that $\gamma_a, \gamma_b \simeq 1$, one could write down F as

$$\begin{aligned}
F &= 4\left[(p_a \cdot p_b)^2 - m_a^2 m_b^2\right]^{\frac{1}{2}} \\
&= 4\left[(E_a E_b - m_a m_b \boldsymbol{\beta}_a \cdot \boldsymbol{\beta}_b)^2 - m_a^2 m_b^2\right]^{\frac{1}{2}} \\
&= 4\left[(E_a E_b - m_a m_b \boldsymbol{\beta}_a \cdot \boldsymbol{\beta}_b)^2 - m_a^2 m_b^2\right]^{\frac{1}{2}} \\
&= 4\left[(E_a E_b - m_a m_b \boldsymbol{\beta}_a \cdot \boldsymbol{\beta}_b)^2 - m_a^2 m_b^2\right]^{\frac{1}{2}}
\end{aligned}$$

Problem 3.8
The Lorentz-invariant flux term for the process $a + b \to 1 + 2$ in the centre-of-mass frame was shown to be $F = 4p_i^* \sqrt{s}$, where p_i^* is the momentum of the initial-state particles. Show that the corresponding expression in the frame where b is at rest is

$$F = 4 m_b p_a.$$

Solution:

Problem 3.9
Show that the momentum in the centre-of-mass frame of the initial-state particles in a two-body scattering process can be expressed as

$$p_i^{*2} = \frac{1}{4s}\left[s-(m_1+m_2)^2\right]\left[s-(m_1-m_2)^2\right]$$

Solution:

Problem 3.10
Repeat the calculation of Section 3.5.2 for the process $e^- p \to e^- p$ where the mass of the electron is no longer neglected.

(a) First show that

$$\frac{\mathrm{d}E}{\mathrm{d}\left(E\cos\theta\right)} = \frac{p_1 p_3^2}{p_3\left(E_1+m_p\right)-E_3 p_1\cos\theta}$$

(b) Then show that

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{1}{64\pi^2}\cdot\frac{p_3^2}{p_1 m_p}\cdot\frac{1}{p_3\left(E_1+m_p\right)-E_3 p_1\cos\theta}\cdot\left|\mathcal{M}_{fi}\right|^2$$

Solution:

---

4 The Dirac Equation

Problem 4.1
Show that

$$\left[\hat{\mathbf{p}}^2, \hat{\mathbf{r}}\times\hat{\mathbf{p}}\right] = 0,$$

and hence the Hamiltonian of the free-particle Schrödinger equation commutes with the angular momentum operator.

Solution: One could expand the given commutator as,

$$\begin{aligned}
\left[\hat{\mathbf{p}}^2, \hat{\mathbf{r}}\times\hat{\mathbf{p}}\right] &= \left[\hat{\mathbf{p}}_a\hat{\mathbf{p}}_a, \epsilon_{abc}r_c\hat{\mathbf{p}}_b\hat{\mathbf{c}}\right] \\
&= \epsilon_{abc}r_c\left[\hat{\mathbf{p}}_a\hat{\mathbf{p}}_a, \hat{\mathbf{p}}_b\hat{\mathbf{c}}\right] \\
&= \epsilon_{abc}r_c\left\{\hat{\mathbf{p}}_a\left[\hat{\mathbf{p}}_a, \hat{\mathbf{p}}_b\hat{\mathbf{c}}\right] + \left[\hat{\mathbf{p}}_a, \hat{\mathbf{p}}_b\hat{\mathbf{c}}\right]\hat{\mathbf{p}}_a\right\} \\
&= \epsilon_{abc}r_c\left\{\hat{\mathbf{p}}_a\left[\hat{\mathbf{p}}_a, \hat{\mathbf{p}}_b\hat{\mathbf{c}}\right] + \left[\hat{\mathbf{p}}_a, \hat{\mathbf{p}}_b\hat{\mathbf{c}}\right]\hat{\mathbf{p}}_a\right\} \qquad \Longleftarrow \quad \left[\hat{\mathbf{p}}_a, \hat{\mathbf{p}}_b\hat{\mathbf{c}}\right] = \delta_{ab}\hat{\mathbf{c}} - i\delta_{ac}\hat{\mathbf{p}}_b
\end{aligned}$$

which will eliminate due to the contraction between ϵ_{abc} and δ_{ab}, δ_{ac}.

Problem 4.2
Show that u_1 and u_2 are orthogonal, i.e. $u_1^\dagger u_2 = 0$.

Solution: Let us first denote

$$u_\uparrow = \begin{pmatrix}1\\0\end{pmatrix} \quad \text{and} \quad u_\downarrow = \begin{pmatrix}0\\1\end{pmatrix}.$$

One should note that $u_\uparrow^\dagger u_\downarrow = 0$. u_1, u_2 could also be expressed in terms of

$$u_1 = \begin{pmatrix} u_\uparrow \\ \frac{\boldsymbol{\sigma}\cdot\mathbf{p}}{E+m} u_\uparrow \end{pmatrix} \quad \text{and} \quad u_2 = \begin{pmatrix} u_\downarrow \\ \frac{\boldsymbol{\sigma}\cdot\mathbf{p}}{E+m} u_\downarrow \end{pmatrix}.$$

Now $u_1^\dagger u_2$ could be written as,

$$\begin{aligned}
u_1^\dagger u_2 &= \begin{pmatrix} u_\uparrow \\ \frac{\boldsymbol{\sigma}\cdot\mathbf{p}}{E+m} u_\uparrow \end{pmatrix}^\dagger \begin{pmatrix} u_\downarrow \\ \frac{\boldsymbol{\sigma}\cdot\mathbf{p}}{E+m} u_\downarrow \end{pmatrix} \\
&= \begin{pmatrix} u_\uparrow^\dagger & \frac{1}{E+m} \left((\boldsymbol{\sigma}\cdot\mathbf{p})\, u_\uparrow \right)^\dagger \end{pmatrix} \begin{pmatrix} u_\downarrow \\ \frac{\boldsymbol{\sigma}\cdot\mathbf{p}}{E+m} u_\downarrow \end{pmatrix} \\
&= u_\uparrow^\dagger u_\downarrow + \frac{1}{(E+m)^2} \left((\boldsymbol{\sigma}\cdot\mathbf{p})\, u_\uparrow \right)^\dagger \left((\boldsymbol{\sigma}\cdot\mathbf{p})\, u_\downarrow \right) \\
&= u_\uparrow^\dagger u_\downarrow + \frac{1}{(E+m)^2} u_\uparrow^\dagger (\boldsymbol{\sigma}\cdot\mathbf{p})^\dagger (\boldsymbol{\sigma}\cdot\mathbf{p})\, u_\downarrow
\end{aligned}$$

One could use the fact that,

$$(\boldsymbol{\sigma}\cdot\mathbf{p})^\dagger (\boldsymbol{\sigma}\cdot\mathbf{p}) = (\boldsymbol{\sigma}\cdot\mathbf{p})^2 = \begin{pmatrix} p_x^2 + p_y^2 + p_z^2 & 0 \\ 0 & p_x^2 + p_y^2 + p_z^2 \end{pmatrix} = \left(E^2 - m^2 \right) I_2$$

Thus it could be tidied up as,

$$\begin{aligned}
u_1^\dagger u_2 &= u_\uparrow^\dagger u_\downarrow + \frac{1}{(E+m)^2} u_\uparrow^\dagger (\boldsymbol{\sigma}\cdot\mathbf{p})^\dagger (\boldsymbol{\sigma}\cdot\mathbf{p})\, u_\downarrow \\
&= \left[1 + \frac{E^2 - m^2}{(E+m)^2} \right] u_\uparrow^\dagger u_\downarrow = 0 \quad \square
\end{aligned}$$

Problem 4.3
Verify the statement that the Einstein energy-momentum relationship is recovered if any of the four Dirac spinors of (4.48) are subtitutes into the Dirac equation written in terms of momentum, $(\gamma^\mu p_\mu - m)\, u = 0$.

Solution: Let us choose u_1 to plug in the Dirac equation. Then it could be expressed as,

$$\begin{aligned}
(\not{p} - m)\, u_1 = 0 &\implies \begin{pmatrix} (E-m)\, I_2 & -\boldsymbol{\sigma}\cdot\mathbf{p} \\ \boldsymbol{\sigma}\cdot\mathbf{p} & -(E+m)\, I_2 \end{pmatrix} \begin{pmatrix} E+m \\ 0 \\ p_z \\ p_x + ip_y \end{pmatrix} = 0 \\
&\implies \begin{pmatrix} E^2 - m^2 \\ 0 \end{pmatrix} + \boldsymbol{\sigma}\cdot\mathbf{p} \begin{pmatrix} E+m-p_z \\ -p_x - ip_y \end{pmatrix} - (E+m) \begin{pmatrix} p_z \\ p_x + ip_y \end{pmatrix} = 0 \\
\text{[first row]} &\implies \left(E^2 - m^2 \right) + (E+m)\, p_z - \left(p_x^2 + p_y^2 + p_z^2 \right) - (E+m)\, p_z = 0 \\
&\implies E^2 = p_x^2 + p_y^2 + p_z^2 + m^2 \quad \square
\end{aligned}$$

Problem 4.4
For a particle with four-momentum $p^\mu = (E, \mathbf{p})$, the general solution to the free-particle Dirac equation can be written

$$\psi(p) = [au_1(p) + bu_2(p)]\, e^{i(\mathbf{p}\cdot\mathbf{x} - Et)}$$

Using the explicit forms for u_1 and u_2, show that the four-vector current $j^\mu = (\rho, \mathbf{j})$ is given by

$$j^\mu = 2p^\mu$$

Furthermore, show that the resulting probability density and probability current are consistent with a particle moving with velocity $\beta = p/E$.

Solution:

Problem 4.5
Writing the four-component spinor u in terms of two two-component vectors

$$u = \begin{pmatrix} u_A \\ u_B \end{pmatrix},$$

show that in the non-relativistic limit, where $\beta \cong v/c \ll 1$, the components of u_B are smaller than those of u_A by a factor v/c.

Solution:

Problem 4.6
By considering the three cases $\mu = \nu = 0$, $\mu = \nu \neq 0$ and $\mu \neq \nu$ show that

$$\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2g^{\mu\nu}.$$

Solution: For brevity, the gamma matrices will be presented in terms of direct products as :

$$\begin{aligned}
\gamma^0 &= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \otimes I_2 \equiv \beta \\
\gamma^i &\equiv \beta\alpha_i \\
&= \left[\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \otimes I_2\right]\left[\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \otimes \sigma_i\right] \\
&= \left[\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \otimes \sigma_i\right]
\end{aligned}$$

Then considering the three cases,

(a) $\mu = \nu = 0$

$$\begin{aligned}
\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu &= 2\gamma^0\gamma^0 \\
&= 2\left[\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \otimes I_2\right]\left[\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \otimes I_2\right] \\
&= 2\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \otimes I_2 = 2I_4 = 2g^{00}I_4
\end{aligned}$$

(b) $\mu \neq 0, \nu \neq 0$

$$\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = \gamma^i\gamma^j + \gamma^j\gamma^i$$

$$= \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}^2 \otimes \sigma_i\sigma_j + \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}^2 \otimes \sigma_j\sigma_i$$

$$= -I_2 \otimes (\sigma_j\sigma_i + \sigma_i\sigma_j)$$

$$= -I_2 \otimes \{\sigma_i, \sigma_j\}$$

$$= -2I_2 \otimes \delta_{ij} I_2 = -2\delta_{ij} I_4 = 2g^{ij} I_4$$

(c) $\mu \neq \nu$

Due to the result from (b), one could easily realize that when both indices are not 0, $\left\{\gamma^i, \gamma^j\right\} = 0$ which is $2g^{ij}I_4$. Consider one of the indices to be 0, then :

$$\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = \gamma^0\gamma^i + \gamma^i\gamma^0$$

$$= \left[\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\right] \otimes \sigma_i = 0 = 2g^{0i} I_4 \quad \square$$

Problem 4.7
By operating on the Dirac equation,

$$(i\gamma^\mu\partial_\mu - m)\,\psi = 0$$

with $\gamma^\nu\partial_\nu$ prove that the components of ψ satisfy the Klein-Gordon equation,

$$\left(\partial^\mu\partial_\mu + m^2\right)\psi = 0.$$

Solution: Straightforwardly following the instructions given by the problem,

$$\begin{aligned}
(i\gamma^\mu\partial_\mu - m)\,\psi = 0 &\implies \gamma^\nu\partial_\nu\,(i\gamma^\mu\partial_\mu - m)\,\psi = 0 \\
&\implies \left[i\gamma^\nu\partial_\nu\,(\gamma^\mu\partial_\mu) - m\gamma^\nu\partial_\nu\right]\psi = 0 \\
&\implies \left[i\gamma^\nu\,(\partial_\nu\gamma^\mu)\,\partial_\mu + i\gamma^\nu\gamma^\mu\partial_\nu\partial_\mu - m\gamma^\nu\partial_\nu\right]\psi = 0 \\
&\implies \left(\frac{i}{2}\{\gamma^\nu, \gamma^\mu\}\,\partial_\nu\partial_\mu - m\gamma^\nu\partial_\nu\right)\psi = 0 \\
&\implies (ig^{\nu\mu}\partial_\nu\partial_\mu - m\gamma^\nu\partial_\nu)\,\psi = 0
\end{aligned}$$

For the latter term, one could utilize the Dirac equation :

$$(i\gamma^\mu\partial_\mu - m)\,\psi = 0 \implies \partial\!\!\!/\psi = -im\psi$$

Thus the above could be tidied up as,

$$\begin{aligned}
(ig^{\nu\mu}\partial_\nu\partial_\mu - m\gamma^\nu\partial_\nu)\,\psi = 0 &\implies \left(i\partial^\mu\partial_\mu - m\partial\!\!\!/\right)\psi = 0 \\
&\implies \left(\partial^\mu\partial_\mu + m^2\right)\psi = 0 \quad \square
\end{aligned}$$

Problem 4.8
Show that

$$(\gamma^\mu)^\dagger = \gamma^0\gamma^\mu\gamma^0.$$

Solution: Let us seperate the cases with indices being 0 and else.

(a) $\mu = 0$

$$\begin{aligned}\gamma^{0\dagger} &= \gamma^0 \\ &= I_4\gamma^0 = \gamma^0\gamma^0\gamma^0\end{aligned}$$

(b) $\mu = k \neq 0$

$$\begin{aligned}\gamma^{k\dagger} &= -\gamma^k \\ &= -I_4\gamma^k = -\gamma^0\gamma^0\gamma^k = \gamma^0\gamma^k\gamma^0 \quad \square\end{aligned}$$

Problem 4.9
Starting from

$$(\gamma^\mu p_\mu - m)\, u = 0,$$

show that the corresponding equation for the adjoint spinor is

$$\bar{u}\,(\gamma^\mu p_\mu - m) = 0.$$

Hence, without using the explicit form for the u spinors, show that the normalisation condition $u^\dagger u = 2E$ leads to

$$\bar{u}u = 2m,$$

and that

$$\bar{u}\gamma^\mu u = 2p^\mu.$$

Solution: Let us first derive the corresponding Dirac equation for the adjoint spinor.

$$\begin{aligned}(\gamma^\mu p_\mu - m)\, u = 0 &\implies u^\dagger\,(\gamma^\mu p_\mu - m)^\dagger = 0 \\ &\implies \bar{u}\gamma^0\left(\gamma^{\mu\dagger} p_\mu - m\right) = 0 \quad \text{from} \quad \bar{u} = u^\dagger\gamma^0 \iff u^\dagger = \bar{u}\gamma^0 \\ &\implies \bar{u}\left(\gamma^0\gamma^{\mu\dagger}\gamma^0 p_\mu - m\gamma^0\gamma^0\right) = 0 \\ &\implies \bar{u}\,(\gamma^\mu p_\mu - m) = 0 \quad \text{from} \quad \gamma^0\gamma^\mu\gamma^0 = \gamma^{\mu\dagger}\end{aligned}$$

In order to obtain the other relations, let us start from evaluating $\bar{u}\gamma^\mu u$ first.

$$\begin{aligned}\bar{u}\gamma^\mu u &= \frac{1}{m}\bar{u}\gamma^\mu \not{p} u \qquad \Longleftarrow \quad \not{p}u = mu \\ &= \frac{1}{m}\bar{u}\gamma^\mu\gamma^\nu p_\nu u = \frac{1}{m}\bar{u}\left[2g^{\mu\nu} - \gamma^\nu\gamma^\mu\right] p_\nu u \\ &= \frac{1}{m}\left[2\bar{u}up^\mu - \bar{u}\gamma^\nu\gamma^\mu p_\nu u\right]\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{m}\left[2\bar{u}up^\mu - \bar{u}\not{p}\gamma^\mu u\right] \qquad \Longleftarrow \quad \bar{u}\not{p} = m\bar{u} \\
&= \frac{1}{m}\left[2\bar{u}up^\mu - m\bar{u}\gamma^\mu u\right] \\
&\Longleftrightarrow \quad \bar{u}\gamma^\mu u = \frac{1}{m}\bar{u}p^\mu u
\end{aligned}
$$

Under such relation letting $\mu = 0$ gives

$$
\begin{aligned}
\bar{u}\gamma^0 u = \frac{1}{m}\bar{u}p^0 u \implies & u^\dagger\gamma^0\gamma^0 u = \frac{E}{m}\bar{u}u \\
\implies & u^\dagger u = \frac{E}{m}\bar{u}u \\
\implies & 2E = \frac{E}{m}\bar{u}u \\
\implies & \bar{u}u = 2m
\end{aligned}
$$

Now plugging in such relation back into $\bar{u}\gamma^\mu u$ gives,

$$
\bar{u}\gamma^\mu u = \frac{1}{m}\bar{u}up^\mu = 2p^\mu. \quad \square
$$

Problem 4.10
Demonstrate that the two relations of Equation (4.45) are consistent by showing that

$$
(\boldsymbol{\sigma}\cdot\mathbf{p})^2 = \mathbf{p}^2.
$$

Solution: One could easily show that

$$
\begin{aligned}
(\boldsymbol{\sigma}\cdot\mathbf{p})^2 &= \sigma^i\sigma^j p_i p_j \\
&= \frac{1}{2}\left(\sigma^i\sigma^j + \sigma^j\sigma^i\right)p_i p_j \quad \Longleftarrow \quad \left\{\sigma^i, \sigma^j\right\} = 2\delta^{ij} \\
&= \delta^{ij} p_i p_j = \mathbf{p}^2. \quad \square
\end{aligned}
$$

One could notice that not only for $\mathbf{p}$ but for any cartesian vector the above should hold. This relation leads to the equivalence of,

$$
\begin{aligned}
u_A = \frac{(\boldsymbol{\sigma}\cdot\mathbf{p})}{E-m}u_B \implies & (\boldsymbol{\sigma}\cdot\mathbf{p})\, u_A = \frac{(\boldsymbol{\sigma}\cdot\mathbf{p})^2}{E-m}u_B \\
\implies & (\boldsymbol{\sigma}\cdot\mathbf{p})\, u_A = \frac{\mathbf{p}^2}{E-m}u_B \\
\implies & \frac{(\boldsymbol{\sigma}\cdot\mathbf{p})}{E+m}u_A = u_B
\end{aligned}
$$

which is the desired result.

