Modern Particle Physics
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1 Introduction

Problem 1.1
Feynman diagrams are constructed out of the Standard Model vertices shown in Figure 1.4. Only the weak charged-current interaction can change the flavour of the particle at the interaction vertex. Explaining your reasoning, state whether each of the sixteen diagrams below represents a valid Standard Model vertex.

Solution:

(a) Valid.

(b) Invalid, due to the fact that ν_e has no electric charge.

(c) Valid.

(d) Valid.

(e) Invalid.

(f) Valid.

(g) Invalid.

(h) Invalid.

(i) Invalid, leptons do not carry color charge.

(j) Valid.

(k) Valid.

(l) Invalid.

(m) Invalid.

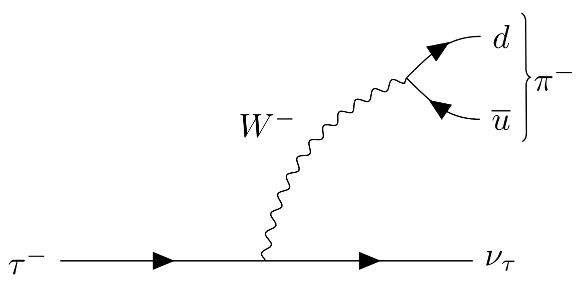
(n) Valid.

(o) Valid.

(p) Invalid.

Problem 1.2
Draw the Feynman diagram for $\tau^- \to \pi^- \nu_\tau$. (The π^- is the lightest $d\bar{u}$ meson)

Solution:



Problem 1.3
Explain why it is not possible to construct a valid Feynman diagram using the Standard Model vertices for the following processes :

(a) $\mu^- \to e^+ e^- e^+$

(b) $\nu_\tau + p \to \mu^- + n$

(c) $\nu_\tau + p \to \tau^+ + n$

(d) $\pi^+(u\bar{d}) + \pi^-(d\bar{u}) \to n(udd) + \pi^0(u\bar{u})$

Solution:

(a) $\mu^- \to e^+ e^- e^+$: Charge is not conserved, as well as lepton numbers.

(b) $\nu_\tau + p \to \mu^- + n$: Charge is not conserved, as well as baryon numbers.

(c) $\nu_\tau + p \to \tau^+ + n$: Both baryon and lepton number is not conserved.

(d) $\pi^+(u\bar{d}) + \pi^-(d\bar{u}) \to n(udd) + \pi^0(u\bar{u})$: Baryon number is not conserved.

Problem 1.4
Draw the Feynman diagram for the decays:

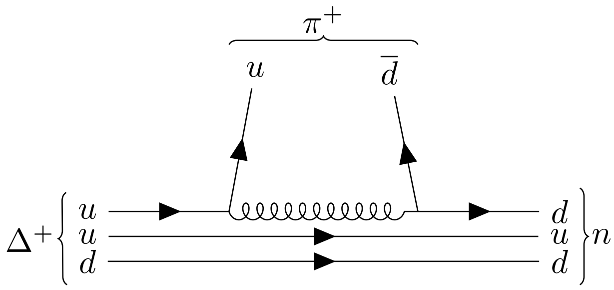
(a) $\Delta^+(uud) \to n(udd)\pi^+(u\bar{d})$

(b) $\Sigma^0(uds) \to \Lambda(uds)\gamma$

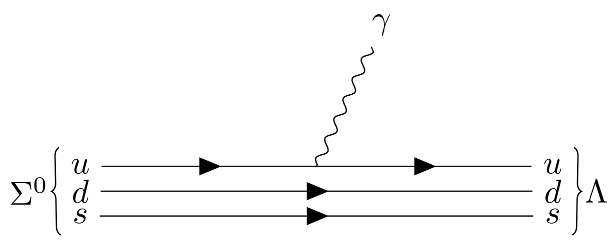
(c) $\pi^+(u\bar{d}) \to \mu^+\nu_\mu$

Solution:

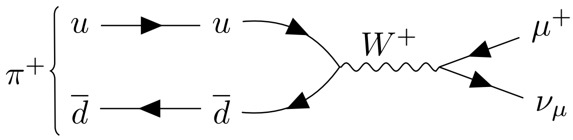
(a) $\Delta^+(uud) \to n(udd)\pi^+(u\bar{d})$



(b) $\Sigma^0(uds) \to \Lambda(uds)\gamma$



(c) $\pi^+(u\bar{d}) \to \mu^+\nu_\mu$



Problem 1.5
Treating the π^0 as a uu bound state, draw the Feynman diagrams for:

(a) $\pi^0 \to \gamma\gamma$

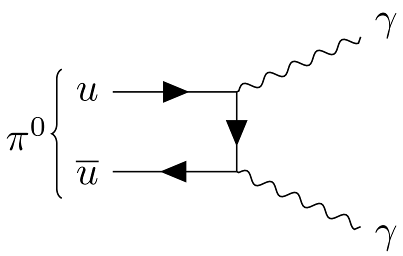
(b) $\pi^0 \to \gamma e^+ e^-$

(c) $\pi^0 \to e^+ e^- e^+ e^-$

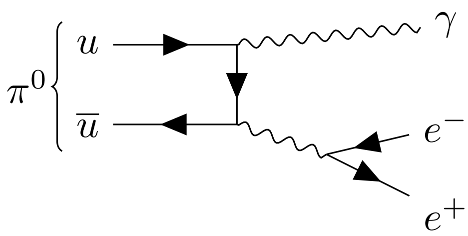
(d) $\pi^0 \to e^+ e^-$

Solution:

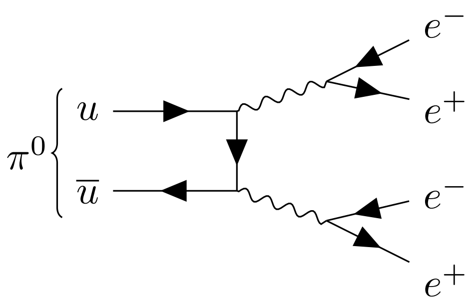
(a) $\pi^0 \to \gamma\gamma$



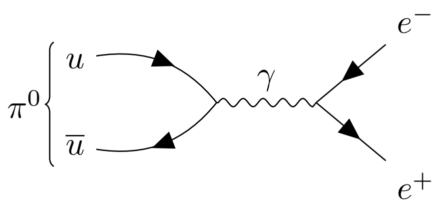
(b) $\pi^0 \to \gamma e^+ e^-$



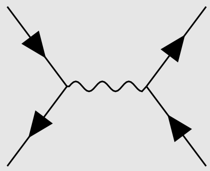
(c) $\pi^0 \to e^+ e^- e^+ e^-$

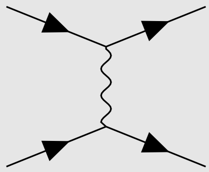


(d) $\pi^0 \to e^+ e^-$



Problem 1.6
Particle interactions fall into two main categories, scattering processes and annihilation processes, as indicated by the Feynman diagrams below.





Draw the lowest-order Feynman diagrams for the scattering and/or annihilation processes:

(a) $e^- e^- \to e^- e^-$

(b) $e^+ e^- \to \mu^+ \mu^-$

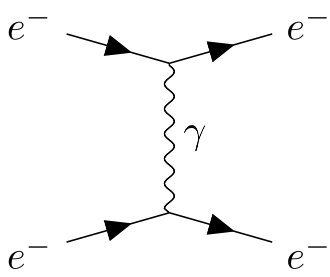
(c) $e^+ e^- \to e^+ e^-$

(d) $e^- \nu_e \to e^- \nu_e$

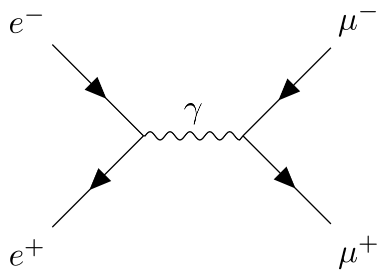
(e) $e^- \overline{\nu_e} \to e^- \overline{\nu_e}$

Solution:

(a) $e^- e^- \to e^- e^-$



(b) $e^+ e^- \to \mu^+ \mu^-$



(c) $e^+ e^- \to e^+ e^-$

(d) $e^- \nu_e \to e^- \nu_e$

(e) $e^- \overline{\nu_e} \to e^- \overline{\nu_e}$

Problem 1.7
High-energy muons traversing matter lose energy according to

$$-\frac{1}{\rho}\frac{\mathrm{d}E}{\mathrm{d}x} \approx a + bE$$

where a is due to ionisation energy loss and b is due to the bremsstrahlung and e^+e^- pair-production processes. For standard rock, taken to have $A = 22, Z = 11$ and $\rho = 2.65 \mathrm{gcm}^{-3}$, the parameters a and b depend only weakly on the muon energy and have values $a \approx 2.5$ MeVg^{-1}cm^2 and $b \approx 3.5 \times 10^{-6}$g^{-1}cm^2.

(a) At what muon energy are the ionisation and bremsstrahlung/pair production processes equally important?

(b) Approximately how far does a 100 GeV cosmic-ray muon propagate in rock?

Solution:

(a) One could assume that ionisation and bremsstrahlung/pair production processes become equally important for a certain energy scale E^* when $a \simeq bE^*$. Such $E^* \simeq a/b$ can be calculated as ~ 700 GeV.

(b) Using the values given,

$$-\frac{\mathrm{d}E}{\mathrm{d}x} \approx a\rho + b\rho E \impliedby \left(a\rho \sim 6.6 \text{ MeV/cm} \text{ , } b\rho \sim 9.275 \times 10^{-6}/\text{cm}\right)$$
$$\simeq 7.52 \text{ MeV/cm}$$

which shows that a 100 GeV muon will go through around 132 metres of rock.

Problem 1.8
Tugsten has a radiation length of $X_0 = 0.35$ cm and a critical energy of $E_c = 7.97$ MeV. Roughly what thickness of tungsten is required to fully contain a 500 GeV electromagnetic shower from an electron?

Solution: Getting x_{max} for the given situation, one obtains :

$$x_{\text{max}} = \frac{1}{\ln 2} \ln\left(\frac{E}{E_c}\right) = \frac{1}{\ln 2} \ln\left(\frac{500 \text{ GeV}}{7.97 \text{ MeV}}\right) \sim 16$$

Thus, roughly around $x_{\text{max}} X_0 \simeq 5.6$cm of tungsten would be able to contain a 500 GeV electromagnetic shower from an electron.

Problem 1.9
The CPLEAR detector consisted of: tracking detectors in a magnetic field of 0.44 T; and electromagnetic calorimeter; and Čerenkov detectors with a radiator of refractive index $n = 1.25$ used to distinguish $\pi^{\pm}$ from $K^{\pm}$.

A charged particle travelling perpendicular to the direction of the magnetic field leaves a track with a measured radius of curvature of $R = 4$m. If it is observed to give a Čerenkov signal, is it possible to distinguish between the particle being a pion or kaon? Take $m_\pi \approx 140$ MeV/c^2 and $m_K \approx 494$ MeV/c^2

Solution: First, the momentum could be extracted from the fact that the charged particles are travelling perpendicular ($\lambda = 0$) to the 0.44 T magnetic field, which eventually gives $p = 0.3BR = 0.528$ GeV. The threshold mass for Čerenkov radiation in this case would be,

$$\sqrt{n^2 - 1}p = 0.75 \times p = 0.396 \text{ GeV}$$

Problem 1.10
In a fixed-target pp experiment, what proton energy would be required to achieve the same centre-of-mass energy as the LHC, which will ultimately operate at 14 TeV.

Solution: Let the four-momentum of the beam proton and the fixed target proton as $p_1 = (E, 0, 0, p)$ and $p_2 = (m_p, 0, 0, 0)$. Using the following expression of the centre-of-mass energy $\sqrt{s}$, the proton energy E to satisfy the required situation would be :

$$\sqrt{s} = (p_1 + p_2)^2 = 2m_p^2 + 2p_1 \cdot p_2$$
$$= 2m_p\left(m_p + E\right) = 14 \text{ TeV} \implies \boxed{E \simeq 7.4 \text{ PeV}}$$

Problem 1.11
At the LEP e^+e^- collider, which had a circumference of 27 km, the electron and positron beam currents were both 1.0 mA. Each beam consisted of four equally spaced bunches of electrons/positrons. The bunches had an effective area of $1.8 \times 10^4 \mu\text{m}^2$. Calculate the instantaneous luminosity on the assumption that the beams collided head-on.

Solution:

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2 Underlying Concepts

Problem 2.1
When expressed in natural units the lifetime of the W boson is approximately $\tau \approx 0.5$ GeV^{-1}. What is the corresponding value in S.I. units?

Solution: In natural units, $\hbar = 1.055 \times 10^{-34}\text{J} \cdot \text{s} = 6.582 \times 10^{-25}\text{GeV} \cdot \text{s}$ which is, $1 \text{ GeV}^{-1} = 6.582 \times 10^{-25}\text{s}$. Thus the lifetime of the W boson in S.I. units can be written as, $\boxed{\tau \simeq 3.291 \times 10^{-25}\text{s}}$.

Problem 2.2
A cross section is measured to be 1 pb; convert this to natural units.

Solution: Taking note that $\hbar c = 0.197$ GeV fm, which is $0.197 \text{ GeV} = 1\text{fm}^{-1}$

$$1 \text{ pb} = 10^{-10} \text{ fm}^2 = 10^{-10} \times \left(\frac{1}{0.197}\right)^2 \text{ GeV}^{-2} = \boxed{2.57 \times 10^{-9} \text{ GeV}^{-2}}$$

Problem 2.3
Show that the process $\gamma \to e^+e^-$ can not occur in vacuum.

Solution: If it were so, such process should occur in any frame. Let such frame as the rest frame of

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Problem 2.4
A particle of mass 3 GeV is travelling in the positive z-direction with momentum 4 GeV. What are its energy and velocity?

Solution: Using the relation of $m^2 = E^2 - |\mathbf{p}|^2$, one gets $E^2 = 25 \text{ GeV}^2$ thus the energy is $\boxed{E = 5 \text{ GeV}}$. Now considering the relation of $|\mathbf{p}| = E\beta$, it is seen that $\beta = |\mathbf{p}|E^{-1} = 0.8$ thus the velocity is $\boxed{0.8c}$.

Problem 2.5
In the laboratory frame, denoted Σ, a particle travelling in the z-direction has momentum $\mathbf{p} = p_z\hat{\mathbf{z}}$ and energy E.

(a) Use the Lorentz transformation to find expressions for the momentum p'_z and energy E' of the particle in a frame Σ' which is moving in a velopcity $\mathbf{v} = +v\hat{\mathbf{z}}$ relative to Σ, and show that $E^2 - p_z^2 = (E')^2 - (p'_z)^2$.

