

1 Introduction

Problem 1.1

Feynman diagrams are constructed out of the Standard Model vertices shown in Figure 1.4. Only the weak charged-current interaction can change the flavour of the particle at the interaction vertex. Explaining your reasoning, state whether each of the sixteen diagrams below represents a valid Standard Model vertex.

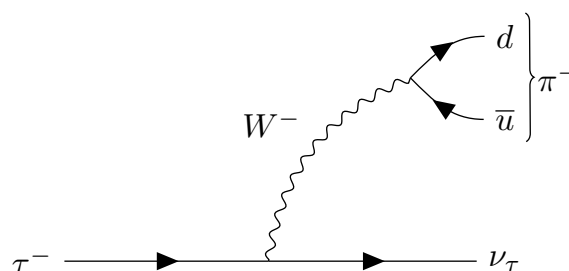
Solution:

- (a) Valid.
- (b) Invalid, due to the fact that ν_e has no electric charge.
- (c) Valid.
- (d) Valid.
- (e) Invalid.
- (f) Valid.
- (g) Invalid.
- (h) Invalid.
- (i) Invalid, leptons do not carry color charge.
- (j) Valid.
- (k) Valid.
- (l) Invalid.
- (m) Invalid.
- (n) Valid.
- (o) Valid.
- (p) Invalid.

Problem 1.2

Draw the Feynman diagram for $\tau^- \rightarrow \pi^- \nu_\tau$. (The π^- is the lightest $d\bar{u}$ meson)

Solution:



Problem 1.3

Explain why it is not possible to construct a valid Feynman diagram using the Standard Model vertices for the following processes :

- (a) $\mu^- \rightarrow e^+ e^- e^+$
- (b) $\nu_\tau + p \rightarrow \mu^- + n$
- (c) $\nu_\tau + p \rightarrow \tau^+ + n$
- (d) $\pi^+(u\bar{d}) + \pi^-(d\bar{u}) \rightarrow n(udd) + \pi^0(u\bar{u})$

Solution:

- (a) $\mu^- \rightarrow e^+ e^- e^+$: Charge is not conserved, as well as lepton numbers.
- (b) $\nu_\tau + p \rightarrow \mu^- + n$: Charge is not conserved, as well as baryon numbers.
- (c) $\nu_\tau + p \rightarrow \tau^+ + n$: Both baryon and lepton number is not conserved.
- (d) $\pi^+(u\bar{d}) + \pi^-(d\bar{u}) \rightarrow n(udd) + \pi^0(u\bar{u})$: Baryon number is not conserved.

Problem 1.4

Draw the Feynman diagram for the decays:

- (a) $\Delta^+(uud) \rightarrow n(udd)\pi^+(u\bar{d})$
- (b) $\Sigma^0(uds) \rightarrow \Lambda(uds)\gamma$
- (c) $\pi^+(u\bar{d}) \rightarrow \mu^+\nu_\mu$

Solution:

- (a) $\Delta^+(uud) \rightarrow n(udd)\pi^+(u\bar{d})$



- (b) $\Sigma^0(uds) \rightarrow \Lambda(uds)\gamma$



(c) $\pi^+(u\bar{d}) \rightarrow \mu^+\nu_\mu$



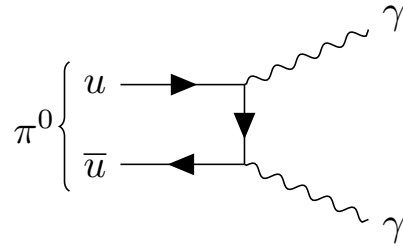
Problem 1.5

Treating the π^0 as a $u\bar{u}$ bound state, draw the Feynman diagrams for:

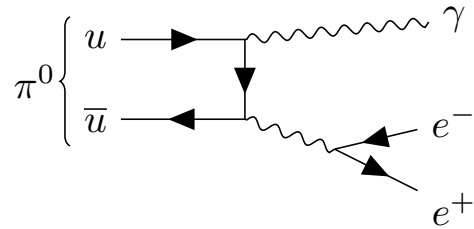
- (a) $\pi^0 \rightarrow \gamma\gamma$
- (b) $\pi^0 \rightarrow \gamma e^+ e^-$
- (c) $\pi^0 \rightarrow e^+ e^- e^+ e^-$
- (d) $\pi^0 \rightarrow e^+ e^-$

Solution:

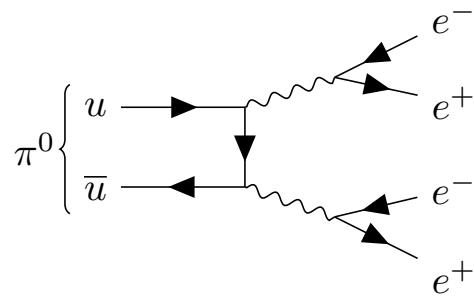
(a) $\pi^0 \rightarrow \gamma\gamma$



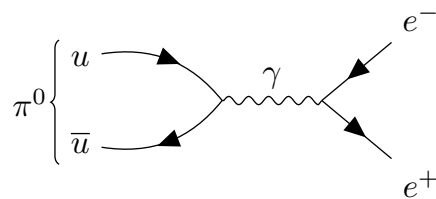
(b) $\pi^0 \rightarrow \gamma e^+ e^-$



(c) $\pi^0 \rightarrow e^+ e^- e^+ e^-$



(d) $\pi^0 \rightarrow e^+ e^-$



Problem 1.6

Particle interactions fall into two main categories, scattering processes and annihilation processes, as indicated by the Feynman diagrams below.



