Vector-valued functions of $\mathbb{R}$

2.1 Introduction

Linear algebra is the study of linear functions from one vector space into another. In this book we shall be concerned with functions from $\mathbb{R}^m$ into $\mathbb{R}^n$ that can be approximated (in a sense to be made precise) by linear functions. This is the main theme of differential vector calculus.

In the present chapter we confine ourselves to the study of vector-valued functions $f: D \subseteq \mathbb{R} \to \mathbb{R}^n$ *where D is an interval in* $\mathbb{R}$ and the dimension n of the codomain is open to choice. We shall see that such functions are significant in both geometry and dynamics. Geometrically, if f is 'continuous' on an interval D, then the image of f is a curve and the 'derivatives' of f describe the way the curve twists and turns in $\mathbb{R}^n$. In dynamics, on the other hand, the position of a particle moving in space is a function of time, so that with respect to suitable frames of reference we have a corresponding function $f: D \subseteq \mathbb{R} \to \mathbb{R}^3$ where at time t the position vector of the particle is $f(t)$. The 'derivatives' of f relate to the velocity and acceleration of the particle. Before having a closer look at geometry and dynamics, however, we must explain precisely what is meant by the continuity and differentiability of vector-valued functions of $\mathbb{R}$.

In the following definitions we introduce three fundamental ideas and then explore their relationship by considering some simple examples.

2.1.1 *Definition.* *Corresponding to any function $f: D \subseteq \mathbb{R} \to \mathbb{R}^n$ we define* coordinate functions $f_i: D \subseteq \mathbb{R} \to \mathbb{R}$, $i = 1, \ldots, n$, *by means of the expression*

$$f(t) = (f_1(t), \ldots, f_n(t)), \qquad t \in D.$$

Our knowledge of calculus applied to the real-valued functions f_i will lead to the calculus of the vector-valued function f.

2.1.2 *Definition.* *The* image *of a function $f: D \subseteq \mathbb{R} \to \mathbb{R}^n$ is the* *subset of* $\mathbb{R}^n$ *given by* $\{f(t) \in \mathbb{R}^n | t \in D\}$. *The expressions*

$$x_1 = f_1(t), \ldots, x_n = f_n(t), \qquad t \in D$$

are said to give a parametrization *of the image of* f *with* parameter t.

2.1.3 *Definition.* *The* graph *of a function* $f : D \subseteq \mathbb{R} \to \mathbb{R}^n$ *is the subset of* $\mathbb{R}^{n+1}$ *given by* $\{(t, f_1(t), \ldots, f_n(t)) | t \in D\}$. *We could also express the graph as the set* $\{(t, f(t)) | t \in D\} \subseteq \mathbb{R}^{n+1}$.

For the case $n = 1$, Definition 2.1.3 reduces to the usual high-school concept of the graph of $f : \mathbb{R} \to \mathbb{R}$ as a subset of $\mathbb{R}^2$. For example, the graph of the sine function consists of all points $(t, \sin t)$ in $\mathbb{R}^2$, that is, all points (t, y) such that $y = \sin t$.

2.1.4 *Example.* The rule

$$f(t) = (1, 2, -3) + t(2, 0, 1), \qquad t \in \mathbb{R}$$

defines a function $f : \mathbb{R} \to \mathbb{R}^3$ whose image is the straight line in $\mathbb{R}^3$ through $(1, 2, -3)$ in direction $(2, 0, 1)$. The coordinate functions of f are given by

$$f_1(t) = 1 + 2t, \qquad f_2(t) = 2, \qquad f_3(t) = t - 3, \qquad t \in \mathbb{R},$$

and the corresponding parametrization of the straight line is $x = 1 + 2t$, $y = 2$, $z = t - 3$.

2.1.5 *Example.* The function $g : \mathbb{R} \to \mathbb{R}^2$ defined by

$$g(t) = (\cos t, \sin t), \qquad t \in \mathbb{R}$$

has image the unit circle $x^2 + y^2 = 1$ in $\mathbb{R}^2$ (see Fig. 2.1).

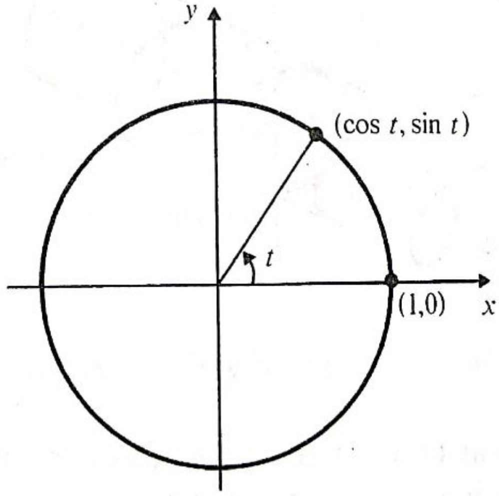


Fig. 2.1 Image of $g(t) = (\cos t, \sin t)$

The coordinate functions of g are given by $g_1(t) = \cos t$ and $g_2(t) = \sin t$, and the corresponding parametrization of the unit circle is $x = \cos t$, $y = \sin t$. The function g is periodic for, since $g(t + 2\pi) = g(t)$ for all t in $\mathbb{R}$, the values of $g(t)$ repeat regularly. The *period* of g is 2π, since this is the smallest positive number p such that $g(t + p) = g(t)$ for all t.

2.1.6 ***Example.*** Compare the function g of Example 2.1.5 with the function $h : \mathbb{R} \to \mathbb{R}^2$ defined by

$$h(t) = (\cos 2t, \sin 2t), \qquad t \in \mathbb{R}.$$

The image of h is the unit circle of Fig. 2.1 and yet g and h are different functions (for example $g(\pi) \neq h(\pi)$). Like g, the function h is periodic but unlike g its period is π. As t passes from 0 to 2π, $g(t)$ performs one revolution of the circle whereas $h(t)$ performs two revolutions. Although the functions g and h have the same image, they can be distinguished pictorially by sketching their respective graphs.

2.1.7 ***Example.*** The graphs of the functions g and h of Example 2.1.5 and 2.1.6 are respectively the set of points $(t, \cos t, \sin t)$ and $(t, \cos 2t, \sin 2t)$, for all t in $\mathbb{R}$. Just as the graph of a real-valued function (such as the sine function) can be sketched in the plane, so we can also sketch the graph of our function $g : \mathbb{R} \to \mathbb{R}^2$ (see Fig. 2.2). The graph of g is called a *circular helix*. The graph of h is also a circular helix, the windings being around the same circular cylinder but with half the pitch.

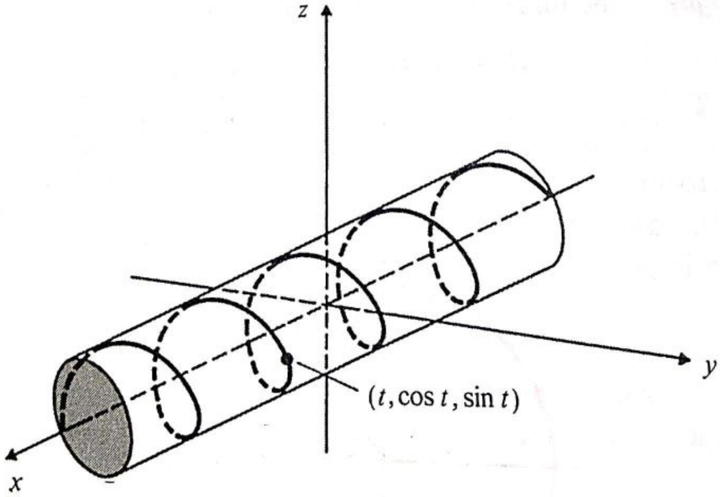


Fig. 2.2 Graph of $g(t) = (\cos t, \sin t)$

We have seen that two different functions can have the same image. It is not difficult to see that there is an infinity of different parametrizations of any non-empty subset of $\mathbb{R}^n$. The unit circle in

$\mathbb{R}^2$, for example, has parametrization

$$x = \cos(kt + c), \qquad y = \sin(kt + c), \qquad t \in \mathbb{R},$$

for any c and $k \neq 0$ in $\mathbb{R}$.

In contrast, a function is uniquely determined by its graph. That is, two different functions have different graphs (Exercise 2.1.8). The following theorem provides a useful way of thinking about the graph of a function.

2.1.8 *Theorem.* *The graph of a function $f : D \subseteq \mathbb{R} \to \mathbb{R}^n$ is the image of the function $f^* : D \subseteq \mathbb{R} \to \mathbb{R}^{n+1}$ given by*

$$f^*(t) = (t, f_1(t), \ldots, f_n(t)) = (t, f(t)) \in \mathbb{R}^{n+1}, \qquad t \in D.$$

Proof. The theorem follows immediately from Definitions 2.1.2 and 2.1.3.

Exercises 2.1

1. Find a vectorial equation of the straight line in $\mathbb{R}^3$ through $(1, 0, -4)$ in the direction $(-1, 2, 3)$. Prove that the point (x, y, z) lies on the line if and only if
$$\frac{x-1}{-1} = \frac{y}{2} = \frac{z+4}{3}.$$
2. The function $g : \mathbb{R} \to \mathbb{R}^2$ defined by
$$g(t) = (2\cos t, \sin t)$$
has as its image the ellipse $\frac{1}{4}x^2 + y^2 = 1$. Sketch the ellipse. Find a second parametrization of the ellipse that corresponds to a function of period 4π.

   Does there exist a non-periodic function $h : \mathbb{R} \to \mathbb{R}^2$ whose image is the ellipse?

Hint: consider, for example, $h(t) = (2\cos(t^2), \sin(t^2))$.

3. Prove that there is an infinity of different parametrizations of any non-empty subset of $\mathbb{R}^n$.

4. Describe the largest possible domain of the function f, where $f(t) = (\sqrt{(4 - t^2)}, t)$. Sketch its image. Compare with the image of g, where $g(t) = (-\sqrt{(4 - t^2)}, t)$.

5. Sketch (a) the image and (b) the graph of $f : \mathbb{R} \to \mathbb{R}^2$ given by $f(t) = (t, t^2)$.

Note: a rough sketch only is possible for the graph of f, since it is a subset of $\mathbb{R}^3$.

6. Find the period of the function $g:\mathbb{R}\to\mathbb{R}^2$ defined by $g(t)=(\cos 2t, \cos 3t)$.
7. Is it true that if the coordinate functions of $f: D\subseteq\mathbb{R}\to\mathbb{R}^n$ are periodic, then f is periodic?
8. Prove that a function $f: D\subseteq\mathbb{R}\to\mathbb{R}^n$ is uniquely determined by its graph, that is that two different functions on D have different graphs.

Hint: If f and g are different functions on D then there exists $t\in D$ such that $f(t)\neq g(t)$.

2.2 Sequences and limits

The reader will find in Chapter 1 a definition (1.6.3) of convergence for sequences of real numbers. The following theorem is an immediate consequence of that definition. Recall that we denote the sequence $a_1, a_2, a_3, \ldots,$ by (a_k).

2.2.1 *Theorem.* *The sequence* (a_k) *of real numbers converges to* $a\in\mathbb{R}$ *if and only if* $\lim_{k\to\infty}|a_k-a|=0$.

This theorem suggests that we can define the convergence of a sequence of vectors in $\mathbb{R}^n$ by generalizing the idea of absolute value to vectors in $\mathbb{R}^n$. The required generalization is the norm or length $\|x\|$ of a vector $\mathbf{x}\in\mathbb{R}^n$ which we defined (Definition 1.2.5) by

$$\|\mathbf{x}\| = \|(x_1, \ldots, x_n)\| = (x_1^2+\cdots+x_n^2)^{1/2}.$$

We note that in $\mathbb{R}^n$

2.2.2 $$\|\mathbf{x}-\mathbf{y}\|^2 = \sum_{r=1}^{n}(x_r-y_r)^2, \qquad \mathbf{x}, \mathbf{y}\in\mathbb{R}^n$$

and that accordingly in $\mathbb{R}$

2.2.3 $$\|x-y\| = |x-y|, \qquad x, y\in\mathbb{R}.$$

We refer to the real number $\|\mathbf{x}-\mathbf{y}\|$ as the *distance* between the vectors $\mathbf{x}$ and $\mathbf{y}$ in $\mathbb{R}^n$. This generalizes the familiar concept of distance in physical space. If $\mathbf{x}$ and $\mathbf{y}$ are the position vectors of physical points P and Q relative to a rectangular coordinate system, then $\mathbf{x}-\mathbf{y}$ is the vector $\overrightarrow{QP}$ (see Fig. 2.3(i)). The distance between P and Q is $\|\mathbf{x}-\mathbf{y}\|$ units.

Intuitively a sequence of vectors $(\mathbf{a}_k)$ in $\mathbb{R}^n$ converges to $\mathbf{a}\in\mathbb{R}^n$ if the

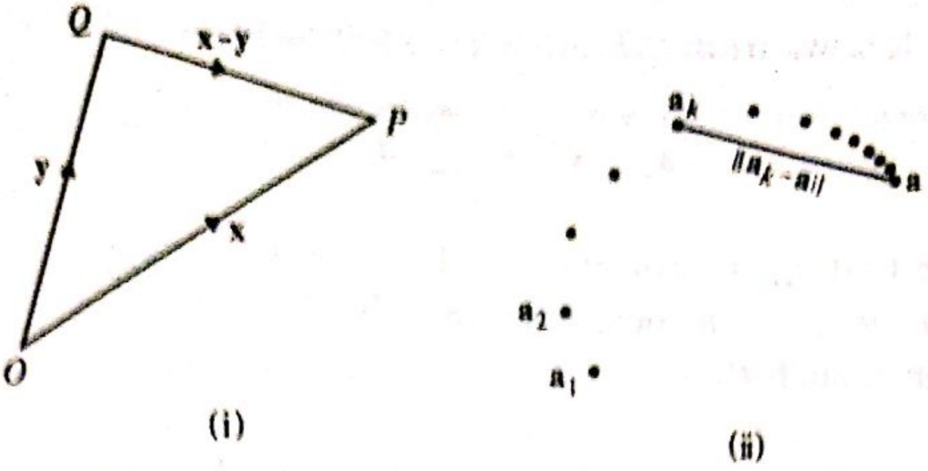


Fig. 2.3

distance between $\mathbf{a}_k$ and $\mathbf{a}$ tends to 0 as k tends to infinity (Fig. 2.3(ii)). The following definition captures this idea and is a natural extension of Theorem 2.2.1.

2.2.4 *Definition.* *The sequence* $(\mathbf{a}_k)$ *in* $\mathbb{R}^n$ *is said to* converge *to* $\mathbf{a}\in\mathbb{R}^n$ *if* $\lim_{k\to\infty}\|\mathbf{a}_k-\mathbf{a}\|=0$ *in* $\mathbb{R}$. *The convergence of* $(\mathbf{a}_k)$ *to* $\mathbf{a}$ *is denoted by* $\lim_{k\to\infty}\mathbf{a}_k=\mathbf{a}$ *or by* $\mathbf{a}_k\to\mathbf{a}$. *If the sequence* $(\mathbf{a}_k)$ *does not converge to* $\mathbf{a}$ *we write* $\mathbf{a}_k\not\to\mathbf{a}$.

2.2.5 *Example.* The sequence defined by $\mathbf{a}_k=(1, k/(k+1), 1/k), k\in\mathbb{N}$ converges to $(1, 1, 0)$ in $\mathbb{R}^3$ since

$$\|\mathbf{a}_k-(1,1,0)\|=\left\|\left(0,\frac{1}{k+1},\frac{1}{k}\right)\right\|=\left(\left(\frac{1}{k+1}\right)^2+\left(\frac{1}{k}\right)^2\right)^{1/2}\to 0.$$

The following theorem is an immediate consequence of the definition of convergence in $\mathbb{R}$.

2.2.6 *Theorem.* *The sequence* $(\mathbf{a}_k)$ *in* $\mathbb{R}^n$ *converges to* $\mathbf{a}\in\mathbb{R}^n$ *if and only if to each* $\varepsilon>0$ *there corresponds a natural number* K *such that*

$$\|\mathbf{a}_k-\mathbf{a}\|<\varepsilon \quad \textit{whenever} \quad k>K.$$

The techniques we have for studying convergence in $\mathbb{R}$ can be used to study the more general situation by appealing to the follow-

Proof. It follows from 2.2.2 that for each $k \in \mathbb{N}$

$$\|\mathbf{a}_k - \mathbf{a}\|^2 = \sum_{i=1}^{n} (a_{ki} - a_i)^2.$$

2.2.8

Suppose that $a_{ki} \to a_i$ for each $i = 1, \ldots, n$. Choose any $\varepsilon > 0$. To each $i = 1, \ldots, n$ there corresponds a number $K_i \in \mathbb{N}$ which depends on ε, such that

$$|a_{ki} - a_i| < \left(\frac{\varepsilon}{n}\right)^{1/2} \quad \text{whenever} \quad k > K_i.$$

Let $K = \max\{K_1, \ldots, K_n\}$. By 2.2.8,

$$\|\mathbf{a}_k - \mathbf{a}\|^2 = \sum_{i=1}^{n} (a_{ki} - a_i)^2 < n\left(\frac{\varepsilon}{n}\right) = \varepsilon \quad \text{whenever} \quad k > K.$$

This establishes that $\|\mathbf{a}_k - \mathbf{a}\|^2 \to 0$ and hence that $\|\mathbf{a}_k - \mathbf{a}\| \to 0$. Therefore $\mathbf{a}_k \to \mathbf{a}$.

