7

Double integrals in $\mathbb{R}^2$

7.1 Integral over a rectangle

Let R be the compact rectangle $[a, b] \times [c, d] = \{(x, y) \in \mathbb{R}^2 \mid a \leqslant x \leqslant b,\ c \leqslant y \leqslant d\}$ in $\mathbb{R}^2$ and let $f : R \subseteq \mathbb{R}^2 \to \mathbb{R}$ be a bounded function. Since f is bounded, there exists (a lower bound) $m \in \mathbb{R}$ and (an upper bound) $M \in \mathbb{R}$ such that

$$m \leqslant f(x, y) \leqslant M, \qquad \text{for all } (x, y) \in R.$$

A partition of $[a, b]$

$$a = x_0 < x_1 < \cdots < x_{k-1} < x_k = b$$

and a partition of $[c, d]$

$$c = y_0 < y_1 < \cdots < y_{l-1} < y_l = d$$

lead to a partition $\mathscr{P}$ of R into kl subrectangles of the form

$$R_{ij} = \{(x, y) \in \mathbb{R}^2 \mid x_{i-1} \leqslant x \leqslant x_i,\ y_{j-1} \leqslant y \leqslant y_j\}$$

where $i = 1, \ldots, k$ and $j = 1, \ldots, l$. The area of the subrectangle R_{ij} (which is shaded in Fig. 7.1(i)) is

$$A_{ij}(x_i - x_{i-1})(y_j - y_{j-1}) > 0.$$

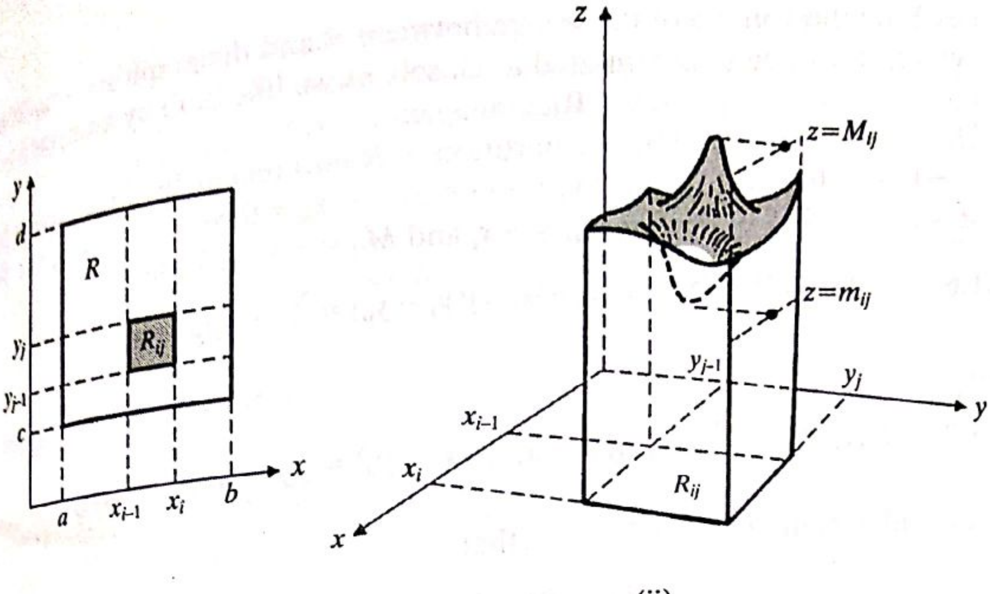
Since f is bounded on R it is also bounded on each subrectangle. For each i and j let m_{ij} and M_{ij} be respectively the greatest lower bound and the least upper bound of f on R_{ij}. For every choice of $\mathbf{p}_{ij} \in R_{ij}$ we have

7.1.1

$$m_{ij}A_{ij} \leqslant f(\mathbf{p}_{ij})A_{ij} \leqslant M_{ij}A_{ij}.$$

When f is positive on R_{ij}, the left-hand side and right-hand side of 7.1.1 are respectively lower and upper approximations to the volume above R_{ij} bounded by the graph $z = f(x, y)$ (see Fig. 7.1(ii)). This suggests that for any bounded function f on R and any partition $\mathscr{P}$ of R we define

7.1.2 $$L(f, \mathscr{P}) = \sum_{i,j} m_{ij}A_{ij}, \qquad \text{the lower Riemann sum}$$



(ii)

(i)

Fig. 7.1

and

7.1.3 $\quad U(f, \mathscr{P}) = \sum_{i,j} M_{ij} A_{ij}, \qquad$ the upper Riemann sum

of f corresponding to the partition $\mathscr{P}$.

It follows from 7.1.1 that

7.1.4 $\quad L(f, \mathscr{P}) \leqslant \sum_{i,j} f(\mathbf{p}_{ij})(x_i - x_{i-1})(y_j - y_{j-1}) \leqslant U(f, \mathscr{P}).$

7.1.5 *Example.* Let $R = [-1, 0] \times [-1, 0] \subseteq \mathbb{R}^2$ and let $f : R \subseteq \mathbb{R}^2 \to \mathbb{R}$ be defined by $f(x, y) = -x$, $(x, y) \in R$. The graph of f is that part of the plane $z = -x$ sketched in Fig. 7.2.

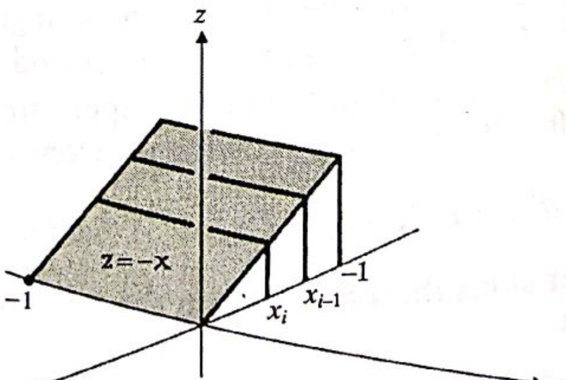


Fig. 7.2

Let V be the volume of the wedge between R and the graph $z=-x$. We show that V can be approximated as closely as we like both by an upper Riemann sum and by a lower Riemann sum.

Given any $\varepsilon>0$ let $\mathscr{P}$ be the partition of R into rectangles determined by $y_0=-1$, $y_1=0$ and $-1=x_0<x_1<\cdots<x_{k-1}<x_k=0$ where $x_i-x_{i-1}<\varepsilon$ for all $i=1,\ldots,k$. For each i, $m_{i1}=-x_i$ and $M_{i1}=-x_{i-1}$. Hence

7.1.6 $$L(f,\mathscr{P})=\sum_i -x_i(x_i-x_{i-1})(y_1-y_0)=\sum_i -x_i(x_i-x_{i-1})$$

and

7.1.7 $$U(f,\mathscr{P})=\sum_i -x_{i-1}(x_i-x_{i-1})(y_1-y_0)=\sum_i -x_{i-1}(x_i-x_{i-1}).$$

Consideration of Fig. 7.2 shows that

$$L(f,\mathscr{P})<V<U(f,\mathscr{P}),$$

whereas 7.1.6 and 7.1.7 imply that

$$U(f,\mathscr{P})-L(f,\mathscr{P})=\sum_i(x_i-x_{i-1})(x_i-x_{i-1})<\varepsilon\sum_i(x_i-x_{i-1})=\varepsilon.$$

Hence both $L(f,\mathscr{P})$ and $U(f,\mathscr{P})$ are within ε of V.

7.1.8 *Example.* Let $R=[0,1]\times[0,1]\subseteq\mathbb{R}^2$ and let $g:R\subseteq\mathbb{R}^2\to\mathbb{R}$ be defined by

$$g(x,y)=\begin{cases}1 & \text{if } x \text{ and } y \text{ are both rational}\\ 0 & \text{otherwise.}\end{cases}$$

Then for any partition $\mathscr{P}$ of R, $L(g,\mathscr{P})=0$ and $U(g,\mathscr{P})=1$. Hence 7.1.4 is satisfied but in contrast to Example 7.1.5 there is no real number which can be approximated arbitrarily closely both by lower sums and by upper sums.

We say that a partition $\mathscr{P}^*$ of R is a *refinement* of a partition $\mathscr{P}$ if every subrectangle of $\mathscr{P}^*$ is a subset of a subrectangle of $\mathscr{P}$. Suppose now that $\mathscr{P}_1$ and $\mathscr{P}_2$ are partitions of R and that $\mathscr{P}^*$ is the refinement of both $\mathscr{P}_1$ and $\mathscr{P}_2$ obtained by superimposing $\mathscr{P}_1$ on top of $\mathscr{P}_2$. It is not difficult to show, following on from 7.1.4, that

$$L(f,\mathscr{P}_1)\leqslant L(f,\mathscr{P}^*)\leqslant U(f,\mathscr{P}^*)\leqslant U(f,\mathscr{P}_2).$$

The set of lower sums therefore lies on the negative side of the set of upper sums.

7.1.9 *Definition.* *A bounded function* $f:R\subseteq\mathbb{R}^2\to\mathbb{R}$ *is said to be* (Riemann) integrable *over the compact rectangle* R *if the least upper* *bound of all lower Riemann sums is equal to the greatest lower bound of all upper Riemann sums. The common value is then called the* double integral of f over R *or simply the* integral of f over R. *It is denoted by*

$$\iint_R f \, \mathrm{d}A \qquad \text{or by} \qquad \iint_R f(x, y) \, \mathrm{d}x \, \mathrm{d}y.$$

Other common notations are

$$\iint_R f \qquad \text{or} \qquad \iint_R f(x, y) \, \mathrm{d}(x, y).$$

7.1.10 Theorem. *A bounded function $f: R \subseteq \mathbb{R}^2 \to \mathbb{R}$ is integrable over the compact rectangle R if and only if to each $\varepsilon > 0$ there correspond partitions $\mathcal{P}_1$ and $\mathcal{P}_2$ of R such that*

$$U(f, \mathcal{P}_1) - L(f, \mathcal{P}_2) < \varepsilon,$$

or equivalently, to each $\varepsilon > 0$ there corresponds a partition $\mathcal{P}$ of R such that

$$U(f, \mathcal{P}) - L(f, \mathcal{P}) < \varepsilon.$$

Proof. Exercise.

7.1.11 *Example.* The function f of Example 7.1.5 is integrable. The function g of Example 7.1.8 is not integrable.

7.1.12 Theorem. *Let $f: R \subseteq \mathbb{R}^2 \to \mathbb{R}$ and $g: R \subseteq \mathbb{R}^2 \to \mathbb{R}$ be integrable bounded functions over the compact rectangle R, and let c be a real number. Then $f + g$ and cf are integrable over R and*

$$\iint_R (f+g) \, \mathrm{d}A = \iint_R f \, \mathrm{d}A + \iint_R g \, \mathrm{d}A \qquad \text{and} \qquad \iint_R cf \, \mathrm{d}A = c \iint_R f \, \mathrm{d}A$$

Proof. Exercise.

When f is a positive continuous function on R its integral corresponds to the volume in $\mathbb{R}^3$ above R in the x, y plane and below the graph $z = f(x, y)$.

corresponds $\delta > 0$ such that

7.1.15 $|f(\mathbf{v}) - f(\mathbf{w})| < \varepsilon$ whenever $\mathbf{v}, \mathbf{w} \in R$ and $\|\mathbf{v} - \mathbf{w}\| < \delta$.

Let $\mathscr{P}$ be a partition of R such that the diagonal length of each subrectangle R_{ij} of $\mathscr{P}$ is less than δ. Then from 7.1.15,

$$M_{ij} - m_{ij} \leqslant \varepsilon,$$

and, from 7.1.2 and 7.1.3,

$$\begin{aligned} U(f, \mathscr{P}) - L(f, \mathscr{P}) &= \sum_{i,j} (M_{ij} - m_{ij})(x_i - x_{i-1})(y_j - y_{j-1}) \\ &\leqslant \varepsilon(b - a)(d - c). \end{aligned}$$

Since $\varepsilon(b - a)(d - c)$ can be chosen arbitrarily small, it follows, by Theorem 7.1.11, that f is integrable over R.

Exercises 7.1

1. Let $R = [0, 1] \times [0, 1] \subseteq \mathbb{R}^2$ and let $f : R \subseteq \mathbb{R}^2 \to \mathbb{R}$ be the continuous function defined by $f(x, y) = xy$, $(x, y) \in R$. For a fixed integer n let $\mathscr{P}$ be the partition of R into n^2 subsquares obtained by partitioning the edges of R into n equal pieces. Show that

$$U(f, \mathscr{P}) = \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{i}{n}\frac{j}{n}\frac{1}{n^2} = \frac{(\frac{1}{2}n(n+1))^2}{n^4},$$

$$L(f, \mathscr{P}) = \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{i-1}{n}\frac{j-1}{n}\frac{1}{n^2} = \frac{(\frac{1}{2}(n-1)n)^2}{n^4}.$$

Deduce that

$$\iint_R f \, \mathrm{d}A = \tfrac{1}{4}.$$

2. Let $R=[-1, 0]\times[-1, 0]\subseteq\mathbb{R}^2$. Evaluate by the method of Exercise 1 the integral $\iint_R -x \, dx \, dy$.

Answer: $\frac{1}{2}$, as was shown geometrically in the text.

3. Evaluate (a) $\iint_R x \, dx \, dy$, (b) $\iint_R y \, dx \, dy$, where $R=[0, 1]\times[1, 3]\subseteq\mathbb{R}^2$

Answer: (a) 1; (b) 4.

7.2 Null sets in $\mathbb{R}^2$

There are two questions that arise naturally following the definition of the double integral in Section 7.1. Firstly, what conditions on a bounded function, more general than continuity, are sufficient to ensure that the double integral exists? Secondly, is there a technique for evaluating the integral which does not involve returning to first principles? This second question will be considered in Section 7.3.

We give one answer to the first question in Theorem 7.2.12. The *null subsets* needed in the statement of the theorem are defined in terms of the partitions of $\mathbb{R}^2$ which we now describe.

Two finite sequences of numbers $x_0<x_1<\cdots<x_k$ and $y_0<y_1<\cdots<y_l$ lead to two families of grid lines $x=x_i$ $(i=0, \ldots, k)$ and $y=y_j$ $(j=0, \ldots, l)$ in $\mathbb{R}^2$. These lines lead to a partition $\mathcal{P}$ of $\mathbb{R}^2$ into $(k+2)(l+2)$ closed rectangles (see Fig. 7.3).

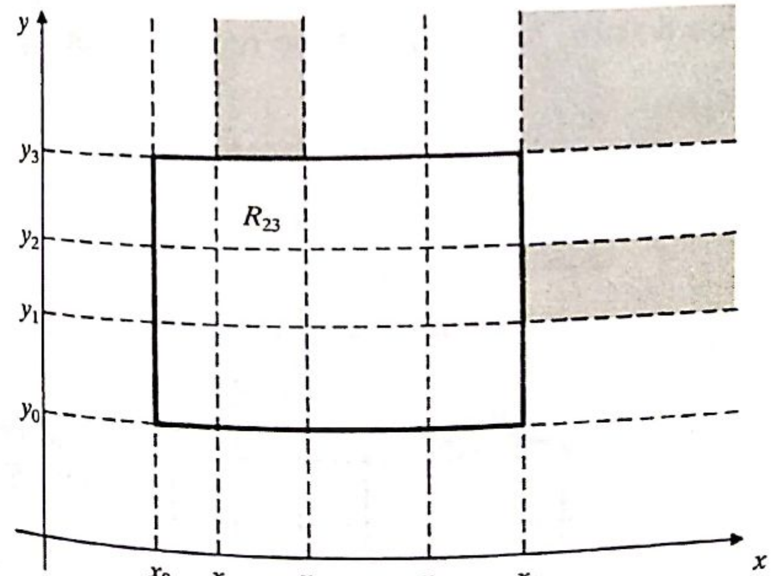


Fig. 7.3

Some of these rectangles are unbounded (the ones shaded in Fig. 7.3 for example). However, kl of them are bounded and therefore compact—each of these being of the form

$$R_{ij} = \{(x, y) \in \mathbb{R}^2 \mid x_{i-1} \leqslant x \leqslant x_i,\ y_{j-1} \leqslant y \leqslant y_j\},$$
$$i = 1, \ldots, k \quad \text{and} \quad j = 1, \ldots, l.$$

7.2.1 *Example.* The compact rectangles in the above partition of $\mathbb{R}^2$ form a partition of $[x_0, x_k] \times [y_0, y_l]$.

Conversely, any partition of a compact rectangle $[a, b] \times [c, d]$ in $\mathbb{R}^2$ has a natural extension to a partition of $\mathbb{R}^2$.

7.2.2 *Example.* Let $S_1, \ldots, S_m$ be compact rectangles in $\mathbb{R}^2$, each of whose sides is parallel to one of the coordinate axes. A partition of $\mathbb{R}^2$ can be formed by extending the edges of S_i for each i.

7.2.3 *Definition.* *If all the grid lines of a partition $\mathcal{P}$ of $\mathbb{R}^2$ are also grid lines of a partition $\mathcal{P}^*$ then $\mathcal{P}^*$ is said to be a refinement of $\mathcal{P}$. Equivalently $\mathcal{P}^*$ refines $\mathcal{P}$ if every rectangle in $\mathcal{P}^*$ is a subset of a rectangle in $\mathcal{P}$.*

7.2.4 *Example.* Let $\mathcal{P}_1$ and $\mathcal{P}_2$ be partitions of $\mathbb{R}^2$ and let $\mathcal{P}^*$ be obtained by using all the grid lines of both $\mathcal{P}_1$ and $\mathcal{P}_2$. Then $\mathcal{P}^*$ is a refinement of both $\mathcal{P}_1$ and $\mathcal{P}_2$.

Let $\mathcal{P}$ be a partition of $\mathbb{R}^2$. For any non-empty subset A of $\mathbb{R}^2$, there are some rectangles in $\mathcal{P}$ which have points in common with A and some which do not. In Fig. 7.4 those rectangles of a partition

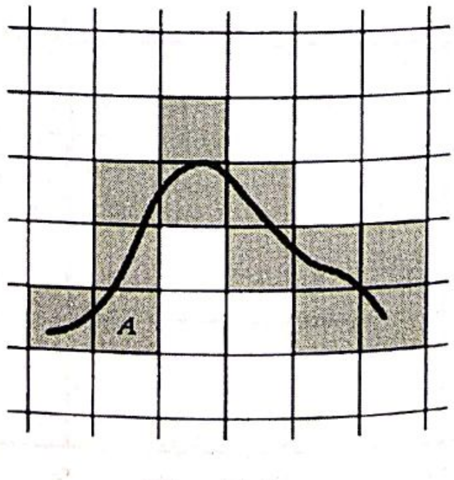


Fig. 7.4

that do meet a simple arc A have been shaded. The following definition is concerned with the total area (possibly infinite) of those rectangles that meet the subset.

7.2.5 Definition. *Let $\mathscr{P}$ be a partition of $\mathbb{R}^2$. For any subset A of $\mathbb{R}^2$, the sum of the areas of those rectangles of $\mathscr{P}$ whose intersection with A is non-empty is called the* contact *of $\mathscr{P}$ with A and is denoted by $\kappa(A, \mathscr{P})$.*

Notice that $\kappa(A, \mathscr{P}) > 0$ when A is not empty. It is clear that for any bounded set A, a partition of $\mathbb{R}^2$ can be chosen having finite contact with A.

7.2.6 Example. Let $R = [0, 1] \times [0, 1] \subseteq \mathbb{R}^2$, and let A be the simple closed curve forming the edge of R. For a given natural number n, let $\mathscr{P}_n$ be the partition of $\mathbb{R}^2$ determined by the grid lines

$$x = \frac{r-1}{n} \quad \text{and} \quad y = \frac{r-1}{n}, \qquad \text{for } r = 0, 1, \ldots, n+2.$$

See Fig. 7.5. The rectangles which meet A are shaded. The total area of the

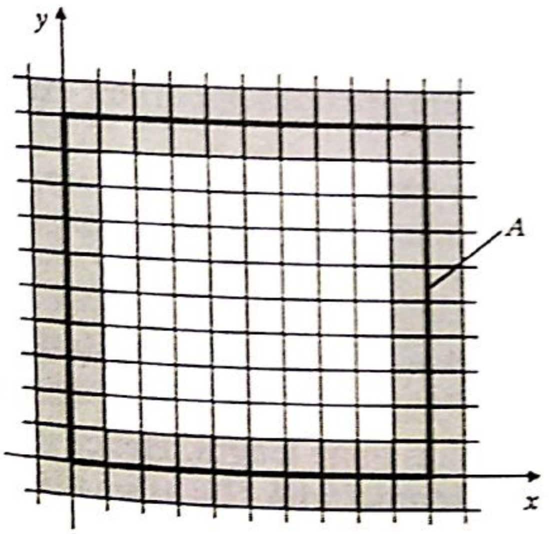


Fig. 7.5

compact rectangles in $\mathscr{P}_n$ is $(1 + 2/n)^2$ and the total area of these compact rectangles that do not meet A is $(1 - 2/n)^2$. Hence

7.2.7 $$\kappa(A, \mathscr{P}_n) = \left(1 + \frac{2}{n}\right)^2 - \left(1 - \frac{2}{n}\right)^2 = \frac{8}{n}.$$

Notice that in Example 7.2.6, by making n large enough we can, by 7.2.7, make $\kappa(A, \mathcal{P}_n)$ as small as we like. This means that A is a null set in the following sense.

7.2.8 *Definition.* *A set $N \subseteq \mathbb{R}^2$ is said to be a* null set *in $\mathbb{R}^2$ if to each $\varepsilon > 0$ there corresponds a partition $\mathcal{P}$ of $\mathbb{R}^2$ such that $\kappa(N, \mathcal{P}) < \varepsilon$.*

In considering properties of null sets it will be helpful to know the following theorems concerning contact.

7.2.9 *Theorem.* *For any two subsets A and B of $\mathbb{R}^2$ and any partition $\mathcal{P}$ of $\mathbb{R}^2$,*

$$\kappa(A, \mathcal{P}) \leqslant \kappa(A \cup B, \mathcal{P}) \leqslant \kappa(A, \mathcal{P}) + \kappa(B, \mathcal{P}).$$

Proof. Any rectangle of $\mathcal{P}$ which meets A also meets $A \cup B$. Any rectangle of $\mathcal{P}$ which meets $A \cup B$ also meets A or B or both.

7.2.10 *Theorem.* *Let A be a subset of $\mathbb{R}^2$, and let $\mathcal{P}^*$ be a refinement of a partition $\mathcal{P}$ of $\mathbb{R}^2$. Then $\kappa(A, \mathcal{P}^*) \leqslant \kappa(A, \mathcal{P})$.*

Proof. Exercise.

The following theorem provides important examples of null sets.

7.2.11 *Theorem.* *The image of a C^1 path $\alpha : [a, b] \subseteq \mathbb{R} \to \mathbb{R}^2$ is a null set in $\mathbb{R}^2$. In particular simple arcs and simple closed curves in $\mathbb{R}^2$ are null sets.*

Proof. We omit the details. The idea behind the proof is as follows. Let α be a C^1 path in $\mathbb{R}^2$ with image C. Since α' is continuous the path α has finite length. Hence it is possible to find a partition of $\mathbb{R}^2$ whose contact with C is as small as we please.

The C^1 condition on α cannot be dropped in Theorem 7.2.11. As we remarked in Section 2.6 there exist continuous functions defined on a compact interval $[a, b]$ whose image in $\mathbb{R}^2$ fills a square and therefore is not a null set.

The significance of null sets lies in the following important result answering the first of the two questions asked at the beginning of the section. The theorem generalizes Theorem 7.1.14.

7.2.12 Theorem. *Let $R = [a, b] \times [c, d]$ be a compact rectangle in $\mathbb{R}^2$, and let $f : R \subseteq \mathbb{R}^2 \to \mathbb{R}$ be a bounded function. If the set of points at which f is discontinuous forms a null set in $\mathbb{R}^2$ then f is integrable over R.*

Proof. Assume that

$$N = \{(x, y) \subseteq \mathbb{R}^2 \mid (x, y) \in \mathbb{R},\ f \text{ is not continuous at } (x, y)\}$$

is a null set. Corresponding to any $\varepsilon > 0$ there is a partition $\mathcal{P}$ of $\mathbb{R}^2$ such that

7.2.13 $$\kappa(N, \mathcal{P}) < \varepsilon.$$

We may assume, by Theorem 7.2.10, that the grid lines of $\mathcal{P}$ include $x = a$, $x = b$ and $y = c$, $y = d$. The partition $\mathcal{P}$ is therefore the extension of a partition $\mathcal{Q}$ of the rectangle R.

Let $Q_1, \ldots, Q_m$ be the subrectangles in $\mathcal{Q}$ that meet N and let $R_1, \ldots, R_n$ be the subrectangles in $\mathcal{Q}$ that do not meet N.