(b) For a system of particles, prove that the total four-momentum squared,

$$p^\mu p_\mu \equiv \left(\sum_i E_i\right)^2 - \left(\sum_i \mathbf{p}_i\right)^2$$

is invariant under Lorentz transformations.

Solution:

(a) Let the four-momentum of the given particle in the frame Σ and Σ' as $p = (E, 0, 0, p_z), p' = (E', \mathbf{p}')$ respectively. Denoting the corresponding matrix representation of the given Lorentz transformation as $\mathbf{\Lambda}$, one could write down the transformation of p as,

$$
\begin{aligned}
p' = \mathbf{\Lambda} p \implies p'^{\mu} &= \Lambda^{\mu}_{\nu} p^{\nu} \\
&= \begin{pmatrix} \gamma & 0 & 0 & -\gamma\beta \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\gamma\beta & 0 & 0 & \gamma \end{pmatrix} \begin{pmatrix} E \\ 0 \\ 0 \\ p_z \end{pmatrix} = \gamma \begin{pmatrix} E - \beta p_z \\ 0 \\ 0 \\ -E\beta + p_z \end{pmatrix}
\end{aligned}
$$

which implies that $E' = \gamma (E - \beta p_z)$ and $p'_z = -\gamma(E\beta - p_z)$. Using such expression of p', one could show that :

$$
\begin{aligned}
(E')^2 - (p'_z)^2 &= \gamma^2 (E - \beta p_z)^2 - \gamma^2 (E\beta - p_z)^2 \\
&= \gamma^2 \left[(E - \beta p_z)^2 - (E\beta - p_z)^2 \right] \\
&= \gamma^2 \left[(E - \beta p_z + E\beta - p_z)(E - \beta p_z - E\beta + p_z) \right] \\
&= \gamma^2 (1 + \beta)(1 - \beta)(E - p_z)(E + p_z) = E^2 - p_z^2 \quad \square
\end{aligned}
$$

(b)

Problem 2.6

For the decay $a \to 1 + 2$, show that the mass of the particle a can be expressed as

$$m_a^2 = m_1^2 + m_2^2 + 2E_1 E_2 (1 - \beta_1 \beta_2 \cos\theta)$$

where β_1 and β_2 are the velocities of the daughter particles and θ is the angle between them.

Solution: Let the four-momenta of the daughters as $p_i = (E_i, \mathbf{p}_i)$ for $i = 1, 2$. Momentum conservation states that $p_a = p_1 + p_2$ where p_a is the four-momentum of the mother particle. Squaring both sides, one obtains

$$
\begin{aligned}
p_a \cdot p_a = m_a^2 &= (p_1 + p_2)^2 \\
&= p_1 \cdot p_1 + p_2 \cdot p_2 + 2p_1 \cdot p_2 \\
&= m_1^2 + m_2^2 + 2(E_1 E_2 - \mathbf{p}_1 \cdot \mathbf{p}_2) \\
&= m_1^2 + m_2^2 + 2(E_1 E_2 - |\mathbf{p}_1||\mathbf{p}_2| \cos\theta) \\
&= m_1^2 + m_2^2 + 2(E_1 E_2 - E_1 \beta_1 E_2 \beta_2 \cos\theta) \\
&= m_1^2 + m_2^2 + 2E_1 E_2 (1 - \beta_1 \beta_2 \cos\theta) \quad \square
\end{aligned}
$$

Problem 2.7

In a collider experiment, Λ baryons can be identified from the decay $\Lambda \to \pi^- p$, which gives rise to a displaced vertex in a tracking detector. In a particular decay, the momenta of the π^+ and p are measured to be 0.75 GeV and 4.25 GeV respectively, and the opening angle between the tracks is 9°. The masses of the pion and proton are 189.6 MeV and 938.3 MeV.

(a) Calculate the mass of the Λ baryon.

(b) On average, Λ baryons of this energy are observed to decay at a distance of 0.35 m from the point of production. Calculate the lifetime of the Λ.

Solution:

(a) Let the four-momenta of π^-, p as $p_\pi = (0.75 \text{ GeV}, \mathbf{p}_\pi), p_p = (4.25 \text{ GeV}, \mathbf{p}_p)$ respectively, which gives $p_\Lambda = p_\pi + p_p = (5 \text{ GeV}, \mathbf{p}_\pi + \mathbf{p}_p)$ as the four-momenta of Λ. Using the mass of the pion and proton, one could obtain

$$|\mathbf{p}_\pi|^2 = (0.75 \text{ GeV})^2 - m_\pi^2 \simeq 0.5265 \text{ GeV}^2 \implies |\mathbf{p}_\pi| \simeq 0.725 \text{ GeV}$$
$$|\mathbf{p}_p|^2 = (4.25 \text{ GeV})^2 - m_p^2 \simeq 17.18 \text{ GeV}^2 \implies |\mathbf{p}_\pi| \simeq 4.144 \text{ GeV}$$

The mass of the Λ baryon m_Λ can be acquired as :

$$\begin{aligned} m_\Lambda^2 = p_\Lambda \cdot p_\Lambda &= (5 \text{ GeV})^2 - |\mathbf{p}_\pi + \mathbf{p}_p|^2 \\ &= (5 \text{ GeV})^2 - \left[|\mathbf{p}_\pi|^2 + |\mathbf{p}_p|^2 + 2\,|\mathbf{p}_\pi|\,|\mathbf{p}_p| \cos 9^\circ\right] \\ &= (5 \text{ GeV})^2 - [0.5265 + 17.18 + 2 \cdot 0.725 \cdot 4.144 \cdot 0.98] \text{ GeV}^2 \\ &\simeq 1.35 \text{ GeV}^2 \implies \boxed{m_\Lambda \simeq 1.16 \text{ GeV}} \end{aligned}$$

which agrees well with experimental values.

(b) Let the lifetime and β of Λ as τ_Λ and β_Λ then one could realize that $c\beta_\Lambda\tau_\Lambda \sim 0.35\text{m}$. β_Λ can be simply derived using $\beta_\Lambda = |\mathbf{p}_\Lambda| / E_\Lambda \simeq 0.97$. Thus the lifetime of Λ becomes $\boxed{\tau_\Lambda \simeq 0.12 \times 10^{-8}\text{s}}$

Problem 2.8
In the laboratory frame, a proton with total energy E collides with proton at rest. Find the minimum proton energy such that process

$$p + p \to p + p + \bar{p} + \bar{p}$$

is kinematically allowed.

Solution:

Problem 2.9
Find the maximum opening angle between the photons produced in the decay $\pi^0 \to \gamma\gamma$ if the energy of the neutral pion is 10 GeV, given that $m_{\pi^0} = 135$ MeV.

Solution: Using the results derived in Problem 2 and taking account on the fact that photons are massless, one could write down

$$m_{\pi_0^2} = 2E_1E_2\left(1 - \beta_1\beta_2\cos\theta\right) = 2E_1E_2\left(1 - \cos\theta\right) \implies \cos\theta = \frac{m_{\pi_0}^2}{2E_1E_2} - 1$$

Taking account that $E_1 + E_2 = 10$ GeV, let $E_1 = E$ and express θ in terms of E as,

$$\cos\theta = \frac{m_{\pi_0}^2}{2E(10 - E)} - 1$$

In the range of $E \in [0, 10]$ GeV the RHS of the above identity will take a local minimum when $E = 5$ GeV which will give the maximum value of θ, which will be denoted as θ^*. One could get θ^* as,

$$\cos\theta^* = \frac{\left(1.35 \times 10^{-1} \text{ GeV}\right)^2}{100 \text{ GeV}^2} - 1 = -0.99981775 \implies \boxed{\theta^* \simeq 178.906099^\circ}$$

which is nearly back-to-back.

Problem 2.10
The maximum of the $\pi^- p$ cross section, which occurs at $p_\pi = 300$ MeV, corresponds to the resonant production of the Δ^0 baryon (i.e. $\sqrt{s} = m_\Delta$). What is the mass of the Δ?

Solution:

Problem 2.11
Tau-leptons are produced in the process $e^+e^- \to \tau^+\tau^-$ at a centre-of-mass energy of 91.2 GeV. The angular distribution of the π^- from the decay $\tau^- \to \pi^-\nu_\tau$ is

$$\frac{\mathrm{d}N}{\mathrm{d}\left(\cos\theta^*\right)} \propto 1 + \cos\theta^*$$

where θ^* is the polar angle of the π^- in the tau-lepton rest frame, relative to the direction defined by the τ spin. Determine the laboratory frame energy distribution of the π^- for the cases where the tau lepton spin is (i) *aligned with* or (ii) *opposite* to its direction of flight.

Solution:

Problem 2.12
For the process $1 + 2 \to 3 + 4$, the Mandelstam variables s, t and u are defined as $s = (p_1 + p_2)^2, t = (p_1 - p_3)^2$ and $u = (p_1 - p_4)^2$. Show that

$$s + t + u = m_1^2 + m_2^2 + m_3^2 + m_4^2.$$

Solution: By definition of the Mandelstam variables, one could express $(s + t + u)$ as

$$\begin{aligned}
s + t + u &= (p_1 + p_2)^2 + (p_1 - p_3)^2 + (p_1 - p_4)^2 \\
&= \sum_i p_i \cdot p_i + 2p_1 \cdot p_1 + 2p_1 \cdot p_2 - 2p_1 \cdot p_3 - 2p_1 \cdot p_4 \\
&= \sum_i m_i^2 + 2p_1 \cdot (p_1 + p_2 - p_3 - p_4) \\
&= \sum_i m_i^2 \quad \square
\end{aligned}$$

The fact that in any frame $p^\mu p_\mu = m^2$ for a particle with mass m is used in the third identity, and in the last step the conservation of momentum $p_1 + p_2 = p_3 + p_4$ is used.

Problem 2.13
At the HERA collider, 27.5 GeV electrons were collided head-on with 820 GeV protons. Calculate the centre-of-mass energy.

Solution: Let the four-momentum of the electron and proton as $p_e = (E_e, \mathbf{p}_e)$, $p_p = (E_p, \mathbf{p}_p)$ respectively. The centre-of-mass energy $\sqrt{s}$ can be expressed as,

$$\begin{aligned}
s &= \left(p_e + p_p\right)^2 = p_e \cdot p_e + p_p \cdot p_p + 2p_e \cdot p_p \\
&= m_e^2 + m_p^2 + 2\left(E_e E_p - \mathbf{p}_e \cdot \mathbf{p}_p\right) \\
&= m_e^2 + m_p^2 + 2\left(E_e E_p + |\mathbf{p}_e|\,|\mathbf{p}_p|\right) \simeq 4E_e E_p \qquad \left(|\mathbf{p}_i|^2 = E_i^2 - m_i^2 \sim E_i^2\right)
\end{aligned}$$

As the collision is occuring head-on, one could say that $\mathbf{p}_e \cdot \mathbf{p}_p = -|\mathbf{p}_e|\,|\mathbf{p}_p|$ which was used in the last identity. Looking upon the order of the variables, $m_e \simeq 0.5$ MeV$, m_p \simeq 93.8$ MeV and $E_e = 27.5$ GeV$, E_p = 820$ GeV for an approximation it is okay to consider $m_e, m_p \sim 0$. Thus the centre-of-mass energy $\boxed{\sqrt{s} \simeq 300 \text{ GeV}}$ when all the needed values are plugged in.

Problem 2.14
Consider the Compton scattering of a photon of momentum $\mathbf{k}$ and energy $E = |\mathbf{k}| = \mathbf{k}$ from an electron at rest. Writing the four-momenta of the scattered photon and electron respectively as k' and p', conservation of four-momentum is expressed as $k + p = k' + p'$. Use the relation $p'^2 = m_e^2$ to show that the energy of the scattered photon is given by

$$E' = \frac{E}{1 + (E/m_e)(1 - \cos\theta)}$$

Solution:

Problem 2.15
Using the commutation relations for position and momentum, prove that

$$\left[\hat{L}_x, \hat{L}_y\right] = i\hat{L}_z$$

Using the commutation relations for the components of angular momenta prove

$$\left[\hat{L}^2, \hat{L}_x\right] = 0$$

and

$$\hat{L}^2 = \hat{L}_-\hat{L}_+ + \hat{L}_z + \hat{L}_z^2$$

Solution:

Problem 2.16
Show that the operators $\hat{S}_i = \frac{1}{2}\sigma_i$, where σ_i are the three Pauli spin-matrices,

$$\hat{S}_x = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \hat{S}_y = \frac{1}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \text{and} \quad \hat{S}_z = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

satisfy the same algebra as the angular momentum operators, namely

$$\left[\hat{S}_x, \hat{S}_y\right] = i\hat{S}_z \quad \left[\hat{S}_y, \hat{S}_z\right] = i\hat{S}_x \quad \text{and} \quad \left[\hat{S}_z, \hat{S}_x\right] = i\hat{S}_y$$

Find the eigenvalue(s) of the operator $\hat{\mathbf{S}}^2 = \frac{1}{4}\left(\hat{S}_x^2 + \hat{S}_y^2 + \hat{S}_z^2\right)$ and deduce that the eigenstates of $\hat{S}_z$ are a suitable representation of a spin-half particle.

Solution:

Problem 2.17
Find the third-order term in the transition matrix element of Fermi's golden rule.

Solution:

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3 Decay Rates and Cross Sections

Problem 3.1
Calculate the energy of the μ^- produced in the decay at rest $\pi^- \to \mu\bar{\nu}_\mu$. Assume $m_\pi = 140$ MeV, $m_\mu = 106$ MeV and take $m_\nu \sim 0$.

Solution: Let the four-momenta of the muon and the neutrino to be $p_1 = (E_1, 0, 0, E_2)$ and $p_2 = (E_2, 0, 0, -E_2)$. In the pion rest frame, $E_1 + E_2 = m_\pi$ and from the muon mass constraint $m_\mu^2 = E_1^2 - E_2^2$. Solving these equation gives

$$\boxed{E_1 = \frac{m_\pi^2 + m_\mu^2}{2m_\pi} = 110.13 \text{ GeV}}$$

Problem 3.2
For the decay $a \to 1 + 2$, show that the momenta of both daughter particles in the centre-of mass frame p^* are

$$p^* = \frac{1}{2m_a}\sqrt{\left[m_a^2 - (m_1^2 + m_2^2)\right]\left[m_a^2 - (m_1^2 - m_2^2)\right]}$$

.

Solution: Let the four-momenta of the mother particle and the daughter particles to be $p_a = (m_a, 0, 0, 0)$, $p_1 = (E_1, 0, 0, p^*)$, $p_2 = (E_2, 0, 0, -p^*)$ From the mass constraints, we get $E_1 + E_2 = m_a$, $E_1^2 - p^{*2} = m_1^2$, and $E_2^2 - p^{*2} = m_2^2$.