Draw the lowest-order Feynman diagrams for the scattering and/or annihilation processes:

- (a) $e^-e^- \rightarrow e^-e^-$
- (b) $e^+e^- \rightarrow \mu^+\mu^-$
- (c) $e^+e^- \rightarrow e^+e^-$
- (d) $e^-\nu_e \rightarrow e^-\nu_e$
- (e) $e^-\bar{\nu}_e \rightarrow e^-\bar{\nu}_e$

Solution:

- (a) $e^-e^- \rightarrow e^-e^-$



- (b) $e^+e^- \rightarrow \mu^+\mu^-$



- (c) $e^+e^- \rightarrow e^+e^-$
- (d) $e^-\nu_e \rightarrow e^-\nu_e$
- (e) $e^-\bar{\nu}_e \rightarrow e^-\bar{\nu}_e$

Problem 1.7

High-energy muons traversing matter lose energy according to

$$-\frac{1}{\rho} \frac{dE}{dx} \approx a + bE$$

where a is due to ionisation energy loss and b is due to the bremsstrahlung and e^+e^- pair-production processes. For standard rock, taken to have $A = 22, Z = 11$ and $\rho = 2.65 \text{ g cm}^{-3}$, the parameters a and b depend only weakly on the muon energy and have values $a \approx 2.5 \text{ MeV g}^{-1} \text{ cm}^2$ and $b \approx 3.5 \times 10^{-6} \text{ g}^{-1} \text{ cm}^2$.

- (a) At what muon energy are the ionisation and bremsstrahlung/pair production processes equally important?

(b) Approximately how far does a 100 GeV cosmic-ray muon propagate in rock?

Solution:

- (a) One could assume that ionisation and bremsstrahlung/pair production processes become equally important for a certain energy scale E^* when $a \simeq bE^*$. Such $E^* \simeq a/b$ can be calculated as ~ 700 GeV.
- (b) Using the values given,

$$-\frac{dE}{dx} \approx a\rho + b\rho E \iff (a\rho \sim 6.6 \text{ MeV/cm}, b\rho \sim 9.275 \times 10^{-6}/\text{cm})$$
$$\simeq 7.52 \text{ MeV/cm}$$

which shows that a 100 GeV muon will go through around 132 metres of rock.

Problem 1.8

Tungsten has a radiation length of $X_0 = 0.35$ cm and a critical energy of $E_c = 7.97$ MeV. Roughly what thickness of tungsten is required to fully contain a 500 GeV electromagnetic shower from an electron?

Solution: Getting x_{\max} for the given situation, one obtains :

$$x_{\max} = \frac{1}{\ln 2} \ln \left(\frac{E}{E_c} \right) = \frac{1}{\ln 2} \ln \left(\frac{500 \text{ GeV}}{7.97 \text{ MeV}} \right) \sim 16$$

Thus, roughly around $x_{\max}X_0 \simeq 5.6$ cm of tungsten would be able to contain a 500 GeV electromagnetic shower from an electron.

Problem 1.9

The CPLEAR detector consisted of: tracking detectors in a magnetic field of 0.44 T; and electromagnetic calorimeter; and Čerenkov detectors with a radiator of refractive index $n = 1.25$ used to distinguish π^\pm from K^\pm .

A charged particle travelling perpendicular to the direction of the magnetic field leaves a track with a measured radius of curvature of $R = 4$ m. If it is observed to give a Čerenkov signal, is it possible to distinguish between the particle being a pion or kaon? Take $m_\pi \approx 140$ MeV/ c^2 and $m_K \approx 494$ MeV/ c^2

Solution: First, the momentum could be extracted from the fact that the charged particles are travelling perpendicular ($\lambda = 0$) to the 0.44 T magnetic field, which eventually gives $p = 0.3BR = 0.528$ GeV. The threshold mass for Čerenkov radiation in this case would be,

$$\sqrt{n^2 - 1}p = 0.75 \times p = 0.396 \text{ GeV}$$

Problem 1.10

In a fixed-target pp experiment, what proton energy would be required to achieve the same centre-of-mass energy as the LHC, which will ultimately operate at 14 TeV.

Solution: Let the four-momentum of the beam proton and the fixed target proton as $p_1 = (E, 0, 0, p)$ and $p_2 = (m_p, 0, 0, 0)$. Using the following expression of the centre-of-mass energy \sqrt{s} , the proton energy E to satisfy the required situation would be :

$$\begin{aligned} \sqrt{s} &= (p_1 + p_2)^2 = 2m_p^2 + 2p_1 \cdot p_2 \\ &= 2m_p(m_p + E) = 14 \text{ TeV} \implies \boxed{E \simeq 7.4 \text{ PeV}} \end{aligned}$$

Problem 1.11

At the LEP e^+e^- collider, which had a circumference of 27 km, the electron and positron beam currents were both 1.0 mA. Each beam consisted of four equally spaced bunches of electrons/positrons. The bunches had an effective area of $1.8 \times 10^4 \mu\text{m}^2$. Calculate the instantaneous luminosity on the assumption that the beams collided head-on.

Solution:

2 Underlying Concepts

Problem 2.1

When expressed in natural units the lifetime of the W boson is approximately $\tau \approx 0.5 \text{ GeV}^{-1}$. What is the corresponding value in S.I. units?

Solution: In natural units, $\hbar = 1.055 \times 10^{-34} \text{ J} \cdot \text{s} = 6.582 \times 10^{-25} \text{ GeV} \cdot \text{s}$ which is, $1 \text{ GeV}^{-1} = 6.582 \times 10^{-25} \text{ s}$. Thus the lifetime of the W boson in S.I. units can be written as, $\tau \simeq 3.291 \times 10^{-25} \text{ s}$.

Problem 2.2

A cross section is measured to be 1 pb; convert this to natural units.

Solution: Taking note that $\hbar c = 0.197 \text{ GeV fm}$, which is $0.197 \text{ GeV} = 1 \text{ fm}^{-1}$

$$1 \text{ pb} = 10^{-10} \text{ fm}^2 = 10^{-10} \times \left(\frac{1}{0.197} \right)^2 \text{ GeV}^{-2} = \boxed{2.57 \times 10^{-9} \text{ GeV}^{-2}}$$

Problem 2.3

Show that the process $\gamma \rightarrow e^+e^-$ can not occur in vacuum.

Solution: If it were so, such process should occur in any frame. Let such frame as the rest frame of

Problem 2.4

A particle of mass 3 GeV is travelling in the positive z-direction with momentum 4 GeV. What are its energy and velocity?

