5.1 Introduction

The first type of integral we shall study is the integral of a real-valued function $f:S\subseteq\mathbb{R}^n\to\mathbb{R}$ (scalar field) along a path $\alpha:[a, b]\subseteq\mathbb{R}\to\mathbb{R}^n$. We use lower-case Greek letters to describe paths of integration in $\mathbb{R}^n$, thus distinguishing them clearly from the function f to be integrated.

We begin by reviewing the definition of the (Riemann) integral of a bounded function $f:[a, b]\subseteq\mathbb{R}\to\mathbb{R}$ over the compact interval $[a, b]$. The definition involves partitions of the interval $[a, b]$ into subintervals. Let one such partition $\mathcal{P}$ be given by $a=x_0<x_1<\cdots<x_k=b$. We associate with $\mathcal{P}$ a *Riemann sum* of the form

5.1.1 $$R(f, \mathcal{P})=\sum_{i=1}^{k} f(p_i)(x_i-x_{i-1})$$

where p_i is any point chosen in the ith subinterval $[x_{i-1}, x_i]$ of $\mathcal{P}$.

Denoting the infimum and the supremum of f on the subinterval $[x_{i-1}, x_i]$ by m_i and M_i respectively and defining the *lower* and *upper Riemann sums* of f corresponding to the partititon $\mathcal{P}$ by

5.1.2

$$L(f, \mathcal{P})=\sum_{i=1}^{k} m_i(x_i-x_{i-1}) \qquad \text{and} \qquad U(f, \mathcal{P})=\sum_{i=1}^{k} M_i(x_i-x_{i-1}),$$

we have the inequality

5.1.3 $$L(f, \mathcal{P})\leqslant\sum_{i=1}^{k} f(p_i)(x_i-x_{i-1})\leqslant U(f, \mathcal{P}).$$

It follows (see Exercise 5.1.4) that for any partitions $\mathcal{P}$ and $\mathcal{Q}$ of $[a, b]$

$$L(f, \mathcal{P})\leqslant U(f, \mathcal{Q}).$$

Let $L(f)$ and $U(f)$ denote respectively the supremum of the set of lower sums $L(f, \mathcal{P})$ and the infimum of the set of upper sums

$U(f, \mathcal{P})$. Then for any partitions $\mathcal{P}$ and $\mathcal{Q}$ of $[a, b]$,

$$L(f, \mathcal{P}) \leqslant L(f) \leqslant U(f) \leqslant U(f, \mathcal{Q}).$$

5.1.4

5.1.5 *Definition.* *The function* $f:[a, b] \subseteq \mathbb{R} \to \mathbb{R}$ *is said to be* (Riemann) integrable over $[a, b]$ *if* $L(f) = U(f)$. *The common value is then called the* (definite) integral of f over $[a, b]$ *and is denoted by* $\int_a^b f(x)\,dx$.

Notice that f is integrable over $[a, b]$ is and only if to each $\varepsilon > 0$ there correspond partitions $\mathcal{P}$ and $\mathcal{Q}$ of $[a, b]$ such that

$$U(f, \mathcal{Q}) - L(f, \mathcal{Q}) < \varepsilon.$$

5.1.6 *Example.* Let $f:[0, b] \subseteq \mathbb{R} \to \mathbb{R}$ be given by $f(x) = x^3$ for all $x \in [0, b]$. Consider the partition $\mathcal{P}$ of $[0, b]$ given, for some $k \in \mathbb{N}$, by

$$0 < \frac{b}{k} < \frac{2b}{k} < \cdots < \frac{(k-1)b}{k} < \frac{kb}{k} = b.$$

Then the ith subinterval in $\mathcal{P}$ is $[(i-1)b/k, ib/k]$ which has length b/k. Since f is monotonic increasing we have, with the above notation, for each $i = 1, \ldots, k$,

$$m_i = \left(\frac{(i-1)b}{k}\right)^3 \quad \text{and} \quad M_i = \left(\frac{ib}{k}\right)^3.$$

Therefore

$$L(f, \mathcal{P}) = \sum_{i=1}^{k} \frac{(i-1)^3 b^3}{k^3} \cdot \frac{b}{k} = \frac{b^4}{k^4} \sum_{i=1}^{k} (i-1)^3$$

and

$$U(f, \mathcal{P}) = \sum_{i=1}^{k} \frac{i^3 b^3}{k^3} \cdot \frac{b}{k} = \frac{b^4}{k^4} \sum_{i=1}^{k} i^3$$

It follows that

$$U(f, \mathcal{P}) - L(f, \mathcal{P}) = \frac{b^4}{k}.$$

Since this can be made arbitrarily small by taking k large enough, the function f is integrable over $[0, b]$.

Applying the well known formula

$$1^3 + 2^3 + \cdots + n^3 = \tfrac{1}{2} n^2 (n+1)^2$$

we find that

$$L(f, \mathcal{P}) = \tfrac{1}{4}b^4\left(\frac{k-1}{k}\right)^2 \quad \text{and} \quad U(f, \mathcal{P}) = \tfrac{1}{4}b^4\left(\frac{k+1}{k}\right)^2.$$

Hence $L(f) \geqslant \frac{1}{4}b^4$ and $U(f) \leqslant \frac{1}{4}b^4$. Therefore the value of the integral $\int_0^b x^3\, \mathrm{d}x$ is $\frac{1}{4}b^4$.

There are bounded functions $f:[a, b] \subseteq \mathbb{R} \to \mathbb{R}$ which are not integrable (see Exercise 5.1.7). However, all *continuous* functions are integrable and, in particular (the Fundamental Theorem), if $f:[a, b] \subseteq \mathbb{R} \to \mathbb{R}$ is a differentiable function whose derivative f' is continuous then

5.1.7
$$\int_a^b f'(x)\, \mathrm{d}x = f(b) - f(a).$$

Taking $f(x) = \frac{1}{4}x^4$ and $a = 0$ in 5.1.7 leads to the integral evaluated in Example 5.1.4.

There is an alternative definition of the integrability of the function $f:[a, b] \subseteq \mathbb{R} \to \mathbb{R}$ in terms of the mesh of partitions of $[a, b]$.

5.1.8 *Definition.* *Let $\mathcal{P}$ be a partition of $[a, b]$ given by $a = x_0 < x_1 < \cdots < x_k = b$. The* mesh *of $\mathcal{P}$ is the length of the longest subinterval in $\mathcal{P}$. It is defined and denoted by*

$$\mu(\mathcal{P}) = \max\{(x_i - x_{i-1}) \mid i = 1, \ldots, k\}.$$

The following theorem tells us in what sense the integral of a function is the limit of the approximating Riemann sums $R(f, \mathcal{P})$ as the mesh of $\mathcal{P}$ tends to 0. Remember that the value of $R(f, \mathcal{P})$ depends upon both the partition $\mathcal{P}$ and the choice of the points p_i in the subintervals of $\mathcal{P}$.

5.1.9 *Theorem.* *A bounded function $f:[a, b] \subseteq \mathbb{R} \leftarrow \mathbb{R}$ is integrable with integral $I = \int_a^b f(x)\, \mathrm{d}x$ if and only if, whenever $\mathcal{P}_1, \mathcal{P}_2, \mathcal{P}_3, \ldots$ is a sequence of partitions of $[a, b]$ whose mesh tends to 0, $R(f, \mathcal{P}_n)$ tends to I, no matter what points are chosen in the subintervals of the partitions $\mathcal{P}_n$ in evaluating the Riemann sums $R(f, \mathcal{P}_n)$.*

5.1.10 *Example.* Consider a straight rod AB of uniform cross sectional area 1 cm^2 and of length b cm. Given that the density of the material of the rod at a point x cm from A is x^3 gm/cm^3, find the mass of the rod.

Consider a partition $\mathcal{P}$ of the rod into a large number of short sections with partition points at $0 = x_0 < x_1 < \cdots < x_k = b$ centimetres from A. For each $i = 1, \ldots, k$ choose a point p_i cm from A where $p_i \in [x_{i-1}, x_i]$. The length of the ith subsection is $(x_i - x_{i-1})$ cm and the density of it is approximately p_i^3 gm/cm^3. The mass of this section is therefore approximately $p_i^3(x_i - x_{i-1})$ gm. The total mass of the rod is correspondingly approximated by

$$\sum_{i=1}^{k} p_i^3(x_i - x_{i-1}) \text{ gm.}$$

5.1.11

This approximation tends to the true mass of the rod as the mesh of the partition $\mathcal{P}$ of the rod tends to 0. But expression 5.1.11 is a Riemann sum, $R(f, \mathcal{P})$, of the function $f(x) = x^3$. Hence by Example 5.1.4, the mass of the rod is $\frac{1}{4}b^4$ gm.

We have spelt out Example 5.1.10 in some detail in order to emphasise that the Riemann sum 5.1.1 is found by first finding the length of each subinterval, second 'weighting' each length by the value of the function at some point on it, and finally adding up these weighted lengths. The integral $\int_a^b f(x)\,dx$ to which the sum approximates is, as the notation suggests, a continuous form of this process. Of course in general, since the function will take both positive and negative values, the weights may be positive or negative.

The evaluation of the mass of the rod in Example 5.1.10 does not depend upon the rod being straight. In later sections we shall extend the definition of the Riemann integral to the integral of real valued functions over curves in $\mathbb{R}^n$ and of vector valued functions along oriented (directed) simple arcs in $\mathbb{R}^n$. But first we define the integral of a real-valued function along a path as a natural extension of the integral considered in this section.

Exercises 5.1

1. Let $f: [0, 1] \subseteq \mathbb{R} \to \mathbb{R}$ be the continuous function defined by $f(x) = x^2$, $x \in [0, 1]$. Let $\mathcal{P}$ be the partition $0 = x_0 < x_1 < \cdots < x_{k-1} < x_k = 1$ of $[0, 1]$ into k equal pieces; so $x_i = i/k$, $i = 0, \ldots, k$. Show that

$$L(f, \mathcal{P}) = \sum_{i=1}^{k} \frac{(i-1)^2}{k^3} \quad \text{and} \quad U(f, \mathcal{P}) = \sum_{i=1}^{k} \frac{i^2}{k^3}.$$

Deduce that f is integrable over $[0, 1]$, and that $\int_0^1 f(x)\,dx = \frac{1}{3}$.

Note: $\sum_{i=1}^{k} i^2 = \frac{1}{6}k(k+1)(2k+1)$.

2. Prove similarly from first principles that $\int_0^b x\,dx = \frac{1}{2}b^2$, and that $\int_a^b x^2\,dx = \frac{1}{3}(b^3 - a^3)$.

3. Let $\mathscr{P}$ and $\mathscr{P}^*$ be partitions of $[a, b]$ given respectively by $a = x_0 < x_1 < \cdots < x_{k-1} < x_k = b$ and $a = x_0^* < x_1^* < \cdots < x_{m-1}^* < x_m^* = b$. The partition $\mathscr{P}^*$ is said to be a *refinement* of $\mathscr{P}$ if each point x_i, $i = 1, \ldots, k-1$, occurs among the points $x_1^*, \ldots, x_{m-1}^*$. Prove that if $\mathscr{P}^*$ is a refinement of the partition $\mathscr{P}$ of $[a, b]$ then for any bounded function $f : [a, b] \subseteq \mathbb{R} \to \mathbb{R}$,

$$L(f, \mathscr{P}) \leqslant L(f, \mathscr{P}^*) \leqslant U(f, \mathscr{P}^*) \leqslant U(f, \mathscr{P}).$$

4. Let $f : [a, b] \subseteq \mathbb{R} \to \mathbb{R}$ be a bounded function. Prove that for any partitions $\mathscr{P}$ and $\mathscr{Q}$ of $[a, b]$,

$$L(f, \mathscr{P}) \leqslant U(f, \mathscr{Q}).$$

Hint: consider a refinement $\mathscr{P}^*$ of both $\mathscr{P}$ and $\mathscr{Q}$ which contains all the points of $\mathscr{P}$ and of $\mathscr{Q}$.

5. Prove that a bounded function $f : [a, b] \subseteq \mathbb{R} \to \mathbb{R}$ is Riemann integrable over $[a, b]$ if and only if to any $\varepsilon > 0$ there corresponds a partition $\mathscr{P}$ of $[a, b]$ such that $U(f, \mathscr{P}) - L(f, \mathscr{P}) < \varepsilon$.

Hint: consider a refinement of the partitions $\mathscr{P}$ and $\mathscr{Q}$ referred to in the text.

6. Prove from first principles that a bounded monotonic function $f : [a, b] \subseteq \mathbb{R} \to \mathbb{R}$ is integrable over $[a, b]$.

Hint: let $\mathscr{P}$ be the partition $a = x_0 < x_1 < \cdots < x_k = b$ of $[a, b]$ into k equal pieces. Prove that

$$U(f, \mathscr{P}) - L(f, \mathscr{P}) = \frac{|f(b) - f(a)|}{k}.$$

7. Let $f : [0, 1] \subseteq \mathbb{R} \to \mathbb{R}$ be the bounded function defined by

$$f(x) = \begin{cases} 0 & \text{when } x \text{ is rational,} \\ 1 & \text{when } x \text{ is irrational.} \end{cases}$$

(a) Prove that f is not integrable over $[0, 1]$. Show in particular that $L(f) = 0$ and $U(f) = 1$.

Note: an interval $[p, q]$ where $p < q$ has both rational and irrational points.

(b) Show that for any sequence $\mathscr{P}_1, \mathscr{P}_2, \mathscr{P}_3, \ldots$ of partitions of $[0, 1]$ such that $\mu(\mathscr{P}_n) \to 0$ there are choices of points in the subintervals such that $R(f, \mathscr{P}_n) \to 0$. Why does this not contradict Theorem 5.1.9?

5.2 Integral of a scalar field along a path

The integral of a real-valued function (scalar field) $f: S \subseteq \mathbb{R}^n \to \mathbb{R}$ along a path $\alpha: [a, b] \subseteq \mathbb{R} \to \mathbb{R}^n$ in S is defined in much the same way as the integral of a real-valued function over an interval, which was reviewed in Section 5.1. In that case the starting point is the Riemann sum 5.1.1 where the length $(x_i - x_{i-1})$ of each subinterval is weighted by multiplying it by the value $f(p_i)$ of the function at some point p_i within it. Similarly, integrating a real-valued function $f: S \subseteq \mathbb{R}^n \to \mathbb{R}$ along a path $\alpha: [a, b] \subseteq \mathbb{R} \to \mathbb{R}^n$ in S is a process of accumulating weighted lengths of path as $\alpha(t)$ traces, and perhaps retraces, its image.

5.2.1 ***Example.*** Imagine a fish swimming, open mouthed, through the sea gathering plankton as it goes. The total mass of plankton that the fish gathers in a given time interval depends upon the path α which describes its motion and the density f of plankton at each point it passes through. In a short time interval the mass gathered is approximately the distance covered multiplied by the linear density of plankton at any one point passed through in that short time. The mass gathered along the whole path α can be approximated by finding a partition of the time interval and adding up all the local approximations. See Fig. 5.1. By considering partitions of arbitrarily small mesh we obtain in the limit the true mass of plankton gathered. This is the integral of the density function f along the path α.

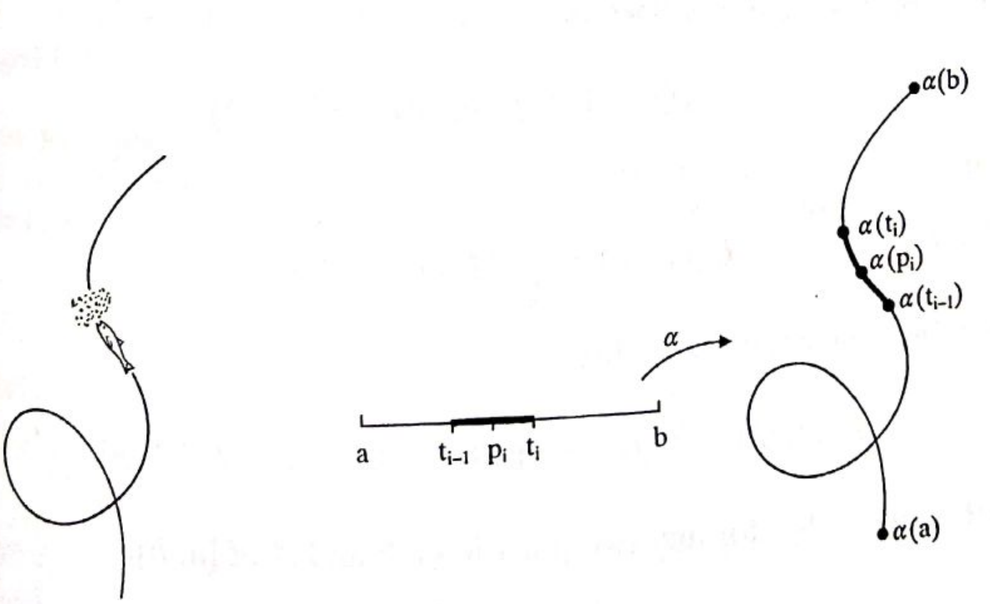

Fig. 5.1

We now establish a formal definition of the path integral of a real valued function which was illustrated in the above example.

Let $\alpha:[a, b] \subseteq \mathbb{R} \to \mathbb{R}^n$ be a C^1 path in $\mathbb{R}^n$. Any partition $a = t_0 < t_1 < \cdots < t_k = b$ of $[a, b]$, call it $\mathscr{P}$, leads to a sequence of points $\alpha(t_0), \alpha(t_1), \ldots, \alpha(t_k)$ which α passes through. (See Fig. 5.1.)

Let $\lambda(t)$ be the length of the path $\alpha:[a, t] \subseteq \mathbb{R} \to \mathbb{R}^n$. Then the distance covered in following the path from $\alpha(t_{i-1})$ to $\alpha(t_i)$ is $\lambda(t_i) - \lambda(t_{i-1})$. Choose $p_i \in [t_{i-1}, t_i]$ for each $i = 1, \ldots, k$.