The proof of the converse is left as an exercise.

2.2.9 ***Example.*** [i] $\lim_{k\to\infty}(k/1+k), k/(1+k^2)) = (1, 0)$.
[ii] The sequence $(\mathbf{a}_k)$ where $\mathbf{a}_k = ((-1)^k, 1/k)$ does not converge in $\mathbb{R}^2$.

Exercises 2.2

1. Show that $\lim_{k\to\infty}(k\sin(1/k), (1/k)\sin k) = (1, 0)$.
2. Prove that a convergent sequence $(\mathbf{a}_k)$ in $\mathbb{R}^n$ has a unique limit in $\mathbb{R}^n$.
3. Let (a_k) be the sequence of real numbers defined by $a_k = 10^{-k}[10^k\sqrt{2}]$, where $[r]$ denotes the integer part of the real number r. Verify that the first four terms of the sequence are $a_1 = 1$, $a_2 = 1.4$, $a_3 = 1.41$, $a_4 = 1.414$.
   Prove that (a_k) is a convergent sequence of rational numbers with limit $\sqrt{2}$.

Hint: given $\varepsilon > 0$, prove that $|a_k - \sqrt{2}| < \varepsilon$ for all k such that $1/10^k < \varepsilon$.

4. Let $(\mathbf{a}_k)$ and $(\mathbf{b}_k)$ be convergent sequences in $\mathbb{R}^n$. Prove that if $\mathbf{a}_k \to \mathbf{a}$ and $\mathbf{b}_k \to \mathbf{b}$ then $\mathbf{a}_k + \mathbf{b}_k \to \mathbf{a} + \mathbf{b}$. Show also that for any constant c, the sequence $(c\mathbf{a}_k)$ is convergent, and $c\mathbf{a}_k \to c\mathbf{a}$.
5. Prove that the definition of distance between $\mathbf{x}$ and $\mathbf{y}$ is symmetric in $\mathbf{x}$ and $\mathbf{y}$. (Show that $\|\mathbf{x} - \mathbf{y}\| = \|\mathbf{y} - \mathbf{x}\|$.)
6. A sequence $(\mathbf{a}_k)$ in $\mathbb{R}^n$ is a Cauchy sequence if to each $\varepsilon > 0$ there corresponds $K \in \mathbb{N}$ such that

$$\|\mathbf{a}_k - \mathbf{a}_l\| < \varepsilon \quad \text{whenever } k, l > K.$$

Prove that a Cauchy sequence in $\mathbb{R}^n$ is convergent.

Hint: Apply the ideas in the proof of Theorem 2.2.7 and Theorem 1.6.4.

2.3 Continuity

A function $f : D \subseteq \mathbb{R} \to \mathbb{R}$ is *discontinuous* at $p \in D$ if we can find a sequence (a_k) in D such that $a_k \to p$, but $f(a_k) \not\to f(p)$. If no such sequence can be found we say that f is continuous at p. Thus we have the following definition of continuity, generalized to functions with codomain $\mathbb{R}^n$ for arbitrary dimension n.

2.3.1 *Definition.* *The function $f : D \subseteq \mathbb{R} \to \mathbb{R}^n$ is said to be* continuous *at $p \in D$ if $f(a_k) \to f(p)$ whenever $a_k \to p$, where $a_k \in D$ for all $k \in \mathbb{N}$.*

The following theorem about continuity corresponds to Theorem 2.2.7 about limits.

2.3.2 *Theorem.* *The function $f : D \subseteq \mathbb{R} \to \mathbb{R}^n$ is continuous at $p \in D$ if and only if for each $i = 1, \ldots, n$, the coordinate function $f_i : D \subseteq \mathbb{R} \to \mathbb{R}$ is continuous at p.*

Proof. Let (a_k) be a sequence in D with $a_k \to p$. By definition of coordinate functions $f(a_k) = (f_1(a_k), \ldots, f_n(a_k))$ for each $k \in \mathbb{N}$, and $f(p) = (f_1(p), \ldots, f_n(p))$. By Theorem 2.2.7, $f(a_k) \to f(p)$ if and only if $f_i(a_k) \to f_i(p)$ for each $i = 1, \ldots, n$. The result follows.

2.3.3 *Example.* The function $f : \mathbb{R} \to \mathbb{R}^2$ given by $f(t) = (t, t^2)$ is continuous at all points in $\mathbb{R}$.

2.3.4 *Example.* The function $f : \mathbb{R} \to \mathbb{R}^2$ given by

$$f(t) = \begin{cases} (2t-1, 2t-1) & \text{when } t \leqslant 2, \\ (3-t, 5-t) & \text{when } t > 2, \end{cases}$$

is not continuous at 2 since the coordinate function f_1 is not continuous there. On the other hand the coordinate function f_2 is continuous everywhere.

2.3.5 *Corollary.* *Consider functions $f : D \subseteq \mathbb{R} \to \mathbb{R}^n$, $g : D \subseteq \mathbb{R} \to \mathbb{R}^n$ and $\phi : D \subseteq \mathbb{R} \to \mathbb{R}$. If f, g and ϕ are continuous at $p \in D$, then so are the functions ϕf, $f + g$, $f \cdot g$ and, provided $n = 3$, so is $f \times g$.*

Proof. Exercise.

2.3.6 *Definition.* *A function* $f: D \subseteq \mathbb{R} \to \mathbb{R}^n$ *is said to be continuous if it is continuous at* p *for every* $p \in D$.

Theorem 2.3.2 establishes that f is continuous if and only if all its coordinate functions are continuous.

Exercises 2.3

1. (a) Let $f: \mathbb{R} \to \mathbb{R}$ be the linear function defined by $f(t) = ct$, where c is a constant. Prove from first principles that f is continuous.
   (b) Prove that a linear function $f: \mathbb{R} \to \mathbb{R}^n$ is continuous.

Hint: consider the coordinate functions of f.

2. Prove that the function $f: \mathbb{R} \to \mathbb{R}^2$ defined by

$$f(t) = \begin{cases} (t, 3t-2) & \text{when } t \leqslant 1 \\ (2t-1, t+a) & \text{when } t > 1 \end{cases}$$

is continuous on $\mathbb{R}$ if and only if $a = 0$.

3. Prove Corollary 2.3.5.

Hint: consider the coordinate functions, and apply Exercise 1.6.2.

2.4 Limits and continuity

We have seen that the continuity of the function $f: D \subseteq \mathbb{R} \to \mathbb{R}^n$ at $p \in D$ depends upon how f behaves near to p as well as upon the value that f takes at p. In this section we study how f behaves near p by considering what f does to sequences (a_k) in D which converge to p but are such that $a_k \neq p$ for all $k \in \mathbb{N}$. *Remember that we are restricting ourselves to functions whose domain is an interval of some sort.*

2.4.1 *Definition.* *The* cluster points *of an interval* D *in* $\mathbb{R}$ *are the points of* D *together with its end points.*

2.4.2. *Example*

(i) The cluster points of $[-2, 1[$ form $[-2, 1]$.
(ii) The cluster points of $]-\infty, 3[$ form $]-\infty, 3]$.

2.4.3 *Theorem.* *A point* $p \in \mathbb{R}$ *is a cluster point of an interval* D *if* *and only if there is a sequence (a_k) in D such that $a_k \to p$ but $a_k \neq p$ for all $k \in \mathbb{N}$.*

Proof. Exercise.

2.4.4 *Example.* Let $D = [-2, 1[$. The sequences $(-2+k^{-1})$, (k^{-1}) and $(1-k^{-1})$ in D converge to cluster points -2, 0, and 1 respectively.

2.4.5 *Definition.* *Consider a function $f : D \subseteq \mathbb{R} \to \mathbb{R}^n$, a cluster point p of an interval D and a point $\mathbf{q}$ in $\mathbb{R}^n$. We write $\lim_{t \to p} f(t) = \mathbf{q}$ if $f(a_k) \to \mathbf{q}$ whenever (a_k) is a sequence in D such that $a_k \to p$ and $a_k \neq p$ for all $k \in \mathbb{N}$.*

For this type of limit we have the coordinate result that we have come to expect.

2.4.6 *Theorem.* *Given $f : D \subseteq \mathbb{R} \to \mathbb{R}^n$ and p a cluster point of D, then $\lim_{t \to p} f(t) = \mathbf{q}$ if and only if $\lim_{t \to p} f_i(t) = q_i$ for each $i = 1, \ldots, n$.*

Proof. From Theorem 2.2.7, for any sequence (a_k) in D, $f(a_k) \to \mathbf{q}$ if and only if $f_i(a_k) \to q_i$ for all $i = 1, \ldots, n$. This equivalence still holds if we confine ourselves to sequences such that $a_k \to p$ and $a_k \neq p$ for all $k \in \mathbb{N}$.

2.4.7 *Example.* Define $f : \mathbb{R} \to \mathbb{R}^2$ by

$$f(t) = \begin{cases} (1+t, 2+t) & \text{when } t \geqslant 0, \\ (1+t, 1-t) & \text{when } t < 0. \end{cases}$$

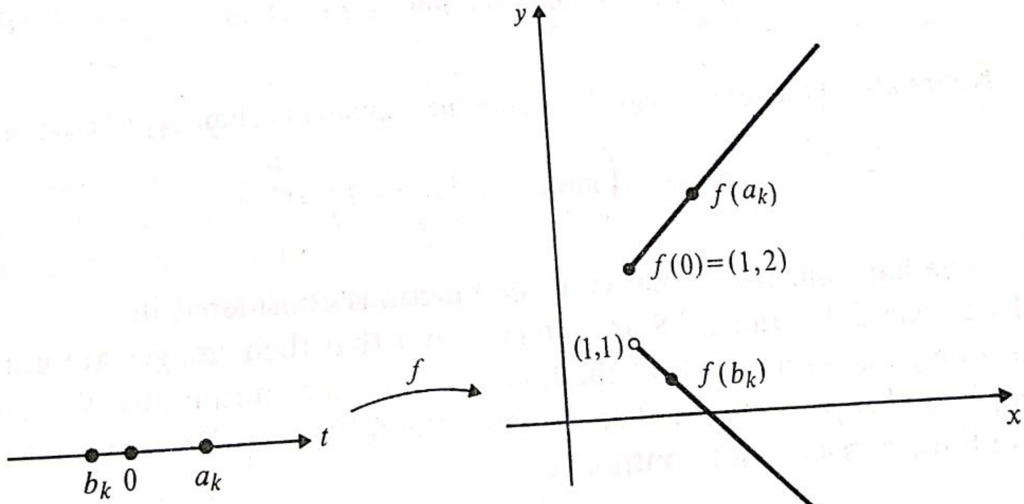


Fig. 2.4 Image of function f (Example 2.4.7)

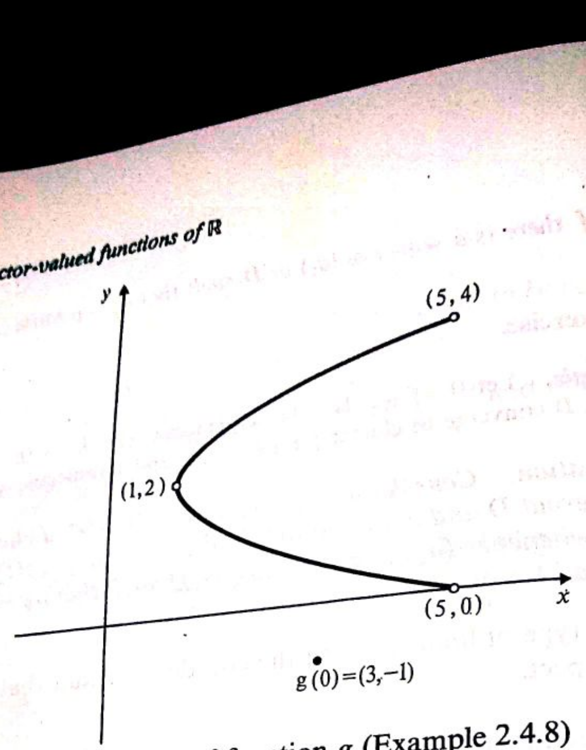


Fig. 2.5 Image of function g (Example 2.4.8)

Let $a_k = k^{-1}$ and $b_k = -k^{-1}$. Then $a_k \to 0$ and $b_k \to 0$ but $\lim_{k\to\infty} f(a_k) = (1, 2)$ and $\lim_{k\to\infty} f(b_k) = (1, 1)$. Hence $\lim_{t\to 0} f(t)$ does not exist. The situation is sketched in Fig. 2.4. In particular $\lim_{t\to 0} f_1(t) = 1$ but $\lim_{t\to 0} f_2(t)$ does not exist.

2.4.8 *Example.* Figure 2.5 is a sketch of the image of the function $g:]-2,2[\to \mathbb{R}^2$ defined by $g(0) = (3, -1)$ and $g(t) = (t^2+1, t+2)$ when $t \neq 0$. The image of 0 is separated from the rest of the image, but $\lim_{t\to 0} g(t)$ exists and, by Theorem 2.4.6,

$$\lim_{t\to 0} g(t) = \left(\lim_{t\to 0} t^2+1,\ \lim_{t\to 0} t+2\right) = (1, 2).$$

Notice also that, even though 2 is not in the domain of g, $\lim_{t\to 2} g(t)$ exists and

$$\lim_{t\to 2} g(t) = \left(\lim_{t\to 2} t^2+1,\ \lim_{t\to 2} t+2\right) = (5, 4).$$

The fact that the domains of the functions considered in Examples 2.4.7 and 2.4.8 are intervals but that their images are not in one piece indicates that the functions are not continuous. We can explore this property further by using the following theorem relating the limit concept and continuity.

2.4.9 *Theorem.* *The function* $f: D \subseteq \mathbb{R} \to \mathbb{R}^n$ *is continuous at* $p \in D$ *if and only if* $\lim_{t\to p} f(t) = f(p)$.

Proof. It is clear that if $f(a_k) \to f(p)$ whenever $a_k \to p$, with $a_k \neq p$ for all $k \in \mathbb{N}$, then $f(a_k) \to f(p)$ whenever $a_k \to p$ without that restriction.

2.4.10 ***Example.*** [i] The function g of Example 2.4.8 is not continuous at 0 since $\lim_{t \to 0} g(t) = (1, 2) \neq (3, -1) = g(0)$. If, however, we redefine the value of g at 0 to be (1, 2), then the new function is continuous at 0.

[ii] The function f of Example 2.4.7 is not continuous at 0 since $\lim_{t \to 0} f(t)$ does not exist. Furthermore, f cannot be made continuous by changing its value at 0.

The following result provides an important tool for manipulating with limits.

2.4.11 ***Theorem.*** *Consider a function $f : D \subseteq \mathbb{R} \to \mathbb{R}^n$, a cluster point p of D and a point $\mathbf{q} \in \mathbb{R}^n$. Then $\lim_{t \to p} f(t) = \mathbf{q}$ if and only if to each $\varepsilon > 0$ there corresponds $\delta > 0$ such that*

2.4.12 $$\|f(t) - \mathbf{q}\| < \varepsilon \quad \textit{whenever} \quad t \in D \textit{ and } 0 < |t - p| < \delta.$$

Proof. Suppose firstly that the ε, δ condition is satisfied. Let (a_k) be any sequence in D such that $a_k \to p$ and $a_k \neq p$ for all $k \in \mathbb{N}$. Given any $\varepsilon > 0$ there corresponds, by 2.4.12, a $\delta > 0$ such that

2.4.13 $$\|f(a_k) - \mathbf{q}\| < \varepsilon \quad \text{whenever} \quad 0 < |a_k - p| < \delta.$$

But $a_k \to p$ and $a_k \neq p$ for all $k \in \mathbb{N}$, and so there exists $k \in \mathbb{N}$ such that

2.4.14 $$0 < |a_k - p| < \delta \quad \text{whenever} \quad k > K.$$

Expressions 2.4.13 and 2.4.14 together establish that $\lim_{k \to \infty} f(a_k) = \mathbf{q}$. Since this is true for all suitable sequences (a_k) it follows from Definition 2.4.5 that $\lim_{t \to p} f(t) = \mathbf{q}$.