Solution: Using the relation of $m^2 = E^2 - |\mathbf{p}|^2$, one gets $E^2 = 25 \text{ GeV}^2$ thus the energy is $\boxed{E = 5 \text{ GeV}}$. Now considering the relation of $|\mathbf{p}| = E\beta$, it is seen that $\beta = |\mathbf{p}|E^{-1} = 0.8$ thus the velocity is $\boxed{0.8c}$.

Problem 2.5

In the laboratory frame, denoted Σ , a particle travelling in the z-direction has momentum $\mathbf{p} = p_z \hat{\mathbf{z}}$ and energy E .

- Use the Lorentz transformation to find expressions for the momentum p'_z and energy E' of the particle in a frame Σ' which is moving in a velocity $\mathbf{v} = +v\hat{\mathbf{z}}$ relative to Σ , and show that $E^2 - p_z^2 = (E')^2 - (p'_z)^2$.
- For a system of particles, prove that the total four-momentum squared,

$$p^\mu p_\mu \equiv \left(\sum_i E_i \right)^2 - \left(\sum_i \mathbf{p}_i \right)^2$$

is invariant under Lorentz transformations.

Solution:

- (a) Let the four-momentum of the given particle in the frame Σ and Σ' as $p = (E, 0, 0, p_z), p' = (E', \mathbf{p}')$ respectively. Denoting the corresponding matrix representation of the given Lorentz transformation as $\mathbf{\Lambda}$, one could write down the transformation of p as,

$$\begin{aligned} p' = \mathbf{\Lambda} p &\implies p'^\mu = \Lambda^\mu_\nu p^\nu \\ &= \begin{pmatrix} \gamma & 0 & 0 & -\gamma\beta \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\gamma\beta & 0 & 0 & \gamma \end{pmatrix} \begin{pmatrix} E \\ 0 \\ 0 \\ p_z \end{pmatrix} = \gamma \begin{pmatrix} E - \beta p_z \\ 0 \\ 0 \\ -E\beta + p_z \end{pmatrix} \end{aligned}$$

which implies that $E' = \gamma(E - \beta p_z)$ and $p'_z = -\gamma(E\beta - p_z)$. Using such expression of p' , one could show that :

$$\begin{aligned} (E')^2 - (p'_z)^2 &= \gamma^2(E - \beta p_z)^2 - \gamma^2(E\beta - p_z)^2 \\ &= \gamma^2 [(E - \beta p_z)^2 - (E\beta - p_z)^2] \\ &= \gamma^2 [(E - \beta p_z + E\beta - p_z)(E - \beta p_z - E\beta + p_z)] \\ &= \gamma^2(1 + \beta)(1 - \beta)(E - p_z)(E + p_z) = E^2 - p_z^2 \quad \square \end{aligned}$$

(b)

Problem 2.6

For the decay $a \rightarrow 1 + 2$, show that the mass of the particle a can be expressed as

$$m_a^2 = m_1^2 + m_2^2 + 2E_1E_2(1 - \beta_1\beta_2\cos\theta)$$

where β_1 and β_2 are the velocities of the daughter particles and θ is the angle between them.

Solution: Let the four-momenta of the daughters as $p_i = (E_i, \mathbf{p}_i)$ for $i = 1, 2$. Momentum conservation states that $p_a = p_1 + p_2$ where p_a is the four-momentum of the mother particle. Squaring both sides, one obtains

$$\begin{aligned} p_a \cdot p_a &= m_a^2 = (p_1 + p_2)^2 \\ &= p_1 \cdot p_1 + p_2 \cdot p_2 + 2p_1 \cdot p_2 \\ &= m_1^2 + m_2^2 + 2(E_1E_2 - \mathbf{p}_1 \cdot \mathbf{p}_2) \\ &= m_1^2 + m_2^2 + 2(E_1E_2 - |\mathbf{p}_1||\mathbf{p}_2|\cos\theta) \\ &= m_1^2 + m_2^2 + 2(E_1E_2 - E_1\beta_1E_2\beta_2\cos\theta) \\ &= m_1^2 + m_2^2 + 2E_1E_2(1 - \beta_1\beta_2\cos\theta) \quad \square \end{aligned}$$

Problem 2.7

In a collider experiment, Λ baryons can be identified from the decay $\Lambda \rightarrow \pi^- p$, which gives rise to a displaced vertex in a tracking detector. In a particular decay, the momenta of the π^+ and p are measured to be 0.75 GeV and 4.25 GeV respectively, and the opening angle between the tracks is 9° . The masses of the pion and proton are 189.6 MeV and 938.3 MeV.

- Calculate the mass of the Λ baryon.
- On average, Λ baryons of this energy are observed to decay at a distance of 0.35 m from the point of production. Calculate the lifetime of the Λ .

Solution:

- (a) Let the four-momenta of π^- , p as $p_\pi = (0.75 \text{ GeV}, \mathbf{p}_\pi)$, $p_p = (4.25 \text{ GeV}, \mathbf{p}_p)$ respectively, which gives $p_\Lambda = p_\pi + p_p = (5 \text{ GeV}, \mathbf{p}_\pi + \mathbf{p}_p)$ as the four-momenta of Λ . Using the mass of the pion and proton, one could obtain

$$\begin{aligned} |\mathbf{p}_\pi|^2 &= (0.75 \text{ GeV})^2 - m_\pi^2 \simeq 0.5265 \text{ GeV}^2 \implies |\mathbf{p}_\pi| \simeq 0.725 \text{ GeV} \\ |\mathbf{p}_p|^2 &= (4.25 \text{ GeV})^2 - m_p^2 \simeq 17.18 \text{ GeV}^2 \implies |\mathbf{p}_p| \simeq 4.144 \text{ GeV} \end{aligned}$$

The mass of the Λ baryon m_Λ can be acquired as :

$$\begin{aligned} m_\Lambda^2 &= p_\Lambda \cdot p_\Lambda = (5 \text{ GeV})^2 - |\mathbf{p}_\pi + \mathbf{p}_p|^2 \\ &= (5 \text{ GeV})^2 - [|\mathbf{p}_\pi|^2 + |\mathbf{p}_p|^2 + 2|\mathbf{p}_\pi||\mathbf{p}_p|\cos 9^\circ] \\ &= (5 \text{ GeV})^2 - [0.5265 + 17.18 + 2 \cdot 0.725 \cdot 4.144 \cdot 0.98] \text{ GeV}^2 \\ &\simeq 1.35 \text{ GeV}^2 \implies \boxed{m_\Lambda \simeq 1.16 \text{ GeV}} \end{aligned}$$

which agrees well with experimental values.