The function f is continuous on the compact subset $K = \bigcup_1^n R_j$ of $\mathbb{R}^2$ and hence it is uniformly continuous on K. Therefore there exists $\delta > 0$ such that

7.2.14 $$|f(\mathbf{v})| - f(\mathbf{w})| < \varepsilon \text{ whenever } \|\mathbf{v} - \mathbf{w}\| < \delta, \qquad \mathbf{v}, \mathbf{w} \in K.$$

Let $\mathcal{Q}^*$ be a refinement of $\mathcal{Q}$ such that the diagonal length of each subrectangle in $\mathcal{Q}^*$ is less than δ. We consider the expression

7.2.15 $$U(f, \mathcal{Q}^*) - L(f, \mathcal{Q}^*).$$

The area of K is less than or equal to $(b - a)(d - c)$ and the variation of f on each subrectangle in $\mathcal{Q}^*$ which is a subset of K is, by 7.2.14, less than ε. Hence the contribution to expression 7.2.15 from the subrectangles in $\mathcal{Q}^*$ which are subsets of K is less than $\varepsilon(b - a)(d - c)$.

Similarly the area of $J = \bigcup_1^n Q_i$ is, by 7.2.13, less than ε since $\mathcal{Q}^*$ is a refinement of $\mathcal{Q}$. the variation of f on J is less than or equal to $M - m$ where m and M are respectively a lower and upper bound of f on R. Hence the contribution to expression 7.2.15 from subrectangles in $\mathcal{Q}^*$ which are subsets of J is less than $\varepsilon(M - m)$. Therefore

7.2.16 $$U(f, \mathcal{Q}^*) - L(f, \mathcal{Q}^*) < \varepsilon(b - a)(d - c) + \varepsilon(M - m).$$

Since, by taking $\varepsilon > 0$ small enough, the right-hand side of 7.2.16

can be made as small as we please, the result follows from Theorem 7.1.10.

7.2.17 *Example.* Let A be the annulus $\{(x, y) \in \mathbb{R}^2 \mid 1 \leqslant x^2 + y^2 < 4\}$ in $\mathbb{R}^2$ and let $R = [-3, 3] \times [-3, 3] \subseteq \mathbb{R}^2$. The function $f: R \subseteq \mathbb{R}^2 \to \mathbb{R}$ defined by

$$f(x, y) = \begin{cases} x^2 + \sin y, & (x, y) \in A, \\ 0, & (x, y) \notin A \end{cases}$$

is bounded and continuous everywhere on R except along the two circles that form the boundary of A. These circles, being simple closed curves, form a null set in $\mathbb{R}^2$ and so f is integrable over R.

7.2.18 *Theorem.* *Let N be a null subset of a compact rectangle R in $\mathbb{R}^2$.*

[i] *If $f: R \subseteq \mathbb{R}^2 \to \mathbb{R}$ is a bounded function with the property that $f(\mathbf{p}) = 0$ for all $\mathbf{p} \in R \backslash N$, then f is integrable over R and $\iint_R f \, dA = 0$.*

[ii] *Let $g: R \subseteq \mathbb{R}^2 \to \mathbb{R}$ be integrable over R and let $h: R \subseteq \mathbb{R}^2 \to \mathbb{R}$ be a bounded function such that $g(\mathbf{p}) = h(\mathbf{p})$ for all $\mathbf{p} \in R \backslash N$. Then h is integrable over R and $\iint_R h \, dA = \iint_R g \, dA$.*

Proof. [*i*] Let M and m be respectively upper and lower bounds of a function $f: R \subseteq \mathbb{R}^2 \to \mathbb{R}$. For any $\varepsilon > 0$ find a partition $\mathscr{P}$ of $\mathbb{R}^2$ such that $\mu = \kappa(N, \mathscr{P}) < \varepsilon$. We can assume that the edges of R are grid lines of $\mathscr{P}$. Let $\mathscr{Q}$ be the partition of R determined by $\mathscr{P}$.

If $f(\mathbf{p}) = 0$ for all $\mathbf{p} \in R \backslash N$ then $U(f, \mathscr{Q}) \leqslant M\mu$ and $L(f, \mathscr{Q}) \geqslant m\mu$. Therefore

$$U(f, \mathscr{Q}) - L(f, \mathscr{Q}) \leqslant (M - m)\mu < (M - m)\varepsilon.$$

Hence f is integrable, by Theorem 7.1.10. Also

$$m\mu \leqslant L(f, \mathscr{Q}) \leqslant \iint_R f \, dA \leqslant U(f, \mathscr{Q}) \leqslant M\mu.$$

Since $0 < \mu < \varepsilon$, therefore $\iint_R f \, dA = 0$.

[*ii*] Let $f = h - g$. Then by part (i) f is integrable. Since $h = f + g$,

$$\iint_R h \, dA = \iint_R f \, dA + \iint_R g \, dA = \iint_R g \, dA.$$

In the final example of this section we show that a function may be integrable over R even though its discontinuities form a set which is not null.

7.2.19 ***Example.*** Let R be the compact rectangle $[0, 1] \times [0, 1]$ in $\mathbb{R}^2$ and define the subset S of R by

$$S = \{(x, y) \in \mathbb{R}^2 \mid (x, y) \in R, x \text{ and } y \text{ both rational}\}.$$

Let $\mathscr{P}$ be any partition of $\mathbb{R}^2$. Any subrectangle of $\mathscr{P}$ that meets R also meets S. Since the area of R is 1, it follows that $\kappa(S, \mathscr{P}) \geqslant 1$. Hence S is not a null set in $\mathbb{R}^2$.

Each point of S can be expressed in standard form $(p/q, r/s)$ where p, q, r and s are non-negative integers and the two fractions are in their lowest terms.

Define a function $h : R \subseteq \mathbb{R}^2 \to \mathbb{R}^2$ as follows

$$h(x, y) = \begin{cases} \dfrac{1}{qs} & \text{when } (x, y) = \left(\dfrac{p}{q}, \dfrac{r}{s}\right) \in S, \text{ in standard form,} \\ 0 & \text{when } (x, y) \in R \backslash S. \end{cases}$$

Clearly, for any given $(a, b) \in S$, the function h takes the value 0 as close as we like to (a, b). Therefore h is not continuous at (a, b). Therefore, since S is not a null set, the set of discontinuities of h is not a null set.

To prove that h is integrable over R we consider the difference $U(h, \mathscr{Q}) - L(h, \mathscr{Q})$ for different partitions $\mathscr{Q}$ of R. From the definition of the function h, $L(h, \mathscr{Q}) = 0$ for all $\mathscr{Q}$. Therefore if the integral exists at all, it is 0.

In order to estimate $U(h, \mathscr{Q})$ we shall need to know that

7.2.20

$$0 \leqslant h(x, y) \leqslant 1,$$

Corresponding to any ε with $1 > \varepsilon > 0$ consider the set $T = \{(x, y) \in \mathbb{R}^2 \mid (x, y) \in R, h(x, y) \geqslant \varepsilon\}$. The definition of h implies that T is a finite subset of S. Suppose that T contains n elements. Find a partition $\mathscr{Q}$ of R such that the edge of each subrectangle in $\mathscr{Q}$ has length less than ε/n. The area of each subrectangle is therefore less than ε^2/n^2.

The contribution to $U(h, \mathscr{Q})$ from those subrectangles of $\mathscr{Q}$ which do not meet T is, by definition of T, less that ε (remember the area of R is 1). Similarly, since each point of T lies in at most 4 subrectangles, the total area of the subrectangles of $\mathscr{Q}$ which do meet T is less than $4n(\varepsilon^2/n^2)$. Hence their contribution to $U(h, \mathscr{Q})$ is less than $4n(\varepsilon^2/n^2)$, by 7.2.20. Therefore

$$U(h, \mathscr{Q}) < \varepsilon + 4n(\varepsilon^2/n^2) < 5\varepsilon.$$

Hence

$$U(h, \mathcal{Q}) - L(h, \mathcal{Q}) < 5\varepsilon$$

and therefore h is integrable over R, and $\iint_R h \, dA = 0$.

Theorem 7.2.12 gives an answer to the first of our two questions about the integral of a bounded function over a rectangle. The full answer is given in Exercise 7.2.2(b).

We turn to the second question, concerning a technique for evaluating the integral, in the next section.

Exercises 7.2

1. Let A be the image of the discontinuous function $\phi : [0, 1] \subseteq \mathbb{R} \to \mathbb{R}^2$ defined by $\phi(0) = (0, 0)$ and

$$\phi(t) = \left(t, \sin\frac{1}{t}\right), \qquad 0 < t \leqslant 1.$$

Is A a null set?

Answer: yes. Notice that for any $0 < \varepsilon < 1$ the function ϕ restricted to $[\varepsilon, 1]$ is C^1.

2. (a) Prove that a subset A of $\mathbb{R}^2$ is a null set if and only if to any $\varepsilon > 0$ there correspond a finite number of compact rectangles R_i, $i = 1, \ldots, n$, of total area not greater than ε such that $A \subseteq \bigcup_1^n R_i$.

Hint: show that it is sufficient to consider rectangles R_i whose sides are parallel to the axes. Given the existence of such rectangles R_i, $i = 1, \ldots, n$, construct a partition $\mathcal{P}$ such that $\kappa(A, \mathcal{P}) \leqslant \varepsilon$.

(*b*) Let R be the compact rectangle $[0, 1] \times [0, 1]$ in $\mathbb{R}^2$ and let

$$S = \{(x, y) \in \mathbb{R}^2 \mid (x, y) \in R, \ x \text{ and } y \text{ both rational}\}.$$

In Example 7.2.19 we saw that S is not a null set. Show, however, that to any $\varepsilon > 0$ there correspond an *infinite* number of compact rectangles $R_i \subseteq R$, $i \in \mathbb{N}$, of total area not exceeding ε, such that $S \subseteq \bigcup_1^\infty R_i$.

Hint: the set S is countable. Assume an enumeration, and include the first point of S in a rectangle R_1 of area $\varepsilon/2$, the second in R_2 of area $\varepsilon/4, \ldots$, the nth in R_n of area $\varepsilon/2^n, \ldots$. In view of the above property, the set S is said to have *measure zero*.

A bounded function $f : R \subseteq \mathbb{R}^2 \to \mathbb{R}$ is integrable if and only if the set of discontinuities of f has measure zero.

3. Let $h : R \subseteq \mathbb{R}^2 \to \mathbb{R}$ be the function defined in Example 7.2.19. Prove

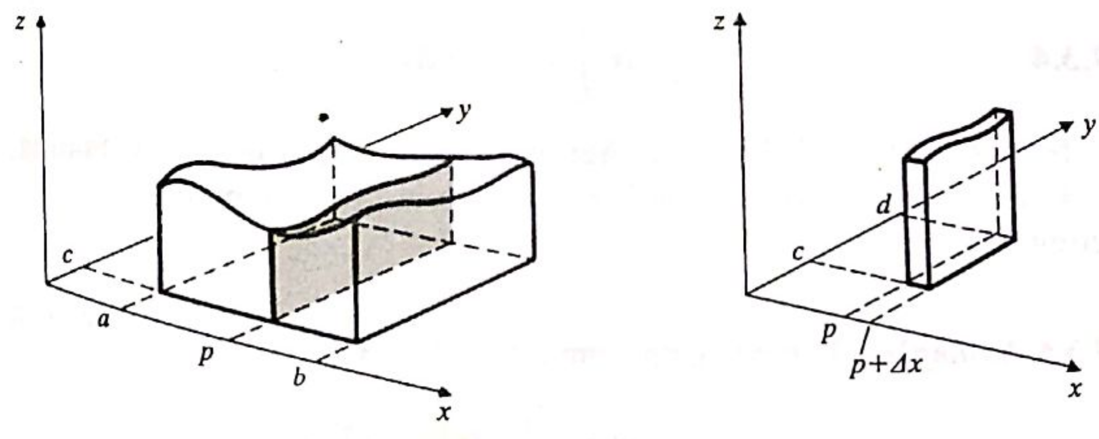
that h is continuous at $(a, b) \in R$ if and only if at least one of a, b is irrational.

Hint: given $\varepsilon > 0$, there are only a finite number of rational pairs (x, y) such that $h(x, y) \geqslant \varepsilon$. Consider a neighbourhood of (a, b) that excludes such rational pairs.

4. Let $R = [0, 1] \times [0, 1] \subseteq \mathbb{R}^2$, let $\mathscr{P}$ be a partition of $\mathbb{R}^2$, and let $S = \{(x, y) \in \mathbb{R}^2 \mid (x, y) \in R,\ x \text{ and } y \text{ both rational}\}$. In Example 7.2.19 we saw that $\kappa(S, \mathscr{P}) \geqslant 1$. Show that also $\kappa(R \backslash S, \mathscr{P}) \geqslant 1$.

7.3 Repeated integrals

Let $f: R \subseteq \mathbb{R}^2 \to \mathbb{R}$ be a positive continuous function on the compact rectangle $R = [a, b] \times [c, d]$ in $\mathbb{R}^2$. By Theorem 7.1.14 the double integral of f over R exists. It measures the volume in $\mathbb{R}^3$ bounded by R in the (x, y)-plane and the graph $z = f(x, y)$. See Fig. 7.6(i).



(i) (ii)

Fig. 7.6

The plane $x = p$ cuts this volume for each $a \leqslant p \leqslant b$. The area of the associated cross-section (shaded in Fig. 7.6(i)) is

7.3.1
$$F(p) = \int_c^d f(p, y)\, \mathrm{d}y.$$

Since this integral exists for every p, 7.3.1 defines a function $F: [a, b] \subseteq \mathbb{R} \to \mathbb{R}$.

7.3.2 ***Example.*** Consider the function $f:R\subseteq\mathbb{R}^2\to\mathbb{R}$ defined by

$$f(x, y) = xy, \qquad 0\leqslant x\leqslant 1,\ 0\leqslant y\leqslant 1$$

where R is the rectangle $[0, 1]\times[0, 1]$ in $\mathbb{R}^2$.

The associated function $F:[a, b]\subseteq\mathbb{R}\to\mathbb{R}$ defined in 7.3.1 is given by

$$F(x) = \int_0^1 xy\ \mathrm{d}y = [\tfrac{1}{2}xy^2]_{y=0}^{y=1} = \tfrac{1}{2}x.$$

In Fig. 7.6(ii) we have sketched a thin slice of the volume which lies between the planes $x = p$ and $x = p + \Delta x$. The volume of this thin slice is approximately $F(p)\ \Delta x$. This suggests that we have the following expression for the total volume

7.3.3 $$\iint_R f(x, y)\ \mathrm{d}x\ \mathrm{d}y = \int_a^b F(x)\ \mathrm{d}x = \int_a^b \left\{\int_c^d f(x, y)\ \mathrm{d}y\right\} \mathrm{d}x.$$

The integral on the right-hand side of 7.3.3 is called a *repeated integral.* It is more conveniently denoted by

7.3.4 $$\int_a^b \mathrm{d}x \int_c^d f(x, y)\ \mathrm{d}y.$$

Notice that if 7.3.3 is true then it gives a technique for evaluating a double integral which involves performing two elementary integrations.

7.3.5 ***Example.*** Following on from Example 7.3.2,

$$\int_0^1 \mathrm{d}x \int_0^1 xy\ \mathrm{d}y = \int_0^1 \tfrac{1}{2}x\ \mathrm{d}x = [\tfrac{1}{4}x^2]_0^1 = \tfrac{1}{4}.$$

This conclusion is in accordance with 7.3.3 since we have seen in Exercise 7.1.1 that the integral of f over R is also $\frac{1}{4}$.

We could repeat the above argument in terms of cross-sections of the volume cut by planes of the form $y = q$, $c\leqslant q\leqslant d$. We would expect that

7.3.6 $$\iint_R f(x, y)\ \mathrm{d}x\ \mathrm{d}y = \int_c^d \left\{\int_a^b f(x, y)\ \mathrm{d}x\right\} \mathrm{d}y.$$

The integral of the right-hand side of 7.3.6 is again a repeated integral. It is denoted by

7.3.7 $$\int_c^d dy \int_a^b f(x, y)\, dx.$$

Comparison of 7.3.3 and 7.3.6 suggests that for well enough behaved functions $f: R \subseteq \mathbb{R}^2 \to \mathbb{R}$,

7.3.8 $$\iint_R f(x, y)\, dx\, dy = \int_a^b dx \int_c^d f(x, y)\, dy = \int_c^d dy \int_a^b f(x, y)\, dx.$$

We shall establish conditions under which 7.3.8 holds later in this section (Theorem 7.3.17). Before we do that, however, it is instructive to contrast the following examples.

7.3.9 *Example.*

$$\int_0^1 dx \int_1^3 (x^2 + y)\, dy = \int_0^1 [x^2y + \tfrac{1}{2}y^2]_{y=1}^{y=3}\, dx = \int_0^1 (2x^2 + 4)\, dx = \tfrac{14}{3},$$

$$\int_1^3 dy \int_0^1 (x^2 + y)\, dx = \int_1^3 (\tfrac{1}{3} + y)\, dy = \tfrac{14}{3}.$$

7.3.10 *Example.* Define $f: R \subseteq \mathbb{R}^2 \to \mathbb{R}$ by

$$f(x, y) = \begin{cases} 2y & \text{if } x \text{ irrational,} \\ 1 & \text{if } x \text{ rational,} \end{cases}$$

where $R = [0, 1] \times [0, 1] \subseteq \mathbb{R}^2$. Then

$$\int_0^1 f(x, y)\, dy = \begin{cases} \displaystyle\int_0^1 2y\, dy = 1 & \text{if } x \text{ irrational,} \\ \displaystyle\int_0^1 1\, dy = 1 & \text{if } x \text{ rational.} \end{cases}$$

Hence

$$\int_0^1 dx \int_0^1 f(x, y)\, dy = \int_0^1 dx = 1.$$

On the other hand, for any fixed y, the integral of $f(x, y)$ with respect to x over $[0, 1]$ does not exist. Hence

$$\int_0^1 dy \int_0^1 f(x, y)\, dx \text{ does not exist.}$$

We turn now to the general theorems which provide the required technique for evaluating a double integral. The notation given in 7.3.4 and 7.3.7 which was motivated by considering positive continuous functions will be adopted generally whenever the repeated integral exists. We use the notation developed in Section 7.1 in relation to a partition of a compact rectangle.

7.3.11 Theorem. *Let R be the compact rectangle $[a, b] \times [c, d]$ in $\mathbb{R}^2$ and let $f: R \subseteq \mathbb{R}^2 \to \mathbb{R}$ be a bounded function which is integrable over R. Suppose that for each fixed $x \in [a, b]$ the integral*

$$F(x) = \int_c^d f(x, y)\, dy$$

exists. Then the function $F: [a, b] \subseteq \mathbb{R} \to \mathbb{R}$ is integrable and

$$\iint_R f(x, y)\, dx\, dy = \int_a^b dx \int_c^d f(x, y)\, dy.$$

Proof. We first prove that the function F is bounded on $[a, b]$. Let m and M be respectively lower and upper bounds for f on R. Then for each given $p \in [a, b]$ and all $y \in [c, d]$,

$$m \leqslant f(p, y) \leqslant M.$$

Hence, by Theorem 1.6.19,

$$m(d-c) \leqslant \int_c^d f(p, y)\, dy \leqslant M(d-c).$$

Therefore $m(d-c) \leqslant F(p) \leqslant M(d-c)$ and so F is bounded. It follows that the upper Riemann sum $U(F, \Pi)$ and lower Riemann sum $L(F, \Pi)$ of F corresponding to any partition Π of $[a, b]$ exist.

Choose any $\varepsilon > 0$. Since f is integrable over R there exists a partition $\mathcal{P}$ of R such that

7.3.12 $$U(f, \mathcal{P}) - L(f, \mathcal{P}) < \varepsilon.$$

Suppose that $\mathcal{P}$ derives from a partition $a = x_0 < \cdots < x_{k-1} < x_k = b$ of $[a, b]$ and a partition $c = y_0 < \cdots < y_{l-1} < y_l = d$ of $[c, d]$, and denote the partition of $[a, b]$ by Π.

For each $i = 1, \ldots, k$ choose a point $p_i \in [x_{i-1}, x_i]$. Then for all $j = 1, \ldots, l$ and all y with $y_{j-1} \leqslant y \leqslant y_j$, we have in the notation of Section 7.1

$$m_{ij} \leqslant f(p_i, y) \leqslant M_{ij}.$$

Hence, by Theorem 1.6.19,

7.3.13 $$m_{ij}(y_j - y_{j-1}) \leqslant \int_{y_{j-1}}^{y_j} f(p_i, y)\,\mathrm{d}y \leqslant M_{ij}(y_j - y_{j-1}).$$

Multiplying the inequalities 7.3.13 by $(x_i - x_{i-1})$ and summing over all i and j we obtain

7.3.14 $$L(f, \mathcal{P}) \leqslant \sum_i F(p_i)(x_i - x_{i-1}) \leqslant U(f, \mathcal{P}).$$

But the inequality 7.3.14 is true for all choices of $p_i \in [x_{i-1}, x_i]$ and so

7.3.15 $$L(f, \mathcal{P}) \leqslant L(F, \Pi) \leqslant U(F, \Pi) \leqslant U(f, \mathcal{P}).$$

Hence, from 7.3.12,

7.3.16 $$U(F, \Pi) - L(F, \Pi) < \varepsilon.$$

Since Π corresponds to the arbitrary $\varepsilon > 0$, 7.3.16 implies that f is integrable over $[a, b]$ and 7.3.15 in turn implies that

$$\int_a^b F(x)\,\mathrm{d}x = \iint_R f(x, y)\,\mathrm{d}x\,\mathrm{d}y,$$

and the proof is complete.

Clearly the above argument can also be used (with the roles of x and y interchanged) to prove the corresponding result concerning a bounded integrable function $f: R \subseteq \mathbb{R}^2 \to \mathbb{R}$ which has the property that

$$\int_a^b f(x, y)\,\mathrm{d}x$$

exists for each $y \in [c, d]$. The reader is recommended to make the necessary alterations to the theorem and its proof.

The two results together give us the following theorem.

7.3.17 Theorem. (*Fubini's Theorem*) *Let $f: R \subseteq \mathbb{R}^2 \to \mathbb{R}$ be a bounded function which is integrable over a compact rectangle R in $\mathbb{R}^2$. If f is integrable as a function of x for each y and integrable as a function of y for each x then the two repeated integrals exist and are both equal to the double integral of f over R.*

The following theorem is an important special case.

7.3.18 Theorem. *Let* $f: R \subseteq \mathbb{R}^2 \to \mathbb{R}$ *be continuous on the compact rectangle* R*, where* $R = [a, b] \times [c, d]$*. Then* f *is integrable over* R *and*

$$\iint_R f(x, y)\,dx\,dy = \int_a^b dx \int_c^d f(x, y)\,dy = \int_c^d dy \int_a^b f(x, y)\,dx.$$

Proof. The continuous function f satisfies the hypothesis of Theorem 7.3.17.

7.3.19 *Example.* The function $f: [-1, 0] \times [-1, 0] \subseteq \mathbb{R}^2 \to \mathbb{R}$ defined by $f(x, y) = -x$, $(x, y) \in [-1, 0] \times [-1, 0] = R$ is continuous. hence

$$\iint_R -x\,dx\,dy = \int_{-1}^0 dx \int_{-1}^0 -x\,dy = \int_{-1}^0 -x\,dx = \tfrac{1}{2}.$$

The double integral is the volume of the wedge sketched in Fig. 7.2 and considered in Example 7.1.5.

Example 7.3.10 shows that care is needed in general when reversing the order of integration in a repeated integral. Furthermore there exist functions which are integrable but which have the property that neither repeated integral exists. See Exercise 7.3.3.

Exercises 7.3

1. Let R be the rectangle $[-1, 2] \times [-1, 1]$. For each of the following continuous functions $f: R \subseteq \mathbb{R}^2 \to \mathbb{R}$ evaluate the double integral $\iint_R f\,dA$ in two ways: (a) as the repeated integral $\int_{-1}^1 dy \int_{-1}^2 f(x, y)\,dx$; (b) as the repeated integral $\int_{-1}^2 dx \int_{-1}^1 f(x, y)\,dy$.
   (i) $f(x, y) = xy(x + y)$;
   (ii) $f(x, y) = x^2 \sin y$;
   (iii) $f(x, y) = |x + y|$.

Hint for (iii): $\int_{-1}^2 |x + y|\,dx = \int_{-1}^{-y} -(x + y)\,dx + \int_{-y}^2 (x + y)\,dx$.

Answers: (i) 1; (ii) 0; (iii) 17/3.

2. Prove that the function f of Example 7.3.10 is not integrable over the square $R = [0, 1] \times [0, 1]$.

Hint: prove that for any partition $\mathcal{P}$ of R, $L(f, \mathcal{P}) < 3/4$ and $U(f, \mathcal{P}) > 5/4$.

3. Define the function $f: R \subseteq \mathbb{R}^2 \to \mathbb{R}$ over the unit square $R = [0, 1] \times [0, 1]$

by the rule

$$f(x, y) = \begin{cases} 1 & \text{when } x = \frac{1}{2},\ y \text{ rational } or\ y = \frac{1}{2},\ x \text{ rational} \\ 0 & \text{otherwise.} \end{cases}$$

Prove that f is integrable over R, but that neither of the repeated integrals $\int_0^1 dy \int_0^1 f(x, y)\, dx$ and $\int_0^1 dx \int_0^1 f(x, y)\, dy$ exists.

Hint: integrability follows from Theorem 7.2.12.

4. Consider the integrable function $h : R \subseteq \mathbb{R}^2 \to \mathbb{R}$ defined in Example 7.2.19. Do the repeated integrals

$$\int_0^1 dy \int_0^1 h(x, y)\, dx \quad \text{and} \quad \int_0^1 dx \int_0^1 h(x, y)\, dy$$

exist?

