Probability Theory and Statistics 3, Assignment 2

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Exercise 1

Objective: $\alpha = P\big[\log(\tilde{X}_{0.3}) \leq 0.477 \;\big|\; X_i \sim GAM(\theta = 2, k = 2)\big]$

Assume that "H_0: $\theta = 2$ and $k = 2$" is true.

Then $X_i \sim GAM(2,2)$ and $f(x) = \frac{1}{4} x e^{-\frac{x}{2}}$, $(x > 0)$

$f(x)$ is continuous and nonzero on the support of x.

$\Rightarrow$ Use Theorem 7.5.1 :

$\tilde{X}_{0.3} = X_{39:130}$ is asymptotically normal with mean $x_{0.3}$ and variance $\frac{c^2}{130}$ where

$$c^2 = \frac{0.3 \cdot 0.7}{\big[f(x_{0.3})\big]^2} \tag{1}$$

First of all, we need $x_{0.3}$ now.

$$X_i \sim GAM(2,2) = \chi^2(4) \quad \Rightarrow \quad x_{0.3} \overset{\text{App. C5}}{\approx} x_{0.301} = 2.2 \tag{2}$$

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Secondly, substitute (2) to (1) to get c^2

$$c^2 = \frac{0.3 \cdot 0.7}{\left[f(2.2)\right]^2} = \frac{0.21}{0.3025 \cdot e^{-2.2}} \tag{3}$$

Thus, $\tilde{X}_{0.3}$ is asymptotically normal with asymptotic mean $x_{0.3} \approx 2.2$ and asymptotic variance $\frac{c^2}{130} = \frac{0.21}{130 \cdot (0.3025 \cdot e^{-2.2})}$.
From now on, c will be written to denote $\sqrt{c^2} = \sqrt{\frac{0.21}{0.3025 \cdot e^{-2.2}}}$.
Then, by Central Limit Theorem,

$$Z_{130} = \sqrt{130} \cdot \frac{(\tilde{X}_{0.3} - 2.2)}{c} \xrightarrow{d} Z \sim N(0,1) \tag{4}$$

$$\begin{aligned}
\alpha &= P\left[\log(\tilde{X}_{0.3}) \leq 0.477 \middle| X_i \sim GAM(\theta = 2, k = 2)\right] && (5) \\
&\quad H_0 \text{ has been assumed to be true at the beginning.} && (6) \\
&= P\left[\log(\tilde{X}_{0.3}) \leq 0.477\right] && (7) \\
&= P\left[\tilde{X}_{0.3} \leq e^{0.477}\right] && (8) \\
&= P\left[\sqrt{130} \cdot \frac{(\tilde{X}_{0.3} - 2.2)}{c} \leq \sqrt{130} \cdot \frac{(e^{0.477} - 2.2)}{c}\right] && (9) \\
&\quad \text{Use (4)} && (10) \\
&\approx P\left[Z \leq -2.68\right] = 1 - P\left[Z \leq 2.68\right] \overset{\text{App. C3}}{=} 1 - 0.9963 = 0.0037 && (11)
\end{aligned}$$

$\therefore$ The probability of the type I error $\alpha \approx 0.0037$

Exercise 2

a.

$$E[X] = \int_0^{\theta_2} \frac{\theta_1}{(\theta_2)^{\theta_1}} \cdot x^{\theta_1} dx \;=\; \frac{\theta_1 \cdot \theta_2}{(\theta_1 + 1)} \tag{12}$$

$$\Rightarrow\; \frac{\tilde{\theta}_1 \cdot \theta_2}{(\tilde{\theta}_1 + 1)} = \bar{X}_n \tag{13}$$

$$\Rightarrow\; \tilde{\theta}_1 = \frac{\bar{X}_n}{\theta_2 - \bar{X}_n} \tag{14}$$

b.

From the subquestion (a), we have got

$$\text{Equation (12)} \;\Rightarrow\; \theta_1 = \frac{\mu}{\theta_2 - \mu}\;,\;\; (\theta_2 \neq \mu) \tag{15}$$

$$\text{Equation (14)} \;\Rightarrow\; \tilde{\theta}_1 = \frac{\bar{X}_n}{\theta_2 - \bar{X}_n}\;,\;\; (\theta_2 \neq \bar{X}_n) \tag{16}$$

We are going to show that the convergence in probability of each nominator and denominator in (16).

First, Theorem 7.6.2 (Law of Large Numbers) can be applied to the nominator, since the distribution of X has finite mean and variance.

$$\therefore \bar{X}_n \xrightarrow{P} \mu \tag{17}$$

Next, Theorem 7.7.3 can be used for the denominator.

$$\bar{X}_n \xrightarrow{P} \mu \tag{18}$$

$$\text{By Theorem 7.7.3.1} \tag{19}$$

$$\Rightarrow \; \theta_2 - \bar{X}_n \xrightarrow{P} \theta_2 - \mu \tag{20}$$

$$\text{By Theorem 7.7.3.4} \tag{21}$$

$$\Rightarrow \; \frac{1}{\theta_2 - \bar{X}_n} \xrightarrow{P} \frac{1}{\theta_2 - \mu} \; , \; (\theta_2 \neq \mu, \bar{X}_n) \tag{22}$$

Now we can use Theorem 7.7.3.2 with (17) and (22).

$$\frac{\bar{X}_n}{\theta_2 - \bar{X}_n} \xrightarrow{P} \frac{\mu}{\theta_2 - \mu} \; , \; (\theta_2 \neq \mu, \bar{X}_n) \tag{23}$$

$$\text{Substitute (15), (16) to (23)} \tag{24}$$

$$\therefore \tilde{\theta}_1 \xrightarrow{P} \theta_1 \tag{25}$$

c.