- (b) Let the lifetime and β of Λ as τ_Λ and β_Λ then one could realize that $c\beta_\Lambda\tau_\Lambda \sim 0.35\text{m}$. β_Λ can be simply derived using $\beta_\Lambda = |\mathbf{p}_\Lambda|/E_\Lambda \simeq 0.97$. Thus the lifetime of Λ becomes $\boxed{\tau_\Lambda \simeq 0.12 \times 10^{-8}\text{s}}$

Problem 2.8

In the laboratory frame, a proton with total energy E collides with proton at rest. Find the minimum proton energy such that process

$$p + p \rightarrow p + p + \bar{p} + \bar{p}$$

is kinematically allowed.

Solution:

Problem 2.9

Find the maximum opening angle between the photons produced in the decay $\pi^0 \rightarrow \gamma\gamma$ if the energy of the neutral pion is 10 GeV, given that $m_{\pi^0} = 135 \text{ MeV}$.

Solution: Using the results derived in Problem 2 and taking account on the fact that photons are massless, one could write down

$$m_{\pi^0}^2 = 2E_1E_2(1 - \beta_1\beta_2\cos\theta) = 2E_1E_2(1 - \cos\theta) \implies \cos\theta = \frac{m_{\pi^0}^2}{2E_1E_2} - 1$$

Taking account that $E_1 + E_2 = 10 \text{ GeV}$, let $E_1 = E$ and express θ in terms of E as,

$$\cos\theta = \frac{m_{\pi^0}^2}{2E(10 - E)} - 1$$

In the range of $E \in [0, 10] \text{ GeV}$ the RHS of the above identity will take a local minimum when $E = 5 \text{ GeV}$ which will give the maximum value of θ , which will be denoted as θ^* . One could get θ^* as,

$$\cos\theta^* = \frac{(1.35 \times 10^{-1} \text{ GeV})^2}{100 \text{ GeV}^2} - 1 = -0.99981775 \implies \boxed{\theta^* \simeq 178.906099^\circ}$$

which is nearly back-to-back.

Problem 2.10

The maximum of the $\pi^- p$ cross section, which occurs at $p_\pi = 300$ MeV, corresponds to the resonant production of the Δ^0 baryon (i.e. $\sqrt{s} = m_\Delta$). What is the mass of the Δ ?

Solution:

Problem 2.11

Tau-leptons are produced in the process $e^+e^- \rightarrow \tau^+\tau^-$ at a centre-of-mass energy of 91.2 GeV. The angular distribution of the π^- from the decay $\tau^- \rightarrow \pi^- \nu_\tau$ is

$$\frac{dN}{d(\cos\theta^*)} \propto 1 + \cos\theta^*$$

where θ^* is the polar angle of the π^- in the tau-lepton rest frame, relative to the direction defined by the τ spin. Determine the laboratory frame energy distribution of the π^- for the cases where the tau lepton spin is (i) *aligned with* or (ii) *opposite* to its direction of flight.

Solution:

Problem 2.12

For the process $1 + 2 \rightarrow 3 + 4$, the Mandelstam variables s, t and u are defined as $s = (p_1 + p_2)^2, t = (p_1 - p_3)^2$ and $u = (p_1 - p_4)^2$. Show that

$$s + t + u = m_1^2 + m_2^2 + m_3^2 + m_4^2.$$

Solution: By definition of the Mandelstam variables, one could express $(s + t + u)$ as

$$\begin{aligned} s + t + u &= (p_1 + p_2)^2 + (p_1 - p_3)^2 + (p_1 - p_4)^2 \\ &= \sum_i p_i \cdot p_i + 2p_1 \cdot p_1 + 2p_1 \cdot p_2 - 2p_1 \cdot p_3 - 2p_1 \cdot p_4 \\ &= \sum_i m_i^2 + 2p_1 \cdot (p_1 + p_2 - p_3 - p_4) \\ &= \sum_i m_i^2 \quad \square \end{aligned}$$

The fact that in any frame $p^\mu p_\mu = m^2$ for a particle with mass m is used in the third identity, and in the last step the conservation of momentum $p_1 + p_2 = p_3 + p_4$ is used.

Problem 2.13

At the HERA collider, 27.5 GeV electrons were collided head-on with 820 GeV protons. Calculate the centre-of-mass energy.

Solution: Let the four-momentum of the electron and proton as $p_e = (E_e, \mathbf{p}_e)$, $p_p = (E_p, \mathbf{p}_p)$ respectively. The centre-of-mass energy \sqrt{s} can be expressed as,

$$\begin{aligned} s &= (p_e + p_p)^2 = p_e \cdot p_e + p_p \cdot p_p + 2p_e \cdot p_p \\ &= m_e^2 + m_p^2 + 2(E_e E_p - \mathbf{p}_e \cdot \mathbf{p}_p) \\ &= m_e^2 + m_p^2 + 2(E_e E_p + |\mathbf{p}_e| |\mathbf{p}_p|) \simeq 4E_e E_p \quad (|\mathbf{p}_i|^2 = E_i^2 - m_i^2 \sim E_i^2) \end{aligned}$$

As the collision is occurring head-on, one could say that $\mathbf{p}_e \cdot \mathbf{p}_p = -|\mathbf{p}_e| |\mathbf{p}_p|$ which was used in the last identity. Looking upon the order of the variables, $m_e \simeq 0.5$ MeV, $m_p \simeq 93.8$ MeV and $E_e = 27.5$ GeV, $E_p = 820$ GeV for an approximation it is okay to consider $m_e, m_p \sim 0$. Thus the centre-of-mass energy $\boxed{\sqrt{s} \simeq 300 \text{ GeV}}$ when all the needed values are plugged in.

