Probability Theory and Statistics 3, Assignment 3

Jiyeon Kim *, Younjoo Mo †

November 13, 2020

Exercise 1

a.

$$Y_i \mid X_i = x_i \sim N(\beta_0 + \beta_1 x_i, \sigma^2) \tag{1}$$

Likelihood function

$$L(\beta_0, \beta_1, \sigma) = \prod_{i=1}^{n} f_{X,Y}(x, y) \ = \ \prod_{i=1}^{n} f_{Y|X}(y \mid x \, ; \, \beta_0, \beta_1, \sigma) f_X(x) \tag{2}$$

$$= \left[\frac{1}{(\sqrt{2\pi}\sigma)^n} \exp\left(-\frac{\sum_{i=1}^{n}(y_i - \beta_0 - \beta_1 x_i)^2}{2\sigma^2} \right) \prod_{i=1}^{n} f_X(x) \right] \tag{3}$$

Log-likelihood function ('log' is the natural logarithm)

$$l(\beta_0, \beta_1, \sigma) \ = \ \log\left[L(\beta_0, \beta_1, \sigma)\right] \tag{4}$$

$$= -n \log(\sqrt{2\pi}\sigma) - \frac{\sum_{i=1}^{n}(y_i - \beta_0 - \beta_1 x_i)^2}{2\sigma^2} + \sum_{i=1}^{n} \log f_X(x) \tag{5}$$

*Student number: 12470236
†Studentnumber: 12475440

b. To find the MLEs of β_0, β_1 and σ, we would check the first order condition for each of the unknown parameters as the first step. Then the term including $f_X(x)$ immediately disappears as we take the partial derivative of the log-likelihood function. Because β_0, β_1 and σ are not a component of $f_X(x)$. In other words, $f_X(x)$ is independent of those parameters. Therefore we do not need to know what $f_X(x)$ is, in order to find the MLEs of β_0, β_1 and σ.

c.

i) The first order condition for β_0

$$\frac{\partial}{\partial \beta_0} l(\beta_0, \beta_1, \sigma) \tag{6}$$

Substitute (5) and omit the terms not including β_0 (7)

$$= \frac{\partial}{\partial \beta_0}\left[-\frac{\sum_{i=1}^{n}(y_i - \beta_0 - \beta_1 x_i)^2}{2\sigma^2} \right] \tag{8}$$

$$= \frac{\partial}{\partial \beta_0}\left[-\frac{1}{2\sigma^2}\left(\sum_{i=1}^{n} {y_i}^2 - 2\beta_0 \sum_{i=1}^{n} y_i - 2\beta_1 \sum_{i=1}^{n} x_i y_i + n{\beta_0}^2 + 2\beta_0\beta_1 \sum_{i=1}^{n} x_i + {\beta_1}^2 \sum_{i=1}^{n} {x_i}^2 \right) \right] \overset{\text{f.o.c}}{=} 0 \tag{9}$$

$$\Rightarrow -\sum_{i=1}^{n} y_i + n\beta_0 + \beta_1 \sum_{i=1}^{n} x_i = 0 \tag{10}$$

$$\Rightarrow \hat{\beta}_0 = \frac{\sum_{i=1}^{n} y_i - \beta_1 \sum_{i=1}^{n} x_i}{n} = \bar{Y}_n - \beta_1 \bar{X}_n \tag{11}$$

ii) The first order condition for β_1

$$\frac{\partial}{\partial \beta_1} l(\beta_0, \beta_1, \sigma) \tag{12}$$

$$\text{Substitute (5) and omit the terms not including } \beta_1 \tag{13}$$

$$= \frac{\partial}{\partial \beta_1}\left[-\frac{\sum_{i=1}^{n}(y_i - \beta_0 - \beta_1 x_i)^2}{2\sigma^2}\right] \tag{14}$$