Hint: in the first case fix a rational number y in $[0, 1]$ and show that $\int_0^1 h(x, y)\, dx$ exists by considering lower and upper Riemann sums of h.

Answer: both repeated integrals exist and are zero, since $\int_0^1 h(x, b)\, dx = \int_0^1 h(a, y)\, dy = 0$ for all a and b in $[0, 1]$.

5. Let $f : R \subseteq \mathbb{R}^2 \to \mathbb{R}$ be defined over the rectangle $R = [a, b] \times [c, d]$. Prove that if the double integral $\iint_R f(x, y)\, dx\, dy$ exists then the two repeated integrals

$$\int_a^b dx \int_c^d f(x, y)\, dy \quad \text{and} \quad \int_c^d dy \int_a^b f(x, y)\, dx$$

cannot exist without being equal.

7.4 Integrals over general subsets of $\mathbb{R}^2$

The integral of a bounded real-valued function over a bounded subset S of $\mathbb{R}^2$ is defined in terms of the integral of a related function over a compact rectangle containing S.

7.4.1 *Definition.* *Let $f : S \subseteq \mathbb{R}^2 \to \mathbb{R}$ be a function defined on a bounded subset S of $\mathbb{R}^2$. Corresponding to any compact rectangle R such that $S \subseteq R$ the* associated function *$f^* : R \subseteq \mathbb{R}^2 \to \mathbb{R}$ is defined to be the extension of f given by*

$$f^*(\mathbf{p}) = \begin{cases} f(\mathbf{p}) & \text{if} \quad \mathbf{p} \in S, \\ 0 & \text{if} \quad \mathbf{p} \in R \backslash S. \end{cases}$$

7.4.2 Definition. *Let* $f: S \subseteq \mathbb{R}^2 \to \mathbb{R}$ *be a bounded function defined on a bounded subset* S *of* $\mathbb{R}^2$ *and let* R *be a compact rectangle such that* $S \subseteq R$. *The function* f *is said to be integrable over* S *if the associated function* $f^*: R \subseteq \mathbb{R}^2 \to \mathbb{R}$ *is integrable over* R. *The* (double) integral of f over S *is then defined to be the integral of* f^* *over* R.

We adopt a natural extension of the notation in Definition 7.1.9 and denote the integral of f over S by

$$\iint_S f \, \mathrm{d}A \qquad \text{or by} \iint_S f(x, y) \, \mathrm{d}x \, \mathrm{d}y.$$

The integral is also denoted by

$$\iint_S f \qquad \text{or by} \iint_S f(x, y) d(x, y).$$

The effect of Definition 7.4.2 is that

$$\iint_S f \, \mathrm{d}A = \iint_R f^* \, \mathrm{d}A.$$

The criterion for integrability and the value of the integral given in Definition 7.4.2 are independent of the choice of the compact rectangle R containing S.

7.4.3 *Example.* Let S be the unit disc $\{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 \leqslant 1\}$ in $\mathbb{R}^2$ and let $f: S \subseteq \mathbb{R}^2 \to \mathbb{R}$ be the constant function given by $f(\mathbf{p}) = 1$ for all $\mathbf{p} \in S$. The disc is a subset of the compact rectangle $R = [-1, 1] \times [-1, 1]$. The associated function $f^*: R \subseteq \mathbb{R}^2 \to \mathbb{R}$ given by

$$f^*(\mathbf{p}) = 1 \text{ if } \mathbf{p} \in S \qquad \text{and} \qquad f^*(\mathbf{p}) = 0 \text{ if } \mathbf{p} \in R \backslash S$$

is continuous everywhere in R except at the points on the boundary of the disc. The boundary is the circle $x^2 + y^2 = 1$ in $\mathbb{R}^2$. The circle, being a simple closed curve, is (by Theorem 7.2.11) a null set. Therefore, by Theorem 7.2.12, f^* is integrable over R and hence f is integrable over S.

We can evaluate the integral by appealing to Theorem 7.3.11 since, for each $x \in [-1, 1]$, $f^*(x, y)$ is integrable as a function of y. Indeed, in the notation adopted there

$$F(x) = \int_{-1}^{1} f^*(x, y) \, \mathrm{d}y = \int_{-1}^{-\sqrt{(1-x^2)}} 0 \, \mathrm{d}y + \int_{-\sqrt{(1-x^2)}}^{\sqrt{(1-x^2)}} 1 \, \mathrm{d}y + \int_{\sqrt{(1-x^2)}}^{1} 0 \, \mathrm{d}y$$
$$= 2\sqrt{(1-x^2)}.$$

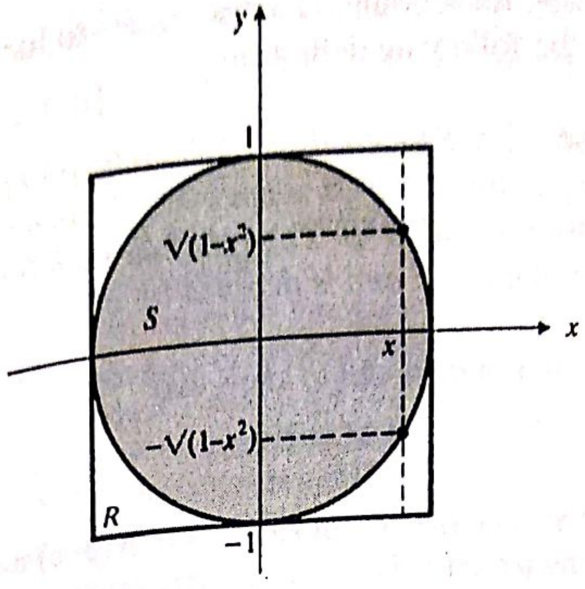


Fig. 7.7

Hence

$$\iint_S f(x, y)\,dx\,dy = \iint_R f^*(x, y)\,dx\,dy = \int_{-1}^{1} dx \int_{-1}^{1} f^*(x, y)\,dy$$
$$= \int_{-1}^{1} F(x)\,dx = \int_{-1}^{1} 2\sqrt{(1-x^2)}\,dx = \pi.$$

As expected, the integral of the constant function 1 over the unit disc is just the area of the disc.

The above example illustrates the following formal definition of the area of a subset of $\mathbb{R}^2$.

7.4.4 *Definition.* *Let S be a bounded subset of* $\mathbb{R}^2$. *The* area *of S is defined to be* $\iint_S 1\,dA$, *if that integral exists.*

7.4.5 *Example.* All null sets in $\mathbb{R}^2$ have zero area. See Theorem 7.2.18(i).

7.4.6 *Example.* If the area of a bounded set S in $\mathbb{R}^2$ exists then its value is

$$\inf\{\kappa(S, \mathcal{P}) \mid \text{partitions } \mathcal{P} \text{ of } \mathbb{R}^2\}.$$

This expression, whether S has area or not, is often called the *outer content* of S.

We shall obtain presently (Theorem 7.4.13) a necessary and

... of S and is denoted by ∂S. ... said ... urhood ... uch boundary points ...

7.4.8 *Example.* It is often useful to know that

7.4.9
$$\partial S = \bar{S} \backslash \operatorname{Int} S.$$

Proof. First $\mathbf{p} \in \bar{S}$ if and only if, for each $\varepsilon > 0$, $N(\mathbf{p}, \varepsilon)$ meets S (Theorem 3.2.28), and second $\mathbf{p} \in \operatorname{Int} S$ if and only if there exists $\varepsilon > 0$ such that $N(\mathbf{p}, \varepsilon)$ does not meet $\mathbb{R}^2 \backslash S$. this proves 7.4.9.

It follows from 7.4.9, since $\operatorname{Int} S \subseteq \bar{S}$, that

7.4.10
$$S \cup \partial S = \bar{S}.$$

7.4.11 *Example.* Let $\phi : \mathbb{R} \to \mathbb{R}$ be a continuous function and let
$$S = \{(x, y) \in \mathbb{R}^2 \mid y \leqslant \phi(x)\}.$$
The set S and the graph G of ϕ are sketched in Fig. 7.8. We show that $\partial S = G$.

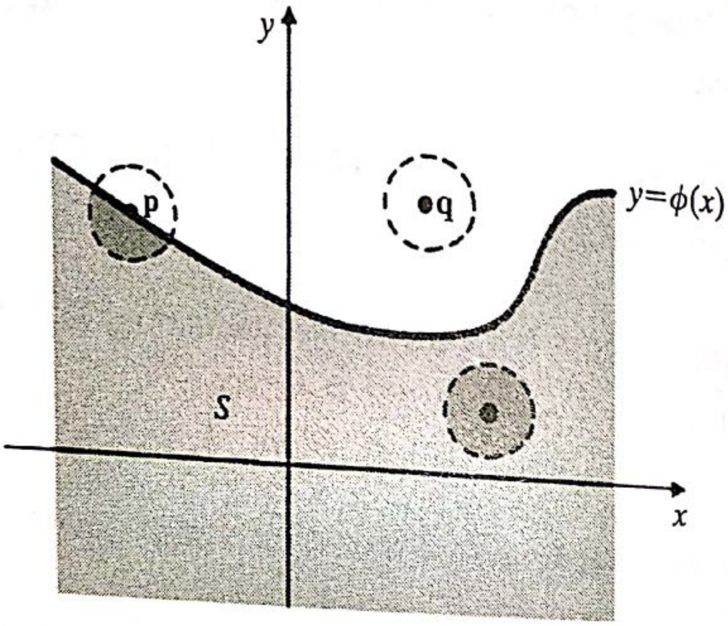


Fig. 7.8

Consider the function $f : \mathbb{R}^2 \to \mathbb{R}$ defined by
$$f(x, y) = \phi(x) - y, \qquad (x, y) \in \mathbb{R}^2.$$

Since ϕ is continuous, the function f is also continuous. Furthermore

$$f^{-1}(0) = G, f^{-1}(]-\infty, 0[) = \mathbb{R}^2\backslash S \quad \text{and} \quad f^{-1}(]0, \infty[) = S\backslash G,$$

and the sets $\mathbb{R}^2\backslash S$ and $S\backslash G$ are open sets in $\mathbb{R}^2$ (Theorem 4.2.10).

First, every point of G lies in ∂S. For let $\mathbf{p} = (t, \phi(t)) \in G$. Then given any $\varepsilon > 0$ the neighbourhood $N(\mathbf{p}, \varepsilon)$ contains the points $\mathbf{p} \in S$ and $(t, \phi(t) + \varepsilon/2) \in \mathbb{R}^2\backslash S$. Hence $\mathbf{p} \in \partial S$.

Second, no point of $\mathbb{R}^2\backslash S$ lies in ∂S. For consider $\mathbf{q} \in \mathbb{R}^2\backslash S$. Since $\mathbb{R}^2\backslash S$ is an open subset of $\mathbb{R}^2$, there exists a neighbourhood of $\mathbf{q}$ lying in $\mathbb{R}^2\backslash S$. This neighbourhood does not meet S and so $\mathbf{q} \notin \partial S$.

Third, by a similar argument, no point of $S\backslash G$ lies in ∂S. Therefore, since $\mathbb{R}^2 = G \cup (\mathbb{R}^2\backslash S) \cup (S\backslash G)$, the boundary of S is G.

The condition of continuity in this example cannot be relaxed. See Exercise 7.4.1.

7.4.12 *Example.* The annulus $S = \{(x, y) \in \mathbb{R}^2 \mid 1 \leqslant x^2 + y^2 < 4\}$ is a bounded set; it lies for example in the rectangle $[-3, 3] \times [-2, 2]$. The boundary of S consists of two concentric circles, $x^2 + y^2 = 1$, which belongs to S, and $x^2 + y^2 = 4$, which does not. Notice that the boundary of S is not in one piece.

The following theorem provides an important class of integrable functions.

7.4.13 *Theorem.* *Let S be a bounded subset of $\mathbb{R}^2$ with null boundary, and let $f: S \subseteq \mathbb{R}^2 \to \mathbb{R}$ be a bounded function which is continuous everywhere except on a null set in $\mathbb{R}^2$. Then f is integrable over S.*

Proof. Let N be the set of those discontinuities of f that lie in the interior of S. Then N is a null set and therefore $N \cup \partial S$ is also a null set. Let R be a compact rectangle such that $\bar{S} \subseteq \operatorname{Int} R$, and let $f^*: R \subseteq \mathbb{R}^2 \to \mathbb{R}$ be the extension of f to R which takes the value zero outside S. Then f^* is continuous on $(\operatorname{Int} S)\backslash N$ and on $R\backslash\bar{S}$. Therefore the discontinuities of f^* lie in $N \cup (\bar{S}\backslash\operatorname{Int} S) = N \cup \partial S$, by 7.4.9. Since this is a null set, it follows by Theorem 7.2.12 that f^* is integrable over R. Therefore f is integrable over S.

The following theorem concerns the existence of area of a bounded subset of $\mathbb{R}^2$.

7.4.14 *Theorem.* *Let S be a bounded subset of $\mathbb{R}^2$. Then the integral $\iint_S 1 \, \mathrm{d}A$ exists if and only if ∂S is a null set.*

Proof. If ∂S is a null set then $\iint_S 1 \, dA$ exists by Theorem 7.4.13.

Suppose conversely that ∂S is not a null set. Let R be a compact rectangle in $\mathbb{R}^2$ such that $\bar{S} \subseteq \operatorname{Int} R$. Then there exists an $\varepsilon > 0$ such that

7.4.15 $\qquad \kappa(\partial S, \mathcal{Q}) > \varepsilon \qquad$ for every partition $\mathcal{Q}$ of R.

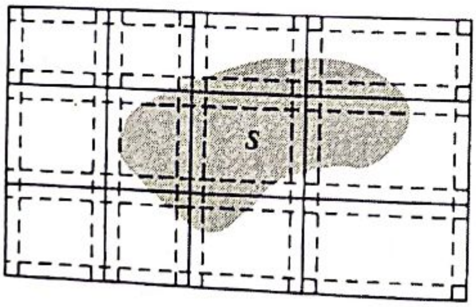
Define $f: R \subseteq \mathbb{R}^2 \to \mathbb{R}$ to be the function such that $f(\mathbf{p}) = 1$ if $p \in S$ and $f(\mathbf{p}) = 0$ if $p \in R \backslash S$. (This is the extension of the constant function 1 on S to take the value zero outside S.)

Let $\mathcal{P}$ be any partition of R. We shall prove that there exists a refinement $\mathcal{P}^*$ of $\mathcal{P}$ such that

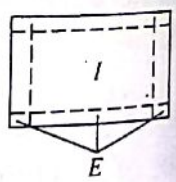
7.4.16 $$U(f, \mathcal{P}) - L(f, \mathcal{P}) \geqslant \tfrac{1}{2}\kappa(\partial S, \mathcal{P}^*)$$

It then follows from 7.4.16, 7.4.15 and Theorem 7.1.10 that f is not integrable over R and hence that $\iint_S 1 \, dA$ does not exist.

It remains to prove 7.4.16 for a suitable refinement $\mathcal{P}^*$ of $\mathcal{P}$. We begin by refining $\mathcal{P}$ through the addition of two grid lines between each pair of adjacent horizontal and vertical grid lines of $\mathcal{P}$ as shown in Fig. 7.9(i), where the new lines are dotted.



(i)



(ii)

Fig. 7.9

Notice that each rectangle of the refinement of $\mathcal{P}$ is either an *edge rectangle* (that is, it meets the edge of a rectangle of $\mathcal{P}$) or an *interior rectangle* (that is, it lies in the interior of a rectangle of $\mathcal{P}$). In fact, each rectangle of $\mathcal{P}$ is partitioned into one interior rectangle and eight edge rectangles (Fig. 7.9(ii)). Now choose $\mathcal{P}^*$ to be such a refinement of $\mathcal{P}$ for which *the sum of the areas of all edge rectangles is less than* $U(f, \mathcal{P}) - L(f, \mathcal{P})$.

Let $E_1, \ldots, E_n$ be the edge rectangles of $\mathscr{P}^*$ that meet ∂S. Then

$$\sum_1^n \text{area } E_j \leqslant U(f, \mathscr{P}) - L(f, \mathscr{P}). \tag{7.4.17}$$

Similarly let $I_1, \ldots, I_k$ be the interior rectangles of $\mathscr{P}^*$ that meet ∂S, and let $B_1, \ldots, B_k$ be the corresponding rectangles of $\mathscr{P}$ in whose interior they lie. Then each B_r has a point of ∂S in its interior and so contains points of both S and $R\backslash S$. Hence $f(B_r) = \{0, 1\}$, and therefore

$$\sum_1^k \text{area } I_r \leqslant \sum_1^k \text{area } B_r \leqslant U(f, \mathscr{P}) - L(f, \mathscr{P}). \tag{7.4.18}$$

Since

$$\kappa(\partial S, \mathscr{P}^*) = \sum_1^n \text{area } E_j + \sum_1^k \text{area } I_r \tag{7.4.19}$$

the inequality 7.4.16 follows from 7.4.17 and 7.4.18. This completes the proof.

In most applications, functions are integrated over bounded subsets of $\mathbb{R}^2$ whose boundaries consist of a finite number of simple closed curves. By Theorem 7.2.11, such boundaries are null sets. Part (i) of the following example shows, however, that in general arguments we have to be careful of the boundary.

7.4.20 *Example.* Let $S = \{(x, y) \in \mathbb{R}^2 \mid 0 \leqslant x \leqslant 1, 0 \leqslant y \leqslant 1, \text{ both } x \text{ and } y \text{ rational}\}$. Then $\partial S = [0, 1] \times [0, 1]$, which is not a null set.

[i] The function $f : S \subseteq \mathbb{R}^2 \to \mathbb{R}$ given by, $f(x, y) = 1$ for all $(x, y) \in S$ is continuous on S but is not integrable over S. See Example 7.1.8.

[ii] The function $g : S \subseteq \mathbb{R}^2 \to \mathbb{R}$ given by $g(x, y) = 0$ for all $(x, y) \in S$ is integrable over S.

[iii] The function $h : S \subseteq \mathbb{R}^2 \to \mathbb{R}$ given by

$$h(x, y) = \frac{1}{qs} \text{ whenever } (x, y) = \left(\frac{p}{q}, \frac{r}{s}\right) \in S, \text{ in standard form,}$$

is not continuous anywhere in S and yet is integrable over S. See Example 7.2.19.

We must be careful, even when the domain of integration is a bounded open set, for such a set can have a non-null boundary. See Exercise 7.4.3.

The following two theorems describe particularly important cases, where the double integral of f over S can be evaluated in terms of repeated single integrals.

7.4.21 Theorem. *Let* $\phi:[a, b]\subseteq\mathbb{R}\to\mathbb{R}$ *and* $\psi:[a, b]\subseteq\mathbb{R}\to\mathbb{R}$ *be continuous functions such that* $\phi(x)<\psi(x)$ *for all* $x\in\,]a, b[$. *Let*

$$S=\{(x, y)\in\mathbb{R}^2\mid a\leqslant x\leqslant b,\ \phi(x)\leqslant y\leqslant\psi(x)\}$$

and suppose that the boundary of S is a simple closed curve. Then any continuous function $f:S\subseteq\mathbb{R}^2\to\mathbb{R}$ *is integrable over S, and*

$$\iint_S f(x, y)\,\mathrm{d}x\,\mathrm{d}y=\int_a^b \mathrm{d}x\int_{\phi(x)}^{\psi(x)} f(x, y)\,\mathrm{d}y.$$

Proof. Consider the region S sketched in Fig. 7.10. Let

$$c=\inf\{\phi(x)\mid a\leqslant x\leqslant b\}\quad\text{and}\quad d=\sup\{\psi(x)\mid a\leqslant x\leqslant b\}.$$

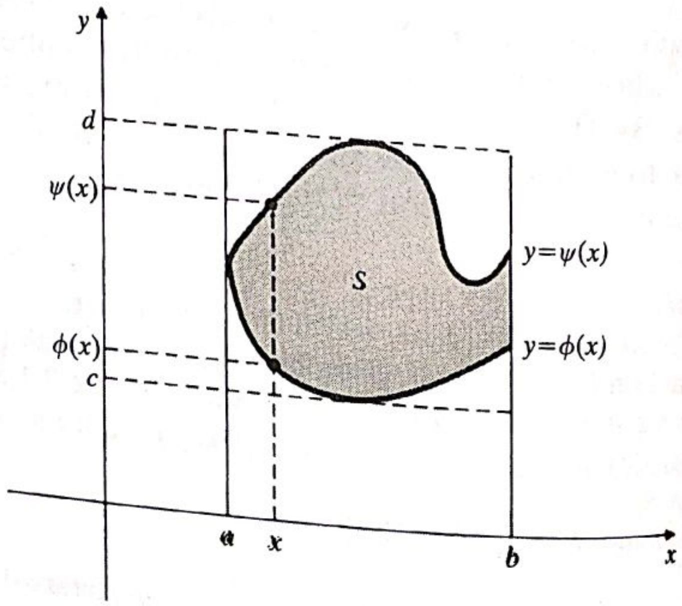


Fig. 7.10 An x-simple region

Then S is a subset of the compact rectangle $R=[a, b]\times[c, d]$. By Theorem 7.2.11 the boundary of S is a null set. Furthermore, f is bounded on S. Therefore, by Theorem 7.4.13 f is integrable over S.

To complete the proof we apply Theorem 7.3.11 to evaluate the integral of f^* over R. We need to know that for each $x\in[a, b]$ the function $f^*(x, y)$ is integrable as a function of y. Now $f^*(x, y) = 0$ if $c \leqslant y < \phi(x)$, $f^*(x, y) = f(x, y)$ if $\phi(x) \leqslant y \leqslant \psi(x)$ and $f^*(x, y) = 0$ if $\psi(x) < y \leqslant d$. Furthermore, for each $x \in [a, b]$, $f(x, y)$ is continuous as a function of $y \in [\phi(x), \psi(x)]$. Therefore $f^*(x, y)$ is integrable as a function of y and

$$\int_c^d f^*(x, y)\, dy = \int_c^{\phi(x)} 0\, dy + \int_{\phi(x)}^{\psi(x)} f(x, y)\, dy + \int_{\psi(x)}^d 0\, dy$$
$$= \int_{\phi(x)}^{\psi(x)} f(x, y)\, dy.$$

Hence, by Theorem 7.3.11,

$$\iint_S f(x, y)\, dx\, dy = \iint_R f^*(x, y)\, dx\, dy = \int_a^b dx \int_{\phi(x)}^{\psi(x)} f(x, y)\, dy,$$

and the proof is complete.

A set S of the type defined in Theorem 7.4.21 is called an *x-simple region* (in $\mathbb{R}^2$). See Fig. 7.10

The following theorem is similar to Theorem 7.4.21 and has much the same proof except that the roles of x and y are interchanged.

7.4.22 Theorem. *Let $\phi : [c, d] \subseteq \mathbb{R} \to \mathbb{R}$ and $\psi : [c, d] \subseteq \mathbb{R} \to \mathbb{R}$ be continuous functions such that $\phi(y) < \psi(y)$ for all $y \in]c, d[$. Let*

$$S = \{(x, y) \in \mathbb{R}^2 \mid c \leqslant y \leqslant d,\ \phi(y) \leqslant x \leqslant \psi(y)\}$$

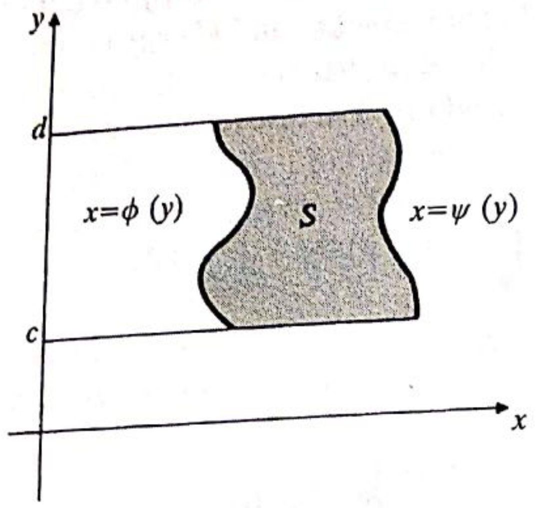
and suppose that the boundary of S is a simple closed curve. Then any continuous function $f : S \subseteq \mathbb{R}^2 \to \mathbb{R}$ is integrable over S and

$$\iint_S f(x, y)\, dx\, dy = \int_c^d dy \int_{\phi(y)}^{\beta(y)} f(x, y)\, dx.$$

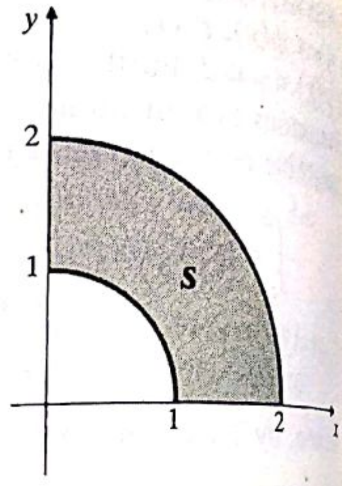
Proof. Exercise.

A set S of the type defined in Theorem 7.4.22 is called a *y-simple region* (in $\mathbb{R}^2$). See Fig. 7.11(i).