Problem 2.14

Consider the Compton scattering of a photon of momentum \mathbf{k} and energy $E = |\mathbf{k}| = \hbar\omega$ from an electron at rest. Writing the four-momenta of the scattered photon and electron respectively as k' and p' , conservation of four-momentum is expressed as $k + p = k' + p'$. Use the relation $p^2 = m_e^2$ to show that the energy of the scattered photon is given by

$$E' = \frac{E}{1 + (E/m_e)(1 - \cos \theta)}$$

Solution:

Problem 2.15

Using the commutation relations for position and momentum, prove that

$$[\hat{L}_x, \hat{L}_y] = i\hat{L}_z$$

Using the commutation relations for the components of angular momenta prove

$$[\hat{L}^2, \hat{L}_x] = 0$$

and

$$\hat{L}^2 = \hat{L}_- \hat{L}_+ + \hat{L}_z + \hat{L}_z^2$$

Solution:

Problem 2.16

Show that the operators $\hat{S}_i = \frac{1}{2}\sigma_i$, where σ_i are the three Pauli spin-matrices,

$$\hat{S}_x = \frac{1}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \hat{S}_y = \frac{1}{2} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \text{and} \quad \hat{S}_z = \frac{1}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

satisfy the same algebra as the angular momentum operators, namely

$$[\hat{S}_x, \hat{S}_y] = i\hat{S}_z \quad [\hat{S}_y, \hat{S}_z] = i\hat{S}_x \quad \text{and} \quad [\hat{S}_z, \hat{S}_x] = i\hat{S}_y$$

Find the eigenvalue(s) of the operator $\hat{\mathbf{S}}^2 = \frac{1}{4}(\hat{S}_x^2 + \hat{S}_y^2 + \hat{S}_z^2)$ and deduce that the eigenstates of \hat{S}_z are a suitable representation of a spin-half particle.

Solution:

Problem 2.17

Find the third-order term in the transition matrix element of Fermi's golden rule.

Solution:

3 Decay Rates and Cross Sections

Problem 3.1

Calculate the energy of the μ^- produced in the decay at rest $\pi^- \rightarrow \mu^- \bar{\nu}_\mu$. Assume $m_\pi = 140$ MeV, $m_\mu = 106$ MeV and take $m_\nu \sim 0$.

Solution: Let the four-momenta of the muon and the neutrino to be $p_1 = (E_1, 0, 0, E_2)$ and $p_2 = (E_2, 0, 0, -E_2)$. In the pion rest frame, $E_1 + E_2 = m_\pi$ and from the muon mass constraint $m_\mu^2 = E_1^2 - E_2^2$. Solving these equation gives

$$E_1 = \frac{m_\pi^2 + m_\mu^2}{2m_\pi} = 110.13 \text{ GeV}$$

Problem 3.2

For the decay $a \rightarrow 1 + 2$, show that the momenta of both daughter particles in the centre-of mass frame p^* are

$$p^* = \frac{1}{2m_a} \sqrt{[m_a^2 - (m_1^2 + m_2^2)][m_a^2 - (m_1^2 - m_2^2)]}$$

Solution: Let the four-momenta of the mother particle and the daughter particles to be $p_a = (m_a, 0, 0, 0)$, $p_1 = (E_1, 0, 0, p^*)$, $p_2 = (E_2, 0, 0, -p^*)$. From the mass constraints, we get $E_1 + E_2 = m_a$, $E_1^2 - p^{*2} = m_1^2$, and $E_2^2 - p^{*2} = m_2^2$.

Since we have three unknown variables E_1, E_2, p^* and three equations, it is possible to get p^* in terms of m_a, m_1 and m_2 , which gives the desired solution. In detail,

$$\begin{aligned} p^{*2} = E_1^2 - m_1^2 = E_2^2 - m_2^2 &\implies E_1^2 = E_2^2 + m_1^2 - m_2^2 \\ &= (m_a - E_1)^2 + m_1^2 - m_2^2 \implies E_1 = \frac{1}{2m_a} (m_a^2 + m_1^2 - m_2^2) \end{aligned}$$

which leads to a similar expression of E_2 using $E_1 + E_2 = m_a$

$$E_2 = m_a - E_1 = \frac{1}{2m_a} (m_a^2 - m_1^2 + m_2^2)$$

Then one could finally write down p^* in terms of m_a, m_1, m_2 as, using the fact that $p^{*2} = E_1^2 - m_1^2 = E_2^2 - m_2^2$

$$\begin{aligned} p^{*2} &= \frac{1}{2} [E_1^2 + E_2^2 - (m_1^2 + m_2^2)] \\ &= \frac{1}{2} [(E_1 + E_2)^2 - 2E_1E_2 - (m_1^2 + m_2^2)] \\ &= \frac{1}{2} \left[m_a^2 - \frac{1}{2m_a^2} [m_a^2 - (m_1^2 + m_2^2)][m_a^2 - (m_1^2 - m_2^2)] - (m_1^2 + m_2^2) \right] \\ &= \frac{1}{2} [m_a^2 - (m_1^2 + m_2^2)] \left[1 - \frac{1}{2m_a^2} [m_a^2 - (m_1^2 - m_2^2)] \right] \\ &= \frac{1}{2} [m_a^2 - (m_1^2 + m_2^2)] \left[1 - \frac{1}{2m_a^2} [m_a^2 - (m_1^2 - m_2^2)] \right] \\ &= \frac{1}{4m_a^2} [m_a^2 - (m_1^2 + m_2^2)] [m_a^2 - (m_1^2 - m_2^2)] \quad \square \end{aligned}$$

Problem 3.3

Calculate the branching ratio for the decay $K^+ \rightarrow \pi^+\pi^0$, given the partial decay width $\Gamma(K^+ \rightarrow \pi^+\pi^0) = 1.2 \times 10^{-8} \text{ eV}$ and the mean kaon lifetime $\tau(K^+) = 1.2 \times 10^{-8} \text{ s}$.