Since we have three unknown variables E_1, E_2, p^* and three equations, it is possible to get p^* in terms of m_a, m_1 and m_2, which gives the desired solution. In detail,

$$\begin{aligned} p^{*2} = E_1^2 - m_1^2 = E_2^2 - m_2^2 \implies E_1^2 &= E_2^2 + m_1^2 - m_2^2 \\ &= (m_a - E_1)^2 + m_1^2 - m_2^2 \implies E_1 = \frac{1}{2m_a}\left(m_a^2 + m_1^2 - m_2^2\right) \end{aligned}$$

which leads to a similar expression of E_2 using $E_1 + E_2 = m_a$

$$E_2 = m_a - E_1 = \frac{1}{2m_a}\left(m_a^2 - m_1^2 + m_2^2\right)$$

Then one could finally write down p^* in terms of m_a, m_1, m_2 as, using the fact that $p^{*2} = E_1^2 - m_1^2 = E_2^2 - m_2^2$

$$\begin{aligned} p^{*2} &= \frac{1}{2}\left[E_1^2 + E_2^2 - \left(m_1^2 + m_2^2\right)\right] \\ &= \frac{1}{2}\left[(E_1 + E_2)^2 - 2E_1E_2 - \left(m_1^2 + m_2^2\right)\right] \\ &= \frac{1}{2}\left[m_a^2 - \frac{1}{2m_a^2}\left[m_a^2 - (m_1^2 + m_2^2)\right]\left[m_a^2 - (m_1^2 - m_2^2)\right] - \left(m_1^2 + m_2^2\right)\right] \\ &= \frac{1}{2}\left[m_a^2 - \left(m_1^2 + m_2^2\right)\right]\left[1 - \frac{1}{2m_a^2}\left[m_a^2 - \left(m_1^2 - m_2^2\right)\right]\right] \\ &= \frac{1}{2}\left[m_a^2 - \left(m_1^2 + m_2^2\right)\right]\left[1 - \frac{1}{2m_a^2}\left[m_a^2 - \left(m_1^2 - m_2^2\right)\right]\right] \\ &= \frac{1}{4m_a^2}\left[m_a^2 - (m_1^2 + m_2^2)\right]\left[m_a^2 - (m_1^2 - m_2^2)\right] \quad \square \end{aligned}$$

Problem 3.3
Calculate the branching ratio for the decay $K^+ \to \pi^+\pi^0$, given the partial decay width $\Gamma(K^+ \to \pi^+\pi^0) = 1.2 \times 10^{-8}$ eV and the mean kaon lifetime $\tau(K^+) = 1.2 \times 10^{-8}$ s.

Solution: Using the given information,

$$\begin{aligned}
\mathrm{BR}\left(K^+ \to \pi^+\pi^0\right) &= \frac{1}{\Gamma_{K^+}} \times \Gamma\left(K^+ \to \pi^+\pi^0\right) \\
&= \tau(K^+) \times \Gamma\left(K^+ \to \pi^+\pi^0\right) \\
&= \left(1.2 \times 10^{-8}\mathrm{s}\right) \times \left(1.2 \times 10^{-8}\mathrm{eV}\right) \\
&= \left(1.2 \times 10^{-8}\right) \times \left(\frac{1}{6.58} \times 10^{16}\ \mathrm{eV}^{-1}\right) \times \left(1.2 \times 10^{-8}\mathrm{eV}\right) \\
&= \frac{1.2^2}{6.58} \simeq \boxed{21\%}
\end{aligned}$$

which is as much as expected from the known branching rate.

Problem 3.4
At a future e^+e^- linear collider operating as a Higgs factory at a centre-of-mass energy of $\sqrt{s} = 250$ GeV, the cross section for the process $e^+e^- \to HZ$ is 250 fb. If the collider has an instantaneous luminosity of $2 \times 10^{34}\mathrm{cm}^{-2}\mathrm{s}^{-1}$ and is operational for 50% of the time, how many Higgs bosons will be produced in five years of running?

Solution: Let the total number of Higgs bosons that will be produced in 5 years of running in such condition as N, then one could calculate N as,

$$\begin{aligned}
N &= \left(2 \times 10^{34}\mathrm{cm}^{-2}\mathrm{s}^{-1}\right) \times (5\ \mathrm{yrs}) \times (250\ \mathrm{fb}) \times 0.5 \\
&= \left(2 \times 10^{34}\mathrm{cm}^{-2}\mathrm{s}^{-1}\right)\left(1.5768 \times 10^{8}\mathrm{s}\right) \times \left(2.5 \times 10^{-41}\mathrm{cm}^2\right) \times 0.5 \\
&= 39.42
\end{aligned}$$

Problem 3.5
The total $e^+e^- \to \gamma \to \mu^+\mu^-$ annihilation cross section is $\sigma = 4\pi\alpha^2/3s$, where $\alpha \simeq 1/137$. Calculate the cross section at $\sqrt{s} = 50$ GeV, expressing your answer in both natural units and in barns. Compare this to the total pp cross section at $\sqrt{s} = 50$ GeV which is approximately 40 mb and comment on the result.

Solution: Plugging in all the values we know in natural units,

$$\sigma = \frac{4\pi}{3 \cdot \left(2.5 \times 10^3\ \mathrm{GeV}^2\right) \cdot 137^2} = 8.9 \times 10^{-8}\ \mathrm{GeV}^{-2}$$

which could be converted into barns using 1 $\mathrm{GeV}^{-2} = 0.3894$ mb, gives $\boxed{\sigma = 346.5\ \mathrm{pb}}$.

Problem 3.6
A 1 GeV muon neutrino is fired at a 1m thick block of iron with density $\rho = 7.874 \times 10^3\mathrm{kg} \cdot \mathrm{m}^{-3}$. If the average neutrino-nucleon interaction cross section is $\sigma = 8 \times 10^{-39}\mathrm{m}^2$, calculate the (small) probability that the neutrino interacts in the block.

Solution: The muon neutrino will pass through $\sim 7.874 \times 10^3\mathrm{kg} \cdot \mathrm{m}^{-2}$ of iron. As iron has atomic mass of 56, around 56 g of iron will contain 6.022×10^{23} number of nucleons, which is nearly 8.43×10^{28} nucleons for 7.874×10^3 kg of iron. This could be considered as a flux of nucleons per area $\sim 8.43 \times 10^{28}\mathrm{m}^{-2}$. Thus the probability could be derived as the neutrino-nucleon interaction cross section multiplied with such flux of nuclei, which gives $\boxed{\sim 6.74 \times 10^{-14}}$.

Problem 3.7
For the process $a+b\to 1+2$ the Lorents-invariant flux term installed

$$F = 4\left[(p_a\cdot p_b)^2 - m_a^2 m_b^2\right]^{\frac{1}{2}}$$

In the non-relativistic limit, $\beta_a, \beta_b \ll 1$, show that

$$F \approx 4 m_a m_b \left|\mathbf{v}_a - \mathbf{v}_b\right|$$

where $\mathbf{v}_a, \mathbf{v}_b$ are the (non-relativistic) velocities of the two particles.

Solution: Let the four-momenta of a,b as $p_a = (E_a, \mathbf{p}_a)$ and $p_b = (E_b, \mathbf{p}_b)$. Under the non-relativistic limit which implies that $\gamma_a, \gamma_b \simeq 1$, one could write down F as

$$\begin{aligned}
F &= 4\left[(p_a\cdot p_b)^2 - m_a^2 m_b^2\right]^{\frac{1}{2}} \\
&= 4\left[(E_aE_b - m_am_b\boldsymbol{\beta}_a\cdot\boldsymbol{\beta}_b)^2 - m_a^2 m_b^2\right]^{\frac{1}{2}} \\
&= 4\left[(E_aE_b - m_am_b\boldsymbol{\beta}_a\cdot\boldsymbol{\beta}_b)^2 - m_a^2 m_b^2\right]^{\frac{1}{2}} \\
&= 4\left[(E_aE_b - m_am_b\boldsymbol{\beta}_a\cdot\boldsymbol{\beta}_b)^2 - m_a^2 m_b^2\right]^{\frac{1}{2}}
\end{aligned}$$

Problem 3.8
The Lorentz-invariant flux term for the process $a+b\to 1+2$ in the centre-of-mass frame was shown to be $F = 4p_i^*\sqrt{s}$, where p_i^* is the momentum of the initial-state particles. Show that the corresponding expression in the frame where b is at rest is

$$F = 4m_b p_a.$$

Solution:

Problem 3.9
Show that the momentum in the centre-of-mass frame of the initial-state particles in a two-body scattering process can be expressed as

$$p_i^{*2} = \frac{1}{4s}\left[s-(m_1+m_2)^2\right]\left[s-(m_1-m_2)^2\right]$$

Solution:

Problem 3.10
Repeat the calculation of Section 3.5.2 for the process $e^-p\to e^-p$ where the mass of the electron is no longer neglected.

(a) First show that

$$\frac{\mathrm{d}E}{\mathrm{d}\left(E\cos\theta\right)} = \frac{p_1p_3^2}{p_3\left(E_1+m_p\right) - E_3p_1\cos\theta}$$

(b) Then show that

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{1}{64\pi^2} \cdot \frac{p_3^2}{p_1 m_p} \cdot \frac{1}{p_3\left(E_1 + m_p\right) - E_3 p_1 \cos\theta} \cdot \left|\mathcal{M}_{fi}\right|^2$$

Solution:

4 The Dirac Equation

Problem 4.1
Show that

$$\left[\hat{\mathbf{p}}^2, \hat{\mathbf{r}} \times \hat{\mathbf{p}}\right] = 0,$$

and hence the Hamiltonian of the free-particle Schrödinger equation commutes with the angular momentum operator.

Solution: One could expand the given commutator as,

$$\begin{aligned}
\left[\hat{\mathbf{p}}^2, \hat{\mathbf{r}} \times \hat{\mathbf{p}}\right] &= \left[\hat{\mathbf{p}}_a \hat{\mathbf{p}}_a, \epsilon_{abc} r_c \hat{\mathbf{p}}_b \hat{\mathbf{c}}\right] \\
&= \epsilon_{abc} r_c \left[\hat{\mathbf{p}}_a \hat{\mathbf{p}}_a, \hat{\mathbf{p}}_b \hat{\mathbf{c}}\right] \\
&= \epsilon_{abc} r_c \left\{\hat{\mathbf{p}}_a \left[\hat{\mathbf{p}}_a, \hat{\mathbf{p}}_b \hat{\mathbf{c}}\right] + \left[\hat{\mathbf{p}}_a, \hat{\mathbf{p}}_b \hat{\mathbf{c}}\right] \hat{\mathbf{p}}_a\right\} \\
&= \epsilon_{abc} r_c \left\{\hat{\mathbf{p}}_a \left[\hat{\mathbf{p}}_a, \hat{\mathbf{p}}_b \hat{\mathbf{c}}\right] + \left[\hat{\mathbf{p}}_a, \hat{\mathbf{p}}_b \hat{\mathbf{c}}\right] \hat{\mathbf{p}}_a\right\} \qquad \Longleftarrow \quad \left[\hat{\mathbf{p}}_a, \hat{\mathbf{p}}_b \hat{\mathbf{c}}\right] = \delta_{ab} \hat{\mathbf{c}} - i \delta_{ac} \hat{\mathbf{p}}_b
\end{aligned}$$

which will eliminate due to the contraction between ϵ_{abc} and δ_{ab}, δ_{ac}.

Problem 4.2
Show that u_1 and u_2 are orthogonal, i.e. $u_1^\dagger u_2 = 0$.

Solution: Let us first denote

$$u_\uparrow = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{and} \quad u_\downarrow = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

One should note that $u_\uparrow^\dagger u_\downarrow = 0$. u_1, u_2 could also be expressed in terms of

$$u_1 = \begin{pmatrix} u_\uparrow \\ \frac{\boldsymbol{\sigma}\cdot\mathbf{p}}{E+m} u_\uparrow \end{pmatrix} \quad \text{and} \quad u_2 = \begin{pmatrix} u_\downarrow \\ \frac{\boldsymbol{\sigma}\cdot\mathbf{p}}{E+m} u_\downarrow \end{pmatrix}.$$

Now $u_1^\dagger u_2$ could be written as,

$$\begin{aligned}
u_1^\dagger u_2 &= \begin{pmatrix} u_\uparrow \\ \frac{\boldsymbol{\sigma}\cdot\mathbf{p}}{E+m} u_\uparrow \end{pmatrix}^\dagger \begin{pmatrix} u_\downarrow \\ \frac{\boldsymbol{\sigma}\cdot\mathbf{p}}{E+m} u_\downarrow \end{pmatrix} \\
&= \begin{pmatrix} u_\uparrow^\dagger & \frac{1}{E+m} \left(\left(\boldsymbol{\sigma}\cdot\mathbf{p}\right) u_\uparrow\right)^\dagger \end{pmatrix} \begin{pmatrix} u_\downarrow \\ \frac{\boldsymbol{\sigma}\cdot\mathbf{p}}{E+m} u_\downarrow \end{pmatrix} \\
&= u_\uparrow^\dagger u_\downarrow + \frac{1}{\left(E+m\right)^2} \left(\left(\boldsymbol{\sigma}\cdot\mathbf{p}\right) u_\uparrow\right)^\dagger \left(\left(\boldsymbol{\sigma}\cdot\mathbf{p}\right) u_\downarrow\right)
\end{aligned}$$

$$= u_\uparrow^\dagger u_\downarrow + \frac{1}{(E+m)^2} u_\uparrow^\dagger (\boldsymbol{\sigma}\cdot\mathbf{p})^\dagger (\boldsymbol{\sigma}\cdot\mathbf{p})\, u_\downarrow$$

One could use the fact that,

$$(\boldsymbol{\sigma}\cdot\mathbf{p})^\dagger (\boldsymbol{\sigma}\cdot\mathbf{p}) = (\boldsymbol{\sigma}\cdot\mathbf{p})^2 = \begin{pmatrix} p_x^2+p_y^2+p_z^2 & 0 \\ 0 & p_x^2+p_y^2+p_z^2 \end{pmatrix} = \left(E^2-m^2\right) I_2$$

Thus it could be tidied up as,

$$u_1^\dagger u_2 = u_\uparrow^\dagger u_\downarrow + \frac{1}{(E+m)^2} u_\uparrow^\dagger (\boldsymbol{\sigma}\cdot\mathbf{p})^\dagger (\boldsymbol{\sigma}\cdot\mathbf{p})\, u_\downarrow$$

$$= \left[1 + \frac{E^2-m^2}{(E+m)^2}\right] u_\uparrow^\dagger u_\downarrow = 0 \quad \square$$

Problem 4.3
Verify the statement that the Einstein energy-momentum relationship is recovered if any of the four Dirac spinors of (4.48) are subtitutes into the Dirac equation written in terms of momentum, $(\gamma^\mu p_\mu - m)\, u = 0$.