$$= \frac{\partial}{\partial \beta_1}\left[-\frac{1}{2\sigma^2}\left(\sum_{i=1}^{n} {y_i}^2 - 2\beta_0 \sum_{i=1}^{n} y_i - 2\beta_1 \sum_{i=1}^{n} x_i y_i + n{\beta_0}^2 + 2\beta_0\beta_1 \sum_{i=1}^{n} x_i + {\beta_1}^2 \sum_{i=1}^{n} {x_i}^2\right)\right] \overset{\text{f.o.c}}{=} 0 \tag{15}$$

$$\Rightarrow -\sum_{i=1}^{n} x_i y_i + \beta_0 \sum_{i=1}^{n} x_i + \beta_1 \sum_{i=1}^{n} {x_i}^2 = 0 \tag{16}$$

$$\text{(11) can be substituted for } \beta_0 \text{ of (16)} \tag{17}$$

$$\Rightarrow -\sum_{i=1}^{n} x_i y_i + (\bar{Y}_n - \beta_1 \bar{X}_n) \sum_{i=1}^{n} x_i + \beta_1 \sum_{i=1}^{n} {x_i}^2 = 0 \tag{18}$$

$$\Rightarrow \hat{\beta}_1 = \frac{-\sum_{i=1}^{n} x_i y_i + \bar{Y}_n \sum_{i=1}^{n} x_i}{\bar{X}_n \sum_{i=1}^{n} x_i - \sum_{i=1}^{n} {x_i}^2} \tag{19}$$

iii) The first order condition for σ

$$\frac{\partial}{\partial \sigma} l(\beta_0, \beta_1, \sigma) \tag{20}$$

$$\text{Substitute (5) and omit the terms not including } \sigma \tag{21}$$

$$= \frac{\partial}{\partial \sigma}\left[-n\log(\sqrt{2\pi}\sigma) - \frac{\sum_{i=1}^{n}(y_i - \beta_0 - \beta_1 x_i)^2}{2\sigma^2}\right] \overset{\text{f.o.c}}{=} 0 \tag{22}$$

$$\Rightarrow -\frac{n}{\sigma} + \frac{\sum_{i=1}^{n}(y_i - \beta_0 - \beta_1 x_i)^2}{\sigma^3} = 0 \tag{23}$$

$$\Rightarrow \sigma^2 = \frac{\sum_{i=1}^{n}(y_i - \beta_0 - \beta_1 x_i)^2}{n} \tag{24}$$

$$\hat{\sigma} = \sqrt{\frac{\sum_{i=1}^{n}(y_i - \beta_0 - \beta_1 x_i)^2}{n}} \tag{25}$$

d.

<code>

```
df<-read.table("PTS3_assignment3_data.txt")
x<-df[,1]
y<-df[,2]
beta1<- (-sum(x*y)+mean(y)*sum(x))/(mean(x)*sum(x)-sum(x^2))
beta1
beta0<- mean(y)-beta1*mean(x)
beta0
sigma<-sqrt(sum((y-beta0-beta1*x)^2)/50)
sigma
```

If we run the code, we get the following result.