Solution: Using the given information,

$$\begin{aligned}
 \text{BR}(K^+ \rightarrow \pi^+\pi^0) &= \frac{1}{\Gamma_{K^+}} \times \Gamma(K^+ \rightarrow \pi^+\pi^0) \\
 &= \tau(K^+) \times \Gamma(K^+ \rightarrow \pi^+\pi^0) \\
 &= (1.2 \times 10^{-8} \text{s}) \times (1.2 \times 10^{-8} \text{eV}) \\
 &= (1.2 \times 10^{-8}) \times \left(\frac{1}{6.58} \times 10^{16} \text{ eV}^{-1} \right) \times (1.2 \times 10^{-8} \text{eV}) \\
 &= \frac{1.2^2}{6.58} \simeq \boxed{21\%}
 \end{aligned}$$

which is as much as expected from the known branching rate.

Problem 3.4

At a future e^+e^- linear collider operating as a Higgs factory at a centre-of-mass energy of $\sqrt{s} = 250$ GeV, the cross section for the process $e^+e^- \rightarrow HZ$ is 250 fb. If the collider has an instantaneous luminosity of $2 \times 10^{34} \text{cm}^{-2} \text{s}^{-1}$ and is operational for 50% of the time, how many Higgs bosons will be produced in five years of running?

Solution: Let the total number of Higgs bosons that will be produced in 5 years of running in such condition as N , then one could calculate N as,

$$\begin{aligned}
 N &= (2 \times 10^{34} \text{cm}^{-2} \text{s}^{-1}) \times (5 \text{ yrs}) \times (250 \text{ fb}) \times 0.5 \\
 &= (2 \times 10^{34} \text{cm}^{-2} \text{s}^{-1}) (1.5768 \times 10^8 \text{s}) \times (2.5 \times 10^{-41} \text{cm}^2) \times 0.5 \\
 &= 39.42
 \end{aligned}$$

Problem 3.5

The total $e^+e^- \rightarrow \gamma \rightarrow \mu^+\mu^-$ annihilation cross section is $\sigma = 4\pi\alpha^2/3s$, where $\alpha \simeq 1/137$. Calculate the cross section at $\sqrt{s} = 50$ GeV, expressing your answer in both natural units and in barns. Compare this to the total pp cross section at $\sqrt{s} = 50$ GeV which is approximately 40 mb and comment on the result.

Solution: Plugging in all the values we know in natural units,

$$\sigma = \frac{4\pi}{3 \cdot (2.5 \times 10^3 \text{ GeV}^2) \cdot 137^2} = 8.9 \times 10^{-8} \text{ GeV}^{-2}$$

which could be converted into barns using $1 \text{ GeV}^{-2} = 0.3894 \text{ mb}$, gives $\sigma = 346.5 \text{ pb}$.

Problem 3.6

A 1 GeV muon neutrino is fired at a 1m thick block of iron with density $\rho = 7.874 \times 10^3 \text{kg} \cdot \text{m}^{-3}$. If the average neutrino-nucleon interaction cross section is $\sigma = 8 \times 10^{-39} \text{m}^2$, calculate the (small) probability that the neutrino interacts in the block.

Solution: The muon neutrino will pass through $\sim 7.874 \times 10^3 \text{kg} \cdot \text{m}^{-2}$ of iron. As iron has atomic mass of 56, around 56 g of iron will contain 6.022×10^{23} number of nucleons, which is nearly 8.43×10^{28} nucleons for $7.874 \times 10^3 \text{ kg}$ of iron. This could be considered as a flux of nucleons per area $\sim 8.43 \times 10^{28} \text{m}^{-2}$. Thus the probability could be derived as the neutrino-nucleon interaction cross section multiplied with such flux of nuclei, which gives $\sim 6.74 \times 10^{-14}$.

Problem 3.7

For the process $a + b \rightarrow 1 + 2$ the Lorents-invariant flux term installed

$$F = 4 \left[(p_a \cdot p_b)^2 - m_a^2 m_b^2 \right]^{\frac{1}{2}}$$

In the non-relativistic limit, $\beta_a, \beta_b \ll 1$, show that

$$F \approx 4m_a m_b |\mathbf{v}_a - \mathbf{v}_b|$$

where $\mathbf{v}_a, \mathbf{v}_b$ are the (non-relativistic) velocities of the two particles.

Solution: Let the four-momenta of a,b as $p_a = (E_a, \mathbf{p}_a)$ and $p_b = (E_b, \mathbf{p}_b)$. Under the non-relativistic limit which implies that $\gamma_a, \gamma_b \simeq 1$, one could write down F as

$$\begin{aligned} F &= 4 \left[(p_a \cdot p_b)^2 - m_a^2 m_b^2 \right]^{\frac{1}{2}} \\ &= 4 \left[(E_a E_b - m_a m_b \beta_a \cdot \beta_b)^2 - m_a^2 m_b^2 \right]^{\frac{1}{2}} \\ &= 4 \left[(E_a E_b - m_a m_b \beta_a \cdot \beta_b)^2 - m_a^2 m_b^2 \right]^{\frac{1}{2}} \\ &= 4 \left[(E_a E_b - m_a m_b \beta_a \cdot \beta_b)^2 - m_a^2 m_b^2 \right]^{\frac{1}{2}} \end{aligned}$$

Problem 3.8

The Lorentz-invariant flux term for the process $a + b \rightarrow 1 + 2$ in the centre-of-mass frame was shown to be $F = 4p_i^* \sqrt{s}$, where p_i^* is the momentum of the initial-state particles. Show that the corresponding expression in the frame where b is at rest is

$$F = 4m_b p_a.$$

Solution:

Problem 3.9

Show that the momentum in the centre-of-mass frame of the initial-state particles in a two-body scattering process can be expressed as

$$p_i^{*2} = \frac{1}{4s} \left[s - (m_1 + m_2)^2 \right] \left[s - (m_1 - m_2)^2 \right]$$

Solution:

Problem 3.10

Repeat the calculation of Section 3.5.2 for the process $e^- p \rightarrow e^- p$ where the mass of the electron is no longer neglected.