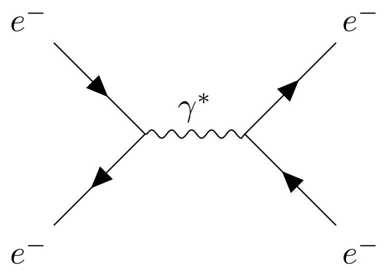
Problem 4.11
Consider the $e^+e^- \to \gamma \to e^+e^-$ annihilation process in the centre-of-mass frame where the energy of the photon is $2E$. Discuss energy and charge conservation for the two cases where:

(a) the negative energy solutions of the Dirac equation are interpreted as negative energy particles propagating backwards in time

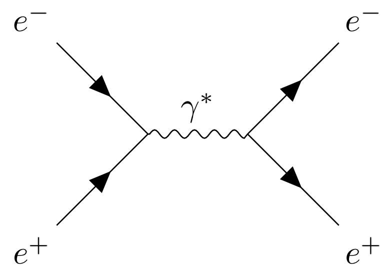
(b) the negative energy solutions of the Dirac equation are interpreted as positive energy antiparticles propagating forwards in time

Solution:

(a) the negative energy solutions of the Dirac equation are interpreted as negative energy particles propagating backwards in time



(b) the negative energy solutions of the Dirac equation are interpreted as positive energy antiparticles propa- gating forwards in time



Problem 4.12
Verify that the helicity operator

$$\hat{h} = \frac{\hat{\boldsymbol{\Sigma}} \cdot \hat{\mathbf{p}}}{2p} = \frac{1}{2p}\begin{pmatrix} \sigma \cdot \hat{\mathbf{p}} & 0 \\ 0 & \sigma \cdot \hat{\mathbf{p}} \end{pmatrix}$$

commutes with the Dirac Hamiltonian,

$$\hat{H}_D = \boldsymbol{\alpha} \cdot \hat{\mathbf{p}} + \beta m$$

Solution: The commutation between $\hat{h}$ and $\hat{H}_D$ could be written as,

$$[\hat{H}_D, \hat{h}] = \frac{1}{2p}\left[\boldsymbol{\alpha} \cdot \hat{\mathbf{p}} + \beta m, \hat{\boldsymbol{\Sigma}} \cdot \hat{\mathbf{p}}\right]$$

$$= \frac{1}{2p}\left[\boldsymbol{\alpha} \cdot \hat{\mathbf{p}}, \hat{\boldsymbol{\Sigma}} \cdot \hat{\mathbf{p}}\right] + \frac{m}{2p}\left[\beta, \hat{\boldsymbol{\Sigma}} \cdot \hat{\mathbf{p}}\right]$$

Let us first evaluate the second commutator :

$$\left[\beta, \hat{\boldsymbol{\Sigma}} \cdot \hat{\mathbf{p}}\right] = [\beta, \Sigma_i \hat{p}_i]$$

$$= \cancelto{0}{[\beta, \Sigma_i]}\,\hat{p}_i + \Sigma_i \cancelto{0}{[\beta, \hat{p}_i]}$$

where the second term is eliminated due to the fact that matrix operation and differentiation could be commuted (one could check oneself by operating on an arbitrary spinor). The first commutator then could be again reduced into,

$$\left[\boldsymbol{\alpha} \cdot \hat{\mathbf{p}}, \hat{\boldsymbol{\Sigma}} \cdot \hat{\mathbf{p}}\right] = [\alpha_i \hat{p}_i, \Sigma_j \hat{p}_j]$$

$$= \alpha_i \cancelto{0}{[\hat{p}_i, \Sigma_j]}\,\hat{p}_j + \alpha_i \Sigma_j \cancelto{0}{[\hat{p}_i, \hat{p}_j]} + [\alpha_i, \Sigma_j]\,\hat{p}_j \hat{p}_i + \Sigma_j \cancelto{0}{[\alpha_i, \hat{p}_j]}\,\hat{p}_i$$

$$= 2i\epsilon_{ijk}\alpha_k\hat{p}_j\hat{p}_i = 2i\epsilon_{ijk}\alpha_k\left(\delta_{ji} - \hat{p}_i\hat{p}_j\right) = 2i\epsilon_{ijk}\alpha_k\hat{p}_i\hat{p}_j$$

$$= 0 \quad \square$$

Problem 4.13
4.13 Show that

$$\mathsf{P}u_\uparrow(\theta,\phi) = u_\downarrow(\pi-\theta,\pi+\phi)$$

and comment on the result.

Solution: One could straightforwardly write down,

$$\mathsf{P}u_\uparrow(\theta,\phi) = \gamma^0 u_\uparrow(\theta,\phi) = N\gamma^0\begin{pmatrix}\cos\frac{\theta}{2}\\ e^{i\phi}\sin\frac{\theta}{2}\\ \frac{p}{E+m}\cos\frac{\theta}{2}\\ \frac{p}{E+m}e^{i\phi}\sin\frac{\theta}{2}\end{pmatrix} = N\begin{pmatrix}\cos\frac{\theta}{2}\\ e^{i\phi}\sin\frac{\theta}{2}\\ -\frac{p}{E+m}\cos\frac{\theta}{2}\\ -\frac{p}{E+m}e^{i\phi}\sin\frac{\theta}{2}\end{pmatrix}$$

$$= N\begin{pmatrix}-\sin\left(\frac{\pi}{2}-\frac{\theta}{2}\right)\\ e^{i(\pi+\phi)}\cos\left(\frac{\pi}{2}-\frac{\theta}{2}\right)\\ \frac{p}{E+m}\sin\left(\frac{\pi}{2}-\frac{\theta}{2}\right)\\ -\frac{p}{E+m}e^{i(\pi+\phi)}\cos\left(\frac{\pi}{2}-\frac{\theta}{2}\right)\end{pmatrix}$$

$$= u_\downarrow(\pi-\theta,\pi+\phi) \quad \square$$

One could notice that in terms of spherical angles, the result above shows that the parity operator actually flips the momentum direction to $-\mathbf{p}$, which will result in the opposite helicity.

Problem 4.14
Under the combined operation of parity and charge conjugation (CP) spinors transform as

$$\psi \to \psi^C = \mathsf{CP}\psi = i\gamma^2\gamma^0\psi^*$$

Show that up to an overall complex phase factor

$$\mathsf{CP}u_\uparrow(\theta,\phi) = v_\downarrow(\pi-\theta,\pi+\phi)$$

Solution: Using the result of Problem 4.13 which shows how P acts on helicity eigen-spinors, one only needs to show that

$$\mathsf{CP}u_\uparrow(\theta,\phi) = \mathsf{C}u_\downarrow(\pi-\theta,\pi+\phi) = v_\downarrow(\pi-\theta,\pi+\phi)$$

$$\implies \mathsf{C}u_\downarrow(\theta',\phi') = v_\downarrow(\theta',\phi')$$

up to a certain complex phase factor. One could directly compute,

$$\mathsf{C}u_\downarrow(\theta,\phi) = i\gamma^2 u_\downarrow(\theta,\phi)^*$$

$$= N\begin{pmatrix}0 & i\sigma_2\\ -i\sigma_2 & 0\end{pmatrix}\begin{pmatrix}-\sin\theta\\ \cos\theta e^{i\phi}\\ r\sin\theta\\ -r\cos\theta e^{i\phi}\end{pmatrix}^* \quad \text{where} \quad r \equiv \frac{p}{E+m}$$

$$= -N \begin{pmatrix} r\cos\theta e^{-i\phi} \\ r\sin\theta \\ \cos\theta e^{-i\phi} \\ \sin\theta \end{pmatrix} = -Ne^{-i\phi} \begin{pmatrix} r\cos\theta \\ r\sin\theta e^{i\phi} \\ \cos\theta \\ \sin\theta e^{i\phi} \end{pmatrix} = -e^{-i\phi} v_{\downarrow}(\theta, \phi) \quad \square$$

Problem 4.15
Starting from the Dirac equation, derive the identity

$$\bar{u}(p')\gamma^\mu u(p) = \frac{1}{2m}\bar{u}(p')(p+p')u(p) + \frac{i}{m}\bar{u}(p')\Sigma^{\mu\nu}q_\nu u(p)$$

where $q = p' - p$ and $\Sigma^{\mu\nu} = \frac{i}{4}[\gamma^\mu, \gamma^\nu]$

Solution: Using the Dirac equation for $u(p)$, one could write down $\bar{u}(p')\gamma^\mu u(p)$ as,

$$\bar{u}(p')\gamma^\mu u(p) = \frac{1}{m}\bar{u}(p')\gamma^\mu \not{p} u(p) \tag{1}$$

$$= \frac{1}{m}\bar{u}(p')\gamma^\mu\gamma^\nu p_\nu u(p)$$

$$= \frac{1}{m}\bar{u}(p')(2g^{\mu\nu} - \gamma^\nu\gamma^\mu)p_\nu u(p)$$

$$= \frac{2}{m}\bar{u}(p')p^\mu u(p) - \frac{1}{m}\bar{u}(p')\not{p}\gamma^\mu u(p) \tag{2}$$

One could do the same using the Dirac equation for the adjoint case,

$$\bar{u}(p')\gamma^\mu u(p) = \frac{1}{m}\bar{u}(p')\not{p}'\gamma^\mu u(p) \tag{3}$$

$$= \frac{1}{m}\bar{u}(p')\gamma^\nu\gamma^\mu p'_\nu u(p)$$

$$= \frac{1}{m}\bar{u}(p')(2g^{\nu\mu} - \gamma^\mu\gamma^\nu)p'_\nu u(p)$$

$$= \frac{2}{m}\bar{u}(p')p'^\mu u(p) - \frac{1}{m}\bar{u}(p')\gamma^\mu\not{p}' u(p) \tag{4}$$

Using the above relations one could again express $\bar{u}(p')\gamma^\mu u(p)$ as,

$$\bar{u}(p')\gamma^\mu u(p) = \frac{1}{2}\left\{\frac{1}{m}\bar{u}(p')\gamma^\mu\not{p}u(p) + \frac{1}{m}\bar{u}(p')\not{p}'\gamma^\mu u(p)\right\}$$

$$= \frac{1}{m}\bar{u}(p')(p+p')^\mu u(p) - \frac{1}{2m}\bar{u}(p')\left\{\not{p}\gamma^\mu + \gamma^\mu\not{p}'\right\}u(p) \tag{5}$$

Before moving on, one could again use another relation that could be derived from (1) to (4) as :

$$(1) = (2) \implies \frac{1}{m}\bar{u}(p')\gamma^\mu\not{p}u(p) = \frac{2}{m}\bar{u}(p')p^\mu u(p) - \frac{1}{m}\bar{u}(p')\not{p}\gamma^\mu u(p)$$

$$\implies 2\bar{u}(p')p^\mu u(p) = \bar{u}(p')\left\{\gamma^\mu\not{p} + \not{p}\gamma^\mu\right\}u(p) \tag{6}$$

$$(3) = (4) \implies \bar{u}(p')\not{p}'\gamma^\mu u(p) = \frac{2}{m}\bar{u}(p')p'^\mu u(p) - \frac{1}{m}\bar{u}(p')\gamma^\mu\not{p}' u(p)$$

$$\implies 2\bar{u}(p')p'^{\mu}u(p) = \bar{u}(p')\left\{\gamma^{\mu}\not{p}' + \not{p}'\gamma^{\mu}\right\}u(p) \tag{7}$$

Adding up both (6) and (7) gives,

$$\bar{u}(p')(p+p')^{\mu}u(p) = \frac{1}{2}\bar{u}(p')\left\{\gamma^{\mu}\not{p}' + \not{p}'\gamma^{\mu} + \gamma^{\mu}\not{p} + \not{p}\gamma^{\mu}\right\}u(p) \tag{8}$$

Plugging (8) into (5) but splitting the first term into half gives,

$$\begin{aligned}
\bar{u}(p')\gamma^{\mu}u(p) &= \frac{1}{2m}\bar{u}(p')(p+p')^{\mu}u(p) + \frac{1}{2m}\bar{u}(p')(p+p')^{\mu}u(p) - \frac{1}{2m}\bar{u}(p')\left\{\not{p}\gamma^{\mu} + \gamma^{\mu}\not{p}'\right\}u(p) \\
&= \frac{1}{2m}\bar{u}(p')(p+p')^{\mu}u(p) + \frac{1}{4m}\bar{u}(p')\left\{\gamma^{\mu}\not{p}' + \not{p}'\gamma^{\mu} + \gamma^{\mu}\not{p} + \not{p}\gamma^{\mu}\right\}u(p) - \frac{1}{2m}\bar{u}(p')\left\{\not{p}\gamma^{\mu} + \gamma^{\mu}\not{p}'\right\}u(p) \\
&= \frac{1}{2m}\bar{u}(p')(p+p')^{\mu}u(p) - \frac{1}{4m}\bar{u}(p')\left\{\gamma^{\mu}\not{p}' - \not{p}'\gamma^{\mu} - \gamma^{\mu}\not{p} + \not{p}\gamma^{\mu}\right\}u(p) \\
&= \frac{1}{2m}\bar{u}(p')(p+p')^{\mu}u(p) - \frac{1}{4m}\bar{u}(p')\left\{\gamma^{\mu}(\not{p}' - \not{p}) - (\not{p}' - \not{p})\gamma^{\mu}\right\}u(p) \\
&= \frac{1}{2m}\bar{u}(p')(p+p')^{\mu}u(p) - \frac{1}{4m}\bar{u}(p')\left[\gamma^{\mu}, \not{q}\right]u(p) \\
&= \frac{1}{2m}\bar{u}(p')(p+p')^{\mu}u(p) - \frac{1}{4m}\bar{u}(p')\left[\gamma^{\mu}, \gamma^{\nu}\right]q_{\nu}u(p) \\
&= \frac{1}{2m}\bar{u}(p')(p+p')^{\mu}u(p) + \frac{i}{m}\bar{u}\left(p'\right)\Sigma^{\mu\nu}q_{\nu}u(p) \quad \square
\end{aligned}$$

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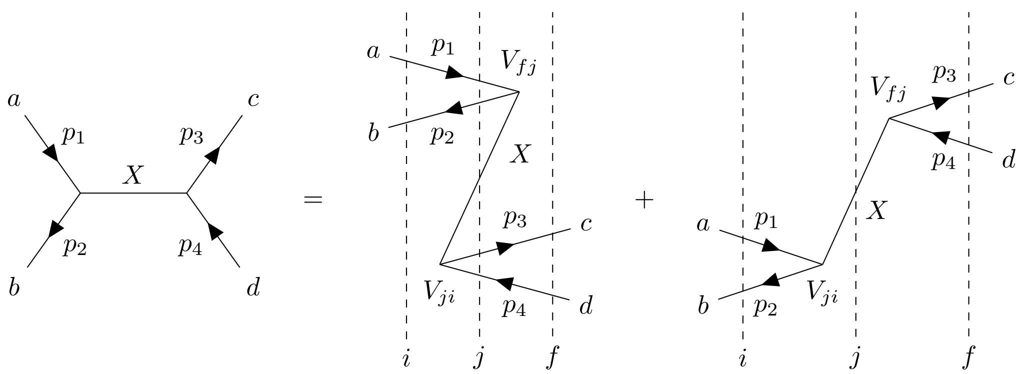
5 Interaction by Particle Exchange

Problem 5.1
Draw the two time-ordered diagrams for the s-channel process shown in Figure 5.5. By repeating the steps of Section 5.1.1, show that the propagator has the same form as obtained for the t-channel process.

Hint: one of the time-ordered diagrams is non-intuitive, remember that in second-order perturbation theory the intermediate state does not conserve energy.

Solution: Similar with the t-channel case introduced in the text, one could draw the time-ordered diagram for the s-channel case as,



For the first diagram, one could write down the second-order perturbation term as :

$$\mathcal{T}_{fi}^{(1)} = \frac{1}{E_i - E_j} V_{fj}^{(1)} V_{ji}^{(1)} = \frac{1}{E_i - E_j} \langle X + a + b | V | 0 \rangle \langle 0 | V | X + c + d \rangle$$

$$= \frac{1}{(E_a + E_b) - (E_a + E_b + E_X + E_c + E_d)} \cdot \frac{\mathcal{M}_{a \to b + X}}{\sqrt{2E_X 2E_a 2E_b}} \cdot \frac{\mathcal{M}_{c \to d + X}}{\sqrt{2E_X 2E_c 2E_d}}$$

$$= -\frac{1}{E_X + E_c + E_d} \cdot \frac{g_a g_c}{2E_X} \cdot \frac{1}{\sqrt{2E_a 2E_b 2E_c 2E_d}}$$

where we again assume scalar LI matrix elements. Similarly the corresponding term for the second diagram could be calculated as :

$$\mathcal{T}_{fi}^{(2)} = \frac{1}{E_i - E_j} V_{fj}^{(2)} V_{ji}^{(2)} = \frac{1}{E_i - E_j} \langle c + d | V | X \rangle \langle X | V | a + b \rangle$$

$$= \frac{1}{(E_a + E_b) - E_X} \cdot \frac{\mathcal{M}_{c \to d + X}}{\sqrt{2E_X 2E_c 2E_d}} \cdot \frac{\mathcal{M}_{a \to b + X}}{\sqrt{2E_X 2E_a 2E_b}}$$

$$= \frac{1}{E_a + E_b - E_X} \cdot \frac{g_a g_c}{2E_X} \cdot \frac{1}{\sqrt{2E_a 2E_b 2E_c 2E_d}}$$

The full LI matrix element then could be written as,

$$\mathcal{M}_{fi} = \sqrt{2E_a 2E_b 2E_c 2E_d} \left\{ \mathcal{T}_{fi}^{(1)} + \mathcal{T}_{fi}^{(2)} \right\}$$

$$= \frac{g_a g_c}{2E_X} \left[\frac{1}{E_a + E_b - E_X} - \frac{1}{E_c + E_d + E_X} \right] \quad \Longleftarrow \quad \begin{array}{c} E_a + E_b = E_c + E_d \\ \text{(from energy conservation)} \end{array}$$

$$= \frac{g_a g_c}{2E_X} \left[\frac{1}{E_a + E_b - E_X} - \frac{1}{E_a + E_b + E_X} \right]$$

$$= \frac{g_a g_c}{2E_X} \cdot \frac{2E_X}{(E_a + E_b)^2 - E_X^2}$$

$$= \frac{g_a g_c}{(E_a + E_b)^2 - (\mathbf{p}_a + \mathbf{p}_b)^2 - m_X^2} \quad \Longleftarrow \quad q \equiv p_a + p_b$$

$$= \frac{g_a g_c}{q^2 - m_X^2} \quad \square$$

which shows that the propagator term for s-channel diagrams also show the same form with

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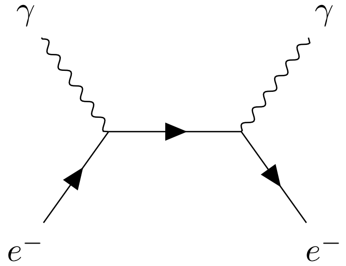
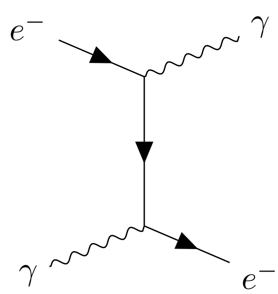
Problem 5.2
Draw the two lowest-order Feynman diagrams for the Compton scattering process $\gamma e^- \to \gamma e^-$.

Solution:

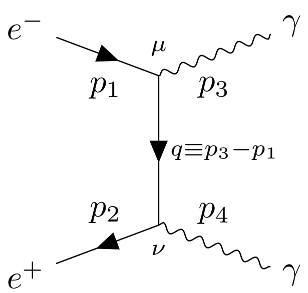




Problem 5.3
Draw the lowest-order t-channel and u-channel Feynman diagrams for $e^+e^- \to \gamma\gamma$ and use the Feynman rules for QED to write down the corresponding matrix elements.

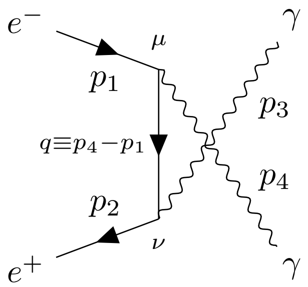
Solution:

(a) t-channel



$$= e^2 \epsilon^*_\mu(p_3)\gamma^\mu u(p_1) \left[\frac{i\left(\not{q}+m\right)}{q^2-m^2}\right] \epsilon^*_\nu(p_4)\gamma^\nu \bar{v}(p_2)$$

(b) u-channel



$$= e^2 \epsilon^*_\mu(p_4)\gamma^\mu u(p_1) \left[\frac{i\left(\not{q}+m\right)}{q^2-m^2}\right] \epsilon^*_\nu(p_3)\gamma^\nu \bar{v}(p_2)$$

6 Electron-Positron Annihilation

Problem 6.1
Using the properties of the γ-matrices of (4.33) and (4.34), and the definition of $\gamma^5 \equiv \gamma^0\gamma^1\gamma^2\gamma^3$, show that

$$\left(\gamma^5\right)^2 = 1 \quad , \quad \gamma^{5\dagger} = i\gamma^5 \quad \text{and} \quad \gamma^5\gamma^\mu = -\gamma^\mu\gamma^5$$

Solution:

(a) $\left(\gamma^5\right)^2 = 1$

$$\begin{aligned}
\left(\gamma^5\right)^2 &= -\gamma^0\gamma^1\gamma^2\gamma^3\gamma^0\gamma^1\gamma^2\gamma^3 \\
&= (-1)^4\,\gamma^0\gamma^0\gamma^1\gamma^2\gamma^3\gamma^1\gamma^2\gamma^3 \\
&= (-1)^6\,\gamma^1\gamma^1\gamma^2\gamma^3\gamma^2\gamma^3 \\
&= (-1)^8\,\gamma^2\gamma^2\gamma^3\gamma^3 = 1 \quad \square
\end{aligned}$$

(b) $\gamma^{5\dagger} = \gamma^5$

$$\begin{aligned}
\gamma^{5\dagger} = \left(i\gamma^0\gamma^1\gamma^2\gamma^3\right)^\dagger &= -i\gamma^{3\dagger}\gamma^{2\dagger}\gamma^{1\dagger}\gamma^{0\dagger} \\
&= -i\,(-1)^3\,\gamma^3\gamma^2\gamma^1\gamma^0 \\
&= -i\,(-1)^3\,(-1)^6\,\gamma^0\gamma^1\gamma^2\gamma^3 = \gamma^5 \quad \square
\end{aligned}$$

(c) $\gamma^5\gamma^\mu = -\gamma^\mu\gamma^5$

$$\begin{aligned}\gamma^5\gamma^\mu &= i\gamma^0\gamma^1\gamma^2\gamma^3\gamma^\mu \\ &= (-1)^3\,\gamma^\mu i\gamma^0\gamma^1\gamma^2\gamma^3 = -\gamma^\mu\gamma^5 \quad \square\end{aligned}$$

Here the fact that μ will be one of $0,1,2,3$ is used to obtain the $(-1)^3$ factor, as one of the indices will be identical, thus not giving an additional -1 factor when switching the position of γ^μ.