$\beta_1 = 1.546367$

$\beta_0 = -1.47139$

$\sigma = 1.191652$

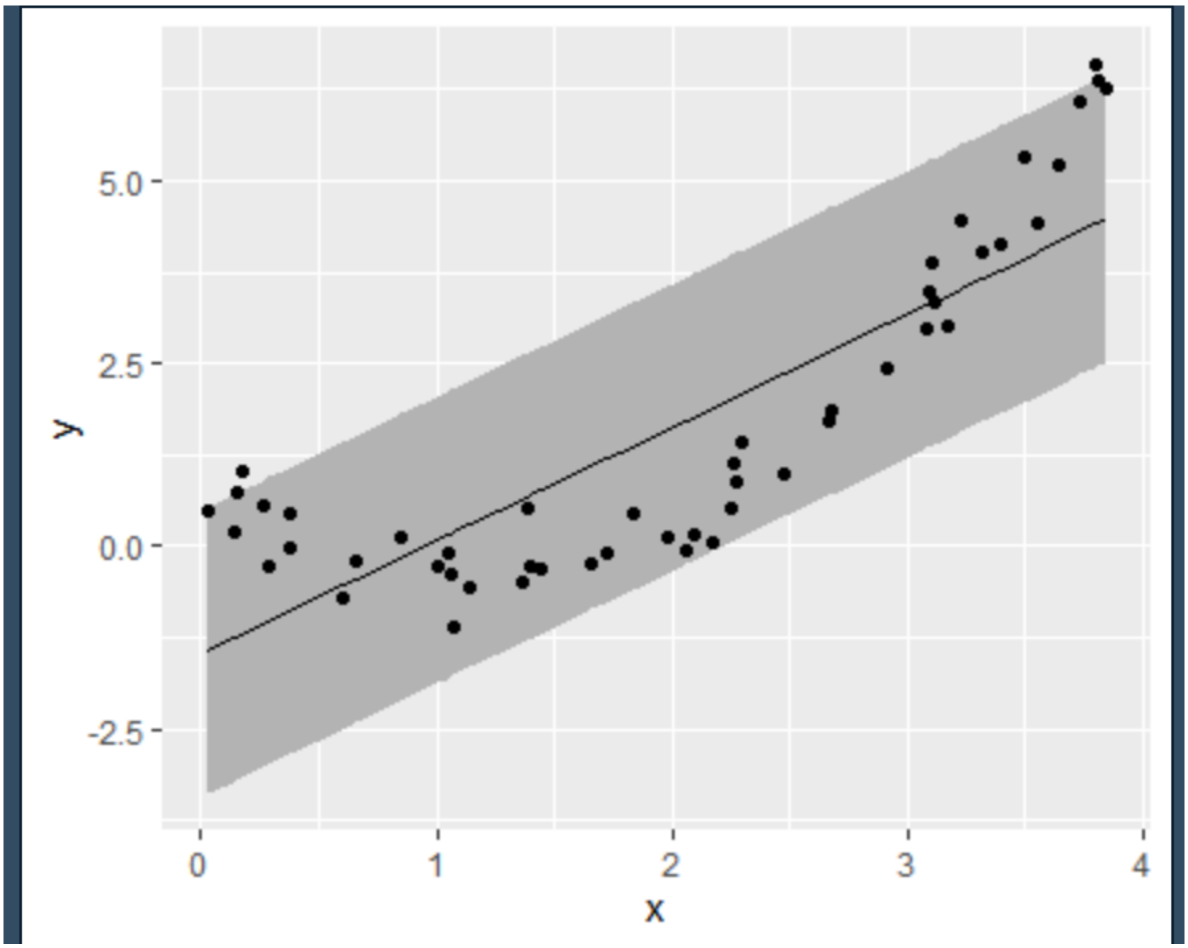
e. In order to compute 95-percent confidence interval for Y, we used $\sigma \approx \hat{\sigma} = 1.191652$

$$P(-1.645 < \frac{Y \mid (X = x) - (\beta_0 + \beta_1 x_i)}{\sigma} < 1.645) \tag{26}$$

$$\Rightarrow P(\beta_0 + \beta_1 x_i - 1.645\sigma < Y \mid (X = x) < \beta_0 + \beta_1 x_i + 1.645\sigma) \tag{27}$$

$$\text{Let } \hat{y} = \hat{\beta}_0 + \hat{\beta}_1 x_i \tag{28}$$

$$\Rightarrow P(\hat{y} - 1.645\hat{\sigma} < Y \mid (X = x) < \hat{y} + 1.645\hat{\sigma}) \tag{29}$$



f. Likelihood function

$$Y_i \mid X_i = x_i \sim N(\beta_0 + \beta_1 x_i + \beta_2 x_i^2, \sigma^2) \tag{30}$$

$$L(\beta_0, \beta_1, \beta_2, \sigma) = \prod_{i=1}^{n} f_{X,Y}(x, y; \beta_0, \beta_1, \beta_2, \sigma) \; = \; \prod_{i=1}^{n} f_{Y|X}(y \mid x; \beta_0, \beta_1, \beta_2, \sigma) f_X(x) \tag{31}$$

$$= \left[\frac{1}{(\sqrt{2\pi}\sigma)^n} \exp\left(-\frac{\sum_{i=1}^{n}(y_i - \beta_0 - \beta_1 x_i - \beta_2 x_i^2)^2}{2\sigma^2} \right) \prod_{i=1}^{n} f_X(x) \right] \tag{32}$$