Solution: Let us choose u_1 to plug in the Dirac equation. Then it could be expressed as,

$$(\not{p} - m)\, u_1 = 0 \implies \begin{pmatrix} (E-m)\, I_2 & -\boldsymbol{\sigma}\cdot\mathbf{p} \\ \boldsymbol{\sigma}\cdot\mathbf{p} & -(E+m)\, I_2 \end{pmatrix} \begin{pmatrix} E+m \\ 0 \\ p_z \\ p_x+ip_y \end{pmatrix} = 0$$

$$\implies \begin{pmatrix} E^2-m^2 \\ 0 \end{pmatrix} + \boldsymbol{\sigma}\cdot\mathbf{p} \begin{pmatrix} E+m-p_z \\ -p_x-ip_y \end{pmatrix} - (E+m) \begin{pmatrix} p_z \\ p_x+ip_y \end{pmatrix} = 0$$

$$[\text{first row}] \implies \left(E^2-m^2\right) + (E+m)\, p_z - \left(p_x^2+p_y^2+p_z^2\right) - (E+m)\, p_z = 0$$

$$\implies E^2 = p_x^2+p_y^2+p_z^2+m^2 \quad \square$$

Problem 4.4
For a particle with four-momentum $p^\mu = (E, \mathbf{p})$, the general solution to the free-particle Dirac equation can be written

$$\psi(p) = [a u_1(p) + b u_2(p)]\, e^{i(\mathbf{p}\cdot\mathbf{x}-Et)}$$

Using the explicit forms for u_1 and u_2, show that the four-vector current $j^\mu = (\rho, \mathbf{j})$ is given by

$$j^\mu = 2p^\mu$$

Furthermore, show that the resulting probability density and probability current are consistent with a particle moving with velocity $\beta = p/E$.

Solution:

Problem 4.5
Writing the four-component spinor u in terms of two two-component vectors

$$u = \begin{pmatrix} u_A \\ u_B \end{pmatrix},$$

show that in the non-relativistic limit, where $\beta \cong v/c \ll 1$, the components of u_B are smaller than those of u_A by a factor v/c.

Solution:

Problem 4.6
By considering the three cases $\mu = \nu = 0$, $\mu = \nu \neq 0$ and $\mu \neq \nu$ show that

$$\gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu = 2g^{\mu\nu}.$$

Solution: For brevity, the gamma matrices will be presented in terms of direct products as :

$$\begin{aligned}
\gamma^0 &= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \otimes I_2 \equiv \beta \\
\gamma^i &\equiv \beta \alpha_i \\
&= \left[\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \otimes I_2 \right] \left[\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \otimes \sigma_i \right] \\
&= \left[\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \otimes \sigma_i \right]
\end{aligned}$$

Then considering the three cases,

(a) $\mu = \nu = 0$

$$\begin{aligned}
\gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu &= 2\gamma^0 \gamma^0 \\
&= 2 \left[\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \otimes I_2 \right] \left[\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \otimes I_2 \right] \\
&= 2 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \otimes I_2 = 2I_4 = 2g^{00} I_4
\end{aligned}$$

(b) $\mu \neq 0, \nu \neq 0$

$$\begin{aligned}
\gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu &= \gamma^i \gamma^j + \gamma^j \gamma^i \\
&= \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}^2 \otimes \sigma_i \sigma_j + \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}^2 \otimes \sigma_j \sigma_i \\
&= -I_2 \otimes (\sigma_j \sigma_i + \sigma_i \sigma_j) \\
&= -I_2 \otimes \{\sigma_i, \sigma_j\} \\
&= -2I_2 \otimes \delta_{ij} I_2 = -2\delta_{ij} I_4 = 2g^{ij} I_4
\end{aligned}$$

(c) $\mu \neq \nu$
Due to the result from (b), one could easily realize that when both indices are not 0, $\left\{\gamma^i, \gamma^j\right\} = 0$ which is $2g^{ij} I_4$. Consider one of the indices to be 0, then :

$$\begin{aligned}
\gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu &= \gamma^0 \gamma^i + \gamma^i \gamma^0 \\
&= \left[\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right] \otimes \sigma_i = 0 = 2g^{0i} I_4 \quad \square
\end{aligned}$$

Problem 4.7
By operating on the Dirac equation,

$$(i\gamma^\mu \partial_\mu - m)\,\psi = 0$$

with $\gamma^\nu \partial_\nu$ prove that the components of ψ satisfy the Klein-Gordon equation,

$$\left(\partial^\mu \partial_\mu + m^2\right)\psi = 0.$$

Solution: Straightforwardly following the instructions given by the problem,

$$\begin{aligned}
(i\gamma^\mu \partial_\mu - m)\,\psi = 0 \implies & \gamma^\nu \partial_\nu \,(i\gamma^\mu \partial_\mu - m)\,\psi = 0 \\
\implies & \left[i\gamma^\nu \partial_\nu \,(\gamma^\mu \partial_\mu) - m\gamma^\nu \partial_\nu\right]\psi = 0 \\
\implies & \left[i\gamma^\nu \,(\partial_\nu \gamma^\mu)\,\partial_\mu + i\gamma^\nu \gamma^\mu \partial_\nu \partial_\mu - m\gamma^\nu \partial_\nu\right]\psi = 0 \\
\implies & \left(\frac{i}{2}\left\{\gamma^\nu, \gamma^\mu\right\}\partial_\nu \partial_\mu - m\gamma^\nu \partial_\nu\right)\psi = 0 \\
\implies & \left(ig^{\nu\mu}\partial_\nu \partial_\mu - m\gamma^\nu \partial_\nu\right)\psi = 0
\end{aligned}$$

For the latter term, one could utilize the Dirac equation :

$$(i\gamma^\mu \partial_\mu - m)\,\psi = 0 \implies \not{\partial}\psi = -im\psi$$

Thus the above could be tidied up as,

$$\begin{aligned}
\left(ig^{\nu\mu}\partial_\nu \partial_\mu - m\gamma^\nu \partial_\nu\right)\psi = 0 \implies & \left(i\partial^\mu \partial_\mu - m\not{\partial}\right)\psi = 0 \\
\implies & \left(\partial^\mu \partial_\mu + m^2\right)\psi = 0 \quad \square
\end{aligned}$$

Problem 4.8
Show that

$$(\gamma^\mu)^\dagger = \gamma^0 \gamma^\mu \gamma^0.$$

Solution: Let us seperate the cases with indices being 0 and else.

(a) $\mu = 0$

$$\begin{aligned}
\gamma^{0\dagger} &= \gamma^0 \\
&= I_4 \gamma^0 = \gamma^0 \gamma^0 \gamma^0
\end{aligned}$$

(b) $\mu = k \neq 0$

$$\begin{aligned}
\gamma^{k\dagger} &= -\gamma^k \\
&= -I_4 \gamma^k = -\gamma^0 \gamma^0 \gamma^k = \gamma^0 \gamma^k \gamma^0 \quad \square
\end{aligned}$$

Problem 4.9
Starting from

$$(\gamma^\mu p_\mu - m)\, u = 0,$$

show that the corresponding equation for the adjoint spinor is

$$\bar{u}\,(\gamma^\mu p_\mu - m) = 0.$$

Hence, without using the explicit form for the u spinors, show that the normalisation condition $u^\dagger u = 2E$ leads to

$$\bar{u}u = 2m,$$

and that

$$\bar{u}\gamma^\mu u = 2p^\mu.$$

Solution: Let us first derive the corresponding Dirac equation for the adjoint spinor.

$$\begin{aligned}
(\gamma^\mu p_\mu - m)\, u = 0 &\implies u^\dagger\,(\gamma^\mu p_\mu - m)^\dagger = 0 \\
&\implies \bar{u}\gamma^0 \left(\gamma^{\mu\dagger} p_\mu - m\right) = 0 \quad \text{from} \quad \bar{u} = u^\dagger\gamma^0 \iff u^\dagger = \bar{u}\gamma^0 \\
&\implies \bar{u}\left(\gamma^0\gamma^{\mu\dagger}\gamma^0 p_\mu - m\gamma^0\gamma^0\right) = 0 \\
&\implies \bar{u}\,(\gamma^\mu p_\mu - m) = 0 \quad \text{from} \quad \gamma^0\gamma^\mu\gamma^0 = \gamma^{\mu\dagger}
\end{aligned}$$

In order to obtain the other relations, let us start from evaluating $\bar{u}\gamma^\mu u$ first.

$$\begin{aligned}
\bar{u}\gamma^\mu u &= \frac{1}{m}\bar{u}\gamma^\mu \not{p} u \qquad \Longleftarrow \not{p}u = mu \\
&= \frac{1}{m}\bar{u}\gamma^\mu\gamma^\nu p_\nu u = \frac{1}{m}\bar{u}\,[2g^{\mu\nu} - \gamma^\nu\gamma^\mu]\, p_\nu u \\
&= \frac{1}{m}\,[2\bar{u}up^\mu - \bar{u}\gamma^\nu\gamma^\mu p_\nu u] \\
&= \frac{1}{m}\left[2\bar{u}up^\mu - \bar{u}\not{p}\gamma^\mu u\right] \qquad \Longleftarrow \bar{u}\not{p} = m\bar{u} \\
&= \frac{1}{m}\,[2\bar{u}up^\mu - m\bar{u}\gamma^\mu u] \\
\iff \bar{u}\gamma^\mu u &= \frac{1}{m}\bar{u}p^\mu u
\end{aligned}$$

Under such relation letting $\mu = 0$ gives

$$\begin{aligned}
\bar{u}\gamma^0 u = \frac{1}{m}\bar{u}p^0 u &\implies u^\dagger\gamma^0\gamma^0 u = \frac{E}{m}\bar{u}u \\
&\implies u^\dagger u = \frac{E}{m}\bar{u}u \\
&\implies 2E = \frac{E}{m}\bar{u}u
\end{aligned}$$

$$\implies \bar{u}u = 2m$$

Now plugging in such relation back into $\bar{u}\gamma^\mu u$ gives,

$$\bar{u}\gamma^\mu u = \frac{1}{m}\bar{u}up^\mu = 2p^\mu. \quad \square$$

Problem 4.10
Demonstrate that the two relations of Equation (4.45) are consistent by showing that

$$(\boldsymbol{\sigma} \cdot \mathbf{p})^2 = \mathbf{p}^2.$$

Solution: One could easily show that

$$\begin{aligned}(\boldsymbol{\sigma} \cdot \mathbf{p})^2 &= \sigma^i \sigma^j p_i p_j \\ &= \frac{1}{2}\left(\sigma^i\sigma^j + \sigma^j\sigma^i\right) p_i p_j \impliedby \left\{\sigma^i, \sigma^j\right\} = 2\delta^{ij} \\ &= \delta^{ij} p_i p_j = \mathbf{p}^2. \quad \square\end{aligned}$$

One could notice that not only for $\mathbf{p}$ but for any cartesian vector the above should hold. This relation leads to the equivalence of,

$$\begin{aligned} u_A = \frac{(\boldsymbol{\sigma} \cdot \mathbf{p})}{E-m} u_B &\implies (\boldsymbol{\sigma} \cdot \mathbf{p})\, u_A = \frac{(\boldsymbol{\sigma} \cdot \mathbf{p})^2}{E-m} u_B \\ &\implies (\boldsymbol{\sigma} \cdot \mathbf{p})\, u_A = \frac{\mathbf{p}^2}{E-m} u_B \\ &\implies \frac{(\boldsymbol{\sigma} \cdot \mathbf{p})}{E+m} u_A = u_B \end{aligned}$$

which is the desired result.

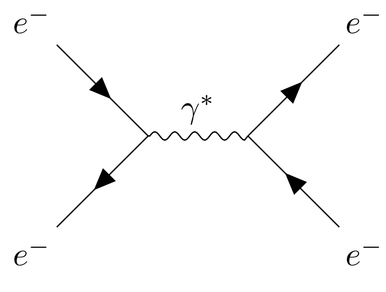
Problem 4.11
Consider the $e^+e^- \to \gamma \to e^+e^-$ annihilation process in the centre-of-mass frame where the energy of the photon is $2E$. Discuss energy and charge conservation for the two cases where:

(a) the negative energy solutions of the Dirac equation are interpreted as negative energy particles propagating backwards in time

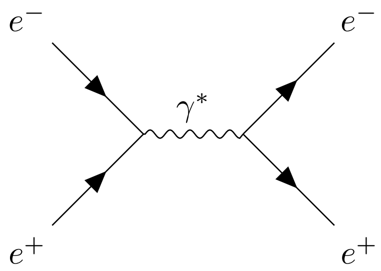
(b) the negative energy solutions of the Dirac equation are interpreted as positive energy antiparticles propagating forwards in time

Solution:

(a) the negative energy solutions of the Dirac equation are interpreted as negative energy particles propagating backwards in time



(b) the negative energy solutions of the Dirac equation are interpreted as positive energy antiparticles propa- gating forwards in time



Problem 4.12
Verify that the helicity operator

$$\hat{h} = \frac{\hat{\mathbf{\Sigma}} \cdot \hat{\mathbf{p}}}{2p} = \frac{1}{2p}\begin{pmatrix} \sigma \cdot \hat{\mathbf{p}} & 0 \\ 0 & \sigma \cdot \hat{\mathbf{p}} \end{pmatrix}$$

commutes with the Dirac Hamiltonian,

$$\hat{H}_D = \boldsymbol{\alpha} \cdot \hat{\mathbf{p}} + \beta m$$

Solution: The commutation between $\hat{h}$ and $\hat{H}_D$ could be written as,

$$\begin{aligned}[\hat{H}_D, \hat{h}] &= \frac{1}{2p}\left[\boldsymbol{\alpha} \cdot \hat{\mathbf{p}} + \beta m, \hat{\mathbf{\Sigma}} \cdot \hat{\mathbf{p}}\right] \\ &= \frac{1}{2p}\left[\boldsymbol{\alpha} \cdot \hat{\mathbf{p}}, \hat{\mathbf{\Sigma}} \cdot \hat{\mathbf{p}}\right] + \frac{m}{2p}\left[\beta, \hat{\mathbf{\Sigma}} \cdot \hat{\mathbf{p}}\right]\end{aligned}$$

Let us first evaluate the second commutator :

$$\begin{aligned}\left[\beta, \hat{\mathbf{\Sigma}} \cdot \hat{\mathbf{p}}\right] &= [\beta, \Sigma_i \hat{p}_i] \\ &= \cancelto{0}{[\beta, \Sigma_i]\,\hat{p}_i} + \Sigma_i \cancelto{0}{[\beta, \hat{p}_i]}\end{aligned}$$

where the second term is eliminated due to the fact that matrix operation and differentiation could be commuted (one could check oneself by operating on an arbitrary spinor). The first commutator then could be again reduced into,

$$\begin{aligned}\left[\boldsymbol{\alpha} \cdot \hat{\mathbf{p}}, \hat{\mathbf{\Sigma}} \cdot \hat{\mathbf{p}}\right] &= [\alpha_i \hat{p}_i, \Sigma_j \hat{p}_j] \\ &= \alpha_i \cancelto{0}{[\hat{p}_i, \Sigma_j]\hat{p}_j} + \alpha_i \Sigma_j \cancelto{0}{[\hat{p}_i, \hat{p}_j]} + [\alpha_i, \Sigma_j]\,\hat{p}_j \hat{p}_i + \Sigma_j \cancelto{0}{[\alpha_i, \hat{p}_j]\hat{p}_i} \\ &= 2i\epsilon_{ijk}\alpha_k \hat{p}_j \hat{p}_i = 2i\epsilon_{ijk}\alpha_k\left(\delta_{ji} - \hat{p}_i \hat{p}_j\right) = 2i\epsilon_{ijk}\alpha_k \hat{p}_i \hat{p}_j \\ &= 0 \quad \square\end{aligned}$$

Problem 4.13
4.13 Show that

$$\mathsf{P}u_\uparrow(\theta, \phi) = u_\downarrow(\pi - \theta, \pi + \phi)$$

and comment on the result.