Problem 6.2
Show that the chiral projection operators

$$P_R = \frac{1}{2}\left(1+\gamma^5\right) \quad \text{and} \quad P_L = \frac{1}{2}\left(1-\gamma^5\right)$$

satisfy

$$P_R + P_L = 1 \quad , \quad P_R P_R = P_R \quad , \quad P_L P_L = P_L \quad \text{and} \quad P_L P_R = 0$$

Solution:

(a) $P_R + P_L = 1$: Trivial

(b) $P_R P_R = P_R$

$$P_R P_R = \frac{1}{4}\left(1+\gamma^5\right)\left(1+\gamma^5\right) = \frac{1}{4}\left(1+2\gamma^5+\gamma^5\gamma^5\right) = \frac{1}{4}\left(2+2\gamma^5\right) = P_R \quad \square$$

(c) $P_L P_L = P_L$

$$P_L P_L = \frac{1}{4}\left(1-\gamma^5\right)\left(1-\gamma^5\right) = \frac{1}{4}\left(1-2\gamma^5+\gamma^5\gamma^5\right) = \frac{1}{4}\left(2-2\gamma^5\right) = P_L \quad \square$$

(d) $P_L P_R = 0$

$$P_L P_R = \frac{1}{4}\left(1-\gamma^5\right)\left(1+\gamma^5\right) = \frac{1}{4}\left(1-\gamma^5\gamma^5\right) = 0 \quad \square$$

Problem 6.3
Show that

$$\Lambda^+ = \frac{m+\not{p}}{2m} \quad \text{and} \quad \Lambda^- = \frac{m-\not{p}}{2m}$$

are also projection operators, and show that they respectively project out particle and antiparticle states, i.e.

$$\Lambda^+ u = u \quad , \quad \Lambda^- v = v \quad \text{and} \quad \Lambda^+ v = \Lambda^- u = 0$$

Solution:

(a) Show that $\Lambda^{\pm}$ are projection operators.

- $\Lambda^+ + \Lambda^- = 1$: Trivial
- $\Lambda^+\Lambda^+ = \Lambda^+$

$$\Lambda^+\Lambda^+ = \frac{1}{4m^2}\left(m + \not{p}\right)\left(m + \not{p}\right) = \frac{1}{4m^2}\left(m^2 + 2m\not{p} + \not{p}\not{p}\right)$$
$$= \frac{1}{4m^2}\left(2m^2 + 2m\not{p}\right) = \Lambda^+$$

where the following identity is used :

$$\not{p}\not{p} = \gamma^\mu p_\mu \gamma^\nu p_\nu = \frac{1}{2}\left\{\gamma^\mu, \gamma^\nu\right\} p_\mu p_\nu = p \cdot p = m^2$$

- $\Lambda^-\Lambda^- = \Lambda^-$

$$\Lambda^-\Lambda^- = \frac{1}{4m^2}\left(m - \not{p}\right)\left(m - \not{p}\right) = \frac{1}{4m^2}\left(m^2 - 2m\not{p} + \not{p}\not{p}\right)$$
$$= \frac{1}{4m^2}\left(2m^2 - 2m\not{p}\right) = \Lambda^-$$

- $\Lambda^+\Lambda^- = 0$

$$\Lambda^+\Lambda^- = \frac{1}{4m^2}\left(m + \not{p}\right)\left(m - \not{p}\right) = \frac{1}{4m^2}\left(m^2 - \not{p}\not{p}\right) = 0$$

(b) Using the Dirac equations, $\left(\not{p} - m\right) u = 0$ and $\left(\not{p} + m\right) v = 0$ one could easily show the projections :

$$\Lambda^+ u = \frac{1}{2m}\left(m + \not{p}\right) u = \frac{1}{2m}\left(mu + \not{p}u\right) = \frac{2mu}{2m} = u$$
$$\Lambda^+ v = \frac{1}{2m}\left(m + \not{p}\right) v = 0$$
$$\Lambda^- v = \frac{1}{2m}\left(m - \not{p}\right) v = \frac{1}{2m}\left(mv - \not{p}v\right) = \frac{2mv}{2m} = v$$
$$\Lambda^- u = \frac{1}{2m}\left(m - \not{p}\right) u = 0 \quad \square$$

Problem 6.4
Show that the helicity operator can be expressed as

$$\hat{h} = -\frac{1}{2}\frac{\gamma^0\gamma^5\boldsymbol{\gamma}\cdot\mathbf{p}}{p}$$

Solution: Using the following form of the helicity operator,

$$\hat{h} = \frac{1}{2p}\hat{\Sigma} \cdot \hat{p} = \frac{1}{2p}\left(1 \otimes \boldsymbol{\sigma} \cdot \mathbf{p}\right)$$

Using the properties of Kronecker prodcuts, one could decompose $1 \otimes \boldsymbol{\sigma} \cdot \mathbf{p}$ as

$$1 \otimes \boldsymbol{\sigma} \cdot \mathbf{p} = -\left(i\sigma^2 \otimes 1\right)\left(i\sigma^2 \otimes \boldsymbol{\sigma} \cdot \mathbf{p}\right) \tag{9}$$

The two terms in the right-hand side of (9) can be again written as,

$$\begin{aligned} i\sigma^2 \otimes 1 &= -\sigma^1\sigma^3 \otimes 1 \\ &= -\left(\sigma^1 \otimes 1\right)\left(\sigma^3 \otimes 1\right) = -\gamma^0\gamma^5 \end{aligned} \tag{10}$$

$$i\sigma^2 \otimes \boldsymbol{\sigma} \cdot \mathbf{p} = \left(i\sigma^2 \otimes \sigma^j\right) p_j = \boldsymbol{\gamma} \cdot \mathbf{p} \tag{11}$$

where the expression of gamma matrices, $\gamma^0 = \sigma^3 \otimes 1$, $\gamma^j = i\sigma^2 \otimes \sigma^j$ and $\gamma^5 = \sigma^1 \otimes 1$ is used. Plugging in (10) and (11) into the original expression of the helicity operator,

$$\hat{h} = \frac{1}{2p}\left(1 \otimes \boldsymbol{\sigma} \cdot \mathbf{p}\right) = -\frac{1}{2p}\gamma^5\gamma^0\boldsymbol{\gamma} \cdot \mathbf{p} = \frac{1}{2p}\gamma^0\gamma^5\boldsymbol{\gamma} \cdot \mathbf{p} \quad \square$$

Problem 6.5
In general terms, explain why high-energy electron-positron colliders must also have high instantaneous luminosities.

Solution: As seen in the text, the cross section σ for electron-positron annihilation decreases as the center-of-mass energy $\sqrt{s}$ increases, from the relation $\sigma \sim s^{-1}$. Thus, such colliders must have high instantaneous luminosities in order to compensate the decreasing effect on the cross section stemming from the high collision energy.

Problem 6.6
For a spin-1 system, the eigenstate of the operator $\hat{S}_n = \mathbf{n} \cdot \hat{\mathbf{S}}$ with eigenvalue $+1$ corresponds to the spin being in the direction $\hat{\mathbf{n}}$. Writing this state in terms of the eigenstates of $\hat{S}_z$, i.e.

$$\left|1, +1\right\rangle_\theta = \alpha\left|1, -1\right\rangle + \beta\left|1, 0\right\rangle + \gamma\left|1, +1\right\rangle$$

and taking $\mathbf{n} = (\sin\theta, 0, \cos\theta)$ show that

$$\left|1, +1\right\rangle_\theta = \frac{1}{2}\left(1 - \cos\theta\right)\left|1, -1\right\rangle + \frac{1}{\sqrt{2}}\sin\theta\left|1, 0\right\rangle + \frac{1}{2}\left(1 + \cos\theta\right)\left|1, +1\right\rangle$$

Hint: Write $\hat{S}_x$ in terms of the ladder operators.

Solution: Using the given $\mathbf{n}$, the operator $\hat{S}_n$ can be written as $\sin\theta\hat{S}_x + \cos\theta\hat{S}_z$.

$$\hat{S}_n = \sin\theta\hat{S}_x + \cos\theta\hat{S}_z = \frac{1}{2}\sin\theta\left(\hat{S}_+ + \hat{S}_-\right) + \cos\theta\hat{S}_z$$

where $\hat{S}_{\pm}$ are the ladder operators which follows $\hat{S}_{\pm}|1,m\rangle = \sqrt{2-m(m\pm 1)}\,|1,m\pm 1\rangle$. Then using the above expression of $\hat{S}_n$, one could write down

$$\begin{aligned}\hat{S}_n|1,+1\rangle_\theta &= \frac{1}{2}\sin\theta\left(\hat{S}_+ + \hat{S}_-\right)|1,+1\rangle_\theta + \cos\theta\hat{S}_z|1,+1\rangle_\theta \\ &= \left(\frac{1}{\sqrt{2}}\sin\theta\beta - \alpha\cos\theta\right)|1,-1\rangle + \frac{1}{\sqrt{2}}(\alpha+\gamma)|1,0\rangle + \left(\frac{1}{\sqrt{2}}\sin\theta\beta + \gamma\cos\theta\right)|1,+1\rangle\end{aligned}$$

and from the definition of $\hat{S}_n$, it should satisfy $\hat{S}_n|1,+1\rangle_\theta = |1,+1\rangle_\theta$ which gives a set of linear equations of α, β and γ :

$$\alpha = \frac{1}{\sqrt{2}}\sin\theta\beta - \alpha\cos\theta$$

$$\beta = \frac{1}{\sqrt{2}}(\alpha+\gamma)$$

$$\gamma = \frac{1}{\sqrt{2}}\sin\theta\beta + \gamma\cos\theta$$

which gives the following solution :

$$\alpha = \frac{1}{2}(1-\cos\theta)$$

$$\beta = \frac{1}{\sqrt{2}}\sin\theta$$

$$\gamma = \frac{1}{2}(1+\cos\theta) \quad \square$$

Problem 6.7
Using helicity amplitudes, calculate the differential cross section for $e^-\mu^- \to e^-\mu^-$ scattering in the following steps :

(a) From the Feynman rules for QED, show that the lowest-order QED matrix element for $e^-\mu^- \to e^-\mu^-$ is

$$\mathcal{M}_{fi} = -\frac{e^2}{(p_1-p_3)^2}g_{\mu\nu}\left[\bar{u}(p_3)\gamma^\mu u(p_1)\right]\left[\bar{u}(p_4)\gamma^\nu u(p_2)\right]$$

where p_1 and p_3 are the four-momenta of the initial and final state e^-, and p_2 and p_4 are the four-momenta of the initial and final state μ^-.

(b) Working in the centre-of-mass frame, and writing the four-momenta of the initial- and final-state e^- as $p_1^\mu = (E_1, 0, 0, p)$ and $p_3^\mu = (E_1, p\sin\theta, 0, p\cos\theta)$ respectively, show that the electron currents for the four possible helicity combinations are

$$\begin{aligned}\bar{u}_\downarrow(p_3)\gamma^\mu u_\downarrow(p_1) &= 2(E_1c, ps, -ips, pc)\\ \bar{u}_\uparrow(p_3)\gamma^\mu u_\downarrow(p_1) &= 2(ms, 0, 0, 0)\\ \bar{u}_\uparrow(p_3)\gamma^\mu u_\uparrow(p_1) &= 2(E_1c, ps, ips, pc)\end{aligned}$$

$$\bar{u}_\downarrow(p_3)\gamma^\mu u_\uparrow(p_1) = -2(ms, 0, 0, 0)$$

where m is the electron mass, $s = \sin(\theta/2)$ and $c = \cos(\theta/2)$.

(c) Explain why the effect of the parity operator $\hat{\mathsf{P}} = \gamma^0$ is

$$\hat{\mathsf{P}} u_\uparrow(p, \theta, \phi) = u_\downarrow(p, \pi - \theta, \pi + \theta)$$

Hence, or otherwise, show that the muon currents for the four helicity combinations are

$$\begin{aligned}
\bar{u}_\downarrow(p_4)\gamma^\mu u_\downarrow(p_2) &= 2(E_2 c, -ps, -ips, -pc)\\
\bar{u}_\uparrow(p_4)\gamma^\mu u_\downarrow(p_2) &= 2(Ms, 0, 0, 0)\\
\bar{u}_\uparrow(p_4)\gamma^\mu u_\uparrow(p_2) &= 2(E_2 c, -ps, ips, -pc)\\
\bar{u}_\downarrow(p_4)\gamma^\mu u_\uparrow(p_2) &= -2(Ms, 0, 0, 0)
\end{aligned}$$

where M is the muon mass.

(d) For the relativistic limit where $E \gg M$, , show that the matrix element squared for the case where the incoming e^- and incoming μ^- are both left-handed is given by

$$|\mathcal{M}_{LL}|^2 = \frac{4e^2 s^2}{(p_1 - p_3)^4}.$$

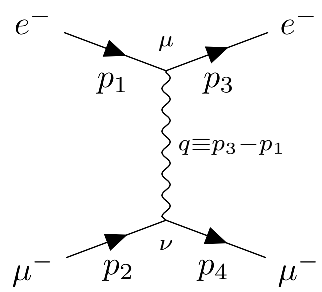
where $s = (p_1 + p_2)^2$. Find the corresponding expressions for $|\mathcal{M}_{RL}|^2, |\mathcal{M}_{RR}|^2$ and $|\mathcal{M}_{LR}|^2$.

(e) In this relativistic limit, show that the differential cross section for unpolarised $e^-\mu^- \to e^-\mu^-$ scattering in the centre-of-mass frame is

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{2\alpha^2}{s} \cdot \frac{1 + \frac{1}{4}(1 + \cos\theta)^2}{(1 - \cos\theta)^2}.$$

Solution:

(a) Obtain lowest-order QED matrix element.



$$= -i\mathcal{M} = \bar{u}(p_3) ie\gamma^\mu u(p_1) \frac{-ig^{\mu\nu}}{(p_3 - p_1)^2} \bar{u}(p_4) ie\gamma^\nu u(p_2)$$

$$= \frac{ie^2}{(p_1 - p_3)^2} g_{\mu\nu} \left[\bar{u}(p_3)\gamma^\mu u(p_1)\right] \left[\bar{u}(p_4)\gamma^\nu u(p_2)\right] \quad \square$$

(b) Letting the incoming and outgoing e^- with $(\theta', \phi') = (0, 0)$ and $(\theta, 0)$ respectively, one could write down the corresponding spinors as ,

$$u_\uparrow(p_1) = \sqrt{E_1+m}\begin{pmatrix} 1 \\ 0 \\ \frac{p}{E_1+m} \\ 0 \end{pmatrix} \quad , \quad u_\downarrow(p_1) = \sqrt{E_1+m}\begin{pmatrix} 0 \\ 1 \\ 0 \\ \frac{-p}{E_1+m} \end{pmatrix}$$

$$u_\uparrow(p_3) = \sqrt{E_1+m}\begin{pmatrix} c \\ s \\ \frac{p}{E_1+m}c \\ \frac{p}{E_1+m}s \end{pmatrix} \quad , \quad u_\downarrow(p_3) = \sqrt{E_1+m}\begin{pmatrix} -s \\ c \\ \frac{p}{E_1+m}s \\ \frac{-p}{E_1+m}c \end{pmatrix}$$

Using such expression of the spinors, one could directly calculate the possible electron currents as :

$$\begin{aligned}
\bar{u}_\downarrow(p_3)\gamma^\mu u_\downarrow(p_1) &= 2(E_1c, ps, -ips, pc) \\
\bar{u}_\uparrow(p_3)\gamma^\mu u_\downarrow(p_1) &= 2(ms, 0, 0, 0) \\
\bar{u}_\uparrow(p_3)\gamma^\mu u_\uparrow(p_1) &= 2(E_1c, ps, ips, pc) \\
\bar{u}_\downarrow(p_3)\gamma^\mu u_\uparrow(p_1) &= -2(ms, 0, 0, 0)
\end{aligned}$$

(Trust me I really did all the calculations lol)

(c) The effect of $\hat{\mathsf{P}}$ on helicity states was once discussed in Problem (4.13), that acting such operator will actually flip the spatial part of p also leading to an opposite helicity state, hence $u_\downarrow$. Also, considering how the four-momenta are set from the electron currents, one could notice that by substituting $E_1 \leftrightarrow E_2$ and $m \leftrightarrow M$ while having all the electron current helicity transformed under $\hat{\mathsf{P}}$ will get us the full set of the muon current helicity combinations.

(d) Denoting the corresponding helicity configuration with momentum p_i as h_i, $\mathcal{M}_{LL}$ can be expressed as,

$$\begin{aligned}
\mathcal{M}_{LL} &= -\frac{e^2}{(p_1-p_3)^2}\sum_{h_3,h_4\in\{L,R\}} j^e_{Lh_3}\cdot j^\mu_{Lh_4} \\
&= -\frac{e^2}{(p_1-p_3)^2}\left[j^e_{LL}\cdot j^\mu_{LL} + j^e_{LL}\cdot j^\mu_{LR} + j^e_{LR}\cdot j^\mu_{LL} + j^e_{LR}\cdot j^\mu_{LR}\right] \qquad (12)
\end{aligned}$$

Noting that under the relativistic limit of $E \gg M$, the only term that survives in (12) is $j^e_{LL}\cdot j^\mu_{LL}$ which can be calculated as,

$$\begin{aligned}
j^e_{LL}\cdot j^\mu_{LL} &= 4\left(E_1\cos\frac{\theta}{2}, p\sin\frac{\theta}{2}, -ip\sin\frac{\theta}{2}, p\cos\frac{\theta}{2}\right)\cdot\left(E_2\cos\frac{\theta}{2}, -p\sin\frac{\theta}{2}, -ip\sin\frac{\theta}{2}, -p\cos\frac{\theta}{2}\right) \\
&= 4\left(E_1E_2\cos^2\frac{\theta}{2} + p^2\sin^2\frac{\theta}{2} + p^2\sin^2\frac{\theta}{2} + p^2\cos^2\frac{\theta}{2}\right) \Longleftarrow p^2 \sim E_1^2 \sim E_2^2 (E \gg M) \\
&= 4E_1E_2 = 2\left(p_1+p_2\right)^2 \equiv 2s
\end{aligned}$$

Plugging this back into (12) and squaring the matrix element gives,

$$|\mathcal{M}_{LL}|^2 = \frac{e^4}{(p_1-p_3)^4}\left(j^e_{LL}\cdot j^\mu_{LL}\right)^2 = \frac{4e^4s^2}{(p_1-p_3)^4} \quad \square$$

For the other helicity combinations, the same could be done.

$$
\begin{aligned}
|\mathcal{M}_{RL}|^2 &= \frac{e^4}{(p_1-p_3)^4}\left[\sum_{h_3,h_4\in\{L,R\}} j^e_{Rh_3}\cdot j^\mu_{Lh_4}\right]^2 \\
&= \frac{e^4}{(p_1-p_3)^4}\left[\cancelto{0}{j^e_{RL}\cdot j^\mu_{LL}} + \cancelto{0}{j^e_{RL}\cdot j^\mu_{LR}} + j^e_{RR}\cdot j^\mu_{LL} + \cancelto{0}{j^e_{RR}\cdot j^\mu_{LR}}\right]^2 \\
&= \frac{16e^4}{(p_1-p_3)^4}\left(E_1E_2\cos^2\frac{\theta}{2} + p^2\sin^2\frac{\theta}{2} - p^2\sin^2\frac{\theta}{2} + p^2\cos^2\frac{\theta}{2}\right)^2 \\
&= \frac{4e^4s^2}{(p_1-p_3)^4}\cos^4\frac{\theta}{2}
\end{aligned}
$$

Again,

$$
\begin{aligned}
|\mathcal{M}_{RR}|^2 &= \frac{e^4}{(p_1-p_3)^4}\left(j^e_{RR}\cdot j^\mu_{RR}\right)^2 \\
&= \frac{16e^4}{(p_1-p_3)^4}\left(E_1E_2\cos^2\frac{\theta}{2} + p^2\sin^2\frac{\theta}{2} + p^2\sin^2\frac{\theta}{2} + p^2\cos^2\frac{\theta}{2}\right)^2 = \frac{4e^4s^2}{(p_1-p_3)^4}
\end{aligned}
$$

And finally,

$$
\begin{aligned}
|\mathcal{M}_{LR}|^2 &= \frac{e^4}{(p_1-p_3)^4}\left(j^e_{LL}\cdot j^\mu_{RR}\right)^2 \\
&= \frac{16e^4}{(p_1-p_3)^4}\left(E_1E_2\cos^2\frac{\theta}{2} + p^2\sin^2\frac{\theta}{2} - p^2\sin^2\frac{\theta}{2} + p^2\cos^2\frac{\theta}{2}\right)^2 = \frac{4e^4s^2}{(p_1-p_3)^4}\cos^4\frac{\theta}{2}
\end{aligned}
$$

(e) Averaging out all the helicity dependent amplitudes from (d), one would obtain

$$
\begin{aligned}
\langle|\mathcal{M}_{fi}|^2\rangle &= \frac{1}{4}\left\{|\mathcal{M}_{LR}|^2 + |\mathcal{M}_{RR}|^2 + |\mathcal{M}_{RL}|^2 + |\mathcal{M}_{LL}|^2\right\} \\
&= \frac{2e^4s^2}{(p_1-p_3)^4}\left(1+\cos^4\frac{\theta}{2}\right) \\
&= \frac{2e^4s^2}{(p_1-p_3)^4}\left[1+\frac{1}{4}\left(1+\cos\theta\right)^2\right] \\
&= \frac{e^4s^2}{2E_1^4\left(1-\cos\theta\right)^2}\left[1+\frac{1}{4}\left(1+\cos\theta\right)^2\right] \Longleftarrow \left(p_1-p_3\right)^2 \simeq -2p_1\cdot p_3 = -2E_1^2\left(1-\cos\theta\right) \\
&\simeq \frac{8e^4}{\left(1-\cos\theta\right)^2}\left[1+\frac{1}{4}\left(1+\cos\theta\right)^2\right]
\end{aligned}
$$

Then, the differential cross section can be written as,

$$
\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{1}{64\pi^2 s}\langle|\mathcal{M}_{fi}|^2\rangle = \frac{1}{8\pi^2 s}\cdot\frac{e^4}{\left(1-\cos\theta\right)^2}\left[1+\frac{1}{4}\left(1+\cos\theta\right)^2\right] \Longleftarrow e^4 = 16\pi^2\alpha^2
$$

$$= \frac{2\alpha^2}{s} \cdot \frac{1}{(1-\cos\theta)^2} \left[1 + \frac{1}{4}(1+\cos\theta)^2\right] \quad \square$$

Problem 6.8
Using $\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2g^{\mu\nu}$, prove that

$$\gamma^\mu\gamma_\mu = 4 \quad , \quad \gamma^\mu \not{a} \gamma_\mu = -2\not{a} \quad \text{and} \quad \gamma^\mu \not{a}\not{b}\gamma_\mu = 4a\cdot b$$

Solution:

(a) $\gamma^\mu\gamma_\mu = 4$

$$\gamma^\mu\gamma_\mu = \frac{1}{2} g_{\nu\mu}(\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu) = g^{\mu\nu}g_{\mu\nu} = 4 \quad \square$$

(b) $\gamma^\mu \not{a} \gamma_\mu = -2\not{a}$

$$\begin{aligned} \gamma^\mu \not{a} \gamma_\mu &= a_\nu \gamma^\mu\gamma^\nu\gamma_\mu = a_\nu (2g^{\mu\nu} - \gamma^\nu\gamma^\mu)\gamma_\mu \\ &= 2a^\mu\gamma_\mu - \not{a}\gamma^\mu\gamma_\mu = 2\not{a} - 4\not{a} = -2\not{a} \quad \square \end{aligned}$$

(c) $\gamma^\mu \not{a}\not{b}\gamma_\mu = 4a\cdot b$

$$\begin{aligned} \gamma^\mu \not{a}\not{b}\gamma_\mu &= a_\nu b_\rho \gamma^\mu\gamma^\nu\gamma^\rho\gamma_\mu = \frac{1}{2} a_\nu b_\rho \gamma^\mu \{\gamma^\nu, \gamma^\rho\}\gamma_\mu \\ &= \frac{1}{2} a_\nu b_\rho 2g^{\nu\rho}\gamma^\mu\gamma_\mu = 4a\cdot b \quad \square \end{aligned}$$

Problem 6.9
Prove the relation $\left(\bar{\psi}\gamma^\mu\gamma^5\phi\right)^\dagger = \bar{\phi}\gamma^\mu\gamma^5\psi$.