Log-likelihood function ('log' is the natural logarithm)

$$l(\beta_0, \beta_1, \beta_2, \sigma) \; = \; \log\left[L(\beta_0, \beta_1, \beta_2, \sigma) \right] \tag{33}$$

$$= -n \log(\sqrt{2\pi}\sigma) - \frac{\sum_{i=1}^{n}(y_i - \beta_0 - \beta_1 x_i - \beta_2 x_i^2)^2}{2\sigma^2} + \sum_{i=1}^{n} \log f_X(x) \tag{34}$$

i) The first order condition for β_0

$$\frac{\partial}{\partial \beta_0} l(\beta_0, \beta_1, \beta_2, \sigma) \tag{35}$$

$$= \frac{\partial}{\partial \beta_0} \left[-\frac{\sum_{i=1}^{n}(y_i - \beta_0 - \beta_1 x_i - \beta_2 x_i^{)2}}{2\sigma^2} \right] \tag{36}$$

$$= \frac{\partial}{\partial \beta_0} \left[-\frac{1}{2\sigma^2} \left(\sum_{i=1}^{n} {y_i}^2 - 2\beta_0 \sum_{i=1}^{n} y_i - 2\beta_1 \sum_{i=1}^{n} x_i y_i - 2\beta_2 \sum_{i=1}^{n} x_i^2 y_i \right. \right. \tag{37}$$

$$\left. \left. + n{\beta_0}^2 + \beta_1^2 \sum_{i=1}^{n} x_i^2 + \beta_2^2 \sum_{i=1}^{n} x_i^4 + 2\beta_0\beta_2 \sum_{i=1}^{n} x_i^2 + 2\beta_1\beta_2 \sum_{i=1}^{n} x_i^3 + 2\beta_0\beta_1 \sum_{i=1}^{n} x_i \right) \right] \overset{\text{f.o.c}}{=} 0 \tag{38}$$

$$\Rightarrow -\sum_{i=1}^{n} y_i + n\beta_0 + \beta_1 \sum_{i=1}^{n} x_i + \beta_2 \sum_{i=1}^{n} x_i^2 = 0 \tag{39}$$

$$\Rightarrow \hat{\beta}_0 = \frac{\sum_{i=1}^{n} y_i - \beta_1 \sum_{i=1}^{n} x_i - \beta_2 \sum_{i=1}^{n} x_i^2}{n} = \bar{y} - \beta_1 \bar{x} - \beta_2 \bar{x}^2 \tag{40}$$

$$\therefore \Rightarrow \bar{y} = \hat{\beta}_0 + \hat{\beta}_1 \bar{x} + \hat{\beta}_2 \bar{x}^2 \tag{41}$$

ii) The first order condition for β_1

$$\frac{\partial}{\partial\beta_1} l(\beta_0,\beta_1,\beta_2,\sigma) \tag{42}$$

$$= \frac{\partial}{\partial\beta_1}\left[-\frac{\sum_{i=1}^{n}(y_i-\beta_0-\beta_1 x_i-\beta_2 c_i^2)^2}{2\sigma^2}\right] \tag{43}$$

$$= \frac{\partial}{\partial\beta_1}\left[-\frac{1}{2\sigma^2}\left(\sum_{i=1}^{n} {y_i}^2 - 2\beta_0\sum_{i=1}^{n} y_i - 2\beta_1\sum_{i=1}^{n} x_i y_i - 2\beta_2\sum_{i=1}^{n} x_i^2 y_i\right.\right. \tag{44}$$