The region sketched in Fig. 7.11(ii) is both x-simple and y-simple. In general, a bounded subset S of $\mathbb{R}^2$ will be neither x-simple nor y-simple. It may however be possible to integrate a function $f : S \subseteq \mathbb{R}^2 \to \mathbb{R}$ over such a set by subdividing S into subsets which are of one or other of the two types. The integral over S can then be found by adding up the integrals of f over these subsets. As an illustration of this remark consider the following example.



(i)



(ii)

Fig. 7.11 (i) A y-simple region; (ii) An x-simple and y-simple region

7.4.23 *Example.* Find the area of the region S enclosed by the parabolas $y^2 = x + 3$ and $y^2 = -2x + 6$ (Fig. 7.12). The parabolas intersect where $x + 3 = -2x + 6$. Hence their points of intersection are $(1, -2)$ and $(1, 2)$. We subdivide S into two x-simple regions A and B by the line $x = 1$, as indicated in Fig. 7.12.

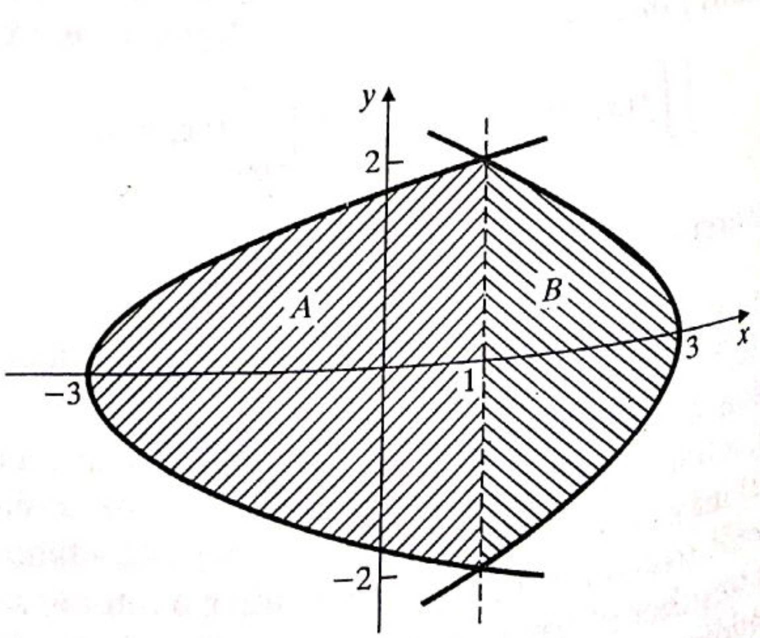


Fig. 7.12

We obtain

$$\text{Area of } S = \iint_A 1 \, dx \, dy + \iint_B 1 \, dx \, dy$$

$$= \int_{-3}^{1} dx \int_{-\sqrt{(x+3)}}^{\sqrt{(x+3)}} 1 \, dy + \int_{1}^{3} dx \int_{-\sqrt{(-2x+6)}}^{\sqrt{(-2x+6)}} 1 \, dy = \tfrac{32}{3} + \tfrac{16}{3} = 16.$$

Alternatively, and more simply, we may regard S as a single y-simple region, and obtain

$$\text{Area of } S = \int_{-2}^{2} dy \int_{y^2-3}^{3-1/2y^2} 1 \, dx = 16.$$

We end this section by recording some useful technical results.

7.4.24 Theorem. *Let* $f : S \subseteq \mathbb{R}^2 \to \mathbb{R}$ *and* $g : S \subseteq \mathbb{R}^2 \to \mathbb{R}$ *be bounded functions which are integrable over the bounded set S, and let c be a real number. Then* $f + g$ *and* cf *are integrable over S, and*

$$\iint_S (f+g) \, dA = \iint_S f \, dA + \iint_S g \, dA \quad \text{and} \quad \iint_S cf \, dA = c \iint_S f \, dA.$$

Proof. Apply Definition 7.4.2 to Theorem 7.1.12.

7.4.25. Theorem. *Let* $f : S \subseteq \mathbb{R}^2 \to \mathbb{R}$ *and* $g : S \subseteq \mathbb{R}^2 \to \mathbb{R}$ *be bounded functions which are integrable over the bounded set S and such that* $f(x, y) \leqslant g(x, y)$ *for all* $(x, y) \in S$. *then*

$$\iint_S f \, dA \leqslant \iint_S g \, dA.$$

Proof. Exercise.

7.4.26 Corollary. *Integral Mean-Value Theorem. Let S be a compact subset of* $\mathbb{R}^2$ *with null boundary, and let* $f : S \subseteq \mathbb{R}^2 \to \mathbb{R}$ *be a continuous function. Then* **[i]**:

$$m(\text{area } S) \leqslant \iint_S f \, dA \leqslant M(\text{area } S)$$

where m and M are respectively the greatest lower bound and the *least upper bound of f on S, and* **[ii]**: *if S is path connected there exists $(a, b) \in S$ such that*

$$\iint_S f\,\mathrm{d}A = f(a, b)(\text{area } S).$$

Proof. Since f is a continuous function on a compact set, both m and M are finite. Furthermore, if S is path connected, then f attains all values between m and M inclusively. Part [i] now follows by applying Theorem 7.4.25 to the inequality $m \leqslant f(x, y) \leqslant M$ for all $(x, y) \in S$. Part [ii] follows from [i].

7.4.27 Theorem. *Let S and T be disjoint bounded subsets of $\mathbb{R}^2$ and let $f: S \cup T \subseteq \mathbb{R}^2 \to \mathbb{R}$ be bounded. If f is integrable over S and f is integrable over T then f is integrable over $S \cup T$ and*

$$\iint_{S \cup T} f\,\mathrm{d}A = \iint_S f\,\mathrm{d}A + \iint_T f\,\mathrm{d}A.$$

Proof. Let R be a compact rectangle such that $S \cup T \subseteq R$. For any subset $B \subseteq S \cap T$ let $f_B: R \subseteq \mathbb{R}^2 \to \mathbb{R}$ be defined by

$$f_B(\mathbf{p}) = \begin{cases} f(\mathbf{p}) & \text{if} \quad \mathbf{p} \in B \\ 0 & \text{if} \quad \mathbf{p} \in R \backslash B. \end{cases}$$

Then, by Definition 7.4.2,

$$\begin{aligned} \iint_S f\,\mathrm{d}A + \iint_T f\,\mathrm{d}A &= \iint_R f_S\,\mathrm{d}A + \iint_R f_T\,\mathrm{d}A \\ &= \iint_R (f_S + f_T)\,\mathrm{d}A && \text{by Theorem 7.4.24} \\ &= \iint_R f_{S \cup T}\,\mathrm{d}A && \text{since } S \cap T \text{ is empty} \\ &= \iint_{S \cup T} f\,\mathrm{d}A && \text{by Definition 7.4.2.} \end{aligned}$$

The next result demonstrates the irrelevance of null subsets in domains of integration.

7.4.28 Theorem. *Let S be a bounded subset of* $\mathbb{R}^2$, *let* $N \subseteq S$ *be a null set, and let* $f: S \subseteq \mathbb{R}^2 \to \mathbb{R}$ *and* $g: S \subseteq \mathbb{R}^2 \to \mathbb{R}$ *be bounded functions which agree on* $S \backslash N$.

[i] *The function f is integrable over S if and only if f is integrable over* $S \backslash N$, *and*

$$\iint_S f\, \mathrm{d}A = \iint_{S \backslash N} f\, \mathrm{d}A$$

when the integrals exist.

[ii] *If f is integrable over S then g is integrable over S, and*

$$\iint_S f\, \mathrm{d}A = \iint_S g\, \mathrm{d}A.$$

Proof. (*i*) Let R be a compact rectangle in $\mathbb{R}^2$ such that $S \subseteq R$. Assume that f is integrable over S. Then, in the notation of the proof of Theorem 7.4.27

$$\iint_S f\, \mathrm{d}A = \iint_R f_S\, \mathrm{d}A.$$

By Theorem 7.2.18[ii]

$$\iint_R f_S\, \mathrm{d}A = \iint_R f_{S \backslash N}\, \mathrm{d}A,$$

since the functions f_S and $f_{S \backslash N}$ disagree on R only on the null set N. The result follows. The converse is proved similarly.

[*ii*] If f is integrable over S then

$$\iint_S f\, \mathrm{d}A = \iint_{S \backslash N} f\, \mathrm{d}A = \iint_{S \backslash N} g\, \mathrm{d}A = \iint_S g\, \mathrm{d}A.$$

Exercises 7.4

1. (a) Let $\phi: \mathbb{R} \to \mathbb{R}$ be the discontinuous function defined by

$$\phi(x) = \begin{cases} 1 & \text{when} \quad x \leqslant 2 \\ 0 & \text{when} \quad x > 2. \end{cases}$$

Define the subset S of $\mathbb{R}^2$ by

7.4.29

$$S = \{(x, y) \in \mathbb{R}^2 \mid y \leqslant \phi(x)\}.$$

Prove that ∂S is the union of the graph of ϕ and the set $\{(2, y) \in \mathbb{R}^2 \mid 0 \leqslant y \leqslant 1\}$. Compare with Example 7.4.11.
(b) Let $\phi: \mathbb{R} \to \mathbb{R}$ be defined by $\phi(0) = 0$ and $\phi(x) = \sin(1/x)$, $x \neq 0$, and let S be given by 7.4.29. Find ∂S.

Answer: ∂S is the union of the graph of ϕ and the set $\{(0, y) \in \mathbb{R}^2 \mid -1 \leqslant y \leqslant 1\}$.

2. Let R be a compact rectangle in $\mathbb{R}^2$ and let S be a subset of $\mathbb{R}^2$. Prove that if R meets both S and $R \backslash S$ then R meets the boundary ∂S of S.

Hint: suppose $\mathbf{p} \in R \cap S$ and $\mathbf{q} \in R \backslash S$. Define the line segment L_x by

$$L_x = \{\mathbf{p} + t(\mathbf{q} - \mathbf{p}).\ 0 \leqslant t \leqslant x\}.$$

then $L_x \subseteq R$, $0 \leqslant x \leqslant 1$. Also $L_0 = \{\mathbf{p}\} \subseteq S$, and L_1 meets $R \backslash S$, so $L_1 \nsubseteq S$. Define

$$c = \sup\{x \in [0, 1] \mid L_x \subseteq S\}.$$

Show that $\mathbf{p} + c(\mathbf{q} - \mathbf{p}) \in (\partial S) \cap R$.

3. Let R be the unit square $[0, 1] \times [0, 1]$ in $\mathbb{R}^2$. Let $\mathbf{p}_1, \ldots, \mathbf{p}_k, \ldots$, be an enumeration of the points in Int R with rational coordinates. For each k let S_k be an open rectangle in R containing $\mathbf{p}_k$ and of area $\leqslant 1/2^{k+1}$. Put $S = \bigcup_k S_k$. It is easy to verify that (i) S is a bounded open subset of $\mathbb{R}^2$; (ii) $\sum_k$ area $S_k \leqslant \frac{1}{2}$; (iii) $\bar{S} = R$. Prove that ∂S, the boundary of S, is not a null set.

Proof: let $\mathcal{Q}$ be any partition of R, and let $R_1, \ldots, R_n$ be the rectangles in $\mathcal{Q}$. Since $\bar{S} = R$, every rectangle R_i meets S. Let $R_1, \ldots, R_m$ be the rectangles such that $R_j \subseteq S$, $j = 1, \ldots, m$. Then $\sum_1^m$ area $R_j \leqslant \frac{1}{2}$. Every other rectangle R_i, $i = m + 1, \ldots, n$, meets both S and $R \backslash S$, and therefore meets ∂S (Exercise 2). Hence $\kappa(\partial S, \mathcal{Q}) \geqslant \frac{1}{2}$.

4. Sketch the elliptical disc

$$S = \{(x, y) \in \mathbb{R}^2 \mid x^2/a^2 + y^2/b^2 \leqslant 1\},$$

where a and b are positive constants.

Find continuous functions $\phi: [-a, a] \subseteq \mathbb{R} \to \mathbb{R}$ and $\psi: [-a, a] \subseteq \mathbb{R} \to \mathbb{R}$ such that

$$S = \{(x, y) \in \mathbb{R}^2 \mid -a \leqslant x \leqslant a,\ \phi(x) \leqslant y \leqslant \psi(x)\}.$$

Compute $\iint_S f(x, y)\, dx\, dy$ as a repeated integral for the cases (a) $f(x, y) = 1$, $(x, y) \in S$; (b) $f(x, y) = x^2$, $(x, y) \in S$. The first integral gives the area of the disc. The second integral measures the second moment of area of the disc about the y-axis.

Answers: $\phi(x, y) = -b\sqrt{(1 - x^2/a^2)}$, $\psi(x, y) = b\sqrt{(1 - x^2/a^2)}$. (a) πab; (b) $\frac{1}{4}\pi ba^3$.

5. By computing a suitable repeated integral, find
   (a) the area enclosed by the parabolas $y = x^2$ and $y = 2 - x^2$;
   (b) the area bounded by the parabola $y^2 = x$ and the line $x = 1$;
   (c) the area enclosed by the parabola $y^2 = 4x$ and the line $x + y = 3$.
   (This is done most simply by a single application of Theorem 7.4.22. As an alternative, perform two applications of Theorem 7.4.21, leading to $\int_0^1 4\sqrt{x}\, dx + \int_1^9 (3 - x + 2\sqrt{x})\, dx$.)

Answers: (a) 8/3; (b) 4/3; (c) 64/3.

6. Let S be a subset of $\mathbb{R}^2$ with area $\mathscr{A} = \iint_S 1\, dA$. Let $f: S \subseteq \mathbb{R}^2 \to \mathbb{R}$ be integrable over S. Provided $\mathscr{A} \neq 0$, the *mean value of f over S* is defined to be the number

$$\bar{f} = \frac{1}{\mathscr{A}} \iint_S f\, dA.$$

In particular, the mean values of $f(x, y) = x$ and $f(x, y) = y$ over S give the coordinates $(\bar{x}, \bar{y})$ of the *centroid* of S. Find the centroids of the regions $S \subseteq \mathbb{R}^2$ whose areas were calculated in Exercise 5(a), (b), (c).

Answers: (a) $(0, 1)$; (b) $(3/5, 0)$; (c) $(17/5, -2)$.

7. (a) Given that f is continuous over $[0, 1] \times [0, 1] \subseteq \mathbb{R}^2$, prove that

$$\int_0^1 dx \int_x^1 f(x, y)\, dy = \int_0^1 dy \int_0^y f(x, y)\, dx.$$

Hint: Show that the repeated integrals are both equal to $\iint_S f\, dA$ where S is a suitable triangle in $\mathbb{R}^2$.

(b) Evaluate $\int_0^1 dx \int_x^1 e^{y^2}\, dy$.

Answer: $\frac{1}{2}(e - 1)$.

8. Sketch the region of integration of

(a) $\displaystyle\int_0^1 dx \int_x^{2x} f(x, y)\, dy$; (b) $\displaystyle\int_{-1}^1 dx \int_0^{|x|} f(x, y)\, dy$.

Deduce that

(a) $$\int_0^1 dx \int_x^{2x} f(x, y)\, dy = \int_0^1 dy \int_{1/2y}^{y} f(x, y)\, dx + \int_1^2 dy \int_{1/2y}^{1} f(x, y)\, dx;$$

(b) $$\int_{-1}^{1} \mathrm{d}x \int_{0}^{|x|} f(x, y)\, \mathrm{d}y = \int_{0}^{1} \mathrm{d}y \int_{y}^{1} (f(x, y) + f(-x, y))\, \mathrm{d}x.$$

Hint for (a): sketch the subset of $\mathbb{R}^2$ enclosed by the lines $x = 0$, $x = 1$, $y = x$ and $y = 2x$.

9. (a) Show that the set S (part of a circular annulus) sketched in Fig. 7.11(ii) is both x-simple and y-simple.

Proof: it is x-simple because $S = \{(x, y) \subseteq \mathbb{R}^2 \mid 0 \leqslant x \leqslant 2,\ \phi(x) \leqslant y \leqslant \psi(x)\}$, where

$$\phi(x) = \begin{cases} \sqrt{(1 - x^2)} & \text{when} \quad 0 \leqslant x \leqslant 1, \\ 0 & \text{when} \quad 1 \leqslant x \leqslant 2. \end{cases}$$

and $\psi(x) = \sqrt{(4 - x^2)}$, $0 \leqslant x \leqslant 2$. It follows similarly that S is a y-simple region.

(b) Prove that the elliptical region $S = \{(x, y) \in \mathbb{R}^2 \mid x^2/a^2 + y^2/b^2 \leqslant 1\}$ is both x-simple and y-simple.

10. Check that the x-simple region sketched in Fig. 7.10 is not y-simple, but that it can be subdivided into two y-simple regions.

11. Sketch the annulus $S = \{(x, y) \in \mathbb{R}^2 \mid 1 \leqslant x^2 + y^2 \leqslant 4\}$. Show that S can be subdivided into four x-simple regions. Applying Theorem 7.4.21, evaluate the integral $\iint_S x^2y^2\, \mathrm{d}x\, \mathrm{d}y$.

Answer: $21\pi/8$.

12. Evaluate the integral $\iint_S x^2y\, \mathrm{d}x\, \mathrm{d}y$, where S is the annulus

$$\{(x, y) \in \mathbb{R}^2 \mid 1 \leqslant x^2 + y^2 \leqslant 4\}.$$

Answer: 0. The result follows directly from the observation that the function $f(x, y) = x^2y$ is skew-symmetric about the x-axis, since $f(x, -y) = -f(x, y)$, and symmetric about the y-axis, since $f(-x, y) = f(x, y)$.

13. Let $f: S \subseteq \mathbb{R}^2 \to \mathbb{R}$ be integrable over S. Show that f may not be integrable over a subset A in S.

Hint: let $S = [0, 1] \times [0, 1]$, let $f(\mathbf{x}) = 1$ for all $\mathbf{x} \in S$ and let $A = \{(x, y) \in \mathbb{R}^2 \mid (x, y) \in S,\ x \text{ and } y \text{ both rational}\}$.

14. Prove the following generalization of Theorem 7.4.27. Let S and T be bounded sets in $\mathbb{R}^2$ which intersect in a null set, and let $f: S \cup T \subseteq \mathbb{R}^2 \to \mathbb{R}$ be bounded. If f is integrable over S and also over T then f is integrable over $S \cup T$, and

$$\iint_{S \cup T} f\, \mathrm{d}A = \iint_S f\, \mathrm{d}A + \iint_T f\, \mathrm{d}A.$$

Hint: put $Q = S \backslash (S \cap T)$. Then Q and T are disjoint subsets of $\mathbb{R}^2$. By Theorem 7.4.28 f is integrable over Q. Apply Theorem 7.4.27 to f over $Q \cup T$.

15. Let S be a bounded set in $\mathbb{R}^2$ with null boundary. Let $f:\bar{S} \subseteq \mathbb{R}^2 \to \mathbb{R}$ be a bounded function. Prove that f is integrable over S if and only if f is integrable over $\bar{S}$, and, if the integrals exist, that

$$\iint_S f\,\mathrm{d}A = \iint_{\bar{S}} f\,\mathrm{d}A.$$

7.5 Green's Theorem

We have seen in Section 6.2 that there are two orientations associated with any simple closed curve. It will be important in this section to be able to distinguish between these orientations without reference to a particular parametrization. The orientations of the unit circle $x^2 + y^2 = 1$ are, for example, counterclockwise (parametrized by $(\cos t, \sin t)$, $t \in [0, 2\pi]$) or clockwise (parametrized by $(\cos t, -\sin t)$, $t \in [0, 2\pi]$). In general, however, such a description is not straightforward. For example, the counterclockwise oriented circle in Fig. 7.13 may be distorted through the plane into either of the two other oriented curves in the figure. In both cases the counterclockwise sense is not so easy to spot.

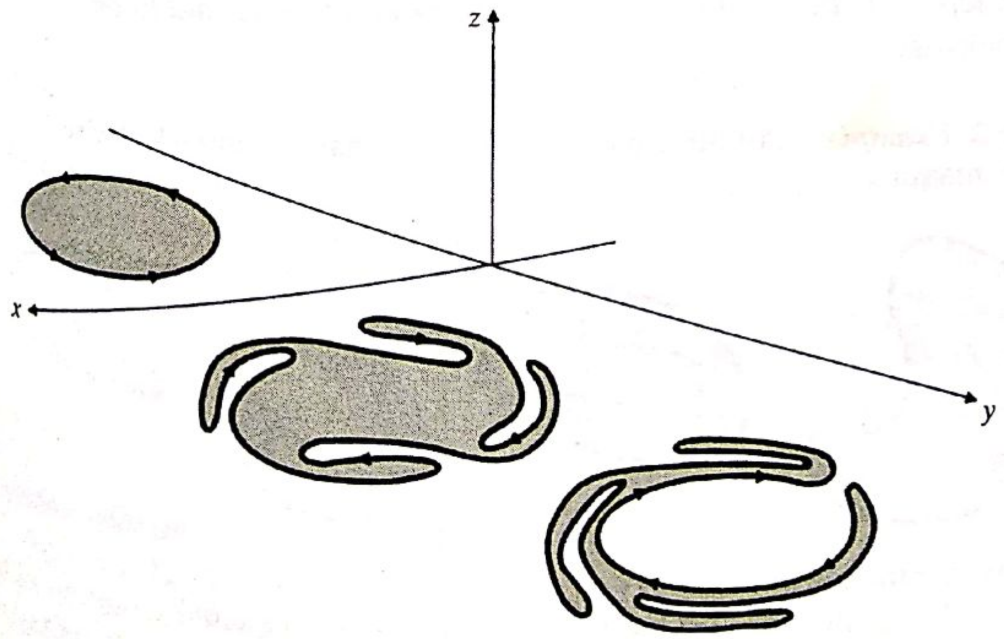


Fig. 7.13

In tackling this problem we shall assume without proof the surprisingly deep theorem, the *Jordan Curve Theorem*, which tells us in particular that a simple closed curve in $\mathbb{R}^2$ separates the plane into two subsets, the *inside* and the *outside*, of which it is the common boundary. That is, we shall assume that if C is a simple closed curve in $\mathbb{R}^2$ then $\mathbb{R}^2 \backslash C$ is the union of two disjoint open sets U and V such that $\partial U = \partial V = C$, and U (the inside) is bounded and V (the outside) is unbounded. Figure 7.13, where the insides of the curves are shaded, indicates some of the difficulties of a general argument. For further details see Apostol, *Mathematical Analysis*, Chapter 8.

One intuitive approach to describing an orientation of C is to sketch $\mathbb{R}^3$ with a right handed set of axes—as in Fig. 7.13—and then to identify $(x, y) \in C$ with $(x, y, 0)$ in the sketch. In this way we obtain a sketch of C in the x, y plane in $\mathbb{R}^3$. If we now imagine ourselves walking around C on the z-positive side of the plane—following a given orientation—then the inside of C will lie always to our left or always to our right. In the case of the three oriented curves sketched in Fig. 7.13 for example, the insides lie to the left. On the other hand if the circle were oriented clockwise, the inside would lie to the right.

7.5.1 *Definition.* *Let C^+ be an oriented simple closed curve in $\mathbb{R}^2$ with inside $U \subseteq \mathbb{R}^2$. Then C^+ is said to have* counterclockwise *orientation if, on traversing C^+ in the above sense, the set U lies to the left. If U lies to the right, then the orientation is said to be* clockwise.

7.5.2 *Example.* All three curves in Fig. 7.13 have counterclockwise orientation.

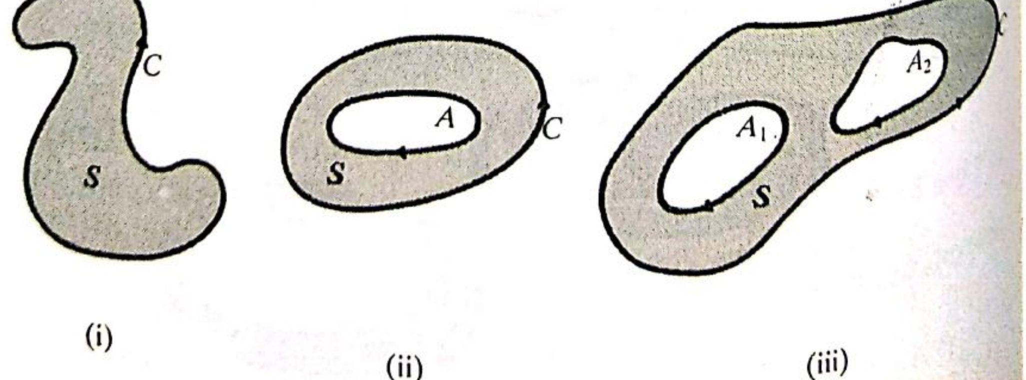


Fig. 7.14

In this section we shall be concerned with integrals over certain subsets S of $\mathbb{R}^2$ whose boundary consists of a number of simple closed curves. See Fig. 7.14.

7.5.3 Definition. *Let C be a simple closed curve in $\mathbb{R}^2$, and let $A_1, \ldots, A_k$ be disjoint simple closed curves lying inside C such that for all $i \neq j$, A_i and A_j are outside each other. The subset $D \subseteq \mathbb{R}^2$ of points lying inside C and outside each of $A_1, \ldots, A_k$ is called an* open region *in $\mathbb{R}^2$. The set $S \subseteq \mathbb{R}^2$,*

$$S = \bar{D} = D \cup C \cup A_1 \cup \cdots \cup A_k$$

is called a compact region.

7.5.4 Example. In Fig. 7.14(i) the compact region S is $\bar{U} = U \cup C$ where U is the inside of the simple closed curve C. Two other compact regions are illustrated in Fig. 7.14(ii) and (iii).

Green's Theorem in the Plane, one of the classical theorems of vector analysis, equates the line integral of a C^1 vector field around the boundary of a compact region and the (double) integral of a related continuous scalar field over the region. The theorem runs as follows.

7.5.5 Green's Theorem in the Plane. *Let S be a compact region in $\mathbb{R}^2$ and let $F : S \subseteq \mathbb{R}^2 \to \mathbb{R}^2$ be a C^1 vector field. Then*

$$\int_{\partial S^+} F \cdot d\mathbf{r} = \iint_S \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) dA$$

where F_1 and F_2 are the coordinate functions of F and ∂S^+ is the boundary of S positively oriented in the following sense.

7.5.6 Definition. *Let the compact region $S \subseteq \mathbb{R}^2$ be formed from simple closed curves $C, A_1, \ldots, A_k$, as in Definition 7.5.3. Then the* positively oriented boundary *∂S^+ of S is*

$$\partial S = C \cup A_1 \cup \cdots \cup A_k$$

together with an orientation of the simple closed curves $C, A_1, \ldots, A_k$ such that traversing any one of them in the direction of its orientation keeps the region S on the left. This involves orienting C counterclockwise and $A_1, \ldots, A_k$ clockwise.

7.5.7 *Example.* In Fig. 7.14 the positively oriented boundaries of the shaded regions S are indicated. Notice that in (ii), for example, the inner edge A is oriented clockwise and the outer edge C counterclockwise.

7.5.8 *Example.* Let $U \subseteq \mathbb{R}^2$ be the inside of a simple closed curve C in $\mathbb{R}^2$. Then $S = \bar{U} = U \cup C$ is a compact region and the positively oriented boundary of S is C with counterclockwise orientation. See for example Figs. 7.13 and 7.14(i).

We can use Green's Theorem to obtain a definition of the counterclockwise orientation of a simple closed curve C which is independent of a physical model of the curve. For let C^+ denote the curve with *either* of its two possible orientations. Then by Green's Theorem, with $F(x, y) = (0, x)$,

$$\int_{C^+} (0, x) \cdot \mathbf{dr} = \pm \iint_S 1 \, dA = \pm \text{area } S$$

where S is the compact region whose interior is the inside of C. We could therefore *define* C^+ as having counterclockwise orientation if the integral $\int_{C^+} (0, x) \cdot \mathbf{dr}$ is positive. In this case the value of the line integral is the area of the inside of C.

We shall not prove Green's Theorem in its full generality. In this section we consider a number of special cases.

We begin by considering Green's Theorem for x-simple regions. Let

$$S = \{(x, y) \in \mathbb{R}^2 \mid a \leqslant x \leqslant b, \ \phi(x) \leqslant y \leqslant \psi(x)\}$$

be an x-simple region. Then S is a compact region, and its boundary is the simple closed curve made up of four simple curves B_1, B_2, B_3,

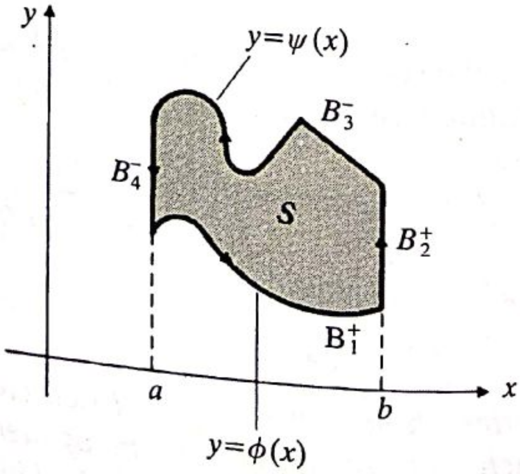


Fig. 7.15

B_4 as illustrated in Fig. 7.15. Assign to these curves orientations as follows:

$B_1^+ = \{(x, y) \in \mathbb{R}^2 \mid y = \phi(x)\}$ oriented from $(a, \phi(a))$ to $(b, \phi(b))$

$B_2^+ = \{(x, y) \in \mathbb{R}^2 \mid x = b, \phi(b) \leqslant y \leqslant \psi(b)\}$

oriented from $(b, \phi(b))$ *to* $(b, \psi(b))$

$B_3^+ = \{(x, y) \in \mathbb{R}^2 \mid y = \psi(x)\}$ oriented from $(a, \psi(a))$ to $(b, \psi(b))$

$B_4^+ = \{(x, y) \in \mathbb{R}^2 \mid x = a, \phi(a) \leqslant y \leqslant \psi(a)\}$

oriented from $(a, \phi(a))$ to $(a, \psi(a))$.

Then the positively oriented boundary ∂S^+ of S is formed from B_1^+, B_2^+, B_3^-, B_4^-.

For any continuous vector field $F: S \subseteq \mathbb{R}^2 \to \mathbb{R}^2$,

7.5.9 $$\int_{\partial S^+} F \cdot d\mathbf{r} = \int_{B_1^+} F \cdot d\mathbf{r} + \int_{B_2^+} F \cdot d\mathbf{r} + \int_{B_3^-} F \cdot d\mathbf{r} + \int_{B_4^-} F \cdot d\mathbf{r}.$$

Suppose now that F is a C^1 vector field on S of the form

$$F(x, y) = (F_1(x, y), 0), \qquad (x, y) \in S$$

where $F_1: S \subseteq \mathbb{R}^2 \to \mathbb{R}$ has continuous partial derivatives.

Let $\alpha: [p, q] \subseteq \mathbb{R} \to \mathbb{R}^2$ be a C^1 parametrization of B_1^+. Then

$$\begin{aligned}\int_{B_1^+} F \cdot d\mathbf{r} &= \int_p^q F_1(\alpha(t))\alpha_1'(t)\, dt \\ &= \int_p^q F_1(\alpha_1(t), \phi(\alpha_1(t)))\alpha_1'(t)\, dt \\ &= \int_a^b F_1(x, \phi(x))\, dx.\end{aligned}$$

Similarly

$$\int_{B_3^+} F \cdot d\mathbf{r} = \int_a^b F_1(x, \psi(x))\, dx.$$

On the other hand, since F is orthogonal to B_2^+ and B_4^+,

$$\int_{B_2^+} F \cdot d\mathbf{r} = \int_{B_4^+} F \cdot d\mathbf{r} = 0.$$

Hence, from 7.5.9,

$$\int_{\partial S^+} (F_1, 0) \cdot \mathrm{d}\mathbf{r} = \int_a^b F_1(x, \phi(x))\, \mathrm{d}x - \int_a^b F_1(x, \psi(x))\, \mathrm{d}x$$

and therefore, since $\partial F_1/\partial y$ is continuous on S,

7.5.10 $$\int_{\partial S^+} (F_1, 0) \cdot \mathrm{d}\mathbf{r} = -\int_a^b \mathrm{d}x \int_{\phi(x)}^{\psi(x)} \frac{\partial F_1}{\partial y}(x, y)\, \mathrm{d}y.$$

We therefore have the following theorem.

7.5.11 *Theorem.* *Let S be an x-simple region in* $\mathbb{R}^2$. *Then for any* C^1 *function* $F_1 : S \subseteq \mathbb{R}^2 \to \mathbb{R}$

$$\int_{\partial S^+} (F_1, 0) \cdot \mathrm{d}\mathbf{r} = -\iint_S \frac{\partial F_1}{\partial y}\, \mathrm{d}A$$

where ∂S^+ *is the positively oriented boundary of S.*

Proof. The theorem follows from 7.5.10 and Theorem 7.4.21.

Theorem 7.5.11 is matched by a similar result in which the roles of x and y are interchanged. It is clear that a y-simple region is also a compact region.

7.5.12 *Theorem.* *Let S be a y-simple region in* $\mathbb{R}^2$. *Then for any* C^1 *function* $F_2 : S \subseteq \mathbb{R}^2 \to \mathbb{R}$

$$\int_{\partial S^+} (0, F_2) \cdot \mathrm{d}\mathbf{r} = \iint_S \frac{\partial F_2}{\partial x}\, \mathrm{d}A$$

where ∂S^+ *is the positively oriented boundary of S.*

Proof. Let $\phi : [c, d] \subseteq \mathbb{R} \to \mathbb{R}$ and $\psi : [c, d] \subseteq R \to R$ be piecewise C^1 functions such that $\phi(y) < \psi(y)$ for all $y \in]c, d[$ and let

$$S = \{(x, y) \in \mathbb{R}^2 \mid c \leqslant y \leqslant d,\ \phi(y) \leqslant x \leqslant \psi(y)\}.$$

An argument similar to the one used for x-simple regions established that

$$\int_{\partial S^+} (0, F_2) \cdot \mathbf{dr} = \int_c^d F_2(\psi(y), y)\, \mathrm{d}y - \int_c^d F_2(\phi(y), y)\, \mathrm{d}y$$
$$= \int_c^d \mathrm{d}y \int_{\phi(y)}^{\psi(y)} \frac{\partial F_2}{\partial x}(x, y)\, \mathrm{d}x.$$

The result follows from Theorem 7.4.22.

We obtain an important special case of Green's Theorem by combining Theorems 7.5.11 and 7.5.12.

7.5.13 *Definition.* *A subset of $\mathbb{R}^2$ is said to be a simple region if it is both x-simple and y-simple.*

An ellipse together with its inside is a simple region, as are the sets sketched in Figures 7.11(ii) and 7.12. A simple region is clearly a compact region.

7.5.14 *Green's Theorem for simple regions.* *Let $S \subseteq \mathbb{R}^2$ be a simple region and let $F: S \subseteq \mathbb{R}^2 \to \mathbb{R}^2$ be a C^1 vector field. Then*

7.5.15
$$\int_{\partial S^+} F \cdot \mathbf{dr} = \iint_S \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) \mathrm{d}A$$

where ∂S^+ is the positively oriented boundary of S.

Proof. For all $(x, y) \in S$

$$F(x, y) = (F_1(x, y), F_2(x, y)) = (F_1(x, y), 0) + (0, F_2(x, y)).$$

Hence

$$\int_{\partial S^+} F \cdot \mathbf{dr} = \int_{\partial S^+} (F_1, 0) \cdot \mathbf{dr} + \int_{\partial S^+} (0, F_2) \cdot \mathbf{dr}.$$

Theorem 7.5.14 now follows from Theorems 7.5.11 and 7.5.12 since both coordinate functions F_1 and F_2 have continuous partial derivatives.

Expression 7.5.15 is commonly written in the form

7.5.16
$$\int_{\partial S^+} (F_1\, \mathrm{d}x + F_2\, \mathrm{d}y) = \iint_S \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) \mathrm{d}x\, \mathrm{d}y.$$

7.5.17 *Example.* Let $R = [-1, 0] \times [-1, 0] \subseteq \mathbb{R}^2$ and let $F: R \to \mathbb{R}^2$ be given by

$$F(x, y) = (xy, 0), \qquad (x, y) \in R.$$

We have seen in Example 7.3.19 that

$$\iint_R \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) \mathrm{d}A = \iint_R -x \, \mathrm{d}x \, \mathrm{d}y = \tfrac{1}{2}.$$

On the other hand it is not difficult to see that

$$\int_{\partial R^+} F \cdot \mathrm{d}\mathbf{r} = \int_\alpha F \cdot \mathrm{d}\mathbf{r} = \int_0^1 (1 - t, 0) \cdot (-1, 0) \, \mathrm{d}t = \tfrac{1}{2}$$

where $\alpha : [0, 1] \subseteq \mathbb{R} \to \mathbb{R}^2$ is the parametrization of the edge of R from $(-1, -1)$ to $(0, -1)$ given by $\alpha(t) = (t - 1, -1)$, $t \in [0, 1]$. The line integrals along the other edges are all zero.

As in the argument following Example 7.5.8, we can find the area of a compact region S by evaluating the line integral of a suitable vector field around its positively oriented boundary ∂S^+. For the case where ∂S is a simple closed curve, if $\alpha : [c, d] \subseteq \mathbb{R} \to \mathbb{R}^2$ is a simple, counterclockwise parametrization of ∂S then

7.5.18 $$\text{Area } S = \int_\alpha (0, x) \cdot \mathrm{d}\alpha = \int_\alpha (-y, 0) \cdot \mathrm{d}\alpha = \tfrac{1}{2} \int_\alpha (-y, x) \cdot \mathrm{d}\alpha.$$

This follows immediately from Theorem 7.5.5 and has as a consequenece the following result.

7.5.19 Theorem. *Let $\alpha : [c, d] \subseteq \mathbb{R} \to \mathbb{R}^2$ be a simple counterclockwise parametrization of the boundary ∂S of a simple region S, and let*

$$\alpha(t) = (x(t), y(t)), \qquad t \in [c, d].$$

Then

$$\text{Area } S = \int_c^d x(t) y'(t) \, \mathrm{d}t = -\int_c^d x'(t) y(t) \, \mathrm{d}t$$
$$= \tfrac{1}{2} \int_c^d (x(t) y'(t) - x'(t) y(t)) \, \mathrm{d}t.$$

Proof. Exercise.

7.5.20 *Example.* The ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

has counterclockwise parametrization $\alpha:[0, 2\pi]\subseteq\mathbb{R}\to\mathbb{R}^2$, where

$$\alpha(t)=(a\cos t,\, b\sin t), \qquad t\in[0, 2\pi].$$

The area of the inside of the ellipse is given by any of the three expressions

$$\int_0^{2\pi} ab\cos^2 t\,\mathrm{d}t,\ -\int_0^{2\pi}(-ab\sin^2 t)\,\mathrm{d}t \quad\text{and}\quad \tfrac{1}{2}ab\int_0^{2\pi}(\cos^2 t+\sin^2 t)\,\mathrm{d}t.$$

They are all equal to πab.

Suppose now that V and W are simple regions with $V\subseteq \operatorname{Int} W$. Then

$$S=W\backslash \operatorname{Int} V$$

is a compact region whose boundary is $\partial W\cup\partial V$. See Fig. 7.16. Let

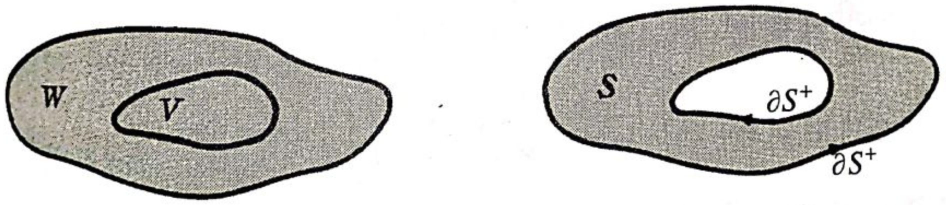


Fig. 7.16

∂V^+ and ∂W^+ be the counterclockwise oriented boundaries of V and W respectively. Then the positively oriented boundary of S is

7.5.21
$$\partial S^+=\partial W^+\cup\partial V^-.$$

For any C^1 vector field $F:W\subseteq\mathbb{R}^2\to\mathbb{R}^2$,

$$\begin{aligned}
\iint_S\left(\frac{\partial F_2}{\partial x}-\frac{\partial F_1}{\partial y}\right)\mathrm{d}A &= \iint_W\left(\frac{\partial F_2}{\partial x}-\frac{\partial F_1}{\partial y}\right)\mathrm{d}A-\iint_{\operatorname{Int} V}\left(\frac{\partial F_2}{\partial x}-\frac{\partial F_1}{\partial y}\right)\mathrm{d}A\\
&= \iint_W\left(\frac{\partial F_2}{\partial x}-\frac{\partial F_1}{\partial y}\right)\mathrm{d}A-\iint_V\left(\frac{\partial F_2}{\partial x}-\frac{\partial F_1}{\partial y}\right)\mathrm{d}A\\
&= \int_{\partial W^+} F\cdot\mathrm{d}\mathbf{r}-\int_{\partial V^+} F\cdot\mathrm{d}\mathbf{r}, \quad\text{by Theorem 7.5.14}\\
&= \int_{\partial W^+} F\cdot\mathrm{d}\mathbf{r}+\int_{\partial V^-} F\cdot\mathrm{d}\mathbf{r}.
\end{aligned}$$

That is, by 7.5.21,

7.5.22
$$\iint\limits_S \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) \mathrm{d}A = \int_{\partial S^+} F \cdot \mathbf{dr}.$$

This establishes Green's Theorem for 'simple regions with a simple hole'. An easy generalization leads to Green's Theorem for simple regions with a finite number of simple holes.

7.5.23 *Example.* Let S be the annulus $\{(x, y) \in \mathbb{R}^2 \mid 1 \leqslant x \leqslant 4\}$. Then ∂S^+, the positively oriented boundary of S is the union of the counterclockwise oriented circle $x^2 + y^2 = 4$ and the clockwise oriented circle $x^2 + y^2 = 1$. Let $F : \mathbb{R}^2 \to \mathbb{R}^2$ be the vector field

$$F(x, y) = (-y, x), \qquad (x, y) \in \mathbb{R}^2.$$

The line integral around a counterclockwise oriented circle centre O in $\mathbb{R}^2$, radius c, is $2\pi c^2$ (Exercise 6.2.3). Hence

$$\int_{\partial S^+} F \cdot \mathbf{dr} = 2\pi(2^2) - 2\pi(1^2) = 6\pi.$$

On the other hand, since the area of S is 3π,

$$\iint\limits_S \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) \mathrm{d}A = \iint\limits_S 2 \, \mathrm{d}A = 6\pi$$

which confirms 7.5.22.

The argument leading to a proof of more general forms of Green's Theorem is more difficult. However, it is often possible to find particular techniques for particular cases. For example, it may be possible to divide S, the region over which we wish to integrate, into a finite number of simple regions by using a finite number of simple closed curves. See Fig. 7.17.

Let the positively oriented boundary of S be ∂S^+ and let the simple regions into which S is subdivided be $S_1, \ldots, S_k$ with positively (counterclockwise) oriented boundaries $\partial S_1^+, \ldots, \partial S_k^+$ respectively.

Notice that if a simple arc lies in just *one* of $\partial S_1, \ldots, \partial S_k$, say ∂S_r, then it also lies in ∂S and is oriented in the same way in both ∂S_r^+ and ∂S^+. this is because if S_r lies on the left on traversing the curve then so does S.

On the other hand, if a simple arc lies in both ∂S_i and ∂S_j $(i \neq j)$ then it is given opposite orientations in ∂S_i^+ and ∂S_j^+. See Fig. 7.17.

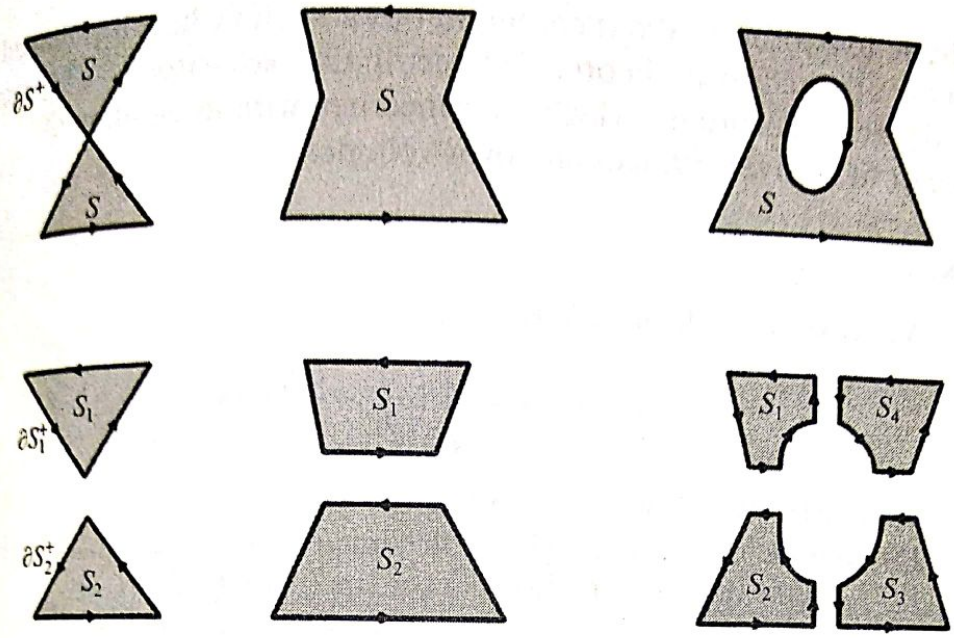


Fig. 7.17 Piecewise simple regions and their decomposition into simple regions

In this case the line integrals of a vector field along that part of ∂S_i^+ and along that part of ∂S_j^+ are equal in magnitude and of opposite sign.

We shall call compact regions that can be subdivided in this way, *piecewise simple regions*.

7.5.24 Green's Theorem for piecewise simple regions. *Let S be a piecewise simple region, and let ∂S^+ be its positively oriented boundary. Then for any C^1 vector field $F : S \subseteq \mathbb{R}^2 \to \mathbb{R}^2$,*

7.5.25
$$\int_{\partial S^+} F \cdot d\mathbf{r} = \iint_S \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) dA.$$

Proof. With the above notation, since the line integrals cancel along common boundaries,

7.5.26
$$\int_{\partial S_1^+} F \cdot d\mathbf{r} + \cdots + \int_{\partial S_k^+} F \cdot d\mathbf{r} = \int_{\partial S^+} F \cdot d\mathbf{r}.$$

Expression 7.5.25 follows by applying Theorem 7.5.14 to each term on the left-hand side of 7.5.26.

In the next section we use Green's Theorem to say something more about irrotational and conservative fields. We also give alternative forms of the theorem, in order to show how it is related to the other classical theorems of vector analysis—Stokes' Theorem and Gauss' Theorem—which are concerned with integration of vector fields over surfaces and over volumes.

Exercises 7.5.

1. Verify Green's Theorem in the plane

$$\int_{\partial S^+} F \cdot \mathrm{d}\mathbf{r} = \iint_S \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) \mathrm{d}A$$

where $F(x, y) = (2xy - x^2, x + y^2)$ and
(a) S is the rectangular region with corners $(0, 0)$, $(0, 2)$, $(1, 2)$, $(1, 0)$;
(b) S is the triangular region with corners $(0, 0)$, $(0, 2)$, $(1, 0)$;
(c) S is the region enclosed by the parabolas $y = x^2$ and $y^2 = x$.

Answers: both integrals are equal to (a) 0; (b) $1/3$; (c) $1/30$.

2. Verify Green's Theorem in the plane where S is the annulus $\{(x, y) \in \mathbb{R}^2 \mid a^2 \leqslant x^2 + y^2 \leqslant b^2\}$ and

(a) $F(x, y) = \left(\dfrac{-y}{\sqrt{(x^2 + y^2)}}, \dfrac{x}{\sqrt{(x^2 + y^2)}}\right)$;

(b) $F(x, y) = \left(\dfrac{-y}{x^2 + y^2}, \dfrac{x}{x^2 + y^2}\right)$;

(c) $F(x, y) = \left(\dfrac{x}{x^2 + y^2}, \dfrac{y}{x^2 + y^2}\right)$.

Answers: (a) $2\pi(b - a)$; (b) 0; (c) 0.

3. Use Green's Theorem to show that the line integral

$$\int_{C^+} \frac{-y}{x^2 + y^2}\,\mathrm{d}x + \frac{x}{x^2 + y^2}\,\mathrm{d}y$$

takes the value
(a) 0, if C^+ is any oriented simple closed curve in $\mathbb{R}^2 \backslash \{(0, 0)\}$ that does not enclose the origin;
(b) 2π, if C^+ is any counterclockwise oriented simple closed curve around the origin.

Hint: put

$$F(x, y) = \left(\frac{-y}{x^2 + y^2}, \frac{x}{x^2 + y^2}\right), \qquad (x, y) \neq (0, 0).$$

To prove (a) apply Green's Theorem to the region $S \subseteq \mathbb{R}^2$ enclosed by

C. To prove (b) apply Green's Theorem to an annulus whose boundary is C and a non-intersecting *circle* centre the origin. Compare Exercise 2(b).

4. Prove that

$$\int_{C^+} \frac{x}{x^2+y^2}\,dx + \frac{y}{x^2+y^2}\,dy = 0,$$

where C^+ is any oriented simple closed curve in $\mathbb{R}^2 \backslash \{(0,0)\}$.

5. Let C^+ be an oriented simple arc in $\mathbb{R}^2$ running from $(0,0)$ to (a, b). Prove, using Green's Theorem that the line integral

$$\int_{C^+} (x + y^2)\,dx + 2xy\,dy$$

is independent of the choice of C^+.

Hint: consider a second oriented simple arc B^+ from $(0,0)$ to (a, b). Prove that $\int_{C^+} + \int_{B^-} = 0$. Compare Exercise 5.5.7.

6. Sketch the astroid $x^{2/3} + y^{2/3} = a^{2/3}$ and use Theorem 7.5.19 to calculate the area of the region enclosed by it.

Answer: $3\pi a^2/8$. A suitable simple parametrization of the astroid is $\alpha(t) = (a\cos^3 t,\ a\sin^3 t)$, $t \in [0, 2\pi]$.

7. Prove that the area of a simple region S in $\mathbb{R}^2$ is given by each of the three line integrals

$$\text{Area of } S = \int_{\partial S^+} x\,dy = \int_{\partial S^+} -y\,dx = \tfrac{1}{2}\int_{\partial S^+} x\,dy - y\,dx,$$

where ∂S^+ is the positively oriented boundary of S.

8. Find the area of one loop of the figure 8 curve $x = \sin t$, $y = \sin 2t$.

Answer: 4/3.

9. Find the area bounded by one arc of the cycloid $x = a(t - \sin t)$, $y = a(1 - \cos t)$, $a > 0$, $t \in [0, 2\pi]$ and the x-axis.

Answer: $3\pi a^2$.

7.6 Rot, curl, and div

In Section 6.3 we justified calling the line integral of a continuous vector field $F : D \subseteq \mathbb{R}^2 \to \mathbb{R}^2$ around an oriented simple closed curve C^+ the *circulation* of F around C^+. We then defined the rotation of

F around C^+ and the rotation of F at a point (a, b) in D. Suppose now that F is a C^1 vector field. Green's Theorem allows us to find an alternative expression for $(\operatorname{rot} F)(a, b)$.

Let S_ε be the closed disc with centre (a, b) and radius $\varepsilon > 0$, where ε is chosen small enough for S_ε to lie in D. From its definition in 6.3.7 and from Green's Theorem 7.5.14, the rotation of F around ∂S_ε^+ is given by

$$\frac{1}{\text{area } S_\varepsilon}\int_{\partial S_\varepsilon^+} F\cdot \mathrm{d}\mathbf{r} = \frac{1}{\text{area } S_\varepsilon}\iint_{S_\varepsilon}\left(\frac{\partial F_2}{\partial x}-\frac{\partial F_1}{\partial y}\right)\mathrm{d}A$$

where ∂S_ε^+ is the boundary circle with counterclockwise orientation. There therefore exists, by the Integral Mean-Value Theorem 7.4.26, a point $(a + h_\varepsilon, b + k_\varepsilon) \in S_\varepsilon$ such that the rotation of F around ∂S_ε^+ is

$$\frac{\partial F_2}{\partial x}(a + h_\varepsilon, b + k_\varepsilon) - \frac{\partial F_1}{\partial y}(a + h_\varepsilon, b + k_\varepsilon).$$

Since $(\operatorname{rot} F)(a, b)$ is the limit of the rotation around S_ε^+ as ε tends to 0, and since F_1 and F_2 have continuous partial derivatives,

7.6.1 $$(\operatorname{rot} F)(a, b) = \frac{\partial F_2}{\partial x}(a, b) - \frac{\partial F_1}{\partial y}(a, b).$$

This useful result, which can of course be taken as a definition of rot F, leads to the following theorem.

7.6.2 *Theorem.* *A C^1 vector field $F : D \subseteq \mathbb{R}^2 \to \mathbb{R}^2$ defined on an open set D in $\mathbb{R}^2$ is irrotational if and only if* $\dfrac{\partial F_2}{\partial x} = \dfrac{\partial F_1}{\partial y}$.

Proof. Immediate from 7.6.1.

The condition of Theorem 7.6.2 is also necessary and sufficient for a field to be conservative provided we suitably restrict the domain of F.

7.6.3 *Definition.* *A subset $D \subseteq \mathbb{R}^2$ is simply connected if it is path connected and if whenever a simple closed curve lies in D then so does its inside.*

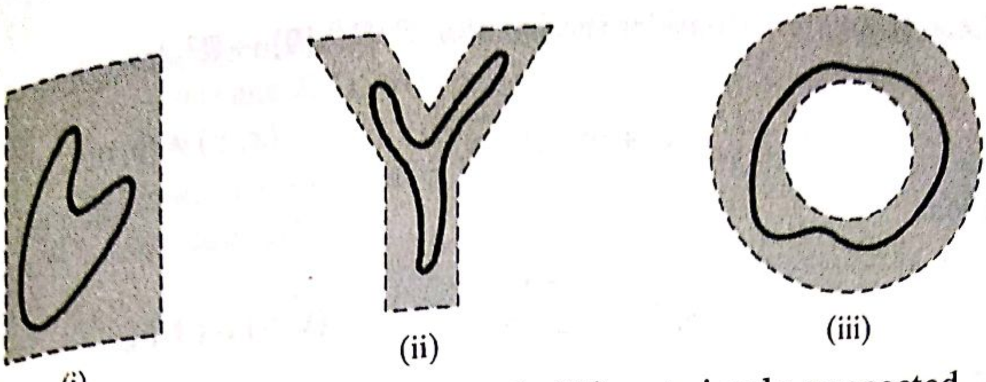


Fig. 7.18 (i), (ii) Simply connected, (iii) not simply connected

7.6.4 Example.
[i] The rectangle and the Y-shape sketched in Fig. 7.18 are simply connected, but the annulus is not.
[ii] The open subset $D = \mathbb{R}^2 \backslash \{\mathbf{0}\}$ of $\mathbb{R}^2$ is not simply connected. The unit circle $x^2 + y^2 = 1$ lies in D but, since $\mathbf{0} \notin D$, its inside does not.
[iii] Let $A \subseteq \mathbb{R}^2$ be the half-axis

$$A = \{(x, 0) \in \mathbb{R}^2 \mid x \leqslant 0\}.$$

Then $\mathbb{R}^2 \backslash A$ is simply connected.

The following theorem generalizes the result of Exercise 5.5.12. We shall assume the general form of Green's Theorem given in 7.5.5.

7.6.5 Theorem. *A C^1 vector field $F : D \subseteq \mathbb{R}^2 \to \mathbb{R}^2$ defined on an open, simply connected subset D of $\mathbb{R}^2$ is conservative if and only if $\partial F_2/\partial x = \partial F_1/\partial y$.*

Proof. First, if F is conservative then $F = \operatorname{grad} f$ for some C^2 scalar field $f : D \subseteq \mathbb{R}^2 \to \mathbb{R}$. By Theorem 3.10.4 the mixed partial derivatives $\partial^2 f/\partial x\, \partial y$ and $\partial^2 f/\partial y\, \partial x$ are equal. It follows that $\partial F_2/\partial x = \partial F_1/\partial y$.

Second, suppose that $\partial F_2/\partial x = \partial F_1/\partial y$. Let C be a simple closed curve in D. Then since D is simply connected, the inside U of C also lies in D. Put $S = U \cup C$ and let C^+ have counterclockwise orientation. Green's Theorem implies that

$$\int_{C^+} F \cdot \mathbf{dr} = \iint_S \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) \mathrm{d}A = 0.$$

Since this is true for all simple closed curves in D, Theorem 5.5.21 implies that F is conservative.

7.6.6 *Example.* Consider the function $F:\mathbb{R}^2\backslash\{\mathbf{0}\}\to\mathbb{R}^2$ defined by

$$F(x, y) = \left(\frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2}\right), \qquad (x, y) \neq (0, 0).$$

Then

$$\frac{\partial F_2}{\partial x} = \frac{\partial F_1}{\partial y} = \frac{y^2-x^2}{(x^2+y^2)^2}, \qquad (x, y) \neq (0, 0).$$

Hence F is irrotational. However, on the domain $\mathbb{R}^2\backslash\{\mathbf{0}\}$, F is *not* conservative. For example, the line integral of F counterclockwise around the unit circle centre $(0, 0)$ is 2π. See also Example 5.5.28.

However, if the domain of F is limited to the simply connected set $\mathbb{R}^2\backslash A$ where $A = \{(x, 0) \in \mathbb{R}^2 \mid x \leqslant 0\}$, then F is conservative. Indeed $F = \operatorname{grad} f$ where $f:\mathbb{R}^2\backslash A \subseteq \mathbb{R}^2 \to \mathbb{R}$ is given in Exercise 5.5.8.

Expression 7.6.1 allows us to rephrase the conclusion of Green's Theorem as

7.6.7
$$\int_{\partial S^+} F \cdot \mathrm{d}\mathbf{r} = \iint_S \operatorname{rot} F \, \mathrm{d}A.$$

We can obtain a physical interpretation of 7.6.7 if, as in Section 6.3, we think of f as representing the flow of a fluid in the (x, y) plane. Remember that $(\operatorname{rot} F)(x, y)$ is positive, negative or zero according to whether the flow near (x, y) is counterclockwise, clockwise or uniform. The identity 7.6.7 tells us that the circulation $\int_{\partial S^+} F \cdot \mathrm{d}\mathbf{r}$ around the positively oriented boundary of S is equal to the aggregate of local rotational effects on S. It may of course be zero despite the presence of local clockwise and counterclockwise rotations.

7.6.8 *Example.* Consider the velocity field $F:\mathbb{R}^2\to\mathbb{R}^2$ defined by

$$F(x, y) = (1, \tfrac{1}{2}x^2), \qquad (x, y) \in \mathbb{R}^2.$$

Then

$$(\operatorname{rot} F)(x, y) = x, \qquad (x, y) \in \mathbb{R}^2.$$

Let S be a circular disc, centre $(0, b)$, radius c, where b and $c > 0$ are arbitrary. Then

$$\int_{\partial S^+} F \cdot \mathrm{d}\mathbf{r} = \iint_S \operatorname{rot} F \, \mathrm{d}A = \iint_S x \, \mathrm{d}x \, \mathrm{d}y = 0.$$

The vanishing of the circulation of F around ∂S^+ is made plausible by reference to Fig. 6.9 (page 337), where the velocity field F is sketched.

We can associate a vector field $F: D \subseteq \mathbb{R}^2 \to \mathbb{R}^2$ with a vector field $F^*: D \times \mathbb{R}^3 \to \mathbb{R}^3$ defined by

7.6.9 $$F^*(x, y, z) = (F_1(x, y), F_2(x, y), 0), \qquad (x, y) \in D.$$

Suppose that F is differentiable. Then so is F^*, and by Definition 4.10.1.

7.6.10 $$(\text{curl } F^*)(x, y, z) = \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)(x, y)\mathbf{k} = (\text{rot } F)(x, y)\mathbf{k},$$

where $\mathbf{k} = (0, 0, 1)$ is the unit vector in the z direction.

Expression 7.6.7 then becomes a statement of Green's Theorem in the form

7.6.11 $$\int_{\partial S^+} F \cdot d\mathbf{r} = \iint_S \text{curl } F^* \cdot \mathbf{k} \, dA.$$

This is the form which generalizes to Stokes' Theorem.

A section of the field F^* with the x, y plane is illustrated in Fig. 7.19. Notice that $(\text{curl } F^*)(\mathbf{p})$ points in the direction of $\mathbf{k}$ if the effective flow around $\mathbf{p}$ is counterclockwise as viewed from 'above' the x, y plane; it points in the direction $-\mathbf{k}$ if the effective flow around the point is clockwise. In Fig. 7.19 the numbers $(\text{rot } F)(\mathbf{p})$ and $(\text{rot } F)(\mathbf{q})$ can be estimated by considering the circulation around small counterclockwise oriented circles in the x, y plane

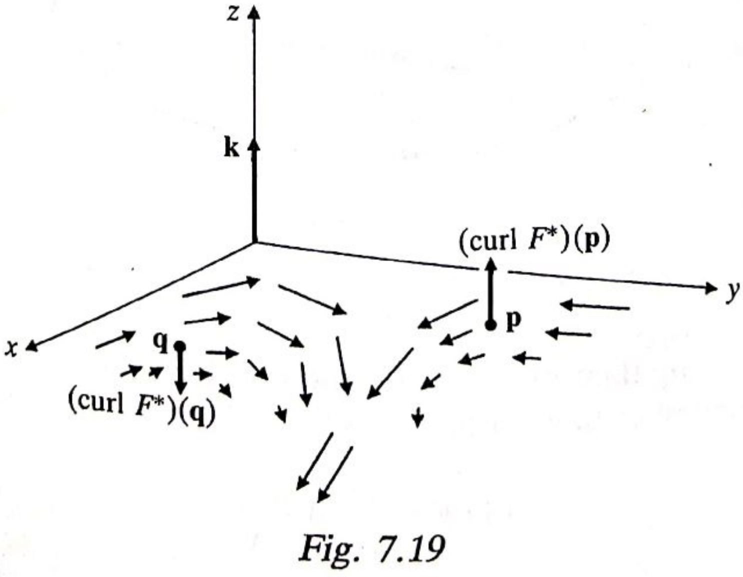


Fig. 7.19

surrounding **p** and **q** respectively. Here (rot F)(**p**) is positive and (rot F)(**q**) is negative.

A picture similar to Fig. 7.19 illustrates the section of F^* with any plane parallel to the x, y plane.

It follows from 7.6.10 that rot $F = 0$ if and only if curl $F^* = \mathbf{0}$. So the field F in $\mathbb{R}^2$ is irrotational (Definition 6.3.9) if and only if the associated field F^* in $\mathbb{R}^3$ is irrotational (Definition 4.10.8).

A second application of Green's Theorem to the fluid flow F in a compact region S relates the net fluid flow across the boundary ∂S of S and the net expansion or contraction of the fluid within S. For this application we suppose that the boundary of S is smooth, that is, it is composed of a finite number of smooth simple closed curves. As usual, let ∂S^+ be the positively oriented boundary of S, and let $T: \partial S \subseteq \mathbb{R}^2 \to \mathbb{R}^2$ be the unit tangent field to ∂S^+. For any $\mathbf{r} \in \partial S$ we define the *unit normal to* ∂S *at* $\mathbf{r}$ *pointing away from* S to be the vector

7.6.12
$$N(\mathbf{r}) = (T_2(\mathbf{r}), -T_1(\mathbf{r})),$$

where $T(\mathbf{r}) = (T_1(\mathbf{r}), T_2(\mathbf{r}))$. So $N(\mathbf{r})$ is obtained by a clockwise rotation through $\frac{1}{2}\pi$ of the unit tangent $T(\mathbf{r})$. It is the unit normal vector to ∂S that points to the right as one follows the oriented boundary ∂S^+. See Fig. 7.20.

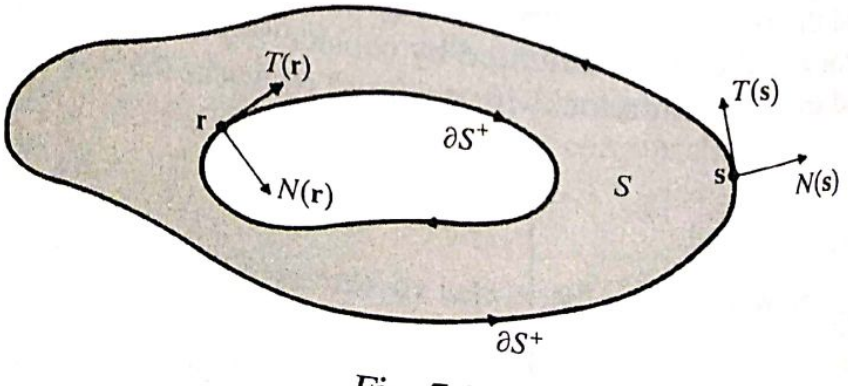


Fig. 7.20

The following theorem provides the relationship referred to above. Remember that for any C^1 vector field $F: S \subseteq \mathbb{R}^2 \to \mathbb{R}^2$

$$\operatorname{div} F = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y}.$$

7.6.13 Divergence Theorem in the Plane. *Let S be a piecewise simple region in $\mathbb{R}^2$ with smooth boundary ∂S. For any C^1 vector field $F : S \subseteq \mathbb{R}^2 \rightarrow \mathbb{R}^2$*

7.6.14
$$\int_{\partial S} F \cdot N \, \mathrm{d}s = \iint_S \operatorname{div} F \, \mathrm{d}A$$

where $N(\mathbf{r})$ is the unit normal to ∂S at $\mathbf{r} \in \partial S$ pointing away from S.

Proof. [*i*] Consider first the case where S is a simple region. Let $\alpha : [a, b] \subseteq \mathbb{R} \rightarrow \mathbb{R}^2$ be a simple (smooth) parametrization of ∂S^+, the positively oriented boundary of S. The unit tangent to ∂S^+ at $\alpha(t)$ is given by

$$T(\alpha(t)) = (\alpha_1'(t), \alpha_2'(t)) / \|\alpha'(t)\|,$$

and the unit normal to ∂S at $\alpha(t)$ pointing away from S is

$$N(\alpha(t)) = (\alpha_2'(t), -\alpha_1'(t)) / \|\alpha'(t)\|.$$

Define a vector field $G : S \subseteq \mathbb{R}^2 \rightarrow \mathbb{R}^2$ by

$$G(x, y) = (-F_2(x, y), F_1(x, y)), \qquad (x, y) \in S.$$

Then G is a C^1 field and

7.6.15 $F(\alpha(t)) \cdot N(\alpha(t)) = G(\alpha(t)) \cdot T(\alpha(t)), \qquad t \in [a, b].$

See Fig. 7.21.

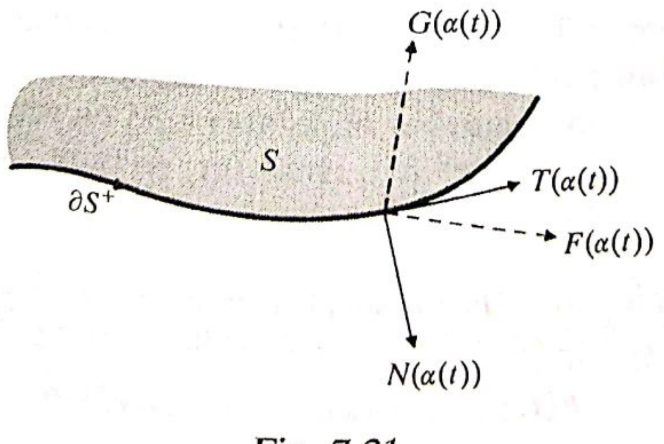


Fig. 7.21

Hence

$$\begin{aligned}
\int_{\partial S} F \cdot N \, \mathrm{d}s &= \int_a^b F(\alpha(t)) \cdot N(\alpha(t)) \, \|\alpha'(t)\| \, \mathrm{d}t \\
&= \int_a^b G(\alpha(t)) \cdot T(\alpha(t)) \, \|\alpha'(t)\| \, \mathrm{d}t \qquad \text{by 7.6.15} \\
&= \int_a^b G(\alpha(t)) \cdot \alpha'(t) \, \mathrm{d}t \\
&= \int_{\partial S^+} G \cdot \mathrm{d}\mathbf{r} = \iint_S \left(\frac{\partial G_2}{\partial x} - \frac{\partial G_1}{\partial y} \right) \mathrm{d}A \text{ by Green's Theorem} \\
&= \iint_S \left(\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} \right) \mathrm{d}A = \iint_S \operatorname{div} F \, \mathrm{d}A.
\end{aligned}$$

[*ii*] Suppose now that S is a piecewise simple region that can be cut up into a finite number of simple regions $S_1, \ldots, S_k$ by a finite number of simple curves. Where two of these simple regions S_i and S_j meet along a simple curve C the associated path integrals satisfy

$$\int_C F \cdot N_i \, \mathrm{d}s + \int_C F \cdot N_j \, \mathrm{d}s = 0,$$

since at any point $\mathbf{p} \in C$, $N_i(\mathbf{p})$ (the unit normal to ∂S_i at $\mathbf{p}$ pointing away from S_i) is in the opposite direction to $N_j(\mathbf{p})$ (the unit normal to ∂S_j at $\mathbf{p}$ pointing away from S_j).

Hence 7.6.14 follows by adding the corresponding expressions relating to each of the simple regions $S_1, \ldots, S_k$.

In the proof of Theorem 7.6.13 the condition that the boundary of S be smooth can be relaxed to piecewise smoothness. In that case there may be a finite number of points $\mathbf{r} \in \partial S$ on which the normal vector $N(\mathbf{r})$ cannot be defined. However, relation 7.6.14 still applies when suitably interpreted.

7.6.16 *Example.* Let S be the simple region $\{(x, y) \in \mathbb{R}^2 \mid |2x| + |y| \leq 2\}$ sketched in Fig. 7.22 and let $F : \mathbb{R}^2 \to \mathbb{R}^2$ be the C^1 vector field defined by

$$F(x, y) = (y, -x), \qquad (x, y) \in \mathbb{R}^2.$$

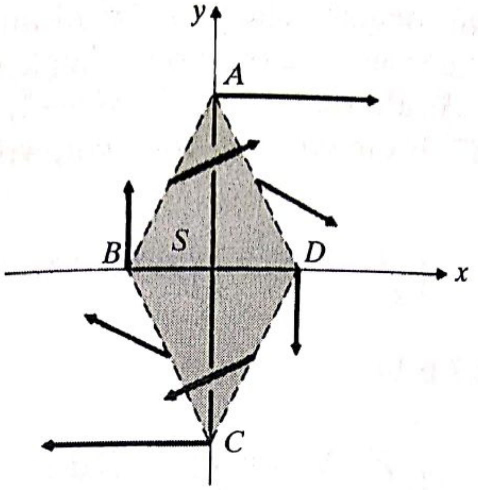


Fig. 7.22 Vector field $F(x, y) = (y, -x)$ on boundary of diamond S

Then $\operatorname{div} F = 0$, and, by 7.6.14,

$$\int_{\partial S} F \cdot N \, ds = \iint_S \operatorname{div} F \, dA = 0.$$

The component $F \cdot N$ is sometimes positive (when $F(x, y)$ is directed away from S) and sometimes negative (when $F(x, y)$ is directed into S). These contributions are balanced in the integral of $F \cdot N$ over ∂S.

Incidentally, by Green's Theorem

$$\int_{\partial S+} F \cdot d\mathbf{r} = \iint_S \operatorname{rot} F \, dA = \iint_S -2 \, dA = -8,$$

since the area of S is 4. This calculation agrees with that in Example 6.2.25.

Concerning the physical application of Theorem 7.6.13, if we again regard F as the velocity field of a fluid flow in a compact region S, then $(\operatorname{div} F)(\mathbf{p})$, $\mathbf{p} \in S$, measures the rate of expansion per unit area of the fluid at $\mathbf{p}$ (see Section 4.11). Hence $\iint_S \operatorname{div} F \, dA$ aggregates the rate of expansion of the fluid in S. From physical considerations, this rate of expansion must equal the rate at which fluid flows out of S across the boundary ∂S, and this is given by the line integral $\int_{\partial S} F \cdot N \, ds$. The equality is confirmed in 7.6.14. The integral $\int_{\partial S} F \cdot N \, ds$ is called the *outward flux of F across S*.

Expression 7.6.14 is the form of Green's Theorem which generalizes to Gauss' Divergence Theorem.

It is interesting to compare Theorem 7.6.13 with the following form of Green's Theorem for a piecewise simple region S with smooth boundary. As above, let $T : \partial S \subseteq \mathbb{R}^2 \to \mathbb{R}^2$ be the unit tangent field to ∂S^+. From Green's Theorem, written in the form 7.6.7, we obtain

7.6.17
$$\int_{\partial S} F \cdot T \, ds = \iint_S \operatorname{rot} F \, dA.$$

Compare this with 7.6.14

$$\int_{\partial S} F \cdot N \, ds = \iint_S \operatorname{div} F \, dA.$$

These formulae when interpreted in terms of fluid flow tell us that the integrals over the boundary of S of the tangential and normal components of F are equal respectively to the aggregate of the local rotational and expansional effects on S.

We can use Theorem 7.6.13 to obtain an expression for the divergence of a vector field which parallels the definition of its rotation given in 6.3.7. Let $F : \mathbb{R}^2 \to \mathbb{R}^2$ be a C^1 vector field and let (a, b) be a point of $\mathbb{R}^2$. Then for any closed disc S_ε, centre (a, b) and radius $\varepsilon > 0$,

$$\frac{1}{\text{area } S_\varepsilon} \int_{\partial S_\varepsilon} F \cdot N \, ds = \frac{1}{\text{area } S_\varepsilon} \iint_{S_\varepsilon} \operatorname{div} F \, dA \qquad \text{by 7.6.14}$$
$$= (\operatorname{div} F)(a + h_\varepsilon, b + k_\varepsilon),$$

for some point $(a + h_\varepsilon, b + k_\varepsilon) \in S_\varepsilon$, by the Integral Mean-Value Theorem. Since $\operatorname{div} F$ is continuous we obtain

7.6.18
$$(\operatorname{div} F)(a, b) = \lim_{\varepsilon \to 0} \frac{1}{\text{area } S_\varepsilon} \int_{\partial S_\varepsilon} F \cdot N \, ds.$$

By comparison, 7.6.17 leads to the following expression for $\operatorname{rot} F$:

7.6.19
$$(\operatorname{rot} F)(a, b) = \lim_{\varepsilon \to 0} \frac{1}{\text{area } S_\varepsilon} \int_{\partial S_\varepsilon} F \cdot T \, ds.$$

Exercises 7.6

1. Let $F : \mathbb{R}^2 \setminus \{\mathbf{0}\} \to \mathbb{R}^2$ be the C^1 vector field defined by

$$F(x, y) = \left(\frac{x}{x^2 + y^2}, \frac{y}{x^2 + y^2} \right), \qquad (x, y) \neq (0, 0).$$

Show that F is irrotational. Is the field F conservative?