Solution: One could show that :

$$\begin{aligned} \left(\bar{\psi}\gamma^\mu\gamma^5\phi\right)^\dagger = \left(\psi^\dagger\gamma^0\gamma^\mu\gamma^5\phi\right)^\dagger &= -\left(\psi^\dagger\gamma^0\gamma^5\gamma^\mu\phi\right)^\dagger \\ &= -\phi^\dagger\gamma^{\mu\dagger}\gamma^{5\dagger}\gamma^{0\dagger}\psi \\ &= -\phi^\dagger\left(\gamma^0\gamma^\mu\gamma^0\right)\gamma^5\gamma^0\psi \\ &= -\phi^\dagger\gamma^0\gamma^\mu\gamma^0\gamma^5\gamma^0\psi = \phi^\dagger\gamma^0\gamma^\mu\gamma^0\gamma^0\gamma^5\psi = \bar{\phi}\gamma^\mu\gamma^5\psi \quad \square \end{aligned}$$

Problem 6.10
Use the trace formalism to calculate the QED spin-averaged matrix element squared for $e^+e^- \to ff$ including the electron mass term.

Solution: The spin-averaged amplitude $\langle\mathcal{M}_{fi}^2\rangle$ for the $e^+e^- \to ff$ without neglecting both mass terms, can be written as

$$\langle \mathcal{M}_{fi}^2 \rangle = \frac{1}{4}\sum_{\text{spins}} |\mathcal{M}_{fi}|^2 = \frac{Q_f^2 e^4}{4q^4}\text{Tr}\left[(\not{p}_2 - m_e)\gamma^\mu(\not{p}_1 + m_e)\gamma^\nu\right] \times \text{Tr}\left[(\not{p}_3 + m_f)\gamma_\mu(\not{p}_4 - m_f)\gamma_\nu\right]$$

The first trace term can be calculated as,

$$\begin{aligned}
\text{Tr}\left[(\not{p}_2 - m_e)\gamma^\mu(\not{p}_1 + m_e)\gamma^\nu\right] &= \text{Tr}\left[\not{p}_2\gamma^\mu\not{p}_1\gamma^\nu + m_e\cancelto{0}{\not{p}_2\gamma^\mu\gamma^\nu} - m_e\cancelto{0}{\gamma^\mu\not{p}_1\gamma^\nu} - m_e^2\gamma^\mu\gamma^\nu\right] \\
&= \text{Tr}\left[p_{1\sigma}p_{2\rho}\gamma^\rho\gamma^\mu\gamma^\sigma\gamma^\nu - m_e^2\gamma^\mu\gamma^\nu\right] \\
&= p_{1\sigma}p_{2\rho}\text{Tr}\left[\gamma^\rho\gamma^\mu\gamma^\sigma\gamma^\nu\right] - m_e^2\text{Tr}\left[\gamma^\mu\gamma^\nu\right] \\
&= 4p_{1\sigma}p_{2\rho}\left(g^{\rho\mu}g^{\sigma\nu} - g^{\rho\sigma}g^{\mu\nu} + g^{\rho\nu}g^{\mu\sigma}\right) - 4m_e^2 g^{\mu\nu} \\
&= 4\left[p_{1\nu}p_{2\mu} + p_{1\mu}p_{2\nu} - \left(p_1\cdot p_2 + m_e^2\right)g^{\mu\nu}\right]
\end{aligned}$$

Observing the second trace term, one could realize that switching $p_1 \leftrightarrow p_3$, $p_2 \leftrightarrow p_4$, $\mu \leftrightarrow \nu$ and $m_e \leftrightarrow m_f$ in the first trace term will be identical to the second one due to the fact that traces remain the same under cyclic permutations. Thus,

$$\text{Tr}\left[(\not{p}_3 + m_f)\gamma_\mu(\not{p}_4 - m_f)\gamma_\nu\right] = 4\left[p_3^\mu p_4^\nu + p_3^\nu p_4^\mu - \left(p_3\cdot p_4 + m_f^2\right)g_{\nu\mu}\right]$$

Using these two trace expressions, the spin-averaged amplitude can be fully written down as

$$\begin{aligned}
\langle \mathcal{M}_{fi}^2 \rangle &= 4Q_f^2\frac{e^4}{q^4}\left[p_{1\nu}p_{2\mu} + p_{1\mu}p_{2\nu} - \left(p_1\cdot p_2 + m_e^2\right)g^{\mu\nu}\right] \times \left[p_3^\mu p_4^\nu + p_3^\nu p_4^\mu - \left(p_3\cdot p_4 + m_f^2\right)g_{\nu\mu}\right] \\
&= 8Q_f^2\frac{e^4}{q^4}\Big[(p_1\cdot p_4)(p_2\cdot p_3) + (p_1\cdot p_3)(p_2\cdot p_4) + m_e^2(p_3\cdot p_4) + m_f^2(p_1\cdot p_2) + 2m_e^2m_f^2\Big] \\
&= \frac{2Q_f^2e^4}{(m_e^2 + p_1\cdot p_2)^2}\Big[(p_1\cdot p_4)(p_2\cdot p_3) + (p_1\cdot p_3)(p_2\cdot p_4) + m_e^2(p_3\cdot p_4) + m_f^2(p_1\cdot p_2) + 2m_e^2m_f^2\Big]
\end{aligned}$$

Under the limit of $m_e \sim 0$ the above coincides with equation (6.63) and going further by neglecting the fermion mass reduces to (6.25), which is as expected.

Problem 6.11
Neglecting the electron mass term, verify that the matrix element for $e^- f \to e^- f$ given in (6.67) can be obtained from the matrix element for $e^+e^- \to f f$ given in (6.63) using crossing symmetry with the substitutions

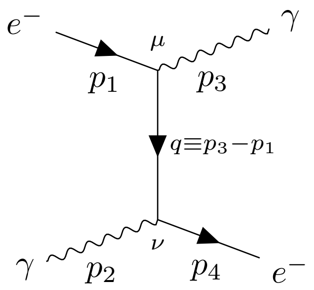
$$p_1 \to p_1 \quad , \quad p_2 \to -p_3 \quad , \quad p_3 \to p_4 \quad \text{and} \quad p_4 \to -p_2$$

Solution: One could directly verify by substituting four-vectors as the problem required,

$$\begin{aligned}
\langle \mathcal{M}_{e^+e^-\to ff}^2 \rangle &= 2\frac{Q_f^2e^4}{(p_1\cdot p_2)^2}\left[(p_1\cdot p_3)(p_2\cdot p_4) + (p_1\cdot p_4)(p_2\cdot p_3) + m_f^2(p_1\cdot p_2)\right] \\
&\longrightarrow 2\frac{Q_f^2e^4}{(p_1\cdot p_3)^2}\left[(p_1\cdot p_4)(-p_3\cdot -p_2) + (p_1\cdot -p_2)(-p_3\cdot p_4) + m_f^2(p_1\cdot -p_3)\right] \\
&= 2\frac{Q_f^2e^4}{(p_1\cdot p_3)^2}\left[(p_1\cdot p_4)(p_2\cdot p_3) + (p_1\cdot p_2)(p_3\cdot p_4) - m_f^2(p_1\cdot p_3)\right] = \langle \mathcal{M}_{e^-f\to e^-f}^2 \rangle \quad \square
\end{aligned}$$

> **Problem 6.12**
> Write down the matrix elements, $\mathcal{M}_1$ and $\mathcal{M}_2$, for the two Feynman diagrams for the Compton scattering process $e^-\gamma \to e^-\gamma$. From first principles, express the spin-averaged matrix element $\langle|\mathcal{M}_1+\mathcal{M}_2|^2\rangle$ as a trace. You will need the completeness relation for the photon polarisation states (see Appendix D).

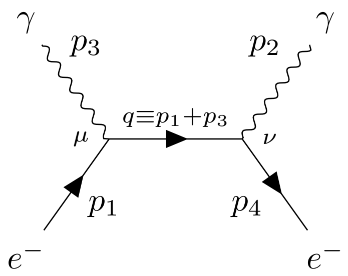
Solution: Starting from the two leading order Feynman diagrams that contribute to the Compton scattering process, using QED Feynman rules one could write down the corresponding matrix elements as,



$$: -i\mathcal{M}_1 = e^2\bar{u}(p_4)\not\epsilon(p_2)\left[\frac{i\,(\not q+m)}{q^2-m^2}\right]\not\epsilon^*(p_3)u(p_1)$$

$$= ie^2\bar{u}(p_4)\not\epsilon(p_2)\left[\frac{\not p_3-\not p_1+m}{(p_3-p_1)^2-m^2}\right]\not\epsilon^*(p_3)u(p_1) \equiv ie^2\bar{u}(p_4)\Gamma^1 u(p_1)$$

And for the other diagram,



$$: -i\mathcal{M}_2 = e^2\bar{u}(p_4)\not\epsilon^*(p_2)\left[\frac{i\,(\not q+m)}{q^2-m^2}\right]\not\epsilon(p_3)u(p_1)$$

$$= ie^2\bar{u}(p_4)\not\epsilon^*(p_2)\left[\frac{\not p_1+\not p_3+m}{(p_1+p_3)^2-m^2}\right]\not\epsilon(p_3)u(p_1) \equiv ie^2\bar{u}(p_4)\Gamma^2 u(p_1)$$

In total, there will be 4 terms in the spin-averaged matrix element,

$$\langle|\mathcal{M}_1+\mathcal{M}_2|^2\rangle = \langle\mathcal{M}_1\mathcal{M}_1^\dagger\rangle + \langle\mathcal{M}_1\mathcal{M}_2^\dagger\rangle + \langle\mathcal{M}_2\mathcal{M}_1^\dagger\rangle + \langle\mathcal{M}_2\mathcal{M}_2^\dagger\rangle$$

which could be calculated term-by-term.

(a) $\langle\mathcal{M}_1\mathcal{M}_1^\dagger\rangle$

$$\langle\mathcal{M}_1\mathcal{M}_1^\dagger\rangle = \frac{e^4}{4}\sum_{\lambda,\lambda'}\sum_{s,s'}\left[\bar{u}_a^s(p_4)\Gamma^1_{ab}(\lambda,\lambda')u_b^{s'}(p_1)\right]\left[\bar{u}_c^{s'}(p_1)\bar{\Gamma}^1_{cd}(\lambda,\lambda')u_d^s(p_4)\right]$$

$$= \frac{e^4}{4}\sum_{\lambda,\lambda'}\left[\sum_s u_d^s(p_4)\bar{u}_a^s(p_4)\right]\left[\sum_{s'} u_b^{s'}(p_1)\bar{u}_c^{s'}(p_1)\right]\Gamma^1_{ab}(\lambda,\lambda')\bar{\Gamma}^1_{cd}(\lambda,\lambda')$$

$$= \frac{e^4}{4}(\not p_4+m)_{da}(\not p_1+m)_{bc}\sum_{\lambda,\lambda'}\Gamma^1_{ab}(\lambda,\lambda')\bar{\Gamma}^1_{cd}(\lambda,\lambda') \tag{13}$$

Using the following expression for Γ^1,

$$\Gamma^1_{ab}(\lambda,\lambda') \equiv \epsilon^{\lambda}_{\mu}\epsilon^{\lambda' *}_{\nu}(\gamma^\mu T\gamma^\nu)_{ab} \quad \text{where} \quad T \equiv \frac{\not{p}_3 - \not{p}_1 + m}{(p_3-p_1)^2 - m^2}$$

which gives the adjoint as,

$$\bar{\Gamma}^1 \equiv \gamma^0\Gamma^{1\dagger}\gamma^0 = \epsilon^{\lambda *}_{\mu'}\epsilon^{\lambda'}_{\nu'}\gamma^0\gamma^{\nu\dagger}T^\dagger\gamma^{\mu\dagger}\gamma^0 = \epsilon^{\lambda *}_{\mu'}\epsilon^{\lambda'}_{\nu'}\gamma^{\nu'}T\gamma^{\mu'}$$

$$\implies \bar{\Gamma}^1_{cd}(\lambda,\lambda') = \epsilon^{\lambda *}_{\mu'}\epsilon^{\lambda'}_{\nu'}(\gamma^{\nu'}T\gamma^{\mu'})_{cd}$$

Plugging in the above expressions back into equation (13) yields,

$$\begin{aligned}
\langle \mathcal{M}_1\mathcal{M}_1^\dagger\rangle &= \frac{e^4}{4}(\not{p}_4+m)_{da}(\not{p}_1+m)_{bc}\sum_{\lambda,\lambda'}\Gamma^1_{ab}(\lambda,\lambda')\bar{\Gamma}^1_{cd}(\lambda,\lambda') \\
&= \frac{e^4}{4}(\not{p}_4+m)_{da}(\not{p}_1+m)_{bc}\left[\sum_{\lambda}\epsilon^{\lambda *}_{\mu'}\epsilon^{\lambda}_{\mu}\right]\left[\sum_{\lambda'}\epsilon^{\lambda' *}_{\nu}\epsilon^{\lambda'}_{\nu'}\right](\gamma^\mu T\gamma^\nu)_{ab}(\gamma^{\nu'}T\gamma^{\mu'})_{cd} \\
&= \frac{e^4}{4}(\not{p}_4+m)_{da}(\not{p}_1+m)_{bc}\left[g_{\mu\mu'}g_{\nu\nu'}\right](\gamma^\mu T\gamma^\nu)_{ab}(\gamma^{\nu'}T\gamma^{\mu'})_{cd} \\
&= \frac{e^4}{4}(\not{p}_4+m)_{da}(\not{p}_1+m)_{bc}(\gamma^\mu T\gamma^\nu)_{ab}(\gamma_\nu T\gamma_\mu)_{cd} \\
&= \frac{e^4}{4}\mathrm{Tr}\left[\gamma^\mu T\gamma^\nu(\not{p}_1+m)\gamma_\nu T\gamma_\mu(\not{p}_4+m)\right]
\end{aligned}$$

Fully expanding T, one could finally obtain

$$\langle \mathcal{M}_1\mathcal{M}_1^\dagger\rangle = \frac{e^4}{4\left[(p_3-p_1)^2-m^2\right]^2}\mathrm{Tr}\left[\gamma^\mu(\not{p}_3-\not{p}_1+m)\gamma^\nu(\not{p}_1+m)\gamma_\nu(\not{p}_3-\not{p}_1+m)\gamma_\mu(\not{p}_4+m)\right] \tag{14}$$

(b) $\langle \mathcal{M}_1\mathcal{M}_2^\dagger\rangle$

$$\begin{aligned}
\langle \mathcal{M}_1\mathcal{M}_2^\dagger\rangle &= \frac{e^4}{4}\sum_{\lambda,\lambda'}\sum_{s,s'}\left[\bar{u}^s_a(p_4)\Gamma^1_{ab}(\lambda,\lambda')u^{s'}_b(p_1)\right]\left[\bar{u}^{s'}_c(p_1)\bar{\Gamma}^2_{cd}(\lambda,\lambda')u^s_d(p_4)\right] \\
&= \frac{e^4}{4}\sum_{\lambda,\lambda'}\left[\sum_s u^s_d(p_4)\bar{u}^s_a(p_4)\right]\left[\sum_{s'}u^{s'}_b(p_1)\bar{u}^{s'}_c(p_1)\right]\Gamma^1_{ab}(\lambda,\lambda')\bar{\Gamma}^1_{cd}(\lambda,\lambda') \\
&= \frac{e^4}{4}(\not{p}_4+m)_{da}(\not{p}_1+m)_{bc}\sum_{\lambda,\lambda'}\Gamma^1_{ab}(\lambda,\lambda')\bar{\Gamma}^2_{cd}(\lambda,\lambda')
\end{aligned} \tag{15}$$

Again, using the similar calculation used before, one could see that

$$\Gamma^2_{cd} = \epsilon^{\lambda' *}_{\mu}\epsilon^{\lambda}_{\nu}(\gamma^\mu T'\gamma^\nu)_{cd} \quad \text{and} \quad \bar{\Gamma}^2_{cd} = \epsilon^{\lambda'}_{\mu}\epsilon^{\lambda *}_{\nu}(\gamma^\nu T'\gamma^\mu)_{cd} \quad \text{where} \quad T' \equiv \frac{\not{p}_1 + \not{p}_3 + m}{(p_1+p_3)^2 - m^2} \tag{16}$$

Then (15) becomes,

$$
\begin{aligned}
\langle \mathcal{M}_1 \mathcal{M}_2^\dagger \rangle &= \frac{e^4}{4} (\not{p}_4 + m)_{da} (\not{p}_1 + m)_{bc} \left(\gamma^\mu T \gamma^\nu\right)_{ab} \left(\gamma_\nu T' \gamma_\mu\right)_{cd} \\
&= \frac{e^4}{4} \mathrm{Tr} \left[\gamma^\mu T \gamma^\nu (\not{p}_1 + m) \gamma_\nu T' \gamma_\mu (\not{p}_4 + m)\right] \\
&= \frac{e^4}{4 \left[(p_1 - p_3)^2 - m^2\right] \left[(p_1 + p_3)^2 - m^2\right]} \mathrm{Tr} \left[\gamma^\mu (\not{p}_3 - \not{p}_1 + m) \gamma^\nu (\not{p}_1 + m) \gamma_\nu (\not{p}_1 + \not{p}_3 + m) \gamma_\mu (\not{p}_4 + m)\right]
\end{aligned} \tag{17}
$$

(c) $\langle \mathcal{M}_2 \mathcal{M}_1^\dagger \rangle$

$$
\begin{aligned}
\langle \mathcal{M}_2 \mathcal{M}_1^\dagger \rangle &= \frac{e^4}{4} \sum_{\lambda, \lambda'} \sum_{s, s'} \left[\bar{u}_a^s(p_4) \Gamma_{ab}^2(\lambda, \lambda') u_b^{s'}(p_1)\right] \left[\bar{u}_c^{s'}(p_1) \bar{\Gamma}_{cd}^1(\lambda, \lambda') u_d^s(p_4)\right] \\
&= \frac{e^4}{4} \sum_{\lambda, \lambda'} \left[\sum_s u_d^s(p_4) \bar{u}_a^s(p_4)\right] \left[\sum_{s'} u_b^{s'}(p_1) \bar{u}_c^{s'}(p_1)\right] \Gamma_{ab}^2(\lambda, \lambda') \bar{\Gamma}_{cd}^1(\lambda, \lambda') \\
&= \frac{e^4}{4} (\not{p}_4 + m)_{da} (\not{p}_1 + m)_{bc} \sum_{\lambda, \lambda'} \Gamma_{ab}^2(\lambda, \lambda') \bar{\Gamma}_{cd}^1(\lambda, \lambda')
\end{aligned} \tag{18}
$$