$$\left.\left.+n{\beta_0}^2+\beta_1^2\sum_{i=1}^{n} x_i^2+\beta_2^2\sum_{i=1}^{n} x_i^4+2\beta_0\beta_2\sum_{i=1}^{n} x_i^2+2\beta_1\beta_2\sum_{i=1}^{n} x_i^3+2\beta_0\beta_1\sum_{i=1}^{n} x_i\right)\right] \overset{\text{f.o.c}}{=} 0 \tag{45}$$

$$\Rightarrow -\sum_{i=1}^{n} x_i y_i + \beta_1\sum_{i=1}^{n} x_i^2 + \beta_0\sum_{i=1}^{n} x_i + \beta_2\sum_{i=1}^{n} {x_i}^3 = 0 \tag{46}$$

$$\Rightarrow \hat{\beta}_1 = \frac{\sum_{i=1}^{n} x_i y_i - \beta_0\sum_{i=1}^{n} x_i - \beta_2\sum_{i=1}^{n} x_i^3}{\sum_{i=1}^{n} x_i^2} = \frac{\overline{xy}-\beta_0\bar{x}-\beta_2\bar{x}^3}{\bar{x}^2} \tag{47}$$

$$\therefore \Rightarrow \overline{xy} = \hat{\beta}_0\bar{x}+\hat{\beta}_1\bar{x}^2+\hat{\beta}_2\bar{x}^3 \tag{48}$$

iii) The first order condition for β_2

$$\frac{\partial}{\partial \beta_1} l(\beta_0, \beta_1, \beta_2, \sigma) \tag{49}$$

$$= \frac{\partial}{\partial \beta_1}\left[-\frac{\sum_{i=1}^n (y_i - \beta_0 - \beta_1 x_i - \beta_2 c_i^2)^2}{2\sigma^2}\right] \tag{50}$$

$$= \frac{\partial}{\partial \beta_1}\left[-\frac{1}{2\sigma^2}\left(\sum_{i=1}^n {y_i}^2 - 2\beta_0 \sum_{i=1}^n y_i - 2\beta_1 \sum_{i=1}^n x_i y_i - 2\beta_2 \sum_{i=1}^n x_i^2 y_i\right.\right. \tag{51}$$

$$\left.\left. + n{\beta_0}^2 + \beta_1^2 \sum_{i=1}^n x_i^2 + \beta_2^2 \sum_{i=1}^n x_i^4 + 2\beta_0\beta_2 \sum_{i=1}^n x_i^2 + 2\beta_1\beta_2 \sum_{i=1}^n x_i^3 + 2\beta_0\beta_1 \sum_{i=1}^n x_i\right)\right] \overset{\text{f.o.c}}{=} 0 \tag{52}$$

$$\Rightarrow -\sum_{i=1}^n x_i^2 y_i + \beta_2 \sum_{i=1}^n x_i^4 + \beta_1 \sum_{i=1}^n x_i^3 + \beta_0 \sum_{i=1}^n {x_i}^2 = 0 \tag{53}$$

$$\Rightarrow \hat{\beta}_2 = \frac{\sum_{i=1}^n x_i^2 y_i - \beta_1 \sum_{i=1}^n x_i^3 - \beta_0 \sum_{i=1}^n x_i^2}{\sum_{i=1}^n x_i^4} = \frac{x^{\bar{2}} y - \beta_1 \bar{x}^3 - \beta_0 \bar{x}^2}{\bar{x}^4} \tag{54}$$

$$\therefore \Rightarrow \overline{x^2 y} \;== \hat{\beta}_0 \bar{x}^2 + \hat{\beta}_1 \bar{x}^3 + \hat{\beta}_2 \bar{x}^4 \tag{55}$$

we get the matrix shown in the question.