Answer: yes. Note that Theorem 7.6.5 cannot be used to prove this, since $\mathbb{R}^2\backslash\{\mathbf{0}\}$ is not simply connected. A direct calculation shows that $F = \operatorname{grad} f$, where $f(x, y) = \frac{1}{2}\ln(x^2 + y^2)$, $(x, y) \neq (0, 0)$. See also Exercise 7.5.4.

2. Verify by applying Theorem 7.6.2 that the field $F: \mathbb{R}^2\backslash\{\mathbf{0}\} \to \mathbb{R}^2$ defined by

$$F(x, y) = \frac{e^x}{x^2 + y^2}(x \sin y - y \cos y,\ x \cos y + y \sin y), \qquad (x, y) \neq (0, 0)$$

is irrotational. (The field F is not conservative. See Exercise 6.3.6.)

3. Let $F: \mathbb{R}^2 \to \mathbb{R}^2$ be the C^1 vector field defined by

$$F(x, y) = (y^2, x^2), \qquad (x, y) \in \mathbb{R}^2.$$

Calculate $(\operatorname{rot} F)(x, y)$. Verify Green's Theorem 7.6.7

$$\int_{\partial S+} F \cdot \mathrm{d}\mathbf{r} = \iint_S \operatorname{rot} F \, \mathrm{d}A$$

where S is the circular disc centre (a, b), radius k, by showing that each integral takes the value $2\pi k^2(a - b)$. Sketch the vector field F.

Note: the field lines of F are $y^3 = x^3 + c$, where c is a constant.

4. Let D be an open set in $\mathbb{R}^2$. Show that the vector field $F: D \subseteq \mathbb{R}^2 \to \mathbb{R}^2$ is conservative if and only if the associated field $F^*: D \times \mathbb{R} \subseteq \mathbb{R}^3 \to \mathbb{R}^3$ (defined in 7.6.9) is conservative.

Hint: suppose $F^* = \operatorname{grad} f^*$ for some scalar field $f^*: D \times \mathbb{R} \subseteq \mathbb{R}^3 \to \mathbb{R}$. Then $\partial f^*/\partial z = 0$, so there exists $f: D \subseteq \mathbb{R}^2 \to \mathbb{R}^2$ such that $f^*(x, y, z) = f(x, y)$ for all $(x, y) \in D$, $z \in \mathbb{R}$. Deduce that $F = \operatorname{grad} f$.

5. Let $Z = \{(0, 0, z)\}$, $z \in \mathbb{R}$ (the z-axis in $\mathbb{R}^3$). The vector field $F^*: \mathbb{R}^3\backslash Z \subseteq \mathbb{R}^3 \to \mathbb{R}^3$ defined by

$$F^*(x, y, z) = \left(\frac{-y}{x^2 + y^2}, \frac{x}{x^2 + y^2}, 0\right), \qquad (x, y) \neq (0, 0)$$

is irrotational (Example 4.10.7). Show that F^* is not conservative. Show, however, that F^* is conservative if its domain is limited to the set $\mathbb{R}^3\backslash H$, where H is the half-plane (containing Z) given by $\{(x, 0, z) \in \mathbb{R}^3 \mid x \leq 0\}$.

Hint: Exercise 4 and Example 7.6.6.

6. In Example 7.6.16, verify directly that $\int_{\partial S} F \cdot N \, \mathrm{d}s = 0$.

Answer: parametrize the directed edge AB (Fig. 7.22) by $\alpha(t) = (-t, 2-2t)$, $t \in [0, 1]$. Then $\alpha'(t) = (-1, -2)$, $N(t) = (-2, 1)$, $F(\alpha(t)) = (2-2t, t)$, and $(F \cdot N)(\alpha(t)) = 5t - 4$. Hence

$$\int_{AB} F \cdot N \, ds = \int_0^1 (F \cdot N)(\alpha(t)) \, \|\alpha'(t)\| \, dt = -3\sqrt{5}/2.$$

The corresponding path integrals over BC, CD, DA are respectively $3\sqrt{5}/2$, $-3\sqrt{5}/2$, and $3\sqrt{5}/2$.

7. Verify the Divergence Theorem in the Plane

$$\int_{\partial S} F \cdot N \, ds = \iint_S \operatorname{div} F \, dA$$

where $F : \mathbb{R}^2 \to \mathbb{R}^2$ is the C^1 vector field defined by

$$F(x, y) = (\tfrac{1}{4}x, \tfrac{1}{4}), \qquad (x, y) \in \mathbb{R}^2$$

(and which is sketched in Fig. 4.18), and S is
(a) the unit square with corners $A(0, 0)$, $B(1, 0)$, $C(1, 1)$, $D(0, 1)$;
(b) the diamond with corners $P(0, 1)$, $Q(-1, 0)$, $R(0, -1)$, $T(1, 0)$;
(c) the circular region $(x-1)^2 + y^2 \leqslant 1$.

Answers: (a) the integrals of $F \cdot N$ over AB, BC, CD, DA are $-\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, 0$;
(b) the integrals of $F \cdot N$ over PQ, QR, RT, TP are $\frac{3}{8}, -\frac{1}{8}, -\frac{1}{8}, \frac{3}{8}$;
(c) parametrize S by $\alpha(t) = (1 + \cos t, \sin t)$, $t \in [0, 2\pi]$. Then $T(\alpha(t)) = (-\sin t, \cos t)$ and $N(\alpha(t)) = (\cos t, \sin t)$.

8. Verify the Divergence Theorem in the Plane where $F(x, y) = (2xy - x^2, x + y^2)$, and S is the rectangular region with corners $(0, 0)$, $(0, 2)$, $(1, 2)$, $(1, 0)$.

9. Verify the Divergence Theorem in the Plane where S is the annulus $0 < a^2 \leqslant x^2 + y^2 \leqslant b^2$ and

(a) $F(x, y) = \left(\dfrac{-y}{x^2+y^2}, \dfrac{x}{x^2+y^2}\right)$; (b) $F(x, y) = \left(\dfrac{x}{x^2+y^2}, \dfrac{y}{x^2+y^2}\right)$.

Interpret your results relative to a sketch of the fields.

7.7 Change of variables in double integrals

The reader will be familiar with the change of variable formula of elementary calculus

7.7.1 $$\int_a^b f(\alpha(t))\alpha'(t) \, dt = \int_{\alpha(a)}^{\alpha(b)} f(x) \, dx.$$

The formula relates two integrals through the change of variable $x = \alpha(t)$. For example, the substitution $x = \alpha(t) = 5 - t^2$ can be used to evaluate

7.7.2
$$\int_{-1}^{2} 2t\sqrt{(5-t^2)}\,\mathrm{d}t = -\int_{4}^{1} \sqrt{x}\,\mathrm{d}x = 14/3.$$

The integrals in 7.7.1 depend on an *orientation* of the intervals of integration. For example, to evaluate the right-hand integral we must integrate 'from $\alpha(a)$ to $\alpha(b)$', where, possibly, $\alpha(a) > \alpha(b)$.

The effect of orientation can be illustrated with the integrals 7.7.2 and the substitution $x = 5 - t^2$. As t increases from -1 to 0 to 1 to 2 the corresponding values of x pass from 4 to 5 to 4 to 1. Considered in this way, the substitution $x = 5 - t^2$ in the left-hand integral of 7.7.2 leads to three pieces

$$\int_{4}^{5} -\sqrt{x}\,\mathrm{d}x + \int_{5}^{4} -\sqrt{x}\,\mathrm{d}x + \int_{4}^{1} -\sqrt{x}\,\mathrm{d}x.$$

The 'doubling back' and consequent cancelling of the first two (oriented) integrals happens because the continuous path α given by $\alpha(t) = 5 - t^2$ is not 1–1 on the interval $[-1, 2]$.

In this section we shall obtain a change of variables formula for double integrals. However, the double integrals we shall consider are integrals over *unoriented* subsets of $\mathbb{R}^2$. Recall that the double integral $\iint_R f(x, y)\,\mathrm{d}x\,\mathrm{d}y$ of a function $f: R \subseteq \mathbb{R}^2 \to \mathbb{R}$ over a rectangle R is defined in terms of Riemann sums associated with partitions of R without reference to an orientation of the subset R of $\mathbb{R}^2$. In the same spirit we may interpret the single integral $\int_a^b g(x)\,\mathrm{d}x$, where $a < b$, as the integral of a function $g: I \subseteq \mathbb{R} \to \mathbb{R}$ over the subset $I = [a, b]$ of $\mathbb{R}$, and denote it by

$$\int_I g\,\mathrm{d}s \qquad \text{or} \qquad \int_I g(x)\,\mathrm{d}x.$$

The following special case of formula 7.7.1 is the basis for generalization to double integrals.

7.7.3 Theorem. *Let $\alpha: I = [a, b] \subseteq \mathbb{R} \to \mathbb{R}$ be a 1–1 C^1 path. Then for any continuous function $f: \alpha(I) \subseteq \mathbb{R} \to \mathbb{R}$*

7.7.4
$$\int_{\alpha(I)} f(x)\,\mathrm{d}x = \int_I f(\alpha(t))\,|\alpha'(t)|\,\mathrm{d}t.$$

Proof. The conditions on α ensure that α is strictly monotonic on $[a, b]$ and that $\alpha([a, b])$ is an interval with end points $\alpha(a)$ and $\alpha(b)$. If α is strictly increasing then $|\alpha'(t)| = \alpha'(t)$ for all $t \in [a, b]$ and $\alpha(I) = [\alpha(a), \alpha(b)]$. If α is strictly decreasing then $|\alpha'(t)| = -\alpha'(t)$ for all $t \in [a, b]$ and $\alpha(I) = [\alpha(b), \alpha(a)]$. In both cases 7.7.4 follows from 7.7.1.

With a view to generalizing Theorem 7.7.3 to double integrals let us consider the integrals in 7.7.4 in terms of their approximating Riemann sums. Since α is strictly monotonic on the interval $I = [a, b]$, there is a 1–1 correspondence between the partitions of I and those of the interval $\alpha(I)$. In particular, a subinterval $[t, t+\Delta t]$ in I of length $\Delta t > 0$ is mapped by α to a subinterval in $\alpha(I)$ of length $|\alpha(t + \Delta t) - \alpha(t)|$. By the Mean-Value Theorem there exists $p \in]t, t + \Delta t[$ such that

$$|\alpha(t + \Delta t) - \alpha(t)| = |\alpha'(p)|\, \Delta t.$$

By the assumed continuity of α' we conclude that for small values of $\Delta t > 0$ the number $|\alpha'(t)|$ measures the magnification of the interval $[t, t + \Delta t]$ under the mapping α.

The following argument suggests that, in a similar way, the magnification factor of a small rectangle in $\mathbb{R}^2$ under a C^1 mapping $G : \mathbb{R}^2 \to \mathbb{R}^2$ is the absolute value of $\det J_{G,(u,v)}$, where (u, v) is a point of the rectangle.

Let R be a rectangle in $\mathbb{R}^2$ with corners $\mathbf{a} = (u, v)$, $\mathbf{b} = (u + \Delta u, v)$, $\mathbf{c} = (u + \Delta u, v + \Delta v)$ and $\mathbf{d} = (u, v + \Delta v)$. See Fig. 7.23. The area of R is $\Delta u\, \Delta v$.

For any C^1 function $G : \mathbb{R}^2 \to \mathbb{R}^2$ the area of $S = G(R)$ is approximately the area of the parallelogram with adjacent sides $\mathbf{b}' - \mathbf{a}'$ and $\mathbf{d}' - \mathbf{a}'$, where $\mathbf{a}'$, $\mathbf{b}'$, $\mathbf{c}'$, and $\mathbf{d}'$ are respectively the

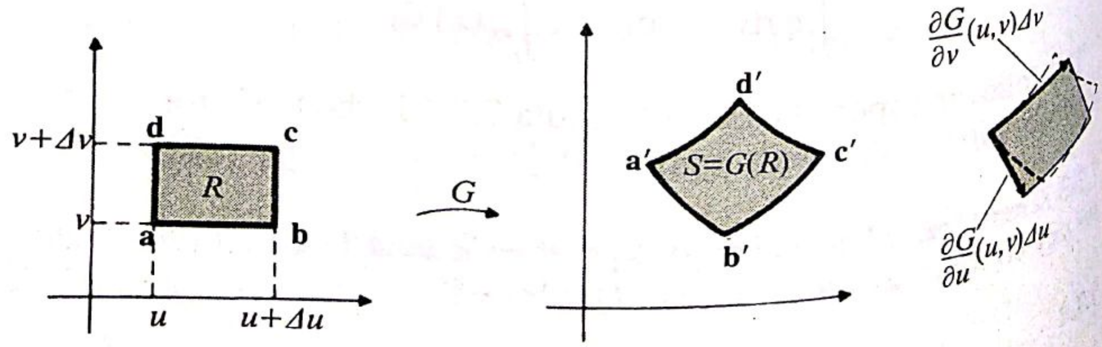


Fig. 7.23

images under G of $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$, and $\mathbf{d}$ (see Fig. 7.23). We observe that

$$\lim_{\Delta u \to 0} \frac{\mathbf{b}' - \mathbf{a}'}{\Delta u} = \lim_{\Delta u \to 0} \frac{G(u + \Delta u, v) - G(u, v)}{\Delta u} = \frac{\partial G}{\partial u}(u, v)$$

and

$$\lim_{\Delta v \to 0} \frac{\mathbf{d}' - \mathbf{a}'}{\Delta v} = \lim_{\Delta v \to 0} \frac{G(u, v + \Delta v) - G(u, v)}{\Delta v} = \frac{\partial G}{\partial v}(u, v).$$

Therefore we have the approximations

$$\mathbf{b}' - \mathbf{a}' \approx \frac{\partial G}{\partial u}(u, v)\, \Delta u \qquad \text{and} \qquad \mathbf{d}' - \mathbf{a}' \approx \frac{\partial G}{\partial v}(u, v)\, \Delta v,$$

and so the area of S is approximated by the area of the parallelogram with adjacent sides

$$\left(\frac{\partial G_1}{\partial u}(u, v)\, \Delta u, \frac{\partial G_2}{\partial u}(u, v)\, \Delta u\right) \quad \text{and}$$

$$\left(\frac{\partial G_1}{\partial v}(u, v)\, \Delta v, \frac{\partial G_2}{\partial v}(u, v)\, \Delta v\right).$$

By Theorem 1.2.23, the area of this parallelogram is equal to the *absolute value* of

$$\det \begin{bmatrix} \dfrac{\partial G_1}{\partial u}(u, v)\, \Delta u & \dfrac{\partial G_1}{\partial v}(u, v)\, \Delta v \\ \dfrac{\partial G_2}{\partial u}(u, v)\, \Delta u & \dfrac{\partial G_2}{\partial v}(u, v)\, \Delta v \end{bmatrix} = \det J_{G,(u,v)}\, \Delta u\, \Delta v.$$

Now let A be a subset of $\mathbb{R}^2$. A function $G : A \subseteq \mathbb{R}^2 \to \mathbb{R}^2$ may reasonably be thought of as representing a change of variables

$$x = G_1(u, v), \qquad y = G_2(u, v)$$

relating $(u, v) \in A$ and $(x, y) \in G(A)$ if, for example, G is 1–1 and both G and G^{-1} are C^1 functions. We would then expect G to transform a rectangular grid covering A into a grid covering $G(A)$ as illustrated in Fig. 7.24.

Let $f : G(A) \subseteq \mathbb{R}^2 \to \mathbb{R}$ be a continuous function. The Change of Variables Theorem for double integrals gives conditions under which $\iint_{G(A)} f(x, y)\, \mathrm{d}x\, \mathrm{d}y$ can be expressed as a double integral over A.

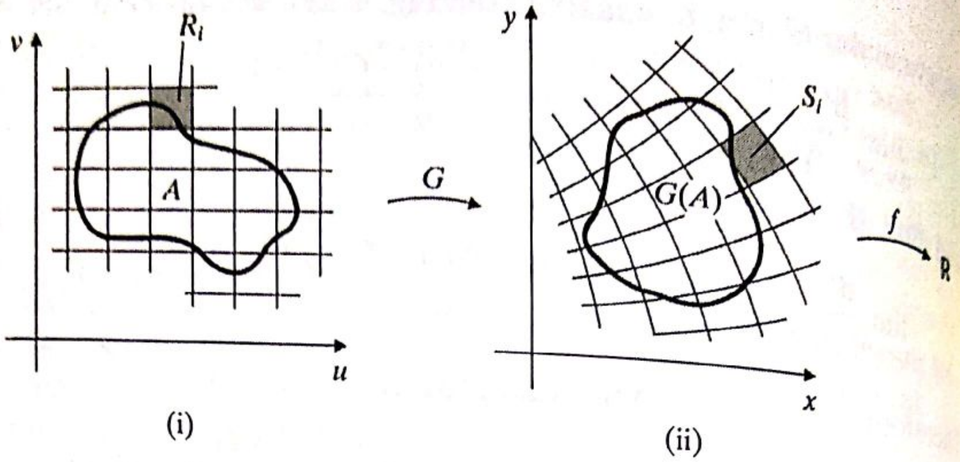


Fig. 7.24 (i) $\mathbb{R}^2: u, v$ plane; (ii) $\mathbb{R}^2: x, y$ plane

We can arrive informally at the appropriate identity by manipulating Riemann sums corresponding to a partition $\mathcal{P}$ of $\mathbb{R}^2$ as follows. Let $R_1, \ldots, R_n$ be the compact rectangles of $\mathcal{P}$ that meet A, and for each $i = 1, \ldots, n$ choose $\mathbf{r}_i \in R_i \cap A$. Let $G(\mathbf{r}_i) = \mathbf{s}_i$ and let $G(R_i) = S_i$. Then we have the following sequence of approximations for the integral of f over $G(A)$.

$$\iint_{G(A)} f(x, y)\, \mathrm{d}x\, \mathrm{d}y \approx \sum_i f(\mathbf{s}_i)(\text{area } S_i)$$

$$\approx \sum_i f(G(\mathbf{r}_i))\, |\det J_{G,\mathbf{r}_i}|\, (\text{area } R_i)$$

$$\approx \iint_A f(G(u, v))\, |\det J_{G,(u,v)}|\, du\, dv.$$

It is surprising how much care is needed in justifying these steps in a formal proof.

7.7.5 Theorem (Change of variables). *Let $G: K \subseteq \mathbb{R}^2 \to \mathbb{R}^2$ be a C^1 function defined on a compact set K in $\mathbb{R}^2$, and let $D \subseteq K$ be an open subset of $\mathbb{R}^2$ such that*

[i] *$K \backslash D$ is a null set,*

[ii] *G is 1–1 on D,*

[iii] $\det J_{G,(u,v)} \neq 0$ *for all $(u, v) \in D$.*

Then, for any bounded function $f: G(K) \subseteq \mathbb{R}^2 \to \mathbb{R}$ which is continuous on $G(D)$

7.7.6 $$\iint_{G(K)} f(x, y)\, \mathrm{d}x\, \mathrm{d}y = \iint_K f(G(u, v))\, |\det J_{G,(u,v)}|\, du\, dv.$$

There are many versions of this theorem. It is often easiest to apply in the form that we have stated it. Notice in particular that G need not be 1–1 on the whole of K.

In traditional terms, Theorem 7.7.5 states that under the change of variables

$$x = G_1(u, v), \qquad y = G_2(u, v)$$

from Cartesian x, y coordinates to curvilinear u, v coordinates, the integral of f over $G(K)$ is transformed according to the rule

$$\iint\limits_{G(K)} f(x, y)\,\mathrm{d}x\,\mathrm{d}y = \iint\limits_{K} \tilde{f}(u, v)\left|\frac{\partial(x, y)}{\partial(u, v)}\right| \mathrm{d}u\,\mathrm{d}v,$$

where $\tilde{f}(u, v) = f(G_1(u, v), G_2(u, v))$, and

$$\frac{\partial(x, y)}{\partial(u, v)} = \det\begin{bmatrix} \dfrac{\partial x}{\partial u}(u, v) & \dfrac{\partial x}{\partial v}(u, v) \\ \dfrac{\partial y}{\partial u}(u, v) & \dfrac{\partial y}{\partial v}(u, v) \end{bmatrix} = \det J_{G,(u,v)}.$$

We end this section with some applications of the theorem and with a brief outline of a method that can be used to prove it. However, a formal proof is beyond the scope of this book.

7.7.7 ***Example.*** Let S be the annulus $S = \{(x, y) \in \mathbb{R}^2 \mid 1 \leqslant x^2 + y^2 \leqslant 4\}$ in $\mathbb{R}^2$. The integral

$$\iint\limits_{S} x^2y^2\,\mathrm{d}x\,\mathrm{d}y$$

can be evaluated as the sum of four integrals over x-simple regions in the x, y plane (Exercise 7.4.11). The following method is much simpler. Consider the transformation to polar coordinates

$$(x, y) = G(r, \theta) = (r\cos\theta, r\sin\theta), \qquad (r, \theta) \in \mathbb{R}^2.$$

Define $K = \{(r, \theta) \in \mathbb{R}^2 \mid 1 \leqslant r \leqslant 2,\ 0 \leqslant \theta \leqslant 2\pi\}$. Then $S = G(K)$. See Fig. 7.25. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be defined by

$$f(x, y) = x^2y^2, \qquad (x, y) \in \mathbb{R}^2.$$

Then the conditions of Theorem 7.7.5 are satisfied (where D is the open rectangle $]1, 2[\times]0, 2[$, and $K \backslash D$ its null boundary). In particular

$$\det J_{G,(r,\theta)} = \frac{\partial(x, y)}{\partial(r, \theta)}(r, \theta) = \det\begin{bmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{bmatrix} = r$$

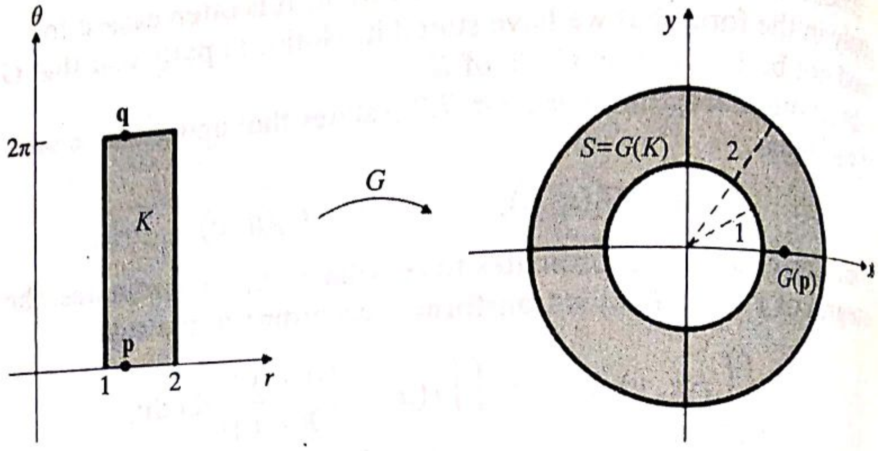


Fig. 7.25 Rectangle K and image under polar transformation $G(r, \theta) = (r \cos \theta, r \sin \theta)$

which is positive for all $(r, \theta) \in D$. Hence, by 7.7.6,

$$\iint_S x^2y^2 \, dx \, dy = \iint_K (r^2 \cos^2 \theta)(r^2 \sin^2 \theta) r \, dr \, d\theta$$

$$= \int_1^2 dr \int_0^2 r^5 \cos^2 \theta \sin^2 \theta \, d\theta.$$

Now

$$\int_0^{2\pi} \cos^2 \theta \sin^2 \theta \, d\theta = \tfrac{1}{4}\int_0^{2\pi} \sin^2 2\theta \, d\theta = \tfrac{1}{4}\pi.$$

Hence

$$\iint_S x^2y^2 \, dx \, dy = \tfrac{1}{4}\pi \int_1^2 r^5 \, dr = \frac{21\pi}{8}.$$

Notice that the transformation G takes the point $\mathbf{p}$ on the boundary of K (Fig. 7.25) to an interior point of $G(K)$. However, it takes points in D, the interior of K, to points in the interior of $G(K)$. The function G is 1–1 on D but not 1–1 on K: the point $\mathbf{q}$ in Fig. 7.25 has the same image as $\mathbf{p}$.

The above example illustrates the following general result.

7.7.8 Theorem. *Let H be the half-plane $H = \{(r, \theta) \in \mathbb{R}^2 \mid r \geqslant 0\}$ and let $G : H \subseteq \mathbb{R}^2 \to \mathbb{R}^2$ be the polar coordinate transformation* *given by*

$$(x, y) = G(r, \theta) = (r\cos\theta, r\sin\theta), \qquad r \geqslant 0.$$

Let D be an open bounded subset of H whose boundary is a null set, and let $K = \bar{D}$. If G is 1–1 on D and if $f: G(K) \subseteq \mathbb{R}^2 \to \mathbb{R}$ is continuous then

7.7.9 $$\iint\limits_{G(K)} f(x, y)\,\mathrm{d}x\,\mathrm{d}y = \iint\limits_{K} f(r\cos\theta, r\sin\theta) r\,\mathrm{d}r\,\mathrm{d}\theta.$$

Proof. The conditions of Theorem 7.7.5 are satisfied.