Using expressions of Γ^1, Γ^2, one could further write down (18) as,

$$
\begin{aligned}
\langle \mathcal{M}_2 \mathcal{M}_1^\dagger \rangle &= \frac{e^4}{4} (\not{p}_4 + m)_{da} (\not{p}_1 + m)_{bc} \left(\gamma^\mu T' \gamma^\nu\right)_{ab} \left(\gamma_\nu T \gamma_\mu\right)_{cd} \\
&= \frac{e^4}{4} \mathrm{Tr} \left[\gamma^\mu T' \gamma^\nu (\not{p}_1 + m) \gamma_\nu T \gamma_\mu (\not{p}_4 + m)\right] \\
&= \frac{e^4}{4 \left[(p_1 - p_3)^2 - m^2\right] \left[(p_1 + p_3)^2 - m^2\right]} \mathrm{Tr} \left[\gamma^\mu (\not{p}_1 + \not{p}_3 + m) \gamma^\nu (\not{p}_1 + m) \gamma_\nu (\not{p}_3 - \not{p}_1 + m) \gamma_\mu (\not{p}_4 + m)\right]
\end{aligned} \tag{19}
$$

(d) $\langle \mathcal{M}_2 \mathcal{M}_2^\dagger \rangle$

$$
\begin{aligned}
\langle \mathcal{M}_2 \mathcal{M}_2^\dagger \rangle &= \frac{e^4}{4} \sum_{\lambda, \lambda'} \sum_{s, s'} \left[\bar{u}_a^{s'}(p_4) \Gamma_{ab}^2(\lambda, \lambda') u_b^s(p_1)\right] \left[\bar{u}_c^s(p_1) \bar{\Gamma}_{cd}^2(\lambda, \lambda') u_d^{s'}(p_4)\right] \\
&= \frac{e^4}{4} \sum_{\lambda, \lambda'} \left[\sum_s u_b^s(p_1) \bar{u}_c^s(p_1)\right] \left[\sum_{s'} u_d^{s'}(p_4) \bar{u}_a^{s'}(p_4)\right] \Gamma_{ab}^2(\lambda, \lambda') \bar{\Gamma}_{cd}^2(\lambda, \lambda') \\
&= \frac{e^4}{4} (\not{p}_1 + m)_{bc} (\not{p}_4 + m)_{da} \sum_{\lambda, \lambda'} \Gamma_{ab}^2(\lambda, \lambda') \bar{\Gamma}_{cd}^2(\lambda, \lambda') \\
&= \frac{e^4}{4} (\not{p}_1 + m)_{bc} (\not{p}_4 + m)_{da} \left(\gamma^\mu T' \gamma^\nu\right)_{ab} \left(\gamma_\nu T' \gamma_\mu\right)_{cd} \\
&= \frac{e^4}{4 \left[(p_1 + p_3)^2 - m^2\right]^2} \mathrm{Tr} \left[\gamma^\mu (\not{p}_1 + \not{p}_3 + m) \gamma^\nu (\not{p}_1 + m) \gamma_\nu (\not{p}_1 + \not{p}_3 + m) \gamma_\mu (\not{p}_4 + m)\right]
\end{aligned} \tag{20}
$$

Then, adding up (14),(17),(19) and (20) one could get the total spin-averaged matrix element as,

$$
\begin{aligned}
\langle|\mathcal{M}_1+\mathcal{M}_2|^2\rangle = {} & \frac{e^4}{4\left[(p_1-p_3)^2-m^2\right]^2}\mathrm{Tr}\left[\gamma^\mu(\not{p}_3-\not{p}_1+m)\gamma^\nu(\not{p}_1+m)\gamma_\nu(\not{p}_3-\not{p}_1+m)\gamma_\mu(\not{p}_4+m)\right] \\
& + \frac{e^4}{4\left[(p_1-p_3)^2-m^2\right]\left[(p_1+p_3)^2-m^2\right]}\Big\{\mathrm{Tr}\left[\gamma^\mu(\not{p}_3-\not{p}_1+m)\gamma^\nu(\not{p}_1+m)\gamma_\nu(\not{p}_1+\not{p}_3+m)\gamma_\mu(\not{p}_4+m)\right] \\
& \qquad + \mathrm{Tr}\left[\gamma^\mu(\not{p}_1+\not{p}_3+m)\gamma^\nu(\not{p}_1+m)\gamma_\nu(\not{p}_3-\not{p}_1+m)\gamma_\mu(\not{p}_4+m)\right]\Big\} \\
& + \frac{e^4}{4\left[(p_1+p_3)^2-m^2\right]^2}\mathrm{Tr}\left[\gamma^\mu(\not{p}_1+\not{p}_3+m)\gamma^\nu(\not{p}_1+m)\gamma_\nu(\not{p}_1+\not{p}_3+m)\gamma_\mu(\not{p}_4+m)\right]
\end{aligned}
$$

---

7 Electron-Proton Elastic Scattering

Problem 7.1
The derivation of (7.8) used the algebraic relation

$$(\gamma+1)^2\left(1-\kappa^2\right)^2 = 4,$$

where

$$\kappa = \frac{\beta\gamma}{\gamma+1} \quad \text{and} \quad \left(1-\beta^2\right)\gamma^2 = 1$$

Show that this holds.

Solution: Using the given relation of κ in terms of β,γ,

$$
\begin{aligned}
(\gamma+1)^2(1-\kappa^2)^2 &= (\gamma+1)^2\left[1-\frac{\beta^2\gamma^2}{(\gamma+1)^2}\right]^2 \\
&= \left[(\gamma+1)-\frac{\beta^2\gamma^2}{\gamma+1}\right]^2 \Longleftarrow \beta^2\gamma^2=\gamma^2-1 \\
&= \left[(\gamma+1)-\frac{\gamma^2-1}{\gamma+1}\right]^2 = \left[(\gamma+1)-(\gamma-1)\right]^2 = 4 \quad \square
\end{aligned}
$$

Problem 7.2
By considering momentum and energy conservation in e^-p elastic scattering from a proton at rest, find an expression for the fractional energy loss of the scattered electron $(E_1-E_3)/E_1$ in terms of the scattering angle and the parameter

$$\kappa = \frac{p}{E_1+m_e} \equiv \frac{\beta\gamma}{\gamma+1}$$

Solution: The author stated in the errata that this problem should be ignored.

Problem 7.3
In an $e^- p$ scattering experiment, the incident electron has energy $E_1 = 529.5$ MeV and the scattered electrons are detected at an angle of $\theta = 75°$ relative to the incoming beam.

(a) At this angle, almost all of the scattered electrons are measured to have an energy of $E_3 \approx 373$ MeV. What can be concluded from this observation?

(b) Find the corresponding value of Q^2.

Solution:

(a) Let us assume that these observed electrons experienced a relativistic elastic scattering with the proton, then under such hypothesis the measured electron energy after the scattering process E_3 shall be :

$$E_3 = \frac{E_1 m_p}{m_p + E_1(1 - \cos\theta)} \simeq \frac{530 \text{ MeV} \cdot 940 \text{ MeV}}{940 \text{ MeV} + 530 \text{ MeV} \cdot (1 - 0.25)} \simeq 372 \text{ MeV}$$

which well corresponds with most of the observed energy of the scattered electrons. This implies that these electrons actually underwent an elastic collision with the proton, and the proton in concern remains intact.

(b) Using the observed values,

$$Q^2 = \frac{2m_p E_1^2(1 - \cos\theta)}{m_p + E_1(1 - \cos\theta)} \simeq \frac{2 \cdot 940 \text{ MeV} \cdot 530^2 \text{ MeV}^2 \cdot (1 - 0.25)}{940 \text{ MeV} + 530 \text{ MeV} \cdot (1 - 0.25)} \simeq \boxed{0.32 \text{ GeV}^2}$$

Problem 7.4
7.4 For a spherically symmetric charge distribution $\rho(r)$, where

$$\int \rho(r)\, \mathrm{d}^3\mathbf{r} = 1,$$

show that the form factor can be expressed as

$$F(\mathbf{q}^2) = \frac{4\pi}{q} \int_0^\infty r \sin(qr)\rho(r)\, \mathrm{d}r \simeq 1 - \frac{1}{6} q^2 \langle R^2 \rangle + \cdots$$

where $\langle R^2 \rangle$ is the mean square charge radius. Hence show that,

$$\langle R^2 \rangle = -6 \left[\frac{\mathrm{d}F(\mathbf{q}^2)}{\mathrm{d}q^2} \right]_{q^2=0}$$

Solution: Starting from the definition of the form factor, and letting θ be the angle between $\mathbf{q}$ and $\mathbf{r}$ while ϕ as the azimuthal angle,

$$\begin{aligned}
F(q^2) &= \int \rho(r) \exp\left[i\mathbf{q} \cdot \mathbf{r}\right] \mathrm{d}^3\mathbf{r} \\
&= -\int \rho(r) \exp\left[iqr\cos\theta\right] r^2\, \mathrm{d}r\, \mathrm{d}\cos\theta\, \mathrm{d}\phi \\
&= 2\pi \int_0^\infty r^2 \rho(r) \int_{-1}^{+1} \exp\left[iqr\cos\theta\right] \mathrm{d}\cos\theta\, \mathrm{d}r \\
&= 2\pi \int_0^\infty r^2 \rho(r) \frac{2}{qr} \sin(qr)\, \mathrm{d}r = \frac{4\pi}{q} \int_0^\infty r \sin(qr)\rho(r)\, \mathrm{d}r
\end{aligned}$$

Using the Taylor expansion of $\sin(qr)$, one could approximate the above expression of the form factor as

$$F(q^2) = \frac{4\pi}{q}\int_0^\infty r\sin(qr)\rho(r)\,\mathrm{d}r = \int_0^\infty 4\pi r^2\rho(r)\,\mathrm{d}r - \frac{1}{3!}q^2\int_0^\infty r^5\rho(r)\,\mathrm{d}r + \mathcal{O}(q^4)$$

$$= \int \rho(r)\,\mathrm{d}\mathbf{r} - \frac{1}{6}q^2\int_0^\infty r^5\rho(r)\,\mathrm{d}r + \mathcal{O}(q^4)$$

$$\simeq 1 - \frac{1}{6}q^2\langle R^2\rangle \quad \square$$

Problem 7.5
Using the answer to the previous question and the data in Figure 7.8a, estimate the root-mean-squared charge radius of the proton.

Solution: From the data in Figure 7.8a, the slope of $G_\mathrm{E}(q^2)$ can be obtained by drawing a tangent at $q^2 = 0$ which has an intercept in the Q^2-axis as 0.32 GeV^2, hence

$$\left.\frac{\mathrm{d}G_\mathrm{E}(q^2)}{\mathrm{d}q^2}\right|_{q^2=0} \simeq -\frac{1}{0.32\ \mathrm{GeV}^2} \simeq -3.125\ \mathrm{GeV}^{-2}$$

which could be interpreted as the root-mean-squared charge radius using the results from Problem (7.4) as,

$$\langle R^2\rangle = -6\cdot\left.\frac{\mathrm{d}G_\mathrm{E}(q^2)}{\mathrm{d}q^2}\right|_{q^2=0} \simeq 18.75\ \mathrm{GeV}^{-2} \sim \boxed{0.72\mathrm{fm}^2}$$

Problem 7.6
From the slope and intercept of the right plot of Figure 7.7, obtain values for $G_\mathrm{M}(0.292\mathrm{GeV}^2)$ and $G_\mathrm{E}(0.292\mathrm{GeV}^2)$

Solution: The value of τ can be first calculated as, $\tau = Q^2/4m_p^2 \simeq 0.082$. From the slope, one could obtain $G_M(Q^2)$ for $Q^2 = 0.292$ GeV^2 as,

$$m \simeq 0.3 = 2\tau\left[G_M(Q^2)\right]^2 \implies G_M(Q^2) \simeq \sqrt{\frac{0.3}{2\tau}} \simeq 1.352$$

and using the intercept which is $\simeq 0.4$ one could get

$$c \simeq 0.4 = \frac{1}{1+\tau}\cdot\left[G_E(Q^2)^2 + \tau G_M(Q^2)^2\right] \implies G_E(Q^2)^2 \simeq 0.4(1+\tau) - \tau G_M(Q^2)^2 \simeq 0.282$$

Thus, one could conclude that:

$$\boxed{G_E(0.292\ \mathrm{GeV}^2) \simeq 0.531 \quad \text{and} \quad G_M(0.292\ \mathrm{GeV}^2) \simeq 1.352}$$

Problem 7.7
Use the data of Figure 7.7 to estimate $G_E(Q^2)$ at $Q^2 = 0.500$ GeV^2.

Solution: In order to calculate $G_E(Q^2)$ at $Q^2 = 0.5$ GeV^2, one would have to extract two points with different θ from the $(E_1, \frac{d\sigma}{d\Omega})$ plot that corresponds to $Q^2 = 0.5$ GeV^2 and transcribe it into $\left(\tan^2\left(\frac{\theta}{2}\right), \frac{d\sigma}{d\Omega}/\left(\frac{d\sigma}{d\Omega}\right)_0\right)$ and get the slope and intercept that would eventually be used to derive $G_E(Q^2)$ and $G_M(Q^2)$. As there are abundant amount of datapoints in $\theta = 75°, 135°$, two of those curves will be used for the calculation mentioned above. Of course one could further obtain additional points and do a linear regression, but here for brevity estimation will be done by just using two points.

(a) $\theta = 75°$ $(\cos\theta \simeq 0.25)$

Using the relation (7.32), one could derive the corresponding E_1 for $Q^2 = 0.5$ GeV^2 as

$$Q^2 = \frac{2 \cdot 0.938 \text{ GeV} \cdot E_1^2(1-0.25)}{0.938 \text{ GeV} + E_1(1-0.25)} = 0.5 \text{ GeV}^2 \implies 2.814E_1^2 - 0.75E_1 - 0.938 = 0$$

Solving the quadratic equation, the physical solution gives $E_1 \simeq 725$ MeV. The corresponding $\frac{d\sigma}{d\Omega}$ at $E_1 \simeq 725$ MeV gives 10^{-32}cm^2/steradian from Fig7.7. One would also need to calculate $\left(\frac{d\sigma}{d\Omega}\right)_0$, which shows to beam

$$\begin{aligned}\left(\frac{d\sigma}{d\Omega}\right)_0 &= \frac{\alpha^2}{4E_1^2\sin^4\frac{\theta}{2}}\left[\frac{m_p}{m_p + E_1(1-\cos\theta)}\right]\cos^2\frac{\theta}{2} \\ &\simeq \frac{5.3\times10^{-5}}{4\cdot 0.725^2 \text{ GeV}^2 \cdot 0.137} \times \frac{0.938}{0.938 + 0.725(1-0.25)} \times 0.629 \simeq 2.871\times10^{-32}\text{cm}^2\end{aligned}$$

Thus from $\theta = 75°$, the following point is obtained :

$$\left(\tan^2\frac{\theta}{2}, \frac{d\sigma}{d\Omega}\cdot\left(\frac{d\sigma}{d\Omega}\right)_0^{-1}\right)_{\theta=75°} \simeq (0.5887, 0.3483) \tag{21}$$

(b) $\theta = 135°$ $(\cos\theta \simeq -0.70)$

Similarly, one could do the same for $\theta = 135°$ by first obtaining the corresponding E_1 value as :

$$Q^2 = \frac{2 \cdot 0.938 \text{ GeV} \cdot E_1^2(1+0.7)}{0.938 \text{ GeV} + E_1(1+0.7)} = 0.5 \text{ GeV}^2 \implies 6.378E_1^2 - 1.7E_1 - 0.938 = 0$$

Again, the quadratic equation gives a physical solution of $E_1 \simeq 539$ MeV and the corresponding point gives $\frac{d\sigma}{d\Omega} \simeq 2\times10^{-33}$cm^2/steradian from Fig7.7, and again

$$\begin{aligned}\left(\frac{d\sigma}{d\Omega}\right)_0 &= \frac{\alpha^2}{4E_1^2\sin^4\frac{\theta}{2}}\left[\frac{m_p}{m_p + E_1(1-\cos\theta)}\right]\cos^2\frac{\theta}{2} \\ &\simeq \frac{5.3\times10^{-5}}{4\cdot 0.539^2 \text{ GeV}^2 \cdot 0.728} \times \frac{0.938}{0.938 + 0.539(1+0.7)} \times 0.146 \simeq 1.792\times10^{-33}\text{cm}^2\end{aligned}$$

Which finally gives,

$$\left(\tan^2\frac{\theta}{2}, \frac{d\sigma}{d\Omega}\cdot\left(\frac{d\sigma}{d\Omega}\right)_0^{-1}\right)_{\theta=135°} \simeq (5.8284, 1.1160) \tag{22}$$

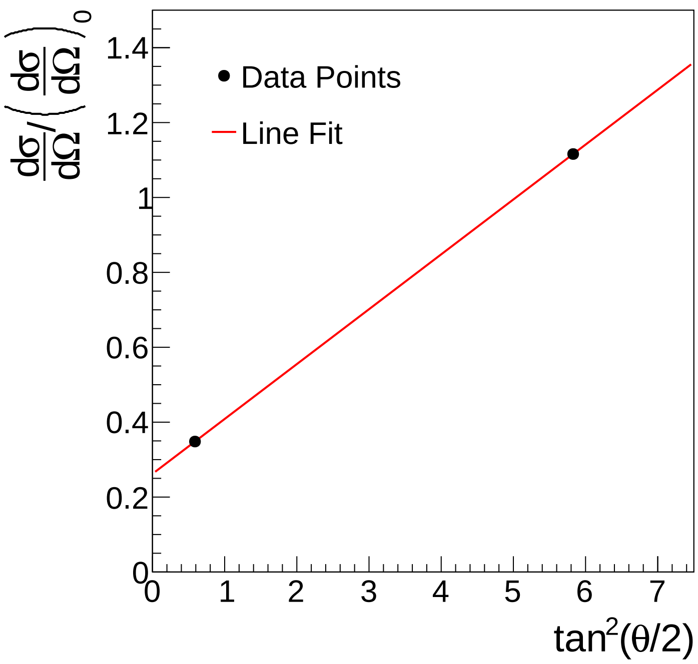


Figure 1: Linear relation obtained for $Q^2 = 0.5$ GeV^2

Using the two points obtained from (21) and (22), a linear relation having a slope of $\simeq 0.1465$ and an intercept of $\simeq 0.2620$ can be derived, which is also shown in Figure 1.

The slope and intercept will be used to compute $G_M(Q^2)$ and $G_E(Q^2)$, where $\tau = Q^2/4m_p^2 \simeq 0.071$:

$$m \simeq 0.1465 = 2\tau \left[G_M(Q^2)\right]^2 \implies G_M(Q^2) \simeq \sqrt{\frac{0.1465}{2 \cdot 0.071}} \simeq 1.015$$

and using the intercept which is $\simeq 0.4$ one could get

$$c \simeq 0.2620 = \frac{1}{1+\tau} \cdot \left[G_E(Q^2)^2 + \tau G_M(Q^2)^2\right] \implies G_E(Q^2)^2 \simeq 0.2620(1+0.071) - 0.071 \cdot G_M(Q^2)^2 \simeq 0.2074$$

Thus one could finally say that,

$$G_E(0.5 \text{ GeV}^2) \simeq \boxed{0.4554}$$

Problem 7.8
The experimental data of Figure 7.8 can be described by the form factor

$$G(Q^2) = \frac{G(0)}{\left(1 + Q^2/Q_0^2\right)^2}$$

with $Q_0^2 = 0.71$ GeV^2. Taking $Q^2 \approx \mathbf{q}^2$, show that this implies that proton has an exponential charge distribution of the form

$$\rho(\mathbf{r}) = \rho_0 e^{-\frac{r}{a}}$$

and find the value of a.