Solution: One could straightforwardly write down,

$$
\begin{aligned}
\mathsf{P}u_{\uparrow}(\theta,\phi) = \gamma^0 u_{\uparrow}(\theta,\phi) = N\gamma^0 \begin{pmatrix} \cos\frac{\theta}{2} \\ e^{i\phi}\sin\frac{\theta}{2} \\ \frac{p}{E+m}\cos\frac{\theta}{2} \\ \frac{p}{E+m}e^{i\phi}\sin\frac{\theta}{2} \end{pmatrix} &= N \begin{pmatrix} \cos\frac{\theta}{2} \\ e^{i\phi}\sin\frac{\theta}{2} \\ -\frac{p}{E+m}\cos\frac{\theta}{2} \\ -\frac{p}{E+m}e^{i\phi}\sin\frac{\theta}{2} \end{pmatrix} \\
&= N \begin{pmatrix} -\sin\left(\frac{\pi}{2}-\frac{\theta}{2}\right) \\ e^{i(\pi+\phi)}\cos\left(\frac{\pi}{2}-\frac{\theta}{2}\right) \\ \frac{p}{E+m}\sin\left(\frac{\pi}{2}-\frac{\theta}{2}\right) \\ -\frac{p}{E+m}e^{i(\pi+\phi)}\cos\left(\frac{\pi}{2}-\frac{\theta}{2}\right) \end{pmatrix} \\
&= u_{\downarrow}(\pi-\theta,\pi+\phi) \quad \square
\end{aligned}
$$

One could notice that in terms of spherical angles, the result above shows that the parity operator actually flips the momentum direction to $-\mathbf{p}$, which will result in the opposite helicity.

Problem 4.14
Under the combined operation of parity and charge conjugation (CP) spinors transform as

$$\psi \to \psi^C = \mathsf{CP}\psi = i\gamma^2\gamma^0\psi^*$$

Show that up to an overall complex phase factor

$$\mathsf{CP}u_{\uparrow}(\theta,\phi) = v_{\downarrow}(\pi-\theta,\pi+\phi)$$

Solution: Using the result of Problem 4.13 which shows how P acts on helicity eigen-spinors, one only needs to show that

$$
\begin{aligned}
\mathsf{CP}u_{\uparrow}(\theta,\phi) = \mathsf{C}u_{\downarrow}(\pi-\theta,\pi+\phi) &= v_{\downarrow}(\pi-\theta,\pi+\phi) \\
\implies \mathsf{C}u_{\downarrow}(\theta',\phi') &= v_{\downarrow}(\theta',\phi')
\end{aligned}
$$

up to a certain complex phase factor. One could directly compute,

$$
\begin{aligned}
\mathsf{C}u_{\downarrow}(\theta,\phi) &= i\gamma^2 u_{\downarrow}(\theta,\phi)^* \\
&= N\begin{pmatrix} 0 & i\sigma_2 \\ -i\sigma_2 & 0 \end{pmatrix} \begin{pmatrix} -\sin\theta \\ \cos\theta e^{i\phi} \\ r\sin\theta \\ -r\cos\theta e^{i\phi} \end{pmatrix}^* \quad \text{where} \quad r \equiv \frac{p}{E+m} \\
&= -N\begin{pmatrix} r\cos\theta e^{-i\phi} \\ r\sin\theta \\ \cos\theta e^{-i\phi} \\ \sin\theta \end{pmatrix} = -Ne^{-i\phi}\begin{pmatrix} r\cos\theta \\ r\sin\theta e^{i\phi} \\ \cos\theta \\ \sin\theta e^{i\phi} \end{pmatrix} = -e^{-i\phi}v_{\downarrow}(\theta,\phi) \quad \square
\end{aligned}
$$

Problem 4.15
Starting from the Dirac equation, derive the identity

$$\bar{u}(p')\gamma^\mu u(p) = \frac{1}{2m}\bar{u}(p')\,(p+p')\,u(p) + \frac{i}{m}\bar{u}\,(p')\,\Sigma^{\mu\nu}q_\nu u(p)$$

where $q = p' - p$ and $\Sigma^{\mu\nu} = \frac{i}{4}\left[\gamma^\mu, \gamma^\nu\right]$

Solution: Using the Dirac equation for $u(p)$, one could write down $\bar{u}(p')\gamma^\mu u(p)$ as,

$$\begin{aligned}
\bar{u}(p')\gamma^\mu u(p) &= \frac{1}{m}\bar{u}(p')\gamma^\mu \not{p} u(p) && (1)\\
&= \frac{1}{m}\bar{u}(p')\gamma^\mu\gamma^\nu p_\nu u(p)\\
&= \frac{1}{m}\bar{u}(p')\left(2g^{\mu\nu} - \gamma^\nu\gamma^\mu\right)p_\nu u(p)\\
&= \frac{2}{m}\bar{u}(p')p^\mu u(p) - \frac{1}{m}\bar{u}(p')\not{p}\gamma^\mu u(p) && (2)
\end{aligned}$$

One could do the same using the Dirac equation for the adjoint case,

$$\begin{aligned}
\bar{u}(p')\gamma^\mu u(p) &= \frac{1}{m}\bar{u}(p')\not{p}'\gamma^\mu u(p) && (3)\\
&= \frac{1}{m}\bar{u}(p')\gamma^\nu\gamma^\mu p'_\nu u(p)\\
&= \frac{1}{m}\bar{u}(p')\left(2g^{\nu\mu} - \gamma^\mu\gamma^\nu\right)p'_\nu u(p)\\
&= \frac{2}{m}\bar{u}(p')p'^\mu u(p) - \frac{1}{m}\bar{u}(p')\gamma^\mu\not{p}' u(p) && (4)
\end{aligned}$$

Using the above relations one could again express $\bar{u}(p')\gamma^\mu u(p)$ as,

$$\begin{aligned}
\bar{u}(p')\gamma^\mu u(p) &= \frac{1}{2}\left\{\frac{1}{m}\bar{u}(p')\gamma^\mu\not{p}u(p) + \frac{1}{m}\bar{u}(p')\not{p}'\gamma^\mu u(p)\right\}\\
&= \frac{1}{m}\bar{u}(p')(p+p')^\mu u(p) - \frac{1}{2m}\bar{u}(p')\left\{\not{p}\gamma^\mu + \gamma^\mu\not{p}'\right\}u(p) && (5)
\end{aligned}$$

Before moving on, one could again use another relation that could be derived from (1) to (4) as :

$$\begin{aligned}
(1) = (2) \implies \frac{1}{m}\bar{u}(p')\gamma^\mu\not{p}u(p) &= \frac{2}{m}\bar{u}(p')p^\mu u(p) - \frac{1}{m}\bar{u}(p')\not{p}\gamma^\mu u(p)\\
\implies 2\bar{u}(p')p^\mu u(p) &= \bar{u}(p')\left\{\gamma^\mu\not{p} + \not{p}\gamma^\mu\right\}u(p) && (6)\\
(3) = (4) \implies \bar{u}(p')\not{p}'\gamma^\mu u(p) &= \frac{2}{m}\bar{u}(p')p'^\mu u(p) - \frac{1}{m}\bar{u}(p')\gamma^\mu\not{p}' u(p)\\
\implies 2\bar{u}(p')p'^\mu u(p) &= \bar{u}(p')\left\{\gamma^\mu\not{p}' + \not{p}'\gamma^\mu\right\}u(p) && (7)
\end{aligned}$$

Adding up both (6) and (7) gives,

$$\bar{u}(p')(p+p')^\mu u(p) = \frac{1}{2}\bar{u}(p')\left\{\gamma^\mu \slashed{p}' + \slashed{p}'\gamma^\mu + \gamma^\mu \slashed{p} + \slashed{p}\gamma^\mu\right\} u(p) \tag{8}$$

Plugging (8) into (5) but splitting the first term into half gives,

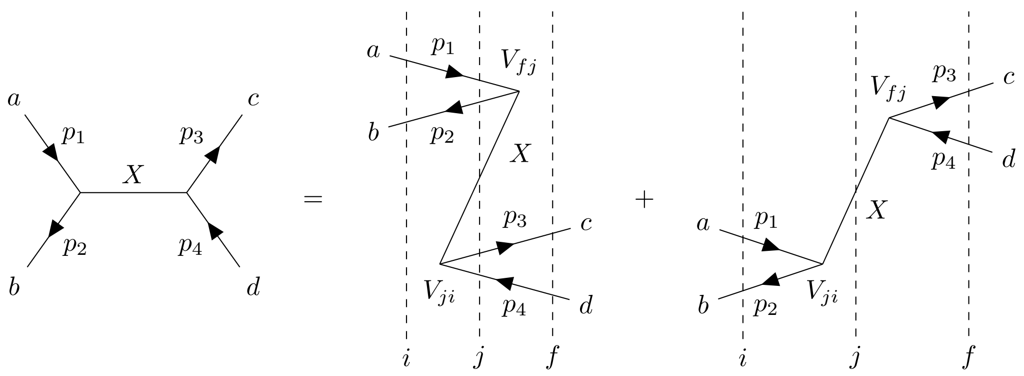
$$\begin{aligned}
\bar{u}(p')\gamma^\mu u(p) &= \frac{1}{2m}\bar{u}(p')(p+p')^\mu u(p) + \frac{1}{2m}\bar{u}(p')(p+p')^\mu u(p) - \frac{1}{2m}\bar{u}(p')\left\{\slashed{p}\gamma^\mu + \gamma^\mu \slashed{p}'\right\} u(p) \\
&= \frac{1}{2m}\bar{u}(p')(p+p')^\mu u(p) + \frac{1}{4m}\bar{u}(p')\left\{\gamma^\mu \slashed{p}' + \slashed{p}'\gamma^\mu + \gamma^\mu \slashed{p} + \slashed{p}\gamma^\mu\right\} u(p) - \frac{1}{2m}\bar{u}(p')\left\{\slashed{p}\gamma^\mu + \gamma^\mu \slashed{p}'\right\} u(p) \\
&= \frac{1}{2m}\bar{u}(p')(p+p')^\mu u(p) - \frac{1}{4m}\bar{u}(p')\left\{\gamma^\mu \slashed{p}' - \slashed{p}'\gamma^\mu - \gamma^\mu \slashed{p} + \slashed{p}\gamma^\mu\right\} u(p) \\
&= \frac{1}{2m}\bar{u}(p')(p+p')^\mu u(p) - \frac{1}{4m}\bar{u}(p')\left\{\gamma^\mu (\slashed{p}' - \slashed{p}) - (\slashed{p}' - \slashed{p})\gamma^\mu\right\} u(p) \\
&= \frac{1}{2m}\bar{u}(p')(p+p')^\mu u(p) - \frac{1}{4m}\bar{u}(p')\left[\gamma^\mu, \slashed{q}\right] u(p) \\
&= \frac{1}{2m}\bar{u}(p')(p+p')^\mu u(p) - \frac{1}{4m}\bar{u}(p')\left[\gamma^\mu, \gamma^\nu\right] q_\nu u(p) \\
&= \frac{1}{2m}\bar{u}(p')(p+p')^\mu u(p) + \frac{i}{m}\bar{u}\,(p')\,\Sigma^{\mu\nu} q_\nu u(p) \quad \square
\end{aligned}$$

5 Interaction by Particle Exchange

Problem 5.1
Draw the two time-ordered diagrams for the s-channel process shown in Figure 5.5. By repeating the steps of Section 5.1.1, show that the propagator has the same form as obtained for the t-channel process.

Hint: one of the time-ordered diagrams is non-intuitive, remember that in second-order perturbation theory the intermediate state does not conserve energy.