(a) First show that

$$\frac{dE}{d(E \cos \theta)} = \frac{p_1 p_3^2}{p_3 (E_1 + m_p) - E_3 p_1 \cos \theta}$$

(b) Then show that

$$\frac{d\sigma}{d\Omega} = \frac{1}{64\pi^2} \cdot \frac{p_3^2}{p_1 m_p} \cdot \frac{1}{p_3 (E_1 + m_p) - E_3 p_1 \cos \theta} \cdot |\mathcal{M}_{fi}|^2$$

Solution:

4 The Dirac Equation

Problem 4.1

Show that

$$[\hat{\mathbf{p}}^2, \hat{\mathbf{r}} \times \hat{\mathbf{p}}] = 0,$$

and hence the Hamiltonian of the free-particle Schrödinger equation commutes with the angular momentum operator.

Solution: One could expand the given commutator as,

$$\begin{aligned} [\hat{\mathbf{p}}^2, \hat{\mathbf{r}} \times \hat{\mathbf{p}}] &= [\hat{\mathbf{p}}_a \hat{\mathbf{p}}_a, \epsilon_{abc} r_c \hat{\mathbf{p}}_b \hat{\mathbf{c}}] \\ &= \epsilon_{abc} r_c [\hat{\mathbf{p}}_a \hat{\mathbf{p}}_a, \hat{\mathbf{p}}_b \hat{\mathbf{c}}] \\ &= \epsilon_{abc} r_c \{ \hat{\mathbf{p}}_a [\hat{\mathbf{p}}_a, \hat{\mathbf{p}}_b \hat{\mathbf{c}}] + [\hat{\mathbf{p}}_a, \hat{\mathbf{p}}_b \hat{\mathbf{c}}] \hat{\mathbf{p}}_a \} \\ &= \epsilon_{abc} r_c \{ \hat{\mathbf{p}}_a [\hat{\mathbf{p}}_a, \hat{\mathbf{p}}_b \hat{\mathbf{c}}] + [\hat{\mathbf{p}}_a, \hat{\mathbf{p}}_b \hat{\mathbf{c}}] \hat{\mathbf{p}}_a \} \quad \Longleftarrow [\hat{\mathbf{p}}_a, \hat{\mathbf{p}}_b \hat{\mathbf{c}}] = \delta_{ab} \hat{\mathbf{c}} - i \delta_{ac} \hat{\mathbf{p}}_b \end{aligned}$$

Problem 4.2

Show that u_1 and u_2 are orthogonal, i.e. $u_1^\dagger u_2 = 0$.

Solution: Let us first denote

$$u_\uparrow = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{and} \quad u_\downarrow = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

One should note that $u_\uparrow^\dagger u_\downarrow = 0$. u_1, u_2 could also be expressed in terms of

$$u_1 = \begin{pmatrix} u_\uparrow \\ \frac{\boldsymbol{\sigma} \cdot \mathbf{p}}{E+m} u_\uparrow \end{pmatrix} \quad \text{and} \quad u_2 = \begin{pmatrix} u_\downarrow \\ \frac{\boldsymbol{\sigma} \cdot \mathbf{p}}{E+m} u_\downarrow \end{pmatrix}.$$

Now $u_1^\dagger u_2$ could be written as,

$$\begin{aligned} u_1^\dagger u_2 &= \begin{pmatrix} u_\uparrow \\ \frac{\boldsymbol{\sigma} \cdot \mathbf{p}}{E+m} u_\uparrow \end{pmatrix}^\dagger \begin{pmatrix} u_\downarrow \\ \frac{\boldsymbol{\sigma} \cdot \mathbf{p}}{E+m} u_\downarrow \end{pmatrix} \\ &= \left(u_\uparrow^\dagger \quad \frac{1}{E+m} ((\boldsymbol{\sigma} \cdot \mathbf{p}) u_\uparrow)^\dagger \right) \begin{pmatrix} u_\downarrow \\ \frac{\boldsymbol{\sigma} \cdot \mathbf{p}}{E+m} u_\downarrow \end{pmatrix} \\ &= u_\uparrow^\dagger u_\downarrow + \frac{1}{(E+m)^2} ((\boldsymbol{\sigma} \cdot \mathbf{p}) u_\uparrow)^\dagger ((\boldsymbol{\sigma} \cdot \mathbf{p}) u_\downarrow) \end{aligned}$$

$$= u_{\uparrow}^{\dagger} u_{\downarrow} + \frac{1}{(E+m)^2} u_{\uparrow}^{\dagger} (\boldsymbol{\sigma} \cdot \mathbf{p})^{\dagger} (\boldsymbol{\sigma} \cdot \mathbf{p}) u_{\downarrow}$$

One could use the fact that,

$$(\boldsymbol{\sigma} \cdot \mathbf{p})^{\dagger} (\boldsymbol{\sigma} \cdot \mathbf{p}) = (\boldsymbol{\sigma} \cdot \mathbf{p})^2 = \begin{pmatrix} p_x^2 + p_y^2 + p_z^2 & 0 \\ 0 & p_x^2 + p_y^2 + p_z^2 \end{pmatrix} = (E^2 - m^2) I_2$$

Thus it could be tidied up as,

$$\begin{aligned} u_1^{\dagger} u_2 &= u_{\uparrow}^{\dagger} u_{\downarrow} + \frac{1}{(E+m)^2} u_{\uparrow}^{\dagger} (\boldsymbol{\sigma} \cdot \mathbf{p})^{\dagger} (\boldsymbol{\sigma} \cdot \mathbf{p}) u_{\downarrow} \\ &= \left[1 + \frac{E^2 - m^2}{(E+m)^2} \right] u_{\uparrow}^{\dagger} u_{\downarrow} = 0 \quad \square \end{aligned}$$

Problem 4.3

Verify the statement that the Einstein energy-momentum relationship is recovered if any of the four Dirac spinors of (4.48) are substitutes into the Dirac equation written in terms of momentum, $(\gamma^{\mu} p_{\mu} - m) u = 0$.