Next, we are going to prove $\hat{\sigma} = \sqrt{\frac{\sum_{i=1}^n (y_i - \beta_0 - \beta_1 x_i - \beta_2 x_i^2)^2}{n}}$

iv) The first order condition for $\hat{\sigma}$

$$\frac{\partial}{\partial \sigma} l(\beta_0, \beta_1, \beta_2, \sigma) \tag{56}$$

Substitute (34) and omit the terms not including σ (57)

$$= \frac{\partial}{\partial \sigma}\left[-n\log(\sqrt{2\pi}\sigma) - \frac{\sum_{i=1}^n (y_i - \beta_0 - \beta_1 x_i - \beta_2 x_i^2)^2}{2\sigma^2}\right] \overset{\text{f.o.c}}{=} 0 \tag{58}$$

$$\Rightarrow -\frac{n}{\sigma} + \frac{\sum_{i=1}^n (y_i - \beta_0 - \beta_1 x_i - \beta_2 x_i^2)^2}{\sigma^3} = 0 \tag{59}$$

$$\Rightarrow \sigma^2 = \frac{\sum_{i=1}^n (y_i - \beta_0 - \beta_1 x_i - \beta_2 x_i^2)^2}{n} \tag{60}$$

$$\hat{\sigma} = \sqrt{\frac{\sum_{i=1}^n (y_i - \beta_0 - \beta_1 x_i - \beta_2 x_i^2)^2}{n}} \tag{61}$$

The second part of Subquestion 1-f : Compute $\hat{\beta}_0, \hat{\beta}_1, \hat{\beta}_2, \sigma$

```
<code>
df<-read.table("PTS3_assignment3_data.txt")
x<-df[,1]
y<-df[,2]

# Use the fist given expression to get the beta hats
# Create the matrix at the left side
mean.of.x <- c(1, mean(x), mean(x^2),
               mean(x), mean(x^2), mean(x^3),
               mean(x^2), mean(x^3), mean(x^4))
matA <- matrix(mean.of.x, nrow=3)

# Create the matrix at the right side
mean.of.xy <- c(mean(y), mean(x*y), mean(y*x^2))
matB <- matrix(mean.of.xy, nrow=3)

# Beta hats
beta <- solve(matA) %*% matB
beta

# Use the second given expression to get the sigma hat
# Sigma hat
sigma <- sqrt(sum((y - beta[1] - beta[2]*x - beta[3]*x^2)^2)/50)
```

If we run the code, we get the following result.