Suppose now that $f: S \subseteq \mathbb{R}^n \to \mathbb{R}$ is a function which is bounded on the image of the path α. We consider the (weighted) sum

5.2.2 $$R_\alpha(f, \mathscr{P}) = \sum_{i=1}^{k} f(\alpha(p_i))(\lambda(t_i) - \lambda(t_{i-1})).$$

Notice the similarlity between 5.2.2 and the Riemann sum 5.1.1. The expression 5.2.2 is a typical *Riemann sum of the function f along the path* α.

By our hypothesis, on the function f the function $f \circ \alpha:[a, b] \subseteq \mathbb{R} \to \mathbb{R}$ is bounded. Denote the infimum and the supremum of $f \circ \alpha$ on the subinterval $[t_{i-1}, t_i]$ by m_i and M_i respectively. So

$$m_i = \inf \{f(\alpha(t)) \in \mathbb{R} \mid t_{i-1} \leqslant t \leqslant t_i\}$$

and

$$M_i = \sup \{f(\alpha(t)) \in \mathbb{R} \mid t_{i-1} \leqslant t \leqslant t_i\}.$$

We define the *lower* and *upper Riemann sums of f along* α *corresponding to the partition* $\mathscr{P}$ *of* $[a, b]$ by

5.2.3 $$L_\alpha(f, \mathscr{P}) = \sum_{i=1}^{k} m_i(\lambda(t_i) - \lambda(t_{i-1}))$$

and

5.2.4 $$U_\alpha(f, \mathscr{P}) = \sum_{i=1}^{k} M_i(\lambda(t_i) - \lambda(t_{i-1})).$$

We then have the inequality

$$L_\alpha(f, \mathscr{P}) \leqslant \sum_{i=1}^{k} f(\alpha(p_i))(\lambda(t_i) - \lambda(t_{i-1})) \leqslant U_\alpha(f, \mathscr{P}).$$

More generally, for any two partitions $\mathscr{P}$ and $\mathscr{Q}$ of $[a, b]$,

$$L_\alpha(f, \mathscr{P}) \leqslant U_\alpha(f, \mathscr{Q}).$$

This is proved, as in the elementary theory, by considering refinements of partitions. Define

5.2.5
$$L_\alpha(f) = \sup\{L(f, \mathscr{P}) \mid \mathscr{P} \text{ a partition of } [a, b]\}$$

and

5.2.6
$$U_\alpha(f) = \inf\{U(f, \mathscr{P}) \mid \mathscr{P} \text{ a partition of } [a, b]\}.$$

Then, for any partitions $\mathscr{P}$ and $\mathscr{Q}$ of $[a, b]$,

5.2.7
$$L_\alpha(f, \mathscr{P}) \leqslant L_\alpha(f) \leqslant U_\alpha(f) \leqslant U_\alpha(f, \mathscr{Q}).$$

The function f is said to be (*Riemann*) *integrable along the path* α if $L_\alpha(f) = U_\alpha(f)$. The common value is then called the *integral of* f *along* α and is denoted by

5.2.8
$$\int_\alpha f \, \mathrm{d}s.$$

It follows from 5.2.7 that f is integrable along α if and only if to each $\varepsilon > 0$ there correspond partitions $\mathscr{P}$ and $\mathscr{Q}$ of $[a, b]$ such that

5.2.9
$$U_\alpha(f, \mathscr{P}) - L_\alpha(f, \mathscr{Q}) < \varepsilon.$$

In the notation 5.2.8, the quantity $\mathrm{d}s$ suggests an element of path-length if we describe the path-length function by $s = \lambda(t)$, as in Section 2.7.

5.2.10 *Example.* In Example 5.2.1, if the fish traverses a path $\alpha:[a, b] \subseteq \mathbb{R} \to \mathbb{R}^3$ and if the linear density of plankton is given by a function $f:\mathbb{R}^3 \to \mathbb{R}$ then the total mass of plankton gathered is the path integral $\int_\alpha f \, \mathrm{d}s$.

5.2.11 *Example.* Let $\alpha:[-1, 1] \subseteq \mathbb{R} \to \mathbb{R}^2$ be a C^1 path given by $\alpha(t) = (\cos 2\pi t, \sin 2\pi t)$, a path which passes twice around the unit circle in $\mathbb{R}^2$ in a counterclockwise direction. Then, with the above notation

$$\lambda(t) = \int_{-1}^{t} \|\alpha'(t)\| \, \mathrm{d}t = \int_{-1}^{t} 2\pi \, \mathrm{d}t = 2\pi(1 + t).$$

Let $f:\mathbb{R}^2 \to \mathbb{R}$ be given by $f(x, y) = x^2$. Then f is bounded on the image of α (unit circle). Consider a partition $\mathscr{P}$ of $[-1, 1]$, say $-1 = t_0 < t_1 < \cdots < t_k = 1$. Taking $p_i \in [t_{i-1}, t_i]$ for each $i = 1, \ldots, k$, form the Riemann sum

$$\begin{aligned} R_\alpha(f, \mathscr{P}) &= \sum_{i=1}^{k} f(\alpha(p_i))(\lambda(t_i) - \lambda(t_{i-1})) \\ &= \sum_{i=1}^{k} (\cos^2 2\pi p_i) 2\pi(t_i - t_{i-1}). \end{aligned}$$

But this gives the Riemann sums of the function $2\pi \cos^2 2\pi t$ over the interval $[-1, 1]$. Since

$$\int_{-1}^{1} 2\pi(\cos^2 2\pi t)\, dt = 2\pi,$$

therefore

$$\int_\alpha f\, ds = 2\pi.$$

The path integral along a path $\alpha : [a, b] \subseteq \mathbb{R} \to \mathbb{R}^n$ has the familiar linear properties

5.2.12 $$\int_\alpha (f+g)\, ds = \int_\alpha f\, ds + \int_\alpha g\, ds \quad \text{and} \quad \int_\alpha kf\, ds = k\int_\alpha f\, ds$$

for all functions $f : D \subseteq \mathbb{R}^n \to \mathbb{R}$ and $g : D \subseteq \mathbb{R}^n \to \mathbb{R}$ that are integrable along α, and for all $k \in \mathbb{R}$.

The following theorem provides a useful formula for the evaluation of path integrals.

5.2.13 *Theorem.* *Let $\alpha : [a, b] \subseteq \mathbb{R} \to \mathbb{R}^n$ be a C^1 path and let $f : S \subseteq \mathbb{R}^n \to \mathbb{R}$ be a function whose domain S contains the image of α. If the composite function $f \circ \alpha : [a, b] \subseteq \mathbb{R} \to \mathbb{R}$ is integrable over $[a, b]$ then f is integrable along α, and its integral is given by*

5.2.14 $$\int_\alpha f\, ds = \int_a^b f(\alpha(t))\|\alpha'(t)\|\, dt.$$

As a special case of Theorem 5.2.13 we have the following important result.

5.2.15 *Corollary.* *Let $f : S \subseteq \mathbb{R}^n \to \mathbb{R}$ be a continuous function and let $\alpha : [a, b] \subseteq \mathbb{R} \to \mathbb{R}^n$ be a C^1 path in S. Then f is integrable along α, and its integral is given by 5.2.14.*

Proof of Theorem 5.2.13. Let $a = t_0 < t_1 < \cdots < t_k = b$ be a partition $\mathcal{P}$ of $[a, b]$. Consider the Riemann sum of f along α

5.2.16 $$R_\alpha(f, \mathcal{P}) = \sum_{i=1}^{k} f(\alpha(p_i))(\lambda(t_i) - \lambda(t_{i-1}))$$

where for each $i = 1, \ldots, k$, the point p_i is chosen arbitrarily in the interval $[t_{i-1}, t_i]$.

Remember that $\lambda(t)$ is the length of the path $\alpha:[a, t] \subseteq \mathbb{R} \to \mathbb{R}^n$,

so

$$\lambda(t) = \int_a^t \|\alpha'(t)\| \, dt.$$

5.2.17

Since $\|\alpha'\|$ is continuous, $\lambda'(t) = \|\alpha'(t)\|$. By the Mean-Value Theorem, there exists $q_i \in [t_{i-1}, t_i]$ for each $i = 1, \ldots, k$ such that

$$\lambda(t_i) - \lambda(t_{i-1}) = \|\alpha'(q_i)\|(t_i - t_{i-1}).$$

Substituting in 5.2.16, we obtain

$$R_\alpha(f, \mathcal{P}) = \sum_{i=1}^{k} f(\alpha(p_i))\|\alpha'(q_i)\|(t_i - t_{i-1}).$$

Hence

5.2.18 $$R_\alpha(f, \mathcal{P}) = \sum_{i=1}^{k} f(\alpha(p_i))\|\alpha'(p_i)\|(t_i - t_{i-1}) + r$$

where

5.2.19 $$r = \sum_{i=1}^{k} f(\alpha(p_i))(\|\alpha'(q_i)\| - \|\alpha'(p_i)\|)(t_i - t_{i-1}).$$

Now the first term in the expression 5.2.18 for $R_\alpha(f, \mathcal{P})$ is a typical Riemann sum over $[a, b]$ of the function $g:[a, b] \subseteq \mathbb{R} \to \mathbb{R}$ given by

5.2.20 $$g(t) = f(\alpha(t))\|\alpha'(t)\|, \qquad t \in [a, b].$$

Since by hypothesis the function $f \circ \alpha$ is integrable over $[a, b]$ and since $\|\alpha'\|$ is continuous (because α' is continuous), the function g is integrable over $[a, b]$. Therefore (in the notation of Definition 5.1.5)

$$L(g) = U(g) = \int_a^b g(t) \, dt = \int_a^b f(\alpha(t))\|\alpha'(t)\| \, dt.$$

We shall complete the proof by deducing from 5.2.18 and 5.2.19 that $L_\alpha(f) = U_\alpha(f) = L(g)$.

Consider first the remainder term r in the Riemann sum 5.2.18. Since $\|\alpha'\|$ is continuous on $[a, b]$ it is also uniformly continuous. Therefore to any given $\varepsilon > 0$ there corresponds $\delta > 0$ such that

$$|\|\alpha'(q)\| - \|\alpha'(p)\|| < \varepsilon \qquad \text{whenever } |q - p| < \delta.$$

Suppose now that the partition $\mathscr{P}$ of $[a, b]$ has mesh less than δ. Then for each $i = 1, \ldots, k$

$$-\varepsilon < \|\alpha'(q_i)\| - \|\alpha'(p_i)\| < \varepsilon$$

for all choices of $p_i, q_i \in [t_{i-1}, t_i]$.

Let M be an upper bound of the function $|f \circ \alpha|$ on $[a, b]$. Then the remainder term r given in 5.2.19 is bounded as follows for all partitions $\mathscr{P}$ of mesh less than δ:

5.2.21
$$-M\varepsilon(b-a) < r < M\varepsilon(b-a).$$

Consider next the Riemann sums $R_\alpha(f, \mathscr{P})$ given by 5.2.18. Taking lower Riemann sums, it follows from 5.2.20 and 5.2.21 that

$$L_\alpha(f, \mathscr{P}) > L(g, \mathscr{P}) - M\varepsilon(b-a)$$

for all partitions $\mathscr{P}$ with $\mu(\mathscr{P}) < \delta$. Hence the least upper bound $L_\alpha(f)$ of the lower sums $L_\alpha(f, \mathscr{P})$ satisfies

$$L_\alpha(f) > L(g, \mathscr{P}) - M\varepsilon(b-a)$$

provided $\mu(\mathscr{P}) < \delta$. From this it follows that the least upper bound $L(g)$ of the lower sums $L(g, \mathscr{P})$ satisfies

$$L_\alpha(f) + M\varepsilon(b-a) \geqslant L(g).$$

Similarly

$$U_\alpha(f) - M\varepsilon(b-a) \leqslant U(g).$$

Now $L(g) = U(g) = \int_a^b g(t)\,\mathrm{d}t$, and so, for all $\varepsilon > 0$,

$$L_\alpha(f) + M\varepsilon(b-a) \geqslant \int_a^b g(t)\,\mathrm{d}t \geqslant U_\alpha(f) - M\varepsilon(b-a).$$

Therefore

$$L_\alpha(f) \geqslant \int_a^b g(t)\,\mathrm{d}t \geqslant U_\alpha(f).$$

But $L_\alpha(f) \leqslant U_\alpha(f)$, by 5.2.7. Hence $L_\alpha(f) = U_\alpha(f)$, and so f is integrable along α and the integral is

$$\int_\alpha f\,\mathrm{d}s = \int_a^b g(t)\,\mathrm{d}t = \int_a^b f(\alpha(t))\|\alpha'(t)\|\,\mathrm{d}t.$$

5.2.22 *Remark.* Again, put $s = \lambda(t)$. Since, by 5.2.17,

$\mathrm{d}s/\mathrm{d}t = \|\alpha'(t)\|$, Theor

which further justifies

5.2.23 *Example.* In E
integral of the function
where $t \in [-1, 1]$. We c
$\|\alpha'(t)\| = 2\pi$ for all t,

$$\int_\alpha f\,\mathrm{d}s = \int_{-1}^{1}$$

The reader should r
Definition 2.7.2 tha
the path integral of

5.2.24

This is to be expec
5.2.16

a Riemann sum c
length of α.

The following
paths.

5.2.25 *Theorem* *and let* $f : D \subseteq \mathbb{R}$
contains the ima
and is given by

where this inte
a sum of integ

Proof. Exer

$ds/dt = \|\alpha'(t)\|$, Theorem 5.2.13 establishes that

$$\int_\alpha f\,\mathrm{d}s = \int_a^b f(\alpha(t))\,\frac{\mathrm{d}s}{\mathrm{d}t}\,\mathrm{d}t,$$

which further justifies our notation for the path integral.

5.2.23 *Example.* In Example 5.2.11 we found from first principles the integral of the function $f(x, y) = x^2$ along the path $\alpha(t) = (\cos 2\pi t, \sin 2\pi t)$, where $t \in [-1, 1]$. We can now find the path integral more directly. Since $\|\alpha'(t)\| = 2\pi$ for all t,

$$\int_\alpha f\,\mathrm{d}s = \int_{-1}^{1} f(\alpha(t))\|\alpha'(t)\|\,\mathrm{d}t = \int_{-1}^{1} (\cos 2\pi t)^2 2\pi\,\mathrm{d}t = 2\pi.$$

The reader should notice in comparing Theorem 5.2.13 and Definition 2.7.2 that the length of a C^1 path $\alpha : [a, b] \subseteq \mathbb{R} \to \mathbb{R}^n$ is the path integral of the constant function $f(x) = 1$ along α. In short

5.2.24
$$l(\alpha) = \int_\alpha \mathrm{d}s.$$

This is to be expected since, for any partition $\mathscr{P}$ of $[a, b]$, from 5.2.16

$$R_\alpha(1, \mathscr{P}) = \sum_{i=1}^{k} (\lambda(t_i) - \lambda(t_{i-1})),$$

a Riemann sum of the path integral, is also an expression for the length of α.

The following result generalizes Corollary 5.2.15 to *piecewise* C^1 paths.

5.2.25 *Theorem.* *Let $\alpha : [a, b] \subseteq \mathbb{R} \to \mathbb{R}^n$ be a piecewise C^1 path and let $f : D \subseteq \mathbb{R}^n \to \mathbb{R}$ be a continuous function whose domain D contains the image of α. Then the path integral of f along α exists and is given by*

$$\int_\alpha f\,\mathrm{d}s = \int_a^b f(\alpha(t))\|\alpha'(t)\|\,\mathrm{d}t,$$

where this integral is interpreted, in the spirit of Definition 2.6.35, as a sum of integrals.

Proof. Exercise.

It follows that if $\beta:[a, b]\subseteq\mathbb{R}\to\mathbb{R}^n$ and $\gamma:[b, c]\subseteq\mathbb{R}\to\mathbb{R}^n$ are C^1 paths such that $\beta(b)=\gamma(b)$, and if $\alpha:[a, c]\subseteq\mathbb{R}\to\mathbb{R}^n$ is defined by $\alpha\mid[a, b]=\beta$ and $\alpha\mid[b, c]=\gamma$ then α is a piecewise C^1 path, and

5.2.26
$$\int_\alpha f\,\mathrm{d}s=\int_\beta f\,\mathrm{d}s+\int_\gamma f\,\mathrm{d}s$$

for any continuous function f whose domain contains the image of α.

5.2.27 *Example.* Let $\alpha:[-1, 1]\subseteq\mathbb{R}\to\mathbb{R}^2$ be the piecewise C^1 path defined by

$$\alpha(t)=(t, |t|), \qquad t\in[-1, 1].$$

and let $f(x, y)=x^2y$. Let $\alpha\mid[-1, 0]=\beta$ and $\alpha\mid[0, 1]=\gamma$. Then, since $\|\alpha'(t)\|=\sqrt{2}$, $t\neq 0$,

$$\begin{aligned}\int_\alpha f\,\mathrm{d}s&=\int_\beta f\,\mathrm{d}s+\int_\gamma f\,\mathrm{d}s\\&=\int_{-1}^{0}-t^3\sqrt{2}\,\mathrm{d}t+\int_0^1 t^3\sqrt{2}\,\mathrm{d}t=\tfrac{1}{2}\sqrt{2}.\end{aligned}$$

The way that length increases as a path traces its image is matched by the way in which the path integral accumulates values. In particular, when a path traces its image and then retraces it in the opposite direction there is a doubling up in the path integral rather than a cancelling out. For example, the path $\alpha(t)=(t^2, t^4)$, $t\in[-1, 1]$ traces the parabolic arc $y=x^2$, $0\leqslant x\leqslant 1$ twice, and $l(\alpha)$ is twice the length of the arc.

Again, if the feeding fish turns tail and retraces its route in the opposite direction it will gather as much plankton as it did on the outward journey. Here is a further illustration.

5.2.28 *Example.* Let S be the unit semi-circle $\{(x, y)\mid x^2+y^2=1, y\geqslant 0\}$ in the upper half-plane in $\mathbb{R}^2$ and let $f:S\subseteq\mathbb{R}^2\to\mathbb{R}$ be the function given by $f(x, y)=y^2$.