Theorem 7.7.8 is useful when one is required to integrate a function f over a bounded set S of the x, y plane which is contained within two polar angles $\theta = \alpha$ and $\theta = \beta$ (see Fig. 7.26). The corresponding r, θ integral can then be evaluated as a repeated integral by finding K in the r, θ plane such that $S = G(K)$. With ψ_1

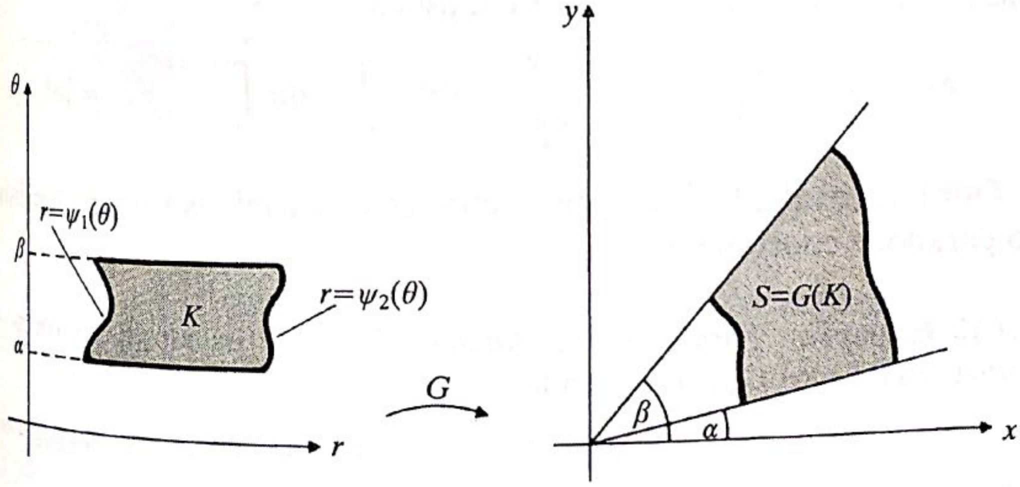


Fig. 7.26 Polar coordinate transformation
$G(r, \theta) = (r\cos\theta, r\sin\theta)$

and ψ_2 as shown in the figure, we then have

$$\iint\limits_{G(K)} f(x, y)\,\mathrm{d}x\,\mathrm{d}y = \int_{\alpha}^{\beta} \mathrm{d}\theta \int_{\psi_1(\theta)}^{\psi_2(\theta)} f(r\cos\theta, r\sin\theta) r\,\mathrm{d}r.$$

7.7.10 *Example.* Find the area in the x, y plane of the set S bounded by the x-axis and the lemniscate which is given in polar coordinates by

7.7.11 $$r^2 = a^2\cos 2\theta, \qquad 0 \leqslant \theta \leqslant \tfrac{1}{4}\pi.$$

The lemniscate consists of the points $(r\cos\theta, r\sin\theta)$ in the x, y plane subject to the condition 7.7.11. See Fig. 7.27.

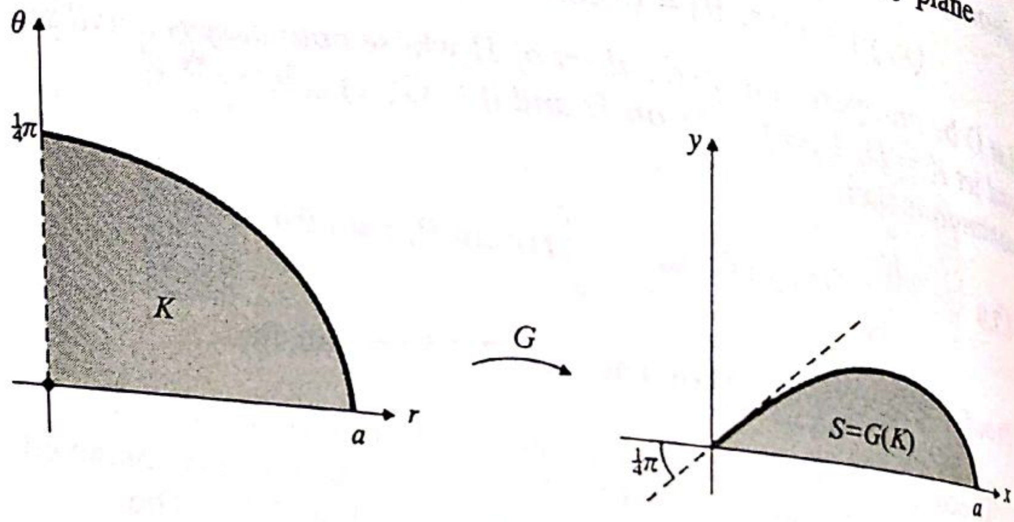


Fig. 7.27

The set S is the image under the polar transformation G of the set K sketched in Fig. 7.27, where K is chosen so that G is 1–1 on its interior D. The conditions of Theorem 7.7.8 are satisfied. Hence

$$\text{Area of } S = \iint_{G(K)} 1 \, \mathrm{d}x \, \mathrm{d}y = \iint_K r \, \mathrm{d}r \, \mathrm{d}\theta = \int_0^{1/4\pi} \mathrm{d}\theta \int_0^{a\sqrt{\cos 2\theta}} r \, \mathrm{d}r = \tfrac{1}{4}a^2.$$

Our final example illustrates a change of variables from Cartesian to parabolic coordinates.

7.7.12 *Example.* *Parabolic Coordinates.* Consider the coordinate transformation $(x, y) = G(u, v)$ given by

$$x = u^2 - v^2, \qquad y = uv.$$

Then

$$\det J_{G(u,v)} = \det \begin{bmatrix} 2u & -2v \\ v & u \end{bmatrix} = 2(u^2 + v^2),$$

which is non-zero except at the origin.

Let U be the subset of the u, v plane consisting of the interior of the rectangle $A(0, 2)$, $B(1, 2)$, $C(1, -2)$, $D(0, -2)$, and let $K = \bar{U}$. See Fig. 7.28. Then G is 1–1 on U and $G(K)$ is closed and bounded by the parabolas

$$x = \tfrac{1}{4}y^2 - 4 \qquad \text{and} \qquad x = 1 - y^2.$$

For example, the edge of K

$$AB = \{(u, 2) \in \mathbb{R}^2 \mid 0 \leqslant u \leqslant 1\}$$

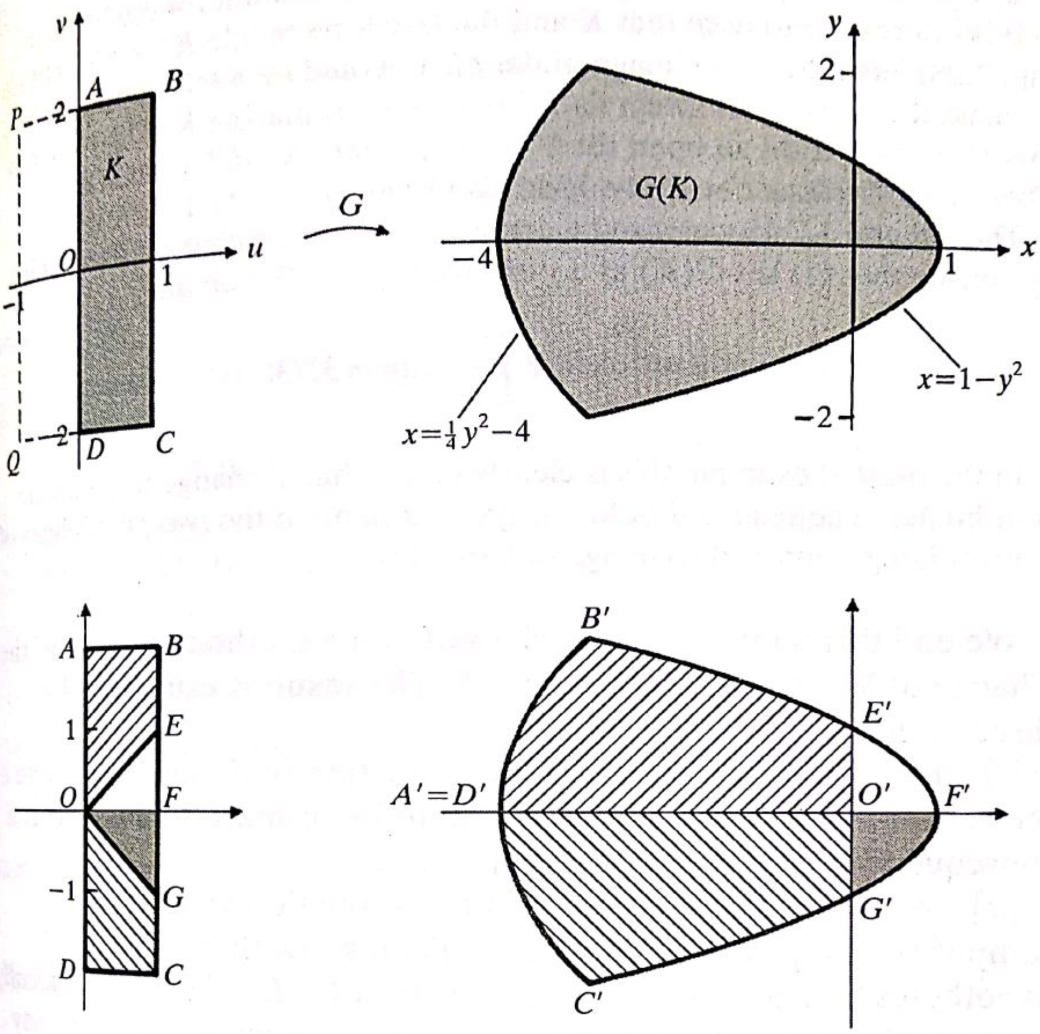


Fig. 7.28 Parabolic coordinate transformation
$G(u, v) = (u^2 - v^2, uv)$

is mapped by G to the boundary curve of $G(K)$

$$A'B' = \{(x, y) \in \mathbb{R}^2 \mid x = u^2 - 4,\ y = 2u,\ 0 \leqslant u \leqslant 1\},$$

and this is a piece of the parabola $x = \frac{1}{4}y^2 - 4$. Note that G is not 1–1 on K, for G takes both OA and OD onto part of the negative x-axis $y = 0$, $0 \geqslant x \geqslant -4$.

Let us apply Theorem 7.7.5 to evaluate the integral

7.7.13

$$I = \iint_{G(K)} y^2 \, dx \, dy.$$

The conditions of the theorem are satisfied. hence, by 7.7.6,

$$I = \iint_K 2u^2v^2(u^2 + v^2)\, du\, dv = 2\int_0^1 du \int_{-2}^{2} (u^4v^2 + u^2v^4)\, dv.$$

The repeated integral is easily evaluated, and we obtain $I = 32/3$.

It is interesting to note that K and the larger rectangle $K^* = PBCQ$ (see Fig. 7.28) have the same image under G. It would be a mistake, however, to claim the equality of integrals 7.7.6 with K^* replacing K. The reason is that G is not 1–1 on an open set $U^* \subseteq K^*$, where $K^* \backslash U^*$ is null. So Theorem 7.7.5 does not apply. (See also Exercise 7.7.12.)

The integral 7.7.13 can also be evaluated without a change of variables by considering the set $G(K)$ as a y-simple region. We obtain

$$I = \int_{-2}^{2} y^2 \, dx \int_{1/4y^2-4}^{1-y^2} dy = 32/3.$$

In the present example this is clearly easier than a change to parabolic coordinates. The reader should always bear in mind the two possibilities of a direct integration and a change of variables.

We end this section with a brief note on a method of proof of the Change of Variables Theorem 7.7.5. The result is established in three stages.

[*i*] First prove the theorem for a function that affects only one of the variables—such a function is said to be *primitive*. This case is a consequence of the single integral result.

[*ii*] Next consider the theorem for a function which is a composition of primitive functions. Then show that under the hypotheses of Theorem 7.7.5 the function $G: D \subseteq \mathbb{R}^2 \to \mathbb{R}^2$ is *locally* a composition of primitive functions, that is, corresponding to any $(u, v) \in D$ there is a neighbourhood U of (u, v) on which G is a composition of primitive functions.

[*iii*] Finally prove Theorem 7.7.5 by finding open sets $U_1, \ldots, U_n$ in D which 'almost' cover K and on each of which G is the composition of primitive functions. The integral on the left hand side of 7.7.6 is approximated by a sum of integrals taken over $G(U_i)$. To achieve this reduction one uses the important concept of a *partition of unity*.

This is the approach given, for example, in Rudin, *Principles of Mathematical Analysis* and in Spivak, *Calculus on Manifolds*.

Exercises 7.7

1. Consider the coordinate transformation $(x, y) = G(u, v)$ given by

$$G(u, v) = (au, bv), \qquad (u, v) \in \mathbb{R}^2$$

where a and b are positive constants.

(a) Prove that G is 1–1 on $\mathbb{R}^2$.
(b) Show that G maps the circular disc $K = \{(u, v) \in \mathbb{R}^2 \mid u^2 + v^2 \leqslant 1\}$ in the u, v plane onto the elliptical disc $G(K) = \{(x, y) \in \mathbb{R}^2 \mid x^2/a^2 + y^2/b^2 \leqslant 1\}$ in the x, y plane.
(c) Calculate $\det J_{G,(u,v)}$.
(d) Calculate, using Theorem 7.7.5, the area of the elliptical disc $G(K)$. (Assume known that the area of K is π.)

Answer: πab.

2. Consider the parabolic coordinate transformation $(x, y) = G(u, v)$, given by

$$x = u^2 - v^2, \qquad y = 2uv \qquad (u, v) \in \mathbb{R}^2.$$

Let k be the compact square in the u, v plane with corners $A(0, 0)$, $B(0, 2)$, $C(2, 2)$, $D(2, 0)$.
(a) Prove that G is 1–1 on K.
(b) Sketch the set $S = G(K)$ in the x, y plane.
(c) Compute the area of S
  (i) by integrating $\iint_S 1 \, dx \, dy$ directly over S;
  (ii) by applying the Change of Variables Theorem 7.7.5.

Answer: 128/3.

3. Three pairs (x, y), (u, v), (w, s) of coordinates of the plane are related by

$$(x, y) = G(u, v) \qquad \text{and} \qquad (u, v) = H(w, s),$$

where $H : D \subseteq \mathbb{R}^2 \to \mathbb{R}^2$ and $G : H(D) \subseteq \mathbb{R}^2 \to \mathbb{R}^2$ are 1–1 C^1 functions defined on open sets D and $H(D)$. Obtain the Chain Rule

$$\frac{\partial(x, y)}{\partial(w, s)}(\mathbf{p}) = \frac{\partial(x, y)}{\partial(u, v)}(H(\mathbf{p}))\frac{\partial(u, v)}{\partial(w, s)}(\mathbf{p}) \qquad \mathbf{p} \in D.$$

In particular, if the function H is inverse to G then the coordinates (x, y) and (u, v) are functionally related by

$$(x, y) = G(u, v) \qquad \text{and} \qquad (u, v) = H(x, y).$$

Deduce in this case that

7.7.14 $$\frac{\partial(x, y)}{\partial(u, v)}(H(\mathbf{p}))\frac{\partial(u, v)}{\partial(x, y)}(\mathbf{p}) = 1, \qquad \mathbf{p} \in D.$$

The rule 7.7.14 is often presented in the abbreviated form

$$\frac{\partial(x, y)}{\partial(u, v)}\frac{\partial(u, v)}{\partial(x, y)} = 1.$$

4. The *hyperbolic coordinates* u, v are implicitly defined in the x, y half-plane $\{(x, y) \in \mathbb{R}^2 \mid x > 0\}$ by

$$u = x^2 - y^2, \qquad v = 2xy.$$

Justify the name by sketching a selection of curves $u = \text{constant}$ and $v = \text{constant}$.

Prove that

$$\frac{\partial(x, y)}{\partial(u, v)}(u, v) = \frac{1}{4\sqrt{(u^2 + v^2)}}.$$

Hint: consider $\partial(u, v)/\partial(x, y)$ and apply the result of Exercise 3.

5. Sketch the set S in the first quadrant of the x, y plane bounded by the curves $x^2 - y^2 = 1$, $x^2 - y^2 = 9$, $xy = 2$, and $xy = 4$. Compute

$$\iint_S (x^2 + y^2)\, dx\, dy$$

by changing to hyperbolic coordinates, given implicitly by

$$x^2 - y^2 = u, \qquad 2xy = v,$$

and applying Theorem 7.7.5.

Hint: use the result of Exercise 4. The alternative method of subdividing S into x-simple regions and applying Theorem 7.4.21 leads to complicated calculations.

Answer: 8.

6. Show that

$$\iint_S x^2 y^2\, dx\, dy = 1/360$$

where S is the triangular region bounded by the lines $y = 0$, $y = x$ and $y = 1 - x$

(a) by computing a repeated integral;
(b) by the change of variables $x = u + v$, $y = u - v$ and then computing a repeated integral. (In the second case the repeated integral takes the simple form $\int_0^{1/2} dv \int_0^v f(u, v)\, du$.)

7. Evaluate $\displaystyle\iint_S \frac{1}{\sqrt{(x^2 + y^2)}}\, dx\, dy$ where S is the annulus $\{(x, y) \in \mathbb{R}^2 \mid a^2 \leqslant x^2 + y^2 \leqslant b^2\}$ by transforming to polar coordinates.

Answer: $2\pi(b - a)$.

8. Find the area in the first quadrant of the x, y plane bounded by the

x-axis, the curve $r = a(1 + \cos\theta)$ (given in polar coordinates) and the circle $r = a$.

Answer: $a^2(\pi + 8)/8$.

9. Compute $\iint\limits_S x \, dx \, dy$, where S is the region defined by $x \geqslant 0$, $y \geqslant 0$, $x^2 + y^2 \geqslant 1$ and $x^2 + y^2 \leqslant 4$. (See Fig. 7.11(ii) where S is sketched.) What are the coordinates of the centroid of S?

Answers: 7/3; (28/9, 28/9). Transform to polar coordinates.

10. Find the area bounded by the lemniscate $(x^2 + y^2)^2 = 2a^2(x^2 - y^2)$ by changing to polar coordinates.

Answer: $2a^2$.

11. (a) Compute the integral

$$\iint\limits_{S(a)} e^{-(x^2+y^2)} \, dx \, dy,$$

where $S(a)$ is the circular disc bounded by the circle $x^2 + y^2 = a^2$. Show that

$$\lim_{a\to\infty} \iint\limits_{S(a)} e^{-(x^2+y^2)} \, dx \, dy = \pi.$$

(b) Let $T(b)$, $b > 0$, be the square in $\mathbb{R}^2$ with corners $(\pm b, \pm b)$. Show that

$$\iint\limits_{T(b)} e^{-(x^2+y^2)} \, dx \, dy = \left(\int_{-b}^{b} e^{-t^2} \, dt\right)^2$$

Hint: apply theorem 7.3.18.

(c) Prove that

$$\lim_{a\to\infty} \iint\limits_{S(a)} e^{-(x^2+y^2)} \, dx \, dy = \lim_{b\to\infty} \iint\limits_{T(b)} e^{-(x^2+y^2)} \, dx \, dy.$$

Deduce that

$$\int_{-\infty}^{\infty} e^{-t^2} \, dt = \sqrt{\pi}.$$

12. Compute the integral of $2u^2v^2(u^2 + v^2)$ over the rectangle $K^* = PBCQ$ described in Example 7.7.12. Compare your result with that of Example 7.7.12.

Answer: 64/3.

7.8 An application. Orientation of simple closed curves

Let D be a simply connected open set in $\mathbb{R}^2$. The Change of Variables Theorem enables us to explore further the significance of the Jacobian of a 1–1 C^1 function $G: D \subseteq \mathbb{R}^2 \to \mathbb{R}^2$. Let C^+ be a counterclockwise oriented simple closed curve in D, and let $\alpha: [a, b] \subseteq \mathbb{R} \to \mathbb{R}^2$ be a simple parametrization of C^+. Then $G \circ \alpha$ is a simple parametrization of the simple closed curve $G(C)$. We shall show that *$G \circ \alpha$ traces $G(C)$ counterclockwise or clockwise whenever $\det J_{G,(u,v)}$ is respectively positive or negative for all $(u, v) \in D$.* Hence the sign of $\det J_{G,(u,v)}$ determines whether G preserves or reverses the orientation of closed curves in D.

Let T be the region interior to C (so $\partial T = C$), and let S be the region interior to $G(C)$. Suppose that $\det J_{G,(u,v)} > 0$ for all $(u, v) \in D$. By the Change of Variables Theorem,

$$\text{Area } S = \iint_S 1 \,\mathrm{d}x\,\mathrm{d}y = \iint_T \det J_{G,(u,v)} \,\mathrm{d}u\,\mathrm{d}v$$

$$= \iint_T \left(\frac{\partial G_1}{\partial u}\frac{\partial G_2}{\partial v} - \frac{\partial G_1}{\partial v}\frac{\partial G_2}{\partial u} \right) \mathrm{d}A$$

$$= \iint_T \left(\frac{\partial}{\partial u}\left(G_1 \frac{\partial G_2}{\partial v} \right) - \frac{\partial}{\partial v}\left(G_1 \frac{\partial G_2}{\partial u} \right) \right) \mathrm{d}A$$

$$= \int_{C^+} G_1 \frac{\partial G_2}{\partial u}\,\mathrm{d}u + G_1 \frac{\partial G_2}{\partial v}\,\mathrm{d}v \qquad \text{by Green's Theorem}$$

$$= \int_{C^+} G_1 \operatorname{grad} G_2 \cdot \mathrm{d}\mathbf{r}$$

$$= \int_a^b G_1(\alpha(t))(\operatorname{grad} G_2)(\alpha(t)) \cdot \alpha'(t)\,\mathrm{d}t, \qquad \text{since } \alpha \text{ parametrizes } C^+$$

$$= \int_a^b (G_1 \circ \alpha)(t)(G_2 \circ \alpha)'(t)\,\mathrm{d}t \qquad \text{by Theorem 3.8.7}$$

$$= \int_a^b \beta_1(t)\beta_2'(t)\,\mathrm{d}t \qquad \text{where } \beta = G \circ \alpha$$

$$= \int_\beta (0, x) \cdot \mathrm{d}\beta$$

$$= \begin{cases} \text{Area } S & \text{if } G(C) \text{ is traced counterclockwise by } G \circ \alpha, \\ -\text{Area } S & \text{if } G(C) \text{ is traced clockwise by } G \circ \alpha \end{cases}$$

by 7.5.18. Therefore $G(C)$ is traced counterclockwise by $G \circ \alpha$. Similarly a negative determinant $\det J_{G,(u,v)}$ for all $(u, v) \in D$ implies that G reverses orientations.

Exercises 7.8

1. Let a and b be non-zero constants. Verify that the function $G: \mathbb{R}^2 \to \mathbb{R}^2$ defined by

$$G(u, v) = (au, bv) \qquad (u, v) \in \mathbb{R}^2$$

is 1–1 and C^1 on $\mathbb{R}^2$, and that

$$\det J_{G,(u,v)} = ab \qquad \text{for all } (u, v) \in \mathbb{R}^2.$$

Let C^+ be the counterclockwise oriented unit circle centred at the origin of $\mathbb{R}^2$. Sketch the simple closed curve $G(C)$, and find the orientation of $G(C^+)$ for the cases (a) $a = 2$, $b = 2$; (b) $a = 2$, $b = -1$.

Answers: (a) circle $x^2 + y^2 = 4$ oriented counterclockwise; (b) ellipse $\frac{1}{4}x^2 + y^2 = 1$ oriented clockwise.

2. The transformation of the plane

$$G(u, v) = \left(\frac{u}{u^2 + v^2}, \frac{v}{u^2 + v^2}\right), \qquad (u, v) \neq (0, 0)$$

is called an *inversion* of the plane with respect to the unit circle (centre the origin). Note that in polar coordinates the point $(r \cos \theta, r \sin \theta)$, $r \neq 0$, is mapped to $\left(\frac{1}{r}\cos \theta, \frac{1}{r}\sin \theta\right)$. In particular, points on the unit circle are fixed, points inside the unit circle are mapped to points outside the unit circle and vice versa.

(a) Show that G transforms a circle not passing through the origin into a circle.

Hint: put $x = u/(u^2 + v^2)$, $y = v/(u^2 + v^2)$. Then $u = x/(x^2 + y^2)$, $v = y/(x^2 + y^2)$. Now consider the image under G of the circle $(u - a)^2 + (v - b)^2 = c^2$.

(b) Prove that

$$\det J_{G,(u,v)} = -\frac{1}{(u^2 + v^2)^2} < 0, \qquad (u, v) \neq (0, 0).$$

(c) Sketch the circle C with equation $(u - 2)^2 + v^2 = 1$ and its image

$G(C)$. Verify that C with counterclockwise orientation is mapped to $G(C)$ with clockwise orientation.

Hint: parametrize C by $\alpha(t) = (2 + \cos t, \sin t)$, $t \in [0, 2\pi]$, for example.

(d) Show that G transforms the counterclockwise oriented circle A^+ of radius a, centre the origin, into the counterclockwise oriented circle $G(A^+)$ of radius $1/a$, centre the origin. Why does this not contradict the results of this section?

Answer: The region of $\mathbb{R}^2$ interior to A contains the origin, which lies outside the domain of G. Note that $\mathbb{R}^2\backslash\{(0, 0)\}$ is not simply connected.