Likelihood function

$$L(\theta_1, \theta_2) \; = \; \prod_{i=1}^{n} f_X(x_i; \theta_1, \theta_2) \; = \; \left[\frac{\theta_1}{(\theta_2)^{\theta_1}} \right]^n (x_1 \times \cdots \times x_n)^{\theta_1 - 1} \tag{26}$$

Log-likelihood function ('log' is the natural logarithm)

$$\begin{aligned} l(\theta_1, \theta_2) &= \log \left[L(\theta_1, \theta_2) \right] && (27) \\ &= n \log \left[\frac{\theta_1}{(\theta_2)^{\theta_1}} \right] + (\theta_1 - 1) \sum_{i=1}^{n} \log x_i && (28) \\ &= n \log(\theta_1) - n\theta_1 \log(\theta_2) + (\theta_1 - 1) \sum_{i=1}^{n} \log x_i && (29) \end{aligned}$$

Take the partial derivative with respect to θ_1 and θ_2, respectively, to find out $\hat{\theta}_1$ and $\hat{\theta}_2$

i) The first order condition for θ_1

$$\begin{align}
\frac{d}{d\theta_1} l(\theta_1, \theta_2) &= \frac{n}{\theta_1} - n\log(\theta_2) + \sum_{i=1}^{n} \log x_i \overset{\text{f.o.c}}{=} 0 \tag{30}\\
\Rightarrow \frac{n}{\theta_1} &= n\log(\theta_2) - \sum_{i=1}^{n} \log x_i \tag{31}\\
\Rightarrow \frac{1}{\theta_1} &= \frac{n\log(\theta_2) - \sum_{i=1}^{n} \log x_i}{n} \tag{32}\\
\Rightarrow \hat{\theta}_1 &= \frac{n}{n\log(\theta_2) - (\sum_{i=1}^{n} \log x_i)} \tag{33}
\end{align}$$

ii) The second order condition for θ_1

$$\frac{d^2}{d\theta_1^2} l(\theta_1, \theta_2) = -\frac{n}{(\theta_1)^2} < 0 \quad (\because n > 0) \tag{34}$$

$$\therefore \hat{\theta}_1 \text{ in (33) corresponds to a maximum.} \tag{35}$$

iii) The first order condition for θ_2

$$\frac{d}{d\theta_2} l(\theta_1, \theta_2) = -\frac{n\theta_1}{\theta_2} < 0 \quad (\because n, \theta_1, \theta_2 > 0) \tag{36}$$

$$\therefore \text{The first order condition is never satisfied.} \tag{37}$$

The first order condition cannot be used for getting $\hat{\theta}_2$. But through (36), we can see that $l(\theta_1, \theta_2)$ becomes larger as θ_2 gets smaller. Therefore, the maximum likelihood estimate(MLE) $\hat{\theta}_2$ should be as small as possible.

On the other hand, Question 2 has given the information $0 < x < \theta_2$. Thus, $\hat{\theta}_2$ should be greater than x as much as possible.

Now let's suppose that there are n samples, $X_1, X_2, ..., X_n$. As we combine two requirements mentioned above, we can take the largest order statistic($X_{n:n}$) as the MLE of θ_2. Because $X_{n:n}$ is the smallest value that we can take to estimate θ_2 considering the samples, and at the same time, it is the greatest value among the samples.

$$\therefore \hat{\theta}_2 = X_{n:n} \tag{38}$$

d.

from c, we got the following equation

$$\hat{\theta}_1 = \frac{n}{n\log(\theta_2) - (\sum_{i=1}^{n} \log x_i)} \tag{39}$$

$$n = 4 \tag{40}$$

$$\log(x_1 \times x_2 \times x_3 \times x_4) = \log(11.6325) \tag{41}$$

$$\Rightarrow \hat{\theta}_1 = \frac{4}{4\log(\theta_2) - \log(11.6325)} \tag{42}$$

If we put the sample in order

$$x_{1:4} = 0.9,\ x_{2:4} = 1.1,\ x_{3:4} = 2.5,\ x_{4:4} = 4.7 \tag{43}$$

$$\text{From (38)},\ \ \hat{\theta}_2 = X_{n:n} \tag{44}$$

$$\therefore \hat{\theta}_2 = x_{4:4} = 4.7 \tag{45}$$

(45) can be substituted for θ_2 of (42)

$$\therefore \hat{\theta}_1 = \frac{4}{4\log(4.7) - \log(11.6325)} \approx 1.0705 \tag{46}$$