To prove the converse we will show that the failure of the ε, δ condition 2.4.12 implies that $\lim_{t \to \infty} f(t) \neq \mathbf{q}$. Failure of the condition implies that there exists an $\varepsilon > 0$ such that no corresponding δ can be found. In particular $\delta = 1/k$ will not do for any $k \in \mathbb{N}$. For any $k \in \mathbb{N}$, therefore, there exists $a_k \in D$ such that

2.4.15 $$0 < |a_k - p| < 1/k \quad \text{but} \quad \|f(a_k) - \mathbf{q}\| \geqslant \varepsilon.$$

The first part of 2.4.15 tells us that the sequence (a_k) converges to p and that $a_k \neq p$ for all $k \in \mathbb{N}$. The second part of 2.4.15 tells us that $\lim_{k \to \infty} f(a_k) \neq \mathbf{q}$. This is the required contradiction.

The ε, δ condition of Theorem 2.4.11 means that given any required level of approximation, the value of $f(t)$ is approximately equal to $\mathbf{q}$ for all $t \neq p$ sufficiently close to p.

2.4.16 *Corollary.* *Given $f: D \subseteq \mathbb{R} \to \mathbb{R}^n$ and $p \in D$, then f is continuous at p if and only if to each $\varepsilon > 0$ there corresponds $\delta > 0$ such that $\|f(t) - f(p)\| < \varepsilon$ whenever $t \in D$ and $|t - p| < \delta$.*

Proof. Immediate from Theorem 2.4.9 and Theorem 2.4.11.

2.4.17 *Remark.* The limit $\lim_{t \to p} f(t)$ defined in Definition 2.4.5 is related to a function f whose domain is an interval D in $\mathbb{R}$. The definition still makes sense if D is *any* non-empty subset of $\mathbb{R}$ and $p \in \mathbb{R}$ is any point such that there is a sequence (a_k) in D for which $a_k \to p$ and $a_k \neq p$ for all $k \in \mathbb{N}$.

2.4.18 *Example.* Define $g: \mathbb{R} \backslash \{0\} \to \mathbb{R}$ by $g(t) = (\sin t)/t$. Then $\lim_{t \to 0} g(t)$ is defined and in fact $\lim_{t \to 0} g(t) = 1$ since, for all small t,

$$\left| t - \frac{t^3}{6} \right| \leqslant |\sin t| \leqslant |t|.$$

Exercises 2.4

1. Apply the test for continuity of Corollary 2.4.16 to prove that
   (a) the function $f: \mathbb{R} \to \mathbb{R}^n$ given by $f(t) = (t, t^2)$ $t \in \mathbb{R}$, is continuous at 0;
   (b) a linear function $f: \mathbb{R} \to \mathbb{R}^2$ is continuous at $p \in \mathbb{R}$.

2. Prove that the real-valued function $f: \mathbb{R} \to \mathbb{R}$ defined by $f(t) = \sin(1/t)$, $t \neq 0$, $f(0) = 0$, is discontinuous at 0 by showing that the test for continuity of Corollary 2.4.16 breaks down.

(*Hint*: choose $\varepsilon = \frac{1}{2}$ and show that for any $\delta > 0$ there exists $0 < x < \delta$ such that $|f(x) - f(0)| > \varepsilon$. In fact, it is possible to choose $0 < x < \delta$ such that $f(x) = 1$.)

3. A function $f: D \subseteq \mathbb{R} \to \mathbb{R}^n$ is said to have a *removable discontinuity* at $p \in D$ if f is not continuous at p but there exists $\mathbf{q} \in \mathbb{R}^n$ such that $\lim_{t \to p} f(t) = \mathbf{q}$. This means that f can be made continuous at p by changing its value there. Show that 0 is a removable discontinuity of the function g of Example 2.4.8 but 0 is not a removable discontinuity of the function f of Example 2.4.7.

4. Given a function $f: D \subseteq \mathbb{R} \to \mathbb{R}^n$, consider the function $\|f\|: D \subseteq \mathbb{R} \to \mathbb{R}$ where $\|f\|(t) = \|f(t)\|$, $t \in D$. Prove that if f is continuous then $\|f\|$ is continuous. Is the converse true?

2.5 Differentiability and tangent lines

The derivative of a function $\phi : \mathbb{R} \to \mathbb{R}$ at $p \in \mathbb{R}$ is defined to be

2.5.1 $$\phi'(p) = \lim_{h \to 0} \frac{\phi(p+h) - \phi(p)}{h},$$

if this limit exists.

In generalizing this definition to a function $f : D \subseteq \mathbb{R} \to \mathbb{R}^n$ at a point $p \in \mathbb{R}$ we consider vectors of the form

2.5.2 $$\frac{1}{h}(f(p+h) - f(p)) \in \mathbb{R}^n,$$

where $h \in \mathbb{R}$ is non-zero and is such that $p + h \in D$. Since the vector given in 2.5.2 is a multiple of $f(p+h) - f(p)$ it is parallel to the line joining $f(p)$ and $f(p+h)$. It is pictured (for the case $h > 0$) in Fig. 2.4(i) as lying along this line and based at $f(p)$.

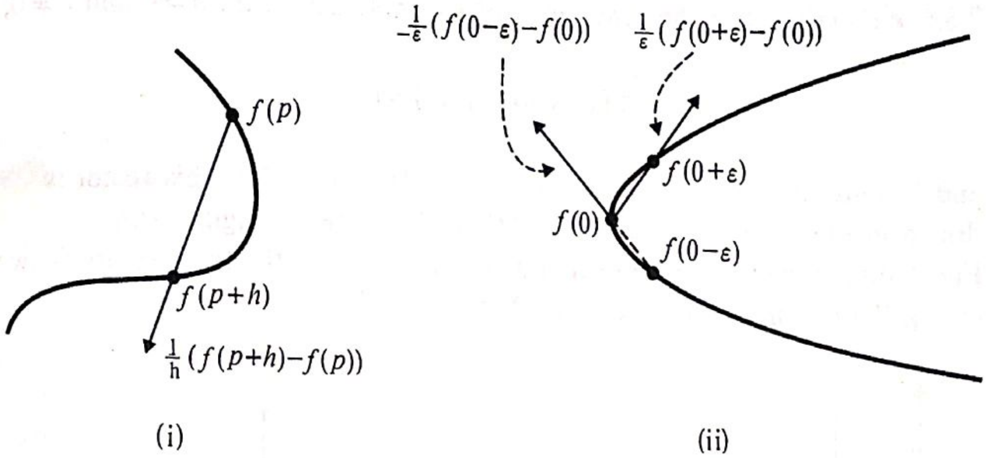


Fig. 2.6

2.5.3 *Example.* Consider the function $f : \mathbb{R} \to \mathbb{R}^2$ defined by $f(t) = (t^2 + 1, t + 2)$. Taking $p = 0$, for any $\varepsilon > 0$ the two vectors in $\mathbb{R}^2$ obtained from 2.5.2 by taking $h = \varepsilon$ and $h = -\varepsilon$ are respectively

$$\frac{1}{\varepsilon}(f(0+\varepsilon) - f(0)) = (\varepsilon, 1) \quad \text{and} \quad \frac{1}{-\varepsilon}(f(0-\varepsilon) - f(0)) = (-\varepsilon, 1).$$

These vectors are drawn, together with the image of f, in Fig. 2.6(ii). The image of f is the parabola $(y-2)^2 = (x-1)$ in $\mathbb{R}^2$.

2.5.4 *Definition.* *Let p be a point in an interval D. Then the interval D_p is defined by*

$$D_p = \{h \in \mathbb{R} \mid p + h \in D\}.$$

The interval D_p is merely the translation of D through $(-p)$. Since p lies in D, 0 lies in D_p. In view of Remark 2.4.17 the following definition makes sense.

2.5.5 *Definition.* *The function* $f: D \subseteq \mathbb{R} \to \mathbb{R}^n$ *is said to be* differentiable *at* $p \in D$ *if the limit*

$$\lim_{h \to 0} \frac{1}{h}(f(p+h) - f(p)), \qquad h \in D_p \backslash \{0\}$$

exists in $\mathbb{R}^n$. *If the limit does exist then it is called the* derivative *of* f *at* p *and is denoted by* $f'(p)$.

We can say, as in the elementary case, that $f'(p)$ measures the rate of change of f at p. Notice, however, that in this case $f'(p)$ is a vector in $\mathbb{R}^n$.

2.5.6 *Example.* For the function f of Example 2.5.3 with $p = 0$ and $h \neq 0$,

$$\frac{1}{h}(f(p+h) - f(p)) = (h, 1)$$

and Definition 2.5.5 gives $f'(0) = \lim_{h \to 0}(h, 1) = (0, 1) \in \mathbb{R}^2$. This vector is drawn in Fig. 2.7(i) based at $f(0) = (1, 2)$. Compare this figure with Fig. 2.6(ii). In particular, note that as ε tends to zero, the two vectors drawn in Fig. 2.6(ii) tend to the vector $f'(0)$ drawn in Fig. 2.7(i).

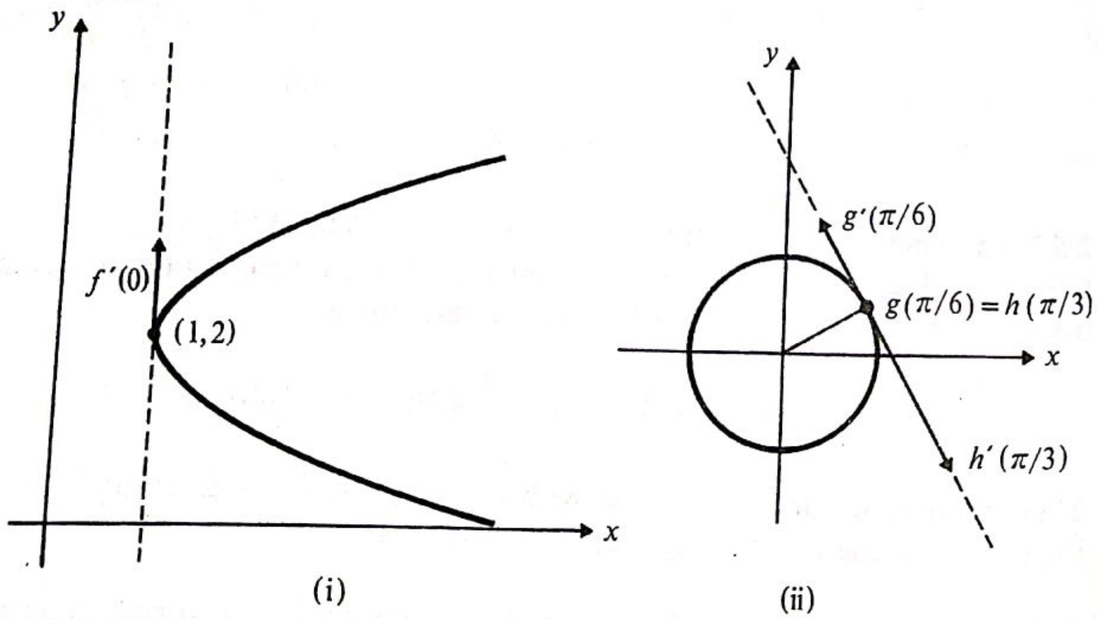


Fig. 2.7 (i) Tangent vector $f'(0) = (0, 1)$;
(ii) Point $g(\pi/6) = h(\pi/3) = (\frac{1}{2}\sqrt{3}, \frac{1}{2})$
Tangent vectors $g'(\pi/6) = \frac{1}{2}(-1, \sqrt{3})$ and $h'(\pi/3) = (1, -\sqrt{3})$

2.5.7 Theorem. *The function $f: D \subseteq \mathbb{R} \to \mathbb{R}^n$ is differentiable at $p \in D$ if and only if for each $i = 1, \ldots, n$ the coordinate function $f_i: D \subseteq \mathbb{R} \to \mathbb{R}$ is differentiable at p. Furthermore, if f is differentiable at p, then*

$$f'(p) = (f_1'(p), \ldots, f_n'(p)).$$

Proof. Apply Theorem 2.4.6.

2.5.8 *Example.* (i) The function $g: \mathbb{R} \to \mathbb{R}^2$ defined by $g(t) = (\cos t, \sin t)$ has as its image the unit circle centre $(0, 0)$ in $\mathbb{R}^2$. The function is differentiable at $\pi/6$ with derivative

$$g'\left(\frac{\pi}{6}\right) = \left(-\sin\frac{\pi}{6}, \cos\frac{\pi}{6}\right) = \left(-\frac{1}{2}, \frac{\sqrt{3}}{2}\right).$$

This vector is drawn in Fig. 2.7(ii) based at $g(\pi/6) = (\sqrt{3}/2, 1/2)$.

(ii) The function $h: \mathbb{R} \to \mathbb{R}^2$ defined by $h(u) = (\sin 2u, \cos 2u)$ has image the same unit circle in $\mathbb{R}^2$. The function h is differentiable at $\pi/3$ with derivative

$$h'\left(\frac{\pi}{3}\right) = \left(2\cos\frac{2\pi}{3}, -2\sin\frac{2\pi}{3}\right) = (1, -\sqrt{3}).$$

This derivative is also drawn in Fig. 2.7(ii) based, as was $g'(\pi/6)$, at $h(\pi/3) = (\sqrt{3}/2, 1/2)$.

In Example 2.5.8 the vectors $g'(\pi/6)$ and $h'(\pi/3)$ are linearly dependent and are orthogonal to the radius vector $g(\pi/6) = h(\pi/3) = (\sqrt{3}/2, 1/2)$. Both derivatives are therefore in the direction of the tangent to the unit circle at $(\sqrt{3}/2, 1/2)$ (see Fig. 2.7(ii)). Similarly, in Example 2.5.3, the vector $f'(0)$ is in the direction of the tangent to the parabola $(y-2)^2 = (x-1)$ at $f(0) = (1, 2)$ (see Fig. 2.7(i)). Such geometrical considerations suggest the following definitions.

2.5.9 *Definition.* *Let $f: D \subseteq \mathbb{R} \to \mathbb{R}^n$ be differentiable at $p \in D$.*

[i] *The derivative $f'(p)$ will also be called the* tangent vector *to f at p.*

[ii] *If $f'(p) \neq \mathbf{0}$, then the straight line in $\mathbb{R}^n$ through $f(p)$ in direction $f'(p)$ will be called the* tangent line *to f at p. It is the set $\{f(p) + sf'(p) \mid s \in \mathbb{R}\} \subseteq \mathbb{R}^n$.*

By excluding the case $f'(p) = 0$ in Definition 2.5.9(ii) we avoid the tangent line degenerating to a single point. Notice that the tangent vector and tangent line are defined in terms of a function and not in terms of its image.

2.5.10 *Example.* The tangent vector to the function f of Example 2.5.6 at 0 is $(0, 1) \in \mathbb{R}^2$. The tangent line at 0 is the line $\{(1, 2) + s(0, 1) | s \in \mathbb{R}\} \subseteq \mathbb{R}^2$. It is the line $x = 1$ sketched as a broken line in Fig. 2.7(i).

2.5.11 *Example.* The functions g and h of Example 2.5.8 both have image the unit circle centre $\mathbf{0}$ in $\mathbb{R}^2$. Although $g(\pi/6) = h(\pi/3)$ the two tangent vectors $g'(\pi/6)$ and $h'(\pi/3)$ are different. The tangent line to g at $\pi/6$ is the set

$$\{(\sqrt{3}/2, 1/2) + s(-1/2, \sqrt{3}/2) | s \in \mathbb{R}\} \subseteq \mathbb{R}^2.$$

It is the straight line $y + \sqrt{3}x = 2$ sketched as a broken line in Fig. 2.7(ii). Since $h(\pi/3) = g(\pi/6)$ and since $h'(\pi/3)$ is a non-zero multiple of $g'(\pi/6)$, the tangent line to h at $\pi/3$ is the same straight line in $\mathbb{R}^2$. Our earlier work on the geometry of Fig. 2.7(ii) shows that this tangent line is the tangent to the circle at $(\sqrt{3}/2, 1/2)$.

We shall have a second look at the relationship between tangent lines to a function and the image of the function in the next section.

2.5.12 *Definition.* *The function* $f : D \subseteq \mathbb{R} \to \mathbb{R}^n$ *is said to be* differentiable *if it is differentiable at each point* $p \in D$. *The function* $f' : D \subseteq \mathbb{R} \to \mathbb{R}^n$ *whose image at* $p \in D$ *is the vector* $f'(p) \in \mathbb{R}^n$ *is called the* derivative *of* f. *If* f' *is itself differentiable, then its derivative is denoted by* $f'' : D \subseteq \mathbb{R} \to \mathbb{R}^n$.