Solution: Similar with the t-channel case introduced in the text, one could draw the time-ordered diagram for the s-channel case as,



For the first diagram, one could write down the second-order perturbation term as :

$$\mathcal{T}_{fi}^{(1)} = \frac{1}{E_i - E_j} V_{fj}^{(1)} V_{ji}^{(1)} = \frac{1}{E_i - E_j} \langle X + a + b|\, V\, |0\rangle \langle 0|\, V\, |X + c + d\rangle$$

$$= \frac{1}{(E_a + E_b) - (E_a + E_b + E_X + E_c + E_d)} \cdot \frac{\mathcal{M}_{a\to b+X}}{\sqrt{2E_X 2E_a 2E_b}} \cdot \frac{\mathcal{M}_{c\to d+X}}{\sqrt{2E_X 2E_c 2E_d}}$$

$$= -\frac{1}{E_X + E_c + E_d} \cdot \frac{g_a g_c}{2E_X} \cdot \frac{1}{\sqrt{2E_a 2E_b 2E_c 2E_d}}$$

where we again assume scalar LI matrix elements. Similarly the corresponding term for the second diagram could be calculated as :

$$\mathcal{T}_{fi}^{(2)} = \frac{1}{E_i - E_j} V_{fj}^{(2)} V_{ji}^{(2)} = \frac{1}{E_i - E_j} \langle c + d| V |X\rangle \langle X| V |a + b\rangle$$

$$= \frac{1}{(E_a + E_b) - E_X} \cdot \frac{\mathcal{M}_{c\to d+X}}{\sqrt{2E_X 2E_c 2E_d}} \cdot \frac{\mathcal{M}_{a\to b+X}}{\sqrt{2E_X 2E_a 2E_b}}$$

$$= \frac{1}{E_a + E_b - E_X} \cdot \frac{g_a g_c}{2E_X} \cdot \frac{1}{\sqrt{2E_a 2E_b 2E_c 2E_d}}$$

The full LI matrix element then could be written as,

$$\mathcal{M}_{fi} = \sqrt{2E_a 2E_b 2E_c 2E_d} \left\{ \mathcal{T}_{fi}^{(1)} + \mathcal{T}_{fi}^{(2)} \right\}$$

$$= \frac{g_a g_c}{2E_X} \left[\frac{1}{E_a + E_b - E_X} - \frac{1}{E_c + E_d + E_X} \right] \quad \Longleftarrow \quad \begin{array}{c} E_a + E_b = E_c + E_d \\ \text{(from energy conservation)} \end{array}$$

$$= \frac{g_a g_c}{2E_X} \left[\frac{1}{E_a + E_b - E_X} - \frac{1}{E_a + E_b + E_X} \right]$$

$$= \frac{g_a g_c}{2E_X} \cdot \frac{2E_X}{(E_a + E_b)^2 - E_X^2}$$

$$= \frac{g_a g_c}{(E_a + E_b)^2 - (\mathbf{p}_a + \mathbf{p}_b)^2 - m_X^2} \quad \Longleftarrow \quad q \equiv p_a + p_b$$

$$= \frac{g_a g_c}{q^2 - m_X^2} \quad \square$$

which shows that the propagator term for s-channel diagrams also show the same form with

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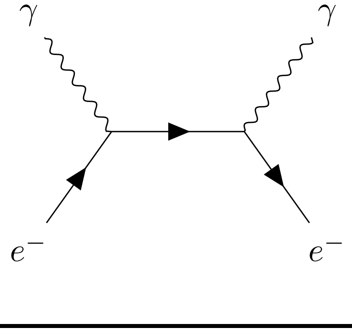
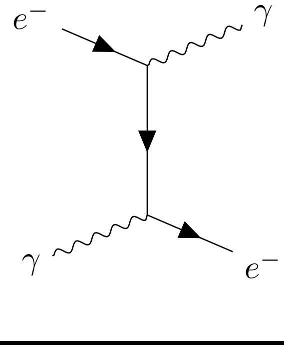
Problem 5.2
Draw the two lowest-order Feynman diagrams for the Compton scattering process $\gamma e^- \to \gamma e^-$.

Solution:

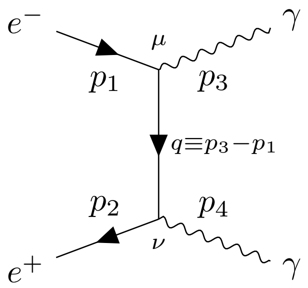




Problem 5.3
Draw the lowest-order t-channel and u-channel Feynman diagrams for $e^+e^- \to \gamma\gamma$ and use the Feynman rules for QED to write down the corresponding matrix elements.

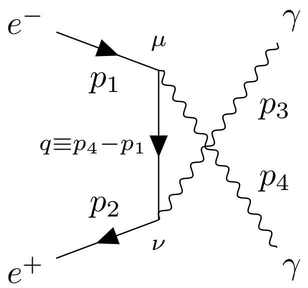
Solution:

(a) t-channel



$$= e^2 \epsilon^*_\mu(p_3)\gamma^\mu u(p_1) \left[\frac{i\left(\not{q}+m\right)}{q^2-m^2}\right] \epsilon^*_\nu(p_4)\gamma^\nu \bar{v}(p_2)$$

(b) u-channel



$$= e^2 \epsilon^*_\mu(p_4)\gamma^\mu u(p_1) \left[\frac{i\left(\not{q}+m\right)}{q^2-m^2}\right] \epsilon^*_\nu(p_3)\gamma^\nu \bar{v}(p_2)$$

6 Electron-Positron Annihilation

Problem 6.1
Using the properties of the γ-matrices of (4.33) and (4.34), and the definition of $\gamma^5 \equiv \gamma^0\gamma^1\gamma^2\gamma^3$, show that

$$\left(\gamma^5\right)^2 = 1 \quad , \quad \gamma^{5\dagger} = i\gamma^5 \quad \text{and} \quad \gamma^5\gamma^\mu = -\gamma^\mu\gamma^5$$

Solution:

(a) $\left(\gamma^5\right)^2 = 1$

$$\begin{aligned}
\left(\gamma^5\right)^2 &= -\gamma^0\gamma^1\gamma^2\gamma^3\gamma^0\gamma^1\gamma^2\gamma^3 \\
&= (-1)^4\, \gamma^0\gamma^0\gamma^1\gamma^2\gamma^3\gamma^1\gamma^2\gamma^3 \\
&= (-1)^6\, \gamma^1\gamma^1\gamma^2\gamma^3\gamma^2\gamma^3 \\
&= (-1)^8\, \gamma^2\gamma^2\gamma^3\gamma^3 = 1 \quad \square
\end{aligned}$$

(b) $\gamma^{5\dagger} = \gamma^5$

$$\begin{aligned}
\gamma^{5\dagger} = \left(i\gamma^0\gamma^1\gamma^2\gamma^3\right)^\dagger &= -i\gamma^{3\dagger}\gamma^{2\dagger}\gamma^{1\dagger}\gamma^{0\dagger} \\
&= -i\,(-1)^3\, \gamma^3\gamma^2\gamma^1\gamma^0 \\
&= -i\,(-1)^3\,(-1)^6\, \gamma^0\gamma^1\gamma^2\gamma^3 = \gamma^5 \quad \square
\end{aligned}$$

(c) $\gamma^5\gamma^\mu = -\gamma^\mu\gamma^5$

$$\begin{aligned}\gamma^5\gamma^\mu &= i\gamma^0\gamma^1\gamma^2\gamma^3\gamma^\mu \\ &= (-1)^3\,\gamma^\mu i\gamma^0\gamma^1\gamma^2\gamma^3 = -\gamma^\mu\gamma^5 \quad \square\end{aligned}$$

Here the fact that μ will be one of $0, 1, 2, 3$ is used to obtain the $(-1)^3$ factor, as one of the indices will be identical, thus not giving an additional -1 factor when switching the position of γ^μ.

Problem 6.2
Show that the chiral projection operators

$$P_R = \frac{1}{2}\left(1+\gamma^5\right) \quad \text{and} \quad P_L = \frac{1}{2}\left(1-\gamma^5\right)$$

satisfy

$$P_R + P_L = 1 \quad , \quad P_R P_R = P_R \quad , \quad P_L P_L = P_L \quad \text{and} \quad P_L P_R = 0$$

Solution:

(a) $P_R + P_L = 1$: Trivial

(b) $P_R P_R = P_R$

$$P_R P_R = \frac{1}{4}\left(1+\gamma^5\right)\left(1+\gamma^5\right) = \frac{1}{4}\left(1+2\gamma^5+\gamma^5\gamma^5\right) = \frac{1}{4}\left(2+2\gamma^5\right) = P_R \quad \square$$

(c) $P_L P_L = P_L$

$$P_L P_L = \frac{1}{4}\left(1-\gamma^5\right)\left(1-\gamma^5\right) = \frac{1}{4}\left(1-2\gamma^5+\gamma^5\gamma^5\right) = \frac{1}{4}\left(2-2\gamma^5\right) = P_L \quad \square$$

(d) $P_L P_R = 0$

$$P_L P_R = \frac{1}{4}\left(1-\gamma^5\right)\left(1+\gamma^5\right) = \frac{1}{4}\left(1-\gamma^5\gamma^5\right) = 0 \quad \square$$

Problem 6.3
Show that

$$\Lambda^+ = \frac{m+\not{p}}{2m} \quad \text{and} \quad \Lambda^- = \frac{m-\not{p}}{2m}$$

are also projection operators, and show that they respectively project out particle and antiparticle states, i.e.

$$\Lambda^+ u = u \quad , \quad \Lambda^- v = v \quad \text{and} \quad \Lambda^+ v = \Lambda^- u = 0$$

Solution:

(a) Show that $\Lambda^{\pm}$ are projection operators.

- $\Lambda^{+} + \Lambda^{-} = 1$: Trivial
- $\Lambda^{+}\Lambda^{+} = \Lambda^{+}$

$$\Lambda^{+}\Lambda^{+} = \frac{1}{4m^2}\left(m + \not{p}\right)\left(m + \not{p}\right) = \frac{1}{4m^2}\left(m^2 + 2m\not{p} + \not{p}\not{p}\right)$$
$$= \frac{1}{4m^2}\left(2m^2 + 2m\not{p}\right) = \Lambda^{+}$$

where the following identity is used :

$$\not{p}\not{p} = \gamma^{\mu}p_{\mu}\gamma^{\nu}p_{\nu} = \frac{1}{2}\left\{\gamma^{\mu}, \gamma^{\nu}\right\}p_{\mu}p_{\nu} = p \cdot p = m^2$$

- $\Lambda^{-}\Lambda^{-} = \Lambda^{-}$

$$\Lambda^{-}\Lambda^{-} = \frac{1}{4m^2}\left(m - \not{p}\right)\left(m - \not{p}\right) = \frac{1}{4m^2}\left(m^2 - 2m\not{p} + \not{p}\not{p}\right)$$
$$= \frac{1}{4m^2}\left(2m^2 - 2m\not{p}\right) = \Lambda^{-}$$

- $\Lambda^{+}\Lambda^{-} = 0$

$$\Lambda^{+}\Lambda^{-} = \frac{1}{4m^2}\left(m + \not{p}\right)\left(m - \not{p}\right) = \frac{1}{4m^2}\left(m^2 - \not{p}\not{p}\right) = 0$$

(b) Using the Dirac equations, $\left(\not{p} - m\right)u = 0$ and $\left(\not{p} + m\right)v = 0$ one could easily show the projections :

$$\Lambda^{+}u = \frac{1}{2m}\left(m + \not{p}\right)u = \frac{1}{2m}\left(mu + \not{p}u\right) = \frac{2mu}{2m} = u$$
$$\Lambda^{+}v = \frac{1}{2m}\left(m + \not{p}\right)v = 0$$
$$\Lambda^{-}v = \frac{1}{2m}\left(m - \not{p}\right)v = \frac{1}{2m}\left(mv - \not{p}v\right) = \frac{2mv}{2m} = v$$
$$\Lambda^{-}u = \frac{1}{2m}\left(m - \not{p}\right)u = 0 \quad \square$$

Problem 6.4
Show that the helicity operator can be expressed as

$$\hat{h} = -\frac{1}{2}\frac{\gamma^0\gamma^5\boldsymbol{\gamma} \cdot \mathbf{p}}{p}$$

Solution: Using the following form of the helicity operator,

$$\hat{h} = \frac{1}{2p}\hat{\Sigma} \cdot \hat{p} = \frac{1}{2p}\left(1 \otimes \boldsymbol{\sigma} \cdot \mathbf{p}\right)$$

Using the properties of Kronecker prodcuts, one could decompose $1 \otimes \boldsymbol{\sigma} \cdot \mathbf{p}$ as

$$1 \otimes \boldsymbol{\sigma} \cdot \mathbf{p} = -\left(i\sigma^2 \otimes 1\right)\left(i\sigma^2 \otimes \boldsymbol{\sigma} \cdot \mathbf{p}\right) \tag{9}$$

The two terms in the right-hand side of (9) can be again written as,

$$\begin{aligned} i\sigma^2 \otimes 1 &= -\sigma^1\sigma^3 \otimes 1 \\ &= -\left(\sigma^1 \otimes 1\right)\left(\sigma^3 \otimes 1\right) = -\gamma^0\gamma^5 \end{aligned} \tag{10}$$

$$i\sigma^2 \otimes \boldsymbol{\sigma} \cdot \mathbf{p} = \left(i\sigma^2 \otimes \sigma^j\right) p_j = \boldsymbol{\gamma} \cdot \mathbf{p} \tag{11}$$

where the expression of gamma matrices, $\gamma^0 = \sigma^3 \otimes 1$, $\gamma^j = i\sigma^2 \otimes \sigma^j$ and $\gamma^5 = \sigma^1 \otimes 1$ is used. Plugging in (10) and (11) into the original expression of the helicity operator,

$$\hat{h} = \frac{1}{2p}\left(1 \otimes \boldsymbol{\sigma} \cdot \mathbf{p}\right) = -\frac{1}{2p}\gamma^5\gamma^0\boldsymbol{\gamma} \cdot \mathbf{p} = \frac{1}{2p}\gamma^0\gamma^5\boldsymbol{\gamma} \cdot \mathbf{p} \quad \square$$

Problem 6.5
In general terms, explain why high-energy electron-positron colliders must also have high instantaneous luminosities.

Solution: As seen in the text, the cross section σ for electron-positron annihilation decreases as the center-of-mass energy $\sqrt{s}$ increases, from the relation $\sigma \sim s^{-1}$. Thus, such colliders must have high instantaneous luminosities in order to compensate the decreasing effect on the cross section stemming from the high collision energy.

Problem 6.6
For a spin-1 system, the eigenstate of the operator $\hat{S}_n = \mathbf{n} \cdot \hat{\mathbf{S}}$ with eigenvalue $+1$ corresponds to the spin being in the direction $\hat{\mathbf{n}}$. Writing this state in terms of the eigenstates of $\hat{S}_z$, i.e.

$$\left|1, +1\right\rangle_\theta = \alpha\left|1, -1\right\rangle + \beta\left|1, 0\right\rangle + \gamma\left|1, +1\right\rangle$$

and taking $\mathbf{n} = (\sin\theta, 0, \cos\theta)$ show that

$$\left|1, +1\right\rangle_\theta = \frac{1}{2}\left(1 - \cos\theta\right)\left|1, -1\right\rangle + \frac{1}{\sqrt{2}}\sin\theta\left|1, 0\right\rangle + \frac{1}{2}\left(1 + \cos\theta\right)\left|1, +1\right\rangle$$

Hint: Write $\hat{S}_x$ in terms of the ladder operators.