Solution: Under the approximation of $Q^2 \approx \mathbf{q}^2$ one could write down

$$G(Q^2) \simeq G(\mathbf{q}^2) = \int e^{i\mathbf{q}\cdot\mathbf{r}} \rho(\mathbf{r})\, \mathrm{d}^3\mathbf{r} \implies \rho(\mathbf{r}) \simeq \int e^{-i\mathbf{q}\cdot\mathbf{r}} G(\mathbf{q}^2)\, \mathrm{d}^3\mathbf{q}$$

From the dipole approximation of the form factors given, the above could be further calculated while defining θ as the angle between $\mathbf{q}$ and $\mathbf{r}$ and ϕ as the remaining azimuthal angle :

$$\begin{aligned}
\rho(\mathbf{r}) &\simeq \int e^{-i\mathbf{q}\cdot\mathbf{r}} G(\mathbf{q}^2)\, \mathrm{d}^3\mathbf{q} \\
&= \int_0^\infty \int_{+1}^{-1} \int_0^{2\pi} \exp\left(-iqr\cos\theta\right) \frac{G(0)}{(1+q^2/Q_0^2)^2} q^2\, \mathrm{d}q\, \mathrm{d}(-\cos\theta)\, \mathrm{d}\phi \\
&= 2\pi G(0) \int_0^\infty \int_{-1}^{+1} \exp\left(-iqr\cos\theta\right) \frac{1}{(1+q^2/Q_0^2)^2} q^2\, \mathrm{d}q\, \mathrm{d}\cos\theta \\
&= 2\pi G(0) \int_0^\infty \frac{\sin(qr)}{qr} \frac{1}{(1+q^2/Q_0^2)^2} q^2\, \mathrm{d}q \\
&= 2\pi G(0) \cdot \frac{1}{r} \int_0^\infty \frac{q\sin(qr)}{(1+q^2/Q_0^2)^2}\, \mathrm{d}q
\end{aligned} \tag{23}$$

By using integration by parts and other computational techniques, the q integration can be done as the following,

$$\int_0^\infty \frac{q\sin(qr)}{(1+q^2/Q_0^2)^2}\, \mathrm{d}q = \frac{1}{4}\pi r Q_0^2 e^{-Q_0 r}$$

and again plugging into (23) gives

$$\begin{aligned}
\rho(\mathbf{r}) &\simeq 2\pi G(0) \cdot \frac{1}{r} \int_0^\infty \frac{q\sin(qr)}{(1+q^2/Q_0^2)^2}\, \mathrm{d}q \\
&= \left[\frac{1}{2}\pi^2 G(0) Q_0^3\right] e^{-Q_0 r} \equiv \rho_0 e^{-\frac{r}{a}} \quad \square
\end{aligned}$$

which succesfully shows that when form factors are approximated as a dipole function the charge distribution will take an exponential form. The value of $a = Q_0^{-1}$ can be directly computed as,

$$a = Q_0^{-1} \simeq 1.1867\ \mathrm{GeV}^{-1} \simeq 1.1867 \times 0.197\mathrm{fm} \simeq \boxed{0.233\mathrm{fm}}$$

which well coincides with the value stated in the textbook.

8 Deep Inelastic Scattering

Problem 8.1
Use the data in Figure 8.2 to estimate the lifetime of the Δ^+ baryon.

Solution: One could easily notice that the corresponding FWHM Γ_{Δ^+} for the Δ^+ baryon which resonates around $W = 1.232$ GeV or $E_3 \approx 4.2$ GeV is around $\Gamma_{\Delta^+} \simeq 0.1$ GeV. Thus the lifetime τ can be simply estimated by inverting Γ_{Δ^+},

$$\tau_{\Delta^+} = \Gamma_{\Delta^+}^{-1} \simeq 10\ \text{GeV}^{-1} = 10 \cdot 6.58 \times 10^{-25}\text{s} \simeq \boxed{0.6 \times 10^{-23}\text{s}}$$

which is a good estimation compared to the value stated in the PDG summary table.

Problem 8.2
8.2 In fixed-target electron-proton elastic scattering

$$Q^2 = 2m_p\,(E_1 - E_2) = 2m_p E_1 y \quad \text{and} \quad Q^2 = 4E_1E_3\sin^2\frac{\theta}{2}$$

(a) Use these relations to show that

$$\sin^2\frac{\theta}{2} = \frac{E_1}{E_3}\frac{m_p^2}{Q^2}y^2 \quad \text{and hence} \quad \frac{E_3}{E_1}\cos^2\frac{\theta}{2} = 1 - y - \frac{m_p^2y^2}{Q^2}.$$

(b) Assuming azimuthal symmetry and using Equations (7.31) and (7.32), show that

$$\frac{\mathrm{d}\sigma}{\mathrm{d}Q^2} = \left|\frac{\mathrm{d}\Omega}{\mathrm{d}Q^2}\right|\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{\pi}{E_3^2}\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}$$

(c) Using the results of (a) and (b) show that the Rosenbluth equation,

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{\alpha^2}{4E_1^2\sin^4\frac{\theta}{2}}\frac{E_3}{E_1}\left(\frac{G_E^2 + \tau G_M^2}{1+\tau}\cos^2\frac{\theta}{2} + 2\tau G_M^2\sin^2\frac{\theta}{2}\right),$$

can be written in the Lorentz-invariant form

$$\frac{\mathrm{d}\sigma}{\mathrm{d}Q^2} = \frac{4\pi\alpha^2}{Q^4}\left[\frac{G_E^2 + \tau G_M^2}{1+\tau}\left(1 - y - \frac{m_p^2y^2}{Q^2}\right) + \frac{1}{2}y^2G_M^2\right]$$

Solution:

(a) It is straightforward to see that,

$$\begin{aligned} Q^2 = 4E_1E_3\sin^2\frac{\theta}{2} \implies \sin^2\frac{\theta}{2} &= \frac{Q^2}{4E_1E_3} = \frac{2m_pE_1y}{2E_3} \\ &= \frac{m_py}{2E_3} \times \frac{2E_1y}{2E_1y} \\ &= \frac{m_py}{2E_3} \times \frac{2E_1y}{Q^2m_p^{-1}} = \frac{E_1}{E_3}\frac{m_p^2}{Q^2}y^2 \quad \square \end{aligned}$$

and also,

$$
\begin{aligned}
\frac{E_3}{E_1}\cos^2\frac{\theta}{2} &= \frac{E_3}{E_1}\left(1-\sin^2\frac{\theta}{2}\right) \\
&= \frac{E_3}{E_1}\left(1-\frac{E_1}{E_3}\frac{m_p^2}{Q^2}y^2\right) \\
&= \frac{E_3}{E_1}-\frac{m_p^2}{Q^2}y^2 = 1-y-\frac{m_p^2}{Q^2}y^2 \quad \square
\end{aligned}
$$

(b) From azimuthal symmetry, one could let $\mathrm{d}\Omega = \mathrm{d}\left(\cos\theta\right)\mathrm{d}\phi = 2\pi\,\mathrm{d}\left(\cos\theta\right)$ which leads to :

$$
\begin{aligned}
\left|\frac{\mathrm{d}Q^2}{\mathrm{d}\Omega}\right| &= \frac{1}{2\pi}\left|\frac{\mathrm{d}Q^2}{\mathrm{d}\left(\cos\theta\right)}\right| \\
&= \frac{1}{2\pi}\left|\frac{\mathrm{d}}{\mathrm{d}\left(\cos\theta\right)}\left[\frac{2m_pE_1^2\left(1-\cos\theta\right)}{m_p+E_1\left(1-\cos\theta\right)}\right]\right| \\
&= \frac{1}{2\pi}\left|\frac{-2m_p^2E_1^2}{\left[m_p+E_1\left(1-\cos\theta\right)\right]^2}\right| = \frac{1}{\pi}E_3^2
\end{aligned}
$$

Thus it could be written as,

$$
\frac{\mathrm{d}\sigma}{\mathrm{d}Q^2} = \left|\frac{\mathrm{d}Q^2}{\mathrm{d}\Omega}\right|^{-1}\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{\pi}{E_3^2}\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} \quad \square \tag{24}
$$

(c) Starting from the relation (24),

$$
\begin{aligned}
\frac{\mathrm{d}\sigma}{\mathrm{d}Q^2} = \frac{\pi}{E_3^2}\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} &= \frac{\pi}{E_3^2}\cdot\frac{\alpha^2}{4E_1^2\sin^4\frac{\theta}{2}}\frac{E_3}{E_1}\left(\frac{G_E^2+\tau G_M^2}{1+\tau}\cos^2\frac{\theta}{2}+2\tau G_M^2\sin^2\frac{\theta}{2}\right) \\
&= \alpha^2\pi\frac{4}{Q^4}\left[\left(\frac{G_E^2+\tau G_M^2}{1+\tau}\right)\frac{E_3}{E_1}\cos^2\frac{\theta}{2}+2\tau G_M^2\frac{E_3}{E_1}\sin^2\frac{\theta}{2}\right] \\
&= \frac{4\alpha^2\pi}{Q^4}\left[\frac{G_E^2+\tau G_M^2}{1+\tau}\left(1-y-\frac{m_p^2}{Q^2}y^2\right)+2\tau G_M^2\frac{m_p^2}{Q^2}y^2\right] \\
&= \frac{4\alpha^2\pi}{Q^4}\left[\frac{G_E^2+\tau G_M^2}{1+\tau}\left(1-y-\frac{m_p^2}{Q^2}y^2\right)+2\frac{Q^2}{4m_p^2}G_M^2\frac{m_p^2}{Q^2}y^2\right] \\
&= \frac{4\pi\alpha^2}{Q^4}\left[\frac{G_E^2+\tau G_M^2}{1+\tau}\left(1-y-\frac{m_p^2y^2}{Q^2}\right)+\frac{1}{2}y^2G_M^2\right] \quad \square
\end{aligned}
$$

Problem 8.3
In fixed-target electron-proton inelastic scattering:

(a) Show that the laboratory frame differential cross section for deep-inelastic scattering is related to the Lorentz-invariant differential cross section of Equation (8.11) by

$$\frac{\mathrm{d}^2\sigma}{\mathrm{d}E_3\,\mathrm{d}\Omega} = \frac{E_1E_3}{\pi}\frac{\mathrm{d}^2\sigma}{\mathrm{d}E_3\,\mathrm{d}Q^2} = \frac{E_1E_3}{\pi}\frac{2m_px^2}{Q^2}\frac{\mathrm{d}^2\sigma}{\mathrm{d}x\,\mathrm{d}Q^2},$$

where E_1, E_3 are the energies of the incoming and outgoing electron.

(b) Show that

$$\frac{2m_px^2}{Q^2}\cdot\frac{y^2}{2} = \frac{1}{m_p}\frac{E_3}{E_1}\sin^2\frac{\theta}{2} \quad \text{and} \quad 1-y-\frac{m_p^2x^2y^2}{Q^2} = \frac{E_3}{E_1}\cos^2\frac{\theta}{2}.$$

(c) Hence, show that the Lorentz-invariant cross section of Equation (8.11) becomes

$$\frac{\mathrm{d}^2\sigma}{\mathrm{d}E_3\,\mathrm{d}\Omega} = \frac{\alpha^2}{4E_1^2\sin^4\frac{\theta}{2}}\left[\frac{F_2}{\nu}\cos^2\frac{\theta}{2} + \frac{2F_1}{m_p}\sin^2\frac{\theta}{2}\right].$$

(d) A fixed-target ep scattering experiment consists of an electron beam of maximum energy 20 GeV and a variable angle spectrometer that can detect scattered electrons with energies greater than 2 GeV. Find the range of values of θ over which deep inelastic scattering events can be studied at $x = 0.2$ and $Q^2 = 2$ GeV^2

Solution:

(a) From the chain rule of derivations, the following holds

$$\begin{aligned}
\frac{\mathrm{d}^2\sigma}{\mathrm{d}E_3\,\mathrm{d}\Omega} &= \left|\frac{\mathrm{d}E_3}{\mathrm{d}x}\right|^{-1}\cdot\left|\frac{\mathrm{d}Q^2}{\mathrm{d}\Omega}\right|^{-1}\frac{\mathrm{d}^2\sigma}{\mathrm{d}x\,\mathrm{d}\Omega} \\
&= \left|\frac{\mathrm{d}E_3}{\mathrm{d}x}\right|^{-1}\cdot\left|\frac{1}{2\pi}\frac{\mathrm{d}}{\mathrm{d}(\cos\theta)}(p_1-p_3)^2\right|^{-1}\frac{\mathrm{d}^2\sigma}{\mathrm{d}x\,\mathrm{d}\Omega} \\
&= \left|\frac{\mathrm{d}E_3}{\mathrm{d}x}\right|^{-1}\cdot\pi\left|\frac{\mathrm{d}}{\mathrm{d}(\cos\theta)}\left[m_e^2 - E_1E_3(1-\cos\theta)\right]\right|^{-1}\frac{\mathrm{d}^2\sigma}{\mathrm{d}x\,\mathrm{d}\Omega} \\
&= \left|\frac{\mathrm{d}E_3}{\mathrm{d}x}\right|^{-1}\cdot\frac{E_1E_3}{\pi}\frac{\mathrm{d}^2\sigma}{\mathrm{d}x\,\mathrm{d}\Omega} \\
&= \left|\frac{\mathrm{d}}{\mathrm{d}x}\left(E_1-\frac{Q^2}{2m_px}\right)\right|^{-1}\cdot\frac{E_1E_3}{\pi}\frac{\mathrm{d}^2\sigma}{\mathrm{d}x\,\mathrm{d}\Omega} \\
&= \frac{2m_px^2}{Q^2}\frac{E_1E_3}{\pi}\frac{\mathrm{d}^2\sigma}{\mathrm{d}x\,\mathrm{d}Q^2} \quad \square
\end{aligned}$$

(b) Starting from the relation of $Q^2 = -(p_1-p_3)^2 \simeq 2E_1E_3(1-\cos\theta)$, one could write down

$$\begin{aligned}
Q^2 = 2E_1E_3(1-\cos\theta) = 4E_1E_3\sin^2\frac{\theta}{2} = 2m_p\nu x &\implies \sin^2\frac{\theta}{2} = \frac{m_p\nu x}{2E_1E_3} = \frac{m_pxy}{2E_3} \\
&\implies \frac{1}{m_p}\frac{E_3}{E_1}\sin^2\frac{\theta}{2} = \frac{1}{m_p}\frac{E_3}{E_1}\frac{m_pxy}{2E_3}
\end{aligned} \tag{25}$$

Also from the relation between Q^2 and E_1

$$Q^2 \simeq \left(s-m_p^2\right)xy = 2E_1m_pxy \implies E_1 = \frac{Q^2}{2m_pxy}$$

Plugging this into (25) yields,

$$\frac{1}{m_p}\frac{E_3}{E_1}\sin^2\frac{\theta}{2} = \frac{1}{m_p}\frac{E_3}{E_1}\frac{m_p xy}{2E_3} = \frac{m_p}{Q^2}x^2y^2 \quad \square$$

Then using the trigonometric relation,

$$\begin{aligned}
\frac{E_3}{E_1}\cos^2\frac{\theta}{2} &= \frac{E_3}{E_1} - \frac{E_3}{E_1}\sin^2\frac{\theta}{2} \\
&= \frac{E_3}{E_1} - \frac{m_p^2}{Q^2}x^2y^2 = 1 - y - \frac{m_p^2}{Q^2}x^2y^2 \quad \square
\end{aligned}$$

(c) From the above results,

$$\begin{aligned}
\frac{\mathrm{d}^2\sigma}{\mathrm{d}E_3\,\mathrm{d}\Omega} &= \frac{2m_p x^2}{Q^2}\frac{E_1E_3}{\pi}\frac{\mathrm{d}^2\sigma}{\mathrm{d}x\,\mathrm{d}Q^2} \\
&= \frac{2m_p x^2}{Q^2}\frac{E_1E_3}{\pi}\frac{4\pi\alpha^2}{Q^4}\left[\left(1 - y - \frac{m_p^2 y^2}{Q^2}\right)\frac{F_2}{x} + y^2F_1\right] \\
&= \frac{2m_p x^2}{Q^2}\frac{E_1E_3}{\pi}\frac{4\pi\alpha^2}{16E_1^2E_3^2\sin^4\frac{\theta}{2}}\left[\frac{E_3}{E_1}\cos^2\frac{\theta}{2}\frac{F_2}{x} + \frac{1}{x^2m_p^2}\frac{E_3}{E_1}\sin^2\frac{\theta}{2}Q^2F_1\right] \\
&= \frac{\alpha^2}{4E_1^2\sin^4\frac{\theta}{2}}\left(\frac{2m_p x^2}{Q^2}\frac{\cancel{E_3}}{\cancel{E_1}}\right)\left[\frac{\cancel{E_3}}{\cancel{E_1}}\cos^2\frac{\theta}{2}\frac{F_2}{x} + \frac{1}{x^2m_p^2}\frac{\cancel{E_3}}{\cancel{E_1}}\sin^2\frac{\theta}{2}Q^2F_1\right] \\
&= \frac{\alpha^2}{4E_1^2\sin^4\frac{\theta}{2}}\left[\cos^2\frac{\theta}{2}F_2\frac{2m_p x}{Q^2} + \sin^2\frac{\theta}{2}F_1\frac{2}{m_p}\right] \\
&= \frac{\alpha^2}{4E_1^2\sin^4\frac{\theta}{2}}\left[\cos^2\frac{\theta}{2}F_2\frac{2m_p x}{2m_p x\nu} + \sin^2\frac{\theta}{2}F_1\frac{2}{m_p}\right] \\
&= \frac{\alpha^2}{4E_1^2\sin^4\frac{\theta}{2}}\left[\frac{F_2}{\nu}\cos^2\frac{\theta}{2} + \frac{2F_1}{m_p}\sin^2\frac{\theta}{2}\right] \quad \square
\end{aligned}$$

(d) From the given values, one could find that

$$\nu = \frac{Q^2}{2m_p x} \simeq \frac{2\ \mathrm{GeV}^2}{2\cdot 0.938\ \mathrm{GeV}\cdot 0.2} \simeq 5.3\ \mathrm{GeV} \implies E_3 = E_1 - 5.3\ \mathrm{GeV} \in [2, 14.7]\ \mathrm{GeV}$$

where the minimum value of 2 GeV is from the detector resolution mentioned in the problem. Using the expression of $\sin^2\theta/2$ derived in (b), one could obtain

$$\sin^2\frac{\theta}{2} = \frac{E_1}{E_3}\frac{m_p^2}{Q^2}x^2y^2 = \frac{1}{E_1E_3}\frac{m_p^2}{Q^2}x^2\nu^2 \simeq \frac{2.5\ \mathrm{GeV}^2}{E_3\,(E_3 + 5.3\ \mathrm{GeV})} \quad \text{for} \quad E_3 \in [2, 14.7]\ \mathrm{GeV}$$

which converts into an expression of angle as,

$$\theta \simeq 2 \arcsin \sqrt{\frac{2.5 \text{ GeV}^2}{E_3\,(E_3 + 5.3 \text{ GeV})}} \quad \text{for} \quad E_3 \in [2, 14.7] \text{ GeV}$$

it can be checked that in the considered region $\theta(E_3)$ decreases monotonically, thus the range for θ can be derived as $\boxed{10.3^\circ \lesssim \theta \lesssim 48.8^\circ}$.

Problem 8.4
If quarks were spin-0 particles, why would $F_1^{ep}(x)/F_2^{ep}(x)$ be zero?

Solution: One should remind that the angular dependence of electron-proton cross sections arise from the helicity conservation of electrons and quarks, which is due to the half-spin structure of themselves. Starting from the differential cross section of an electron-quark collision,

$$\begin{aligned}
\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega^*} &= \frac{Q_q^2 e^4}{8\pi^2 s} \frac{1}{(1-\cos\theta^*)^2} \left[1 + \cancel{\frac{1}{4}(1+\cos\theta^*)^2}\right] \\
&= \frac{Q_q^2 e^4}{8\pi^2 s} \frac{1}{(1-\cos\theta^*)^2} = \frac{Q_q^2}{8\pi^2 s} \cdot 16\pi^2\alpha^2 \cdot \frac{E^4}{q^4} = \frac{2Q_q^2\alpha^2}{q^4}\frac{E^4}{s} = \frac{Q_q^2\alpha^2 s}{8q^4}
\end{aligned}$$

where the second angular dependent term is neglected as spin-0 is assumed for quarks. The corresponding Lorentz-invariant expression can be then achieved using chain rules,

$$\begin{aligned}
\frac{\mathrm{d}\sigma}{\mathrm{d}q^2} = \frac{\mathrm{d}\sigma}{\mathrm{d}\Omega^*}\left|\frac{\mathrm{d}q^2}{\mathrm{d}\Omega^*}\right|^{-1} &= \frac{\mathrm{d}\sigma}{\mathrm{d}\Omega^*}\left|\frac{1}{2\pi}\frac{\mathrm{d}}{\mathrm{d}(\cos\theta^*)}E^2\left[(1-\cos\theta^*)\right]\right|^{-1} \\
&= \frac{Q_q^2\alpha^2 s}{8q^4} \cdot \frac{2\pi}{E^2} = \frac{\pi\alpha^2 Q_q^2}{q^4}
\end{aligned}$$

Now considering the quark-parton model, one could interpret the above differential cross section in terms of proton structure functions :

$$\frac{\mathrm{d}\sigma}{\mathrm{d}Q^2} = \frac{\pi\alpha^2 Q_q^2}{Q^4}\left[1 + 0\cdot(1-y)^2\right] \implies \frac{\mathrm{d}^2\sigma}{\mathrm{d}x\,\mathrm{d}Q^2} = \frac{\pi\alpha^2 Q_q^2}{Q^4}\sum_i Q_i^2 q_i^p(x)$$

which implies that as the coefficient for y^2 has to be 0, $F_1^{\mathrm{ep}}(x)$ has to be 0.

Problem 8.5
What is the expected value of $\int_0^1 u(x) - \bar{u}(x)\,\mathrm{d}x$ for the proton?

Solution: Splitting the up quark PDF into valence and sea contribution, one obtains

$$\int_0^1 u(x) - \bar{u}(x)\,\mathrm{d}x = \int_0^1 \left[\left[u_V(x) + u_S(x)\right] - u_S(x)\right]\mathrm{d}x = \int_0^1 u_V(x)\,\mathrm{d}x = \boxed{2}$$

where the last identity holds as the proton is being considered here.