[i] The path $\alpha:[0, \pi]\subseteq\mathbb{R}\to\mathbb{R}^2$ given by $\alpha(t)=(\cos t, \sin t)$ for all t traces S once from $(1, 0)$ to $(-1, 0)$. Since $\|\alpha'(t)\|=1$ for all $t\in[0, \pi]$,

$$\int_\alpha f\,\mathrm{d}s=\int_0^\pi \sin^2 t\,\mathrm{d}t=\tfrac{1}{2}\pi.$$

[ii] The path $\beta:[-\sqrt{\pi}, 0]\subseteq\mathbb{R}\to\mathbb{R}^2$ given by $\beta(t)=(\cos t^2, \sin t^2)$ for all t traces S once from $(-1, 0)$ to $(1, 0)$. Since $\|\beta'(t)\|=2\,|t|$ for all

$t \in [-\sqrt{\pi}, 0]$,

$$\int_\beta f\,\mathrm{d}s = \int_{-\sqrt{\pi}}^{0} 2\,|t|\sin^2(t^2)\,\mathrm{d}t = -\int_{-\sqrt{\pi}}^{0} 2t\sin^2(t^2)\,\mathrm{d}t = \tfrac{1}{2}\pi.$$

[iii] The path $\gamma:[-\sqrt{\pi}, \sqrt{\pi}] \subseteq \mathbb{R} \to \mathbb{R}^2$ given by $\gamma(t) = (\cos t^2, \sin t^2)$ for all t traces S from $(-1, 0)$ to $(1, 0)$ and back again. This time the path integral is

$$\int_\gamma f\,\mathrm{d}s = \int_{-\sqrt{\pi}}^{\sqrt{\pi}} 2\,|t|\sin^2(t^2)\,\mathrm{d}t = \int_0^{\sqrt{\pi}} 4t^2\sin^2(t^2)\,\mathrm{d}t = \pi.$$

[iv] The path $\delta:[-2\sqrt{\pi}, 2\sqrt{\pi}] \subseteq \mathbb{R} \to \mathbb{R}^2$ given by $\delta(t) = (\cos \frac{1}{4}t^2, \sin \frac{1}{4}t^2)$ for all t also traces S from $(-1, 0)$ to $(1, 0)$ and back again. We obtain

$$\int_\delta f\,\mathrm{d}s = \int_{-2\sqrt{\pi}}^{2\sqrt{\pi}} \tfrac{1}{2}\,|t|\sin^2(\tfrac{1}{4}t^2)\,\mathrm{d}t = \int_0^{2\sqrt{\pi}} t\sin^2(\tfrac{1}{4}t^2)\,\mathrm{d}t = \pi.$$

In Example 5.2.28 the equality of the path integrals of f along α and β and along γ and δ illustrates the following two theorems. Notice that the paths α and β are both 1–1. The paths γ and δ, although not 1–1, are equivalent, since $\delta = \gamma \circ \phi$, where ϕ is the smooth function from $[-2\sqrt{\pi}, 2\sqrt{\pi}]$ on to $[-\sqrt{\pi}, \sqrt{\pi}]$ given by $\phi(u) = \frac{1}{2}u$.

5.2.29 Theorem. *Let $\alpha:[a, b] \subseteq \mathbb{R} \to \mathbb{R}^n$ and $\beta:[c, d] \subseteq \mathbb{R} \to \mathbb{R}^n$ be two 1–1 piecewise C^1 paths having the same image C in $\mathbb{R}^n$. Then*
[i] *α and β have the same length,*
[ii] *for any continuous function $f: C \subseteq \mathbb{R}^n \to \mathbb{R}$, the path integrals of f along α and β are the same.*

Proof. Let $a = t_0 < t_1 < \cdots < t_k = b$ be a partition $\mathscr{P}$ of $[a, b]$. Since β is 1–1, there are unique points $u_0, u_1, \ldots, u_k$ in $[c, d]$ such that

5.2.30 $$\alpha(t_i) = \beta(u_i), \qquad i = 0, \ldots, k.$$

Furthermore, by Theorem 4.2.13, β^{-1} is continuous and therefore $\beta^{-1} \circ \alpha:[a, b] \to [c, d]$ is continuous. Since $\beta^{-1} \circ \alpha$ is also 1–1 it is monotonic, and so the points $u_0, u_1, \ldots, u_k$ form either an increasing or a decreasing partition $\mathscr{Q}$ of $[c, d]$, depending on whether β traces C from $\alpha(a)$ to $\alpha(b)$ or from $\alpha(b)$ to $\alpha(a)$. By a similar process, any partition $\mathscr{Q}$ of $[c, d]$ gives rise to a unique partition $\mathscr{P}$ of $[a, b]$.

In this way we establish a 1–1 correspondence between partitions $\mathscr{P}$ of $[a, b]$ and partitions $\mathscr{Q}$ of $[c, d]$ such that 5.2.30 is satisfied.

[i] For any pair of partitions $\mathcal{P}$ and $\mathcal{Q}$ that match in the above sense,

$$\sum_{i=1}^{k} \|\alpha(t_i) - \alpha(t_{i-1})\| = \sum_{i=1}^{k} \|\beta(u_i) - \beta(u_{i-1})\|.$$

Therefore, by the Mean-Value Theorem

$$\sum_{i=1}^{k} \|\alpha'(p_i)\|(t_i - t_{i-1}) = \sum_{i=1}^{k} \|\beta'(q_i)\| \, (u_i - u_{i-1})$$

for suitable $p_i \in]t_{i-1}, t_i[$ and $q_i \in]u_{i-1}, u_i[$. Hence, by the definition of the length of a path, $l(\alpha) = l(\beta)$.

[ii] Again let $\mathcal{P}$ and $\mathcal{Q}$ be matching partitions in the sense of 5.2.30. Suppose that $c = u_0 < u_1 < \cdots < u_k = d$. Then, by [i], for each $i = 1, \ldots, k$ the lengths of the paths

$$\alpha : [t_{i-1}, t_i] \subseteq \mathbb{R} \to \mathbb{R}^n \quad \text{and} \quad \beta : [u_{i-1}, u_i] \subseteq \mathbb{R} \to \mathbb{R}^n$$

are the same. Let this length be l_i. Then

5.2.31 $$R_\alpha(f, \mathcal{P}) = \sum_{i=1}^{k} f(\alpha(p_i)) l_i = \sum_{i=1}^{k} f(\beta(q_i)) l_i = R_\beta(f, \mathcal{Q}),$$

where $p_i \in [t_{i-1}, t_i]$ and $q_i \in [u_{i-1}, u_i]$ are chosen so that $\alpha(p_i) = \beta(q_i)$. It follows from 5.2.31 and the definition of the path integral that

$$\int_\alpha f \, ds = \int_\beta f \, ds.$$

A simple adjustment in the proof leads to the same conclusion when the partition $\mathcal{Q}$ that matches $\mathcal{P}$ is given by the decreasing sequence $d = u_0 > u_1 > \cdots > u_k = c$.

The following important theorem provides another case where the integrals of f along two related paths are equal.

5.2.32 *Theorem.* *Let $\alpha : [a, b] \subseteq \mathbb{R} \to \mathbb{R}^n$ and $\beta : [c, d] \subseteq \mathbb{R} \to \mathbb{R}^n$ be equivalent C^1 paths in $\mathbb{R}^n$ with image the curve C. Then for any continuous function $f : C \subseteq \mathbb{R}^n \to \mathbb{R}$ the path integrals of f along α and β are the same.*

Proof. Exercise 5.2.2. Recall (Definition 2.6.9) that if β is equivalent to α there exists a differentiable function ϕ such that $\beta = \alpha \circ \phi$ and ϕ' takes either positive values only or negative values only.

We close this section by illustrating Theorem 5.2.29 for the case $n=1$.

5.2.33 Example. Let $f:[a, b]\subseteq \mathbb{R}\to\mathbb{R}$ be a continuous function. The *identity path* $\alpha:[a, b]\subseteq\mathbb{R}\to\mathbb{R}$ given by $\alpha(t)=t$ is a 1–1 C^1 path in $\mathbb{R}$. Since $|\alpha'(t)|=1$, we have

$$\int_\alpha f\ \mathrm{d}s=\int_a^b f(t)\ \mathrm{d}t.$$

Suppose that $\beta:[c, d]\subseteq\mathbb{R}\to\mathbb{R}$ is a 1–1 C^1 path in $\mathbb{R}$ whose image is $[a, b]$. Then

$$\int_\beta f\ \mathrm{d}s=\int_c^d f(\beta(u))\,|\beta'(u)|\ \mathrm{d}u.$$

Hence by Theorem 5.2.29

5.2.34
$$\int_a^b f(t)\ \mathrm{d}t=\int_c^d f(\beta(u))\,|\beta'(u)|\ \mathrm{d}u.$$

This is a particular case of the change of variable rule for elementary integrals. Note that since a 1–1 path in $\mathbb{R}$ is strictly monotonic, the path β is either strictly increasing or strictly decreasing. Suppose the latter. Then $\beta(c)=b$, $\beta(d)=a$ and $\beta'(u)\leqslant 0$ for all $u\in[c, d]$. The change of variable $t=\beta(u)$ then gives

$$\int_a^b f(t)\ \mathrm{d}t=\int_d^c f(\beta(u))\frac{\mathrm{d}t}{\mathrm{d}u}\ \mathrm{d}u=\int_c^d f(\beta(u))(-\beta'(u))\ \mathrm{d}u,$$

which agrees with 5.2.34.

Exercises 5.2

1. Let $f:\mathbb{R}^2\to\mathbb{R}$ be defined by $f(x, y)=x^2y^2$. Evaluate the path integrals $\int_\alpha f\ \mathrm{d}s$, $\int_\beta f\ \mathrm{d}s$, and $\int_\gamma f\ \mathrm{d}s$, where α, β, and γ are the C^1 paths defined by

$$\begin{aligned} \alpha(t)&=(\cos t, \sin t), & t&\in[0, 4\pi],\\ \beta(t)&=(\cos 2t, \sin 2t), & t&\in[-2\pi, 0],\\ \gamma(t)&=(\cos t, -\sin t), & t&\in[0, 2\pi].\end{aligned}$$

Note that α, β, and γ all have the same image. Describe how these paths trace their image and relate this to the values of the path integrals of f along α, β, and γ.

Answer: 2π, 2π, π. The image C of α, β, and γ is the unit circle

$x^2+y^2=1$. The paths α and β trace C twice counterclockwise, and γ traces C once clockwise.

2. Let $\alpha:[a, b]\subseteq\mathbb{R}\to\mathbb{R}^n$ and $\beta:[c, d]\subseteq\mathbb{R}\to\mathbb{R}^n$ be C^1 paths such that $\beta=\alpha\circ\phi$ where ϕ is a C^1 function from $[c, d]$ *onto* $[a, b]$.
(a) Prove that α and β have the same image.
(b) Given that ϕ is monotonic, prove that for any continuous function $f:S\subseteq\mathbb{R}^n\to\mathbb{R}$ such that im $\alpha\subseteq S$,

$$\int_\beta f\,\mathrm{d}s=\int_\alpha f\,\mathrm{d}s.$$

Hint: in the integral of f along β, put $\beta(u)=\alpha(\phi(u))$, $u\in[c, d]$, and use the Chain Rule $\beta'(u)=\alpha'(\phi(u))\phi'(u)$.
(c) Illustrate part (b) with the paths α and β in Exercise 1. Does the result remain valid when ϕ is not necessarily monotonic?

Answer: in Exercise 1, $\beta=\alpha\circ\phi$, where $\phi:[-2\pi, 0]\to[0, 4\pi]$ is defined by $\phi(t)=2t+4\pi$. The result does not remain valid for non-monotonic ϕ. For example, if $\phi:[c, d]\to[a, b]$ traces the interval $[a, b]$ twice, then $\int_\beta f\,\mathrm{d}s=2\int_\alpha f\,\mathrm{d}s$. Verify this for the case $\phi(u)=u^2$, $u\in[-1, 1]$, $\alpha(t)=(t, t)$, $t\in[0, 1]$, and $f(x, y)=1$, all x, y.

3. Show that the C^1 paths δ and μ defined by

$$\delta(t)=(\cos\pi(t^2), \sin\pi(t^2)) \qquad t\in[-1, 2],$$
$$\mu(t)=(-\cos\pi(2t-t^2), \sin\pi(2t-t^2)), \qquad t\in[-1, 2],$$

have the same image, and describe how it is traced. Let $f:\mathbb{R}^2\to\mathbb{R}$ be defined by $f(x, y)=x^2+y^2$. Predict the values of the path integrals $\int_\delta f\,\mathrm{d}s$ and $\int_\mu f\,\mathrm{d}s$, and check by integrating.
Verify also that $\mu=\delta\circ\phi$, where ϕ is the monotonic decreasing function from $[-1, 2]$ to $[-1, 2]$ defined by $\phi(u)=1-u$.

4. Let $f:S\subseteq\mathbb{R}^n\to\mathbb{R}$ be continuous, and let $\alpha:[a, b]\subseteq\mathbb{R}\to\mathbb{R}^n$ be a piecewise C^1 path in S. Define the *mean value of f along the path α* by

$$\bar{f}_\alpha=\frac{1}{l(\alpha)}\int_\alpha f\,\mathrm{d}s,$$

where $l(\alpha)$ is the length of α.
(a) Calculate $\bar{f}_\alpha$ in the following cases.

(i)	$\alpha(t)=(\cos t, \sin t)$,	$t\in[0, \pi]$,	$f(x, y)=x$
(ii)	$\alpha(t)=(\cos t, \sin t)$,	$t\in[0, \pi]$,	$f(x, y)=y$,
(iii)	$\alpha(t)=(\cos t^2, \sin t^2)$,	$t\in[-\sqrt{\pi}, \sqrt{\pi}]$,	$f(x, y)=y$,
(iv)	$\alpha(t)=(t, \lvert t\rvert)$,	$t\in[-1, 1]$,	$f(x, y)=x^2y$.

Answers: (i) 0, (ii) $2/\pi$, (iii) $2/\pi$, (iv) 1/4.

(b) The equality of the values $\bar{f}$ in Exercises 4(a) (ii) and (iii) is no coincidence. Consider the path $\alpha(t) = (\cos t^2, \sin t^2)$, $t \in [-\sqrt{\pi}, \sqrt{\pi}]$ as composed of two 1–1 C^1 pieces: $\beta = \alpha \mid [-\sqrt{\pi}, 0]$ and $\gamma = \alpha \mid [0, \sqrt{\pi}]$. Compare the integral of f along these paths with the integral of f along the path 4(a) (ii) in the manner of Exercise 2.

5. Let α, β, and γ be the 1–1 C^1 paths from $(1,1,1)$ to $(0,0,0)$ defined by

$$\alpha(t) = (-t, -t, -t), \qquad t \in [-1, 0],$$
$$\beta(t) = (t^2, t^2, t^2), \qquad t \in [-1, 0],$$
$$\gamma(t) = \begin{cases} (t^2, t^2, 1), & t \in [-1, 0] \\ (0, 0, 1 - t^2), & t \in [0, 1]. \end{cases}$$

Evaluate the path integrals $\int_\alpha f\, ds$, $\int_\beta f\, ds$, and $\int_\gamma f\, ds$, where $f(x, y, z) = xy + z - 1$.

Answers: $-\sqrt{3}/6$, $-\sqrt{3}/6$, $(2\sqrt{2} - 3)/6$. The equality of the first two integrals illustrates Theorem 5.2 29. See also Exercise 2.

6. Let $f : S \subseteq \mathbb{R}^n \to \mathbb{R}$ be continuous on S, and let $\alpha : [a, b] \subseteq \mathbb{R} \to \mathbb{R}^n$ and $\beta : [c, d] \subseteq \mathbb{R} \to \mathbb{R}^n$ be piecewise C^1 paths in S such that $\alpha(b) = \beta(c)$. Prove the following integral formulae concerning the inverse path α^- and the product path $\alpha\beta$

$$\int_{\alpha^-} f\, ds = \int_\alpha f\, ds,$$

$$\int_{\alpha\beta} f\, ds = \int_\alpha f\, ds + \int_\beta f\, ds.$$

5.3 Integral of a vector field along a path

In this section we define the integral of a continuous vector field $F : S \subseteq \mathbb{R}^m \to \mathbb{R}^m$ along a path in S. Recall that a necessary and sufficient condition for the continuity of a vector field is that its coordinate functions are continuous.

5.3.1 *Example.* The function $F : \mathbb{R}^2 \to \mathbb{R}^2$ given by $F(x, y) = \frac{1}{4}(-y, x)$ defines a continuous vector field on $\mathbb{R}^2$. The coordinate functions of F are $F_i : \mathbb{R}^2 \to \mathbb{R}$, where $F_1(x, y) = -\frac{1}{4}y$ and $F_2(x, y) = \frac{1}{4}x$.

When the domain of a vector field F is a subset of $\mathbb{R}^2$ then we can sketch the field by drawing for various values of x, y the vector

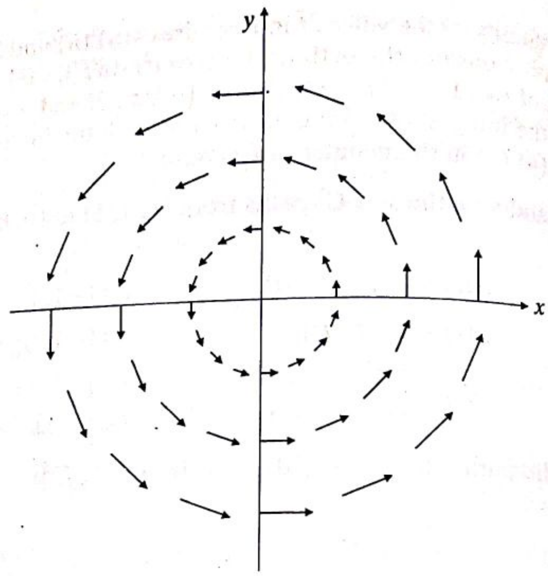

Fig. 5.2 Field $F(x, y) = \frac{1}{4}(-y, x)$

$F(x, y)$ based at $(x, y) \in \mathbb{R}^2$. Figure 5.2 is such a sketch of the field defined in Example 5.3.1. In this case, for each (x, y), the vector $F(x, y)$ is at right angles to the straight line through $(0, 0)$ and (x, y), since the dot product

5.3.2 $$(-y, x) \cdot (x, y) = -yx + xy$$

is zero for all x, y. The overall effect of the vector field is a counterclockwise flow.

In many applications, when m is 2 or 3, vector fields are used to model the effect of a force field on a particle. Examples include gravitational (or magnetic or electrostatic) fields acting on a point mass (pole, charge). In such a model, the path integral (to be defined below) corresponds to the work done by the force field on a particle when the particle moves, or is moved, from one position to another. We discuss this application first as a way of introducing the definition of the integral. In order to explain it we need to introduce some terms from mechanics.

Let $\mathbf{F}$ be a force acting on a particle in space. Then $\mathbf{F}$ is a vector quantity—it has both magnitude and direction. In Standard International (SI) units the magnitude of $\mathbf{F}$, which we denote by $\|\mathbf{F}\|$, is measured in newtons.

Let $\mathbf{u}$ be any non-zero vector and let the angle between $\mathbf{F}$ and $\mathbf{u}$

be θ. Then **F** is uniquely expressible as the sum of two mutually perpendicular vectors

$$\mathbf{F}=\mathbf{F}_{\mathbf{u}}+\mathbf{F}_{\mathbf{u}}^{\perp},$$

where $\mathbf{F}_{\mathbf{u}}$ is a scalar multiple of the vector **u**. See Fig. 5.3. The vector $\mathbf{F}_{\mathbf{u}}$ is called the (*perpendicular*) *projection of* **F** *in direction* **u**. From Fig. 5.3 it is clear that the magnitude of $\mathbf{F}_{\mathbf{u}}$ is $\|\mathbf{F}\|\cos\theta$ when $0\leq\theta\leq\frac{1}{2}\pi$, and $-\|\mathbf{F}\|\cos\theta$ when $\frac{1}{2}\pi\leqslant\theta\leqslant\pi$. The quantity $\|\mathbf{F}\|\cos\theta$ is called the *component of* **F** *in direction* **u**. The component of **F** in a particular direction may therefore be positive, negative, or (when **F** is perpendicular to the given direction) zero.

Consider now a particle moving in a constant force field, that is wherever the particle is in space it experiences the same (constant) force **F**. In this case, when the particle is moved in a given direction along a straight line, the *work* done on it by the force field is defined to be the product of the component of **F** in the direction of motion and the distance the particle moves. In SI units work is measured in newton metres—one newton metre being called a joule. Notice that the work done, in joules, may be positive, negative, or zero.

Now consider the work done by a variable force field on a particle that moves through the field along a curve. We can construct a mathematical model of this situation by choosing rectangular axes for space and an axis for time, and using the scales prescribed by SI units. The force field leads to a vector field $F:S\subseteq\mathbb{R}^3\rightarrow\mathbb{R}^3$ and the moving particle to a path $\alpha:[a,b]\subseteq\mathbb{R}\rightarrow\mathbb{R}^3$ whose image lies in S. At time t seconds the particle has position vector $\alpha(t)$ and experiences a force of $\|F(\alpha(t))\|$ newtons in the direction of $F(\alpha(t))$. We assume that F is continuous and that α is C^1. Denote

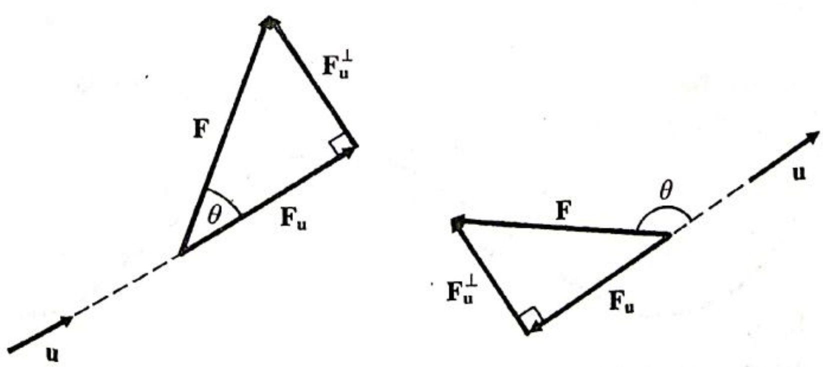

Fig. 5.3

the length of the path $\alpha:[a, t] \subseteq \mathbb{R} \to \mathbb{R}^3$ by $\lambda(t)$. Then $\lambda'(t) = \|\alpha'(t)\|$.

Let $a = t_0 < t_1 < \ldots < t_k = b$ be a partition of $[a, b]$. We approximate the work done by the field on the particle by adding up approximations to the work done in each of the time intervals $[t_{i-1}, t_i]$. Let the length of the path $\alpha:[t_{i-1}, t_i] \subseteq \mathbb{R} \to \mathbb{R}^3$ be l_i for each $i = 1, \ldots, k$. So $l_i = \lambda(t_i) - \lambda(t_{i-1})$. By the Mean-Value Theorem there exists $p_i \in [t_{i-1}, t_i]$ such that

5.3.3 $$l_i = \lambda'(p_i)(t_i - t_{i-1}) = \|\alpha'(p_i)\|(t_i - t_{i-1}).$$

If $l_i = 0$, then the work done in $[t_{i-1}, t_i]$ is 0 joules. When $l_i \neq 0$ the vector $\alpha'(p_i)$ is non-zero and gives the direction of motion of the particle at time p_i seconds. In this case, let the angle between $F(\alpha(p_i))$ and $\alpha'(p_i)$ be $\theta(p_i)$. See Fig. 5.4.

At time p_i seconds, the component of the force on the particle in the direction of its motion is $\|F(\alpha(p_i))\| \cos\theta(p_i)$ newtons. The work done by the field on the particle in the time interval $[t_{i-1}, t_i]$ is therefore approximately

$$\|F(\alpha(p_i))\| \cos\theta(p_i).l_i \text{ joules.}$$

The total work done by the field on the particle in the time interval $[a, b]$ is approximated (in joules) by

5.3.4 $$\sum_{i=1}^{k} \|F(\alpha(p_i))\| \cos\theta(p_i).l_i.$$

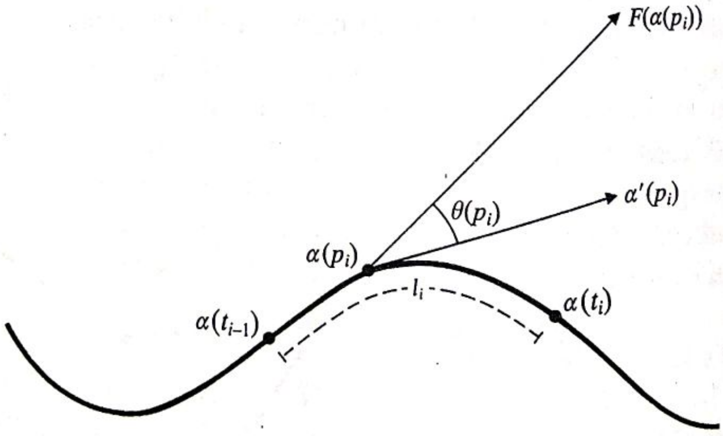

Fig. 5.4

Using 5.3.3 and 1.2.17, this approximation becomes

5.3.5
$$\sum_{i=1}^{k} F(\alpha(p_i)) \cdot \alpha'(p_i)(t_i - t_{i-1}).$$

The work done by the force field on the particle in the time interval $[a, b]$ is defined to be (in joules) the limit of this expression as the mesh of the partition tends to 0. But 5.3.5 is a Riemann sum of the function $F(\alpha(t)) \cdot \alpha'(t)$. Hence the work done by the field on the particle in the time interval $[a, b]$ is given (in joules) by

$$\int_a^b F(\alpha(t)) \cdot \alpha'(t)\, \mathrm{d}t.$$

This integral is very important in both the theory and applications of vector analysis.

5.3.6 *Definition.* *Let $F: S \subseteq \mathbb{R}^n \to \mathbb{R}^n$ be a continuous vector field, and let $\alpha: [a, b] \subseteq \mathbb{R} \leftarrow \mathbb{R}^n$ be a C^1 path in S. The* path integral of F along α *is defined and denoted by*

5.3.7
$$\int_\alpha F \cdot \mathrm{d}\alpha = \int_a^b F(\alpha(t)) \cdot \alpha'(t)\, \mathrm{d}t.$$

5.3.8 *Remark.* In the following examples, and whenever the path integral of F along α occurs, it will be helpful to remember that the Riemann sums for the integral $\int_\alpha F \cdot \mathrm{d}\alpha$ are formed from the expression 5.3.4. They are the sums of lengths along the path, where each length is weighted by the component of the vector field *in the direction in which the length is traversed.*

5.3.9 *Example.* Consider the vector field $F: \mathbb{R}^2 \to \mathbb{R}^2$ given by $F(x, y) = \frac{1}{4}(-y, x)$, and sketched in Fig. 5.2.

[i] Let $\alpha: [-1, 1] \subseteq \mathbb{R} \to \mathbb{R}^2$ be the path $\alpha(t) = (t, t^2)$ which traces a segment of the parabola $y = x^2$ from $(-1, 1)$ to $(1, 1)$. See Fig. 5.5. Then $F(\alpha(t)) = \frac{1}{4}(-t^2, t)$ and $\alpha'(t) = (1, 2t)$. The path integral of F along α is

$$\int_\alpha F \cdot \mathrm{d}\alpha = \tfrac{1}{4}\int_{-1}^{1} (-t^2, t) \cdot (1, 2t)\, \mathrm{d}t = \tfrac{1}{4}\int_{-1}^{1} t^2\, \mathrm{d}t = 1/6.$$

[ii] Let $\beta: [0, 3\pi/2] \subseteq \mathbb{R} \to \mathbb{R}^2$ be the path $\beta(t) = (\cos t, -\sin t)$ which traces the unit circle from $(1, 0)$ to $(0, 1)$ in a clockwise direction. The component of $F(\beta(t))$ in direction $\beta'(t)$ is negative for all t (see Fig. 5.6),

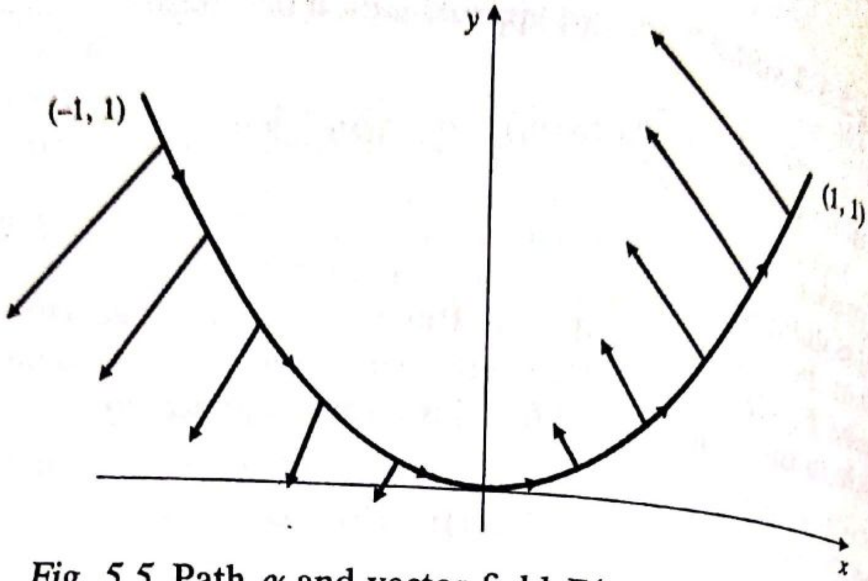

Fig. 5.5 Path α and vector field $F(x, y) = \frac{1}{4}(-y, x)$

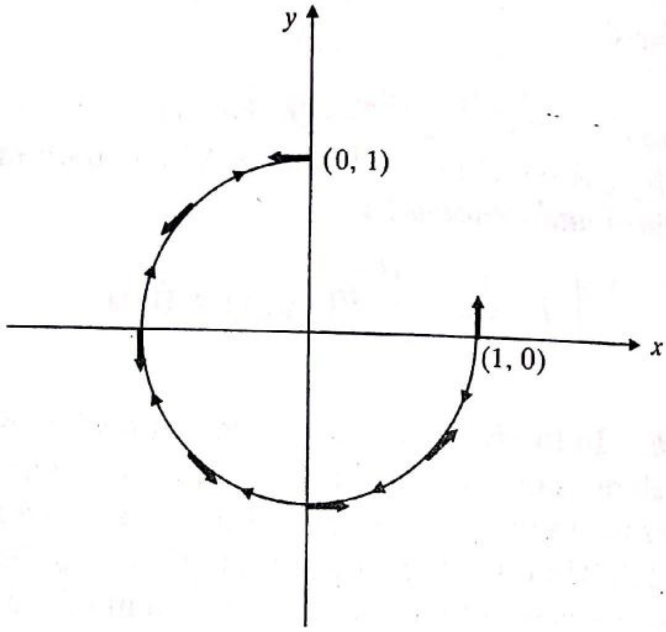

Fig. 5.6 Path β and vector field $F(x, y) = \frac{1}{4}(-y, x)$

and we expect the path integral to be negative. Indeed

$$\int_\beta F \cdot \mathrm{d}\beta = \frac{1}{4}\int_0^{3\pi/2} (\sin t, \cos t) \cdot (-\sin t, -\cos t)\, \mathrm{d}t = -3\pi/8.$$

[iii] Let $\gamma : [-1, 1] \subseteq \mathbb{R} \to \mathbb{R}^2$ be the path $\gamma(t) = (t, t)$ which runs along the straight line $y = x$ from $(-1, -1)$ to $(1, 1)$. The vectors $F(\gamma(t))$ and $\gamma'(t)$ are orthogonal for all t, and so the component of $F(\gamma(t))$ in direction $\gamma'(t)$ is always 0 (see Fig. 5.7). The path integral is

$$\int_\gamma F \cdot \mathrm{d}\gamma = \frac{1}{4}\int_{-1}^{1} (-t, t) \cdot (1, 1)\, \mathrm{d}t = 0.$$

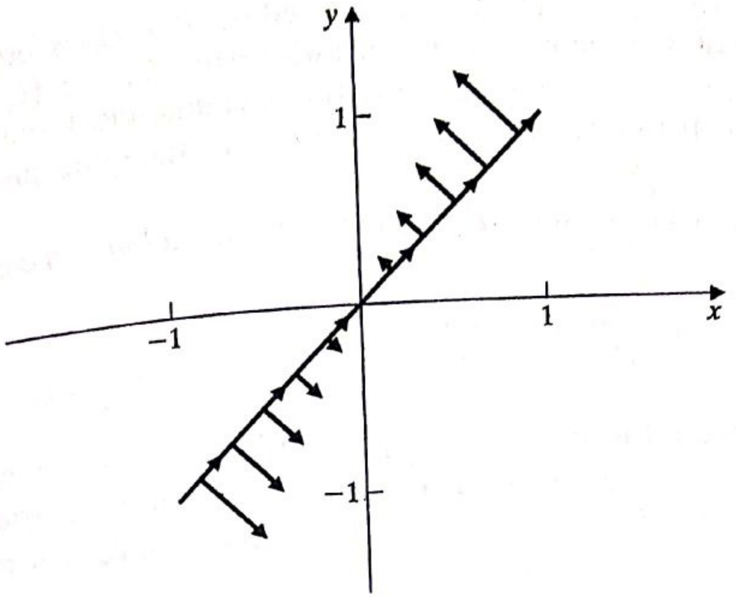

Fig. 5.7 Path γ and vector field $F(x, y) = \frac{1}{4}(-y, x)$

We saw in Section 5.2 first that the path integral of a *scalar field* is independent of the direction in which a path traces its image and second that it is accumulative, in the sense that if part of the image is traced twice in opposite directions then there is a doubling up in the path integral. In contrast, the physical interpretation of the path integral of a *vector field* and the content of Remark 5.3.8 suggest that reversing the direction of the path will change the sign of the integral, and that if part of the image is traced twice in opposite directions then there will be a cancelling out in the path integral. This is because when the direction of tracing is reversed, the component of the vector field in the direction of tracing changes sign.

5.3.10 ***Example*** As in Example 5.3.9 let $F:\mathbb{R}^2 \to \mathbb{R}^2$ be the vector field $F(x, y) = \frac{1}{4}(-y, x)$.

[i] The path $\mu:[\frac{1}{2}\pi, 2\pi] \subseteq \mathbb{R} \to \mathbb{R}^2$ given by $\mu(t) = (\cos t, \sin t)$ traces the segment of the unit circle sketched in Fig. 5.6 from $(0, 1)$ to $(1, 0)$ in a counterclockwise direction. The component of the field in the direction of tracing is now always positive. The path integral is

$$\int_\mu F \cdot \mathrm{d}\mu = \tfrac{1}{4}\int_{\frac{1}{2}\pi}^{2\pi} (-\sin t, \cos t) \cdot (-\sin t, \cos t)\,\mathrm{d}t = 3\pi/8.$$

Compare this answer with the path integral of F along β as defined in Example 5.3.9 [ii].

[ii] The path $\rho:[-1, 2]\subseteq\mathbb{R}\to\mathbb{R}^2$ defined by $\rho(t)=(\cos\frac{1}{2}\pi t^2, \sin\frac{1}{2}\pi t^2)$ traces a quadrant of the unit circle clockwise from $(0, 1)$ to $(1, 0)$ and then back to $(0, 1)$ before continuing to $(1, 0)$ in a counterclockwise direction. We expect the path integrals along μ and ρ to be the same, and in fact

$$\begin{aligned}\int_\rho F\cdot d\rho &= \tfrac{1}{4}\int_{-1}^{2}(-\sin\tfrac{1}{2}\pi t^2, \cos\tfrac{1}{2}\pi t^2)\cdot(-\pi t\sin\tfrac{1}{2}\pi t^2, \pi t\cos\tfrac{1}{2}\pi t^2)\,dt\\ &= \tfrac{1}{4}\int_{-1}^{2}\pi t\,dt = 3\pi/8.\end{aligned}$$

The following theorem sums up the 'direction dependence' of the path integral of a vector field. It should be seen in contrast to Exercise 5.2.2. Note that the condition given there that ϕ be monotonic is now dropped.

5.3.11 ***Theorem.*** *Let $F:S\subseteq\mathbb{R}^n\to\mathbb{R}^n$ be a continuous vector field. Let $\alpha:[a, b]\subseteq\mathbb{R}\to\mathbb{R}^n$ and $\beta:[c, d]\subseteq\mathbb{R}\to\mathbb{R}^n$ be two C^1 paths in S, and suppose that there is a C^1 function $\phi:[c, d]\to[a, b]$ such that $\beta=\alpha\circ\phi$.*

[i] *If $\phi(c)=a$ and $\phi(d)=b$ then $\int_\alpha F\cdot d\alpha=\int_\beta F\cdot d\beta$.*

[ii] *If $\phi(c)=b$ and $\phi(d)=a$ then $\int_\alpha F\cdot d\alpha=-\int_\beta F\cdot d\beta$.*

Proof. By the Chain Rule,

$$\begin{aligned}\int_\beta F\cdot d\beta &= \int_c^d F(\beta(u))\cdot\beta'(u)\,du\\ &= \int_c^d [F(\alpha(\phi(u))\cdot\alpha'(\phi(u))]\phi'(u)\,du\\ &= \int_{\phi(c)}^{\phi(d)} F(\alpha(t))\cdot\alpha'(t)\,dt.\end{aligned}$$

The result follows.

The following example illustrates the two parts of Theorem 5.3.11.

5.3.12 ***Example.*** In Examples 5.3.9 and 5.3.10 the vector field $F(x, y)=\frac{1}{4}(-y, x)$ was integrated along paths β, μ, and ρ.

[i] If $\phi:[-1, 2]\to[\frac{1}{2}\pi, 2\pi]$ is defined by $\phi(u)=\frac{1}{2}\pi u^2$ then $\rho=\mu\circ\phi$, and $\phi(-1)=\frac{1}{2}\pi$, and $\phi(2)=2\pi$. The path integrals of F along ρ and μ were found to be equal.

[ii] If $\phi:[0, 3\pi/2]\to[\frac{1}{2}\pi, 2\pi]$ is defined by $\phi(u)=2\pi-u$ then $\beta=\mu\circ\phi$, and $\phi(0)=2\pi$, and $\phi(3\pi/2)=\frac{1}{2}\pi$. The path integrals of F along β and μ were found to be $-3\pi/8$ and $3\pi/8$ respectively.

The path integrals along a path $\alpha:[a, b]\subseteq\mathbb{R}\to\mathbb{R}^n$ of vector fields F and G have the familiar linear properties

5.3.13 $$\int_\alpha (F+G)\cdot d\alpha = \int_\alpha F\cdot d\alpha + \int_\alpha G\cdot d\alpha$$

$$\text{and} \qquad \int_\alpha kF\cdot d\alpha = k\int_\alpha F\cdot d\alpha$$

for all $k\in\mathbb{R}$, whenever the image of α lies in the domains of F and G.

Definition 5.3.6 of the path integral of a continuous vector field $F:S\subseteq\mathbb{R}^n\to\mathbb{R}^n$ along a path $\alpha:[a, b]\subseteq\mathbb{R}\to\mathbb{R}^n$ in S is given in terms of a C^1 path α. However, expression 5.3.7 makes sense even if α is piecewise C^1. We can use it to define the path integral in that case also.

5.3.14 *Example.* Let $F:\mathbb{R}^2\to\mathbb{R}^2$ be the continuous vector field $F(x, y)=(x^2+y^2, 1)$ and let $\alpha:[-1, 1]\subseteq\mathbb{R}\to\mathbb{R}^2$ be the piecewise C^1 path $\alpha(t)=(t, |t|)$. Then

$$\alpha'(t)=(1, -1) \qquad \text{for all} \qquad t\in[-1, 0[$$

and

$$\alpha'(t)=(1, 1) \qquad \text{for all} \qquad t\in]0, 1].$$

Hence

$$\int_\alpha F\cdot d\alpha = \int_{-1}^{0}(2t^2, 1)\cdot(1, -1)\,dt + \int_0^1 (2t^2, 1)\cdot(1, 1)\,dt = 4/3.$$

Finally in this section we describe two other common notations for the path integral of a vector field that will be very useful in later sections. Let $F:S\subseteq\mathbb{R}^2\to\mathbb{R}^2$ be a continuous vector field with coordinate functions given by $F=(P, Q)$ and let $\alpha:[a, b]\subseteq\mathbb{R}\to\mathbb{R}^2$ be a (piecewise) C^1 path in S having coordinate functions $\alpha(t)=(x(t), y(t))$, $t\in[a, b]$. Then the path integral of F along α is

$$\int_\alpha F\cdot d\alpha = \int_a^b P(x(t), y(t))\frac{dx}{dt}\,dt + \int_a^b Q(x(t), y(t))\frac{dy}{dt}\,dt.$$

For this reason the path integral is also denoted by

5.3.15 $$\int_\alpha P(x, y)\,dx + Q(x, y)\,dy \qquad \text{or simply by} \qquad \int_\alpha P\,dx + Q\,dy.$$

Denoting the vector (x, y) by $\mathbf{r}$ and putting $(dx, dy) = d\mathbf{r}$ leads to the alternative notation

5.3.16
$$\int_\alpha F \cdot d\mathbf{r}$$

for the path integral of F along α.

In general, for a path α in $\mathbb{R}^n$

5.3.17
$$\int_\alpha F_1(x_1, \ldots, x_n)\, dx_1 + \cdots + F_n(x_1, \ldots, x_n)\, dx_n = \int_\alpha F \cdot d\mathbf{r}$$

is the path integral $\int_\alpha F \cdot d\alpha$, where F is the vector field in $\mathbb{R}^n$ given by $F = (F_1, \ldots, F_n)$.

5.3.18 ***Example.*** **[i]** For any path α in $\mathbb{R}^2$, $\int_\alpha -y\, dx + x\, dy$ is the path integral along α of the vector field $F(x, y) = (-y, x)$.

[ii] For any path α in $\mathbb{R}^3$, $\int_\alpha (x^2 + y)\, dz$ is the path integral along α of the vector field $F(x, y, z) = (0, 0, x^2 + y)$. The expression $\int_\alpha (x^2 + y)\, dz$ is an abbreviation for $\int_\alpha 0\, dx + 0\, dy + (x^2 + y)\, dz$.

Exercises 5.3

1. Let $\alpha : [0, 1] \subseteq \mathbb{R} \to \mathbb{R}^3$ be the path defined by

$$\alpha(t) = (1 + t, 1 - t, t^2), \qquad t \in [0, 1].$$

Evaluate the path integral $\int_\alpha F \cdot d\alpha$, where $F : \mathbb{R}^3 \to \mathbb{R}^3$ is the vector field given by

(a) $F(x, y, z) = (xy, yz, zx)$,
(b) $F(x, y, z) = (xyz, 0, 0)$,
(c) $F(x, y, z) = (0, 0, xyz)$.

Answers: (a) 89/60, (b) 2/15, (c) 1/6.

2. Let L be a straight line through the origin in $\mathbb{R}^2$, and let α be a piecewise C^1 path in L. Let F be the vector field in $\mathbb{R}^2$ defined by

$$F(x, y) = (y, -x), \qquad (x, y) \in \mathbb{R}^2.$$

Prove that $\int_\alpha F \cdot d\alpha = 0$. Illustrate with a sketch.

Hint: put $\alpha(t) = (f(t), mf(t))$, m constant.

3. Let $\alpha : [a, b] \subseteq \mathbb{R} \to \mathbb{R}^2$ be a piecewise C^1 path such that $\alpha(a) = \alpha(b)$. Let F be the vector field in $\mathbb{R}^2$ defined by

$$F(x, y) = (y, x), \qquad (x, y) \in \mathbb{R}^2.$$

Prove that the path integral of F along α is zero.

Hint: put $\alpha(t) = (x(t), y(t))$, $t \in [a, b]$, and prove that

$$\int_\alpha y\,dx + x\,dy = 0.$$

4. Evaluate the path integral

$$\int_\alpha yz\,dx + xz\,dy + xy\,dz$$

along the path $\alpha(t) = (1+t, -1+t, 1+2t)$, $t \in [0, 1]$.

Answer: 1.

5. Let $F: \mathbb{R}^3 \to \mathbb{R}^3$ be the field given by

$$F(x, y, z) = (x, yz, xyz), \qquad (x, y, z) \in \mathbb{R}^3.$$

Find the work done by the field on a particle as it moves from the origin to the point $(1, 1, 1)$:
(a) along the path $\alpha(t) = (t, t^2, t)$, $t \in [0, 1]$; (b) along the path $\beta(t) = (t^2, t, t)$, $t \in [0, 1]$. (Calculate $\int_\alpha F \cdot d\alpha$ from the formula 5.3.7.)

Answers: (a) 11/10; (b) 31/30. This exercise illustrates that the work done by a field on a particle in moving from one point to another depends on te route chosen.

6. Let $F: \mathbb{R}^3 \to \mathbb{R}^3$ be any continuous vector field. Show that the total work done by the field F on a particle as it moves along the path $\alpha(t) = (t^2, 0, t^4)$, $t \in [-1, 1]$ is zero.

Hint: the particle moves from $(1, 0, 1)$ to the origin and then moves back to $(1, 0, 1)$ along the same route. Express the work done in terms of two integrals whose sum is zero.

7. Let $F: S \subseteq \mathbb{R}^n \to \mathbb{R}^n$ be a continuous vector field, and let $\alpha: [a, b] \subseteq \mathbb{R} \to \mathbb{R}^n$ be a 1–1 C^1 path in S such that $\alpha'(t)$ is non-zero for all $t \in [a, b]$. Prove that the path integral of F along α can be expressed as the path integral of a scalar field along α.

Solution: Put $T(\alpha(t)) = \alpha'(t)/\|\alpha'(t)\|$, $t \in [a, b]$, so $T(\alpha(t))$ is the unit vector in the direction of $\alpha'(t)$ (unit tangent to α at t). Consider the real-valued function $f(\alpha(t)) = (F \cdot T)(\alpha(t))$, $t \in [a, b]$. Note that if α is not 1–1, say $\alpha(t_1) = \alpha(t_2)$ for $t_1 \neq t_2$, there is a problem in the definition of $f(\alpha(t_1))$ when $\alpha'(t_1) \neq \alpha'(t_2)$.

8. Let $F: D \subseteq \mathbb{R}^n \to \mathbb{R}$ be a continuous vector field, and let $\alpha: [a, b] \subseteq \mathbb{R} \to \mathbb{R}^n$ and $\beta: [c, d] \subseteq \mathbb{R} \to \mathbb{R}^n$ be piecewise C^1 paths in D such that $\alpha(b) = \beta(c)$. Prove the following integral formulae

concerning the inverse path α^- and the product path $\alpha\beta$

$$\int_{\alpha^-} F \cdot \mathrm{d}(\alpha^-) = -\int_\alpha F \cdot \mathrm{d}\alpha,$$

$$\int_{\alpha\beta} F \cdot \mathrm{d}(\alpha\beta) = \int_\alpha F \cdot \mathrm{d}\alpha + \int_\beta F \cdot \mathrm{d}\beta.$$

9. Let $\alpha:[0, 1] \subseteq \mathbb{R} \to \mathbb{R}^2$ and $\beta:[-1, 1] \subseteq \mathbb{R} \to \mathbb{R}^2$ be the C^1 paths in $\mathbb{R}^2$ defined by $\alpha(t) = (t, t)$, $t \in [0, 1]$, and $\beta(u) = (u^2, u^2)$, $u \in [-1, 1]$. Verify that $\beta = \alpha \circ \phi$ where the C^1 function $\phi:[-1, 1] \to [0, 1]$ is given by $\phi(u) = u^2$, $u \in [-1, 1]$. Let $F:\mathbb{R}^2 \to \mathbb{R}^2$ be the vector field $F(x, y) = (1, 0)$ for all $(x, y) \in \mathbb{R}^2$. Show that

$$\int_\alpha F \cdot \mathrm{d}\alpha = 1, \quad \text{and} \quad \int_\beta F \cdot \mathrm{d}\beta = 0.$$

Why does this not contradict Theorem 5.3.11?

5.4 The Fundamental Theorem of Calculus

The Fundamental Theorem of elementary calculus establishes a method of returning from a derivative to the original function. Specifically, if $f:[a, b] \subseteq \mathbb{R} \to \mathbb{R}$ is continuously differentiable then

5.4.1 $$\int_a^b f'(t)\,\mathrm{d}t = f(b) - f(a).$$

We have seen how closely the properties of the gradient reflect the properties of the elementary derivative and we would therefore expect an identity corresponding to 5.4.1 in the higher dimensional setting. It concerns path integrals of the vector field $\operatorname{grad} f$.

5.4.2 ***The Fundamental Theorem of Calculus.*** *Let $f:D \subseteq \mathbb{R}^n \to \mathbb{R}$ be a continuously differentiable function defined on an open subset D of $\mathbb{R}^n$, and let $\mathbf{p}$ and $\mathbf{q}$ be two points of D. If there exists a piecewise C^1 path α from $\mathbf{p}$ to $\mathbf{q}$ in D then*

5.4.3 $$\int_\alpha (\operatorname{grad} f) \cdot \mathrm{d}\alpha = f(\mathbf{q}) - f(\mathbf{p}).$$

Proof. Suppose first that $\alpha:[a, b] \subseteq \mathbb{R} \to \mathbb{R}^n$ is a C^1 path in D. Then the composite function $f \circ \alpha:[a, b] \subseteq \mathbb{R} \to \mathbb{R}$ is also C^1. Hence by the Chain Rule 3.8.7

$$(f \circ \alpha)'(t) = (\operatorname{grad} f)(\alpha(t)) \cdot \alpha'(t), \qquad t \in [a, b].$$

Therefore

$$\int_a^b (f \circ \alpha)'(t)\, dt = \int_a^b (\operatorname{grad} f)(\alpha(t)) \cdot \alpha'(t)\, dt.$$

That is,

5.4.4
$$f(\alpha(b)) - f(\alpha(a)) = \int_\alpha (\operatorname{grad} f) \cdot d\alpha.$$

This expression is equivalent to 5.4.3 if α runs from $\mathbf{p}$ to $\mathbf{q}$. The case where α is a piecewise C^1 path is now easily proved by expressing the interval $[a, b]$ as a union of subintervals on which α is C^1.

The proof of this important theorem is remarkably short and perhaps does not explain why the result is true. However, if $a = t_0 < t_1 < \cdots < t_k = b$ is a partition of $[a, b]$ then, since $\operatorname{grad} f : D \subseteq \mathbb{R}^n \to \mathbb{R}^n$ is a continuous vector field the Mean-Value Theorem 3.9.1 establishes the following approximation for each $i = 1, \ldots, k$

$$f(\alpha(t_i)) - f(\alpha(t_{i-1})) \approx (\operatorname{grad} f)(\alpha(t_{i-1})) \cdot (\alpha(t_i) - \alpha(t_{i-1})).$$

Therefore, if α is a C^1 path,

5.4.5 $$f(\alpha(t_i)) - f(\alpha(t_{i-1})) \approx (\operatorname{grad} f)(\alpha(t_{i-1})) \cdot \alpha'(t_{i-1})(t_i - t_{i-1}).$$

Summing the left-hand side of 5.4.5 from $i = 1$ to k leads to the left-hand side of 5.4.4. Summing the right-hand side of 5.4.5 from $i = 1$ to k gives a Riemann sum approximating the right-hand side of 5.4.4.

Two striking consequences of Theorem 5.4.2 are first that the integrals of $\operatorname{grad} f$ along all paths from $\mathbf{p}$ to $\mathbf{q}$ are the same and second that the integral of $\operatorname{grad} f$ along any path from $\mathbf{p}$ to $\mathbf{p}$ is 0. Both these properties are very significant, as we shall show in the next section. Notice that Theorem 5.4.2 applies to any piecewise C^1 path α running from $\mathbf{p}$ to $\mathbf{q}$. In particular, α need not be 1–1.

5.4.6 ***Example.*** Let $f : \mathbb{R}^2 \to \mathbb{R}$ be defined by

$$f(x, y) = xy + x^2,$$

and let $\alpha : [-1, 2] \subseteq \mathbb{R} \to \mathbb{R}^2$ be the path $\alpha(t) = (t^2, t^4)$. Then

$$(\operatorname{grad} f)(x, y) = (y + 2x, x),$$

and

$$\int_\alpha \operatorname{grad} f \cdot d\alpha = \int_{-1}^{2} (\operatorname{grad} f)(\alpha(t)) \cdot \alpha'(t)\, dt = \int_{-1}^{2} (t^4 + 2t^2, t^2) \cdot (2t, 4t^3)\, dt$$
$$= \int_{-1}^{2} (6t^5 + 4t^3)\, dt = 78.$$

Putting $\alpha(-1) = (1, 1) = \mathbf{p}$ and $\alpha(2) = (4, 16) = \mathbf{q}$ we obtain

$$f(\mathbf{q}) - f(\mathbf{p}) = 80 - 2 = 78.$$

Notice that the path α is not 1–1. The vector $\alpha(t)$ runs along the parabola $y = x^2$ from $(1, 1)$ to $(0, 0)$ (as t increases from -1 to 0), and then back along the parabola from $(0, 0)$ to $(1, 1)$ (as t increases from 0 to 1), and then along the parabola to $(4, 16)$. The contribution to the integral due to the doubling back $(1, 1) \to (0, 0) \to (1, 1)$ is zero since

$$\int_{-1}^{1} (\operatorname{grad} f)(\alpha(t)) \cdot \alpha'(t)\, dt = \int_{-1}^{1} (6t^5 + 4t^3)\, dt = 0.$$

5.4.7 ***Example.*** Consider the function $f : \mathbb{R}^2 \backslash \{(0, 0)\} \to \mathbb{R}$ given by

$$f(x, y) = \frac{1}{x^2 + y^2} \qquad (x, y) \neq (0, 0),$$

and let $\alpha : [0, 2\pi] \subseteq \mathbb{R} \to \mathbb{R}^2$ be the path $\alpha(t) = (2\cos t, \sqrt{2}\sin t)$ whose image traces the ellipse $x^2 + 2y^2 = 4$ in a counterclockwise direction from $(2, 0)$ to $(2, 0)$. See Fig. 5.8. Then

$$(\operatorname{grad} f)(x, y) = \left(\frac{-2x}{(x^2+y^2)^2}, \frac{-2y}{(x^2+y^2)^2}\right) \qquad (x, y) \neq (0, 0)$$

and

$$\int_\alpha \operatorname{grad} f \cdot d\alpha = \int_0^{2\pi} \frac{4 \sin t \cos t}{(2 + 2\sin^2 t)^2}\, dt = \int_0^{2\pi} \frac{2 \sin 2t}{(3 - \cos 2t)^2}\, dt$$
$$= [-(3 - \cos 2t)^{-1}]_0^{2\pi} = 0.$$

Using the same argument, the integral along α restricted to $[0, \pi/2]$, $[\pi/2, \pi]$, $[\pi, 3\pi/2]$, $[3\pi/2, 2\pi]$ is $1/4$, $-1/4$, $1/4$ and $-1/4$ respectively. The relationship between these values might be expected from the symmetry of Fig. 5.8.

Surprisingly, the Fundamental Theorem in the case $n = 1$ reduces, not to the elementary Fundamental Theorem 5.4.1, but to the elementary Substitution Rule. For, when $n = 1$, we have a C^1 function $f : D \subseteq \mathbb{R} \to \mathbb{R}$ and a piecewise C^1 'path' $\alpha : [a, b] \subseteq \mathbb{R} \to \mathbb{R}$

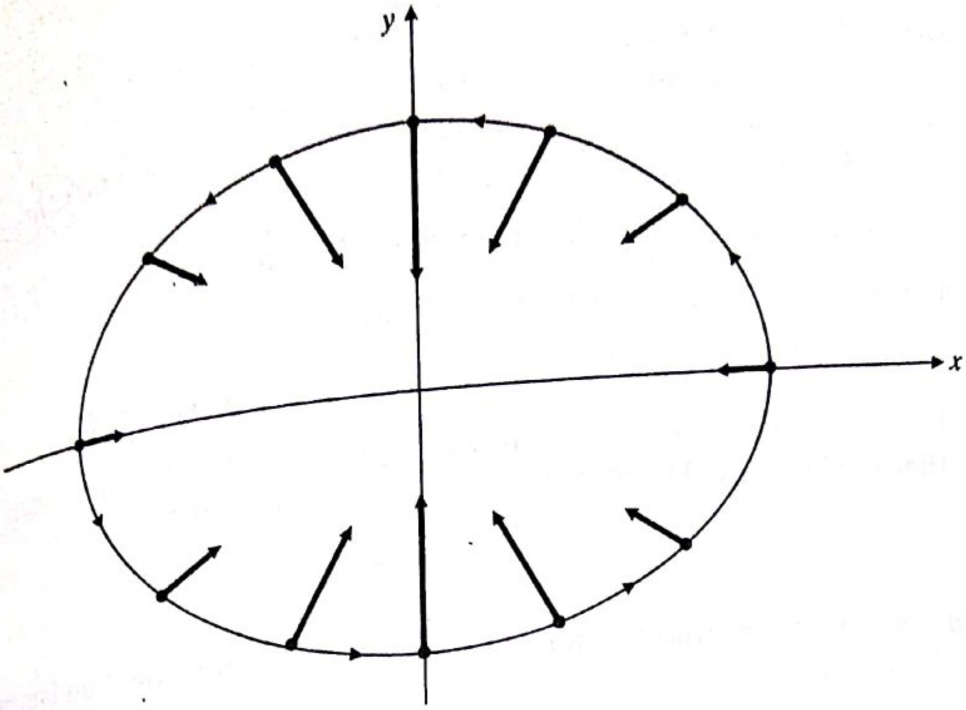

Fig. 5.8

in D. Then Theorem 5.4.2 implies that

5.4.8 $$\int_a^b f'(\alpha(t))\alpha'(t)\,\mathrm{d}t = f(\alpha(b)) - f(\alpha(a)).$$

This usually is given in the form (obtained by 'letting $u = \alpha(t)$ and $\mathrm{d}u = \alpha'(t)\,\mathrm{d}t$')

5.4.9 $$\int_a^b f'(\alpha(t))\alpha'(t)\,\mathrm{d}t = \int_{\alpha(a)}^{\alpha(b)} f'(u)\,\mathrm{d}u.$$

The Fundamental Theorem 5.4.1 is obtained by letting α be the identity path $\alpha(t) = t$ in 5.4.8. Notice that the same result is obtained by letting f be the identity function $f(t) = t$ in 5.4.8.

Exercises 5.4

1. Let $f: \mathbb{R}^2 \to \mathbb{R}$ be defined by $f(x, y) = x^2y + x^3$. Let $\mathbf{p} = (1, 0)$ and $\mathbf{q} = (-1, 0)$. Verify the Fundamental Theorem

$$\int_\alpha (\operatorname{grad} f) \cdot \mathrm{d}\alpha = f(\mathbf{q}) - f(\mathbf{p})$$

for the following C^1 paths α from $\mathbf{p}$ to $\mathbf{q}$:

(a) $\alpha(t) = (\cos t, \sin t)$, $\quad t \in [0, \pi]$;

(b) $\alpha(t) = (\cos t, \sin t)$, $\quad t \in [0, 3\pi]$;

(c) $\alpha(t) = (t, 0)$, $t \in [1, -1]$;
(d) $\alpha(t) = (\sin t, \sin 2t)$, $t \in [\pi/2, 3\pi/2]$;
(e) $\alpha(t) = (\sin t, \sin 2t)$, $t \in [\pi/2, 7\pi/2]$.
In each case sketch how the path α traces its image, and mark the vector $(\text{grad} f)(\alpha(t))$ at various points $\alpha(t)$.

Answer: in (e) the path α makes $1\frac{1}{2}$ circuits of a figure 8 curve.

2. Let $F:\mathbb{R}^3 \to \mathbb{R}^3$ be the vector field defined by

$$F(x, y, z) = (2xyz, x^2z, x^2y), \qquad (x, y, z) \in \mathbb{R}^3.$$

Let $\alpha:[a, b] \subseteq \mathbb{R} \to \mathbb{R}^3$ be an arbitrary piecewise C^1 path in $\mathbb{R}^3$ such that $\alpha(a) = (1, 1, 1)$ and $\alpha(b) = (-1, 2, 2)$. Prove that

$$\int_\alpha F \cdot \mathrm{d}\alpha = 3.$$

Hint: note that $F = \text{grad} f$, where $f(x, y, z) = x^2yz$. Apply the Fundamental Theorem 5.4.2.

3. Solve Exercise 5.3.3 by the above method.

5.5 Potential functions. Conservative fields.

We shall find throughout the rest of this book that the theory of vector fields is rich in physical applications, and conversely that studying the applications helps considerably in the understanding of the pure theorems. The terms and results of this section, for example, follow the close relationship in mechanics between work and energy.

5.5.1 ***Example.*** Let $F:\mathbb{R}^3 \to \mathbb{R}^3$ be the vector field $F(x, y, z) = (0, 0, -mg)$ which models the gravitational force on a particle of mass m kilogrammes. For any piecewise C^1 path $\alpha:[a, b] \subseteq \mathbb{R} \to \mathbb{R}^3$, the path integral of F along $\alpha = (\alpha_1, \alpha_2, \alpha_3)$ is

5.5.2

$$\begin{aligned}\int_\alpha F \cdot \mathrm{d}\alpha &= \int_a^b (0, 0, -mg) \cdot (\alpha_1'(t), \alpha_2'(t), \alpha_3'(t))\, \mathrm{d}t \\ &= -mg \int_a^b \alpha_3'(t)\, \mathrm{d}t = mg\alpha_3(a) - mg\alpha_3(b).\end{aligned}$$

This example shows that, for any two points $\mathbf{p}$ and $\mathbf{q}$ in $\mathbb{R}^3$, the work done by the gravitational field on the particle when it is moved down any piecewise C^1 path from $\mathbf{p}$ to $\mathbf{q}$ is just the drop in the particle's potential energy, $mgp_3 - mgq_3$.

It is significant in Example 5.5.1 that the path integral of the vector field along the path $\alpha:[a, b] \subseteq \mathbb{R} \to \mathbb{R}^2$ depends only upon the end points $\alpha(a)$ and $\alpha(b)$. We have seen this behaviour in the Fundamental Theorem 5.4.2 when integrating a gradient field along a path.

5.5.3 *Example.* Consider the vector field $F:\mathbb{R}^3 \to \mathbb{R}^3$ given by $F(x, y, z) = (yz, xz, xy)$. Then $F = \operatorname{grad} f$ where $f(x, y, z) = xyz$. For any piecewise C^1 path $\alpha:[a, b] \subseteq \mathbb{R} \to \mathbb{R}^3$, the path integral of F along $\alpha = (\alpha_1, \alpha_2, \alpha_3)$ is, by 5.4.3,

$$\int_\alpha F \cdot \mathrm{d}\alpha = \int_\alpha \operatorname{grad} f \cdot \mathrm{d}\alpha = f(\alpha(b)) - f(\alpha(a))$$
$$= \alpha_1(b)\alpha_2(b)\alpha_3(b) - \alpha_1(a)\alpha_2(a)\alpha_3(a).$$

Again we find that the path integral of F along α depends only upon the end points of α.

The following theorem explores the nature of vector fields which have the property that their path integral along any piecewise C^1 path α depends only upon the end points of α. The theorem tells us in particular that a continuous vector field has this property if and only if it is a gradient field.

5.5.4 *Definition.* *A subset S of $\mathbb{R}^n$ is* path connected *if for each pair of points $\mathbf{p} \in S$ and $\mathbf{q} \in S$ there is a path from $\mathbf{p}$ to $\mathbf{q}$ whose image lies in S.*

5.5.5 *Example.* The annulus $\{(x, y) \in \mathbb{R}^2 \mid 1 \leqslant x^2 + y^2 \leqslant 4\}$ in $\mathbb{R}^2$ is path connected (Fig. 5.9(i)). The set $\{(x, y) \in \mathbb{R}^2 \mid xy > 1\}$ in $\mathbb{R}^2$ is not path connected (Fig. 5.9(ii)).

5.5.6 *Theorem.* *Let $F: D \subseteq \mathbb{R}^n \to \mathbb{R}^n$ be a continuous vector field on an open subset D of $\mathbb{R}^n$. Then the following four properties are equivalent.*

[i] *For any two piecewise C^1 paths α and α^* in D from $\mathbf{p}$ to $\mathbf{q}$*

$$\int_\alpha F \cdot \mathrm{d}\alpha = \int_{\alpha^*} F \cdot \mathrm{d}\alpha^*.$$

That is, the path integral of F along any piecewise C^1 path from $\mathbf{p}$ to $\mathbf{q}$ depends only upon the endpoints $\mathbf{p}$ and $\mathbf{q}$.

[ii] *There exists a C^1 function $f: D \subseteq \mathbb{R}^n \to \mathbb{R}$ such that $F = \operatorname{grad} f$.*

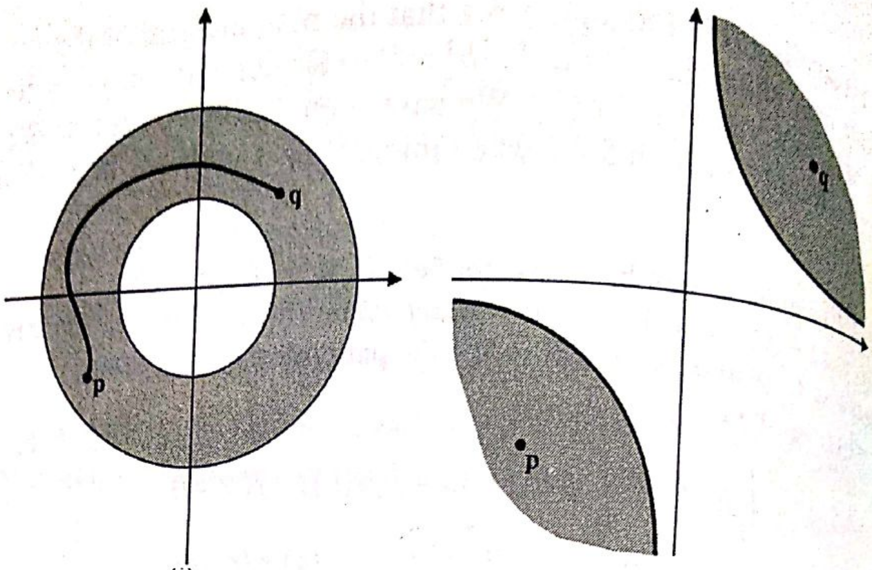

Fig. 5.9 (i) Path connected subset of $\mathbb{R}^2$
(ii) Not path connected

[iii] *There exists a C^1 function $h: D \subseteq \mathbb{R}^n \to \mathbb{R}$ such that for any* $\mathbf{p} \in D$ *and* $\mathbf{q} \in D$ *and any piecewise C^1 path α from* $\mathbf{p}$ *to* $\mathbf{q}$

$$\int_\alpha F \cdot \mathrm{d}\alpha = h(\mathbf{p}) - h(\mathbf{q}).$$

[iv] *For any* $\mathbf{p} \in D$ *and any piecewise C^1 path γ in D from* $\mathbf{p}$ *to* $\mathbf{p}$,

$$\int_\gamma F \cdot \mathrm{d}\gamma = 0.$$

Proof. We shall prove the theorem assuming that D is path connected. For the general case see Exercises 5.5.9 and 5.5.10.

[i] *implies* **[ii]**. Assume property **[i]**. Choose a 'base point' $\mathbf{k}$ in D. Since D is assumed path connected we can define a function $f: D \subseteq \mathbb{R}^n \to \mathbb{R}$ by

5.5.7
$$f(\mathbf{r}) = \int_\phi \mathrm{F} \cdot \mathrm{d}\phi$$

where ϕ is any C^1 path from $\mathbf{k}$ to $\mathbf{r}$ in D. We prove that $F = \operatorname{grad} f$. Choose any point $\mathbf{p}$ in D and let $\mathbf{e}_1 = (1, 0, \ldots, 0) \in \mathbb{R}^n$. Since D is an open set, there exists an interval $I \subseteq \mathbb{R}$, centre 0, such that $\mathbf{p} + u\mathbf{e}_1 \in D$ for all $u \in I$. Choose a particular $u \neq 0$ in I and consider the path $\alpha: [0, 1] \subseteq \mathbb{R} \to \mathbb{R}^n$ defined by $\alpha(t) = \mathbf{p} + tu\mathbf{e}_1$. Then α is a C^1 path in D from $\mathbf{p}$ to $\mathbf{p} + u\mathbf{e}_1$. The path integral of

$F = (F_1, \ldots, F_n)$ along α is

$$\int_\alpha F \cdot \mathrm{d}\alpha = \int_0^1 F(\mathbf{p} + tu\mathbf{e}_1) \cdot u\mathbf{e}_1 \, \mathrm{d}t = u \int_0^1 F_1(\mathbf{p} + tu\mathbf{e}_1) \, \mathrm{d}t.$$

By the Integral Mean-Value Theorem, there exists u^* between 0 and u such that

5.5.8
$$\int_\alpha F \cdot \mathrm{d}\alpha = uF_1(\mathbf{p} + u^*\mathbf{e}_1).$$

Let β and γ be piecewise C^1 paths in D from $\mathbf{k}$ to $\mathbf{p}$ and from $\mathbf{k}$ to $\mathbf{p} + u\mathbf{e}_1$ respectively. See Fig. 5.10. Then, by property **[i]** and Exercise 5.3.8,

$$\int_\gamma F \cdot \mathrm{d}\gamma = \int_{\beta\alpha} F \cdot \mathrm{d}(\beta\alpha) = \int_\beta F \cdot \mathrm{d}\beta + \int_\alpha F \cdot \mathrm{d}\alpha.$$

Hence, from 5.5.7,

$$f(\mathbf{p} + u\mathbf{e}_1) = f(\mathbf{p}) + \int_\alpha F \cdot \mathrm{d}\alpha.$$

From 5.5.8,

$$\frac{f(\mathbf{p} + u\mathbf{e}_1) - f(\mathbf{p})}{u} = F_1(\mathbf{p} + u^*\mathbf{e}_1).$$

Since F is continuous at $\mathbf{p}$, the coordinate function $F_1 : D \subseteq \mathbb{R}^n \to \mathbb{R}$

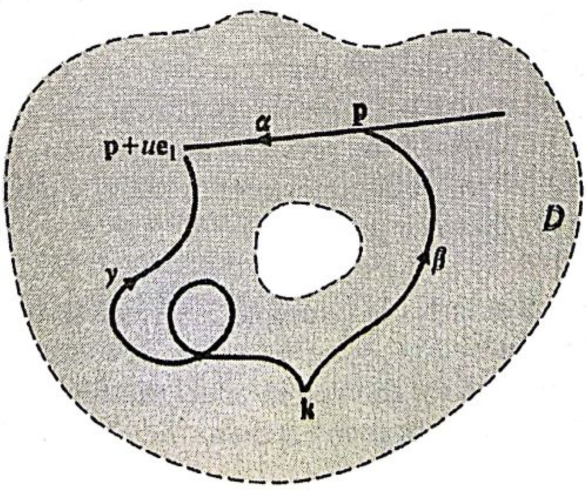

Fig. 5.10

is continuous at $\mathbf{p}$. Therefore

$$\frac{\partial f}{\partial x_1}(\mathbf{p}) = \lim_{u\to 0} \frac{f(\mathbf{p}+u\mathbf{e}_1)-f(\mathbf{p})}{u} = \lim_{u\to 0} F_1(\mathbf{p}+u^*\mathbf{e}_1) = F_1(\mathbf{p}).$$

In the same way, taking other unit directions $\mathbf{e}_i$, we can show that all the partial derivatives of f exist, are continuous and satisfy

$$(F_1, \ldots, F_n) = \left(\frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n}\right).$$

So $F = \operatorname{grad} f$.

[ii] *implies* **[iii]**. Assume **[ii]**. Put $h = -f$. Let $\alpha:[a, b]\subseteq\mathbb{R}\to\mathbb{R}^n$ be a piecewise C^1 path in D from $\alpha(a) = \mathbf{p}$ to $\alpha(b) = \mathbf{q}$. Then, by the Fundamental Theorem 5.4.2,

$$\int_\alpha F\cdot \mathrm{d}\alpha = \int_\alpha \operatorname{grad} f \cdot \mathrm{d}\alpha = f(\mathbf{q}) - f(\mathbf{p}) = h(\mathbf{p}) - h(\mathbf{q}).$$

[iii] *implies* **[iv]**. Set $\mathbf{p} = \mathbf{q}$ in (iii).

[iv] *implies* **[i]**. Let $\alpha:[a, b]\subseteq\mathbb{R}\to\mathbb{R}^n$ and $\alpha^*:[a^*, b^*]\subseteq\mathbb{R}\to\mathbb{R}^n$ be two piecewise C^1 paths in D, both running from $\mathbf{p}$ to $\mathbf{q}$. The product path $\gamma = \alpha^*\alpha^-$ runs from $\mathbf{p}$ to $\mathbf{p}$. Therefore, if property **[iv]** holds then

$$0 = \int_\gamma F\cdot \mathrm{d}\gamma = \int_{\alpha^*} F\cdot \mathrm{d}\alpha^* - \int_\alpha F\cdot \mathrm{d}\alpha,$$

by Exercise 5.3.8. Hence condition **[iv]** implies condition **[i]**.

5.5.9 *Example.* In Example 5.5.1 we considered the vector field $F:\mathbb{R}^3\to\mathbb{R}^3$ given by $F(x, y, z) = (0, 0, -mg)$. We showed in 5.5.2 that F satisfies property **[i]** of the above theorem and that a suitable function $h:\mathbb{R}^3\to\mathbb{R}$ satisfying property **[iii]** is the potential energy function $h(x, y, z) = mgz$. In terms of the proof of the theorem, this function is obtained by taking a base point anywhere in the x, y plane. The reader can verify (property (ii)) that $F = \operatorname{grad}(-h)$.

5.5.10 *Example.* Similarly, in Example 5.5.3, we showed that the vector field $F:\mathbb{R}^3\to\mathbb{R}^3$ given by $F(x, y, z) = (yz, xz, xy)$ satisfies property **[i]** of the theorem. In this case we showed in effect that if $\mathbf{p}$ and $\mathbf{q}$ lie in $\mathbb{R}^3$ and if α is a piecewise C^1 path from $\mathbf{p}$ and $\mathbf{q}$ then

$$\int_\alpha F\cdot \mathrm{d}\alpha = h(\mathbf{p}) - h(\mathbf{q})$$

where $h:\mathbb{R}^3\to\mathbb{R}$ is given by $h(x, y, z) = -xyz$.

Not all continuous vector fields have the properties given in the theorem.

5.5.11 *Example.* Consider the vector field $F:\mathbb{R}^2 \to \mathbb{R}^2$ given by $F(x, y) = \frac{1}{4}(-y, x)$. This is the field sketched in Fig. 5.2.
The C^1 paths $\alpha:[0, \pi/2] \subseteq \mathbb{R} \to \mathbb{R}^2$ and $\beta:[0, 3\pi/2] \subseteq \mathbb{R} \to \mathbb{R}^2$ given by

$$\alpha(t) = (\cos t, \sin t) \qquad \text{and} \qquad \beta(t) = (\cos t, -\sin t)$$

run respectively counterclockwise and clockwise around the unit circle in $\mathbb{R}^2$ from $(1, 0)$ to $(0, 1)$. Now

$$\int_\alpha F \cdot \mathrm{d}\alpha = \int_0^{\pi/2} \tfrac{1}{4}(-\sin t, \cos t) \cdot (-\sin t, \cos t)\,\mathrm{d}t = \frac{\pi}{8}$$

and

$$\int_\beta F \cdot \mathrm{d}\beta = \int_0^{3\pi/2} \tfrac{1}{4}(\sin t, \cos t) \cdot (-\sin t, -\cos t)\,\mathrm{d}t = -\frac{3\pi}{8}.$$

Since these path integrals are different, property **[i]**, and therefore properties **[ii]**, **[iii]** and **[iv]** of the theorem do not hold. In particular, the field F is not a gradient field.

With the above examples in mind we make the following definition.

5.5.12 *Definition.* *Let $F: D \subseteq \mathbb{R}^n \to \mathbb{R}^n$ be a continuous vector field. If there exists a function $h: D \subseteq \mathbb{R}^n \to \mathbb{R}$ such that for all $\mathbf{p} \in D$ and $\mathbf{q} \in D$ and all piecewise C^1 paths α from $\mathbf{p}$ to $\mathbf{q}$ in D*

5.5.13
$$\int_\alpha F \cdot \mathrm{d}\alpha = h(\mathbf{p}) - h(\mathbf{q})$$

then h is said to be a potential function *for F.*

5.5.14 *Example.* **[i]** $h(x, y, z) = mgz$ is a potential function for the vector field $F(x, y, z) = (0, 0, -mg)$. See Example 5.5.9.
[ii] $h(x, y, z) = -xyz$ is a potential function for the vector field $F(x, y, z) = (yz, xz, xy)$. See Example 5.5.10.

We can now restate part of Theorem 5.5.6 in terms of the potential function.

5.5.15 *Theorem.* *Let $F: D \subseteq \mathbb{R}^n \to \mathbb{R}^n$ be a continuous vector field in an open set D. Then there exists a potential function for F if and only if F is a gradient field $F = \operatorname{grad} f$ for some C^1 function*

$f: D \subseteq \mathbb{R}^n \to \mathbb{R}$. *In this case the function* $h = -f$ *is a potential function for* F, *and* $F = -\text{grad}\, h$.

Potential functions for a continuous vector field, when they exist, are not unique.

5.5.16 *Theorem.* *Let* $h: D \subseteq \mathbb{R}^n \to \mathbb{R}$ *be a potential function for a continuous vector field* $F: D \subseteq \mathbb{R}^n \to \mathbb{R}^n$ *on a path-connected subset* D *of* $\mathbb{R}^n$. *Then*

[i] *the function* $g: D \subseteq \mathbb{R}^n \to \mathbb{R}$ *is also a potential function for* F *if and only if* $g = h + c$ *for some* $c \in \mathbb{R}$,

[ii] *if* $g: D \subseteq \mathbb{R}^n \to \mathbb{R}$ *is a potential function for* F *then every level set of* h *is a level set of* g *and vice versa.*

Proof. Exercise.

5.5.17 *Definition.* *Let* $h: D \subseteq \mathbb{R}^n \to \mathbb{R}$ *be a potential function for a continuous vector field* $F: D \subseteq \mathbb{R}^n \to \mathbb{R}^n$. *For each* $c \in \mathbb{R}$ *the level set* $h^{-1}(c)$ *is called an* equipotential set *of the field* F.

Knowing the equipotential sets of a continuous gradient field F enables us to evaluate path integrals painlessly. For example, if the sets are as given in Fig. 5.11 then the path integral of F along any piecewise C^1 path from $\mathbf{p}$ to $\mathbf{q}$ is $1 - (-2) = 3$.

Furthermore, if h is a potential function for F then, according to Theorem 3.8.8, $(\text{grad}\, h)(\mathbf{p})$ is orthogonal at $\mathbf{p}$ to the equipotential set through $\mathbf{p}$ and points in the direction of *increasing* h. Since $F = -\text{grad}\, h$, the vector $F(\mathbf{p})$ is therefore also orthogonal at $\mathbf{p}$ to

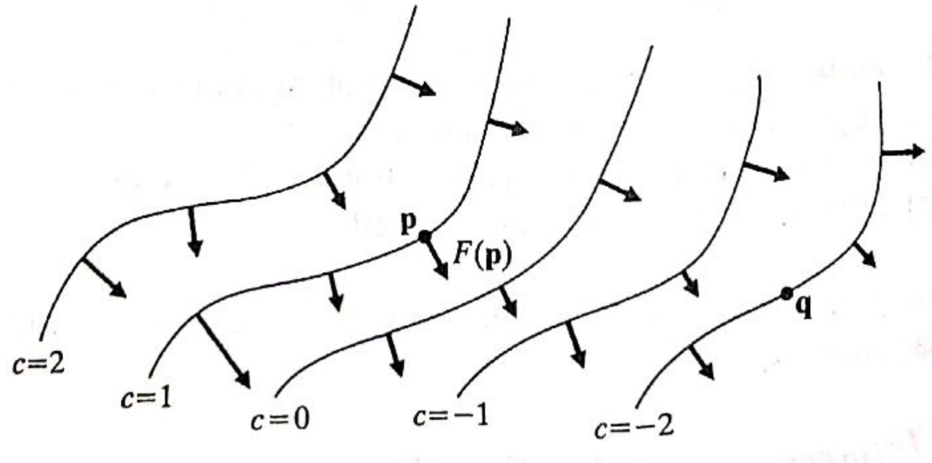

Fig. 5.11 Equipotential sets $h^{-1}(c)$ of field $F = -\text{grad}\, h$

the equipotential set through **p**, *but points in the direction in which h decreases.*

The significance of potential functions in applications is underlined by considering the energy of a particle of mass m kilogrammes moving through space under the influence of a force field. With respect to SI units, let the C^2 path $\alpha:[a, b] \subseteq \mathbb{R} \to \mathbb{R}^3$ model the motion of the particle. At time t in $[a, b]$, the derivatives $\alpha'(t)$ and $\alpha''(t)$ correspond to the particle's velocity and acceleration respectively. The norm $\|\alpha'(t)\|$ corresponds to the particle's speed and $\frac{1}{2}m\ \|\alpha'(t)\|^2$ to its kinetic energy. We shall need to know that, since $\|\alpha'(t)\|^2 = \alpha'(t) \cdot \alpha'(t)$,

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\alpha'(t)\|^2 = 2\alpha'(t) \cdot \alpha''(t).$$

Suppose that the continuous vector field $F:\mathbb{R}^3 \to \mathbb{R}^3$ models the force field. By Newton's Second Law of Motion, for all $t \in [a, b]$

$$F(\alpha(t)) = m\alpha''(t).$$

The path integral of F along α is therefore

5.5.18
$$\int_\alpha F \cdot \mathrm{d}\alpha = m\int_a^b \alpha''(t) \cdot \alpha'(t)\ \mathrm{d}t$$
$$= \tfrac{1}{2}m\ \|\alpha'(b)\|^2 - \tfrac{1}{2}m\ \|\alpha'(a)\|^2.$$

The work done by the field on the particle is therefore the increase in the kinetic energy of the particle.

Suppose now that there exists a potential function $h:\mathbb{R}^3 \to \mathbb{R}$ for the vector field F. Then, by Definition 5.5.12,

$$\int_\alpha F \cdot \mathrm{d}\alpha = h(\alpha(a)) - h(\alpha(b)).$$

It follows from 5.5.18 that

5.5.19 $$h(\alpha(a)) + \tfrac{1}{2}m\ \|\alpha'(a)\|^2 = h(\alpha(b)) + \tfrac{1}{2}m\ \|\alpha'(b)\|^2.$$

In particular, as in Example 5.5.9, when F models the gravitational field acting on the particle then 5.5.19 is the *Principle of Conservation of Energy* which states that the sum of the kinetic and potential energies of the particle remains constant along its path.

More generally, we have shown that the Principle of

Conservation of Energy 5.5.19 applies to all force fields which admit a potential function.

5.5.20 *Definition.* *A continuous vector field $F: D \subseteq \mathbb{R}^n \to \mathbb{R}^n$ is said to be* conservative *if there is a potential function $h: D \subseteq \mathbb{R}^n \to \mathbb{R}$ for F.*

5.5.21 *Theorem.* *Let $F: D \subseteq \mathbb{R}^n \to \mathbb{R}^n$ be a continuous vector field defined on an open set D in $\mathbb{R}^n$. Then*

[i] *the field F is conservative if and only if the work done in moving a particle around any closed contour in D is zero;*

[ii] *the field F is conservative if and only if $F = \operatorname{grad} f$ for some C^1 function $f: D \subseteq \mathbb{R}^n \to \mathbb{R}$.*

Proof. Immediate from Theorem 5.5.6.

Part **[i]** or part **[ii]** of Theorem 5.5.21 is sometimes taken as the definition of a conservative field.

5.5.22 *Example.* The vector field

$$F(x, y, z) = (2xyz + z, x^2z + 1, x^2y + x), \qquad (x, y, z) \in \mathbb{R}^3$$

is conservative. To show this we find a function $f: \mathbb{R}^3 \to \mathbb{R}$ such that $F = \operatorname{grad} f$, that is

$$\frac{\partial f}{\partial x}(x, y, z) = 2xyz + z, \qquad \frac{\partial f}{\partial y}(x, y, z) = x^2z + 1, \qquad \frac{\partial f}{\partial z}(x, y, z) = x^2y + x$$

for all $(x, y, z) \in \mathbb{R}^3$. Integrating the first identity, we obtain

$$f(x, y, z) = x^2yz + xz + g(y, z)$$

where g is some function of y and z still to be determined.

Similarly, integrating the second and third identities, we obtain

$$f(x, y, z) = x^2yz + y + h(x, z)$$

$$f(x, y, z) = x^2yz + xz + k(x, y)$$

Hence by choosing

$$g(y, z) = y, \qquad h(x, z) = xz, \qquad k(x, y) = y,$$

we find

$$f(x, y, z) = x^2yz + xz + y.$$

5.5.23 *Example.* The vector field

$$F(x, y) = (2y, x + y), \qquad (x, y) \in \mathbb{R}^2$$

is not conservative. Proceeding as in Example 5.5.22, let us attempt to find a function $f: \mathbb{R}^2 \to \mathbb{R}$ such that $F = \operatorname{grad} f$. Then, for all $(x, y) \in \mathbb{R}^2$,

$$\frac{\partial f}{\partial x}(x, y) = 2y \quad \text{and} \quad \frac{\partial f}{\partial y}(x, y) = x + y.$$

Hence

$$f(x, y) = 2xy + g(y) \quad \text{and} \quad f(x, y) = xy + \tfrac{1}{2}y^2 + h(x)$$

for suitable functions g and h. Clearly no such functions can be found, and therefore F is not conservative.

5.5.24 *Example.* The vector field

$$F(x, y) = \tfrac{1}{4}(-y, x), \qquad (x, y) \in \mathbb{R}^2$$

is not conservative. This was shown in Example 5.5.11 by comparing two path integrals from $(1, 0)$ to $(0, 1)$. Alternatively, one could demonstrate that F is not a gradient field by the method of Example 5.5.23.

5.5.25 *Example.* The gravitational field due to a particle of mass M is modelled by the vector field $F: \mathbb{R}^3 \backslash \{\mathbf{0}\} \to \mathbb{R}^3$ where

5.5.26
$$F(\mathbf{r}) = -\frac{GM}{r^3}\mathbf{r}, \qquad \mathbf{r} \neq \mathbf{0}.$$

Here $\mathbf{r} = (x, y, z)$ is measured relative to axes with origin at the particle and $r = \|\mathbf{r}\| = \sqrt{(x^2 + y^2 + z^2)}$. The vector $F(\mathbf{r})$ indicates the magnitude and direction of the force of attraction exerted by the mass M (supposed at the origin) on a particle of unit mass with position vector $\mathbf{r}$. See Fig. 5.12. The constant G is called the *gravitational constant.* Expression 5.5.26 is Newton's *Inverse Square Law.* It is so called because the magnitude of the attractive force at $\mathbf{r}$ is proportional to the inverse of the square of r:

$$\|F(\mathbf{r})\| = \frac{GM}{r^3}\|\mathbf{r}\| = \frac{GM}{r^2}.$$

The gravitational field F is conservative, because $F = \operatorname{grad} f$, where the function $f: \mathbb{R}^3 \backslash \{\mathbf{0}\} \to \mathbb{R}$ is defined by

5.5.27
$$f(\mathbf{r}) = \frac{GM}{r}, \qquad \text{for all } r \neq 0.$$

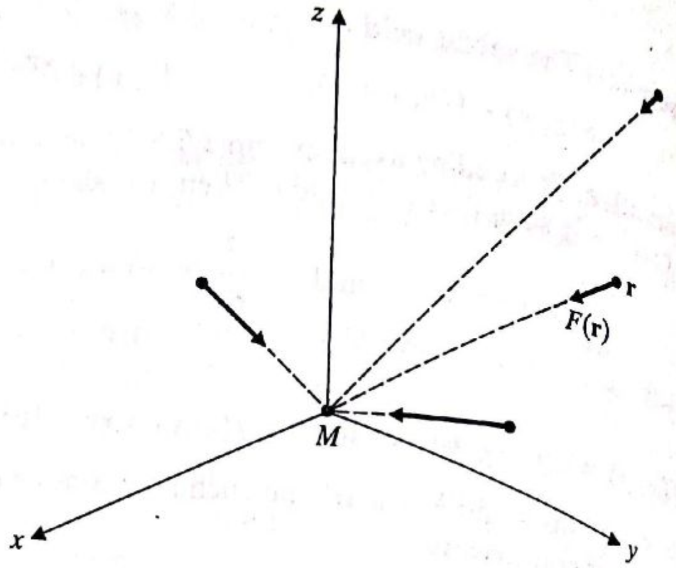

Fig. 5.12

This follows from the calculation

$$\frac{\partial f}{\partial x}(\mathbf{r}) = -\frac{GM}{r^2}\frac{\partial r}{\partial x} = -\frac{GM}{r^3}x,$$

and similar expressions for $(\partial f/\partial y)(\mathbf{r})$ and $(\partial f/\partial z)(\mathbf{r})$.

Finally in this section we consider an example illustrating how the property of a vector field being conservative may depend on the domain on which the field is defined.

5.5.28 *Example.* In $\mathbb{R}^2$ let A be the half axis $\{(x, 0) \in \mathbb{R}^2 \mid x \leqslant 0\}$. Then for each $(x, y) \in \mathbb{R}^2 \backslash A$ there is a unique $r > 0$ and a unique θ in the range $-\pi < \theta < \pi$ such that

5.5.29 $$x = r\cos\theta \qquad \text{and} \qquad y = r\sin\theta.$$

See Fig. 5.13. Let D be the open set $\{(r, \theta) \in \mathbb{R}^2 \mid -\pi < \theta < \pi, r > 0\}$. Define the function $G: D \subseteq \mathbb{R}^2 \to \mathbb{R}^2$ by

$$G(r, \theta) = (r\cos\theta, r\sin\theta), \qquad (r, \theta) \in D.$$

Then G is 1–1 from D onto $\mathbb{R}^2 \backslash A$. Therefore the expressions 5.5.29 establish r and θ as functions of x and y on the domain $\mathbb{R}^2 \backslash A$ and taking values in D. Clearly $r = \sqrt{(x^2 + y^2)}$, but the expression for θ is a bit more troublesome to write down (see Exercise 5.5.8(a)). In any case, since

$$\det J_{G, (r, \theta)} = \det \begin{bmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{bmatrix} = r,$$

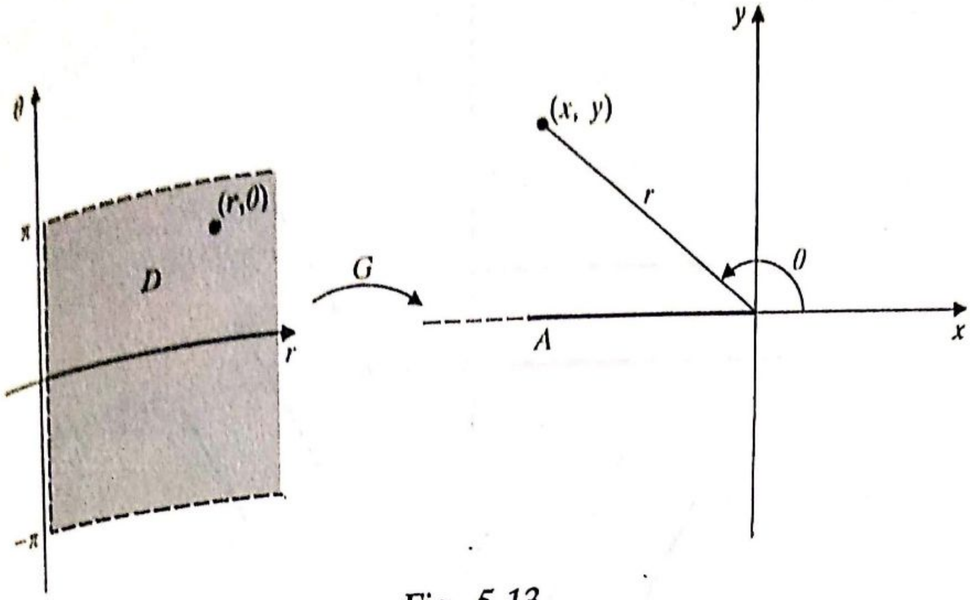

Fig. 5.13

and $r \neq 0$ on D, it follows by the Inverse Function Theorem 4.6.7 that r and θ are differentiable functions of x and y on $\mathbb{R}^2\backslash A$. Differentiating both r and θ partially with respect to x, we obtain from 5.5.29

$$1 = \frac{\partial r}{\partial x}\cos\theta - r\frac{\partial\theta}{\partial x}\sin\theta,$$

$$0 = \frac{\partial r}{\partial x}\sin\theta + r\frac{\partial\theta}{\partial x}\cos\theta.$$

Therefore $\partial\theta/\partial x = -y/(x^2+y^2)$ and, by a similar argument, $\partial\theta/\partial y = x/(x^2+y^2)$.

Let $f:\mathbb{R}^2\backslash A \to \mathbb{R}$ be the real-valued function defined by $f(x, y) = \theta$, where θ is the polar angle given by 5.5.29, and $-\pi < \theta < \pi$. Then $\operatorname{grad} f = (\partial\theta/\partial x, \partial\theta/\partial y)$. Hence $F = \operatorname{grad} f : \mathbb{R}^2\backslash A \to \mathbb{R}^2$ is the conservative vector field such that for each $(x, y) \in \mathbb{R}^2\backslash A$,

5.5.30
$$F(x, y) = \left(\frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2}\right).$$

See Fig. 5.14 for a sketch of this field. The potential function is $h(x, y) = -\theta$.

Notice that if the domain of F is extended to $\mathbb{R}^2\backslash\{\mathbf{0}\}$ and 5.5.30 is taken as defining F over this new domain then F is not a conservative field. To see this consider the closed path $\alpha:[0, 2\pi] \subseteq \mathbb{R} \to \mathbb{R}^2$ defined by $\alpha(t) = (\cos t, \sin t)$, $t \in [0, 2\pi]$. Then

$$\int_\alpha F\cdot d\alpha = \int_0^{2\pi} F(\alpha(t))\cdot\alpha'(t)\,dt = \int_0^{2\pi}(-\sin t, \cos t)\cdot(-\sin t, \cos t)\,dt$$

$$= \int_0^{2\pi} dt = 2\pi.$$

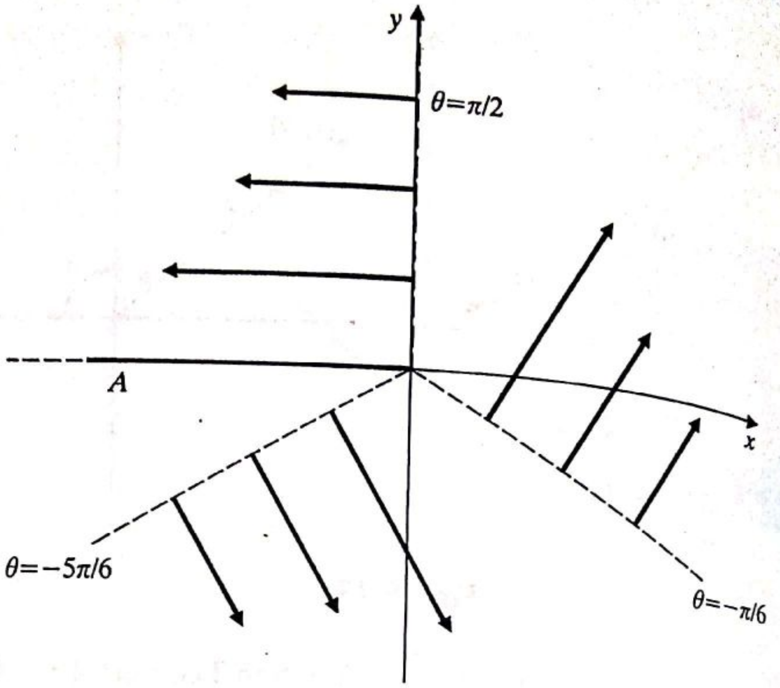

Fig. 5.14 Field $F(x, y) = \left(\frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2}\right)$ and equipotential sets $\theta = \text{constant}$

Since the image of α lies in $\mathbb{R}^2\backslash\{\mathbf{0}\}$, Theorem 5.5.21(i) tells us that F is not a conservative field over $\mathbb{R}^2\backslash\{\mathbf{0}\}$. We are prevented from considering such a path α in $\mathbb{R}^2\backslash A$ by the cut along the negative x-axis. See Fig. 5.15.

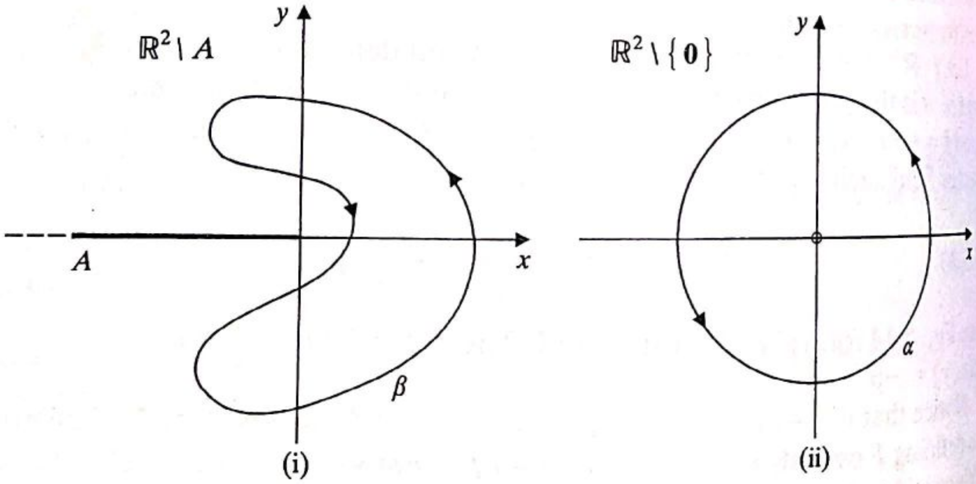

Fig. 5.15 (i) $\int_\beta F \cdot \mathrm{d}\beta = 0$; (ii) $\int_\alpha F \cdot \mathrm{d}\alpha = 2\pi$

Exercises 5.5

1. Let $F: \mathbb{R}^2 \to \mathbb{R}^2$ be the vector field defined by

$$F(x, y) = (2xy, x^2 + 1), \qquad (x, y) \in \mathbb{R}^2.$$

Verify that F is conservative, and find a potential function h for F (such that $F = -\text{grad } h$).

Answer: $h(x, y) = -x^2y - y$

Verify also by direct calculation that $\int_\alpha F \cdot d\alpha = 0$ where α is the closed figure 8 path defined by

$$\alpha(t) = (\sin t, \sin 2t), \qquad t \in [0, 2\pi].$$

2. Sketch the vector field $F: \mathbb{R}^2 \to \mathbb{R}^2$ defined by

$$F(x, y) = (x^2 + y^2, 0), \qquad (x, y) \in \mathbb{R}^2.$$

Note in particular that the field is constant on any circle centred at the origin.

Let α be the closed circular path in $\mathbb{R}^2$ defined by

$$\alpha(t) = (\cos t, \sin t), \qquad t \in [0, 2\pi].$$

Prove that $\int_\alpha F \cdot d\alpha = 0$. Show, however, that F is not a conservative field

(a) by the method of Example 5.5.23,

(b) by constructing a closed piecewise C^1 path β such that $\int_\beta F \cdot d\beta \neq 0$.

Answer: possible closed paths β are (i) the composite 'half-moon' path consisting of the semicircular path $(\cos t, \sin t)$, $t \in [0, \pi]$ followed by the path $(t, 0)$, $t \in [-1, 1]$ along the x-axis, or (ii) a composite path tracing the unit square, vertices $(0, 0)$, $(0, 1)$, $(1, 1)$, $(1, 0)$, for example as follows: $(t, 0)$, $t \in [0, 1]$, then $(1, t)$, $t \in [0, 1]$, then $(1 - t, 1)$, $t \in [0, 1]$, then $(0, 1 - t)$, $t \in [0, 1]$. The integrals $\int_\beta F \cdot d\beta$ take the values (i) $-4/3$, (ii) -1.

3. Let α be any piecewise C^1 path in $\mathbb{R}^3$ running from $(1, -1, 1)$ to $(2, 0, 3)$. Prove that

$$\int_\alpha yz \, dx + xz \, dy + xy \, dz = 1.$$

This generalizes Exercise 5.3.4.

Hint: the field $F(x, y, z) = (yz, xz, xy)$ is conservative.

4. Prove that the vector field $F(x, y, z) = (xy, yz, zx)$ is not conservative.

5. For each of the following conservative fields F find a function f such that $F = \text{grad } f$.

(a) $F(x, y, z) = (yz + 2xz, xz + 1, x^2 + xy + 2)$;

(b) $F(x, y, z) = (x/r, y/r, z/r)$, where $r = \sqrt{(x^2 + y^2 + z^2)} \neq 0$.

Answers: (a) $xyz + x^2z + y + 2z$; (b) r.

6. Define the vector field $F:\mathbb{R}^3\backslash\{\mathbf{0}\}\to\mathbb{R}^3$ by

$$F(\mathbf{r}) = kr^m\mathbf{r} \qquad m\in\mathbb{Z},$$

where k is a constant. Prove that F is conservative and find a suitable potential function for F.

Hint: generalize Example 5.5.25 (where $m=-3$). The case $m=-2$ needs special treatment, logarithms!

7. Let $F:\mathbb{R}^2\to\mathbb{R}^2$ be the vector field given by

$$F(x, y) = (x + y^2, 2xy), \qquad (x, y)\in\mathbb{R}^2.$$

Let α be a piecewise C^1 path in $\mathbb{R}^2$ running from $(0, 0)$ to (a, b). Prove that the path integral $\int_\alpha F\cdot d\alpha$ is independent of the choice of α, and find its value.

Answer: $\frac{1}{2}a^2 + ab^2$. Note that $F = \operatorname{grad} f$, where $f(x, y) = \frac{1}{2}x^2 + xy^2$.

Show also that the path integral $\int_\alpha G\cdot d\alpha$, where

$$G(x, y) = (x + y, 2xy), \qquad (x, y)\in\mathbb{R}^2,$$

depends on the choice of α.

Answer: compare, for example, the straight line path from $(0, 0)$ to (a, b) and the polygonal path from $(0, 0)$ to $(a, 0)$ to (a, b). The corresponding values of the line integral are $\frac{1}{2}a^2 + \frac{1}{2}ab + \frac{2}{3}ab^2$ and $a^2 + ab^2$ respectively. The field F is conservative, but the field G is not conservative.

8. Let $F:\mathbb{R}^2\backslash\{\mathbf{0}\}\to\mathbb{R}^2$ be the vector field defined by

$$F(x, y) = \left(\frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2}\right), \qquad (x, y)\neq(0, 0).$$

(a) Prove by direct integration that F is conservative on the restricted domain given by the half plane $\{(x, y)\in\mathbb{R}^2 \mid x>0\}$.

Answer: $F = \operatorname{grad} f$, where $f(x, y) = \arctan(y/x)$, $x>0$.

Compare with Example 5.5.28. In fact F is conservative on the extended domain $\mathbb{R}^2\backslash A$, where A is the half axis $\{(x, 0)\in\mathbb{R}^2 \mid x\leq 0\}$, and $F = \operatorname{grad} f$ where

$$f(x, y) = \begin{cases} \arctan(y/x) & \text{when } x>0,\\ \tfrac{1}{2}\pi & \text{when } x=0, y>0,\\ \pi + \arctan(y/x) & \text{when } x<0, y>0,\\ -\tfrac{1}{2}\pi & \text{when } x=0, y<0.\\ -\pi + \arctan(y/x) & \text{when } x<0, y<0.\end{cases}$$

Verify that this agrees with the definition $f(x, y) = \theta$, $(x, y) \in \mathbb{R}^2 \setminus A$, given in Example 5.5.28.
(b) Given $(a, b) \neq (0, 0)$ let H be the half line $\{t(a, b) \in \mathbb{R}^2 \mid t \leqslant 0\}$. Prove that F is conservative on the domain $\mathbb{R}^2 \setminus H$.
(c) Let α be a C^1 path running once round the circle centre (a, b) with radius $c < \sqrt{(a^2 + b^2)}$. Prove that $\int_\alpha F \cdot d\alpha = 0$.

9. Let $D = R \cup L$ be the union of open half-planes in $\mathbb{R}^2$

$$R = \{(x, y) \in \mathbb{R}^2 \mid x > 0\} \qquad \text{and} \qquad L = \{(x, y) \in \mathbb{R}^2 \mid x < 0\}.$$

Let $F: D \subseteq \mathbb{R}^2 \to \mathbb{R}^2$ be the continuous vector field defined by

$$F(x, y) = \begin{cases} (y, x) & \text{when} \quad (x, y) \in R, \\ (2x, 0) & \text{when} \quad (x, y) \in L. \end{cases}$$

Prove that the field F is conservative. Verify that the four properties of Theorem 5.5.6 are satisfied. In particular, express the field F in the form $F = \operatorname{grad} f$.

Answer: F is the gradient of the C^1 function $f: D \subseteq \mathbb{R}^2 \to \mathbb{R}$ defined by

$$f(x, y) = \begin{cases} xy & \text{when} \quad (x, y) \in R, \\ x^2 & \text{when} \quad (x, y) \in L. \end{cases}$$

10. Let D be the disjoint union of two path connected open subsets A and B in $\mathbb{R}^n$. There is then no path in D from a point of A to a point of B. Prove Theorem 5.5.6 for this case.

Hint: in proving that (i) implies (ii) choose a base point in each of the path components A and B.

11. With the open subset D of $\mathbb{R}^2$ and the conservative field F defined as in Exercise 9, find two potential functions g and h for F that do not differ by a constant. Why does this not contradict Theorem 5.5.16(i)?

Answer: choose for example $g = -f$, where f is as defined in Exercise 9, and define

$$h(x, y) = \begin{cases} -xy & \text{when} \quad (x, y) \in R. \\ -x^2 + 1 & \text{when} \quad (x, y) \in L. \end{cases}$$

The set D is not path connected.

12. Let $F: D \subseteq \mathbb{R}^2 \to \mathbb{R}^2$ be a C^1 vector field defined on an open disc D in $\mathbb{R}^2$. Prove that a necessary and sufficient condition for F to be conservative is that $\partial F_1/\partial y = \partial F_2/\partial x$ on D, where F_1, F_2 are the coordinate functions of F.

Proof of sufficiency. assume that $\partial F_1/\partial y = \partial F_2/\partial x$. Fix (a, b) in D and define

$$f_1(x, y) = \int_a^x F_1(t, y)\,\mathrm{d}t, \qquad \text{for all } (x, y) \in D.$$

Show that $\partial f_1/\partial x = F_1$ and that $g = F_2 - \partial f_1/\partial y$ is independent of x. Define

$$f_2(y) = \int_b^y g(t)\,\mathrm{d}t,$$

and put $f(x, y) = f_1(x, y) + f_2(y)$. Then $F = \operatorname{grad} f$.

13. Let $F: D \subseteq \mathbb{R}^3 \to \mathbb{R}^3$ be a C^1 vector field defined on an open ball D in $\mathbb{R}^3$. Prove that F is conservative if and only if the following functional relations hold on D:

$$\frac{\partial F_1}{\partial y} = \frac{\partial F_2}{\partial x}, \quad \frac{\partial F_1}{\partial z} = \frac{\partial F_3}{\partial x}, \quad \frac{\partial F_2}{\partial z} = \frac{\partial F_3}{\partial y}.$$

14. Show that a field is conservative if and only if the work done by the field in moving a particle is recoverable by returning the particle to its original position.

15. Show that a conservative C^2 vector field is irrotational.

Proof: Theorem 4.10.9.

16. Let D be a path connected subset of $\mathbb{R}^m$. Let $f: D \subseteq \mathbb{R}^m \to \mathbb{R}^n$ be continuous. Show that $f(D)$ is path connected.