With the help of Theorem 2.5.7 many theorems of elementary calculus are readily generalized.

2.5.13 *Theorem.* *Consider functions* $f : D \subseteq \mathbb{R} \to \mathbb{R}^n$, $g : D \subseteq \mathbb{R} \to \mathbb{R}^n$ *and* $\phi : D \subseteq \mathbb{R} \to \mathbb{R}$. *If* f, g *and* ϕ *are differentiable, then so are the following functions with the stated derivatives.*

[i] *The sum function*: $(f + g)' = f' + g'$.
[ii] *The dot product* $(f \cdot g)' = (f' \cdot g) + (f \cdot g')$
[iii] (*For the case* $n = 3$) *the cross product*:
$(f \times g)' = (f' \times g) + (f \times g')$
[iv] *The product function*: $(\phi f)' = \phi' f + \phi f'$.

Proof. For each $t \in D$ express $(f + g)(t)$, $(f \cdot g)(t)$, $(f \times g)(t)$ and $(\phi f)(t)$ in terms of coordinate functions of f and of g. Now apply Theorem 2.5.7 and use results from elementary calculus to obtain the required results.

The following theorem is an important application of Theorem 2.5.13(ii).

2.5.14 *Theorem.* *Let $h: D \subseteq \mathbb{R} \to \mathbb{R}^n$ be a differentiable function such that $\|h(t)\| = 1$ for all $t \in D$. Then $h(t)$ and $h'(t)$ are orthogonal for all $t \in D$.*

Proof. The image of h lies on the unit sphere centre $\mathbf{0}$ in $\mathbb{R}^n$. For all $t \in D$

2.5.15 $$h(t) \cdot h(t) = \|h(t)\|^2 = 1.$$

Using Theorem 2.5.13(ii) we find, on differentiating 2.5.15, that

$$h(t) \cdot h'(t) + h'(t) \cdot h(t) = 2h(t) \cdot h'(t) = 0, \qquad \text{for all } t \in D.$$

Therefore the tangent vector $h'(t)$ is orthogonal to the radius vector $h(t)$.

In particular, if $n = 2$ and if $h'(p) \neq \mathbf{0}$ then the tangent line to h at p is just the tangent to the unit circle centre $\mathbf{0} \in \mathbb{R}^2$ at $h(p)$. We have already seen this to be true in the particular cases considered in Example 2.5.11.

Despite their geometrical significance, differentiability, tangent vectors and tangent lines are defined in terms of functions rather than in terms of their images. The following two examples should serve to warn the reader against jumping to conclusions about a function merely on the basis of looking at its image.

2.5.16 *Example.* Consider the continuous functions f, g and h from $\mathbb{R}$ into $\mathbb{R}^2$ defined for each $t \in \mathbb{R}$ by

$$g(t) = (t, t^2), \quad f(t) = (t^3, t^6), \quad h(t) = \begin{cases} (t, t^2) & \text{for } t \geq 0, \\ (t^3, t^6) & \text{for } t < 0. \end{cases}$$

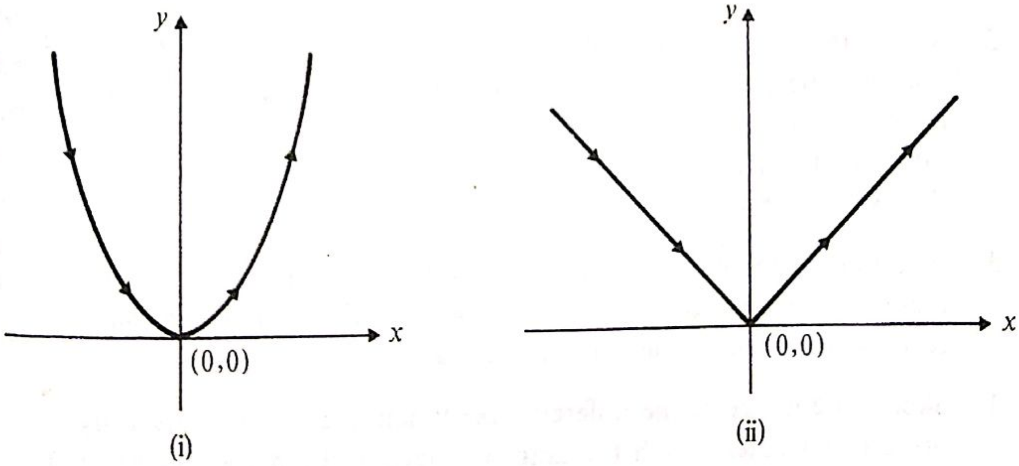


Fig. 2.8

All three functions have the same image (the parabola $y=x^2$ in $\mathbb{R}^2$ — see Fig. 2.8(i)) and all take the value $(0, 0)$ at 0. Furthermore they cover the points of this curve in the same order as t increases (indicated by the arrows in Fig. 2.8(i)). The function g is differentiable at 0 with tangent vector $g'(0)=(1, 0)$ and tangent line $y=0$ there. The function f is also differentiable but f has no tangent line at 0 since $f'(0)=(0, 0)$. Even more striking is the fact that, since the coordinate function h_1 is not differentiable at 0, the function h is not differentiable at 0. This is so despite the smoothness of the image of h.

2.5.17 ***Example.*** The image of a continuous function $g:\mathbb{R}\to\mathbb{R}^2$ where $g(t)=(t|t|, t^2), t\in\mathbb{R}$, is sketched in Fig. 2.8(ii). It is the set $y=|x|$ in $\mathbb{R}^2$. The coordinate functions of g are given by

$$g_1(t)=\begin{cases} t^2 & \text{when } t\geqslant 0 \\ -t^2 & \text{when } t<0 \end{cases} \quad \text{and} \quad g_2(t)=t^2, \qquad t\in\mathbb{R}.$$

Both g_1 and g_2 are differentiable at 0. Therefore g is differentiable at 0 and has a tangent vector there despite the fact that the image has a 'corner' at $g(0)=(0, 0)$. Since the tangent vector at 0 is $g'(0)=(0, 0)$, there is no associated tangent line.

In elementary calculus the differentiability of a function $\phi:\mathbb{R}\to\mathbb{R}$ is usually thought of in terms of the smoothness of the graph of ϕ. It is very important in deducing properties of $f:D\subseteq\mathbb{R}\to\mathbb{R}^n$ from a sketch of its image to remember that it is just the image and not the graph of f. This point is amply supported in Examples 2.5.16 and 2.5.17.

Exercises 2.5

1. Find the tangent vector (a) at 0 (b) at $\frac{1}{2}\pi$ to the function $g:\mathbb{R}\to\mathbb{R}^2$ defined by $g(t)=(2\cos t, \sin t)$. Determine the corresponding tangent lines. Sketch these and the image of g (the ellipse of Exercise 2.1.2).
2. Sketch the images of the following differentiable functions $f:\mathbb{R}\to\mathbb{R}^2$ and indicate the tangent vectors and tangent lines (if they exist) to f (i) at $t=0$; (ii) at $t=1$.
   (a) $f(t)=(t+t^2, t-t^2)$
   (b) $f(t)=(t^2, t^3)$
3. Illustrate Theorem 2.5.14 and its proof with the function $h:\mathbb{R}\to\mathbb{R}^2$ defined by $h(t)=(\cos(t^2), \sin(t^2))$. Calculate $h'(t)$ and sketch the corresponding tangent lines for various values of t.
4. Sketch the image of the differentiable function $g:\mathbb{R}\to\mathbb{R}^2$ given by $g(t)=(t, t^3), t\in\mathbb{R}$. Sketch the tangent vector and tangent line at $t=0$ and at $t=1$.

Using the idea of Example 2.5.16, construct a function h with the same image as g and such that $g(0)=h(0)$, but h is not differentiable at 0.

Answer: for example $h(t)=(t^{1/3}, t)$, $t\in\mathbb{R}$.

5. Prove that the function $f:\mathbb{R}\to\mathbb{R}$ defined by $f(t)=|t|$ is continuous but not differentiable at 0.

6. Verify Theorem 2.5.13 for the functions $f:\mathbb{R}\to\mathbb{R}^3$, $g:\mathbb{R}\to\mathbb{R}^3$ and $\phi:\mathbb{R}\to\mathbb{R}$, where

$$f(t)=(e^t, t, 1),\quad g(t)=(0, -t, t^2+1),\quad \phi(t)=t^2.$$

7. Sketch the image of the differentiable function $f:\mathbb{R}\to\mathbb{R}^2$ defined by

$$f(t)=(e^{kt}\cos t, e^{kt}\sin t),\qquad t\in\mathbb{R},$$

where k is a constant. Prove that

$$\frac{f'(t)\cdot f(t)}{\|f'(t)\|\ \|f(t)\|}=\frac{k}{\sqrt{(1+k^2)}}\qquad t\in\mathbb{R}.$$

Deduce that the angle between the tangent vector $f'(t)$ and the line joining the origin and the point $f(t)$ is the same for all t. The image of f is called an *equiangular spiral.*

2.6 Curves and simple arcs. Orientation

Cartesian geometry and calculus developed during the first half of the seventeenth century through the study of some special curves and in particular through attempts to solve problems concerning their tangents, their length and the areas associated with them.

Intuitively one thinks of a curve as a wavy copy of an interval, with possible self-intersections and corners. It seems reasonable to define a curve in $\mathbb{R}^n$ as the image of an interval D under a continuous function $f:D\subseteq\mathbb{R}\to\mathbb{R}^n$. The point $f(t)\in\mathbb{R}^n$ would then trace the curve as the parameter t moves along the interval. This definition, however, is too general to be useful. There is, for example, such a continuous function defined on a compact interval $[a, b]$ whose image fills out a square in $\mathbb{R}^2$. For an illustration, see Liu Wen, *American Math. Monthly* **90** (1983) p. 283. We therefore impose extra conditions on f, although many of our results are also true if they are weakened.

2.6.1 ***Definition.*** *A function $f:D\subseteq\mathbb{R}\to\mathbb{R}^n$ is said to be* continuously differentiable, *or a* C^1 function, *if f is differentiable and if*

f' is continuous. *In general a* C^k function *is one whose* k*th derivative exists and is continuous.*

2.6.2 *Definition.* *A subset* C *of* $\mathbb{R}^n$ *is a* curve *if there is a* C^1 *function* $f: D \subseteq \mathbb{R} \to \mathbb{R}^n$, *where* D *is an interval, such that* $f(D) = C$. *The function* f *is called a* C^1 parametrization *of the curve.*

2.6.3 *Example.* The function $f(t) = (t^2 + 1, t + 2)$ of Example 2.5.3 defines a C^1 parametrization of the curve (parabola) $(y - 2)^2 = (x - 1)$ in $\mathbb{R}^2$. The functions $g(t) = (\cos t, \sin t)$ and $h(t) = (\sin 2t, \cos 2t)$ of Example 2.5.8 define two different C^1 parametrizations of the curve (circle) $x^2 + y^2 = 1$ in $\mathbb{R}^2$. Notice that g and h trace the circle in opposite directions.

2.6.4 *Example.* The function $f: \mathbb{R} \to \mathbb{R}^3$ defined by $f(t) = (t, \cos t, \sin t)$ is a C^1 parametrization of the circular helix curve sketched in Fig. 2.2.

2.6.5 *Example.* The function $g: \mathbb{R} \to \mathbb{R}^2$ defined by $g(t) = (\sin t, \sin 2t)$ is a C^1 parametrization of the 'figure 8' curve sketched in Fig. 2.9. The arrows in Fig. 2.9 indicate the direction in which $g(t)$ traces the curve as t increases. Clearly g has period 2π and indeed the curve is traced just once in each interval of length 2π.

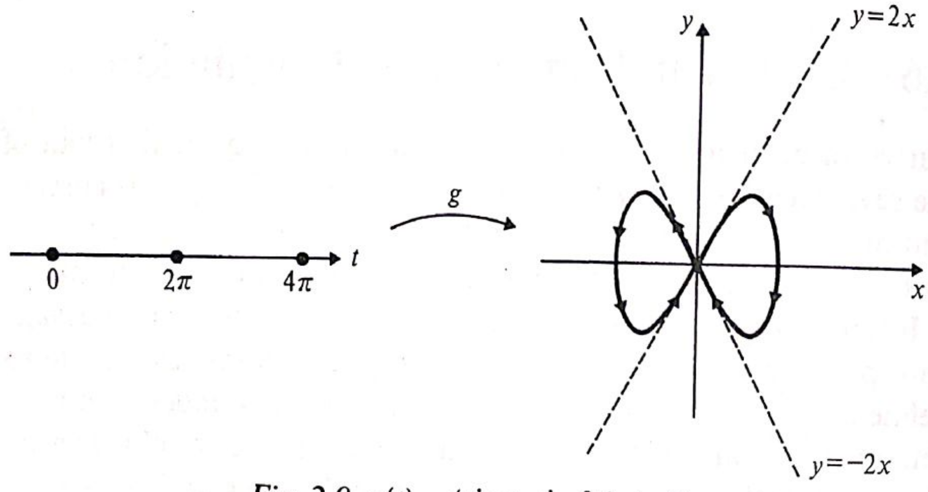


Fig. 2.9 $g(t) = (\sin t, \sin 2t)$, $t \in \mathbb{R}$

In this section we shall consider the possibility of using a parametrization of a curve to define the tangent lines to it as we did in Examples 2.5.10 and 2.5.11. Example 2.5.16 indicates that care is needed, as do the following examples.

2.6.6 *Example.* The function $g: \mathbb{R} \to \mathbb{R}^2$ defined by $g(t) = (t|t|, t^2)$ as in Example 2.5.17 is a C^1 parametrization of the curve $y = |x|$ in $\mathbb{R}^2$ sketched in Fig. 2.8(ii). There is no tangent line at 0.

2.6.7 *Example.* Consider the C^1 parametrization $g(t)=(\sin t, \sin 2t)$ of the 'figure 8' curve defined in Example 2.6.5. In considering the properties of this curve at the cross-over point $(0, 0)$ we observe that $g(0)=g(\pi)=(0, 0)$. The derivative of g is given by

$$g'(t)=(\cos t, 2\cos 2t), \qquad t\in\mathbb{R}.$$

Hence $g'(0)=(1, 2)$ and the tangent line to g at 0 is $y=2x$. On the other hand $g'(\pi)=(-1, 2)$ and the tangent line to g at π is $y=-2x$. These two lines are sketched in Fig. 2.9 as dotted lines through $g(0)=g(\pi)=(0, 0)$.

In general, $y=2x$ is the tangent line to g at points of the form $2k\pi\in\mathbb{R}$, $k\in\mathbb{Z}$, and $y=-2x$ is the tangent line to g at points of the form $(2k+1)\pi$, $k\in\mathbb{Z}$.

2.6.8 *Example.* (i) The function g of Example 2.6.7 restricted to the open interval $]-\pi, \pi[$ leads to a 1–1 C^1 function $f:]-\pi, \pi[\subseteq\mathbb{R}\to\mathbb{R}^2$ defined by $f(t)=(\sin t, \sin 2t)$. The image of f is the same 'figure 8' curve sketched in Fig. 2.9. However, the only value of t for which $f(t)=(0, 0)$ is $t=0$. The tangent line to f at 0 is $y=2x$. The other significant line through $(0, 0)$, $y=-2x$, is not revealed by this parametrization of the curve. See Fig. 2.10 (i).

(ii) By restricting g to the open interval $]0, 2\pi[$, we obtain a 1–1 C^1 function $h:]0, 2\pi[\subseteq\mathbb{R}\to\mathbb{R}^2$ defined by $h(u)=(\sin u, \sin 2u)$. Again the image of h is the 'figure 8' curve. The only value of u for which $h(u)=(0, 0)$ is $u=\pi$. The tangent line to h at π is $y=-2x$. This time the line $y=2x$ is hidden by the parametrization. See Fig. 2.10(ii).