Problem 8.6
Figure 8.18 shows the raw measurements of the structure function $F_2(x)$ in low-energy electron-deuterium scattering. When combined with the measurements of Figure 8.11, it is found that

$$\frac{\int_0^1 F_2^{eD}(x)\,\mathrm{d}x}{\int_0^1 F_2^{ep}(x)\,\mathrm{d}x} \simeq 0.84$$

Write down the quark-parton model prediction for this ratio and determine the relative fraction of the momentum of proton carried by down-/anti-down-quarks compared to that carried by the up-/anti-up-quarks, f_d/f_u

Solution: Starting from writing down $F_2^{\mathrm{eD}}(x)$ in terms of quark PDFs in the scattered deuterium,

$$F_2^{\mathrm{eD}}(x) = x\sum_i Q_i^2 q_i^{\mathrm{D}}(x) = x\left[\frac{4}{9}u^{\mathrm{D}}(x) + \frac{1}{9}d^{\mathrm{D}}(x) + \frac{4}{9}\bar{u}^{\mathrm{D}}(x) + \frac{1}{9}\bar{d}^{\mathrm{D}}(x)\right] \tag{26}$$

and upon the fact that deuterium is consisted of one proton and neutron each, the following expressions for $q_i^{\mathrm{D}}(x)$ holds :

$$u^{\mathrm{D}}(x) = \frac{1}{2}\left[u^p(x) + u^n(x)\right] = \frac{1}{2}\left[u(x) + d(x)\right] \quad \text{and} \quad d^{\mathrm{D}}(x) = \frac{1}{2}\left[d^p(x) + d^n(x)\right] = \frac{1}{2}\left[d(x) + u(x)\right]$$

$$\bar{u}^{\mathrm{D}}(x) = \frac{1}{2}\left[\bar{u}^p(x) + \bar{u}^n(x)\right] = \frac{1}{2}\left[\bar{u}(x) + \bar{d}(x)\right] \quad \text{and} \quad \bar{d}^{\mathrm{D}}(x) = \frac{1}{2}\left[\bar{d}^p(x) + \bar{d}^n(x)\right] = \frac{1}{2}\left[\bar{d}(x) + \bar{u}(x)\right]$$

Inserting these relations into (26) gives,

$$\begin{aligned}
F_2^{\mathrm{eD}}(x) &= x\frac{1}{2}\left[\frac{4}{9}u^{\mathrm{D}}(x) + \frac{1}{9}d^{\mathrm{D}}(x) + \frac{4}{9}\bar{u}^{\mathrm{D}}(x) + \frac{1}{9}\bar{d}^{\mathrm{D}}(x)\right] \\
&= x\frac{1}{2}\left[\frac{4}{9}\left\{u(x) + d(x)\right\} + \frac{1}{9}\left\{u(x) + d(x)\right\} + \frac{4}{9}\left\{\bar{u}(x) + \bar{d}(x)\right\} + \frac{1}{9}\left\{\bar{u}(x) + \bar{d}(x)\right\}\right] \\
&= x\frac{5}{18}\left[u(x) + d(x) + \bar{u}(x) + \bar{d}(x)\right]
\end{aligned}$$

Then, the ratio between the integrated structure functions can be expressed as the following upon the quark-parton model as,

$$\frac{\int_0^1 F_2^{\mathrm{eD}}(x)}{\int_0^1 F_2^{\mathrm{ep}}(x)} = \frac{\frac{5}{18}f_u + \frac{5}{18}f_d}{\frac{4}{9}f_u + \frac{1}{9}f_d} = \frac{5(1+r)}{8+2r} \simeq 0.84 \quad \text{where} \quad r \equiv \frac{f_d}{f_u}$$

solving the above in terms of r gives $\boxed{r \equiv f_d/f_u \simeq 0.51}$ which is as expected.

Problem 8.7
Including the contribution from strange quarks:

(a) Show that $F_2^{ep}(x)$ can be written

$$F_2^{\mathrm{ep}}(x) = \frac{4}{9}x\left[u(x) + \bar{u}(x)\right] + \frac{1}{9}x\left[d(x) + \bar{d}(x) + s(x) + \bar{s}(x)\right],$$

where $s(x)$ and $\bar{s}(x)$ are the strange quark-parton distribution functions of the proton.

(b) Find the corresponding expression for $F_2^{en}(x)$ and show that

$$\int_0^1 \frac{F_2^{\mathrm{ep}}(x) - F_2^{\mathrm{en}}(x)}{x}\,\mathrm{d}x \approx \frac{1}{3} + \frac{2}{3}\int_0^1 \left[\bar{u}(x) - \bar{d}(x)\right]\mathrm{d}x$$

and interpret the measured value of 0.24 ± 0.03

Solution:

(a) It could be easily seen that from the definition of F_2,

$$F_2^{\mathrm{ep}}(x) = x \sum_i Q_i^2 q_i^p(x) = x \left[\frac{4}{9} \left(u^p(x) + \bar{u}^p(x) \right) + \frac{1}{9} \left(d^p(x) + \bar{d}^p(x) \right) + \frac{1}{9} \left(s^p(x) + \bar{s}^p(x) \right) \right]$$

$$= \frac{4}{9} x \left[u(x) + \bar{u}(x) \right] + \frac{1}{9} x \left[d(x) + \bar{d}(x) + s(x) + \bar{s}(x) \right] \quad \square$$

(b) The same can be done for the case of neutrons by considering isospin symmetry :

$$F_2^{\mathrm{en}}(x) = x \sum_i Q_i^2 q_i^p(x) = x \left[\frac{4}{9} \left(u^n(x) + \bar{u}^n(x) \right) + \frac{1}{9} \left(d^n(x) + \bar{d}^n(x) \right) + \frac{1}{9} \left(s^n(x) + \bar{s}^n(x) \right) \right]$$

$$= \frac{1}{9} x \left[u(x) + \bar{u}(x) + s(x) + \bar{s}(x) \right] + \frac{1}{4} x \left[d(x) + \bar{d}(x) \right]$$

Then one could calculate the given integral as,

$$\begin{aligned}
\int_0^1 \left[F_2^{\mathrm{ep}}(x) - F_2^{\mathrm{en}}(x) \right] \frac{\mathrm{d}x}{x} &= \frac{1}{3} \int_0^1 \left[u(x) + \bar{u}(x) - d(x) - \bar{d}(x) \right] \mathrm{d}x \\
&= \frac{1}{3} \int_0^1 \left[u_V(x) + u_S(x) + \bar{u}(x) - d_V(x) - d_S(x) - \bar{d}(x) \right] \mathrm{d}x \\
&\simeq \frac{1}{3} \int_0^1 \left[u_V(x) + 2\bar{u}(x) - d_V(x) - 2\bar{d}(x) \right] \mathrm{d}x \quad \left(u_S(x) \simeq \bar{u}(x) \quad \text{and} \quad d_S(x) \simeq \bar{d}(x) \right) \\
&= \frac{1}{3} \int_0^1 \left[u_V(x) - d_V(x) \right] \mathrm{d}x - \frac{2}{3} \int_0^1 \left[\bar{u}(x) - \bar{d}(x) \right] \mathrm{d}x \\
&= \frac{1}{3} - \frac{2}{3} \int_0^1 \left[\bar{u}(x) - \bar{d}(x) \right] \mathrm{d}x \quad \square
\end{aligned}$$

The observed value of the above expression, which is 0.24 ± 0.03 being nearly close to $1/3$ can be interpreted as the second term being 0, which is that $\bar{u}(x) \simeq \bar{d}(x)$ in the measured region.

Problem 8.8
At the HERA collider, electrons of energy $E_1 = 27.5$ GeV collided with protons of energy $E_2 = 820$ GeV. In deep inelastic scattering events at HERA, show that the Bjorken x is given by

$$x = \frac{E_3}{E_2} \left[\frac{1 - \cos\theta}{2 - \frac{E_3}{E_1}(1 + \cos\theta)} \right]$$

where θ is the angle through which the electron has scattered and E_3 is the energy of the scattered electron. Estimate x and Q^2 for the event shown in Figure 8.13 assuming that the energy of the scattered electron is 250 GeV.

Solution: Setting the four-vectors for the given collision, as

$$\begin{aligned} p_1 &= (E_1, 0, 0, E_1) & p_2 &= (E_2, 0, 0, -E_2) \\ p_3 &= (E_3, E_3 \sin\theta, 0, E_3\cos\theta) & q &= (E_1 - E_3, -E_3\sin\theta, 0, E_1 - E_3\cos\theta) \end{aligned}$$

which could be used to calculate the Bjorken x as

$$\begin{aligned} x = \frac{Q^2}{2p_2 \cdot q} &= -\frac{(p_1 - p_3)^2}{2p_2 \cdot q} \simeq \frac{p_1 \cdot p_3}{p_2 \cdot q} \\ &= \frac{E_1 E_3 (1 - \cos\theta)}{E_2 (E_1 - E_3) + E_2 (E_1 - E_3 \cos\theta)} \\ &= \frac{E_1 E_3 (1 - \cos\theta)}{E_2 [2E_1 - E_3 (1 + \cos\theta)]} \\ &= \frac{E_3}{E_2} \left[\frac{1 - \cos\theta}{2 - \frac{E_3}{E_1}(1 + \cos\theta)} \right] \quad \square \end{aligned}$$

From Figure 8.13 one could notice that $\theta \simeq 150°$, and using the values of $E_1 = 27.5$ GeV , $E_2 = 820$ GeV and $E_3 = 250$ GeV which was given in the problem, Q^2 and x can be computed as,

$$\begin{aligned} Q^2 &\simeq 2E_1E_3 (1 - \cos\theta) \\ &\simeq 2 \cdot 27.5 \text{ GeV} \cdot 250 \text{ GeV} \cdot (1 + 0.86) \simeq \boxed{2.55 \times 10^4 \text{ GeV}^2} \\ x &\simeq \frac{E_3}{E_2} \left[\frac{1 - \cos\theta}{2 - \frac{E_3}{E_1}(1 + \cos\theta)} \right] \\ &\simeq \frac{250}{820} \left[\frac{1 + 0.86}{2 - \frac{250}{27.5}(1 - 0.86)} \right] \simeq \boxed{0.77} \end{aligned}$$

---

9 Symmetries and the Quark Model

Problem 9.1
By writing down the general term in the binomial expansion of

$$\left(1 + i\frac{1}{n}\boldsymbol{\alpha} \cdot \hat{\boldsymbol{G}}\right)^n,$$

show that

$$\hat{U}(\boldsymbol{\alpha}) = \lim_{n\to\infty} \left(1 + i\frac{1}{n}\boldsymbol{\alpha} \cdot \hat{\boldsymbol{G}}\right)^n = \exp\left(i\boldsymbol{\alpha} \cdot \hat{\boldsymbol{G}}\right).$$

Solution: From the general term of the binomial expansion,

$$
\begin{aligned}
\left(1+i\frac{1}{n}\boldsymbol{\alpha}\cdot\hat{\boldsymbol{G}}\right)^n &= \sum_{k=0}^{n}\frac{n!}{k!\,(n-k)!}\left(i\frac{1}{n}\boldsymbol{\alpha}\cdot\hat{\boldsymbol{G}}\right)^k \\
&= \sum_{k=0}^{n}\frac{(n-k+1)\cdots(n-1)\,n}{k!}\,\frac{1}{n^k}\left(i\boldsymbol{\alpha}\cdot\hat{\boldsymbol{G}}\right)^k \\
&= \sum_{k=0}^{n}\frac{n-k+1}{n}\,\frac{n-k+2}{n}\cdots\frac{n-1}{n}\,\frac{n}{n}\,\frac{1}{k!}\left(i\boldsymbol{\alpha}\cdot\hat{\boldsymbol{G}}\right)^k \\
&= \sum_{k=0}^{n}\left[\prod_{j=1}^{k}\left(1-\frac{k-j}{n}\right)\right]\frac{1}{k!}\left(i\boldsymbol{\alpha}\cdot\hat{\boldsymbol{G}}\right)^k
\end{aligned}
$$

Now taking the limit of $n\to\infty$ gives,

$$
\begin{aligned}
\hat{U}(\boldsymbol{\alpha}) &= \lim_{n\to\infty}\left(1+i\frac{1}{n}\boldsymbol{\alpha}\cdot\hat{\boldsymbol{G}}\right)^n \\
&= \lim_{n\to\infty}\sum_{k=0}^{n}\left[\cancelto{1}{\prod_{j=1}^{k}\left(1-\frac{k-j}{n}\right)}\right]\frac{1}{k!}\left(i\boldsymbol{\alpha}\cdot\hat{\boldsymbol{G}}\right)^k \\
&= \sum_{k=0}^{n}\frac{1}{k!}\left(i\boldsymbol{\alpha}\cdot\hat{\boldsymbol{G}}\right)^k = \exp\left(i\boldsymbol{\alpha}\cdot\hat{\boldsymbol{G}}\right) \quad \square
\end{aligned}
$$

Problem 9.2
For an infinitesimal rotation about the z-axis through an angle ϵ show that

$$
\hat{U} = 1 - i\epsilon\hat{J}_z,
$$

where $\hat{J}_z$ is the angular momentum operator $\hat{J}_z = x\hat{p}_y - y\hat{p}_x$.

Solution: One could write down the new coordinates (x', y') after an infinitesimal counter clockwise z-axis rotation of ϵ as

$$
\begin{aligned}
x' &= x\cos\epsilon + y\sin\epsilon = x + y\epsilon + \mathcal{O}(\epsilon^2) \\
y' &= -x\sin\epsilon + y\cos\theta = -x\epsilon + y + \mathcal{O}(\epsilon^2)
\end{aligned}
$$

which transforms the wavefunction as,

$$
\begin{aligned}
\psi(x,y,z)\to\psi'(x',y',z') &= \psi(x+y\epsilon, -x\epsilon+y, z) + \mathcal{O}(\epsilon^2) \\
&= \psi(x,y,z) + y\epsilon\frac{\partial\psi}{\partial x} - x\epsilon\frac{\partial\psi}{\partial y} + \mathcal{O}(\epsilon^2) \\
&\simeq \left[1-\epsilon\left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right)\right]\psi(x,y,z) = \left[1 - i\epsilon\left(x\hat{p}_y - y\hat{p}_x\right)\right]\psi \quad \square
\end{aligned}
$$

Problem 9.3
By considering the isospin states, show that the rates for the following strong interaction decays occur in the ratios

$$\begin{aligned}\Gamma\left(\Delta^- \to \pi^- n\right) \ &: \Gamma\left(\Delta^0 \to \pi^- p\right) \ : \ \Gamma\left(\Delta^0 \to \pi^0 n\right) \ : \ \Gamma\left(\Delta^+ \to \pi^+ n\right) \ : \\ &\Gamma\left(\Delta^+ \to \pi^0 p\right) \ : \ \Gamma\left(\Delta^{++} \to \pi^+ p\right) \ = \ 3:1:2:1:2:3\end{aligned}$$

Solution: One should first note that branching ratio of certain decay modes depend on their cross-section which is proportional to the squared matrix element. Considering isospin states, the matrix element $\mathcal{M}$ can be expressed in terms of initial and final isospin states as,

$$\mathcal{M} = \langle \psi_i | A_{if} | \psi_f \rangle$$

where A_{if} is the isospin operator. Upon the fact that for any total isospin I, the strong force acts the same, it is good to specify $\mathcal{M}_I$ for each total isospin I which is defined as

$$\mathcal{M}_I = \langle \psi_f^I | A_I | \psi_i^I \rangle$$

where A_I corresponds to the isospin operator for total isospin I. The first step is to identify the isospin states. This could be achieved by realizing that the isospin states can be composed using Clebsch-Gordan coefficients, which is summarized in Figure 2

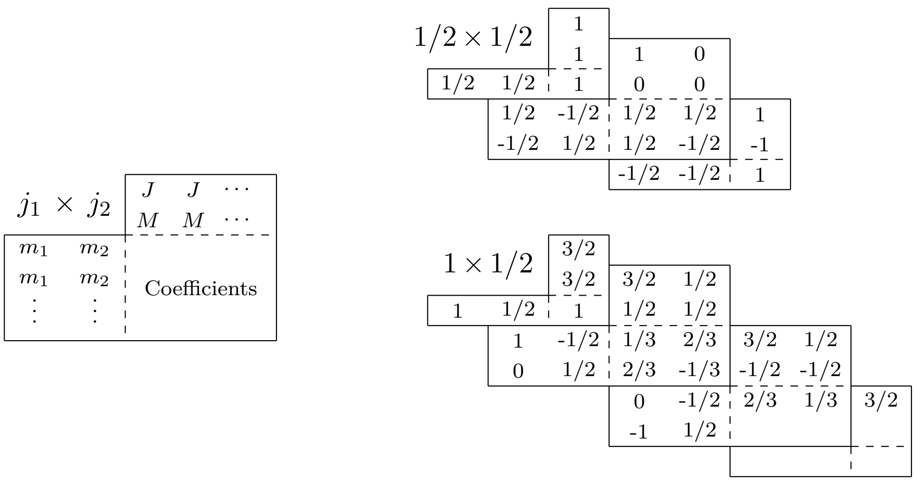


Figure 2: Clebsch-Gordan coefficients where square roots are understood on each coefficient, that is, $-1/2$ meaning $-\sqrt{1/2}$.

The initial state which are solely Δ baryons, can be represented by the following isospin states where each states are displayed in the form of $|I \ \ I_3\rangle$:

$$\psi_{\Delta^{++}} = \left|\frac{3}{2} \ \ \frac{3}{2}\right\rangle \quad , \quad \psi_{\Delta^{+}} = \left|\frac{3}{2} \ \ \frac{1}{2}\right\rangle \quad , \quad \psi_{\Delta^{0}} = \left|\frac{3}{2} \ \ -\frac{1}{2}\right\rangle \quad , \quad \psi_{\Delta^{-}} = \left|\frac{3}{2} \ \ -\frac{3}{2}\right\rangle$$

The isospin states for the π mesons are simply derived using Figure 2,

$$\psi_{\pi^+} = \psi_{uu} = \left|\frac{1}{2} \ \ \frac{1}{2}\right\rangle \otimes \left|\frac{1}{2} \ \ \frac{1}{2}\right\rangle = |1 \ \ 1\rangle$$

$$\psi_{\pi^-} = \psi_{dd} = \left|\frac{1}{2} \;\; -\frac{1}{2}\right\rangle \otimes \left|\frac{1}{2} \;\; -\frac{1}{2}\right\rangle = |1 \;\; -1\rangle$$

$$\psi_{\pi^0} = \psi_{\frac{1}{\sqrt{2}}(ud+du)} = \frac{1}{\sqrt{2}}\left[\left|\frac{1}{2} \;\; \frac{1}{2}\right\rangle \otimes \left|\frac{1}{2} \;\; -\frac{1}{2}\right\rangle + \left|\frac{1}{2} \;\; -\frac{1}{2}\right\rangle \otimes \left|\frac{1}{2} \;\; \frac{1}{2}\right\rangle\right] = |1 \;\; 0\rangle$$

And for the decayed final state, again using the Clebsch-Gordan coefficients from Figure 2, the isospin states can be obtained :

$$\psi_{\pi^+,p} = |1 \;\; 1\rangle \otimes \left|\frac{1}{2} \;\; \frac{1}{2}\right\rangle = \left|\frac{3}{2} \;\; \frac{3}{2}\right\rangle$$

$$\psi_{\pi^+,n} = |1 \;\; 1\rangle \otimes \left|\frac{1}{2} \;\; -\frac{1}{2}\right\rangle = \sqrt{\frac{1}{3}}\left|\frac{3}{2} \;\; \frac{1}{2}\right\rangle + \sqrt{\frac{2}{3}}\left|\frac{1}{2} \;\; \frac{1}{2}\right\rangle$$

$$\psi_{\pi^0,p} = |1 \;\; 0\rangle \otimes \left|\frac{1}{2} \;\; \frac{1}{2}\right\rangle = \sqrt{\frac{2}{3}}\left|\frac{3}{2} \;\; \frac{1}{2}\right\rangle - \sqrt{\frac{1}{3}}\left|\frac{1}{2} \;\; \frac{1}{2}\right\rangle$$

$$\psi_{\pi^0,n} = |1 \;\; 0\rangle \otimes \left|\frac{1}{2} \;\; -\frac{1}{2}\right\rangle = \sqrt{\frac{2}{3}}\left|\frac{3}{2} \;\; -\frac{1}{2}\right\rangle + \sqrt{\frac{1}{3}}\left|\frac{1}{2} \;\; -\frac{1}{2}\right\rangle$$

$$\psi_{\pi^-,p} = |1 \;\; -1\rangle \otimes \left|\frac{1}{2} \;\; \frac{1}{2}\right\rangle = \sqrt{\frac{1}{3}}\left|\frac{3}{2} - \;\; \frac{1}{2}\right\rangle + \sqrt{\frac{2}{3}}\left|\frac{1}{2} \;\; -\frac{1}{2}\right\rangle$$

$$\psi_{\pi^-,n} = |1 \;\; -1\rangle \otimes \left|\frac{1}{2} \;\; -\frac{1}{2}\right\rangle = \left|\frac{3}{2} \;\; -\frac{3}{2}\right\rangle$$

Now using the above isospin expression of each initial and final states, one could estimate the respective cross section and their ratios. As the initial states are all represented with pure $I = 3/2$ states, one would only have to compare the corresponding coefficients which is to only consider $\mathcal{M}_{3/2}$. For instance, consider the $\Delta^- \to \pi^- n$ decay mode,

$$\mathcal{M}_{3/2}\left(\Delta^- \to \pi^- n\right) \sim \langle\psi_{\Delta^-}|\psi^{3/2}_{\pi^-,n}\rangle = \left\langle\frac{3}{2} - \;\; \frac{3}{2}\middle|\frac{3}{2} - \;\; \frac{3}{2}\right\rangle$$

and considering that $\Gamma \sim \mathcal{M}^2$, one could say that

$$\Gamma\left(\Delta^- \to \pi^- n\right) \sim \mathcal{M}_{3/2}\left(\Delta^- \to \pi^- n\right)^2 \sim 1$$

up to some constant factor. Doing the same for all of the other processes, one could obtain

$$\Gamma\left(\Delta^0 \to \pi^- p\right) \sim \mathcal{M}_{3/2}\left(\Delta^0 \to \pi^- p\right)^2 = \left[\sqrt{\frac{1}{3}}\left\langle\frac{3}{2} \;\; -\frac{1}{2}\middle|\frac{3}{2} \;\; -\frac{1}{2}\right\rangle\right]^2 \sim \frac{1}{3}$$

$$\Gamma\left(\Delta^0 \to \pi^0 n\right) \sim \mathcal{M}_{3/2}\left(\Delta^0 \to \pi^0 n\right)^2 = \left[\sqrt{\frac{2}{3}}\left\langle\frac{3}{2} \;\; -\frac{1}{2}\middle|\frac{3}{2} \;\; -\frac{1}{2}\right\rangle\right]^2 \sim \frac{2}{3}$$

$$\Gamma\left(\Delta^+ \to \pi^+ n\right) \sim \mathcal{M}_{3/2}\left(\Delta^+ \to \pi^+ n\right)^2 = \left[\sqrt{\frac{1}{3}}\left\langle\frac{3}{2} \;\; \frac{1}{2}\middle|\frac{3}{2} \;\; \frac{1}{2}\right\rangle\right]^2 \sim \frac{1}{3}$$

$$\Gamma\left(\Delta^+ \to \pi^0 p\right) \sim \mathcal{M}_{3/2}\left(\Delta^+ \to \pi^0 p\right)^2 = \left[\sqrt{\frac{2}{3}}\left\langle\frac{3}{2} \;\; \frac{1}{2}\middle|\frac{3}{2} \;\; \frac{1}{2}\right\rangle\right]^2 \sim \frac{2}{3}$$

$$\Gamma\left(\Delta^{++} \to \pi^+ p\right) \sim \mathcal{M}_{3/2}\left(\Delta^{++} \to \pi^+ p\right)^2 = \left[\left\langle\frac{3}{2} \;\; \frac{3}{2}\middle|\frac{3}{2} \;\; \frac{3}{2}\right\rangle\right]^2 \sim 1$$

by multiplying all with a constant factor of 3, one would finally obtain the ratio of $\boxed{3:1:2:1:2:3}$ for the given branching ratios.