Solution: Using the given $\mathbf{n}$, the operator $\hat{S}_n$ can be written as $\sin\theta\hat{S}_x + \cos\theta\hat{S}_z$.

$$\hat{S}_n = \sin\theta\hat{S}_x + \cos\theta\hat{S}_z = \frac{1}{2}\sin\theta\left(\hat{S}_+ + \hat{S}_-\right) + \cos\theta\hat{S}_z$$

where $\hat{S}_{\pm}$ are the ladder operators which follows $\hat{S}_{\pm}|1,m\rangle = \sqrt{2-m(m\pm 1)}\,|1,m\pm 1\rangle$. Then using the above expression of $\hat{S}_n$, one could write down

$$\begin{aligned}\hat{S}_n|1,+1\rangle_\theta &= \frac{1}{2}\sin\theta\left(\hat{S}_+ + \hat{S}_-\right)|1,+1\rangle_\theta + \cos\theta\hat{S}_z|1,+1\rangle_\theta \\ &= \left(\frac{1}{\sqrt{2}}\sin\theta\beta - \alpha\cos\theta\right)|1,-1\rangle + \frac{1}{\sqrt{2}}(\alpha+\gamma)|1,0\rangle + \left(\frac{1}{\sqrt{2}}\sin\theta\beta + \gamma\cos\theta\right)|1,+1\rangle\end{aligned}$$

and from the definition of $\hat{S}_n$, it should satisfy $\hat{S}_n|1,+1\rangle_\theta = |1,+1\rangle_\theta$ which gives a set of linear equations of α, β and γ :

$$\begin{aligned}\alpha &= \frac{1}{\sqrt{2}}\sin\theta\beta - \alpha\cos\theta \\ \beta &= \frac{1}{\sqrt{2}}(\alpha+\gamma) \\ \gamma &= \frac{1}{\sqrt{2}}\sin\theta\beta + \gamma\cos\theta\end{aligned}$$

which gives the following solution :

$$\begin{aligned}\alpha &= \frac{1}{2}(1-\cos\theta) \\ \beta &= \frac{1}{\sqrt{2}}\sin\theta \\ \gamma &= \frac{1}{2}(1+\cos\theta) \quad \square\end{aligned}$$

Problem 6.7
Using helicity amplitudes, calculate the differential cross section for $e^-\mu^- \to e^-\mu^-$ scattering in the following steps :

(a) From the Feynman rules for QED, show that the lowest-order QED matrix element for $e^-\mu^- \to e^-\mu^-$ is

$$\mathcal{M}_{fi} = -\frac{e^2}{(p_1-p_3)^2}g_{\mu\nu}\left[\bar{u}(p_3)\gamma^\mu u(p_1)\right]\left[\bar{u}(p_4)\gamma^\nu u(p_2)\right]$$

where p_1 and p_3 are the four-momenta of the initial and final state e^-, and p_2 and p_4 are the four-momenta of the initial and final state μ^-.

(b) Working in the centre-of-mass frame, and writing the four-momenta of the initial- and final-state e^- as $p_1^\mu = (E_1, 0, 0, p)$ and $p_3^\mu = (E_1, p\sin\theta, 0, p\cos\theta)$ respectively, show that the electron currents for the four possible helicity combinations are

$$\begin{aligned}\bar{u}_\downarrow(p_3)\gamma^\mu u_\downarrow(p_1) &= 2(E_1 c, ps, -ips, pc) \\ \bar{u}_\uparrow(p_3)\gamma^\mu u_\downarrow(p_1) &= 2(ms, 0, 0, 0) \\ \bar{u}_\uparrow(p_3)\gamma^\mu u_\uparrow(p_1) &= 2(E_1 c, ps, ips, pc)\end{aligned}$$

$$\bar{u}_{\downarrow}(p_3)\gamma^{\mu}u_{\uparrow}(p_1) = -2(ms, 0, 0, 0)$$

where m is the electron mass, $s = \sin(\theta/2)$ and $c = \cos(\theta/2)$.

(c) Explain why the effect of the parity operator $\hat{\mathsf{P}} = \gamma^0$ is

$$\hat{\mathsf{P}}u_{\uparrow}(p, \theta, \phi) = \hat{\mathsf{P}}u_{\downarrow}(p, \pi - \theta, \pi + \theta)$$

Hence, or otherwise, show that the muon currents for the four helicity combinations are

$$\begin{aligned}
\bar{u}_{\downarrow}(p_4)\gamma^{\mu}u_{\downarrow}(p_2) &= 2(E_2c, -ps, -ips, -pc)\\
\bar{u}_{\uparrow}(p_4)\gamma^{\mu}u_{\downarrow}(p_2) &= 2(Ms, 0, 0, 0)\\
\bar{u}_{\uparrow}(p_4)\gamma^{\mu}u_{\uparrow}(p_2) &= 2(E_2c, -ps, ips, -pc)\\
\bar{u}_{\downarrow}(p_4)\gamma^{\mu}u_{\uparrow}(p_2) &= -2(Ms, 0, 0, 0)
\end{aligned}$$

where M is the muon mass.

(d) For the relativistic limit where $E \gg M$, , show that the matrix element squared for the case where the incoming e^- and incoming μ^- are both left-handed is given by

$$|\mathcal{M}_{LL}|^2 = \frac{4e^2s^2}{(p_1 - p_3)^4}.$$

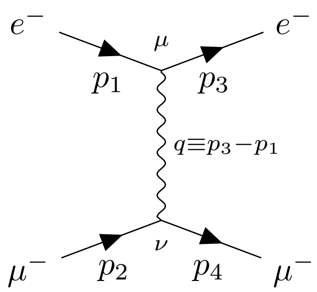
where $s = (p_1 + p_2)^2$. Find the corresponding expressions for $|\mathcal{M}_{RL}|^2$, $|\mathcal{M}_{RR}|^2$ and $|\mathcal{M}_{LR}|^2$.

(e) In this relativistic limit, show that the differential cross section for unpolarised $e^-\mu^- \to e^-\mu^-$ scattering in the centre-of-mass frame is

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{2\alpha^2}{s} \cdot \frac{1 + \frac{1}{4}(1 + \cos\theta)^2}{(1 - \cos\theta)^2}.$$

Solution:

(a) Obtain lowest-order QED matrix element.



$$\begin{aligned}
= -i\mathcal{M} &= \bar{u}(p_3)ie\gamma^{\mu}u(p_1)\frac{-ig^{\mu\nu}}{(p_3 - p_1)^2}\bar{u}(p_4)ie\gamma^{\nu}u(p_2)\\
&= \frac{ie^2}{(p_1 - p_3)^2}g_{\mu\nu}\left[\bar{u}(p_3)\gamma^{\mu}u(p_1)\right]\left[\bar{u}(p_4)\gamma^{\nu}u(p_2)\right] \quad \square
\end{aligned}$$

(b) Letting the incoming and outgoing e^- with $(\theta', \phi') = (0, 0)$ and $(\theta, 0)$ respectively, one could write down the corresponding spinors as ,

$$u_{\uparrow}(p_1) = \sqrt{E_1+m}\begin{pmatrix} 1 \\ 0 \\ \frac{p}{E_1+m} \\ 0 \end{pmatrix} \quad , \quad u_{\downarrow}(p_1) = \sqrt{E_1+m}\begin{pmatrix} 0 \\ 1 \\ 0 \\ \frac{-p}{E_1+m} \end{pmatrix}$$

$$u_{\uparrow}(p_3) = \sqrt{E_1+m}\begin{pmatrix} c \\ s \\ \frac{p}{E_1+m}c \\ \frac{p}{E_1+m}s \end{pmatrix} \quad , \quad u_{\downarrow}(p_3) = \sqrt{E_1+m}\begin{pmatrix} -s \\ c \\ \frac{p}{E_1+m}s \\ \frac{-p}{E_1+m}c \end{pmatrix}$$

Using such expression of the spinors, one could directly calculate the possible electron currents as :

$$\begin{aligned}
\bar{u}_{\downarrow}(p_3)\gamma^{\mu}u_{\downarrow}(p_1) &= 2(E_1 c, ps, -ips, pc) \\
\bar{u}_{\uparrow}(p_3)\gamma^{\mu}u_{\downarrow}(p_1) &= 2(ms, 0, 0, 0) \\
\bar{u}_{\uparrow}(p_3)\gamma^{\mu}u_{\uparrow}(p_1) &= 2(E_1 c, ps, ips, pc) \\
\bar{u}_{\downarrow}(p_3)\gamma^{\mu}u_{\uparrow}(p_1) &= -2(ms, 0, 0, 0)
\end{aligned}$$

(Trust me I really did all the calculations lol)

(c)

(d) Denoting the corresponding helicity configuration with momentum p_i as h_i, $\mathcal{M}_{LL}$ can be expressed as,

$$\begin{aligned}
\mathcal{M}_{LL} &= -\frac{e^2}{(p_1-p_3)^2}\sum_{h_3,h_4\in\{L,R\}} j^e_{Lh_3}\cdot j^{\mu}_{Lh_4} \\
&= -\frac{e^2}{(p_1-p_3)^2}\left[j^e_{LL}\cdot j^{\mu}_{LL} + j^e_{LL}\cdot j^{\mu}_{LR} + j^e_{LR}\cdot j^{\mu}_{LL} + j^e_{LR}\cdot j^{\mu}_{LR}\right] \qquad (12)
\end{aligned}$$

Noting that under the relativistic limit of $E \gg M$, the only term that survives in (12) is $j^e_{LL}\cdot j^{\mu}_{LL}$ which can be calculated as,

$$\begin{aligned}
j^e_{LL}\cdot j^{\mu}_{LL} &= 4\left(E_1\cos\frac{\theta}{2}, p\sin\frac{\theta}{2}, -ip\sin\frac{\theta}{2}, p\cos\frac{\theta}{2}\right)\cdot\left(E_2\cos\frac{\theta}{2}, -p\sin\frac{\theta}{2}, -ip\sin\frac{\theta}{2}, -p\cos\frac{\theta}{2}\right) \\
&= 4\left(E_1E_2\cos^2\frac{\theta}{2} + p^2\sin^2\frac{\theta}{2} + p^2\sin^2\frac{\theta}{2} + p^2\cos^2\frac{\theta}{2}\right) \\
&= 4E_1E_2 = 2\left(p_1+p_2\right)^2 \equiv 2s^2
\end{aligned}$$

Plugging this back into (12) and squaring the matrix element gives,

$$|\mathcal{M}_{LL}|^2 = \frac{e^4}{(p_1-p_3)^4}\left(j^e_{LL}\cdot j^{\mu}_{LL}\right)^2 = \frac{4e^4s^4}{(p_1-p_3)^4} \quad \square$$

For the other helicity combinations, the same could be done.

$$
\begin{aligned}
|\mathcal{M}_{RL}|^2 &= \frac{e^4}{(p_1-p_3)^4}\left[\sum_{h_3,h_4\in\{L,R\}} j^e_{Rh_3}\cdot j^\mu_{Lh_4}\right]^2 \\
&= \frac{e^4}{(p_1-p_3)^4}\left[\cancelto{0}{j^e_{RL}\cdot j^\mu_{LL}} + \cancelto{0}{j^e_{RL}\cdot j^\mu_{LR}} + j^e_{RR}\cdot j^\mu_{LL} + \cancelto{0}{j^e_{RR}\cdot j^\mu_{LR}}\right]^2 \\
&= \frac{16e^4}{(p_1-p_3)^4}\left(E_1E_2\cos^2\frac{\theta}{2} + p^2\sin^2\frac{\theta}{2} - p^2\sin^2\frac{\theta}{2} + p^2\cos^2\frac{\theta}{2}\right)^2 \\
&= \frac{4e^4s^4}{(p_1-p_3)^4}\cos^4\frac{\theta}{2}
\end{aligned}
$$

Again,

$$
\begin{aligned}
|\mathcal{M}_{RR}|^2 &= \frac{e^4}{(p_1-p_3)^4}\left(j^e_{RR}\cdot j^\mu_{RR}\right)^2 \\
&= \frac{16e^4}{(p_1-p_3)^4}\left(E_1E_2\cos^2\frac{\theta}{2} + p^2\sin^2\frac{\theta}{2} + p^2\sin^2\frac{\theta}{2} + p^2\cos^2\frac{\theta}{2}\right)^2 = \frac{4e^4s^4}{(p_1-p_3)^4}
\end{aligned}
$$

And finally,

$$
\begin{aligned}
|\mathcal{M}_{LR}|^2 &= \frac{e^4}{(p_1-p_3)^4}\left(j^e_{LL}\cdot j^\mu_{RR}\right)^2 \\
&= \frac{16e^4}{(p_1-p_3)^4}\left(E_1E_2\cos^2\frac{\theta}{2} + p^2\sin^2\frac{\theta}{2} - p^2\sin^2\frac{\theta}{2} + p^2\cos^2\frac{\theta}{2}\right)^2 = \frac{4e^4s^4}{(p_1-p_3)^4}\cos^4\frac{\theta}{2}
\end{aligned}
$$

(e) Averaging out all the helicity dependent amplitudes from (d), one would obtain

$$
\begin{aligned}
\langle|\mathcal{M}_{fi}|^2\rangle &= \frac{1}{4}\left\{|\mathcal{M}_{LR}|^2 + |\mathcal{M}_{RR}|^2 + |\mathcal{M}_{RL}|^2 + |\mathcal{M}_{LL}|^2\right\} \\
&= \frac{2e^4s^4}{(p_1-p_3)^4}\left(1+\cos^4\frac{\theta}{2}\right) \\
&= \frac{2e^4s^4}{(p_1-p_3)^4}\left[1+\frac{1}{4}\left(1+\cos\theta\right)^2\right] \\
&= \frac{e^4s^4}{2E_1^4\left(1-\cos\theta\right)^2}\left[1+\frac{1}{4}\left(1+\cos\theta\right)^2\right] \iff \left(p_1-p_3\right)^2 \simeq -2p_1\cdot p_3 = -2E_1^2\left(1-\cos\theta\right)
\end{aligned}
$$

Problem 6.8
Using $\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2g^{\mu\nu}$, prove that

$$
\gamma^\mu\gamma_\mu = 4 \quad , \quad \gamma^\mu \not{a}\gamma_\mu = -2\not{a} \quad \text{and} \quad \gamma^\mu\not{a}\not{b}\gamma_\mu = 4a\cdot b
$$

Solution:

(a) $\gamma^\mu \gamma_\mu = 4$

$$\gamma^\mu \gamma_\mu = \frac{1}{2} g_{\nu\mu} \left(\gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu \right) = g^{\mu\nu} g_{\mu\nu} = 4 \quad \square$$

(b) $\gamma^\mu \not{a} \gamma_\mu = -2\not{a}$

$$\begin{aligned} \gamma^\mu \not{a} \gamma_\mu &= a_\nu \gamma^\mu \gamma^\nu \gamma_\mu = a_\nu \left(2g^{\mu\nu} - \gamma^\nu \gamma^\mu \right) \gamma_\mu \\ &= 2a^\mu \gamma_\mu - \not{a} \gamma^\mu \gamma_\mu = 2\not{a} - 4\not{a} = -2\not{a} \quad \square \end{aligned}$$

(c) $\gamma^\mu \not{a} \not{b} \gamma_\mu = 4a \cdot b$

$$\begin{aligned} \gamma^\mu \not{a} \not{b} \gamma_\mu = a_\nu b_\rho \gamma^\mu \gamma^\nu \gamma^\rho \gamma_\mu &= \frac{1}{2} a_\nu b_\rho \gamma^\mu \left\{ \gamma^\nu, \gamma^\rho \right\} \gamma_\mu \\ &= \frac{1}{2} a_\nu b_\rho 2 g^{\nu\rho} \gamma^\mu \gamma_\mu = 4a \cdot b \quad \square \end{aligned}$$

Problem 6.9
Prove the relation $\left(\bar{\psi} \gamma^\mu \gamma^5 \phi \right)^\dagger = \bar{\phi} \gamma^\mu \gamma^5 \psi$.