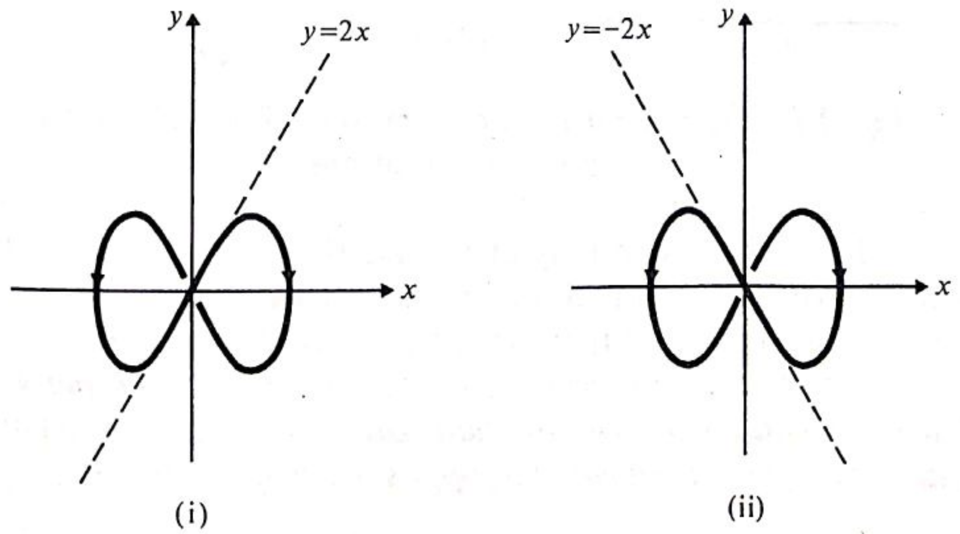


Fig. 2.10 (i) $f(t)=(\sin t, \sin 2t)$, $t\in]-\pi, \pi[$;
(ii) $h(u)=(\sin u, \sin 2u)$, $u\in]0, 2\pi[$

Nevertheless, we would expect that if two parametrizations of a curve are related in some way then they will associate the same tangent lines with points of the curve. Accordingly we define the following equivalence relation between parametrizations.

2.6.9. Definition. *Let $f : D \subseteq \mathbb{R} \to \mathbb{R}^n$ and $h : E \subseteq \mathbb{R} \to \mathbb{R}^n$ be C^1 parametrizations of a curve C. Then h is equivalent to f if there is a differentiable function ϕ from E onto D such that*

[i] *$h = f \circ \phi$ and*

[ii] *either $\phi'(u) > 0$ for all $u \in E$ or $\phi'(u) < 0$ for all $u \in E$.*

Condition [ii] of Definition 2.6.9 ensures that ϕ is strictly monotonic and hence 1–1 from E onto D. By Theorem 1.6.7 the inverse function ϕ^{-1} from D onto E is dfferentiable. It follows that if h is equivalent to f via the differentiable function ϕ then f is equivalent to h via the differentiable function ϕ^{-1}. We leave it to the reader to complete the argument that Definition 2.6.9 establishes an equivalence relation on the class of all C^1 parametrizations of C.

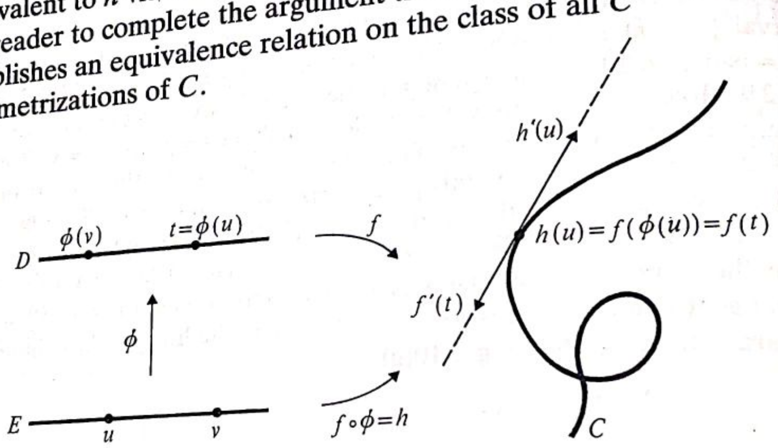


Fig. 2.11 Equivalent parametrizations of f and h with opposite orientations

Notice that ϕ may be strictly increasing (in which case f and h trace the curve in the same direction) or ϕ may be strictly decreasing (in which case they trace the curve in opposite directions). In the former case we say that f and h are *properly equivalent* or *equivalent with the same orientation* and in the latter case that they are *equivalent with opposite orientations*. See Fig. 2.11.

2.6.10 *Example.* The functions g and h considered in Example 2.5.8 were given by $g(t) = (\cos t, \sin t)$, $t \in \mathbb{R}$, and $h(u) = (\sin 2u, \cos 2u)$, $u \in \mathbb{R}$. They are both C^1 parametrizations of the unit circle in $\mathbb{R}^2$. Let $\phi : \mathbb{R} \to \mathbb{R}$ be defined by $\phi(u) = \frac{1}{2}\pi - 2u$, $u \in \mathbb{R}$. Then $h = g \circ \phi$ and $\phi'(u) < 0$ for all $u \in \mathbb{R}$. Hence g and h are equivalent parametrizations with opposite orientation – they trace the circle in opposite directions.

In order to relate the properties of equivalent parametrizations we need to know how their derivatives are related. The required result is a Chain Rule of the following type.

2.6.11 ***Theorem.*** (Chain Rule). *Let $\phi : E \subseteq \mathbb{R} \to \mathbb{R}$ and $f : D \subseteq \mathbb{R} \to \mathbb{R}^n$ be differentiable functions and let $\phi(E) \subseteq D$. Then $f \circ \phi : E \subseteq \mathbb{R} \to \mathbb{R}^n$ is differentiable, and for each $u \in E$,*

2.6.12
$$(f \circ \phi)'(u) = (f'(\phi(u)))\phi'(u).$$

Remark. We have written the scalar $\phi'(u)$ on the right of the vector $f'(\phi(u))$ in 2.6.12 in order to relate this theorem firstly to the elementary Chain Rule (Theorem 1.6.11) and secondly to generalizations in subsequent chapters.

Proof. For each $i = 1, \ldots, n$ the coordinate functions of $f \circ \phi$ and f satisfy

2.6.13
$$(f \circ \phi)_i(u) = f_i(\phi(u)), \qquad \text{for all } u \in E.$$

We are given that ϕ is differentiable at $u \in E$, and that f (and therefore f_i) is differentiable at $\phi(u)$. Expression 2.6.13 and the elementary Chain Rule 1.6.11 together imply that $(f \circ \phi)_i$ is differentiable at u and that

$$(f \circ \phi)_i'(u) = f_i'(\phi(u))\phi'(u).$$

The required conclusion follows from Theorem 2.5.7.

2.6.14 ***Example.*** Let $\phi : \mathbb{R} \to \mathbb{R}$ and $f : \mathbb{R} \to \mathbb{R}^3$ be defined by $\phi(u) = \sin u$, $u \in \mathbb{R}$ and $f(t) = (2t, t^3, 1)$, $t \in \mathbb{R}$. Then $(f \circ \phi)(u) = (2\sin u, \sin^3 u, 1)$, $u \in \mathbb{R}$. We have by way of illustration of the Chain Rule,

$$f'(\phi(u)) = (2, 3\sin^2 u, 0), \quad \phi'(u) = \cos u, \text{ and}$$
$$(f \circ \phi)'(u) = (2\cos u, 3\sin^2 u \cos u, 0), \qquad u \in \mathbb{R}.$$

The relationship $h = f \circ \phi$ will occur frequently when we compare the properties of two parametrizations f and h of a curve, as in the following theorem.

2.6.15 ***Theorem.*** *Let $f : D \subseteq \mathbb{R} \to \mathbb{R}^n$ and $h : E \subseteq \overset{\mathbb{R}}{\cancel{D}} \to \mathbb{R}^n$ be equivalent C^1 parametrizations of a given curve C in $\mathbb{R}^n$ related by a differentiable function $\phi : E \to D$ with the properties given in Definition 2.6.9. Then, whenever $\phi(u) = t$,*

[i] *the tangent vector to f at t, if non-zero, is in the same (opposite) direction as the tangent vector to h at u when f and h have the same (opposite) orientations;*

[ii] *the tangent lines to f at t and to h at u are the same if either of them exists*

Proof. Immediate from the Chain Rule 2.6.11. The theorem is illustrated in Fig. 2.11, where f and h have opposite orientations, the tangent vectors being in opposite directions.

2.6.16 ***Example.*** The functions

$$g(t) = (\cos t, \sin t) \qquad \text{and} \qquad h(u) = (\sin 2u, \cos 2u)$$

considered in Example 2.6.10 are C^1 parametrizations of the unit circle with opposite orientations. For all $u \in \mathbb{R}$, $h(u) = g(\phi(u))$, where $\phi(u) = \frac{1}{2}\pi - 2u$, and the tangent vectors

$$h'(u) = (2\cos 2u, -2\sin 2u) \quad \text{and} \quad g'(\phi(u)) = (-\cos 2u, \sin 2u)$$

are in opposite directions. The tangent lines to h at u and to g at $\phi(u)$ are the same.

2.6.17 ***Example.*** Consider the functions

$$f(t) = (\sin t, \sin 2t),\ t \in]-\pi, \pi[$$

and

$$h(u) = (\sin u, \sin 2u),\ u \in]0, 2\pi[$$

of Example 2.6.8 which parametrize the 'figure 8' curve. If there were a differentiable function $\phi:]0, 2\pi[\to]-\pi, \pi[$ such that $h = f \circ \phi$ then, since $(0,0) = h(\pi) = f(\phi(\pi))$, we would have to have $\phi(\pi) = 0$. Furthermore by Theorem 2.6.15 (ii) the tangent line to h at π would be the same as the tangent line to f at $\phi(\pi) = 0$. But $h'(\pi) = (-1, 2)$ and $f'(0) = (1, 2)$, hence the tangent lines are different (see Fig. 2.10). Therefore no such ϕ exists.

So far, both in this section and the previous one, we have considered tangent lines to functions which parametrize curves. Intuitively, however, provided a curve C does not have a 'corner' at $\mathbf{q} \in C$ and provided C does not cross itself at $\mathbf{q}$, the tangent line to C at $\mathbf{q}$ exists and is independent of parametrization. We can exclude the possibility of corners by requiring that the parametrization should have non-zero derivative (see Example 2.5.17). We can exclude the possibility of crossovers by firstly requiring the parametrization to be 1–1 (see Example 2.6.7) and secondly requiring

The terminology 'smooth' used here is incorrect. It's better to replace it by 'regular.'

that its domain should be a compact interval $[a, b]$ (compare Example 2.6.8.).

2.6.18 Definition. *A function $f : D \subseteq \mathbb{R} \to \mathbb{R}^n$ is said to be* smooth *if it is C^1 (continuously differentiable) and if $f'(t) \neq 0$ for all $t \in D$.*

2.6.19 Example. There is no smooth parametrization of the curve $y = |x|$ in $\mathbb{R}^2$. See Fig. 2.8(ii).

2.6.20 Definition. *A curve C in $\mathbb{R}^n$ is a* (smooth) simple arc *if C has a 1–1 (smooth) C^1 parametrization of the form $f : [a, b] \subseteq \mathbb{R} \to \mathbb{R}^n$. The points $f(a)$ and $f(b)$ are then called the* end points *of the arc. The function f is called a* simple parametrization *of C.*

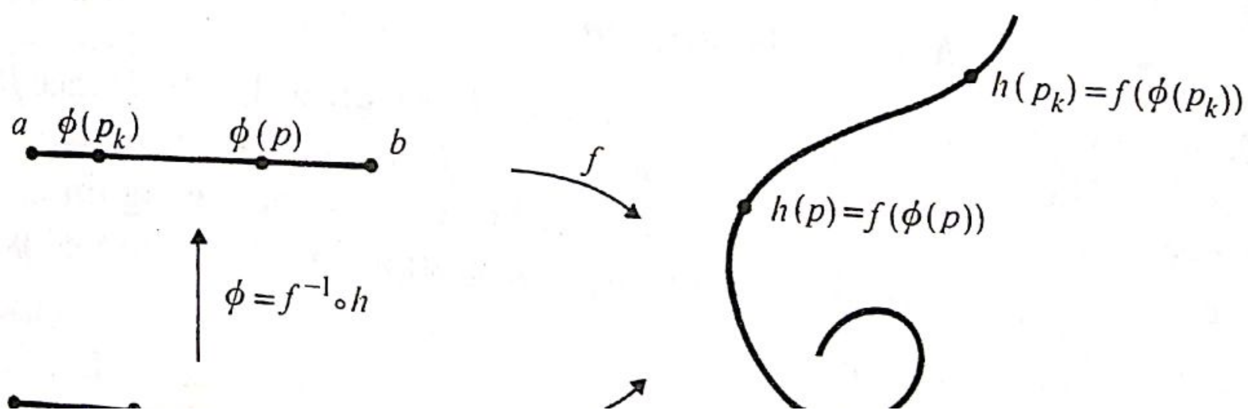
2.6.21 Example. Let $g : [a, b] \subseteq \mathbb{R} \to \mathbb{R}$ be a C^1 function. The graph G of g is a smooth simple arc in $\mathbb{R}^2$ parametrized by the smooth, 1–1 C^1 function $f : [a, b] \subseteq \mathbb{R} \to \mathbb{R}^2$ defined by

2.6.22
$$f(t) = (t, g(t)), \qquad t \in [a, b].$$

The end points of G are $(a, g(a))$ and $(b, g(b))$.

2.6.23 Theorem. *Let C be a smooth simple arc in $\mathbb{R}^n$ simply parametrized by a smooth 1–1 function $f : [a, b] \subseteq \mathbb{R} \to \mathbb{R}^n$. Then any smooth parametrization $h : [c, d] \subseteq \mathbb{R} \to \mathbb{R}^n$ of C is 1–1 and is equivalent to f.*

Proof. The function $\phi = f^{-1} \circ h$ is well defined, maps $[c, d]$ onto $[a, b]$ and satisfies $h = f \circ \phi$. See Fig. 2.12. Choose any $p \in [c, d]$.



(i) We begin by showing that ϕ is continuous at p. If it were not then there would be a sequence $p_k \to p$ in $[c, d]$ such that $\phi(p_k) \not\to \phi(p)$ in $[a, b]$. Since $[a, b]$ is compact we may assume (by taking a subsequence if necessary – see Theorem 1.6.5) that

2.6.24 $$\phi(p_k) \to r \quad \text{in} \quad [a, b] \quad \text{and} \quad r \neq \phi(p).$$

But f is 1–1 and continuous, and $f \circ \phi = h$. Therefore, applying f to 2.6.24,

$$h(p_k) \to f(r) \quad \text{in} \quad C \quad \text{and} \quad f(r) \neq h(p).$$

This contradicts the continuity of h at p.

(ii) We next consider the differentiability of ϕ at p. Consider a sequence

2.6.25 $$p_k \to p \quad \text{in} \quad [c, d] \quad \text{with } p_k \neq p \quad \text{for all} \quad k \in \mathbb{N}.$$

We need to know that, for large k, $\phi(p_k) \neq \phi(p)$. This follows from the smoothness of h by considering the limit

$$\lim_{k \to \infty} \frac{f(\phi(p_k)) - f(\phi(p))}{p_k - p} = \lim_{k \to \infty} \frac{h(p_k) - h(p)}{p_k - p} \neq 0$$

Now, since f is smooth, we assume without loss of generality that $f_1'(\phi(p)) \neq 0$. Consider the identity

2.6.26 $$\frac{h_1(p_k) - h_1(p)}{p_k - p} = \frac{f_1(\phi(p_k)) - f_1(\phi(p))}{\phi(p_k) - \phi(p)} \, \frac{\phi(p_k) - \phi(p)}{p_k - p}.$$

Since ϕ is continuous at p, letting k tend to infinity in 2.6.26 establishes that ϕ is differentiable at p.

(iii) The above argument shows that ϕ is differentiable throughout $[a, b]$. Since $h = f \circ \phi$ the Chain Rule implies that

$$h'(u) = f'(\phi(u))\phi'(u), \qquad u \in [c, d].$$

Since h is smooth, it follows that $\phi'(u) \neq 0$ for all $u \in [c, d]$. Hence (by Rolle's Theorem) the functions ϕ and $h = f \circ \phi$ are 1–1. Finally, since ϕ is continuous and 1–1, ϕ is either strictly increasing (in which case ϕ' is positive) or strictly decreasing (in which case ϕ' is negative).

2.6.27 Corollary. *Let* $\mathbf{q}$ *be a point on a smooth simple arc* C *in* $\mathbb{R}^n$. *Then all smooth parametrizations of* C *associate the same tangent line with* $\mathbf{q}$.

Remark. In view of this corollary we can talk about the tangent line to a smooth simple arc C at a point $\mathbf{q}$ on C.

Proof. Theorem 2.6.23 implies that two smooth parametrizations of C are equivalent and so the result follows from Theorem 2.6.15.

2.6.28 ***Example.*** Consider again the smooth simple arc G which is the graph of a C^1 function $g:[a, b]\subseteq\mathbb{R}\to\mathbb{R}$ (see Example 2.6.21). From 2.6.22 the tangent line to G at $(t, g(t))$ is in direction $(1, g'(t))$.

If G is smoothly parametrized by $(x(u), y(u))$, then the tangent line at $(x(u), y(u))$ has direction $(x'(u), y'(u))$. Hence

2.6.29
$$g'(x(u))=\frac{y'(u)}{x'(u)}.$$

Informally, 2.6.29 expresses the fact that if a graph $y=g(x)$ is smoothly parametrized by $(x(u), y(u))$ then

$$\frac{dy}{dx}=\frac{dy/du}{dx/du}.$$

Since, by Theorem 2.6.23, any two smooth parametrizations of a smooth simple arc are equivalent, the smooth parametrizations of such an arc divide into two classes according to their orientation.

2.6.30 ***Definition.*** *Let C be a smooth simple arc in $\mathbb{R}^n$ parametrized by a smooth function $f:[a, b]\subseteq\mathbb{R}\to\mathbb{R}^n$. The pair $\{C, [f]\}$, where $[f]$ is the set of all smooth parametrizations of C which are properly equivalent to f, is called an* oriented smooth simple arc.

To every smooth simple arc C there correspond precisely two oriented simple arcs. If $\mathbf{a}$ and $\mathbf{b}$ are the end points of C, the two possible orientations can be described as being from $\mathbf{a}$ to $\mathbf{b}$ and from $\mathbf{b}$ to $\mathbf{a}$ respectively. See Fig. 2.13.

We have seen how Definition 2.6.9 establishes an equivalence relation on the set of all parametrizations of a curve in $\mathbb{R}^n$. In any equivalence class either all parametrizations are smooth or none is smooth. In general there may be a large number of equivalence classes of smooth parametrizations. For example, a parametrization of a circle that traces the circle k times cannot be equivalent to one that traces it l times, where $k\neq l$. However, when the curve is a *smooth simple arc,* there are just two classes of properly equivalent smooth parametrizations. These parametrizations are 1–1. They

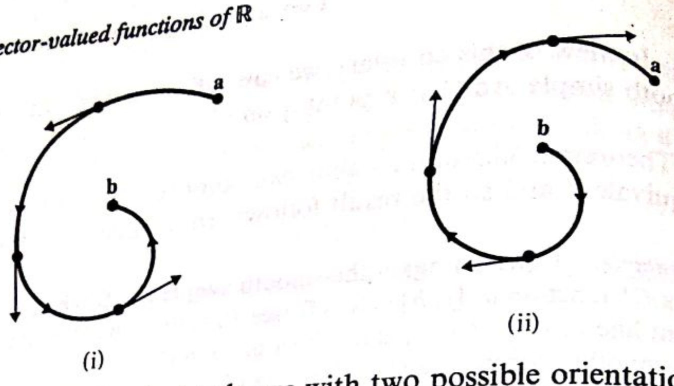


Fig. 2.13 Smooth simple arc with two possible orientations
(i) Orientation from **a** to **b**
(ii) Orientation from **b** to **a**

correspond to the two orientations of the curve from one end point to the other.

A simple arc that is not smooth also has two possible orientations. Correspondingly we might expect just two classes of properly equivalent 1–1 parametrizations of a (not necessarily smooth) simple arc. However, the equivalence relation suggested by Definition 2.6.9 is not the appropriate tool for the classifications of 1–1 parametrizations of a simple arc into just two classes.

2.6.31 ***Example.*** Let C be the simple arc $y = x^2$, $x \in [-1, 1]$. (The curve C happens to be smooth, but this is not relevant to this example.) Then no two of the following four 1–1 parametrizations of C are properly equivalent:

$$f(t) = (t, t^2); \quad g(t) = (-t, t^2); \quad h(t) = (t^3, t^6); \quad k(t) = (-t^3, t^6)$$

for all $t \in [-1, 1]$.

Clearly f and g are not properly equivalent since they trace C in opposite directions. Suppose next that $h = f \circ \phi$. Then $\phi(u) = u^3$, and so $\phi'(0) = 0$. Thus by Definition 2.6.9 f and h are not equivalent. The remaining cases are treated similarly.

The appropriate equivalence relation that reflects the two possible orientations of a simple arc is obtained by dropping the differentiability conditions on ϕ in Definition 2.6.9 and replacing them by continuity and monotonicity. The following theorem sums up the situation.

2.6.32 Theorem. *Let $f : [a, b] \subseteq \mathbb{R} \to \mathbb{R}^n$ and $h : [c, d] \subseteq \mathbb{R} \to \mathbb{R}^n$ be*

1–1 parametrizations of a simple arc C in $\mathbb{R}^n$. Then there exists a unique function ϕ from $[a, b]$ onto $[c, d]$ such that $h = f \circ \phi$. Moreover, ϕ is continuous and strictly monotonic. In particular,
[i] if ϕ is strictly increasing then $f(a) = h(c)$ and $f(b) = h(d)$, and
[ii] if ϕ is strictly decreasing then $f(a) = h(d)$ and $f(b) = h(c)$.

Proof. By the proof of Theorem 2.6.23 the function $\phi = f^{-1} \circ h$ is 1–1, continuous and strictly monotonic.

2.6.33 ***Example.*** The functions

$f:[-1, 1] \subseteq \mathbb{R} \to \mathbb{R}^2$ defined by $f(t) = (t, \sqrt{(1-t^2)})$, $\quad t \in [-1, 1]$,
$g:[0, \pi] \subseteq \mathbb{R} \to \mathbb{R}^2$ defined by $g(t) = (\cos t, \sin t)$, $\quad t \in [0, \pi]$,
$h:[-1, 0] \subseteq \mathbb{R} \to \mathbb{R}^2$ defined by $h(t) = (\cos \pi t, -\sin \pi t)$, $\quad t \in [-1, 0]$

are all 1–1 parametrizations of the simple arc $C = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1, y \geqslant 0\}$, the upper half of the unit circle centred at the origin. The end points of C are $(-1, 1)$ and $(1, 1)$. The parametrizations f and h run from $(-1, 0)$ to $(1, 0)$ while g runs from $(1, 0)$ to $(-1, 0)$.

We find that $h = f \circ \phi$ where $\phi(u) = \cos(\pi u)$, $u \in [-1, 0]$, which is strictly increasing. On the other hand $g = f \circ \psi$, where $\psi(\omega) = \cos \omega$, $\omega \in [0, \pi]$, which is strictly decreasing.

Notice that g and h are C^1 parametrizations of C but f is not C^1.

Finally in this section we consider how the above material can be adjusted to cope with parametrizations which fail to be C^1 at a finite number of points.

2.6.34 ***Example.*** Consider the function $f:[-1, 1] \subseteq \mathbb{R} \to \mathbb{R}^2$ defined by

$$f(t) = (t, |t|), \qquad t \in [-1, 1].$$

whose image is shown in Fig. 2.8(ii). The function f is not C^1 because f is not differentiable at 0. However, we may regard f as being composed of two C^1 pieces $f_1:[-1, 0] \subseteq \mathbb{R} \to \mathbb{R}^2$ and $f_2:[0, 1] \subseteq \mathbb{R} \to \mathbb{R}^2$, where $f_1(t) = f(t) = (t, -t)$ when $t \in [-1, 0]$ and $f_2(t) = f(t) = (t, t)$ when $t \in [0, 1]$.

Example 2.6.34 illustrates a piecewise C^1 path. The general definition is as follows.

2.6.35 ***Definition.*** *A function $f:[a, b] \subseteq \mathbb{R} \to \mathbb{R}^n$ is said to be* piecewise C^1 (piecewise smooth) *if there is a subdivision $a = p_0 < p_1 < \cdots < p_r = b$ of $[a, b]$ such that for each $k = 1, \ldots, r$ the function f restricted to the sub-interval $[p_{k-1}, p_k]$ is C^1 (smooth).*

Notice that the piecewise C^1 function f considered in Definition 2.6.35 may not be differentiable at p_k for $k=1,\ldots,r-1$. However, the fact that f is differentiable on $[p_{k-1}, p_k]$ and on $[p_k, p_{k+1}]$ requires that both the left-hand and right-hand derivatives of f at p_k must exist.

2.6.36 ***Example.*** The function $f:[-1,1]\subseteq\mathbb{R}\to\mathbb{R}^2$ considered in Example 2.6.34 is piecewise C^1 (consider the partition $-1<0<1$ of $[-1,1]$). It is also piecewise smooth. The function f is not differentiable at 0. However, the left-hand derivative of f at 0 is $(1,-1)$ and the right-hand derivative of f at 0 is $(1,1)$.

The image curve C of the function f is sketched in Fig. 2.8(ii). Compare this parametrization of C with the parametrization $g:[-1,1]\subseteq\mathbb{R}\to\mathbb{R}^2$ of C given by

$$g(u)=(u^3, |u^3|), \qquad u\in[-1,1].$$

The function g is a C^1 parametrization of C.

Furthermore f and g have a weaker form of proper equivalence, since if $\phi:[-1,1]\to[-1,1]$ is defined by

$$\phi(t)=t^{1/3}, \qquad t\in[-1,1]$$

then ϕ is strictly increasing and differentiable except at 0 and $f=g\circ\phi$. Notice that $g=f\circ\phi^{-1}$, where ϕ^{-1} is strictly increasing. Incidentally, ϕ^{-1} is differentiable.

2.6.37 ***Remark.*** The process adopted in Example 2.6.36 may be thought of as transforming a piecewise C^1 parametrization to give a 'properly equivalent' C^1 parametrization of the same curve in the weaker sense described above. In fact, given any piecewise C^1 parametrization of a curve there is a C^1 parametrization which is 'properly equivalent' to it in this sense. Consequently a curve which has a 1–1 piecewise C^1 parametrization is automatically a simple arc (Definition 2.6.20).

Exercises 2.6

1. Verify that the following subsets C of $\mathbb{R}^2$ are curves in $\mathbb{R}^2$ according to Definition 2.6.2. In each case find a C^1 parametrization of C and sketch the curve.
   (a) The y-axis, $x=0$;
   (b) part of the y-axis, $x=0,\ -1\leqslant y\leqslant 2$;
   (c) the parabola $y+1=(x-2)^2$;
   (d) the circle $(x-1)^2+(y-2)^2=4$;
   (e) the ellipse $4x^2+y^2=1$;
   (f) the subset C defined by $x=|y|$;

(g) the subset C defined by $x = |y|$, $0 \leqslant x \leqslant 1$;
(h) the subset C defined by $y^2 = x^3$, $0 \leqslant x \leqslant 1$;
(i) the subset C parametrized by

$$f(t) = (\sin 2t \cos t, \sin 2t \sin t), \qquad 0 \leqslant t \leqslant 2\pi.$$

Answers: (a) $(0, t)$, $t \in \mathbb{R}$, alternatively $(0, t^3)$, $t \in \mathbb{R}$. For each example there are many possibilities; (b) $(0, t)$, $-1 \leqslant t \leqslant 2$; (c) $(t, (t-2)^2 - 1)$, alternatively $(t+2, t^2 - 1)$; (d) $(1 + 2\cos t, 2 + 2\sin t)$; (e) $(\frac{1}{2}\cos t, \sin t)$; (f) $(t^2, t|t|)$; (g) $(t^2, t|t|)$, $-1 \leq t \leq 1$; (h) (t^2, t^3), $-1 \leq t \leq 1$; (i) C^1 parametrization as given. The curve has four 'petals' meeting at the origin.

2. Which of the curves in Exercise 1 can be smoothly parametrized?

Answer: all except (f), (g), (h).

Which of the curves in Exercise 1 are simple arcs? Which are smooth simple arcs?

Answer: (b), (g), (h) are simple arcs; of these, only (b) is smooth.

3. For each of the curves C of Exercise 1 find, where appropriate, the equation of the tangent line to C at the point $\mathbf{q} = (a, b)$ on C. Indicate points on C where no tangent line exists.

(*Hint*: using an appropriate parametrization, apply the method of Section 2.5.)

Answers: (a) $x = 0$,
(c) $y - b = (2a - 4)(x - a)$, where $b = (a-2)^2 + 1$;
(e) $(x, y) = (\frac{1}{2}\cos\alpha, \sin\alpha) + s(-\frac{1}{2}\sin\alpha, \cos\alpha)$, $s \in \mathbb{R}$, where $a = \frac{1}{2}\cos\alpha$, $b = \sin\alpha$. For example, the tangent line to C at $(\frac{1}{2}, 0)$ is $x = \frac{1}{2}$, and the tangent line to C at $(\frac{1}{4}, -\frac{1}{2}\sqrt{3})$ is $y + \frac{1}{2}\sqrt{3} = (2/\sqrt{3})(x - \frac{1}{4})$. No tangent lines to C at $(0, 0)$ in examples (f) – (i) inclusive.)

4. Indicate on your sketches of the curves C of Exercise 1 the orientation assigned to C by your choice of C^1 parametrization $f : D \subseteq \mathbb{R} \to \mathbb{R}^n$. (Use arrows to indicate the direction in which C is traced.)

   Obtain in each case an equivalent parametrization $h = f \circ \phi$ where $\phi(u) = -u$, and verify that the orientation assigned to C by h is opposite to that assigned to C by f.

5. Sketch the simple arc C in $\mathbb{R}^2$ defined by

$$x = y^2, \qquad x \leqslant 4.$$

(a) Verify that the following functions define 1–1 smooth parametrizations of C:
(i) $f : [-3, 1] \subseteq \mathbb{R} \to \mathbb{R}^2$ defined by

$$f(t) = (t^2 + 2t + 1, t + 1), \qquad t \in [-3, 1]$$

(ii) $h:[0, 1]\subseteq\mathbb{R}\to\mathbb{R}^2$ defined by

$$h(u)=(16u^2-16u+4, 2-4u), \qquad u\in[0, 1].$$

Show that f and h are equivalent parametrizations of C by constructing a suitable differentiable function ϕ from $[0, 1]$ on to $[-3, 1]$ such that $h=f\circ\phi$. Are f and h *properly* equivalent or not?

Find the equation of the tangent line to C at the point $(\frac{1}{4}, -\frac{1}{2})$ on C (expressing your answer in the form $ax+by=c$), (1) from the parametrization f, (2) from the parametrization h.

(b) Now consider the function $g:[-1, 1]\subseteq\mathbb{R}\to\mathbb{R}^2$ defined by

$$g(w)=(4w^6, 2w^3), \qquad w\in[-1, 1].$$

Show that g is a C^1 parametrization of C. Is g 1–1? Smooth? Are f and g equivalent parametrizations of C (in the sense of Definition 2.6.9)? Find functions $\phi:[-1, 1]\to[-3, 1]$ and $\psi:[-3, 1]\to[-1, 1]$ such that $g=f\circ\phi$ and $f=g\circ\psi$. Is ϕ differentiable? Is ψ differentiable?

Answer: $\phi(w)=2w^3-1$, $\psi(t)=(\frac{1}{2}(t+1))^{1/3}$, not differentiable at $t=-1$.

6. Sketch the simple arc C parametrized by the (not smooth) C^1 function $f:[-1, 1]\subseteq\mathbb{R}\to\mathbb{R}^2$ defined by $f(t)=(t^3, t^5)$. By finding a suitable smooth C^1 re-parametrization of C, show that C is a smooth simple arc.

Answer: $h(u)=(u, u^{5/3})$, $u\in[-1, 1]$.

7. Sketch the curve C in $\mathbb{R}^2$ smoothly parametrized by

$$g(t)=((2\cos t-1)\cos t, (2\cos t-1)\sin t), t\in[0,2\pi].$$

Show that $(0, 0)=g(\pi/3)=g(5\pi/3)$, but $g'(\pi/3)\neq g'(5\pi/3)$. Interpret this on your sketch.

Using the method of Examples 2.6.7 and 2.6.8, obtain 1–1 C^1 parametrizations f and h of C such that (a) for $p\in\mathbb{R}$ such that $f(p)=(0, 0)$, $f'(p)=g'(\pi/3)$, (b) for $q\in\mathbb{R}$ such that $h(q)=(0, 0)$, $h'(q)=g'(5\pi/3)$.

8. Prove that the equivalence of C^1 parametrizations given in Definition 2.6.9 is an equivalence relation on the class of all C^1 parametrizations of curves in $\mathbb{R}^n$.

Hint: Prove symmetry by applying Theorem 1.6.7.

9. Given that $\phi(u)=\sin u$, $u\in E\subseteq\mathbb{R}$, and $f(t)=(t^2, \sqrt{t})$, $t\geqslant 0$, find a suitable interval E such that $f\circ\phi$ is defined and differentiable on E. Apply the Chain Rule 2.6.11 to calculate $(f\circ\phi)'(u)$, $u\in E$.

10. Apply the Chain Rule 2.6.11 to calculate $h'(u)$, where
(a) $h(u)=(\exp(\sin u), 2(\sin u)^2)$, $\quad u\in\mathbb{R}$;
(b) $h(u)=(2\cos\sqrt{u}, \sin\sqrt{u})$, $\quad u>0$.

11. Sketch the curve C p
$f(t)=(t-\sin t, 1-$ c
fixed on the rim of
x-axis.) Prove that

Hint: consider the poi

12. Let C be a curve
$(f_1(t), f_2(t))$, $t\in D$,
graph of a C^1 fu

(*Hint*: apply Theore
$f_2\circ f_1^{-1}$.)

Deduce that t
f on the open i

13. The following
G which is the

Indicate the
$t=\frac{1}{2}$.
Verify tha

and that

2.7 Pa

As a partic
function o
enables us
interval b
position
be no ins
function

2.7.1 D
path *in*
if the f

(ii) $h:[0, 1]\subseteq\mathbb{R}\to\mathbb{R}^2$ defined by

$$h(u)=(16u^2-16u+4, 2-4u), \qquad u\in[0, 1].$$

Show that f and h are equivalent parametrizations of C by constructing a suitable differentiable function ϕ from $[0, 1]$ on to $[-3, 1]$ such that $h=f\circ\phi$. Are f and h *properly* equivalent or not?

Find the equation of the tangent line to C at the point $(\frac{1}{4}, -\frac{1}{2})$ on C (expressing your answer in the form $ax+by=c$), (1) from the parametrization f, (2) from the parametrization h.

(b) Now consider the function $g:[-1, 1]\subseteq\mathbb{R}\to\mathbb{R}^2$ defined by

$$g(w)=(4w^6, 2w^3), \qquad w\in[-1, 1].$$

Show that g is a C^1 parametrization of C. Is g 1–1? Smooth? Are f and g equivalent parametrizations of C (in the sense of Definition 2.6.9)? Find functions $\phi:[-1, 1]\to[-3, 1]$ and $\psi:[-3, 1]\to[-1, 1]$ such that $g=f\circ\phi$ and $f=g\circ\psi$. Is ϕ differentiable? Is ψ differentiable?

Answer: $\phi(w)=2w^3-1$, $\psi(t)=(\frac{1}{2}(t+1))^{1/3}$, not differentiable at $t=-1$.

6. Sketch the simple arc C parametrized by the (not smooth) C^1 function $f:[-1, 1]\subseteq\mathbb{R}\to\mathbb{R}^2$ defined by $f(t)=(t^3, t^5)$. By finding a suitable smooth C^1 re-parametrization of C, show that C is a smooth simple arc.

Answer: $h(u)=(u, u^{5/3})$, $u\in[-1, 1]$.

7. Sketch the curve C in $\mathbb{R}^2$ smoothly parametrized by

$$g(t)=((2\cos t-1)\cos t, (2\cos t-1)\sin t), t\in[0,2\pi].$$

Show that $(0, 0)=g(\pi/3)=g(5\pi/3)$, but $g'(\pi/3)\neq g'(5\pi/3)$. Interpret this on your sketch.

Using the method of Examples 2.6.7 and 2.6.8, obtain 1–1 C^1 parametrizations f and h of C such that (a) for $p\in\mathbb{R}$ such that $f(p)=(0, 0)$, $f'(p)=g'(\pi/3)$, (b) for $q\in\mathbb{R}$ such that $h(q)=(0, 0)$, $h'(q)=g'(5\pi/3)$.

8. Prove that the equivalence of C^1 parametrizations given in Definition 2.6.9 is an equivalence relation on the class of all C^1 parametrizations of curves in $\mathbb{R}^n$.

Hint: Prove symmetry by applying Theorem 1.6.7.

9. Given that $\phi(u)=\sin u$, $u\in E\subseteq\mathbb{R}$, and $f(t)=(t^2,\sqrt{t})$, $t\geqslant 0$, find a suitable interval E such that $f\circ\phi$ is defined and differentiable on E. Apply the Chain Rule 2.6.11 to calculate $(f\circ\phi)'(u)$, $u\in E$.

10. Apply the Chain Rule 2.6.11 to calculate $h'(u)$, where
(a) $h(u)=(\exp(\sin u), 2(\sin u)^2)$, $\quad u\in\mathbb{R}$;
(b) $h(u)=(2\cos\sqrt{u}, \sin\sqrt{u})$, $\quad u>0$.

11. Sketch the curve C parametrized by $f:\mathbb{R}\to\mathbb{R}^2$, where $f(t)=(t-\sin t,\ 1-\cos t), t\in\mathbb{R}$ (The curve C is called a *cycloid*. A point fixed on the rim of a unit circle traces C when the circle rolls along the x-axis.) Prove that C is not the graph of a differentiable function.

Hint: consider the points $t=2k\pi$, $k\in\mathbb{N}$.

12. Let C be a curve smoothly parametrized by a C^1 function $f(t)=(f_1(t), f_2(t))$, $t\in D$, where $f'_1(t)>0$ for all $t\in D$. Prove that C is the graph of a C^1 function $g:D_1\subseteq\mathbb{R}\to\mathbb{R}$, where $D_1=f_1(D)$.

(*Hint*: apply Theorem 1.6.7. The required function is the composite function $f_2\circ f_1^{-1}$.)

Deduce that the part of the cycloid (Exercise 11) which is the image of f on the open interval $]0, 2\pi[$ is the graph of a differentiable function.

13. The following exercise illustrates Example 2.6.28. Sketch the simple arc G which is the graph of the C^1 function

$$g(t)=\sqrt{(1-t^2)}, \qquad t\in[-\tfrac{1}{2}, \tfrac{1}{2}].$$

Indicate the tangent lines to G at $(t, g(t))$ for the cases $t=-\frac{1}{2}$, $t=0$ and $t=\frac{1}{2}$.

Verify that G is smoothly parametrized by

$$(x(\theta), y(\theta))=(\cos\theta, \sin\theta), \qquad \theta\in[\pi/3, 2\pi/3],$$

and that

$$g'(x(\theta))=\frac{y'(\theta)}{x'(\theta)}=-\cot\theta.$$

2.7 Path length and length of simple arcs

As a particle moves about in space its position can be regarded as a function of time. Choosing coordinate systems for space and time enables us to describe the motion of the particle over a finite time interval by a function $f:[a, b]\subseteq\mathbb{R}\to\mathbb{R}^3$, where the particle has position vector $f(t)$ at time t units. In most applications there will be no instantaneous jumps in the motion and so the corresponding function will be continuous. This leads us to the following definition.

2.7.1 *Definition.* *A continuous function $f:[a, b]\subseteq\mathbb{R}\to\mathbb{R}^n$ is called a* path *in $\mathbb{R}^n$ from $f(a)$ to $f(b)$. The path is differentiable, C^1 or smooth if the function f is respectively differentiable, C^1 or smooth.*

The image of a C^1 path is a curve and the image of a 1–1 smooth path is a smooth simple arc.

When a C^1 path f describes the motion of a particle in $\mathbb{R}^n$, the distance travelled by the particle in the time interval between t and $t+\Delta t$, for small positive Δt, is approximately $\|f(t+\Delta t)-f(t)\|$. This distance, by definition of the derivative, is approximately $\|f'(t)\|\Delta t$. Accordingly, the length of the path is defined as follows.

2.7.2 *Definition.* *Let* $f:[a,b]\subseteq\mathbb{R}\to\mathbb{R}^n$ *be a* C^1 *path in* $\mathbb{R}^n$. *The* **length** *of* f *is defined to be*

$$l(f)=\int_a^b \|f'(t)\|\,\mathrm{d}t.$$

2.7.3 *Example.* The function $g:[-r,r]\subseteq\mathbb{R}\to\mathbb{R}^2$ defined by

$$g(u)=(u,\sqrt{(r^2-u^2)}),\qquad u\in[-r,r]$$

is a path in $\mathbb{R}^2$ whose image is a semi-circle of radius $r>0$. However, g is not a C^1 path (why?).

We can reparametrize the semi-circle by defining a C^1 path $f:[0,\pi]\subseteq\mathbb{R}\to\mathbb{R}^2$ by

$$f(t)=(-r\cos t, r\sin t),\qquad t\in[0,\pi].$$

We have $f'(t)=(r\sin t, r\cos t)$, and so $\|f'(t)\|=r$. Hence

$$l(f)=\int_0^\pi r\,\mathrm{d}t=\pi r.$$

2.7.4 *Example.* The function $f:[0,2\pi]\subseteq\mathbb{R}\to\mathbb{R}^2$ defined by

$$f(t)=(\cos 2t, \sin 2t),\qquad t\in[0,2\pi]$$

is a C^1 path parametrizing the unit circle, and

$$\|f'(t)\|=2\|(-\sin 2t, \cos 2t)\|=2.$$

Hence

$$l(f)=\int_0^{2\pi} 2\,\mathrm{d}t=4\pi.$$

As expected, $l(f)$ is twice the circumference of the unit circle which $f(t)$ traverses twice.

2.7.5 *Example.* The 1–1 C^1 path $f:[-1,1]\subseteq\mathbb{R}\to\mathbb{R}^2$ defined by

$$f(t)=(t^2,t^3),\qquad t\in[-1,1]$$

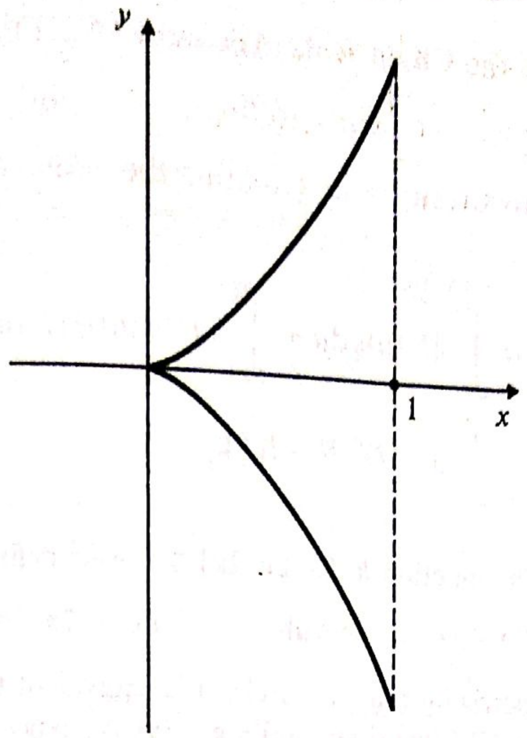


Fig. 2.14 Image of $f(t)=(t^2, t^3)$, $t\in[-1, 1]$

has image curve lying in the semi-cubical parabola $y^2=x^3$ (see Fig. 2.14). We have $f'(t)=(2t, 3t^2)$, and so

$$l(f)=\int_{-1}^{1}\sqrt{(4t^2+9t^4)}\mathrm{d}t=2\int_0^1 t\sqrt{4+9t^2}\,\mathrm{d}t.$$

The substitution $u(t)=4+9t^2$ leads to $l(f)=\frac{2}{27}(13\sqrt{13}-8)$.

We would expect that if two C^1 paths trace the points of a curve in the same order, then their lengths would be the same. Similarly, we would expect that the length of a C^1 path is the same as the length of the C^1 path which traces the points of the curve in the reverse order. These impressions are confirmed by the following theorem.

2.7.6 *Theorem.* *Let $f:[a, b]\subseteq\mathbb{R}\to\mathbb{R}^n$ and $h:[c, d]\subseteq\mathbb{R}\to\mathbb{R}^n$ be two equivalent C^1 paths parametrizing a curve C in $\mathbb{R}^n$. Then $l(f)=l(h)$.*

Proof. Since f and h are equivalent, there exists a 1–1 differentiable function ϕ mapping $[c, d]$ onto $[a, b]$ such that $h=f\circ\phi$. Furthermore, either $\phi'(u)>0$ for all $u\in[c, d]$, in which case $a=\inf\phi=\phi(c)$ and $b=\sup\phi=\phi(d)$, or $\phi'(u)<0$ for all $u\in[c, d]$, in which case $b=\phi(c)$ and $a=\phi(d)$.

It follows from the Chain Rule (Theorem 2.6.11) that

$$\|h'(u)\| = \|f'(\phi(u))\| \, |\phi'(u)|, \qquad u \in [c, d].$$

Hence, by the substitution rule (treating the above alternatives separately),

$$l(h) = \int_c^d \|h'(u)\| \, du = \int_c^d \|f'(\phi(u))\| \, |\phi'(u)| du$$
$$= \int_a^b \|f'(t)\| \, dt = l(f).$$

2.7.7 *Example.* The function $h : [-2\pi, 2\pi] \subseteq \mathbb{R} \to \mathbb{R}^2$ defined by

$$h(u) = (\cos u, -\sin u), \qquad u \in [-2\pi, 2\pi],$$

is a C^1 path parametrizing the unit circle. It is equivalent to the C^1 path f defined in Example 2.7.4 (consider $\phi(u) = \pi - \frac{1}{2}u$). As expected

$$l(h) = \int_{-2\pi}^{2\pi} \|h'(u)\| \, du = 4\pi = l(f).$$

In the previous example, $l(h)$ is twice the circumference of the unit circle because h traces the circle twice. In general the length of a C^1 path cannot be judged by considering its image curve C. However, if C is a smooth simple arc then by Theorem 2.6.23 all smooth parametrizations of C are 1–1. In this case we expect the following result.

2.7.8 *Corollary.* *All smooth parametrizations of a smooth simple arc in $\mathbb{R}^n$ have the same length.*

Proof. Theorem 2.6.23 implies that all smooth parametrizations of a smooth simple arc are equivalent. By Theorem 2.7.6, they therefore have the same length.

2.7.9 *Definition.* *The* length $l(C)$ *of a smooth simple arc C in $\mathbb{R}^n$ is the length of any smooth parametrization of C.*

2.7.10 *Example.* Let G be the graph of the C^1 function $g : [a, b] \subseteq \mathbb{R} \to \mathbb{R}$ considered in Example 2.6.28. Suppose that G is smoothly parametrized by $(x(u), y(u))$, $u \in [c, d]$. Then

$$l(G) = \int_a^b \sqrt{(1 + g'(x)^2)} dx = \int_c^d \sqrt{(x'(u)^2 + y'(u)^2)} du.$$

The natural parametrization of a curve (the one which would be chosen by a small creature living on the curve!) is one in which the parameter s measures the distance along the curve from some fixed base point p. Such a parametrization $h : E \subseteq \mathbb{R} \to \mathbb{R}^n$, if it exists, is readily identified since it requires that for all a and b in E the distance traversed from $s = a$ to $s = b$ is given by

$$\int_a^b \|h'(s)\| \, ds = b - a.$$

An equivalent requirement is that $\|h'(s)\| = 1$ for all $s \in E$ (Exercise 2.7.9). This motivates the following definition.

2.7.11 ***Definition.*** *A C^1 parametrization $h : E \subseteq \mathbb{R} \to \mathbb{R}^n$ of a curve C in $\mathbb{R}^n$ is a* path-length parametrization *of C if $\|h'(s)\| = 1$ for all $s \in E$.*
Clearly such a parametrization is smooth.

2.7.12 ***Example.*** The function $h : \mathbb{R} \to \mathbb{R}^2$ defined by

$$h(s) = \left(r\cos\frac{s}{r}, r\sin\frac{s}{r} \right), \qquad s \in \mathbb{R},$$

is a path-length parametrization of the circle in $\mathbb{R}^2$ with centre $(0, 0)$ and radius r. The length of the path between $s = a$ and $s = b$ is $b - a$.

We shall now show that general smooth parametrizations can be studied by considering equivalent path-length parametrizations.

2.7.13 ***Definition.*** *Let $f : D \subseteq \mathbb{R} \to \mathbb{R}^n$ be a smooth parametrization of a curve C in $\mathbb{R}^n$. Choose a base point $p \in D$. The* path-length *function* based at p *is the function $\lambda : D \subseteq \mathbb{R} \to \mathbb{R}$ defined by*

2.7.14 $$\lambda(t) = \int_p^t \|f'(u)\| \, du, \qquad t \in D.$$

Thinking of f as representing the motion of a particle in $\mathbb{R}^n$ over a time interval D, the number $s = \lambda(t)$ indicates the distance travelled by the particle in the time interval between p and t (with the convention that if $t < p$ then the distance is negative). We find from 2.7.14 that the speed of f at t is given by

2.7.15 $$\lambda'(t) = \|f'(t)\| > 0, \qquad t \in D.$$

This derivative clearly does not depend upon the base point chosen.

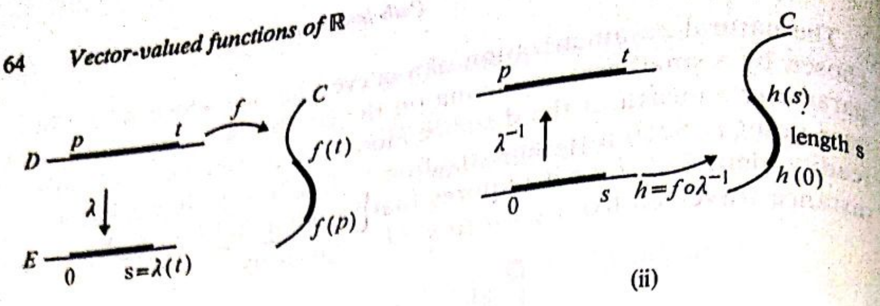


Fig. 2.15

Definition 2.7.13 is illustrated in Fig. 2.15(i). Since λ is continuous, the set $E=\lambda(D)$ is an interval in $\mathbb{R}$. Furthermore, by 2.7.15, λ is a 1–1 function whose inverse function $\lambda^{-1}:E\to D$ has positive derivative throughout E. Comparison of Fig. 2.15(i) and Fig. 2.11 suggests that we can use s as a parameter in a parametrization of C that is properly equivalent to f.

2.7.16 *Theorem.* *For any smooth parametrization $f:D\subseteq\mathbb{R}\to\mathbb{R}^n$ of a curve C in $\mathbb{R}^n$ there is a path-length parametrization $h:E\subseteq\mathbb{R}\to\mathbb{R}^n$ of C which is properly equivalent to f.*

Proof. The proof is illustrated in Fig. 2.15(ii). Chose a base point $p\in D$ and let $\lambda:D\subseteq\mathbb{R}\to\mathbb{R}$ be the path-length function based at p given by 2.7.14. Let $E=\lambda(D)$ and define $h=f\circ\lambda^{-1}:E\subseteq\mathbb{R}\to\mathbb{R}^n$. By 2.7.15, h is properly equivalent to f. Also, since $f=h\circ\lambda$, it follows from the Chain Rule and 2.7.15 that

$$\|f'(t)\|=\|h'(\lambda(t))\|\,|\lambda'(t)|=\|h'(\lambda(t))\|\,\|f'(t)\|.$$

Therefore $\|h'(s)\|=1$ for all $s\in E$.

2.7.17 *Example.* The image of the continuously differentiable function $f:\mathbb{R}\to\mathbb{R}^3$ defined by

$$f(t)=(at,\, b\cos\omega t,\, b\sin\omega t),\qquad t\in\mathbb{R},$$

where $b>0$ and $\omega>0$, is a helix wound around a cylinder of radius b whose axis is the x-axis in $\mathbb{R}^3$. The helix corresponding to $a=b=\omega=1$ is sketched in Fig. 2.2. Now

$$f'(t)=(a,\,-b\omega\sin\omega t,\, b\omega\cos\omega t),\qquad t\in\mathbb{R},$$

and

$$\|f'(t)\|=(a^2+b^2\omega^2)^{1/2},\qquad t\in\mathbb{R}.$$

Hence f is smooth, since $a^2+b^2\omega^2 \neq 0$. Choose $0 \in \mathbb{R}$ as base point. The path-length function $\lambda:\mathbb{R}\to\mathbb{R}$ is defined by

$$\lambda(t)=(a^2+b^2\omega^2)^{1/2}t, \qquad t\in\mathbb{R}.$$

For example, the image of any closed interval of length $2\pi/\omega$ is a single twist of the helix, and hence the length of a single twist is $2\pi(a^2+b^2\omega^2)^{1/2}\omega$. The inverse function $\lambda^{-1}:\mathbb{R}\to\mathbb{R}$ is given by

$$\lambda^{-1}(s)=(a^2+b^2\omega^2)^{-1/2}s, \qquad s\in\mathbb{R}.$$

Let $c=(a^2+b^2\omega^2)^{1/2}$. A path-length parametrization properly equivalent to f is given by

$$h(s)=f(\lambda^{-1}(s))=\left[\frac{as}{c},\, b\cos\frac{\omega s}{c},\, b\sin\frac{\omega s}{c}\right], \qquad s\in\mathbb{R}.$$

As expected (from Theorem 2.7.16), for all $s\in\mathbb{R}$,

$$\|h'(s)\|=\left\|\left(\frac{a}{c}, -\frac{b\omega}{c}\sin\frac{\omega s}{c}, \frac{b\omega}{c}\cos\frac{\omega s}{c}\right)\right\|=\left[\frac{a^2+b^2\omega^2}{c^2}\right]^{1/2}=1.$$

2.7.18 *Example.* We saw in Example 2.6.34 that the piecewise C^1 path

$$f(t)=(t, |t|), \qquad t\in[-1,1]$$

can be regarded as being composed of two C^1 pieces $f_1:[-1,0]\subseteq\mathbb{R}\to\mathbb{R}^2$ and $f_2:[0,1]\subseteq\mathbb{R}\to\mathbb{R}^2$, where $f_1(t)=(t,-t)$ when $t\in[-1,0]$ and $f_2(t)=(t,t)$ when $t\in[0,1]$.

We define the *length* of the path f by

$$l(f)=l(f_1)+l(f_2)=\int_{-1}^{0}\|f_1'(t)\|\,dt+\int_0^1\|f_2'(t)\|\,dt$$
$$=\int_{-1}^{1}\|f'(t)\|\,dt,$$

where in the last integral the integrand is undefined at $t=0$. Since $\|f'(t)\|=\sqrt{2}$, $t\neq 0$, it follows that $l(f)=2\sqrt{2}$.

Finally in this section we introduce two important path constructions.

2.7.19 *Definition.* *Let* $f:[a,b]\subseteq\mathbb{R}\to\mathbb{R}^n$ *be a path. The* inverse path *is defined to be the function* $f^-:[a,b]\subseteq\mathbb{R}\to\mathbb{R}^n$ *where*

$$f^-(t)=f(a+b-t), \qquad t\in[a,b].$$

Clearly f^- is a path. It traces the image of f from $f(b)$ to $f(a)$ by reversing the action of f. Moreover f^- is C^1 (smooth) if and only if f is C^1 (smooth). The paths f and f^- have the same length.

The second construction is based on the observation that if $f:[a, b]\subseteq\mathbb{R}\to\mathbb{R}^n$ and $g:[c, d]\subseteq\mathbb{R}\to\mathbb{R}^n$ are paths in $\mathbb{R}^n$ such that the end point $f(b)$ of f is the initial point $g(c)$ of g, then there is a composite path 'f followed by g' that runs along f from $f(a)$ to $f(b)$ and then runs along g from $f(b)=g(c)$ to $g(d)$. The following definition formalizes this idea.

2.7.20 *Definition.* *Let $f:[a, b]\subseteq\mathbb{R}\to\mathbb{R}^n$ and $g:[c, d]\subseteq\mathbb{R}\to\mathbb{R}^n$ be paths in $\mathbb{R}^n$, where $f(b)=g(c)$. The* product path *$h=fg:[a, b+d-c]\subseteq\mathbb{R}\to\mathbb{R}^n$ is defined by*

$$h(t)=\begin{cases} f(t) & \text{when} \quad a\leqslant t\leqslant b \\ g(c+(t-b)) & \text{when} \quad b\leqslant t\leqslant b+d-c. \end{cases}$$

Clearly if f and g are C^1 (smooth) paths then the product path is piecewise C^1 (smooth).

Exercises 2.7

1. Calculate the lengths of the following smooth simple arcs in $\mathbb{R}^3$.
   (a) The circular helix parametrized by
   $$f(t)=(t, \cos t, \sin t), \qquad t\in[a, b];$$
   (b) the curve parametrized by
   $$f(t)=(e^t\cos t, e^t\sin t, e^t) \qquad t\in[0, k].$$

Answers: (a) $\sqrt{2}(b-a)$; (b) $\sqrt{3}(e^k-1)$.

2. Sketch the following smooth simple arcs in $\mathbb{R}^2$ and calculate their length.
   (a) The curve parametrized by $f(t)=(e^t\cos t, e^t\sin t)$, $t\in[1, 2]$;
   (b) the graph (catenary) $y=\cosh x=\frac{1}{2}(e^x+e^{-x})$ between $x=-1$ and $x=1$;
   (c) the portion of the parabola $y^2=16x$ which lies between the lines $x=0$ and $x=4$.

Answers: (a) $\sqrt{2}(e^2-e)$; (b) $e-e^{-1}$; (c) $2\sqrt{2}+2\ln(\sqrt{2}+1)$.

3. Which of the following two paths in $\mathbb{R}^3$ from $(0, 0, 0)$ to $(1, 1, 1)$ is the longer? (a) $f(t)=(t, t, t^2)$, $0\leqslant t\leqslant 1$; (b) $g(t)=(t, t^2, t^2)$, $0\leqslant t\leqslant 1$.

4. The semi-ellipse E with equation
$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1, \qquad y\geqslant 0,$$

is smoothly parametrized by $f(t) = (a \cos t, b \sin t)$, $0 \leqslant t \leqslant \pi$. Prove that if $a \geqslant b > 0$ then the length of E is given by

$$l(E) = 2a \int_0^{\pi/2} \sqrt{(1 - k^2 \cos^2 t)}\,dt,$$

where $k^2 = (a^2 - b^2)/a^2$. The integral is called an *elliptic integral*. There exist tables of elliptic integrals from which its value can be obtained for various values of k.

5. (a) Sketch the simple arc C

$$x^{2/3} + y^{2/3} = 1, \qquad 0 \leqslant y \leqslant 1.$$

Verify that C is parametrized by the 1–1 C^1 path

$$f(t) = (\cos^3 t, \sin^3 t), \qquad 0 \leqslant t \leqslant \pi.$$

Show that C is not a smooth simple arc. (Consider the point $f(\frac{1}{2}\pi)$.)
(b) Call a simple arc C in $\mathbb{R}^n$ piecewise smooth if it has a piecewise smooth parametrization. Prove that all piecewise smooth parametrizations of C have the same length. Hence frame a definition of the length $l(C)$ of C.
(c) Find the length of the simple arc of part (a) of this exercise.

Answer: 3.

6. Calculate the length of an arch of the cycloid $f(t) = (t - \sin t, 1 - \cos t)$, $t \in [0, 2\pi]$.

Answer: 8.

7. A curve C in $\mathbb{R}^n$ is a (smooth) *simple closed curve* if C has a (smooth) C^1 parametrization of the form $f : [a, b] \subseteq \mathbb{R} \to \mathbb{R}^n$ such that $f(a) = f(b)$ and f is 1–1 on the half-open interval $[a, b[$. The length of C is then unambiguously defined (Section 6.2) by

$$l(C) = l(f) = \int_a^b \|f'(t)\|\,dt.$$

Sketch the following simple closed curves and find their length.
(a) The circle $x^2 + y^2 = 9$ (parametrize by $f(t) = (3\cos t, 3\sin t)$, $0 \leqslant t \leqslant 2\pi$);
(b) the *cardioid* (a heart-shaped curve) parametrized by

$$f(t) = ((1 + \cos t)\cos t, (1 + \cos t)\sin t), \qquad 0 \leqslant t \leqslant 2\pi.$$

(c) the curve (circle!) parametrized by

$$f(t) = (2 - 2\cos t, 2\sin t), \qquad 0 \leqslant t \leqslant 2\pi.$$

(d) the curve (circle!) parametrized by

$$f(t)=(\sin t\cos t, \sin^2 t) \qquad 0\leqslant t\leqslant \pi.$$

(e) the four-pointed star (*astroid*) $x^{2/3}+y^{2/3}=1$.

Answers: (a) 6; (b) 8; (c) 4; (d) π; (e) 6.

8. Let $f:[a, b]\subseteq\mathbb{R}\to\mathbb{R}^2$ be a C^1 path in $\mathbb{R}^2$ defined by

$$f(t)=(r(t)\cos t, r(t)\sin t), \qquad t\in[a, b],$$

where $r(t)\geqslant 0$ for all $t\in[a, b]$.
Prove that the length of f is

$$l(f)=\int_a^b \sqrt{[r(t)^2+r'(t)^2]}\,dt.$$

Hence find the lengths of the following curves of Exercise 7:
(a) the circle $r(t)=3$;
(b) the cardioid $r(t)=1+\cos t$;
(c) the circle $r(t)=\sin t$.

(*Note*: the path f is said to be in *polar coordinate* form. The point $f(t)$ lies on the circle centre the origin, radius $r(t)$, at an angle t from the x-axis.)

9. Let $h:E\subseteq\mathbb{R}\to\mathbb{R}^n$ be a C^1 function. Prove that if for all a, b in E

$$\int_a^b \|h'(s)\|\,ds=b-a,$$

then $\|h'(s)\|=1$ for all $s\in E$.

(*Hint*: fix a, and treat b as variable. Differentiate with respect to b.)

10. Find path-length parametrizations h that are properly equivalent to the given smooth parametrizations f of the following curves.
(a) The unit circle $f(t)=(\cos 2t, \sin 2t)$, $t\in\mathbb{R}$;
(b) the smooth simple arc in $\mathbb{R}^3$ parametrized by

$$f(t)=(e^t\cos t, e^t\sin t, e^t), \qquad t\in[0, \pi].$$

Answers: with base point 0
(a) $\lambda(t)=2t$, $h(s)=f(\lambda^{-1}(s))=(\cos s, \sin s)$;
(b) $\lambda(t)=3(e^t-1)$, $\lambda^{-1}(s)=\ln(1+(s/\sqrt{3}))$, $s\in[0, \sqrt{3}(e^\pi-1)]$; $h=f\circ\lambda^{-1}$.

11. Let $f:[0, 1]\subseteq\mathbb{R}\to\mathbb{R}^2$ and $g:[-1, 1]\subseteq\mathbb{R}\to\mathbb{R}^2$ be the C^1 paths defined by

$$f(t)=(-t, t), \qquad t\in[0, 1],$$
$$g(t)=(t, t^2), \qquad t\in[-1, 1].$$

Verify that $f(1) = g(-1)$. Construct the inverse path f^- and the product path fg. Describe by means of a sketch how they trace their images.

2.8 Differential geometry

In this section we shall show how the twisting and turning of a curve in $\mathbb{R}^n$ can be expressed in terms of the derivatives of a path-length parametrization. We shall then use Theorem 2.7.16 to extend the results to more general parametrizations.

Let $h : E \subseteq \mathbb{R} \to \mathbb{R}^n$ be a twice-differentiable path-length parametrization of the curve $C = h(E)$. Since $\|h'(s)\| = 1$ for all $s \in E$, the vector $h''(s)$ indicates the way in which the unit tangent vector $h'(s)$ is changing direction near s. The larger the value of $\|h''(s)\|$, the greater is the 'curving' of C at s.

By Theorem 2.5.14 $h''(s)$ is orthogonal to $h'(s)$.

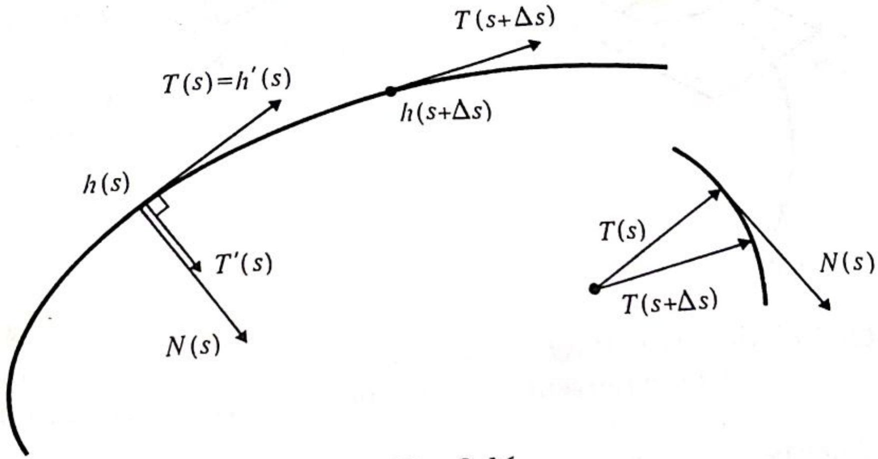


Fig. 2.16

2.8.1 *Definition.* *Let $h : E \subseteq \mathbb{R} \to \mathbb{R}^n$ be a twice-differentiable path-length parametrization of a curve $h(E)$ in $\mathbb{R}^n$.*

[i] *The unit tangent vector $h'(s)$ at $s \in E$ will also be denoted by $T(s)$.*

[ii] *The non-negative number*

$$\kappa(s) = \|T'(s)\| = \|h''(s)\|, \qquad s \in E,$$

is called the curvature of h at s.

[iii] *If $\kappa(s) \neq 0$, then the unit vector $N(s)$ such that*

$$T'(s) = \kappa(s) N(s)$$