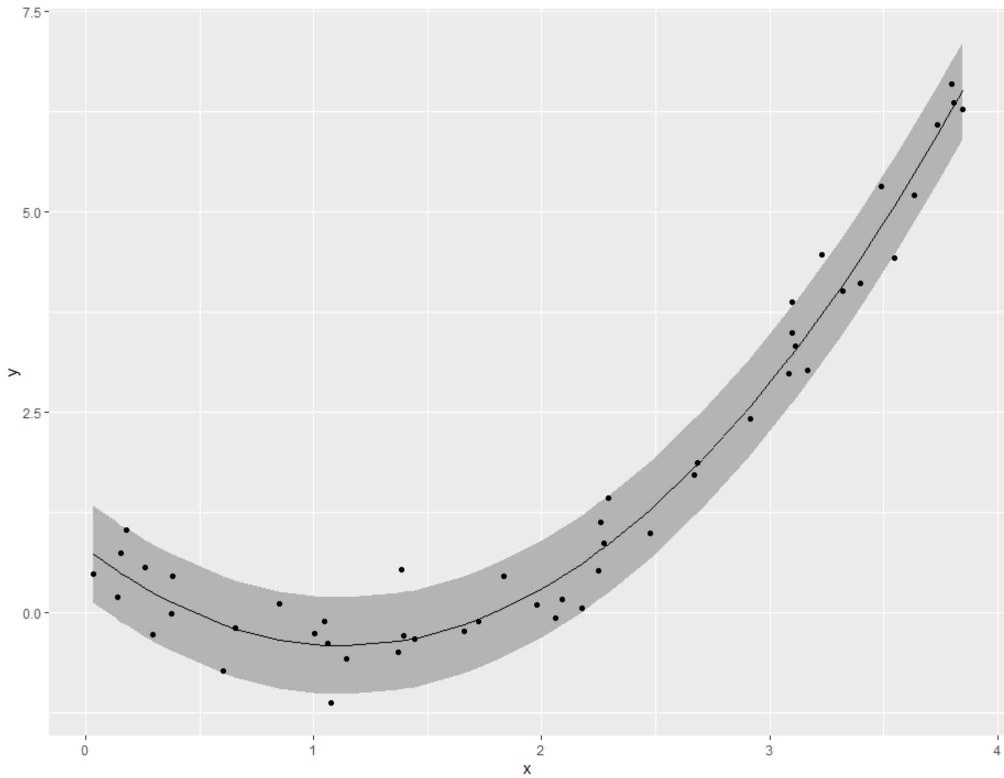
$\beta_0 = 0.7963816$

$\beta_1 = -2.1290201$

$\beta_2 = 0.9402489$

$\sigma = 0.3672581$

The Last part of Subquestion 1-f : Create a plot for the quadratic model



Exercise 2

a.

$$X_i \sim POI(\mu) \tag{62}$$

$$E[X] \ = \ Var[X] \ = \ \mu \tag{63}$$

$$\text{The MLE of } \mu = \bar{X}_n \tag{64}$$

$$X_i \sim POI(\mu) \approx POI(\bar{X}_n) \tag{65}$$

$$\hat{p} = \frac{Y_n}{n} = e^{-\bar{X}_n} + \bar{X}_n \cdot e^{-\bar{X}_n} \tag{66}$$

$$E[\hat{p}] = E[e^{-\bar{X}_n} + \bar{X}_n \cdot e^{-\bar{X}_n}] \tag{67}$$

$$= E[e^{-\bar{X}_n}] + E[\bar{X}_n] \cdot E[e^{-\bar{X}_n}] \tag{68}$$

$$\text{By CLT} \tag{69}$$

$$\sqrt{n} \cdot \frac{\bar{X}_n - \mu}{\sigma} \xrightarrow{\text{d}} Z \sim N(0,1) \tag{70}$$

$$\Rightarrow \text{MGF of Z is } M_X(t) = E[e^{Zt}] = e^{\frac{1}{2}t^2} \tag{71}$$

$$\Rightarrow \text{When } n \to \infty, \quad E\left[e^{\sqrt{n} \cdot \frac{\bar{X}_n - \mu}{\sigma} \cdot t}\right] = e^{\frac{1}{2}t^2} \tag{72}$$

$$\text{Substitute } t = -\frac{\sigma}{\sqrt{n}} \text{ to (72)} \tag{73}$$

$$\Rightarrow E[e^{-\bar{X}_n + \mu}] = E[e^{-\bar{X}_n}] \cdot E[e^{\mu}] = E[e^{-\bar{X}_n}] \cdot e^{\mu} = e^{\frac{\sigma^2}{2n}} \tag{74}$$

$$\Rightarrow E[e^{\bar{X}_n}] = e^{\frac{\sigma^2}{2n} - \mu} \tag{75}$$

$$\frac{\sigma^2}{2n} \to 0 \text{ as } n \to \infty \tag{76}$$

$$\therefore E[e^{\bar{X}_n}] = e^{-\mu} \tag{77}$$

$$\text{Substitute (77) to (68)} \tag{78}$$

$$E[\hat{p}] = e^{-\mu} + \mu \cdot e^{-\mu} = e^{-\mu}(1+\mu) = p \tag{79}$$

$$\therefore \hat{p} \text{ is an unbiased estimator of } p. \tag{80}$$

b.