Solution: Let us choose u_1 to plug in the Dirac equation. Then it could be expressed as,

$$\begin{aligned} (\not{p} - m) u_1 &= 0 \implies \begin{pmatrix} (E - m) I_2 & -\boldsymbol{\sigma} \cdot \mathbf{p} \\ \boldsymbol{\sigma} \cdot \mathbf{p} & -(E + m) I_2 \end{pmatrix} \begin{pmatrix} E + m \\ 0 \\ p_z \\ p_x + ip_y \end{pmatrix} = 0 \\ &\implies \begin{pmatrix} E^2 - m^2 \\ 0 \end{pmatrix} + \boldsymbol{\sigma} \cdot \mathbf{p} \begin{pmatrix} E + m - p_z \\ -p_x - ip_y \end{pmatrix} - (E + m) \begin{pmatrix} p_z \\ p_x + ip_y \end{pmatrix} = 0 \\ \text{[first row]} &\implies (E^2 - m^2) + (E + m) p_z - (p_x^2 + p_y^2 + p_z^2) - (E + m) p_z = 0 \\ &\implies E^2 = p_x^2 + p_y^2 + p_z^2 + m^2 \quad \square \end{aligned}$$

Problem 4.4

For a particle with four-momentum $p^{\mu} = (E, \mathbf{p})$, the general solution to the free-particle Dirac equation can be written

$$\psi(p) = [a u_1(p) + b u_2(p)] e^{i(\mathbf{p} \cdot \mathbf{x} - Et)}$$

Using the explicit forms for u_1 and u_2 , show that the four-vector current $j^{\mu} = (\rho, \mathbf{j})$ is given by

$$j^{\mu} = 2p^{\mu}$$

Furthermore, show that the resulting probability density and probability current are consistent with a particle moving with velocity $\beta = p/E$.

Solution:

Problem 4.5

Writing the four-component spinor u in terms of two two-component vectors

$$u = \begin{pmatrix} u_A \\ u_B \end{pmatrix},$$

show that in the non-relativistic limit, where $\beta \cong v/c \ll 1$, the components of u_B are smaller than those of u_A by a factor v/c .

Solution:

Problem 4.6

By considering the three cases $\mu = \nu = 0$, $\mu = \nu \neq 0$ and $\mu \neq \nu$ show that

$$\gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu = 2g^{\mu\nu}.$$

Solution:

Problem 4.7

By operating on the Dirac equation,

$$(i\gamma^\mu \partial_\mu - m) \psi = 0$$

with $\gamma^\nu \partial_\nu$ prove that the components of ψ satisfy the Klein-Gordon equation,

$$(\partial^\mu \partial_\mu + m^2) \psi = 0.$$

Solution:

Problem 4.8

Show that

$$(\gamma^\mu)^\dagger = \gamma^0 \gamma^\mu \gamma^0.$$

Solution:

Problem 4.9

Starting from

$$(\gamma^\mu p_\mu - m) u = 0,$$

show that the corresponding equation for the adjoint spinor is

$$\bar{u} (\gamma^\mu p_\mu - m) = 0.$$

Hence, without using the explicit form for the u spinors, show that the normalisation condition $u^\dagger u = 2E$ leads to

$$u \bar{u} = 2m,$$

and that

$$\bar{u} \gamma^\mu u = 2p^\mu.$$

Solution:

Problem 4.10

Demonstrate that the two relations of Equation (4.45) are consistent by showing that

$$(\boldsymbol{\sigma} \cdot \mathbf{p})^2 = \mathbf{p}^2.$$

Solution:

Problem 4.11

Consider the $e^+e^- \rightarrow \gamma \rightarrow e^+e^-$ annihilation process in the centre-of-mass frame where the energy of the photon is $2E$. Discuss energy and charge conservation for the two cases where:

- (a) the negative energy solutions of the Dirac equation are interpreted as negative energy particles propagating backwards in time
- (b) the negative energy solutions of the Dirac equation are interpreted as positive energy antiparticles propagating forwards in time

Solution:

Problem 4.12

Verify that the helicity operator

$$\hat{h} = \frac{\hat{\Sigma} \cdot \hat{\mathbf{p}}}{2p} = \frac{1}{2p} \begin{pmatrix} \sigma \cdot \hat{\mathbf{p}} & 0 \\ 0 & \sigma \cdot \hat{\mathbf{p}} \end{pmatrix}$$

commutes with the Dirac Hamiltonian,

$$\hat{H}_D = \boldsymbol{\alpha} \cdot \hat{\mathbf{p}} + \beta m$$

Solution:

Problem 4.13

Show that

$$\hat{p}_{u\uparrow}(\theta, \phi) = u_{\downarrow}(\pi - \theta, \pi + \phi)$$

and comment on the result.

Solution:

Problem 4.14

Under the combined operation of parity and charge conjugation ($\hat{C}\hat{P}$) spinors transform as

$$\psi \rightarrow \psi^C = \hat{C}\hat{P}\psi = i\gamma^2\gamma^0\psi^*$$

Show that up to an overall complex phase factor

$$\hat{C}\hat{p}_{u\uparrow}(\theta, \phi) = v_{\downarrow}(\pi - \theta, \pi + \phi)$$

Solution:

Problem 4.15

Starting from the Dirac equation, derive the identity

$$\bar{u}(p')\gamma^\mu u(p) = \frac{1}{2m}\bar{u}(p')(p+p')u(p) + \frac{i}{m}\bar{u}(p')\Sigma^{\mu\nu}q_\nu u(p)$$

where $q = p' - p$ and $\Sigma^{\mu\nu} = \frac{i}{4}[\gamma^\mu, \gamma^\nu]$

Solution:
