6.1 Line integrals of scalar fields and vector fields

In Section 5.2 we considered the integral of a real-valued function (scalar field) $f: S \subseteq \mathbb{R}^n \to \mathbb{R}$ along a path $\alpha: [a, b] \subseteq \mathbb{R} \to \mathbb{R}^n$ in S. The value of the integral depends on the way the path traces and retraces its image. In contrast, elementary Riemann integration concerns the integral of a real-valued function over a compact interval $[a, b] \subseteq \mathbb{R}$ which is regarded as a subset in $\mathbb{R}$. As a direct generalization of this we shall now consider the integral of a scalar field in $\mathbb{R}^n$ over certain subsets of $\mathbb{R}^n$ that generalize the compact intervals in $\mathbb{R}$, in fact simple arcs.

6.1.1 *Definition.* *Let $f: S \subseteq \mathbb{R}^n \to \mathbb{R}$ be a continuous scalar field and let C be a simple arc in S. The* integral of f over C *is defined and denoted by*

$$\int_C f \, ds = \int_\alpha f \, ds = \int_a^b f(\alpha(t)) \, \|\alpha'(t)\| \, dt$$

where $\alpha: [a, b] \subseteq \mathbb{R} \to \mathbb{R}^n$ is any simple parametrization of C.

The above integral is often called the *line integral of f over C*, although 'curve integral' or 'arc integral' would perhaps be a better description. Definition 6.1.1 is unambiguous, since by Theorem 5.2.29, for any two simple parametrizations α and β of C the path integrals of f along α and β take the same value. It is important to note that only 1–1 parametrizations of C should be used in evaluating the line integral of f over C. The discussion in Section 5.2 shows how this integral may differ in value from the integral of f along a path that traces C more than once.

6.1.2 *Example.* Let C be the simple arc $y = |x|$, $-1 \leq x \leq 1$, and let $f: \mathbb{R}^2 \to \mathbb{R}$ be the continuous scalar field defined by $f(x, y) = x^2 y$. The line

integral of f over C can be evaluated as a path integral by using any simple parametrization of C. Choose for example the parametrization

$$\alpha(t) = \begin{cases} (t^2, t^2) & \text{when} \quad -1 \leqslant t \leqslant 0 \\ (-t^2, t^2) & \text{when} \quad 0 \leqslant t \leqslant 1. \end{cases}$$

Then $\|\alpha'(t)\| = 2\sqrt{2}\,|t|$ and hence

$$\oint_C f\,\mathrm{d}s = \int_\alpha f\,\mathrm{d}s = \int_{-1}^{0} t^6(-2\sqrt{2}t)\,\mathrm{d}t + \int_0^1 t^6(2\sqrt{2}t)\,\mathrm{d}t = \tfrac{1}{2}\sqrt{2}.$$

More simply, regarding C as a *piecewise simple arc,* we can evaluate the integral of f over C using the piecewise simple parametrization $\beta(t) = (t, |t|)$, $-1 \leqslant t \leqslant 1$. Then $\|\beta'(t)\| = \sqrt{2}$, $t \neq 0$, and

$$\int_C f\,\mathrm{d}s = \int_{-1}^{0} -t^3\sqrt{2}\,\mathrm{d}t + \int_0^1 t^3\sqrt{2}\,\mathrm{d}t = \tfrac{1}{2}\sqrt{2}.$$

Notice that α traces the arc C from $(1, 1)$ to $(-1, 1)$, whereas β traces it in the opposite direction.

We now turn to the line integral of a vector field. We observed in Section 5.3 that the integral of a vector field along a path depends on the direction in which the path traces its image. Reversing the direction changes the sign of the integral. Accordingly we define the line integral of a vector field along an oriented simple arc. We shall need to know that if α and β are properly equivalent simple parametrizations of a simple arc C (so they trace C in the same direction) then the integrals of a vector field along α and along β are equal.

6.1.3 *Theorem.* *Let C be a simple arc in $\mathbb{R}^n$ and let $F: S \subseteq \mathbb{R}^n \to \mathbb{R}^n$ be a continuous vector field whose domain S contains C. Let $\alpha: [a, b] \subseteq \mathbb{R} \to \mathbb{R}^n$ and $\beta: [c, d] \subseteq \mathbb{R} \to \mathbb{R}^n$ be properly equivalent simple parametrizations of C. Then*

$$\int_\alpha F \cdot \mathrm{d}\alpha = \int_\beta F \cdot \mathrm{d}\beta.$$

Proof. We shall prove Theorem 6.1.4 for the important special case where C is a *smooth* simple arc and α and β are properly equivalent smooth simple parametrizations of C. Let $\phi: [c, d] \to [a, b]$ be the differentiable function such that $\beta = \alpha \circ \phi$. By the Chain Rule 2.6.11

$$\beta'(u) = \alpha'(\phi(u))\phi'(u), \qquad u \in [c, d].$$

Taking coordinate functions, we obtain, for $i = 1, \ldots, n$,

6.1.4 $$\beta_i'(u) = \alpha_i'(\phi(u))\phi'(u) \qquad u \in [c, d].$$

Now fix u in $[c, d]$. Since β is smooth, β' is continuous and non-zero, and so at least one of the coordinate function β_k', $1 \leq k \leq n$ is continuous and non-zero in some neighbourhood of u. Since α' is continuous it follows by 6.1.4 that ϕ' is continuous at u. Hence ϕ is C^1. By Theorem 5.3.11 the integrals of F along α and along β are equal.

The proof of the general result 6.1.3 which we omit, is based on the comparison of Riemann sums.

Before giving the formal definition of the line integral of a vector field along an oriented simple arc C we introduce a useful notation. Recall (2.7.19) that if a path α traces a curve from **a** to **b** then the inverse path α^- traces it from **b** to **a**.

6.1.5. *Definition.* *Let C be a simple arc in $\mathbb{R}^n$ simply parametrized by α, and let $[\alpha]$ be the set of all simple parametrizations of C which are properly equivalent to α.*

[i] *The oriented simple arc $\{C, [\alpha]\}$ will be denoted either by C^+ or by C^-. If $C^+ = \{C, [\alpha]\}$ then $C^- = \{C, [\alpha^-]\}$ and vice versa.*

[ii] *Let $C^+ = \{C, [\alpha]\}$. Any parametrization in the equivalence class $[\alpha]$ is called a* simple parametrization *of C^+.*

6.1.6. *Example.* In Example 6.1.2, if $C^+ = \{C, [\alpha]\}$ then $C^- = \{C, [\beta]\}$.

6.1.7 *Definition.* *Let $F : S \subseteq \mathbb{R}^n \to \mathbb{R}^n$ be a continuous vector field, and let C^+ be an oriented simple arc in S. The* integral of F along C^+ *is defined and denoted by*

$$\int_{C^+} F \cdot d\mathbf{r} = \int_{\alpha} F \cdot d\alpha$$

where α is any simple parametrization of C^+.

The above integral is often called the *line integral of F along C*. Notice that we speak of integrating a scalar field f *over* a simple arc C, but a vector field F *along* an oriented arc C^+. Theorem 6.1.3 ensures that Definition 6.1.7 is unambiguous. It follows from Exercise 5.3.8 that

6.1.8 $$\int_{C^-} F \cdot d\mathbf{r} = -\int_{C^+} F \cdot d\mathbf{r}.$$

6.1.9 *Example* Define $F:\mathbb{R}^2 \to \mathbb{R}^2$ by $F(x, y) = (xy, -x)$, and let C^+ be the parabolic arc $y = x^2$, $-1 \leqslant x \leqslant 0$ oriented from $(-1, 1)$ to $(0, 0)$. The line integral of F along C^+ can be evaluated as a path integral by using any simple parametrization of C^+. Choose for example the parametrization $\alpha(t) = (t, t^2)$, $-1 \leqslant t \leqslant 0$. Hence

$$\int_{C^+} F \cdot \mathrm{d}\mathbf{r} = \int_{\alpha} F \cdot \mathrm{d}\alpha = \int_{-1}^{0} (t^3, -t) \cdot (1, 2t)\, \mathrm{d}t = -\tfrac{11}{12}.$$

Also the integral of F along C^- is, by 6.1.8,

$$\int_{C^-} F \cdot \mathrm{d}\mathbf{r} = \tfrac{11}{12}.$$

It can be evaluated as the path integral $\int_\beta F \cdot \mathrm{d}\beta$ where β is the simple parametrization of C^- given by $\beta(u) = (-u, u^2)$, $0 \leqslant u \leqslant 1$.

There is a useful relationship between the line integral of a vector field along an oriented smooth simple arc C^+ in $\mathbb{R}^n$ and the line integral of a related scalar field over C.

Let $\alpha:[a, b] \subseteq \mathbb{R} \to \mathbb{R}^n$ be a simple parametrization of C^+. Since α is smooth, for each $t \in [a, b]$ the vector $\alpha'(t)$, tangent to α at t, is non-zero. We can use α to define a continuous unit tangent field $T: C \subseteq \mathbb{R}^n \to \mathbb{R}^n$ by putting, for each $\mathbf{r} \in C$,

6.1.10 $$T(\mathbf{r}) = \frac{\alpha'(t)}{\|\alpha'(t)\|} \qquad \text{where } \mathbf{r} = \alpha(t), \qquad t \in [a, b].$$

It follows from Section 2.6 that any other simple parametrization of C^+ leads to the same unit tangent field T. The oriented smooth curve $C^+ = \{C, [\alpha]\}$ could equally be defined by the pair (C, T). Consequently the following definition is unambiguous.

6.1.11 *Definition.* *Let C^+ be an oriented smooth simple arc in $\mathbb{R}^n$, and let $\alpha:[a, b] \subseteq \mathbb{R} \to \mathbb{R}^n$ be any simple parametrization of C^+. The vector field $T: C \subseteq \mathbb{R}^n \to \mathbb{R}^n$ defined in 6.1.10 is called the* unit tangent field *to the oriented curve C^+. For any $\mathbf{r} \in C$, the unit vector $T(\mathbf{r})$ is called the* unit tangent to C^+ at $\mathbf{r}$.

Suppose now that $F: C \subseteq \mathbb{R}^n \to \mathbb{R}^n$ is a continuous vector field on C. For each $\mathbf{r} \in C$ the component of $F(\mathbf{r})$ in the direction $T(\mathbf{r})$ is the dot product $F(\mathbf{r}) \cdot T(\mathbf{r}) \in \mathbb{R}$. We therefore have a continuous scalar field $F \cdot T: C \subseteq \mathbb{R}^n \to \mathbb{R}$ which can be integrated over C. We call $F \cdot T$ the *component of F along C^+* (where T is the unit tangent field to C^+).

6.1.12 Theorem. *Let C^+ be an oriented smooth simple arc in $\mathbb{R}^n$ with unit tangent field T. Let $F: C \subseteq \mathbb{R}^n \to \mathbb{R}^n$ be a continuous vector field on C. Then*

$$\int_{C^+} F \cdot d\mathbf{r} = \int_C F \cdot T \, ds.$$

That is, the integral of F along C^+ is equal to the integral of the component of F along C^+ over the arc C.

Proof. Let $\alpha: [a, b] \subseteq \mathbb{R} \to \mathbb{R}^n$ be any simple parametrization of C^+. then

$$\int_{C^+} F \cdot d\mathbf{r} = \int_a^b F(\alpha(t)) \cdot \alpha'(t) \, dt$$

$$= \int_a^b F(\alpha(t)) \cdot T(\alpha(t)) \, \|\alpha'(t)\| \, dt = \int_C F \cdot T \, ds.$$

It is clear that if $T: C \subseteq \mathbb{R}^n \to \mathbb{R}^n$ is the unit tangent field to an oriented smooth simple arc C^+ then $-T: C \subseteq \mathbb{R}^n \to \mathbb{R}^n$ is the unit tangent field to C^-. Applying Theorem 6.1.12 to C^-, we obtain

$$\int_{C^-} F \cdot d\mathbf{r} = \int_C F \cdot (-T) \, ds = -\int_{C^+} F \cdot d\mathbf{r},$$

which confirms 6.1.8. The above relation explains why the integral of a vector field along a smooth simple arc changes sign when its orientation is reversed: the corresponding component scalar fields are $F \cdot T$ and $F \cdot (-T)$.

The work of this section generalizes easily to piecewise C^1 and piecewise smooth simple arcs. We leave the details to the reader.

6.1.13 Example. Let C be the piecewise smooth simple arc $y = |x|$, $-1 \leqslant x \leqslant 1$. Let $f: \mathbb{R}^2 \to \mathbb{R}$ be the scalar field defined by $f(x, y) = x^2 y$. The integral of f over C can be evaluated by using any simple parametrization of C. Choose for example the parametrization $\alpha(t) = (t, |t|)$, $-1 \leqslant t \leqslant 1$. Since $\|\alpha'(t)\| = \sqrt{2}$, $t \neq 0$, we obtain

$$\int_C f \, ds = \int_{-1}^0 t^2(-t)\sqrt{2} \, dt + \int_0^1 t^2(t)\sqrt{2} \, dt = \tfrac{1}{2}\sqrt{2}.$$

Now let C^+ be the same arc oriented from $(1, 1)$ to $(-1, 1)$ and let $F: \mathbb{R}^2 \to \mathbb{R}^2$ be the vector field defined by $F(x, y) = (xy, -x)$. The path α

defined above is a simple parametrization of C^-. We can use it to evaluate the integral of F along C^+ as follows. Since

$$\int_\alpha F \cdot d\alpha = \int_{-1}^{0} (-t^2, -t) \cdot (1, -1)\, dt + \int_0^1 (t^2, -t) \cdot (1, 1)\, dt = -1,$$

it follows that

$$\int_{C^+} F \cdot d\mathbf{r} = -\int_{C^-} F \cdot d\mathbf{r} = -\int_\alpha F \cdot d\alpha = 1.$$

A sketch of the oriented arc C^+ and the vector field F at points on C will confirm that the integral of F along C^+ is a positive quantity.

A simple parametrization α of a piecewise smooth simple arc C in $\mathbb{R}^n$ determines a unit tangent field $T : C \subseteq \mathbb{R}^n \to \mathbb{R}^n$ defined and continuous at all but a finite number of points of C. The pair (C, T) could be taken as the definition of the oriented piecewise smooth simple arc $C^+ = \{C, [\alpha]\}$, and the expression

$$\int_{C^+} F \cdot d\mathbf{r} = \int_C F \cdot T\, ds$$

could be taken as a definition of the line integral of f along (C, T).

It is interesting to note in this connection that if G is any continuous vector field on C we can define the integral of F over the pair (C, G) as $\int_C F \cdot G\, ds$. For an application using a unit normal field to a curve see Theorem 7.6.13.

The alternative notation for the integral of a vector field along a path, which we described at the end of Section 5.3, is also used for line integrals. Let $F = (F_1, \ldots, F_n)$ be a vector field on $\mathbb{R}^n$. If C^+ is an oriented simple arc in $\mathbb{R}^n$ parametrized by $\mathbf{r}(t) = (x_1(t), \ldots, x_n(t))$, $t \in [a, b]$, then we have

6.1.14
$$\begin{aligned}\int_{C^+} F \cdot d\mathbf{r} &= \int_a^b F(x_1(t), \ldots, x_n(t)) \cdot (x_1'(t), \ldots, x_n'(t))\, dt \\ &= \int_a^b F_1(\mathbf{r}(t))x_1'(t)\, dt + \cdots + \int_a^b F_n(\mathbf{r}(t))x_n'(t)\, dt \\ &= \int_{C^+} F_1\, dx_1 + \cdots + \int_{C^+} F_n\, dx_n,\end{aligned}$$

where

6.1.15
$$\int_{C^+} f\, dx_k = \int_{C^+} (0, \ldots, 0, f, 0, \ldots, 0) \cdot d\mathbf{r},$$

the function f standing in the kth coordinate place. The integral 6.1.14 is also often denoted by

$$\int_{C^+} F \cdot \mathrm{d}\mathbf{r} = \int_{C^+} F_1\,\mathrm{d}x_1 + F_2\,\mathrm{d}x_2 + \cdots + F_n\,\mathrm{d}x_n.$$

The skill in evaluating the integral 6.1.14 lies in finding a suitable parametrization of C^+. If C^+ is piecewise smooth, it is often best to parametrize each smooth part separately.

6.1.16 ***Example.*** Let C^+ be the piecewise smooth curve in $\mathbb{R}^2$ formed by straight line segments PQ, QR and oriented from P to R, where P, Q, R are the points $(0, -1)$, $(1, -1)$, $(1, 1)$. See Fig. 6.1. Let $F: \mathbb{R}^2 \to \mathbb{R}^2$ be the vector field defined by $F(x, y) = \frac{1}{2}(-xy, y)$. Using the natural parametrizations

$$(x(t), y(t)) = (t, -1), \qquad 0 \leqslant t \leqslant 1 \qquad \text{along } PQ$$

and

$$(x(y), y(t)) = (1, -1 + t), \qquad 0 \leqslant t \leqslant 2 \qquad \text{along } QR$$

we obtain

$$\begin{aligned}\int_{C^+} F \cdot \mathrm{d}\mathbf{r} &= \tfrac{1}{2}\int_0^1 (t, -1) \cdot (1, 0)\,\mathrm{d}t + \tfrac{1}{2}\int_0^2 (1 - t, -1 + t) \cdot (0, 1)\,\mathrm{d}t \\ &= \tfrac{1}{2}\int_0^1 t\,\mathrm{d}t + \tfrac{1}{2}\int_0^2 (-1 + t)\,\mathrm{d}t = \tfrac{1}{4} + 0 = \tfrac{1}{4}.\end{aligned}$$

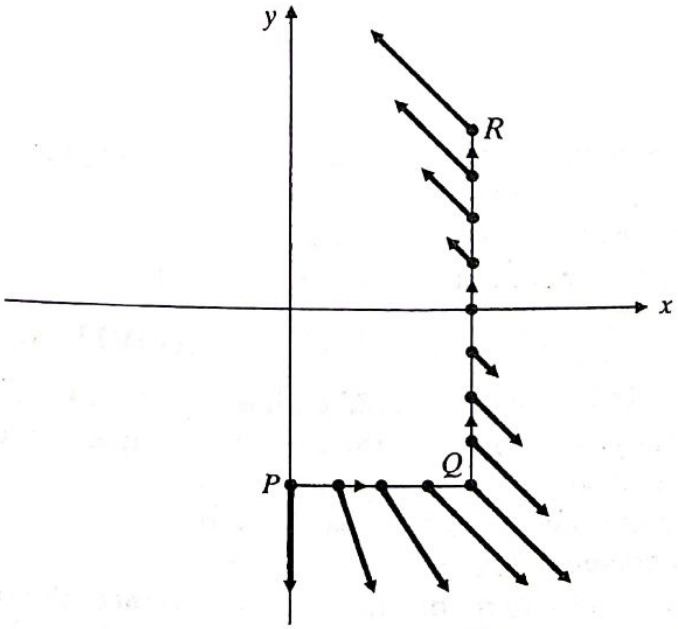

Fig. 6.1 Vector field $F(x, y) = \frac{1}{2}(-xy, y)$ along PQR

The calculations confirm what is clear from Fig. 6.1:

[i] that the line integral of F along PQ is positive (since the component of F along PQ is positive, except at P);

[ii] that the line integral of F along QR is zero.

6.1.17 ***Example.*** Let C^+ be the unit semicircle in the lower half-plane $y \leqslant 0$, centred at $(2, 0)$, and oriented from $(1, 0)$ to $(3, 0)$. Let $F(x, y) = (-xy, y)$. Then

$$\int_{C^+} F \cdot \mathrm{d}\mathbf{r} = \int_{C^+} (-xy)\,\mathrm{d}x + y\,\mathrm{d}y = \int_{C^+} (-xy)\,\mathrm{d}x + \int_{C^+} y\,\mathrm{d}y.$$

The equation $(x-2)^2 + y^2 = 1$, $y \leqslant 0$ for C suggests the parametrization of C^+

$$(x(t), y(t)) = (2 - \cos t, -\sin t) \qquad t \in [0, \pi].$$

We can evaluate separately

$$\int_{C^+} (-xy)\,\mathrm{d}x = -\int_0^\pi x(t)y(t)x'(t)\,\mathrm{d}t$$

$$= \int_0^\pi (2 - \cos t)\sin^2 t\,\mathrm{d}t = \pi,$$

$$\int_{C^+} y\,\mathrm{d}y = \int_0^\pi y(t)y'(t)\,\mathrm{d}t = \int_0^\pi (-\sin t)(-\cos t)\,\mathrm{d}t = 0.$$

Therefore

$$\int_{C^+} F \cdot \mathrm{d}\mathbf{r} = \pi.$$

Exercises 6.1.

1. Sketch the following simple arcs and find their lengths.
   (a) Semicircle $x^2 + y^2 = 4$, $x \geqslant 0$;
   (b) parabolic arc $y = x^2$, $0 \leqslant x \leqslant 1$;
   (c) semicubical parabola $y^2 = x^3$, $-1 \leqslant x \leqslant 1$;

Answers: (a) 2π; (b) $\frac{1}{2}\sqrt{5} + \frac{1}{4}\ln(2 + \sqrt{5})$; (c) $\frac{2}{27}(13\sqrt{13} - 8)$.

2. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be the scalar field defined by $f(x, y) = x^2 y$. Evaluate the line integral $\int_C f\,\mathrm{d}s$ of f over the following simple arcs C.
   (a) The unit semicircle $x^2 + y^2 = 1$, $y \geqslant 0$;
   (b) the unit semicircle $x^2 + y^2 = 1$, $y \leqslant 0$;
   (c) the unit semicircle $x^2 + y^2 = 1$, $x \geqslant 0$;
   (d) the two sides of a unit square which connect the point $(1, 1)$ in $\mathbb{R}^2$ with the points $(1, 0)$ and $(0, 1)$.

Hint for (d): consider the 1–1C^1 paths $(1, t)$, $t \in [0, 1]$ and $(t, 1)$, $t \in [0, 1]$.

Answers: (a) $\frac{2}{3}$; (b) $-\frac{2}{3}$; (c) 0; (d) $\frac{5}{6}$.

3. Sketch the simple arc C in $\mathbb{R}^2$ composed of the parabolic arc $y = x^2$, $-1 \leqslant x \leqslant 0$, and the line segment $y = x$, $0 \leqslant x \leqslant 2$.
   (a) Find a C^1 parametrization of C.
   (b) Calculate $\int_C f\, ds$, where f is the scalar field defined by $f(x, y) = x$.

Answers: (a) The method of Example 2.6.36 gives the C^1 parametrization $\beta : [-1, 2^{1/3}] \subseteq \mathbb{R} \to \mathbb{R}^2$, where

$$\beta(t) = \begin{cases} (t^3, t^6) & \text{when} \quad -1 \leqslant t \leqslant 0, \\ (t^3, t^3) & \text{when} \quad 0 \leqslant t \leqslant 2^{1/3}. \end{cases}$$

(b) $2\sqrt{2} - ((\sqrt{5})^3 - 1)/12$.

4. Let $F : \mathbb{R}^2 \to \mathbb{R}^2$ be the vector field defined by $F(x, y) = (-y, x)$. Evaluate $\int_{C^+} F \cdot d\mathbf{r}$ where
   (a) C^+ is the unit semicircle $\{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1, y \geqslant 0\}$ oriented from $(1, 0)$ to $(-1, 0)$;

Hint: use the parametrization $(x(t), y(t)) = (\cos t, \sin t)$, $t \in [0, \pi]$.

   (b) C^+ is the same semicircle oriented from $(-1, 0)$ to $(1, 0)$;
   (c) C^+ is the piecewise smooth simple arc formed by straight line segments from $P(0, 0)$ to $Q(-1, 1)$ to $R(0, 2)$;
   (d) C^+ is the straight line segment from $P(0, 0)$ to $R(0, 2)$.

Answers: (a) π; (b) $-\pi$; (c) -2; (d) 0.

5. Let C^+ be the oriented semicircle as in Exercise 4(a). Compute the line integral

$$\int_{C^+} -y\, dx + x\, dy$$

by using (a) the C^1 parametrization $(\cos(t^2), \sin(t^2))$, $t \in [0, \sqrt{\pi}]$, (b) the parametrization $(-t, \sqrt{(1 - t^2)})$, $t \in [-1, 1]$. This parametrization is not C^1. It leads to an improper integral which converges to the line integral.

6. Compute the line integral $\int_{C^+} (x^2 - y)\, dx + (y^2 + x)\, dy$ where C^+ is the parabolic arc $y = x^2 + 1$, $0 \leqslant x \leqslant 1$ oriented from $(0, 1)$ to $(1, 2)$.

Answer: 2.

7. A particle moves in a vector field given by

$$F(x, y, z) = (y, z, x), \qquad (x, y, z) \in \mathbb{R}^3.$$

The work done on the particle by F as it moves along an oriented

simple arc C^+ from one end to the other is given by the line integral $\int_{C^+} F \cdot \mathrm{d}\mathbf{r}$, where the orientation of C^+ agrees with the direction of motion of the particle.

Calculate the work done on the particle by F as the particle moves

(a) along straight line segments from $P(0, 0, 0)$ to $Q(0, 1, 1)$ to $R(-1, -1, -2)$;

(b) along the straight line segment from $P(0, 0, 0)$ to $R(-1, -1, -2)$.

Hint: (a) from Q to R a convenient parametrization is $(-t, 1-2t, 1-3t)$, $t \in [0, 1]$.

Answers: (a) 6; (b) $\frac{5}{2}$.

8. Let $f: \mathbb{R}^2 \to \mathbb{R}$ be the scalar field defined by $f(x, y) = x^2 y$, and let C^+ be the unit semicircle $x^2 + y^2 = 1$, $y \geqslant 0$, oriented from $(1, 0)$ to $(-1, 0)$. Compare the integral $\int_C f \, \mathrm{d}s$ of f over C (Definition 6.1.1) and the integral $\int_{C^+} f \, \mathrm{d}x$.

Answer: $\frac{2}{3}$, $-\pi/8$. The second integral is the integral over C of the component of the vector field $F = (f, 0)$ along C^+ (Fig. 6.2).

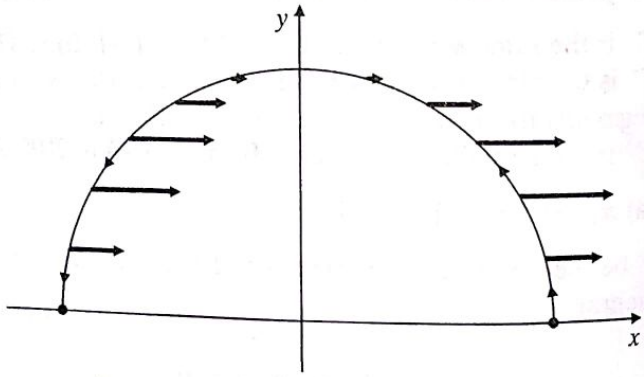

Fig. 6.2 Oriented semicircle C^+ and vector field $(x^2 y, 0)$

9. Let C^+ be a smooth simple arc, oriented from $\mathbf{v}$ to $\mathbf{w}$ in $\mathbb{R}^n$, let $\sigma: [c, d] \subseteq \mathbb{R} \to \mathbb{R}^n$ be a smooth parametrization of C^+, and let $\alpha: [a, b] \subseteq \mathbb{R} \to \mathbb{R}^n$ be *any* (not necessarily 1–1) C^1 path in C from $\mathbf{v}$ to $\mathbf{w}$.

(a) Prove that (i) there exists a unique C^1 function ϕ from $[a, b]$ onto $[c, d]$ such that $\alpha = \sigma \circ \phi$, and that (ii) if α is a smooth parametrization of C^+ then $\phi'(p) > 0$ for all $p \in [a, b]$.

(b) Show that, for any continuous vector field $F: C \subseteq \mathbb{R}^n \to \mathbb{R}^n$,

$$\int_{C^+} F \cdot \mathrm{d}\mathbf{r} = \int_{\alpha} F \cdot \mathrm{d}\alpha.$$

Hints: (a) Let $\phi = \sigma^{-1} \circ \alpha$. So $\alpha = \sigma \circ \phi$. Adapt the proof of Theorem 2.6.23 to show that ϕ is differentiable. (b) Using the Chain Rule, Exercise 4.4.8, show that

$$\int_\alpha F \cdot d\alpha = \int_\sigma F \cdot d\sigma = \int_{C^+} F \cdot d\mathbf{r}.$$

This result is illustrated in the following exercise.

10. Let C^+ be the parabolic arc $y = x^2$, $0 \leqslant x \leqslant 1$ oriented from $(1, 1)$ to $(0, 0)$. Show that
(a) the path $\sigma : [-1, 0] \subseteq \mathbb{R} \to \mathbb{R}^2$ defined by

$$\sigma(u) = (-u, u^2), \qquad u \in [-1, 0]$$

is a smooth parametrization of C^+;
(b) the path $\alpha : [0, 3\pi/2] \subseteq \mathbb{R} \to \mathbb{R}^2$ defined by

$$\alpha(t) = (\cos^2 t, \cos^4 t), \qquad t \in [0, 3\pi/2]$$

is a C^1 path that traces C three times, with initial point $(1, 1)$ and terminal point $(0, 0)$.

Verify that

$$\int_{C^+} F \cdot d\mathbf{r} = \int_\sigma F \cdot d\sigma = \int_\alpha F \cdot d\alpha$$

where $F : \mathbb{R}^2 \to \mathbb{R}^2$ is the vector field defined by $F(x, y) = (xy, x)$.

11. Let $f : [a, b] \subseteq \mathbb{R} \to \mathbb{R}$ be a continuous real-valued function. Interpret the Riemann integral $\int_a^b f(x)\, dx$ as the line integral of a vector field along an oriented arc.

Answer: Let I^+ be the interval $I = [a, b]$ oriented from a to b. Then, in the notation of 6.1.15, $\int_a^b f(x)\, dx = \int_{I^+} f\, dx$.

6.2 Integration around simple closed curves

A simple closed curve in $\mathbb{R}^n$ is a closed loop without any self-intersections. It differs from a simple arc only in that it has no end points. In this section we adapt the methods of Sections 2.6 and 6.1 to define the orientation of simple closed curves and to study the integrals associated with them.

6.2.1 Definition. *A subset C of $\mathbb{R}^n$ is called a* simple closed curve *if there is a piecewise C^1 path $\alpha : [a, a + h] \subseteq \mathbb{R} \to \mathbb{R}^n$, whose image is C, such that $\alpha(a) = \alpha(a + h)$ and α is 1–1 on $[a, a + h[$. The*

function α *is then called a* simple parametrization *of* C based at $\alpha(a)$ in C.

6.2.2 ***Example.*** The circle $x^2 + y^2 = 1$ is a simple closed curve in $\mathbb{R}^2$.

The simple parametrization of the circle $\alpha(t) = (\cos t, \sin t)$, $t \in [0, 2\pi]$, is based at $(1, 0)$. As t increases from 0 to 2π, the point $\alpha(t)$ completes a single counterclockwise trip around the circle. See Fig. 6.3.

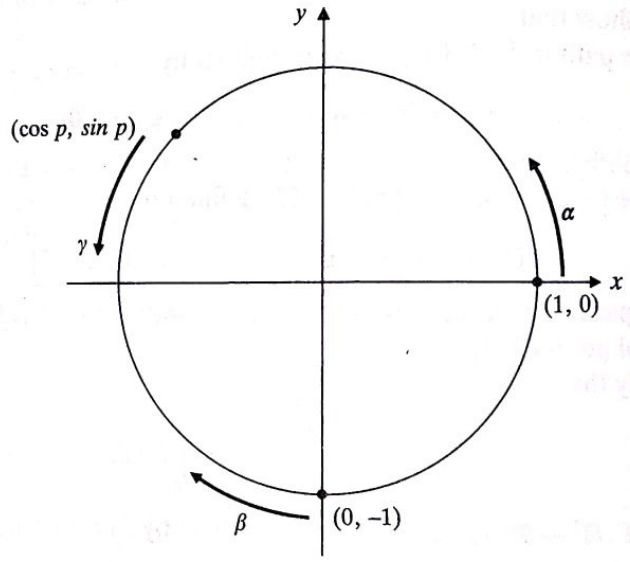

Fig. 6.3

The simple parametrization $\beta(t) = (\sin 2t, \cos 2t)$, $t \in [-\frac{1}{2}\pi, \frac{1}{2}\pi]$ traces the circle clockwise and is based at $(0, -1)$.

For any fixed $p \in \mathbb{R}$, $\gamma(t) = (\cos(p + t), \sin(p + t))$, $t \in [0, 2\pi]$ is a simple parametrization of the circle associated with a counterclockwise direction. In this case the parametrization is based at $(\cos p, \sin p)$.

6.2.3 ***Example.*** The function

$$\alpha(t) = ((2 + \cos 3t)\cos 2t, (2 + \cos 3t)\sin 2t, \sin 3t), \qquad t \in [0, 2\pi]$$

is a simple parametrization, based at $(3, 0, 0)$, of the *trefoil knot* illustrated in Fig. 6.4. Its projection onto the x, y plane is the image of the path $\alpha_{1,2}(t) = r(t)(\cos 2t, \sin 2t)$, $t \in [0, 2\pi]$, where $r(t) = 2 + \cos 3t$.

As in the case of simple arcs, a simple closed curve may have corners.

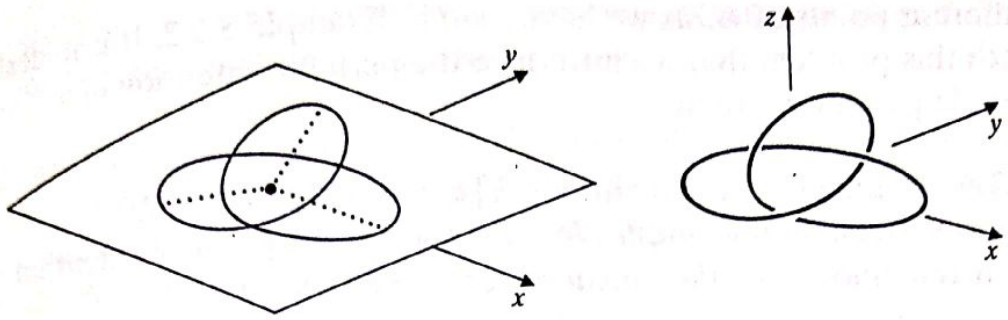

Fig. 6.4 Trefoil knot and projection onto x, y plane

6.2.4 *Example.* The astroid $x^{2/3} + y^{2/3} = 1$ is a simple closed curve in $\mathbb{R}^2$, simply parametrized by

$$\alpha(t) = (\cos^3 t, \sin^3 t), \qquad t \in [0, 2\pi].$$

See Fig. 6.5.

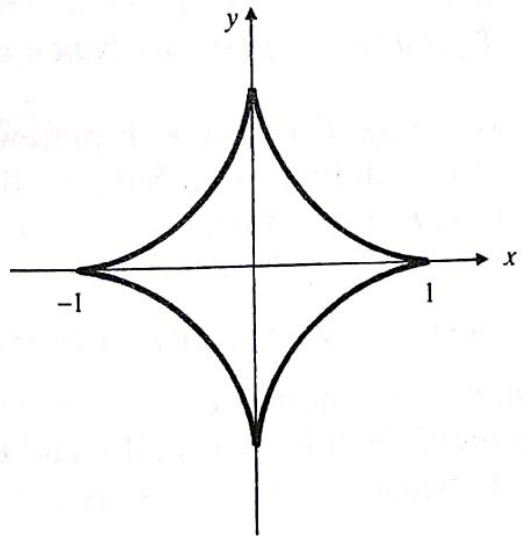

Fig. 6.5 Astroid $x^{2/3} + y^{2/3} = 1$

6.2.5 *Example.* The function

$$\alpha(t) = (\cos t, \sin t, |t|), \qquad t \in [-\pi, \pi]$$

is a simple parametrization of a simple closed curve in $\mathbb{R}^3$ based at $(-1, 0, \pi)$ and lying on the cylinder $x^2 + y^2 = 1$.

In defining the orientation of a simple closed curve C we shall need to know how different parametrizations of C are related. One awkward feature is that different parametrizations may be based at

different points of C, as we have seen in Example 6.2.2. It is to deal with this problem that we introduce the periodic extension of a simple parametrization.

6.2.6 *Definition.* *Let $\alpha:[a, a+h] \subseteq \mathbb{R} \to \mathbb{R}^n$ be a simple parametrization of a simple closed curve C. The periodic extension of α is defined to be the function $\alpha_*:\mathbb{R} \to \mathbb{R}^n$, where*

$$\alpha_*(t+mh) = \alpha(t) \qquad \textit{for all } t \in [a, a+h] \textit{ and all } m \in \mathbb{Z}.$$

Since $\alpha(a) = \alpha(a+h)$ and α is continuous, the function α_* is well defined and continuous. It has period h. It copies the effect of α on each interval $[a+mh, a+mh+h]$ in $\mathbb{R}$.

The following lemma shows how α_* copies the effect of α on *any* interval of length h in $\mathbb{R}$.

6.2.7 *Lemma.* ***Let*** *$\alpha:[a, a+h] \subseteq \mathbb{R} \to \mathbb{R}^n$ be a simple parametrization of a simple closed curve C in $\mathbb{R}^n$, and let $\alpha_*:\mathbb{R} \to \mathbb{R}^n$ be the periodic extension of α. Then for any $p \in \mathbb{R}$, the function α_* restricted to $[p, p+h]$ is a simple parametrization of C.*

Proof. Since $\alpha_*(p) = \alpha_*(p+h)$ and α_* is piecewise C^1, we need only prove that α_* is 1–1 on $[p, p+h[$. Suppose that $\alpha_*(u) = \alpha_*(v)$. Choose $n, m \in \mathbb{Z}$ such that $u - nh$ and $v - mh$ lie in $[a, a+h[$. Then

$$\alpha(u-nh) = \alpha_*(u) = \alpha_*(v) = \alpha(v-mh).$$

Now α is 1–1 on $[a, a+h[$. Therefore $u - nh = v - mh$, and so $u - v$ is an integral multiple of h. Hence, if u and v both lie in $[p, p+h[$ then $u = v$, which proves that α_* is 1–1 on $[p, p+h[$.

6.2.8 *Example.* The circle $x^2 + y^2 = 1$ in $\mathbb{R}^2$ is simply parametrized by $\alpha:[0, 2\pi] \subseteq \mathbb{R} \to \mathbb{R}^2$ given by $\alpha(t) = (\cos t, \sin t)$, $t \in [0, 2\pi]$. The periodic extension $\alpha_*:\mathbb{R} \to \mathbb{R}^2$ is given by $\alpha_*(u) = (\cos u, \sin u)$, $u \in \mathbb{R}$. For any fixed $p \in \mathbb{R}$, the function α_* restricted to $[p, p+2\pi]$ is a simple parametrization of C. It is based at $(\cos p, \sin p)$, and is illustrated as γ in Fig. 6.3.

6.2.9 *Example.* Let $\alpha:[-\pi, \pi] \subseteq \mathbb{R} \to \mathbb{R}^2$ be given by $\alpha(t) = (\cos t, \sin t, |t|)$, $t \in [-\pi, \pi]$. The periodic extension $\alpha_*:\mathbb{R} \to \mathbb{R}^2$ of α is defined for $u \in \mathbb{R}$ by

$$\alpha_*(u) = (\cos u, \sin u, |u - 2m_u\pi|)$$

where $m_u \in \mathbb{Z}$ is so chosen that $-\pi \leqslant u - 2m_u\pi < \pi$.

We shall compare simple parametrizations of a simple closed curve by comparing their periodic extensions.

6.4.10 Theorem. *Let $\alpha : [a, a + h] \subseteq \mathbb{R} \to \mathbb{R}^n$ and $\beta : [b, b + k] \subseteq \mathbb{R} \to \mathbb{R}^n$ be simple parametrizations of a simple closed curve C in $\mathbb{R}^n$, and let α_0 and β_0 be their periodic extensions. Then there exists a strictly monotonic continuous function ϕ_0 from $\mathbb{R}$ onto $\mathbb{R}$ such that $\beta_0 = \alpha_0 \circ \phi_0$. Moreover, if α and β are piecewise smooth then ϕ_0 is piecewise smooth.*

Proof. Let $\alpha(a) = x \in C$. Then there exists a unique $c \in [b, b + k]$ such that $\beta(c) = x$. Let $\gamma : [c, c + k] \subseteq \mathbb{R} \to \mathbb{R}^n$ be the restriction of β to $[c, c + k]$. By 6.4.4 assume α, β, γ is a simple parametrization of C. Since $\gamma(c) = x = \alpha(a)$, both α and γ are based at x.

Fig. 6.6

Let γ_0 and α_0 denote respectively the functions γ restricted to the open interval $]c, c + k[$ and the function α restricted to $]a, a + h[$. Both γ_0 and β_0 are 1–1 and their image is $C \setminus \{x\}$. Therefore the function $\phi_0 = \alpha_0^{-1} \circ \gamma_0$ is 1–1 from $]c, c + k[$ onto $]a, a + h[$. The method used in part (i) of the proof of Theorem 3.6.24 shows that ϕ_0 is continuous as a function from $]c, c + k[$ into the compact interval $[a, a + h]$. Therefore ϕ_0 is strictly monotonic.

Since the image of ϕ_0 is $]a, a + h[$ we can extend ϕ_0 to a continuous strictly monotonic function ϕ from $[c, c + k]$ onto $[a, a + h]$ by taking $\phi(c) = a$ and $\phi(c + k) = a + h$ if ϕ_0 is increasing, and $\phi(c) = a + h$, $\phi(c + k) = a$ if ϕ_0 is decreasing. Since $\gamma_0 = \alpha_0 \circ \phi_0$ on $]c, c + k[$, it is clear that $\gamma = \alpha \circ \phi$ on $[c, c + k]$.

We shall compare simple parametrizations of a simple closed curve by comparing their periodic extensions.

6.2.10 Theorem. *Let $\alpha:[a, a+h] \subseteq \mathbb{R} \to \mathbb{R}^n$ and $\beta:[b, b+k] \subseteq \mathbb{R} \to \mathbb{R}^n$ be simple parametrizations of a simple closed curve C in $\mathbb{R}^n$, and let α_* and β_* be their periodic extensions. Then there exists a strictly monotonic continuous function ϕ_* from $\mathbb{R}$ onto $\mathbb{R}$ such that $\beta_* = \alpha_* \circ \phi_*$. Moreover, if α and β are piecewise smooth then ϕ_* is piecewise smooth.*

Proof. Put $\alpha(a) = \mathbf{r} \in C$. Then there exists a unique $c \in [b, b+k[$ such that $\beta(c) = \mathbf{r}$. Let $\gamma:[c, c+k] \subseteq \mathbb{R} \to \mathbb{R}^n$ be the restriction of β_* to $[c, c+k]$. By, Lemma 6.2.7, γ is a simple parametrization of C. Since $\gamma(c) = \mathbf{r} = \alpha(a)$, both α and γ are based at $\mathbf{r}$.

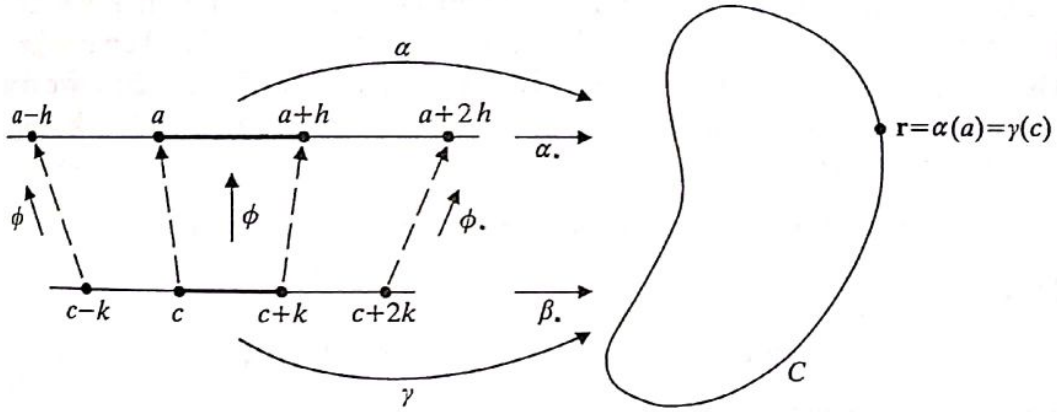

Fig. 6.6

Let γ_0 and α_0 denote respectively the functions γ restricted to the open interval $]c, c+k[$ and the function α restricted to $]a, a+h[$. Both γ_0 and β_0 are 1–1 and their image is $C \backslash \{\mathbf{r}\}$. Therefore the function $\phi_0 = \alpha_0^{-1} \circ \gamma_0$ is 1–1 from $]c, c+k[$ onto $]a, a+h[$. The method used in part (i) of the proof of Theorem 2.6.23 shows that ϕ_0 is continuous as a function from $]c, c+k[$ into the compact interval $[a, a+h]$. Therefore ϕ_0 is strictly monotonic.

Since the image of ϕ_0 is $]a, a+h[$ we can extend ϕ_0 to a continuous strictly monotonic function ϕ from $[c, c+k]$ onto $[a, a+h]$ by taking $\phi(c) = a$ and $\phi(c+k) = a+h$ if ϕ_0 is increasing, and $\phi(c) = a+h$, $\phi(c+k) = a$ if ϕ_0 is decreasing. Since $\gamma_0 = \alpha_0 \circ \phi_0$ on $]c, c+k[$, it is clear that $\gamma = \alpha \circ \phi$ on $[c, c+k]$.

Passing to the periodic extensions γ_*, α_* and β_*, we construct a strictly monotonic continuous function ϕ_* from $\mathbb{R}$ on to $\mathbb{R}$ such that $\beta_* = \alpha_* \circ \phi_*$ as follows:

[i] if ϕ is increasing then define

$$\phi_*(t + mk) = \phi(t) + mk \qquad \text{for all } t \in [c, c+k] \text{ and all } m \in \mathbb{Z}.$$

[ii] if ϕ is decreasing then define

$$\phi_*(t + mk) = \phi(t) - mk \qquad \text{for all } t \in [c, c+k] \text{ and all } m \in \mathbb{Z}.$$

The desired properties of ϕ_* can be verified with reference to Fig. 6.6, where the case ϕ_* increasing is illustrated.

Finally, the piecewise smoothness condition is proved by the method of part (ii) of Theorem 2.6.23.

6.2.11 ***Example.*** Consider the two simple parametrizations of the circle $x^2 + y^2 = 1$ in $\mathbb{R}^2$ given by $\alpha(t) = (\cos t, \sin t)$, $t \in [0, 2\pi]$ and $\beta(t) = (\sin 2t, \cos 2t)$, $t \in [-\frac{1}{2}\pi, \frac{1}{2}\pi]$. The effect is sketched in Fig. 6.3.

Following through the argument of Theorem 6.2.10 we find that $h = 2\pi$, $k = \pi$, that α is based at $\alpha(0) = (1, 0)$, and that $\beta(c) = (1, 0)$ when $c = \frac{1}{4}\pi$. This leads to $\gamma : [\frac{1}{4}\pi, \frac{5}{4}\pi] \subseteq \mathbb{R} \to \mathbb{R}^2$ given by $\gamma(u) = (\sin 2u, \cos 2u)$. We find the monotonic continuous function ϕ from $[\frac{1}{4}\pi, \frac{5}{4}\pi]$ onto $[0, 2\pi]$ such that $\gamma = \alpha \circ \phi$ by solving

$$(\sin 2u, \cos 2u) = (\cos \phi(u), \sin \phi(u)), \qquad u \in [\tfrac{1}{4}\pi, \tfrac{5}{4}\pi].$$

This gives

$$\phi(u) = \tfrac{5}{2}\pi - 2u, \qquad u \in [\tfrac{1}{4}\pi, \tfrac{5}{4}\pi].$$

Notice that ϕ is monotonic decreasing.

Finally, the function ϕ_* such that $\alpha_* = \beta_* \circ \phi_*$ will be found to satisfy

$$\phi_*(u) = \tfrac{5}{2}\pi - 2u, \qquad u \in \mathbb{R}.$$

The function ϕ_* of Theorem 6.2.10 is not unique. In the above example we could equally well choose

$$\phi_*(u) = \tfrac{1}{2}\pi - 2u, \qquad u \in \mathbb{R}.$$

Referring to Fig. 6.6, this effectively corresponds to a periodic shift of the image of ϕ_* (the domain of α_*) through $h = 2\pi$. The possibilities for ϕ_* are related as follows.

6.2.12 ***Theorem.*** *Let $\alpha : [a, a+h] \subseteq \mathbb{R} \to \mathbb{R}^n$ and $\beta : [b, b+k] \subseteq \mathbb{R} \to \mathbb{R}^n$ be simple parametrizations of a simple closed curve C in $\mathbb{R}^n$.*

If $\phi_*:\mathbb{R}\to\mathbb{R}$ and $\psi_*:\mathbb{R}\to\mathbb{R}$ are continuous functions such that

6.2.13 $$\beta_* = \alpha_* \circ \phi_* \quad \text{and} \quad \beta_* = \alpha_* \circ \psi_*$$

then there exists $m \in \mathbb{Z}$ such that

6.2.14 $$\phi_*(u) = \psi_*(u) + mh, \qquad \text{for all } u \in \mathbb{R}.$$

In particular ϕ_* is strictly increasing (decreasing) if and only if ψ_* is strictly increasing (decreasing).

Proof. By 6.2.13, $\alpha_*(\phi_*(u)) = \alpha_*(\psi_*(u))$ for each $u \in \mathbb{R}$. Therefore, by the definition of α_*, corresponding to each $u \in \mathbb{R}$ there is $m_u \in \mathbb{Z}$ such that

$$\phi_*(u) - \psi_*(u) = m_u h.$$

Now $h > 0$ and so the integer m_u depends continuously upon u. Therefore m_u is constant, and 6.2.14 follows. The rest of the proof is left as exercise.

In view of Theorems 6.2.10 and Theorem 6.2.12 the following relationship between parametrizations of a simple closed curve C is well defined.

6.2.15 *Definition.* *Let α and β be two simple parametrizations of a simple closed curve and let ϕ_* be a strictly monotonic continuous function from $\mathbb{R}$ onto $\mathbb{R}$ such that $\beta_* = \alpha_* \circ \phi_*$. Then α and β have the same (opposite) orientation if ϕ_* is strictly increasing (decreasing).*

6.2.16 *Example.* The parametrizations α and β of the unit circle considered in Example 6.2.11 have opposite orientation since ϕ_* is strictly decreasing. Notice that $\alpha(t)$ traces the circle counterclockwise and $\beta(t)$ traces it clockwise.

6.2.17 *Example.* The simple parametrization α and the inverse parametrization α^- of a simple closed curve have opposite orientation (Exercise 6.2.7).

The relationship of having the same orientation is an equivalence relation on the set of all simple parametrizations of a simple closed curve. There are two equivalence classes—the one containing a parametrization α is denoted by $[\alpha]$.

6.2.18. *Definition.* *Let C be a simple closed curve and let α be a simple parametrization of C.*

[i] *The pair $\{C, [\alpha]\}$ is called an* oriented simple closed curve. *It will be denoted either by C^+ or by C^-. If $C^+ = \{C, [\alpha]\}$ then $C^- = \{C, [\alpha^-]\}$ and vice versa.*

[ii] *Let $C^+ = \{C, [\alpha]\}$. Any parametrization in $[\alpha]$ is called a* parametrization *of C^+.*

There are therefore, as expected, two orientations of a simple closed curve. The simple parametrizations of the circle $x^2 + y^2 = 1$ in $\mathbb{R}^2$ for example divide into two classes, those that trace the circle clockwise and those that trace the circle counterclockwise.

The following theorems lead to the definition of the integrals of scalar and vector fields around simple closed curves. As in the case of simple arcs the definitions depend upon the integrals being independent of parametrization of the curve.

6.2.19 *Theorem.* *Let $\alpha : [a, a+h] \subseteq \mathbb{R} \to \mathbb{R}^n$ and $\beta : [b, b+k] \subseteq \mathbb{R} \to \mathbb{R}^n$ be simple parametrizations of a simple closed curve C in $\mathbb{R}^n$ with the same orientation, and let S be a subset of $\mathbb{R}^n$ containing C.*

[i] *For any continuous scalar field $f : S \subseteq \mathbb{R}^n \to \mathbb{R}$*

$$\int_\alpha f \, \mathrm{d}s = \int_\beta f \, \mathrm{d}s.$$

[ii] *For any continuous vector field $F : S \subseteq \mathbb{R}^n \to \mathbb{R}^n$*

$$\int_\alpha F \cdot \mathrm{d}\alpha = \int_\beta F \cdot \mathrm{d}\beta.$$

Proof. We prove the formula for the vector field, leaving the case of the scalar field to the reader.

Find a strictly increasing continuous function $\phi_* : \mathbb{R} \to \mathbb{R}$ such that $\beta_* = \alpha_* \circ \phi_*$.

Let $\phi_*(c) = a$; then $\phi_*(c+k) = a + h$ and taking $\phi_*(c + \frac{1}{2}k) = p$ we have $a < p < a + h$.

The paths $\alpha \mid [a, p]$ and $\beta_* \mid [c, c + \frac{1}{2}k]$ are simple parametrizations of the same simple arc C_1 in S. Since they have the same orientation, by Theorem 6.1.3

6.2.20 $$\int_a^p F(\alpha(t)) \cdot \alpha'(t) \, \mathrm{d}t = \int_c^{c+1/2k} F(\beta_*(u)) \cdot \beta_*'(u) \, \mathrm{d}u.$$

Similarly,

6.2.21 $$\int_p^{a+h} F(\alpha(t)) \cdot \alpha'(t)\,\mathrm{d}t = \int_{c+1/2k}^{c+k} F(\beta_*(u)) \cdot \beta'_*(u)\,\mathrm{d}u.$$

Adding 6.2.20 and 6.2.21 we find that

$$\int_\alpha F \cdot \mathrm{d}\alpha = \int_\beta F \cdot \mathrm{d}\beta.$$

6.2.22 Corollary. *Let α and β be simple parametrizations of a simple closed curve C in $\mathbb{R}^n$ with opposite orientations, and let S be a subset of $\mathbb{R}^n$ containing C.*

[i] *For any continuous scalar field $f : S \subseteq \mathbb{R}^n \to \mathbb{R}$*

$$\int_\alpha f\,\mathrm{d}s = \int_\beta f\,\mathrm{d}s.$$

[ii] *For any continuous vector field $F : S \subseteq \mathbb{R}^n \to \mathbb{R}^n$*

$$\int_\alpha F \cdot \mathrm{d}\mathbf{r} = -\int_\beta F \cdot \mathrm{d}\mathbf{r}.$$

Proof. Apply Theorem 6.2.19 to the parametrizations α and β^-.

Theorem 6.2.19 and its corollary show that the following definitions are not ambiguous.

6.2.23 Definition. *Let $f : S \subseteq \mathbb{R}^n \to \mathbb{R}$ be a continuous scalar field, and let C be a simple closed curve in S. The* integral of f over C *is defined and denoted by*

$$\int_C f\,\mathrm{d}s = \int_\alpha f\,\mathrm{d}s$$

where α is any simple parametrization of C.

6.2.24 Definition. *Let $F : S \subseteq \mathbb{R}^n \to \mathbb{R}^n$ be a continuous vector field, and let C^+ be an oriented simple closed curve in S. The* integral of F around C^+ *is defined and denoted by*

$$\int_{C^+} F \cdot \mathrm{d}\mathbf{r} = \int_\alpha F \cdot \mathrm{d}\alpha$$

where α is any simple parametrization of C^+.

Corollary 6.2.22(ii) implies that

$$\int_{C^-} F \cdot \mathrm{d}\mathbf{r} = -\int_{C^+} F \cdot \mathrm{d}\mathbf{r}.$$

The integral of a vector field F around an oriented simple closed curve C^+ is often denoted by

$$\oint_C F \cdot \mathrm{d}\mathbf{r},$$

the circle through the integral sign indicating that the curve C is closed. As in the case of the integral of F along an oriented simple arc in $\mathbb{R}^n$, the expression $F \cdot \mathrm{d}\mathbf{r}$ can be written $F_1\,\mathrm{d}x_1 + \cdots + F_n\,\mathrm{d}x_n$, as we have done in Exercises 6.2.

6.2.25 ***Example.*** The points $A = (0, 2)$, $B = (-1, 0)$, $C = (0, -2)$, $D = (1, 0)$ lie at the corners of a diamond shape consisting of the straight edges AB, BC, CD and DA. Let C^+ be the oriented simple closed curve consisting of these edges, with counterclockwise orientation. See Fig. 6.7.

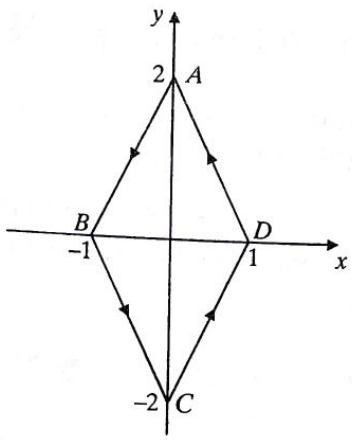

Fig. 6.7

If F is a continuous force field in $\mathbb{R}^2$ then the work done by F on a particle as it moves once around the diamond in a counterclockwise sense is given by $\int_{C^+} F \cdot \mathrm{d}\mathbf{r}$. Let us calculate this for the case

$$F(x, y) = (y, -x) \qquad (x, y) \in \mathbb{R}^2.$$

We consider the oriented simple closed curve C^+ as consisting of four

oriented simple arcs with appropriate simple parametrizations as follows:

[i] C_1^+ from A to B, parametrized by $\alpha:[0, 1] \subseteq \mathbb{R} \to \mathbb{R}^2$, where $\alpha(t) = (-t, 2-2t)$, $t \in [0, 1]$;

[ii] C_2^+ from B to C, parametrized by, $\beta:[0, 1] \subseteq \mathbb{R} \to \mathbb{R}^2$, where $\beta(t) = (-1+t, -2t)$, $t \in [0, 1]$;

[iii] C_3^+ from C to D, parametrized by $\gamma:[0, 1] \subseteq \mathbb{R} \to \mathbb{R}^2$, where $\gamma(t) = (t, -2+2t)$, $t \in [0, 1]$;

[iv] C_4^+ from D to A, parametrized by $\mu:[0, 1] \subseteq \mathbb{R} \to \mathbb{R}^2$, where $\mu(t) = (1-t, 2t)$, $t \in [0, 1]$.

Hence

$$\int_{C^+} F \cdot \mathrm{d}\mathbf{r} = \int_\alpha F \cdot \mathrm{d}\alpha + \int_\beta F \cdot \mathrm{d}\beta + \int_\gamma F \cdot \mathrm{d}\gamma + \int_\mu F \cdot \mathrm{d}\mu$$

$$= \int_0^1 -2\,\mathrm{d}t + \int_0^1 -2\,\mathrm{d}t + \int_0^1 -2\,\mathrm{d}t + \int_0^1 -2\,\mathrm{d}t = -8.$$

Similarly, by considering the inverse paths of α, β, γ, and μ we obtain

$$\int_{C^-} F \cdot \mathrm{d}\mathbf{r} = 8.$$

As in the case of a simple arc, the integral of scalar field may be used to define the length of a simple closed curve.

6.2.26 *Definition.* *Let C be a simple closed curve in $\mathbb{R}^n$. The length of C is defined by $l(C) = \int_C 1\,\mathrm{d}s$.*

6.2.27 *Example.* *Let C* be the ellipse $x^2/a^2 + y^2/b^2 = 1$ in $\mathbb{R}^2$ where $a \geq b > 0$. The length of C may be found by using the simple parametrization $\alpha(t) = (a \cos t, b \sin t)$, $t \in [0, 2\pi]$. We obtain

$$l(C) = l(\alpha) = 4a \int_0^{\pi/2} \sqrt{(1 - k^2 \cos^2 t)}\,\mathrm{d}t,$$

where $k^2 = (a^2 - b^2)/a^2$.

Exercises 6.2

1. Calculate the line integral $\int_{C^+} F \cdot \mathrm{d}\mathbf{r}$ where $F(x, y) = (2xy - x^2, x + y^2)$ and

   (a) C^+ is the rectangle with corners $(0, 0)$ $(1, 0)$, $(1, 2)$, $(0, 2)$ oriented counterclockwise;

   (b) C^+ is the triangle with corners $(0, 0)$, $(1, 0)$, $(0, 2)$ oriented counterclockwise;

(c) C^+ is the closed curve formed by the parabolas $y = x^2$ and $y^2 = x$, $0 \leq x \leq 1$, $0 \leq y \leq 1$, with clockwise orientation.

Answers: (a) 0; (b) 1/3; (c) $-1/30$.

2. Let $f: \mathbb{R}^2 \to \mathbb{R}$ be the scalar field defined by $f(x, y) = 2xy$. Calculate the integrals $\int_C f\, ds$ for the simple closed curves C of Exercise 1.

Answers: (a) 6; (b) $4\sqrt{5}/3$; (c) $(25\sqrt{5} + 1)/30$.

3. Calculate $\int_{C^+} -y\, dx + x\, dy$, where
   (a) C^+ is the unit circle $x^2 + y^2 = 1$ with clockwise orientation;
   (b) C^+ is the circle $x^2 + y^2 = c^2$, $c > 0$, with counterclockwise orientation;
   (c) C^+ is the counterclockwise oriented ellipse $\frac{1}{2}x^2 + y^2 = 1$;
   (d) C^+ is the clockwise oriented unit square, vertices $(0, 0)$, $(0, 1)$, $(1, 1)$, $(1, 0)$.

Answers: (a) -2π; (b) $2\pi c^2$; (c) 4π; (d) -2.

4. A particle moves in $\mathbb{R}^2$ in a force field given by

$$F(x, y) = \left(\frac{-y}{x^2 + y^2}, \frac{x}{x^2 + y^2}\right), \qquad (x, y) \neq (0, 0).$$

   Calculate the work done on the particle by F as the particle
   (a) performs one counterclockwise revolution of the unit circle $x^2 + y^2 = 1$;
   (b) performs one clockwise revolution of the circle $x^2 + y^2 = c^2$;
   (c) performs one clockwise revolution of the circle $(x - 2)^2 + y^2 = 1$.

Answers: compute $\int_{C^+} F \cdot d\mathbf{r}$ for appropriately chosen oriented simple closed curves C^+. (a) 2π; (b) -2π; (c) 0.

5. Calculate $\int_{C^+} y\, dx + z\, dy + x\, dz$, where C^+ is the ellipse in $\mathbb{R}^3$ $\{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 = 1,\ x + y + z = 1\}$ with orientation $(0, 1, 0) \to (-1, 0, 2) \to (1, 0, 0)$ (counterclockwise orientation viewed from 'above' the ellipse on the z axis).

Answer: -3π. An appropriate simple parametrization of the oriented ellipse is $\alpha(t) = (-\sin t, \cos t, 1 + \sin t - \cos t)$, $t \in [0, 2\pi]$.

6. Let $f: S \subseteq \mathbb{R}^n \to \mathbb{R}^n$ be a continuous vector field, and let C^+ be an oriented simple closed curve in S. Define the *mean component of F around C^+* by

$$\frac{1}{l(C)} \int_{C^+} F \cdot d\mathbf{r},$$

   where $l(C)$ is the length of C. Find the mean component of the following fields around the circle C^+ centre the origin, radius c,

oriented counterclockwise.

(a) $F(x, y) = (-y, x)$;

(b) $F(x, y) = \left(\dfrac{y}{x^2+y^2}, \dfrac{-x}{x^2+y^2}\right)$;

(c) $F(x, y) = (y, x)$.

Answers: (a) c; (b) $-1/c$; (c) 0 (sketch the field!).

7. Let $\alpha : [a, a+h] \subseteq \mathbb{R} \to \mathbb{R}^n$ be a simple parametrization of a simple closed curve C. Prove that the inverse parametrization $\alpha^- : [a, a+h] \subseteq \mathbb{R} \to \mathbb{R}^n$ of C defined by

$$\alpha^-(u) = \alpha(2a + h - u), \qquad u \in [a, a+h]$$

has opposite orientation to that of α.

Hint: find the (unique) continuous function ϕ from $[a, a+h]$ onto $[a, a+h]$ such that $\alpha^- = \alpha \circ \phi$, and show that ϕ is strictly decreasing.

10. Let α and β be the simple parametrizations of the ellipse $x^2/a^2 + y^2/b^2 = 1$ given by

$$\alpha(t) = (a \cos t, b \sin t), \qquad t \in [0, 2\pi],$$
$$\beta(t) = (a \sin 2t, -b \cos 2t), \qquad t \in [\pi/3, 4\pi/3]$$

and let α_* and β_* be their periodic extensions. Find a strictly monotonic continuous function ϕ_* from $\mathbb{R}$ onto $\mathbb{R}$ such that $\beta_* = \alpha_* \circ \phi_*$

Answer: $\phi_*(u) = \frac{1}{4}\pi + 2u + m\pi$ for any choice of $m \in \mathbb{Z}$.

6.3 Rotation. Irrotational fields

An important application of the line integral of a vector field around a simple closed curve arises in fluid dynamics. The motion of a fluid (a stream of water, for example) is described by its velocity field F, where $F(\mathbf{r})$ is the velocity of the fluid at the point with position vector $\mathbf{r}$. Imagine a small cork moving with the fluid. Then, besides moving downstream, the cork may also rotate about a vertical axis. The rotation is pronounced close to the outlet of an emptying bath, for example. It occurs less dramatically in a stream as illustrated in Fig. 6.8 where the velocity of the fluid is rather greater near one bank than near the other. It does not occur at all in an evenly flowing stream.

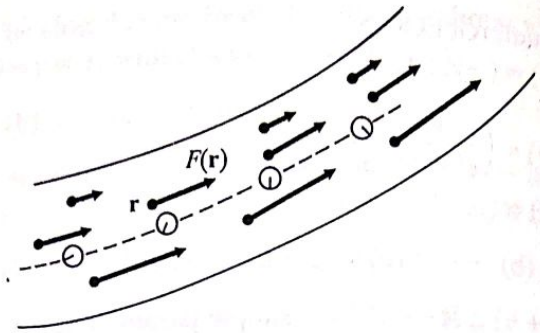

Fig. 6.8

Let $F : S \subseteq \mathbb{R}^2 \to \mathbb{R}^2$ be a continuous vector field modelling the *velocity* on the surface of a fluid. Let C^+ be an oriented simple closed curve in S. The line integral $\int_{C^+} F \cdot d\mathbf{r}$ can be interpreted as the *integral of the tangential component of F around* C^+. If C is the simple closed curve where the cork breaks the surface of the liquid we would expect the cork to rotate in the direction of the orientation of C^+ if $\int_{C^+} F \cdot d\mathbf{r}$ is positive, and in the opposite direction (or not at all) if this integral is negative (or zero). The integral $\int_{C^+} F \cdot d\mathbf{r}$ is called the *circulation of F around* C^+.

Suppose that the curve C is a circle, centre O, radius c. We obtain an absolute measure of the cork's rate of rotation (about a vertical axis through O) by calculating the mean angular velocity about O of the fluid around the circle C^+. Using the relation $v = c\omega$ between the tangential component of velocity v at a point P on C and the angular velocity ω of P about O, we obtain

$$\text{mean angular velocity around } C^+ = \frac{1}{l(C)} \int_{C^+} \left(\frac{F}{c}\right) \cdot d\mathbf{r}$$

$$= \frac{1}{2\pi c^2} \int_{C^+} F \cdot d\mathbf{r}.$$

Notice that πc^2 is the area enclosed by the circle C.

Generalizing, let C^+ be an arbitrary oriented simple closed curve in S. The integral

$$\frac{1}{\text{area enclosed by } C} \int_{C^+} F \cdot d\mathbf{r}$$

is called the *rotation of F around* C^+. It provides a measure in radians per unit time of twice the net rate of turning of the fluid around C^+.

6.3.1 *Example.* Let $F:\mathbb{R}^2 \to \mathbb{R}^2$ be defined by

$$F(x, y) = (1, \tfrac{1}{2}x^2) \qquad (x, y) \in \mathbb{R}^2.$$

Let C^+ be the circle in $\mathbb{R}^2$ centred at (a, b), radius c, with counterclockwise orientation. Using the natural parametrization of C^+ given by

$$(x(t), y(t)) = (a + c\cos t, b + c\sin t), \qquad t \in [0, 2\pi]$$

the circulation of F around C^+ is found to be

$$\int_{C^+} F \cdot \mathrm{d}\mathbf{r} = \int_0^{2\pi} (1, \tfrac{1}{2}(a + c\cos t)^2) \cdot (-c\sin t, c\cos t)\,\mathrm{d}t$$
$$= ac^2\pi.$$

Hence the rotation of F around C^+ is a. A circular cork centred at (a, b) and moving in the velocity field F would experience a mean angular velocity $2a$ about its centre, the turning effect being clockwise or counterclockwise according as $a < 0$ or $a > 0$. See Fig. 6.9.

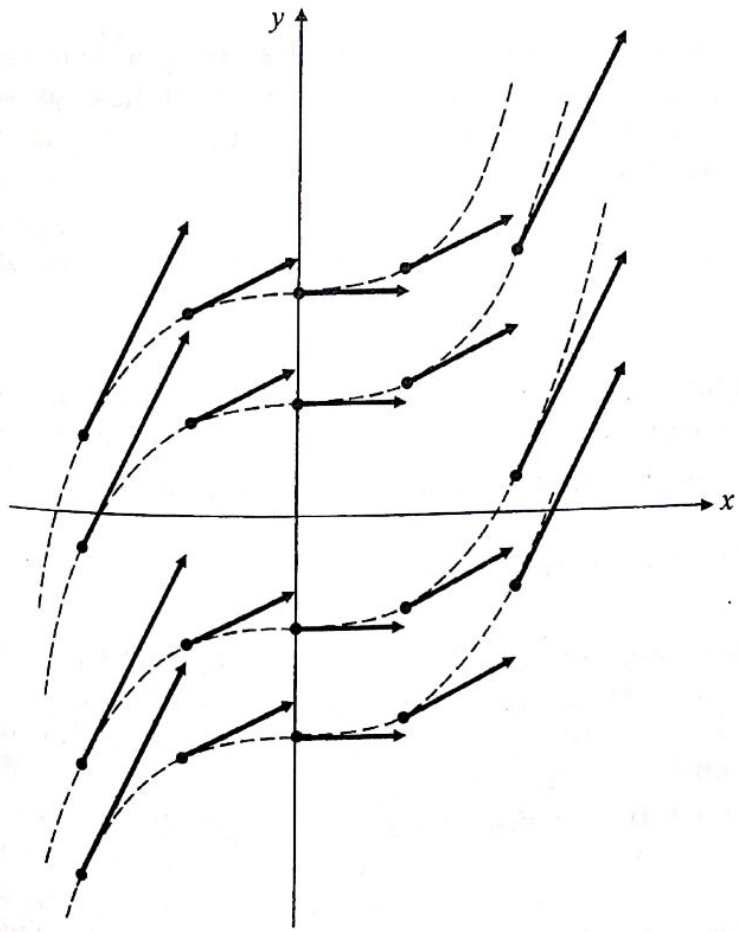

Fig. 6.9 Velocity field $F(x, y) = (1, \tfrac{1}{2}x^2)$

6.3.2 *Example.* Let $F:\mathbb{R}^2\backslash\{\mathbf{0}\}\to\mathbb{R}^2$ be defined by

$$F(x, y)=\left(\frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2}\right), \qquad (x, y)\neq(0, 0).$$

[i] Let C_1^+ be the circle centred at the origin, radius c, with counterclockwise orientation. We find that

6.3.3
$$\int_{C_1^+} F\cdot d\mathbf{r}=\int_0^{2\pi} 1\, dt=2\pi.$$

Hence the rotation of F around C_1^+ is $2/c^2$. A sketch of the field F confirms the positive (counterclockwise) rotation. (See Fig. 5.14, page 308.) The rotation increases unboundedly with decreasing radius c. There is a whirlpool effect at the origin.

[ii] Let C_2 be the circle centred at $(a, b)\neq(0, 0)$ and radius $c<\sqrt{(a^2+b^2)}$. So C_2 does not enclose the origin. In this case we find that, for either orientation of C_2,

6.3.4
$$\int_{C_2^+} F\cdot d\mathbf{r}=0.$$

Hence the rotation of F around C_2^+ is zero. Referring again to Fig. 5.14, the apparent counterclockwise rotation of the field about the origin is balanced on the circle C_2 by an apparent clockwise effect produced by the diminution of speed with increasing distance from the origin.

The integral 6.3.4 can be established directly from the following theorem.

6.3.5 *Theorem.* *Let $F:D\subseteq\mathbb{R}^2\to\mathbb{R}^2$ be a conservative vector field defined on an open subset D of $\mathbb{R}^2$. Then the circulation of F around any oriented simple closed curve C^+ in D is zero.*

Proof. Immediate from Theorem 5.5.21.

6.3.6 *Example.* Returning to Example 6.3.2, the field F given there is conservative on $\mathbb{R}^2\backslash H$, where H is the half-line $\{t(a, b)\in\mathbb{R}^2 \mid t\leq 0\}$. See Exercise 5.5.8. This establishes 6.3.4. On the other hand, F is not conservative on $\mathbb{R}^2\backslash\{\mathbf{0}\}$. Therefore we cannot use Theorem 6.3.5 to obtain information about the rotation of F around simple closed curves enclosing the origin.

Let $F:D\subseteq\mathbb{R}^2\to\mathbb{R}^2$ be a continuous velocity field defined on an open subset D of $\mathbb{R}^2$. Fix a point $\mathbf{p}\in D$. We define the *rotation of F*

at $\mathbf{p}$ by a limiting process. Let C_ε^+ be the circle centred at $\mathbf{p}$, radius ε, with counterclockwise orientation, and let S_ε be the circular disc enclosed by C_ε. Then the rotation of F at $\mathbf{p}$ is defined by

6.3.7
$$(\operatorname{rot} F)(\mathbf{p}) = \lim_{\varepsilon\to 0} (\text{rotation of } F \text{ around } C_\varepsilon^+)$$
$$= \lim_{\varepsilon\to 0} \frac{1}{\text{area } S_\varepsilon} \int_{C_\varepsilon^+} F \cdot d\boldsymbol{r}.$$

provided this limit exists. The number $(\operatorname{rot} F)(\mathbf{p})$ is positive, negative or zero according as the turning effect at $\mathbf{p}$ is counterclockwise, clockwise or zero.

6.3.8 *Example.* In Example 6.3.1, where $F(x, y) = (1, \frac{1}{2}x^2)$,

$$(\operatorname{rot} F)(a, b) = a.$$

In Example 6.3.2, where $F(x, y) = (-y/(x^2+y^2), x/(x^2+y^2))$, $(x, y) \neq (0, 0)$,

$$(\operatorname{rot} F)(x, y) = 0 \qquad \text{for all } (x, y) \in \mathbb{R}^2\backslash\{\mathbf{0}\}.$$

This follows from 6.3.4.

6.3.9 *Definition.* *Let $F : D \subseteq \mathbb{R}^2 \to \mathbb{R}^2$ be a continuous (velocity) field defined on an open subset D of $\mathbb{R}^2$. the field F is said to be irrotational if*

$$(\operatorname{rot} F)(\mathbf{p}) = 0 \qquad \textit{for all } \mathbf{p} \in D.$$

It follows from Theorem 6.3.5 that a conservative field $F : D \subseteq \mathbb{R}^2 \to \mathbb{R}^2$, defined on an open subset D of $\mathbb{R}^2$, is irrotational. Example 6.3.8 shows the converse to be false: the field $F : \mathbb{R}^2\backslash\{\mathbf{0}\} \to \mathbb{R}^2$ given by

6.3.10
$$F(x, y) = \left(\frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2}\right), \qquad (x, y) \neq (0, 0)$$

is irrotational, but it is not conservative (Example 5.5.28). It is true, however, that if $F : D \subseteq \mathbb{R}^2 \to \mathbb{R}^2$ is a C^1 irrotational field then it is *locally conservative* in the sense that to each $\mathbf{p} \in D$ there corresponds a neighbourhood of $\mathbf{p}$ in which F is conservative. We cannot expect a more general converse, since the property of a field being irrotational concerns its local structure (individual points and their neighbourhoods), whereas a field being conservative concerns

its global structure: it depends on the work done by the field along various simple arcs joining points in its domain.

In this section we have introduced the basic ideas relating to rotational flow in two-dimensional fluids. A large part of the remainder of this book is devoted to developing these ideas. In Chapter 7 we shall consider two-dimensional flow in terms of Green's Theorem. The main result of Chapter 9 (Stokes' Theorem) has important applications to three-dimensional fluid flow. We shall use it to assign to each point $\mathbf{p}$ in a C^1 velocity field $F: D \subseteq \mathbb{R}^3 \to \mathbb{R}^3$ an absolute measure of the rotational tendency there. This is achieved by the vector-valued function curl $F: D \subseteq \mathbb{R}^3 \to \mathbb{R}^3$, which generalizes the (scalar-valued) function rot F introduced in this section.

Exercises 6.3.

1. Let $F: \mathbb{R}^2 \to \mathbb{R}^2$ be defined by $F(x, y) = \frac{1}{4}(-y, x)$. (The field is illustrated in Fig. 5.2, page 282.) Find the rotation of F around the oriented simple curve C^+, where
   (a) C^+ is the circle $x^2 + y^2 = c^2$ oriented clockwise;
   (b) C^+ is the circle centred at (a, b), radius c, oriented counterclockwise.

Answers: (a) $-\frac{1}{2}$; (b) $\frac{1}{2}$.

2. Repeat Exercise 1 for the field $F(x, y) = (-y(x^2 + y^2), x(x^2 + y^2))$.

Answers: (a) $-2c^2$; (b) $4a^2 + 4b^2 + 2c^2$.

3. For the field $F(x, y) = (-y(x^2 + y^2), x(x^2 + y^2))$, find $(\text{rot } F)(\mathbf{p})$ at $\mathbf{p} = (a, b) \in \mathbb{R}^2$.

Answer: $4a^2 + 4b^2$.

4. Let $F: D \subseteq \mathbb{R}^2 \to \mathbb{R}^2$ be a velocity field. A curve in D whose tangent at each of its points $\mathbf{p}$ is in the direction of $F(\mathbf{p})$ is called a *stream line* or a *field line of* the field F.

   Find the stream lines of the field $F(x, y) = (1, \frac{1}{2}x^2)$. (The field is sketched in Fig. 6.9.)

Answer: suppose that α parametrizes a stream line C of F. Then

$$\alpha'(t) = (\alpha_1'(t), \alpha_2'(t)) = F(\alpha_1(t), \alpha_2(t)) = (1, \tfrac{1}{2}(\alpha_1(t))^2).$$

Integrating each coordinate equation in turn, deduce that the equation of C is $y = \frac{1}{6}x^3 + k$ for some constant k.

5. (a) Sketch the following velocity fields, indicating some stream lines.

Prove that the fields are irrotational.

(i) The constant field $F:\mathbb{R}^2\to\mathbb{R}^2$ given by $F(x, y) = (a, b)$, for all $(x, y) \in \mathbb{R}^2$;

(ii) The field $F(x, y) = (\frac{1}{4}x, \frac{1}{4})$, $(x, y) \in \mathbb{R}^2$.

(iii) The field $F:\mathbb{R}^2\backslash\{\mathbf{0}\}\to\mathbb{R}^2$ given by

$$F(\mathbf{r}) = \frac{1}{r}\mathbf{r}, \qquad \mathbf{r} = (x, y) \in \mathbb{R}^2\backslash\{\mathbf{0}\},$$

where $r = \|\mathbf{r}\|$.

Answer: the fields are all conservative and therefore irrotational.

(b) Let $g:\mathbb{R}\to\mathbb{R}$ be a continuous real-valued function. Prove that the field $F:\mathbb{R}^2\to\mathbb{R}^2$ defined by

$$F(\mathbf{r}) = g(r)\mathbf{r}, \qquad \mathbf{r} \in \mathbb{R}^2$$

is irrotational.

Answer: $F = \operatorname{grad} f$ where $f(\mathbf{r}) = \int rg(r)\,dr$. So F is conservative.

6. Show that the field $F:\mathbb{R}^2\backslash\{\mathbf{0}\}\to\mathbb{R}^2$ defined by

$$F(x, y) = \frac{e^x}{x^2+y^2}(x \sin y - y \cos y, x \cos y + y \sin y),$$
$$(x, y) \neq (0, 0),$$

is irrotational but not conservative.

Answer: to prove that F is not conservative consider the oriented unit circle C^+ parametrized by $\alpha(t) = (\cos t, \sin t)$, $t \in [0, 2\pi]$. Then

$$\int_{C^+} F \cdot d\mathbf{r} = \int_0^{2\pi} e^{\cos t} \cos(\sin t)\,dt > 0.$$

The appararently mammoth task of proving that F is irrotational will reduce to a simple calculation, once we have Green's Theorem. See Exercise 7.6.2.

Readers familiar with Complex Variable theory will appreciate the origin of the above exercise. The integral $\int_{C^+} F \cdot d\mathbf{r}$ is the imaginary part of the contour integral of e^z/z around C^+. The conclusions of the exercise follow from the observation that

$$\int_{C^+} \frac{e^z}{z}\,dz = \begin{cases} 2\pi i & \text{if } C \text{ encloses the origin,} \\ 0 & \text{otherwise.} \end{cases}$$

Similarly, the field 6.3.10 considered in the text arises from the complex function $1/z$.