Solution: One could show that :

$$\begin{aligned} \left(\bar{\psi} \gamma^\mu \gamma^5 \phi \right)^\dagger = \left(\psi^\dagger \gamma^0 \gamma^\mu \gamma^5 \phi \right)^\dagger &= - \left(\psi^\dagger \gamma^0 \gamma^5 \gamma^\mu \phi \right)^\dagger \\ &= -\phi^\dagger \gamma^{\mu\dagger} \gamma^{5\dagger} \gamma^{0\dagger} \psi \\ &= -\phi^\dagger \left(\gamma^0 \gamma^\mu \gamma^0 \right) \gamma^5 \gamma^0 \psi \\ &= -\phi^\dagger \gamma^0 \gamma^\mu \gamma^0 \gamma^5 \gamma^0 \psi = \phi^\dagger \gamma^0 \gamma^\mu \gamma^0 \gamma^0 \gamma^5 \psi = \bar{\phi} \gamma^\mu \gamma^5 \psi \quad \square \end{aligned}$$

Problem 6.10
Use the trace formalism to calculate the QED spin-averaged matrix element squared for $e^+ e^- \to f f$ including the electron mass term.

Solution: The spin-averaged amplitude $\langle \mathcal{M}_{fi}^2 \rangle$ for the $e^+ e^- \to f f$ without neglecting both mass terms, can be written as

$$\langle \mathcal{M}_{fi}^2 \rangle = \frac{1}{4} \sum_{\text{spins}} \left| \mathcal{M}_{fi} \right|^2 = \frac{Q_f^2 e^4}{4q^4} \mathrm{Tr} \left[(\not{p}_2 - m_e) \gamma^\mu (\not{p}_1 + m_e) \gamma^\nu \right] \times \mathrm{Tr} \left[(\not{p}_3 + m_f) \gamma_\mu (\not{p}_4 - m_f) \gamma_\nu \right]$$

The first trace term can be calculated as,

$$\begin{aligned} \mathrm{Tr} \left[(\not{p}_2 - m_e) \gamma^\mu (\not{p}_1 + m_e) \gamma^\nu \right] &= \mathrm{Tr} \left[\not{p}_2 \gamma^\mu \not{p}_1 \gamma^\nu + m_e \cancelto{0}{\not{p}_2 \gamma^\mu \gamma^\nu} - m_e \cancelto{0}{\gamma^\mu \not{p}_1 \gamma^\nu} - m_e^2 \gamma^\mu \gamma^\nu \right] \\ &= \mathrm{Tr} \left[p_{1\sigma} p_{2\rho} \gamma^\rho \gamma^\mu \gamma^\sigma \gamma^\nu - m_e^2 \gamma^\mu \gamma^\nu \right] \end{aligned}$$

$$= p_{1\sigma} p_{2\rho} \mathrm{Tr}\left[\gamma^\rho \gamma^\mu \gamma^\sigma \gamma^\nu\right] - m_e^2 \mathrm{Tr}\left[\gamma^\mu \gamma^\nu\right]$$

$$= 4 p_{1\sigma} p_{2\rho} \left(g^{\rho\mu} g^{\sigma\nu} - g^{\rho\sigma} g^{\mu\nu} + g^{\rho\nu} g^{\mu\sigma}\right) - 4 m_e^2 g^{\mu\nu}$$

$$= 4\left[p_{1\nu} p_{2\mu} + p_{1\mu} p_{2\nu} - \left(p_1 \cdot p_2 + m_e^2\right) g^{\mu\nu}\right]$$

Observing the second trace term, one could realize that switching $p_1 \leftrightarrow p_3$, $p_2 \leftrightarrow p_4$, $\mu \leftrightarrow \nu$ and $m_e \leftrightarrow m_f$ in the first trace term will be identical to the second one due to the fact that traces remain the same under cyclic permutations. Thus,

$$\mathrm{Tr}\left[(\not{p}_3 + m_f)\gamma_\mu(\not{p}_4 - m_f)\gamma_\nu\right] = 4\left[p_3^\mu p_4^\nu + p_3^\nu p_4^\mu - \left(p_3 \cdot p_4 + m_f^2\right) g_{\nu\mu}\right]$$

Using these two trace expressions, the spin-averaged amplitude can be fully written down as

$$\langle \mathcal{M}_{fi}^2 \rangle = 4 Q_f^2 \frac{e^4}{q^4}\left[p_{1\nu} p_{2\mu} + p_{1\mu} p_{2\nu} - \left(p_1 \cdot p_2 + m_e^2\right) g^{\mu\nu}\right] \times \left[p_3^\mu p_4^\nu + p_3^\nu p_4^\mu - \left(p_3 \cdot p_4 + m_f^2\right) g_{\nu\mu}\right]$$

$$= 8 Q_f^2 \frac{e^4}{q^4}\Big[(p_1 \cdot p_4)(p_2 \cdot p_3) + (p_1 \cdot p_3)(p_2 \cdot p_4) + m_e^2 (p_3 \cdot p_4) + m_f^2 (p_1 \cdot p_2) + 2 m_e^2 m_f^2\Big]$$

$$= \frac{2 Q_f^2 e^4}{(m_e^2 + p_1 \cdot p_2)^2}\Big[(p_1 \cdot p_4)(p_2 \cdot p_3) + (p_1 \cdot p_3)(p_2 \cdot p_4) + m_e^2 (p_3 \cdot p_4) + m_f^2 (p_1 \cdot p_2) + 2 m_e^2 m_f^2\Big]$$

Under the limit of $m_e \sim 0$ the above coincides with equation (6.63) and going further by neglecting the fermion mass reduces to (6.25), which is as expected.

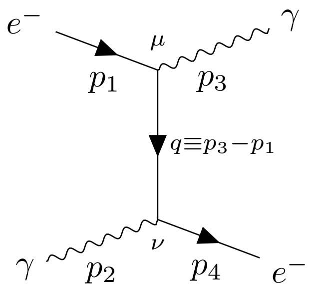
Problem 6.11
Neglecting the electron mass term, verify that the matrix element for $e^- f \to e^- f$ given in (6.67) can be obtained from the matrix element for $e^+ e^- \to f f$ given in (6.63) using crossing symmetry with the substitutions

$$p_1 \to p_1 \quad , \quad p_2 \to -p_3 \quad , \quad p_3 \to p_4 \quad \text{and} \quad p_4 \to -p_2$$

Solution:

Problem 6.12
Write down the matrix elements, $\mathcal{M}_1$ and $\mathcal{M}_2$, for the two Feynman diagrams for the Compton scattering process $e^-\gamma \to e^-\gamma$. From first principles, express the spin-averaged matrix element $\langle |\mathcal{M}_1 + \mathcal{M}_2|^2 \rangle$ as a trace. You will need the completeness relation for the photon polarisation states (see Appendix D).

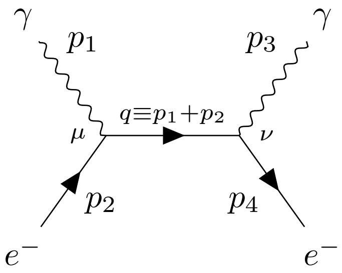
Solution: Starting from the two leading order Feynman diagrams that contribute to the Compton scattering process, using QED Feynman rules one could write down the corresponding matrix elements as,



$$: -i\mathcal{M}_1 = e^2 \bar{u}(p_4) \not{\epsilon}(p_2)\left[\frac{i\left(\not{q} + m\right)}{q^2 - m^2}\right] \not{\epsilon}^*(p_3) u(p_1)$$

$$= i e^2 \bar{u}(p_4) \not{\epsilon}(p_2)\left[\frac{\not{p}_3 - \not{p}_1 + m}{(p_3 - p_1)^2 - m^2}\right] \not{\epsilon}^*(p_3) u(p_1) \equiv i e^2 \bar{u}(p_4) \Gamma_1 u(p_1)$$

And for the other diagram,



$$
\begin{aligned}
: -i\mathcal{M}_2 &= e^2 \bar{u}(p_4) \not{\epsilon}^*(p_3) \left[\frac{i\left(\not{q}+m\right)}{q^2-m^2} \right] \not{\epsilon}(p_1) u(p_2) \\
&= ie^2 \bar{u}(p_4) \not{\epsilon}^*(p_3) \left[\frac{\not{p}_1+\not{p}_2+m}{(p_1+p_2)^2-m^2} \right] \not{\epsilon}(p_1) u(p_2) \equiv ie^2 \bar{u}(p_4) \Gamma_2 u(p_2)
\end{aligned}
$$

In total, there will be 4 terms in the spin-averaged matrix element,

$$
\langle |\mathcal{M}_1+\mathcal{M}_2|^2 \rangle = \langle \mathcal{M}_1\mathcal{M}_1^\dagger \rangle + \langle \mathcal{M}_1\mathcal{M}_2^\dagger \rangle + \langle \mathcal{M}_2\mathcal{M}_1^\dagger \rangle + \langle \mathcal{M}_2\mathcal{M}_2^\dagger \rangle
$$

which could be calculated term-by-term.

(a) $\langle \mathcal{M}_1\mathcal{M}_1^\dagger \rangle$

$$
\begin{aligned}
\langle \mathcal{M}_1\mathcal{M}_1^\dagger \rangle &= \frac{e^4}{4} \sum_{\lambda,\lambda'} \sum_{s,s'} \left[\bar{u}_a^s(p_4) \Gamma_{ab}^1(\lambda,\lambda') u_b^{s'}(p_1) \right] \left[\bar{u}_c^{s'}(p_1) \bar{\Gamma}_{cd}^1(\lambda,\lambda') u_d^s(p_4) \right] \\
&= \frac{e^4}{4} \sum_{\lambda,\lambda'} \left[\sum_s u_d^s(p_4) \bar{u}_a^s(p_4) \right] \left[\sum_{s'} u_b^{s'}(p_1) \bar{u}_c^{s'}(p_1) \right] \Gamma_{ab}^1(\lambda,\lambda') \bar{\Gamma}_{cd}^1(\lambda,\lambda') \\
&= \frac{e^4}{4} (\not{p}_4+m)_{da} (\not{p}_1+m)_{bc} \sum_{\lambda,\lambda'} \Gamma_{ab}^1(\lambda,\lambda') \bar{\Gamma}_{cd}^1(\lambda,\lambda') \qquad (13)
\end{aligned}
$$

Using the following expression for Γ^1,

$$
\Gamma_{ab}^1(\lambda,\lambda') \equiv \epsilon_\mu^\lambda \epsilon_\nu^{\lambda'*} (\gamma^\mu T \gamma^\nu)_{ab} \quad \text{where} \quad T \equiv \frac{\not{p}_3-\not{p}_1+m}{(p_3-p_1)^2-m^2}
$$

which gives the adjoint as,

$$
\begin{aligned}
\bar{\Gamma}^1 \equiv \gamma^0 \Gamma^{1\dagger} \gamma^0 = \epsilon_{\mu'}^{\lambda*} \epsilon_{\nu'}^{\lambda'} \gamma^0 \gamma^{\nu\dagger} T^\dagger \gamma^{\mu\dagger} \gamma^0 = \epsilon_{\mu'}^{\lambda*} \epsilon_{\nu'}^{\lambda'} \gamma^{\nu'} T \gamma^{\mu'} \\
\implies \bar{\Gamma}_{cd}^1(\lambda,\lambda') = \epsilon_{\mu'}^{\lambda*} \epsilon_{\nu'}^{\lambda'} (\gamma^{\nu'} T \gamma^{\mu'})_{cd}
\end{aligned}
$$

Plugging in the above expressions back into equation (13) yields,

$$
\begin{aligned}
\langle \mathcal{M}_1\mathcal{M}_1^\dagger \rangle &= \frac{e^4}{4} (\not{p}_4+m)_{da} (\not{p}_1+m)_{bc} \sum_{\lambda,\lambda'} \Gamma_{ab}^1(\lambda,\lambda') \bar{\Gamma}_{cd}^1(\lambda,\lambda') \\
&= \frac{e^4}{4} (\not{p}_4+m)_{da} (\not{p}_1+m)_{bc} \left[\sum_\lambda \epsilon_{\mu'}^{\lambda*} \epsilon_\mu^\lambda \right] \left[\sum_{\lambda'} \epsilon_\nu^{\lambda'*} \epsilon_{\nu'}^{\lambda'} \right] (\gamma^\mu T \gamma^\nu)_{ab} (\gamma^{\nu'} T \gamma^{\mu'})_{cd}
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{e^4}{4}(\not{p}_4 + m)_{da}(\not{p}_1 + m)_{bc}\left[g_{\mu\mu'}g_{\nu\nu'}\right](\gamma^\mu T\gamma^\nu)_{ab}(\gamma^{\nu'}T\gamma^{\mu'})_{cd} \\
&= \frac{e^4}{4}(\not{p}_4 + m)_{da}(\not{p}_1 + m)_{bc}(\gamma^\mu T\gamma^\nu)_{ab}(\gamma_\nu T\gamma_\mu)_{cd} \\
&= \frac{e^4}{4}\mathrm{Tr}\left[\gamma^\mu T\gamma^\nu(\not{p}_1 + m)\gamma_\nu T\gamma_\mu(\not{p}_4 + m)\right]
\end{aligned}
$$

Fully expanding T, one could finally obtain

$$
\langle \mathcal{M}_1\mathcal{M}_1^\dagger\rangle = \frac{e^4}{4\left[(p_3 - p_1)^2 - m^2\right]^2}\mathrm{Tr}\left[\gamma^\mu(\not{p}_3 - \not{p}_1 + m)\gamma^\nu(\not{p}_1 + m)\gamma_\nu(\not{p}_3 - \not{p}_1 + m)\gamma_\mu(\not{p}_4 + m)\right] \tag{14}
$$

(b) $\langle \mathcal{M}_1\mathcal{M}_2^\dagger\rangle$

(c) $\langle \mathcal{M}_2\mathcal{M}_1^\dagger\rangle$

(d) $\langle \mathcal{M}_2\mathcal{M}_2^\dagger\rangle$