The formula of CRLB is

$$\frac{[\tau^{'}(\mu)]^2}{-nE[\frac{\partial^2 \log f_x(x;\mu)}{\partial \mu^2}]} \tag{81}$$

$\because$ When certain differentiability requirements holds,

$$E\left[\left(\frac{\partial}{\partial\theta}\log f_X(x;\mu)\right)^2\right] \; = \; -E\left[\frac{\partial^2}{\partial\theta^2}\log f_X(x;\mu)\right] \tag{82}$$

quantity equation is as follows

$$[\tau(\mu)] \; = \; P[X \leq 1] \; = \; e^{-\mu}(1+\mu) \tag{83}$$

Take the partial derivative respect to μ

$$[\tau^{'}(\mu)] \; = \; -e^{-\mu}(1+\mu) + e^{-\mu} \; = \; -\mu e^{-\mu} \tag{84}$$

We get the discrete pdf $f_X(X;\mu)$ from the table B.1

$$f_X(X;\mu) \; = \; \frac{e^{-\mu}\mu^X}{X!} \tag{85}$$

$$log f_X(X;\mu) \; = \; -\mu + X\log\mu - \log X! \tag{86}$$

Take the partial derivative respect to μ

$$\frac{\partial}{\partial \mu} \log f_X(X;\mu) \;=\; -1 + \frac{X}{\mu} \tag{87}$$

The second order condition for μ

$$\frac{\partial^2}{\partial \mu^2} \log f_X(X;\mu) \;=\; -\frac{X}{\mu^2} \tag{88}$$

$$E[\frac{\partial^2}{\partial \mu^2} \log f_X(X;\mu)] \;=\; \frac{-E[X]}{\mu^2} \;=\; -\frac{1}{\mu} \tag{89}$$

(8) $\rightarrow \because E(X) = \mu$ in poisson distribution

$$Var(T) \geq \frac{[\tau^{'}(\mu)]^2}{-nE[\frac{\partial^2}{\partial \theta^2} \log f_X(X;\mu)]} \;=\; \frac{\mu^3 exp^{-2\mu}}{n} \tag{90}$$

c.

From the previous sub question, we got the following formulas

$$f_X(X;\mu) \;=\; \frac{e^{-\mu}\mu^X}{X!} \tag{91}$$

'Log' is natural logarithm.

$$log f_X(X;\mu) \;=\; -\mu + X\log\mu - \log X! \tag{92}$$

$$\frac{\partial}{\partial\mu}\log f_X(X;\mu) \;=\; -1+\frac{X}{\mu} \tag{93}$$

We are going to use definition that If unbiased estimator attains CRLB, there should be a linear function of $\sum_{i=1}^{n}\frac{\partial}{\partial\mu}\log f_X(X;\mu)$

$$\sum_{i=1}^{n}\frac{\partial}{\partial\mu}\log f_X(x;\mu) \;=\; (-1+\frac{X_1}{\mu})+(-1+\frac{X_1}{\mu})+\cdots+(-1+\frac{X_n}{\mu}) \tag{94}$$

$$= -n+\frac{X_1+X_2+\cdots+X_n}{\mu} \tag{95}$$

$$[\tau(\mu)] \;=\; P[X\le 1] \;=\; e^{-\mu}(1+\mu) \tag{96}$$

Substitute $\hat{p}$ in equation (19).

$$[\tau(\hat{p})] \;=\; \exp^{-\hat{p}}(1+\hat{p}) \;=\; \exp^{-\bar{X}_n}(1+\bar{X}_n) \tag{97}$$

$$\neq c\left[-n+\frac{X_1+X_2+\cdots+X_n}{\mu}\right], c\in R\backslash\{0\} \tag{98}$$

There is no coefficient c that makes equation (20) and (21) linear

$\therefore\ \hat{p}$ doesnt attain CRLB for unbiased estimators of p.

d.

From the sub-question (a), $\hat{p}$ is an unbiased estimator of p. In addition to this, if $Var(\hat{p})$ is equal to CRLB, then we can tell $\hat{p}$ is an UMVUE of p. It happens only when $\hat{p}$ is a linear function of $\sum_{i=1}^{n} \frac{\partial}{\partial \mu} \log f_X(x;\mu)$. However, we learnt that they do not have a linear relationship, from the sub-question (c).
We can't find UMVUE by using the CRLB-method. However, we don't know whether $\hat{p}$ is UMVUE of p only based on the previous sub-questions and findings. We can't be certain that $\hat{p}$ is not an UMVUE. There are possibilities that if we use other methods from the next lecture or more, probably we could be able to find this UMVUE.