Problem 9.4
If quarks and antiquarks were spin-zero particles, what would be the multiplicity of the $L = 0$ multiplet(s). Remember that the overall wavefunction for bosons must be symmetric under particle exchange.

Solution: As mentioned in the textbook, it is known that the colour wavefunction ξ_{colour} in the overall wavefunction $\psi = \phi_{\text{flavour}}\chi_{\text{spin}}\xi_{\text{colour}}\eta_{\text{space}}$ is necessarily totally antisymmetric, plus only mesonic and baryonic multiplets are available due to colour confinement.

(i) Meson ($q\bar{q}$)

As quarks and antiquarks are distinguishable, there is no restriction on the exchange symmetry of the overall wavefunction. Thus the multiplicity will simply be $\boxed{9}$ in this case.

(ii) Baryon (qqq)

If (anti)quarks were spin-zero particles, the spin-statistics theorem states that the overall wavefunction $\psi = \phi_{\text{flavour}}\chi_{\text{spin}}\xi_{\text{colour}}\eta_{\text{space}}$ of the multiplets should be symmetric under the interchange of any two of the quarks. Also, as $L = 0$ is considered the quarks are descirbed by $l = 0$ s-waves, thus the exchange symmetry factor is given as $(-1)^l = 1$ for any two quarks which implies that η_{space} is symmetric. As $\xi_{\text{colour}}\eta_{\text{space}}$ is antisymmetric, $\phi_{\text{flavour}}\chi_{\text{spin}}$ must be antisymmetric too in order to make ψ totally symmetric. For the spin wavefunction, any multiplet with arbitrary multiplicity will end up being a singlet as it would be a combination of spin zero particles. This boils down to the requirement that if ϕ_{flavour} is totally antisymmetric, the overall wavefunction will be symmetric. As there is only one totally antisymmetric flavour state for a qqq bound state, in this case the multiplicity will be $\boxed{1}$.

Problem 9.5
The neutral vector mesons can decay leptonically through a virtual photon, for example by $V(q\bar{q}) \to \gamma \to e^+e^-$. The matrix element for this decay is proportional to $\langle\psi|\hat{Q}_q|\psi\rangle$ where ψ is the meson flavour wavefunction and $\hat{Q}_q$ is an operator that is proportional to the quark charge. Neglecting the relatively small differences in phase space, show that

$$\Gamma\left(\rho^0 \to e^+e^-\right) \;:\; \Gamma\left(\omega \to e^+e^-\right) \;:\; \Gamma\left(\phi \to e^+e^-\right) \;\approx\; 9:1:2$$

Solution: Let the QED matrix element of $q\bar{q} \to e^+e^-$ as

$$\mathcal{M}(q\bar{q} \to e^+e^-) \sim \langle e^+e^-|\hat{Q}_q|q\bar{q}\rangle \equiv \alpha Q_q$$

where α is an arbitrary constant and Q_q is the charge of the quark q. Using the flavour states of $|\rho^0\rangle$, $|\omega\rangle$ and $|\phi\rangle$ one could express the matrix element of each decay in terms of α as :

$$\begin{aligned}\mathcal{M}(\rho^0 \to e^+e^-) \sim \langle e^+e^-|\hat{Q}_q|\rho^0\rangle &= \frac{1}{\sqrt{2}}\left[\langle e^+e^-|\hat{Q}_q|u\bar{u}\rangle - \langle e^+e^-|\hat{Q}_d|d\bar{d}\rangle\right] \\ &= \frac{\alpha}{\sqrt{2}}\left(Q_u - Q_d\right) = \frac{1}{\sqrt{2}}\alpha \end{aligned} \tag{27}$$

$$\begin{aligned}\mathcal{M}(\omega \to e^+e^-) \sim \langle e^+e^-|\hat{Q}_q|\omega\rangle &= \frac{1}{\sqrt{2}}\left[\langle e^+e^-|\hat{Q}_q|u\bar{u}\rangle + \langle e^+e^-|\hat{Q}_d|d\bar{d}\rangle\right] \\ &= \frac{\alpha}{\sqrt{2}}\left(Q_u + Q_d\right) = \frac{1}{3\sqrt{2}}\alpha \end{aligned} \tag{28}$$

$$\mathcal{M}(\phi \to e^+e^-) \sim \langle e^+e^-|\hat{Q}_q|\phi\rangle = \langle e^+e^-|\hat{Q}_q|s\bar{s}\rangle = \alpha Q_s = -\frac{1}{3}\alpha \tag{29}$$

As the amplitude is proportional to the square of the matrix element, the ratio between corresponding branching ratios can be obtained by squaring (27), (28) and (29) which gives,

$$\Gamma\left(\rho^0 \to e^+e^-\right) \; : \; \Gamma\left(\omega \to e^+e^-\right) \; : \; \Gamma\left(\phi \to e^+e^-\right) \; \approx \frac{1}{2}\alpha^2 : \frac{1}{18}\alpha^2 : \frac{1}{9}\alpha^2 = \boxed{9:1:2} \quad \square$$

Problem 9.6
Using the meson mass formulae of (9.37) and (9.38), obtain predictions for the masses of the $\pi^\pm, \pi^0, \eta, \eta', \rho^0, \rho^\pm, \omega$ and ϕ. Compare the values obtained to the experimental values listed in Table 9.1.

Solution: As the problem is simply requiring to plug in values into the mass formulae for prseudoscalar and vector mesons, a summarized table with both predicted and observed experimental values is presented :

Pseudoscalar			Vector		
Meson	Prediction	Experimental	Meson	Prediction	Experimental
π^0	137 MeV	135 MeV	ρ^0	773 MeV	775 MeV
$\pi^\pm$	137 MeV	140 MeV	$\rho^\pm$	773 MeV	775 MeV
η	512 MeV	548 MeV	ω	773 MeV	783 MeV
η'	400 MeV	958 MeV	ϕ	1042 MeV	1020 MeV

Here the value of $m_u = m_d = 0.307$ GeV and $A = 0.06$ GeV^3 is used, and the averaged constituent mass is used.

Problem 9.7
Compare the experimentally measured values of the masses of the $J_P = \frac{3}{2}^+$ baryons, given in Table 9.2, with the predictions of (9.41). You will need to consider the combined spin of any two quarks in a spin-3/2 baryon state.

Solution: From the $L = 0$ baryon mass prediction formula,

$$m(q_1q_2q_3) = m_1 + m_2 + m_3 + A'\left(\frac{\langle \mathbf{S}_1 \cdot \mathbf{S}_2\rangle}{m_1m_2} + \frac{\langle \mathbf{S}_1 \cdot \mathbf{S}_3\rangle}{m_1m_3} + \frac{\langle \mathbf{S}_2 \cdot \mathbf{S}_3\rangle}{m_2m_3}\right) \tag{30}$$

and using the expression of $\mathbf{S} = \mathbf{S}_1 + \mathbf{S}_2 + \mathbf{S}_3$ one could obtain,

$$\mathbf{S}^2 = \mathbf{S}_1^2 + \mathbf{S}_2^2 + \mathbf{S}_3^2 + 2\left(\mathbf{S}_1 \cdot \mathbf{S}_2 + \mathbf{S}_1 \cdot \mathbf{S}_3 + \mathbf{S}_2 \cdot \mathbf{S}_3\right)$$

$$\implies \langle \mathbf{S}_1 \cdot \mathbf{S}_3\rangle + \langle \mathbf{S}_1 \cdot \mathbf{S}_3\rangle + \langle \mathbf{S}_1 \cdot \mathbf{S}_3\rangle = \frac{1}{2}\left(\langle \mathbf{S}^2\rangle - \langle \mathbf{S}_1^2\rangle - \langle \mathbf{S}_2^2\rangle - \langle \mathbf{S}_3^2\rangle\right) = \frac{1}{2}\left[\frac{3}{2}\cdot\frac{5}{2} - 3\cdot\frac{1}{2}\cdot\frac{3}{2}\right] = \frac{3}{4}$$

and assuming that $m_1 = m_2 = m_3 = \overline{m} = \frac{1}{3}(m_1 + m_2 + m_3)$ one could write down the mass formula (30) as

$$m(q_1q_2q_3) = 3\overline{m} + \frac{3}{4}\cdot\frac{A'}{\overline{m}^2}$$

Plugging in the different $\overline{m}$ values for each baryons which are : $\overline{m}_\Delta = m_d = m_u = 0.365$ GeV , $\overline{m}_\Sigma = \frac{1}{3}(2m_u + m_s) = 0.423$ GeV , $\overline{m}_\Xi = \frac{1}{3}(m_u + 2m_s) = 0.481$ GeV and $\overline{m}_\Omega = m_s = 0.540$ GeV , one could obtain the following predictions which has pretty good agreement within an error of $\mathcal{O}(1\%)$:

Baryon	Prediction	Experimental
Δ	1241 MeV	1230 MeV
Σ	1377 MeV	1385 MeV
Ξ	1527 MeV	1533 MeV
Ω	1686 MeV	1670 MeV

Problem 9.8
Starting from the wavefunction for the Σ^- baryon :

(a) obtain the wavefunction for the Σ^0 and therefore find the wavefunction for the Λ

(b) using (9.41), obtain predictions for the masses of the Σ^0 and the Λ baryons and compare these to the measured values.

Solution:

Problem 9.9
Show that the quark model predictions for the magnetic moments of the Σ^+, Σ^- and Ω^- baryons are

$$\mu(\Sigma^+) = \frac{1}{3}(4\mu_u - \mu_s) \quad , \quad \mu(\Sigma^-) = \frac{1}{3}(4\mu_d - \mu_s) \quad \text{and} \quad \mu(\Omega^-) = 3\mu_s.$$

What values of the quark constituent masses are required to give the best agreement with the measured values of

$$\mu(\Sigma^+) = (2.46 \pm 0.01)\,\mu_N \quad , \quad \mu(\Sigma^-) = (-1.16 \pm 0.03)\,\mu_N \quad \text{and} \quad \mu(\Omega^-) = (-2.02 \pm 0.06)\,\mu_N.$$

Solution: One should first be able to write down the wavefunctions for the given baryons in the spin-up state, $|\Sigma^+\uparrow\rangle, |\Sigma^-\uparrow\rangle$ and $|\Omega^-\uparrow\rangle$,

$$|\Sigma^+\uparrow\rangle = \frac{1}{\sqrt{6}}(2u\uparrow u\uparrow s\downarrow - u\uparrow u\downarrow s\uparrow - u\downarrow u\uparrow s\uparrow)$$

$$|\Sigma^-\uparrow\rangle = \frac{1}{\sqrt{6}}(2d\uparrow d\uparrow s\downarrow - d\uparrow d\downarrow s\uparrow - d\downarrow d\uparrow s\uparrow)$$

$$|\Omega^-\uparrow\rangle = \frac{1}{\sqrt{3}}(s\uparrow s\uparrow s\downarrow - s\uparrow s\downarrow s\uparrow - s\downarrow s\uparrow s\uparrow)$$

As the total magnetic moment of a baryon can be expressed in terms of the vector sum of the magnetic moments of the constituent quarks,

$$\begin{aligned}
\mu(\Sigma^+) &= \langle\Sigma^+\uparrow|\hat{\mu}_z|\Sigma^+\uparrow\rangle \\
&= \frac{4}{6}\langle u\uparrow u\uparrow s\downarrow|\,\hat{\mu}_z\,|u\uparrow u\uparrow s\downarrow\rangle + \frac{1}{6}\langle u\uparrow u\downarrow s\uparrow|\,\hat{\mu}_z\,|u\uparrow u\downarrow s\uparrow\rangle + \frac{1}{6}\langle u\downarrow u\uparrow s\uparrow|\,\hat{\mu}_z\,|u\downarrow u\uparrow s\uparrow\rangle \\
&= \frac{4}{6}(\mu_u + \mu_u - \mu_s) + \frac{1}{6}(\mu_u - \mu_u + \mu_s) + \frac{1}{6}(-\mu_u + \mu_u + \mu_s) \\
&= \frac{1}{3}(4\mu_u - \mu_s)
\end{aligned}$$

and for Ω^-,

$$\mu(\Omega^-) = \langle\Omega^-\uparrow|\hat{\mu}_z|\Omega^-\uparrow\rangle$$

$$= \frac{1}{3}\langle s{\uparrow} s{\uparrow} s{\downarrow}|\,\hat{\mu}_z\,|s{\uparrow} s{\uparrow} s{\downarrow}\rangle + \frac{1}{3}\langle s{\uparrow} s{\downarrow} s{\uparrow}|\,\hat{\mu}_z\,|s{\uparrow} s{\downarrow} s{\uparrow}\rangle + \frac{1}{3}\langle s{\downarrow} s{\uparrow} s{\uparrow}|\,\hat{\mu}_z\,|s{\downarrow} s{\uparrow} s{\uparrow}\rangle$$

$$= \frac{1}{3}(\mu_s + \mu_s - \mu_s) + \frac{1}{3}(\mu_s - \mu_s + \mu_s) + \frac{1}{3}(-\mu_s + \mu_s + \mu_s)$$

$$= \frac{1}{3}(4\mu_u - \mu_s)$$

by switching $u \leftrightarrow d$ in $|\Sigma^+{\uparrow}\rangle$ gives $|\Sigma^-{\uparrow}\rangle$ thus switching $\mu_u \leftrightarrow \mu_d$ for $\mu(\Sigma^+)$ will give $\mu(\Sigma^-)$

Problem 9.10
If the colour did not exist, baryon wavefunctions would be constructed from

$$\psi = \phi_{\text{flavour}}\chi_{\text{spin}}\eta_{\text{space}}.$$

Taking $L = 0$ and using the flavour and spin wavefunctions derived in the text:

(a) show that it is still possible to construct a wavefunction for a spin-up proton for which $\phi_{\text{flavour}}\chi_{\text{spin}}$ is totally antisymmetric;

(b) predict the baryon multiplet structure for this model;

(c) for this colourless model, show that μ_p is negative and that the ratio of the neutron and proton magnetic moments would be

$$\frac{\mu_n}{\mu_p} = -2.$$

Solution:

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10 Quantum Chromodynamics

Problem 10.1
By considering the symmetry of the wavefunction, explain why the existence of the $\Omega^-(sss)$ $L = 0$ baryon provides evidence for a degree of freedom in addition to space $\times$ spin $\times$ flavour.

Solution:

Problem 10.2
From the expression for the running of α_S with $N_f = 3$, determine the value of q^2 at which α_S appears to become infinite. Comment on this result.

Solution:

Problem 10.3
Find the overall “colour factor” for $qq \to qq$ if QCD corresponded to a SU(2) colour symmetry.

Solution:

Problem 10.4
Calculate the non-relativistic QCD potential between quarks q_1 and q_2 in a $q_1q_2q_3$ baryon with colour wave-function

$$\psi = \frac{1}{\sqrt{6}}(rgb - grb + gbr - bgr + brg - rbg)$$

Solution:

Problem 10.5
Draw the lowest-order QCD Feynman diagrams for the process $p\bar{p} \to$ two-jets + X, where X represents the remnants of the colliding hadrons.

Solution:

Problem 10.6
The observed events in the process $pp \to$ two-jets at the LHC can be described in terms of the jet p_T and the jet rapidities y_3 and y_4.

(a) Assuming that the jets are massless, $E^2 = p_T^2 + p_z^2$, show that the four-momenta of the final-state jets can be written as

$$\begin{aligned} p_3 &= (p_T \cosh y_3, +p_T \sin\phi, +p_T \cos\phi, p_T \sinh y_3) \\ p_4 &= (p_T \cosh y_4, -p_T \sin\phi, -p_T \cos\phi, p_T \sinh y_4) \end{aligned}$$

(b) By writing the four-momenta of the colliding partons in a pp collision as

$$p_1 = \frac{\sqrt{s}}{2}(x_1, 0, 0, x_1) \quad \text{and} \quad p_2 = \frac{\sqrt{s}}{2}(x_2, 0, 0, -x_1)$$

show that conservation of energy and momentum implies

$$x_1 = \frac{p_T}{\sqrt{s}}\left(e^{+y_3} + e^{+y_4}\right) \quad \text{and} \quad x_2 = \frac{p_T}{\sqrt{s}}\left(e^{-y_3} + e^{-y_4}\right)$$

(c) Hence show that

$$Q^2 = p_T^2\left(1 + e^{y_4 - y_3}\right).$$

Solution:

Problem 10.7
Using the results of the previous question show that the Jacobian

$$\frac{\partial\left(y_3, y_4, p_T^2\right)}{\partial\left(x_1, x_2, q^2\right)} = \frac{1}{x_1 x_2}.$$

Solution:

Problem 10.8
The total cross section for the Drell-Yan process $p\bar{p} \to \mu^+\mu^- X$ was shown to be

$$\sigma_{\mathrm{DY}} = \frac{4\pi\alpha^2}{81 s}\int_0^1\int_0^1 \frac{1}{x_1 x_2}\left[4u(x_1)u(x_2) + 4\bar{u}(x_1)\bar{u}(x_2) + d(x_1)d(x_2) + \bar{d}(x_1)\bar{d}(x_2)\right]\mathrm{d}x_1\,\mathrm{d}x_2$$

(a) Express this cross section in terms of the valence quark PDFs and a single PDF for the sea contribution, where $S(x) = \bar{u}(x) = \bar{d}(x)$.

(b) Obtain the corresponding expression for $pp \to \mu^+\mu^- X$.

(c) Sketch the region in the $x_1 - x_2$ plane corresponding $S_{q\bar{q}} > s/4$. Comment on the expected ratio of the Drell-Yan cross sections in pp and $p\bar{p}$ collisions (at the same centre-of-mass energy) for the two cases: (i) $\hat{s} \ll s$ and (ii) $\hat{s} > s/4$, where $\hat{s}$ is the centre-of-mass energy of the colliding partons.

Solution:

Problem 10.9
Drell-Yan production of $\mu^+\mu^-$ pairs with an invariant mass Q^2 has been studied in $\pi^\pm$ interactions with carbon (which has equal numbers of protons and neutrons). Explain why the ratio

$$\frac{\sigma\left(\pi^+\mathrm{C} \to \mu^+\mu^- X\right)}{\sigma\left(\pi^-\mathrm{C} \to \mu^+\mu^- X\right)}$$

tends to unity for small Q^2 and tends to $\frac{1}{4}$ as Q^2 approaches s.

Solution:
