Faculty of Science
Dept. of Mathematics

# Notes on Calculus

# Mat 111

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## Department of Mathematics

## For

## First Level Students
## Biology

## 2023

# Contents

# Chapter 1

# Functions

## 1.1 Basic Definitions and Examples

We begin with a short discussion of real numbers. This gives us the opportunity to recall some basic properties and standard notation.

**Definition 1.1.** *A **real number** is a number represented by a decimal or "decimal expansion."*

There are three types of decimal expansions: finite, repeating, and infinite but nonrepeating. For example,

$$\frac{3}{8} = 0.375, \quad \frac{1}{7} = 0.142857142857\ldots = 0.\overline{142857}, \ \pi = 3.141592653589793\ldots$$

The number $\frac{3}{8}$ is represented by a finite decimal, whereas $\frac{1}{7}$ is represented by a repeating or periodic decimal. The bar over 142857 indicates that this sequence repeats indefinitely. The decimal expansion of $\pi$ is infinite but nonrepeating.

The set of all real numbers is denoted by a boldface **R**. When there is no risk of confusion, we refer to a real number simply as a number. We also use the standard symbol $\in$ for the phrase "belongs to." Thus, $a \in \mathbf{R}$ reads " $a$ belongs to $\mathbf{R}$ ".

The set of integers is commonly denoted by the letter $\mathbf{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$. A whole number is a nonnegative integer-that is, one of the numbers $0, 1, 2, \ldots$

**Definition 1.2.** *A real number is called **rational** if it can be represented by a fraction $p/q$, where $p$ and $q$ are integers with $q \neq 0$.*

The set of rational numbers is denoted **Q** (for "quotient"). Numbers that are not rational, such as $\pi$ and $\sqrt{2}$, are called irrational.

We can tell whether a number is rational from its decimal expansion: Rational numbers have finite or repeating decimal expansions, and irrational numbers have infinite, non- repeating decimal expansions. Furthermore, the decimal expansion of a number is unique, apart from the following exception: Every finite decimal is equal to an infinite decimal in which the digit 9 repeats. For example,

$$1 = 0.999\ldots, \quad \frac{3}{8} = 0.375 = 0.374999\ldots, \quad \frac{47}{20} = 2.35 = 2.34999\ldots$$

We visualize real numbers as points on a line (Figure 1.43). For this reason, real numbers are often referred to as points. The point corresponding to 0 is called the origin.

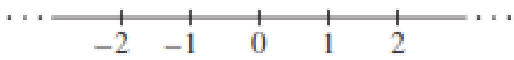



FIGURE 1.1: The set of real numbers represented as a line.

**Definition 1.3.** *The* ***absolute value*** *of a real number $a$, denoted $|a|$, is defined by (Figure 1.2)*

$$|a| = \text{ distance from the origin } = \begin{cases} a & \text{if } a \geq 0 \\ -a & \text{if } a < 0 \end{cases}$$

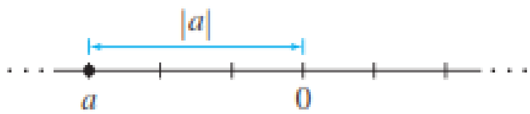



FIGURE 1.2: $|a|$ is the distance from $a$ to the origin.

For example, $|1.2| = 1.2$ and $|-8.35| = 8.35$. The absolute value satisfies

$$|a| = |-a|, \quad |ab| = |a||b|$$

The distance between two real numbers $a$ and $b$ is $|b-a|$, which is the length of the line segment joining $a$ and $b$ (Figure 1.3). Beware that $|a+b|$ is not equal to $|a|+|b|$ unless $a$ and $b$ have the same sign or at least one of $a$ and $b$ is zero. If they have opposite signs, cancellation occurs in the sum $a+b$, and $|a+b| < |a|+|b|$.

For example, $|2+5| = |2|+|5|$ but $|-2+5| = 3$, which is less than $|-2|+|5| = 7$. In any case, $|a+b|$ is never larger than $|a|+|b|$ and this gives us the simple but important **triangle inequality:**

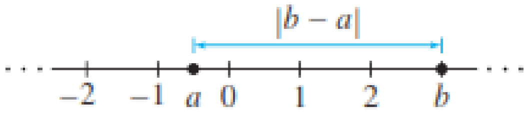



FIGURE 1.3: The distance from $a$ to $b$ is $|b - a|$.

$$\boxed{|a+b| \le |a| + |b|}$$

We use standard notation for intervals. Given real numbers $a < b$, there are four intervals with endpoints $a$ and $b$ (Figure 1.4). They all have length $b - a$ but differ according to which endpoints are included.

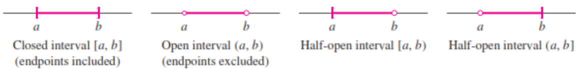



FIGURE 1.4: The four intervals with endpoints $a$ and $b$.

**Definition 1.4.** *The **closed interval** $[a, b]$ is the set of all real numbers $x$ such that $a \le x \le b$ :*

$$[a, b] = \{x \in \mathbf{R} : a \le x \le b\}$$

We usually write this more simply as $\{x : a \le x \le b\}$, it being understood that $x$ belongs to $\mathbf{R}$. The open and half-open intervals are the sets

$$\underbrace{(a, b) = \{x : a < x < b\}}_{\text{Open interval (endpoints excluded)}}, \underbrace{[a, b) = \{x : a \le x < b\}}_{\text{Half-open interval}}, \underbrace{(a, b] = \{x : a < x \le b\}}_{\text{Half-open interval}}$$

The infinite interval $(-\infty, \infty)$ is the entire real line $\mathbf{R}$. A half-infinite interval is closed if it contains its finite endpoint and is open otherwise (Figure 1.5 ):

$$[a, \infty) = \{x : a \le x\}, \quad (-\infty, b] = \{x : x \le b\}$$

Open and closed intervals may be described by inequalities. For example, the interval $(-r, r)$ is described by the inequality $|x| < r$ (Figure 1.6 ):

$$\boxed{|x| < r \quad \Leftrightarrow \quad -r < x < r \quad \Leftrightarrow \quad x \in (-r, r)}$$

More generally, for an interval symmetric about the value $c$ (Figure 1.7 ),

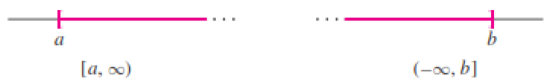



FIGURE 1.5: Closed half-infinite intervals.

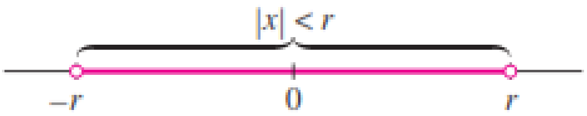



FIGURE 1.6: $|x| < r$.

$$\boxed{|x-c| < r \quad \Leftrightarrow \quad c-r < x < c+r \quad \Leftrightarrow \quad x \in (c-r, c+r)}$$

Closed intervals are similar, with $<$ replaced by $\leq$. We refer to $r$ as the radius and to $c$ as the midpoint or center. The intervals $(a, b)$ and $[a, b]$ have midpoint $c = \frac{1}{2}(a+b)$ and radius $r = \frac{1}{2}(b-a)$ (Figure 1.7).

**Example 1.1.1.** *Describe* $[7, 13]$ *using inequalities.*

***solution:*** *The midpoint of the interval* $[7, 13]$ *is* $c = \frac{1}{2}(7+13) = 10$ *and its radius is* $r = \frac{1}{2}(13-7) = 3$ *(Figure 1.8). Therefore,*

$$[7, 13] = \{x \in \mathbf{R} : |x - 10| \leq 3\}$$

**Example 1.1.2.** *Describe the set*

$$S = \left\{ x : \left| \frac{1}{2}x - 3 \right| > 4 \right\}$$

*in terms of intervals.*

***solution:*** *It is easier to consider the opposite inequality* $\left|\frac{1}{2}x - 3\right| \leq 4$ *.*

$$\begin{aligned} \left| \frac{1}{2}x - 3 \right| \leq 4 \quad \Leftrightarrow \quad & -4 \leq \frac{1}{2}x - 3 \leq 4 \\ & -1 \leq \frac{1}{2}x \leq 7 \quad \textit{(add 3)} \\ & -2 \leq x \leq 14 \quad \textit{(multiply by 2)} \end{aligned}$$

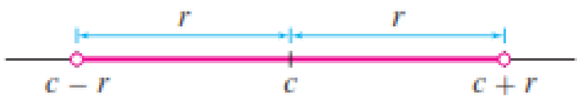



FIGURE 1.7: $|x - c| < r$.

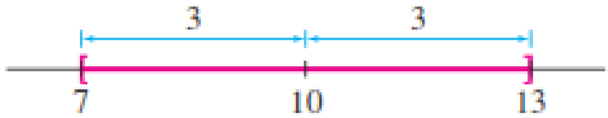



FIGURE 1.8: The interval $[7, 13]$ is described by $|x - 10| \leq 3$.

*Thus,* $\left|\frac{1}{2}x - 3\right| \leq 4$ *is satisfied when* $x$ *belongs to* $[-2, 14]$. *The set* $S$ *is the complement, consisting of all numbers* $x$ *not in* $[-2, 14]$. *We can describe* $S$ *as the union of two intervals:* $S = (-\infty, -2) \cup (14, \infty)$ *(Figure 1.9).*


FIGURE 1.9: The set $S = \left\{x : \left|\frac{1}{2}x - 3\right| > 4\right\}$.

We now review some definitions and notation concerning functions.

**Definition 1.5.** *A function* $f$ *from a set* $D$ *to a set* $Y$ *is a rule that assigns, to each element* $x$ *in* $D$*, a unique element* $y = f(x)$ *in* $Y$*. We write*

$$f : D \to Y$$

The set **D** , called the **domain** of $f$, is the set of "allowable inputs." For $x \in D$, $f(x)$ is called the value of $f$ at $x$, and the set $Y$ is the co-domain of $f$. (Figure 1.10).

**The range R** of $f$ is the subset of $Y$ consisting of all values $f(x)$ :

$$R = \{y \in Y : f(x) = y \text{ for some } x \in D\}$$

Informally, we think of $f$ as a "machine" that produces an output $y$ for every input $x$ in the domain $D$ (Figure 1.11). The domain and range of a function can be any sets of objects, but often in calculus they are sets of real numbers interpreted as points of a coordinate line. Often a function is given by a formula that describes how to calculate the output value from the input variable.

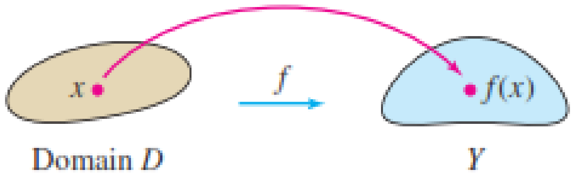



FIGURE 1.10: A function assigns an element $f(x)$ in $Y$ to each $x \in D$ .

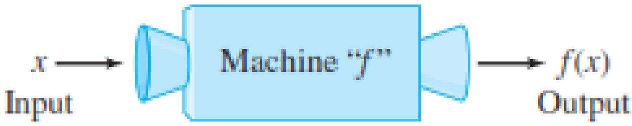



FIGURE 1.11: Think of f as a "machine" that takes the input $x$ and produces the output $f(x)$ .

For instance, the equation $A = pr^2$ is a rule that calculates the area $A$ of a circle from its radius $r$ (so $r$, interpreted as a length, can only be positive in this formula). When we define a function $y = f(x)$ with a formula and the domain is not stated explicitly or restricted by context, the domain is assumed to be the largest set of real $x$-values for which the formula gives real $y$-values, which is called the **natural domain**. If we want to restrict the domain in some way, we must say so. The domain of $y = x^2$ is the entire set of real numbers. To restrict the domain of the function to, say, positive values of $x$, we would write ı$y = x^2, x > 0$.

When the range of a function is a set of real numbers, the function is said to be **real-valued**. The domains and ranges of most real-valued functions of a real variable we consider are intervals or combinations of intervals. The intervals may be open, closed, or half open, and may be finite or infinite. Sometimes the range of a function is not easy to find.

A function $f$ is like a machine that produces an output value $f(x)$ in its range whenever we feed it an input value $x$ from its domain (Figure 1.11). The function keys on a calculator give an example of a function as a machine. For instance, the $\sqrt{x}$ key on a calculator gives an output value (the square root) whenever you enter a nonnegative number $x$ and press the $\sqrt{x}$ key. A function can also be pictured as an arrow diagram (Figure 1.2 ). Each arrow associates an element of the domain $D$ with a unique or single element in the set $Y$. In Figure 1.12, the arrows indicate that $f(a)$ is associated with $a$, $f(x)$ is associated with $x$, and so on. Notice that a function can have the same value at two different input elements in the domain (as occurs with $f(a)$ in Figure 1.2 ), but each input element $x$ is assigned a single output value $f(x)$.

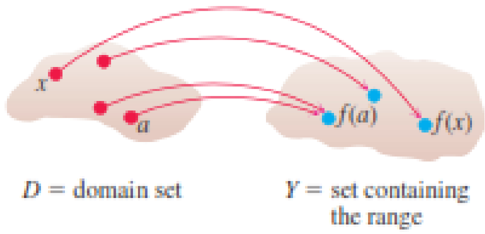



FIGURE 1.12

**Example 1.1.3.** *Let's verify the natural domains and associated ranges of some simple functions.* ***solution:*** *The domains in each case are the values of x for which the formula makes sense.*

| Function | Domain $(x)$ | Range $(y)$ |
|---|---|---|
| $y = x^2$ | $(-\infty, \infty)$ | $[0, \infty)$ |
| $y = 1/x$ | $(-\infty, 0) \cup (0, \infty)$ | $(-\infty, 0) \cup (0, \infty)$ |
| $y = \sqrt{x}$ | $[0, \infty)$ | $[0, \infty)$ |
| $y = \sqrt{4-x}$ | $(-\infty, 4]$ | $[0, \infty)$ |
| $y = \sqrt{1-x^2}$ | $[-1, 1]$ | $[0, 1]$ |

*The formula $y = x^2$ gives a real y-value for any real number $x$, so the domain is $(-\infty, \infty)$. The range of $y = x^2$ is $[0, \infty)$* ***because*** *the square of any real number is nonnegative and every nonnegative number $y$ is the square of its own square root, $y = (\sqrt{y})^2$ for $y \geq 0$.*

*The formula $y = 1/x$ gives a real y-value for every $x$ except $x = 0$. For consistency in the rules of arithmetic, we cannot divide any number by zero. The range of $y = 1/x$, the set of reciprocals of all nonzero real numbers, is the set of all nonzero real numbers, since $y = 1/(1/y)$. That is, for $y \neq 0$ the number $x = 1/y$ is the input assigned to the output value $y$.*

*The formula $y = \sqrt{x}$ gives a real y-value only if $x \geq 0$. The range of $y = \sqrt{x}$ is $[0, \infty)$ because every nonnegative number is some number's square root (namely, it is the square root of its own square).*

*In $y = \sqrt{4-x}$, the quantity $4 - x$ cannot be negative. That is, $4 - x \geq 0$, or $x \leq 4$. The formula gives real y-values for all $x \leq 4$. The range of $\sqrt{4-x}$ is $[0, \infty)$, the set of all nonnegative numbers.*

*The formula $y = \sqrt{1-x^2}$ gives a real y-value for every $x$ in the closed interval from $-1$ to $1$ . Outside this domain, $1 - x^2$ is negative and its square root is not a real number. The values of $1 - x^2$ vary from 0 to 1 on the given domain, and the square roots of these values do the same. The range of $\sqrt{1-x^2}$ is $[0, 1]$.*

**Example 1.1.4.** 1. *Find the range of the function $f : [-1, 8] \to Y$ where $f(x) = 3x + 4$.*

2. *Find the domain of the function $f(x) = \frac{3}{x^2-2x}$*

3. *Find the domain of the function* $g(x) = \sqrt{x^2 - 2x - 8}$

***solution:***

1.

$$\begin{aligned}\text{Im}(f) &= \{f(x) : -1 \leq x \leq 6\} \\ &= \{f(x) : -2 \leq 2x \leq 12\} \\ &= \{5 \leq f(x) = 2x + 3 \leq 19\} \\ &= [5, 19]\end{aligned}$$

2. *Note that* $f(x)$ *is a real number if and only if the denominator* $x^2 - 2x \neq 0$ *So,*

$$\begin{aligned}D(f) &= \left\{x \in R; x^2 - 2x \neq 0\right\} \\ &= \{x \in R : x(x-2) \neq 0\} \\ D(f) &= \{x \in R; x \neq 0, x \neq 2\} \\ &= (-\infty, 0) \cup (0, 2) \cup (2, \infty) \\ &= R \backslash \{0, 2\}\end{aligned}$$

3. *Note that* $f(x)$ *is a real number if and only if the radicand* $x^2 - 2x - 8$ *is nonnegative. Thus,* $f(x)$ *exists if and only if* $x^2 - 2x - 8 \geq 0$ *or , equivalently,* $(x-4)(x+2) \geq 0$. *By solving this inequality, we obtain* $x \geq 4$ *or* $x \leq -2$ *Hence,* $D(f) = (-\infty, -2] \cup [4, \infty)$

**Example 1.1.5.** *Let*

$$f(x) = \frac{\sqrt{4+x}}{1-x}$$

***(a)*** *Find the domain of* $f$.

***(b)*** *Find* $f(5), f(-2), f(-a)$, *and* $-f(a)$.

***solution : (a)*** *Note that* $f(x)$ *exists if and only if* $4 + x \geq 0$ *and* $1 - x \neq 0$ *or, equivalently,* $x \geq -4$ *and* $x \neq -1$ *Hence,* $D(f) = [-4, 1) \cup (1, \infty)$.

***(b)*** *Left to the reader.*

## 1.2 Graphs of Functions

If $f$ is a function with domain $D$, its graph consists of the points in the Cartesian plane whose coordinates are the input-output pairs for $f$. In set notation, the graph is

$$\{(x, f(x)) \mid x \in D\}$$

The graph of the function $f(x) = x + 2$ is the set of points with coordinates $(x, y)$ for which $y = x + 2$. Its graph is the straight line sketched in Figure 1.13. The graph of a function $f$ is a

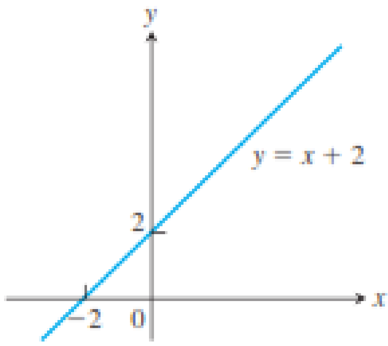



FIGURE 1.13

useful picture of its behavior. If $(x, y)$ is a point on the graph, then $y = f(x)$ is the height of the graph above (or below) the point $x$. The height may be positive or negative, depending on the sign of $f(x)$ (Figure 1.14).

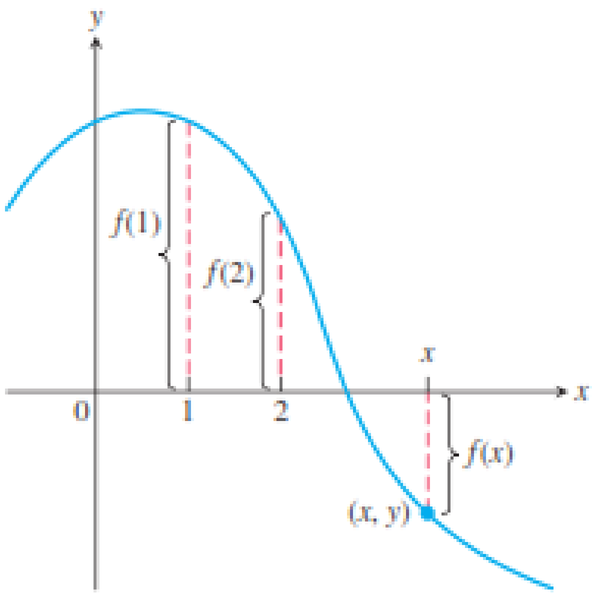



FIGURE 1.14

**Example 1.2.1.** *Graph the function $y = x^2$ over the interval $[-2, 2]$.*

***solution :*** *Make a table of xy-pairs that satisfy the equation $y = x^2$. Plot the points $(x, y)$ whose coordinates appear in the table, and draw a smooth curve (labeled with its equation) through the plotted points (see Figure 1.15).*

| $x$ | $y = x^2$ |
|---|---|
| $-2$ | 4 |
| $-1$ | 1 |
| 0 | 0 |
| 1 | 1 |
| $\frac{3}{2}$ | $\frac{9}{4}$ |
| 2 | 4 |

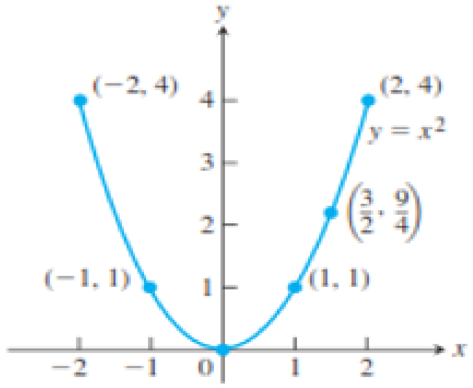
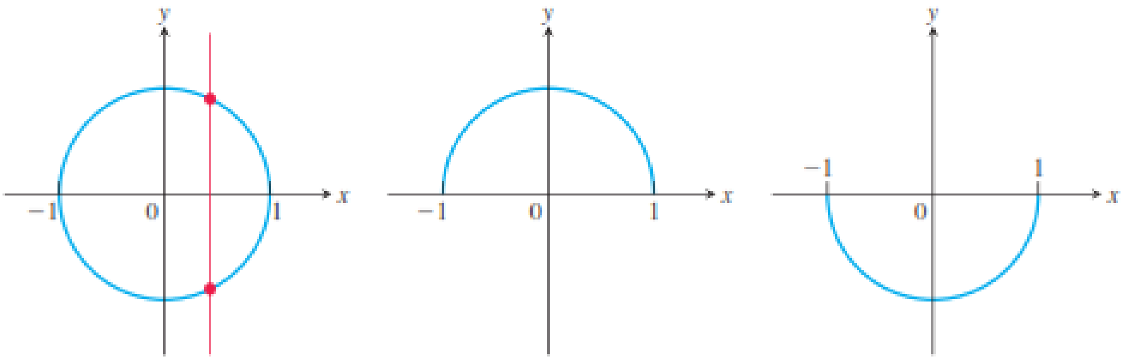



FIGURE 1.15

**The Vertical Line Test for a Function**

Not every curve in the coordinate plane can be the graph of a function. A function $f$ can have only one value $f(x)$ for each $x$ in its domain, so no *vertical* line can intersect the graph of a function more than once. If $a$ is in the domain of the function $f$, then the vertical line $x = a$ will intersect the graph of $f$ at the single point $(a, f(a))$.

A circle cannot be the graph of a function, since some vertical lines intersect the circle, twice. The circle graphed in Figure 1.17a, however, does contain the graphs of functions of $x$, such as the upper semicircle defined by the function $f(x) = \sqrt{1 - x^2}$ and the lower semicircle defined by the function $g(x) = -\sqrt{1 - x^2}$ (Figures 1.17 b and 1.17c ).




(a) $x^2 + y^2 = 1$ (b) $y = \sqrt{1 - x^2}$ (c) $y = -\sqrt{1 - x^2}$

FIGURE 1.16

**Piecewise-Defined Functions**

Sometimes a function is described in pieces by using different formulas on different parts of its domain. One example is the **absolute value function**

$$|x| = \begin{cases} x, & x \geq 0 \quad \text{First formula} \\ -x, & x < 0, \quad \text{Second formula} \end{cases}$$

The right-hand side of the equation means that the function equals $x$ if $x \geq 0$, and equals $-x$ if $x < 0$. Piecewise-defined functions often arise when real-world data are modeled. Here are some other examples.

**Example 1.2.2.** *The function*

$$f(x) = \begin{cases} -x, & x < 0 & \textit{First formula} \\ x^2, & 0 \leq x \leq 1 & \textit{Second formula} \\ 1, & x > 1 & \textit{Third formula} \end{cases}$$

*is defined on the entire real line but has values given by different formulas, depending on the position of $x$. The values of $f$ are given by $y = -x$ when $x < 0, y = x^2$ when $0 \leq x \leq 1$, and $y = 1$ when $x > 1$. The function, however, is just one function whose domain is the entire set of real numbers (Figure 1.17).*

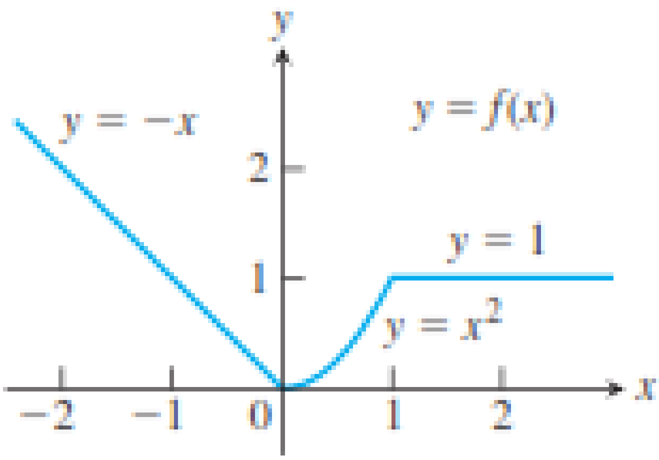



FIGURE 1.17

**Example 1.2.3.** *The function whose value at any number $x$ is the greatest integer less than or equal to $x$ is called the greatest integer function or the integer floor function. It is denoted $\lfloor x \rfloor$. Figure 1.17* *shows the graph. Observe that*

$$\lfloor 2.4 \rfloor = 2, \quad \lfloor 1.9 \rfloor = 1, \quad \lfloor 0 \rfloor = 0, \quad \lfloor -1.2 \rfloor = -2$$
$$\lfloor 2 \rfloor = 2, \quad \lfloor 0.2 \rfloor = 0, \quad \lfloor -0.3 \rfloor = -1, \quad \lfloor -2 \rfloor = -2.$$

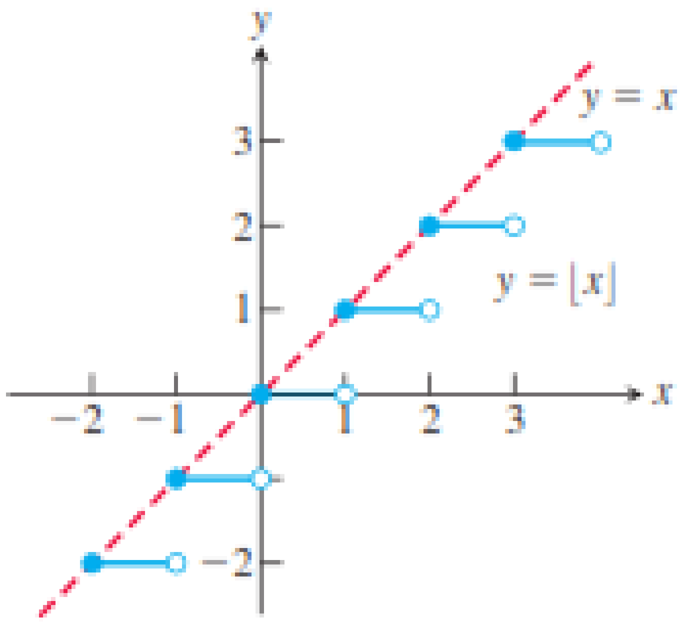



FIGURE 1.18

## 1.3 Combining Functions

In this section we look at the main ways functions are combined or transformed to form new functions. We often complicated functions from simpler functions by combining them in various ways, using arithmetic operations and composition.

Like numbers, functions can be added, subtracted, multiplied, and divided (except where the denominator is zero) to produce new functions.

### 1.3.1 Algebra of Functions

**Definition 1.6.** *If $f$ and $g$ are functions, then for every $x$ that belongs to the domains of both $f$ and $g$ (that is, for $x \in D(f) \cap D(g)$), we define functions $f+g$, $f-g$, and $fg$ by the formulas*

$$(f+g)(x) = f(x) + g(x)$$
$$(f-g)(x) = f(x) - g(x)$$
$$(fg)(x) = f(x)g(x)$$

Notice that the $+$ sign on the left-hand side of the first equation represents the operation of addition of functions, whereas the $+$ on the right-hand side of the equation means addition of the real numbers $f(x)$ and $g(x)$.

**Definition 1.7.** *At any point of $D(f) \cap D(g)$ at which $g(x) \neq 0$, we can also define the function $f/g$. by the formula*

$$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)} \quad (\text{ where } g(x) \neq 0)$$

**Definition 1.8.** *Functions can also be multiplied by constants: If $c$ is a real number, then the function $cf$ is defined for all $x$ in the domain of $f$ by*

$$(cf)(x) = cf(x)$$

**Example 1.3.1.** *The functions defined by the formulas*

$$f(x) = \sqrt{x} \quad \text{and} \quad g(x) = \sqrt{1-x}$$

*have domains $D(f) = [0, \infty)$ and $D(g) = (-\infty, 1]$. The points common to these domains are the points*

$$[0, \infty) \cap (-\infty, 1] = [0, 1]$$

**Example 1.3.2.** *Let*

$$f(x) = \sqrt{9-x^2} \quad \text{and} \quad g(x) = \sqrt{x^2-1}$$

***Solution:*** *Note that, the domain of $f$ is the closed interval $[-3, 3]$ and the domain of $g$ is $(-\infty, -1] \cup [1, \infty)$.*

*Consequently, the domain of $f+g$, $f-g$, $fg$ is $[-3, -1]U[1, 3]$, but the domain of $f/g$ is $[-3, -1) \cup (1, 3]$. So,*

$$(f+g)(x) = f(x) + g(x) = \sqrt{9-x^2} + \sqrt{x^2-1}$$

$$(f-g)(x) = f(x) - g(x) = \sqrt{9-x^2} - \sqrt{x^2-1}$$
$$(fg)(x) = f(x)g(x) = \sqrt{9-x^2}\sqrt{x^2-1}$$
$$(f/g)(x) = \frac{f(x)}{g(x)} = \frac{\sqrt{9-x^2}}{\sqrt{x^2-1}}$$

*The following table summarizes the formulas and domains for the various algebraic combinations of the two functions. We also write $f \cdot g$ for the product function $fg$.*

| Function | Formula | Domain |
|---|---|---|
| $f+g$ | $(f+g)(x) = \sqrt{x} + \sqrt{1-x}$ | $[0,1] = D(f) \cap D(g)$ |
| $f-g$ | $(f-g)(x) = \sqrt{x} - \sqrt{1-x}$ | $[0,1]$ |
| $g-f$ | $(g-f)(x) = \sqrt{1-x} - \sqrt{x}$ | $[0,1]$ |
| $f \cdot g$ | $(f \cdot g)(x) = f(x)g(x) = \sqrt{x(1-x)}$ | $[0,1]$ |
| $f/g$ | $\frac{f}{g}(x) = \frac{f(x)}{g(x)} = \sqrt{\frac{x}{1-x}}$ | $[0,1)(x=1$ *excluded)* |
| $g/f$ | $\frac{g}{f}(x) = \frac{g(x)}{f(x)} = \sqrt{\frac{1-x}{x}}$ | $(0,1](x=0$ *excluded)* |

## 1.3.2 Shifting a Graph of a Function

A common way to obtain a new function from an existing one is by adding a constant to each output of the existing function, or to its input variable. The graph of the new function is the graph of the original function shifted vertically or horizontally, as follows.

**Shift Formulas:**

$$\begin{cases} Vertical Shifts, \quad y = f(x) + k & Shifts the graph of f up \text{k} units if \text{k>0} \\ & Shifts it down \text{ | k | } \quad units if \text{k<0} \\ horizontal Shifts, \quad y = f(x+h) & Shifts the graph of f left \text{h} units if \text{h>0} \\ & Shifts it right \text{ | h | } units if \text{h<0.} \end{cases}$$

**Example 1.3.3.** *(a) Adding 1 to the right-hand side of the formula $y = x^2$ to get $y = x^2 + 1$ shifts the graph up 1 unit (Figure 1.19).*

*(b) Adding $-2$ to the right-hand side of the formula $y = x^2$ to get $y = x^2 - 2$ shifts the graph down 2 units (Figure 1.19).*

*(c) Adding 3 to $x$ in $y = x^2$ to get $y = (x+3)^2$ shifts the graph 3 units to the left, while adding $-2$ shifts the graph 2 units to the right (Figure 1.20). (d) Adding $-2$ to $x$ in $y = |x|$, and then adding $-1$ to the result, gives $y = |x-2| - 1$ } and shifts the graph 2 units to the right and 1 unit down (Figure 1.21).*

**Scaling and Reflecting a Graph of a Function**

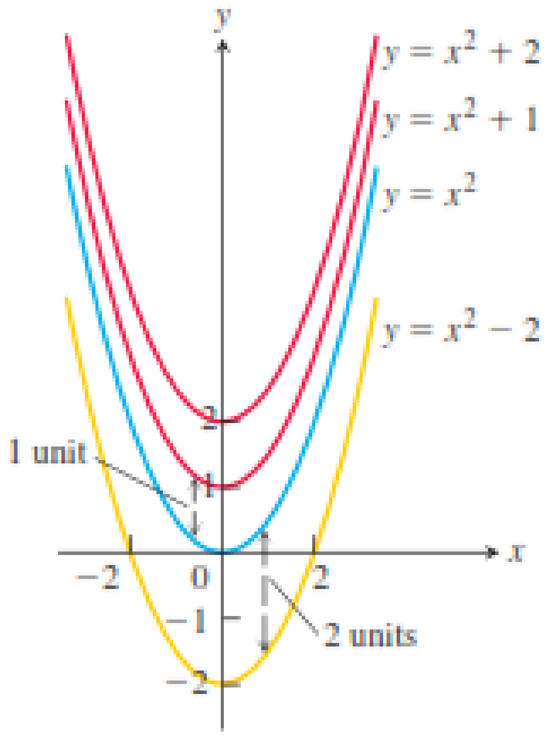



FIGURE 1.19

To scale the graph of a function $y = f(x)$ is to stretch or compress it, vertically or horizontally. This is accomplished by multiplying the function $f$, or the independent variable $x$, by an appropriate constant $c$. Reflections across the coordinate axes are special cases where $c = -1$.

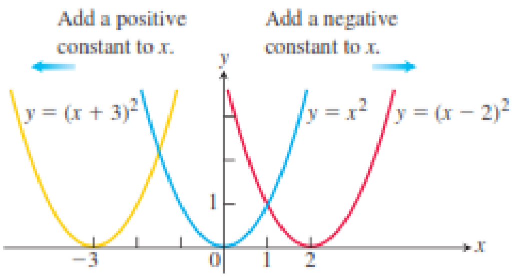



FIGURE 1.20

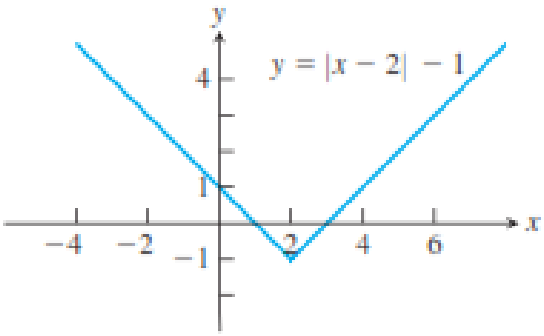



FIGURE 1.21

**Vertical and Horizontal Scaling and Reflecting Formulas**

**For $c > 1$, the graph is scaled:**

$y = cf(x)$ Stretches the graph of $f$ vertically by a factor of $c$

$y = \frac{1}{c}f(x)$ Compresses the graph of $f$ vertically by a factor of $c$

$y = f(cx)$ Compresses the graph of $f$ horizontally by a factor of $c$.

$y = f(x/c)$ Stretches the graph of $f$ horizontally by a factor of $c$

**For $c = -1$, the graph is reflected:**

$y = -f(x)$ **Reflects the graph of** $f$ across the $x$-axis.

$y = \frac{1}{c}f(x)$ Compresses the graph of $f$ vertically by a factor of $c$

$y = f(-x)$ Reflects the graph of $f$ across the $y$-axis.

**Example 1.3.4.** *Here we scale and reflect the graph of* $y = \sqrt{x}$.

***(a) Vertical:*** *Multiplying the right-hand side of* $y = \sqrt{x}$ *by 3 to get* $y = 3\sqrt{x}$ *stretches the graph vertically by a factor of 3 , whereas multiplying by* $1/3$ *compresses the graph by a factor of 3 (Figure 1.22).*

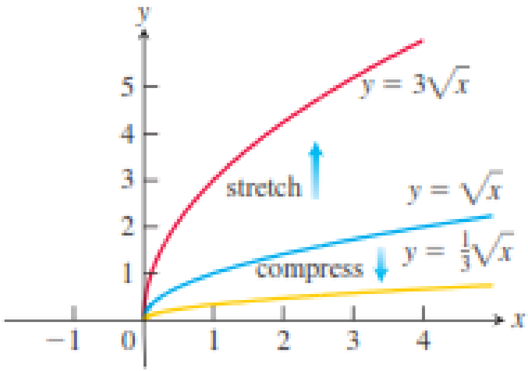



FIGURE 1.22

***(b) Horizontal:*** *The graph of* $y = \sqrt{3x}$ *is a horizontal compression of the graph of* $y = \sqrt{x}$ *by a factor of 3 , and* $y = \sqrt{x/3}$ *is a horizontal stretching by a factor of 3 (Figure 1.23).*

*Note that* $y = \sqrt{3x} = \sqrt{3}\sqrt{x}$ *so a horizontal compression may correspond to a vertical stretching by a different scaling factor. Likewise, a horizontal stretching may correspond to a vertical compression by a different scaling factor.*

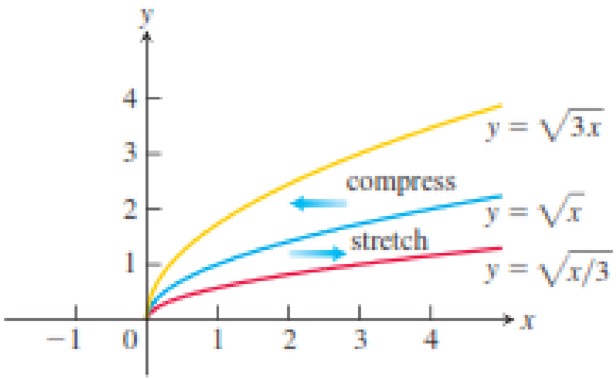



FIGURE 1.23

***(c) Reflection:*** *The graph of* $y = -\sqrt{x}$ *is a reflection of* $y = \sqrt{x}$ *across the x-axis, and* $y = \sqrt{-x}$ *is a reflection across the y-axis (Figure 1.24).*

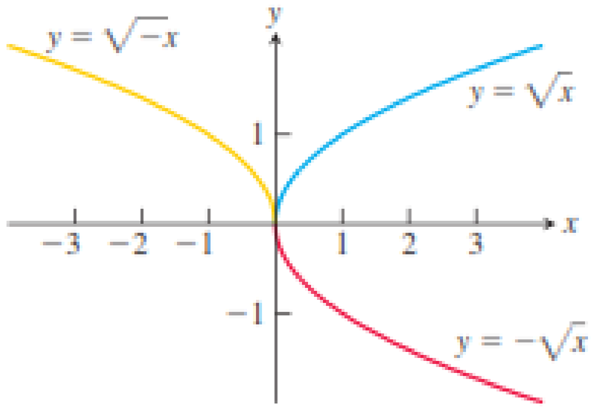



FIGURE 1.24

### 1.3.3 The composite functions

**Composition** is another important way of constructing new functions.

> **Definition 1.9.** *The composition of $f$ and $g$ is the function $f \circ g$ ("$f$ composed with $g$") is defined by $(f \circ g)(x) = f(g(x))$.*
>
> **The domain** of $f \circ g$ is the set of values of $x$ in the domain of $g$ such that $g(x)$ lies in the domain of $f$.

The definition implies that $fog$ can be formed when the range of $g$ lies in the domain of $f$. To find $(fog)(x)$, first find $g(x)$ and second find $f(g(x))$. )Figure 1.25 ) pictures $fog$ as a machine diagram, and (Figure 1.25) shows the composite as an arrow diagram.

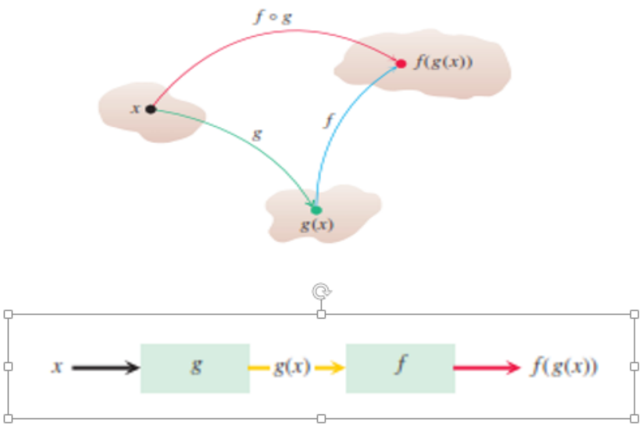



FIGURE 1.25

To evaluate the composite function $gof$ (when defined), we find $(x)$ first and then $g((x))$. The domain of $gof$ is the set of numbers $x$ in the domain of such that $f(x)$ lies in the domain of$g$. The functions $fog$ and $gof$ are usually quite different.

**Example 1.3.5.** *If $f(x) = \sqrt{x}$ and $g(x) = x + 1$, find*

- *(a)* $(f \circ g)(x)$
- *(b)* $(g \circ f)(x)$
- *(c)* $(f \circ f)(x)$
- *(d)* $(g \circ g)(x)$

***Solution***

***Composite*** *(a)* $(f \circ g)(x) = f(g(x)) = \sqrt{g(x)} = \sqrt{x+1}$
***Domain*** $[-1, \infty)$
***Composite*** *(b)* $(g \circ f)(x) = g(f(x)) = f(x) + 1 = \sqrt{x} + 1$
***Domain*** $[0, \infty)$
***Composite****(c)* $(f \circ f)(x) = f(f(x)) = \sqrt{f(x)} = \sqrt{\sqrt{x}} = x^{1/4}$
***Domain***$[0, \infty)$
***Composite****(d)* $(g \circ g)(x) = g(g(x)) = g(x) + 1 = (x+1) + 1 = x + 2$
***Domain*** $(-\infty, \infty)$.

*To see why the domain of $f \circ g$ is $[-1, \infty)$, notice that $g(x) = x + 1$ is defined for all real $x$ but belongs to the domain of $f$ only if $x + 1 \geq 0$, that is to say, when $x \geq -1$.*

***Notice*** *that if $f(x) = x^2$ and $g(x) = \sqrt{x}$, then $(f \circ g)(x) = (\sqrt{x})^2 = x$. However, the domain of $f \circ g$ is $[0, \infty)$, not $(-\infty, \infty)$, since $\sqrt{x}$ requires $x \geq 0$.*

**Example 1.3.6.** *Compute the composite functions $f \circ g$ and $g \circ f$ and discuss their domains, where*

$$f(x) = \sqrt{x}, \quad g(x) = 1 - x$$

***Solution:*** *We have*

$$(f \circ g)(x) = f(g(x)) = f(1-x) = \sqrt{1-x}$$

*The square root $\sqrt{1-x}$ is defined if $1 - x \geq 0$ or $x \leq 1$, so the domain of $f \circ g$ is $\{x : x \leq 1\}$. On the other hand,*

$$(g \circ f)(x) = g(f(x)) = g(\sqrt{x}) = 1 - \sqrt{x}$$

*The domain of $g \circ f$ is $\{x : x \geq 0\}$.*

**Example 1.3.7.** *If $f(x) = \frac{x}{x+2}, g(x) = \frac{x-1}{x}$, Find*

$$(f \circ g)(x), (g \circ f)(x), D(f \circ g), and D(g \circ f)$$

***Solution**: Note that $D(f) = R - \{2\}$, and $D(g) = R - \{0\}$*

$$\begin{aligned}
(f \circ g)(x) &= f(g(x)) = f\left(\frac{x-1}{x}\right) \\
&= \frac{\frac{x-1}{x}}{\frac{x-1}{x} + 2} = \frac{x-1}{3x-1} \\
(g \circ f)(x) &= g(f(x)) = g\left(\frac{x}{x+2}\right) \\
&= \frac{\frac{x}{x+2} - 1}{\frac{x}{x+2}} = \frac{-2}{x} \\
D(f \circ g)(x) &= \{x : x \in D(g), g(x) \in D(f)\} \\
&= \left\{x : x \neq 0, \frac{x-1}{x} \neq -2\right\} \\
&= \left\{x : x \neq 0, x \neq \frac{1}{3}\right\} \\
&= R - \left\{0, \frac{1}{3}\right\} \\
D(g \circ f)(x) &= \{x : x \in D(f), f(x) \in D(g)\} \\
&= \left\{x : x \neq -2, \frac{x}{x+2} \neq 0\right\} \\
&= \{x : x \neq -2, x \neq 0\} \\
&= R - \{-2, 0\}
\end{aligned}$$

**Exercise**
If $f(x) = x^2 - 9$ and $g(x) = \sqrt{x}$, find $(f \circ g)(x), (g \circ f)(x), D(f \circ g)$, and $D(g \circ f)$

## 1.4 Classifying Functions

### 1.4.1 Increasing and Decreasing Functions

If the graph of a function climbs or rises as you move from left to right, we say that the function is increasing. If the graph descends or falls as you move from left to right, the function is decreasing.

1. If $f(x_2) > f(x_1)$ whenever $x_1 < x_2$, then $f$ is said to be **increasing** on $I$.
2. If $f(x_2) < f(x_1)$ whenever $x_1 < x_2$, then $f$ is said to be **decreasing** on $I$.

It is important to realize that the definitions of increasing and decreasing functions must be satisfied for every pair of points $x_1$ and $x_2$ in $I$ with $x_1 < x_2$. Because we use the inequality $<$ to compare the function values, instead of $\leq$, it is sometimes said that $f$ is strictly increasing or decreasing on $I$. The interval $I$ may be finite (also called bounded) or infinite (unbounded) and by definition never consists of a single point .

**Example 1.4.1.** *The function graphed in Figure 1.17 is decreasing on $(-\infty, 0]$ and increasing on $[0, 1]$. The function is neither increasing nor decreasing on the interval $[1, \infty)$ because of the strict inequalities used to compare the function values in the definitions.*

### 1.4.2 Even Functions and Odd Functions: Symmetry

The graphs of even and odd functions have characteristic symmetry properties.

A function $y = f(x)$ is an

- **Even function of** $x$ if $f(-x) = f(x)$.
- **Odd function of** $x$ if $f(-x) = -f(x)$.

for every $x$ in the function's domain.

The names even and odd come from powers of $x$. If $y$ is an even power of $x$, as in $y = x^2$ or $y = x^4$, it is an even function of $x$ because $(-x)^2 = x^2$ and $(-x)^4 = x^4$. If $y$ is an odd power of $x$, as in $y = x$ or $y = x^3$, it is an odd function of $x$ because $(-x)^1 = -x$ and $(-x)^3 = -x^3$

The graph of an even function is **symmetric about the $y$-axis**. Since $f(-x) = f(x)$, a point $(x, y)$ lies on the graph if and only if the point $(-x, y)$ lies on the graph (Figure 1.26 a ). A reflection across the $y$-axis leaves the graph unchanged.

The graph of an odd function is **symmetric about the origin**. Since $f(-x) = -f(x)$, a point $(x, y)$ lies on the graph if and only if the point $(-x, -y)$ lies on the graph (Figure 1.26 b ). Equivalently, a graph is symmetric about the origin if a rotation of $180°$ about the origin leaves the graph unchanged. Notice that the definitions imply that both $x$ and $-x$ must be in the domain of $f$.

**Example 1.4.2.** *Here are several functions illustrating the definition.*

- $f(x) = x^2$ *Even function: $(-x)^2 = x^2$ for all $x$; symmetry about $y$-axis.*

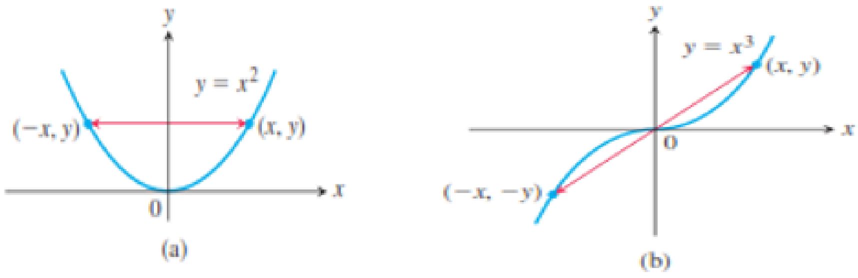



FIGURE 1.26

- $f(x) = x^2 + 1$ *Even function:* $(-x)^2 + 1 = x^2 + 1$ *for all* $x$*; symmetry about* $y$*-axis .*
- $f(x) = x$ *Odd function:* $(-x) = -x$ *for all* $x$*; symmetry about the origin.*
- $f(x) = x + 1$ *Not odd:* $f(-x) = -x + 1$*, but* $-f(x) = -x - 1$*. The two are not equal. Not even:* $(-x) + 1 \neq x + 1$ *for all* $x \neq 0$

**Example 1.4.3.** *Determine whether the function is even, odd, or neither.*

- *(a)* $f(x) = x^4$
- *(b)* $g(x) = x^{-1}$
- *(c)*$h(x) = x^2 + x$

***Solution:***

- *(a)* $f(-x) = (-x)^4 = x^4$*. Thus,* $f(x) = f(-x)$*, and* $f$ *is even.*
- *(b)* $g(-x) = (-x)^{-1} = -x^{-1}$*. Thus,* $g(-x) = -g(x)$*, and* $g$ *is odd.*
- *(c)* $h(-x) = (-x)^2 + (-x) = x^2 - x$*. We see that* $h(-x)$ *is not equal to* $h(x)$ *or to* $-h(x) = -x^2 - x$*. Therefore,* $h$ *is neither even nor odd.*

### 1.4.3 Linear and Quadratic Functions

Linear functions are the simplest of all functions, and their graphs (lines) are the simplest of all curves. However, linear functions and lines play an enormously important role in calculus. For this reason, you should be thoroughly familiar with the basic properties of linear functions and the different ways of writing an equation of a line.

Let's recall that a **linear function** is a function of the form

$$f(x) = mx + b \quad (m \quad \text{and} \quad b \quad \text{constants})$$

The graph of $f$ is a line of slope $m$, and since $f(0) = b$, the graph intersects the $y$-axis at the point $(0, b)$ (Figure 1). The number $b$ is called the $y$-intercept.

The slope-intercept form of the line with slope $m$ and $y$-intercept $b$ is given by

$$y = mx + b$$

We use the symbols $\Delta x$ and $\Delta y$ to denote the change (or increment) in $x$ and $y = f(x)$ over an interval $[x_1, x_2]$ (Figure 1.27):

$$\Delta x = x_2 - x_1, \quad \Delta y = y_2 - y_1 = f(x_2) - f(x_1)$$

The slope $m$ of a line is equal to the ratio

$$m = \frac{\Delta y}{\Delta x} = \frac{\text{vertical change}}{\text{horizontal change}} = \frac{\text{rise}}{\text{run}}$$

This follows from the formula $y = mx + b$ :

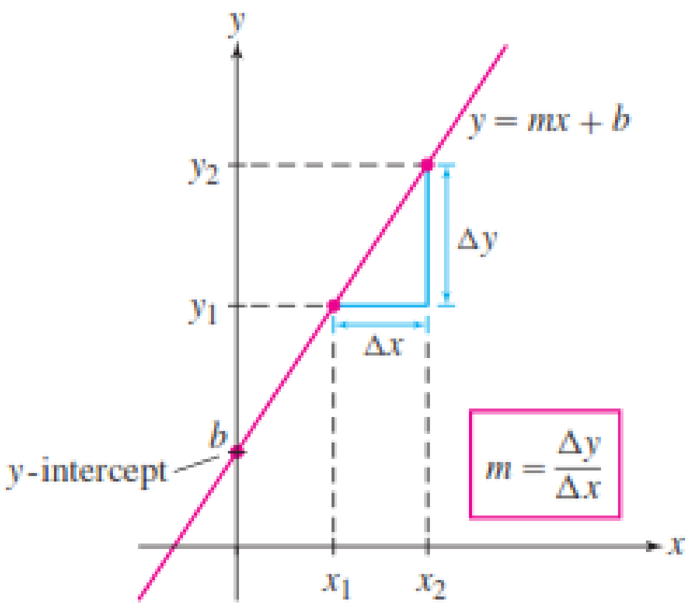



FIGURE 1.27: The slope m is the ratio "rise over run."

$$\frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{(mx_2 + b) - (mx_1 + b)}{x_2 - x_1} = \frac{m(x_2 - x_1)}{x_2 - x_1} = m$$

The slope $m$ measures the rate of change of $y$ with respect to $x$. In fact, by writing

$$\Delta y = m\Delta x$$

we see that a 1 -unit increase in $x$ (i.e., $\Delta x = 1$ ) produces an $m$-unit change $\Delta y$ in $y$. For example, if $m = 5$, then $y$ increases by 5 units per unit increase in $x$. The rate-of-change interpretation of the slope is fundamental in calculus.

**Graphically**, the slope $m$ measures the steepness of the line $y = mx + b$. (Figure 1.28 $A$). shows lines through a point of varying slope $m$.

Note the following properties:

- **Steepness**: The larger the absolute value $|m|$, the steeper the line.

- **Positive slope**: If $m > 0$, the line slants upward from left to right.

- **Negative slope:** If $m < 0$, the line slants downward from left to right. - $f(x) = mx + b$ is increasing if $m > 0$ and decreasing if $m < 0$.

- **The horizontal line** $y = b$ has slope $m = 0$ (Figure 1.28 $B$).

- A **vertical line** has equation $x = c$, where $c$ is a constant. The slope of a vertical line is undefined. It is not possible to write the equation of a vertical line in slope-intercept form $y = mx + b$. A vertical line is not the graph of a function (Figure 1.28 $B$).

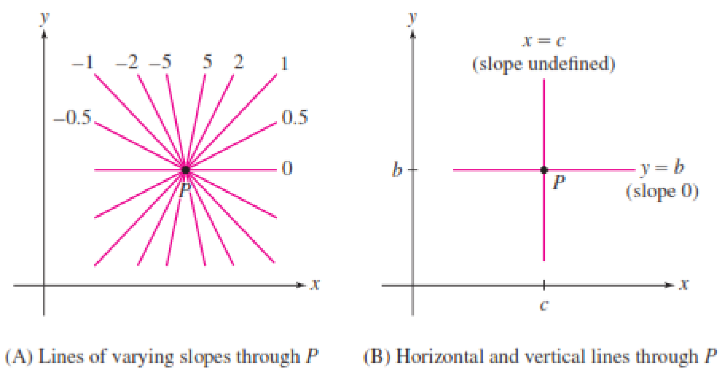



FIGURE 1.28: The slope m is the ratio “rise over run.”

Next, we recall the relation between the slopes of **parallel** and **perpendicular** lines (Figure 1.29):

- Lines of slopes $m_1$ and $m_2$ are parallel if and only if $m_1 = m_2$.

- Lines of slopes $m_1$ and $m_2$ are perpendicular if and only if

$$m_1 = -\frac{1}{m_2} \quad (\text{ or } m_1 m_2 = -1)$$

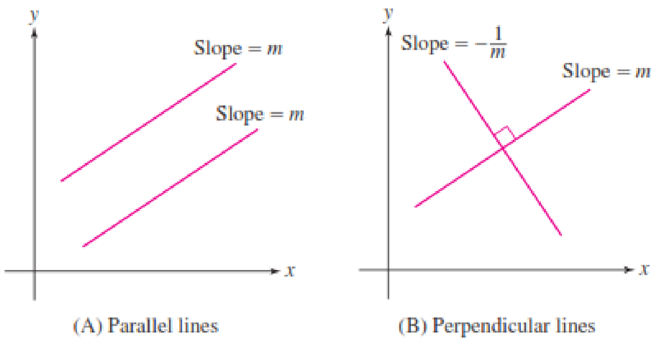



FIGURE 1.29: The slope m is the ratio "rise over run."

As mentioned above, it is important to be familiar with the standard ways of writing the equation of a line. The general linear equation is

$$\boxed{ax + by = c \qquad (1)}$$

where $a$ and $b$ are not both zero. For $b = 0$, we obtain the vertical line $ax = c$. When $b \neq 0$, we can rewrite Eq. (1) in slope-intercept form. For example, $-6x + 2y = 3$ can be rewritten as $y = 3x + \frac{3}{2}$.

Two other forms we will use frequently are the point-slope and point-point forms. Given a point $P = (a, b)$ and a slope $m$, the equation of the line through $P$ with slope $m$ is $y - b = m(x - a)$. Similarly, the line through two distinct points $P = (a_1, b_1)$ and $Q = (a_2, b_2)$ has slope (Figure 7)

$$m = \frac{b_2 - b_1}{a_2 - a_1}$$

Therefore, we can write its equation as $y - b_1 = m(x - a_1)$.

**Additional Equations for Lines:**

1. Point-slope form of the line through $P = (a, b)$ with slope $m$ :

$$y - b = m(x - a)$$

2. Point-point form of the line through

$$P = (a_1, b_1) \text{ and} Q = (a_2, b_2) : y - b_1 = m(x - a_1), \text{ where } m = \frac{b_2 - b_1}{a_2 - a_1}$$

**Example 1.4.4.** *Find the equation of the line through* $(9, 2)$ *with slope* $-\frac{2}{3}$

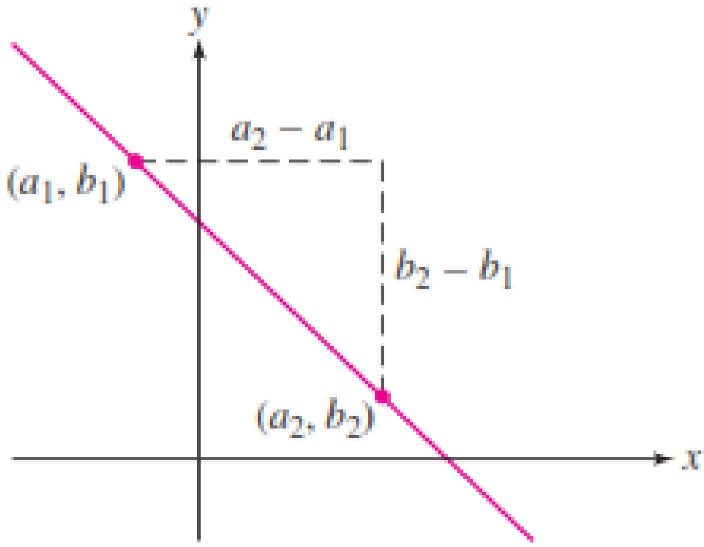



FIGURE 1.30

***Solution*** *In point-slope form:*

$$y - 2 = -\frac{2}{3}(x - 9)$$

*In slope-intercept form:*

$$y = -\frac{2}{3}(x - 9) + 2 \quad or \quad y = -\frac{2}{3}x + 8.$$

**Example 1.4.5.** *Find the equation of the line through* $(2, 1)$ *and* $(9, 5)$.

***Solution*** *The line has slope*

$$m = \frac{(5 - 1)}{(9 - 2)} = \frac{4}{7}$$

*Because* $(2, 1)$ *lies on the line, its equation in point-slope form is*

$$y - 1 = -\frac{4}{7}(x - 2).$$

**A quadratic function:** is a function defined by a quadratic polynomial

$$f(x) = ax^2 + bx + c \quad (a, b, c, \text{ constants with } a \neq 0)$$

The graph of $f$ is a parabola (Figure 1.31). The parabola opens upward if the leading coefficient $a$ is positive and downward if $a$ is negative. The discriminant of $f(x)$ is the quantity

$$D = b^2 - 4ac$$

The roots of $f$ are given by the quadratic formula :

$$\text{Roots of } f = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-b \pm \sqrt{D}}{2a}$$

The sign of $D$ determines whether or not $f$ has real roots (Figure 1.31). If $D > 0$, then $f$ has two real roots, and if $D = 0$, it has one real root (a "double root"). If $D < 0$, then $\sqrt{D}$ is imaginary and $f$ has no real roots.

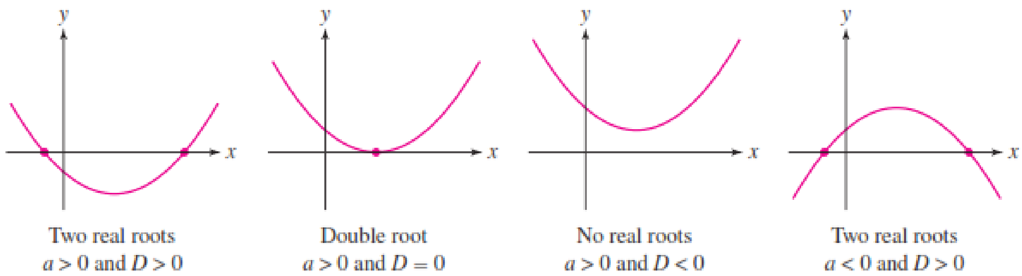



FIGURE 1.31

When $f$ has two real roots $r_1$ and $r_2$, then $f(x)$ factors as

$$f(x) = a\,(x - r_1)\,(x - r_2)$$

For example, $f(x) = 2x^2 - 3x + 1$ has discriminant $D = b^2 - 4ac = 9 - 8 = 1 > 0$ and by the quadratic formula, its roots are $(3 \pm 1)/4$ or 1 and $\frac{1}{2}$. Therefore,

$$f(x) = 2x^2 - 3x + 1 = 2(x-1)\left(x - \frac{1}{2}\right)$$

The technique of **completing the square** consists of writing a quadratic polynomial as a multiple of a square plus a constant. Then

$$x^2 + bx + c = x^2 + bx + \left(\frac{b}{2}\right)^2 - \left(\frac{b}{2}\right)^2 + c = \left(x + \frac{b}{2}\right)^2 - \left(\frac{b}{2}\right)^2 + c$$

If there is a constant $a$ multiplying the $x^2$ term, we factor that out first, as demonstrated in the following example.

**Example 1.4.6.** *Completing the Square Complete the square for the quadratic polynomial* $f(x) = 4x^2 - 12x + 3$

***Solution:*** *First factor out the leading coefficient:*

$$4x^2 - 12x + 3 = 4\left(x^2 - 3x + \frac{3}{4}\right)$$

*Then complete the square for the term* $x^2 - 3x$ *:*

$$x^2 - 3x = x^2 - 3x + \left(\frac{3}{2}\right)^2 - \left(\frac{3}{2}\right)^2 = \left(x - \frac{3}{2}\right)^2 - \frac{9}{4}$$

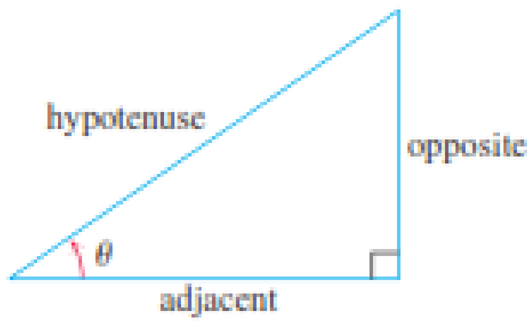
*Therefore,*

$$4x^2 - 12x + 3 = 4\left(\left(x - \frac{3}{2}\right)^2 - \frac{9}{4} + \frac{3}{4}\right) = 4\left(x - \frac{3}{2}\right)^2 - 6$$

### 1.4.4 Trigonometric Functions

**The Six Basic Trigonometric Functions**

You are probably familiar with defining the trigonometric functions of an acute angle in terms of the sides of a right triangle (Figure 1.32).




$$\sin\theta = \frac{\text{opp}}{\text{hyp}} \qquad \csc\theta = \frac{\text{hyp}}{\text{opp}}$$

$$\cos\theta = \frac{\text{adj}}{\text{hyp}} \qquad \sec\theta = \frac{\text{hyp}}{\text{adj}}$$

$$\tan\theta = \frac{\text{opp}}{\text{adj}} \qquad \cot\theta = \frac{\text{adj}}{\text{opp}}$$

FIGURE 1.32: Trigonometric ratios of an acute angle.

We extend this definition to obtuse and negative angles by first placing the angle in standard position in a circle of radius r. We then define the trigonometric functions in terms of the coordinates of the point P(x, y) where the angle's terminal ray intersects the circle (Figure 1.32).

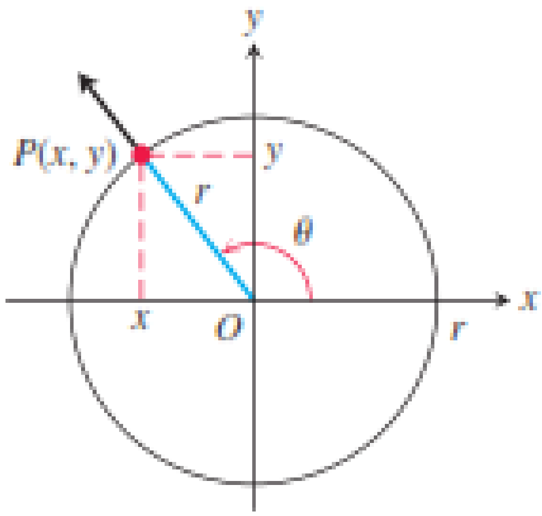



FIGURE 1.33

$$
\begin{array}{llll}
\text{sine:} & \sin\theta = \dfrac{y}{r} & \text{cosecant:} & \csc\theta = \dfrac{r}{y} \\
\text{cosine:} & \cos\theta = \dfrac{x}{r} & \text{secant:} & \sec\theta = \dfrac{r}{x} \\
\text{tangent:} & \tan\theta = \dfrac{y}{x} & \text{cotangent:} & \cot\theta = \dfrac{x}{y}
\end{array}
$$

These extended definitions agree with the right-triangle definitions when the angle is acute. Notice also that whenever the quotients are defined,

$$
\begin{array}{ll}
\tan\theta = \dfrac{\sin\theta}{\cos\theta} & \cot\theta = \dfrac{1}{\tan\theta} \\
\sec\theta = \dfrac{1}{\cos\theta} & \csc\theta = \dfrac{1}{\sin\theta}
\end{array}
$$

As you can see, $\tan\theta$ and $\sec\theta$ are not defined if $x = \cos\theta = 0$. This means they are not defined if $\theta$ is $\pm\pi/2, \pm 3\pi/2, \ldots$. Similarly, $\cot\theta$ and $\csc\theta$ are not defined for values of $\theta$ for which $y = 0$, namely $\theta = 0, \pm\pi, \pm 2\pi, \ldots$.

The exact values of these trigonometric ratios for some angles can be read from the triangles in Figure 1.41. For instance,

$$\begin{array}{lll} \sin\frac{\pi}{4} = \frac{1}{\sqrt{2}} & \sin\frac{\pi}{6} = \frac{1}{2} & \sin\frac{\pi}{3} = \frac{\sqrt{3}}{2} \\ \cos\frac{\pi}{4} = \frac{1}{\sqrt{2}} & \cos\frac{\pi}{6} = \frac{\sqrt{3}}{2} & \cos\frac{\pi}{3} = \frac{1}{2} \\ \tan\frac{\pi}{4} = 1 & \tan\frac{\pi}{6} = \frac{1}{\sqrt{3}} & \tan\frac{\pi}{3} = \sqrt{3} \end{array}$$

The CAST rule (Figure 1.42 ) is useful for remembering when the basic trigonometric functions are positive or negative. For instance, from the triangle in (Figure **??**), we see that

$$\sin\frac{2\pi}{3} = \frac{\sqrt{3}}{2}, \quad \cos\frac{2\pi}{3} = -\frac{1}{2}, \quad \tan\frac{2\pi}{3} = -\sqrt{3}$$

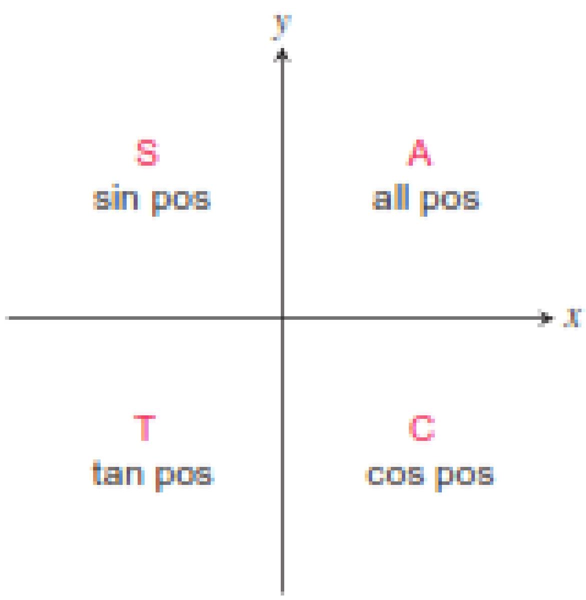



FIGURE 1.34: The CAST rule.

**Periodicity and Graphs of the Trigonometric Functions**

When an angle of measure $\theta$ and an angle of measure $\theta + 2\pi$ are in standard position, their terminal rays coincide. The two angles therefore have the same trigonometric function values: $\sin(\theta + 2\pi) = \sin\theta, \tan(\theta + 2\pi) = \tan\theta$, and so on. Similarly, $\cos(\theta - 2\pi) = \cos\theta, \sin(\theta - 2\pi) = \sin\theta$, and so on. We describe this repeating behavior by saying that the six basic trigonometric functions are periodic.

**Definition 1.10.** *A function $f(x)$ is **periodic** if there is a positive number $p$ such that $f(x+p) = f(x)$ for every value of $x$. The smallest such value of $p$ is the period of $f$*

When we graph trigonometric functions in the coordinate plane, we usually denote the independent variable by $x$ instead of $\theta$. (Figure 1.37) shows that the tangent and cotangent functions have period $p = \pi$, and the other four functions have period $2\pi$. Also, the symmetries in these graphs reveal that the cosine and secant functions are even and the other four functions are odd (although this does not prove those results).

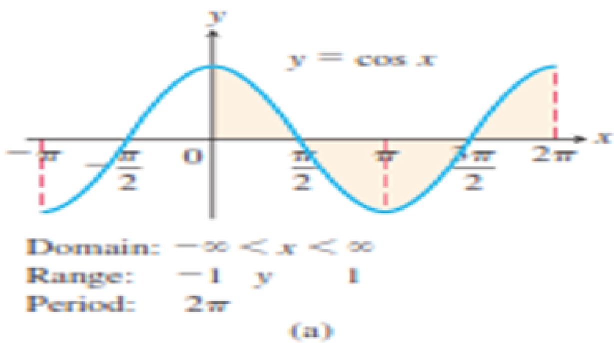



FIGURE 1.35

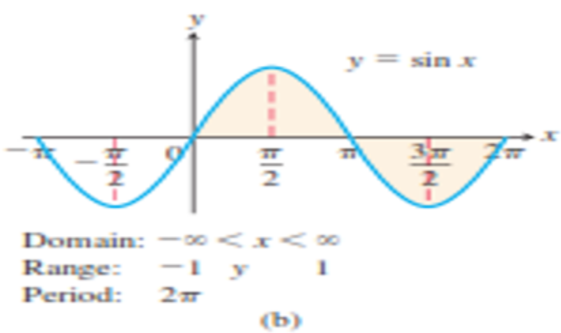



FIGURE 1.36

**Example 1.4.7.** *Find the values of the six trigonometric functions at $x = 4\pi/3$*

***Solution:*** *The point $P$ on the unit circle corresponding to the angle $x = 4\pi/3$ lies opposite the point with angle $\pi/3$ (Figure 1.37).*

$$\sin\frac{4\pi}{3} = -\sin\frac{\pi}{3} = -\frac{\sqrt{3}}{2}, \quad \cos\frac{4\pi}{3} = -\cos\frac{\pi}{3} = -\frac{1}{2}$$

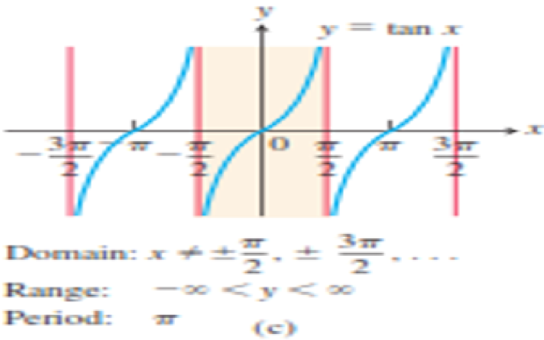



FIGURE 1.37

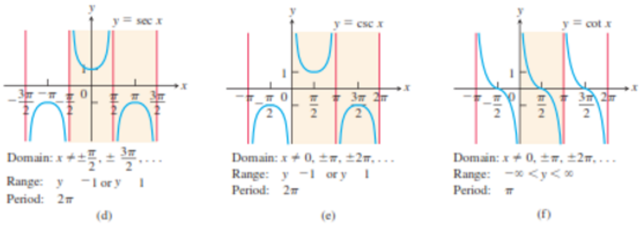



FIGURE 1.38

*The remaining values are*

$$\tan\frac{4\pi}{3} = \frac{\sin 4\pi/3}{\cos 4\pi/3} = \frac{-\sqrt{3}/2}{-1/2} = \sqrt{3}, \quad \cot\frac{4\pi}{3} = \frac{\cos 4\pi/3}{\sin 4\pi/3} = \frac{\sqrt{3}}{3}$$
$$\sec\frac{4\pi}{3} = \frac{1}{\cos 4\pi/3} = \frac{1}{-1/2} = -2, \quad \csc\frac{4\pi}{3} = \frac{1}{\sin 4\pi/3} = \frac{-2\sqrt{3}}{3}$$

**Example 1.4.8.** *Find the angles $x$ such that* $\sec x = 2$

***Solution:*** *Because* $\sec x = 1/\cos x$, *we must solve* $\cos x = \frac{1}{2}$. *From (Figure 1.37). we see that $x = \pi/3$ and $x = -\pi/3$ are solutions. We may add any integer multiple of $2\pi$, so the general solution is $x = \pm\pi/3 + 2\pi k$ for any integer $k$.*

**Trigonometric Identities**

A key feature of trigonometric functions is that they satisfy a large number of identities. First and foremost, sine and cosine satisfy a fundamental identity, which is equivalent to the **Pythagorean Theorem:**

$$\boxed{\sin^2 x + \cos^2 x = 1 \qquad (1)}$$

Equivalent versions are obtained by dividing Eq. (1) by $\cos^2 x$ or $\sin^2 x$ :

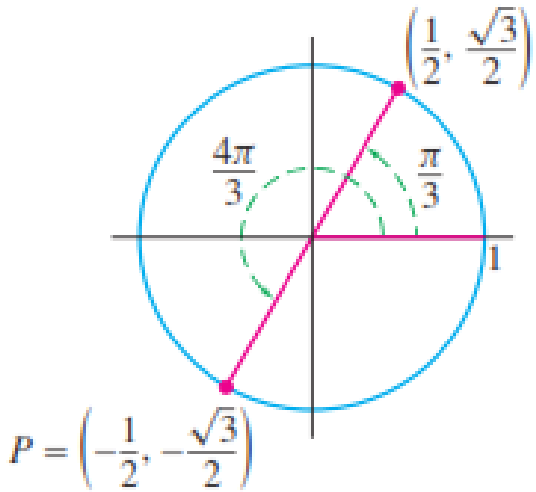



FIGURE 1.39

$$\boxed{\tan^2 x + 1 = \sec^2 x, \quad 1 + \cot^2 x = \csc^2 x}$$

**Example 1.4.9.** *Suppose that* $\cos\theta = \frac{2}{5}$. *Calculate* $\tan\theta$ *in the following two cases:*

***(a)*** $0 < \theta < \frac{\pi}{2}$ *and*

***(b)*** $\pi < \theta < 2\pi$

***Solution:***

*First, using the identity* $\cos^2\theta + \sin^2\theta = 1$, *we obtain*

$$\sin\theta = \pm\sqrt{1-\cos^2\theta} = \pm\sqrt{1-\frac{4}{25}} = \pm\frac{\sqrt{21}}{5}$$

***(a)*** *If* $0 < \theta < \frac{\pi}{2}$, *then* $\sin\theta$ *is positive and we take the positive square root:*

$$\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{\sqrt{21}/5}{2/5} = \frac{\sqrt{21}}{2}$$

*To visualize this computation, draw a right triangle with angle* $\theta$ *such that* $\cos\theta = \frac{2}{5}$ . *The opposite side then has length* $\sqrt{21} = \sqrt{5^2 - 2^2}$ *by the Pythagorean Theorem.*

***(b)*** *If* $\pi < \theta < 2\pi$, *then* $\sin\theta$ *is negative and* $\tan\theta = -\frac{\sqrt{21}}{2}$.

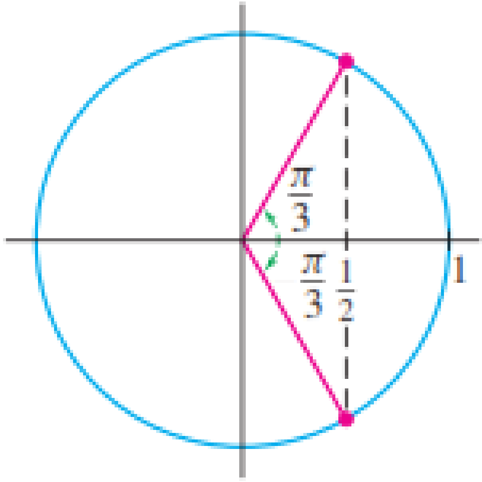



FIGURE 1.40

### 1.4.5 Exponential and Logarithmic Functions

**Exponential Functions**

**Definition 1.11.** *Functions of the form $f(x) = a^T$, where the base $a > 0$ is a positive constant and $a \neq 1$, are called **exponential** functions.*

All exponential functions have domain $(-\infty, \infty)$ and range $(0, \infty)$, so an exponential function never assumes the value 0 . The graphs of some exponential functions are shown in (Figure 1.41) . The exponential functions has the following properties

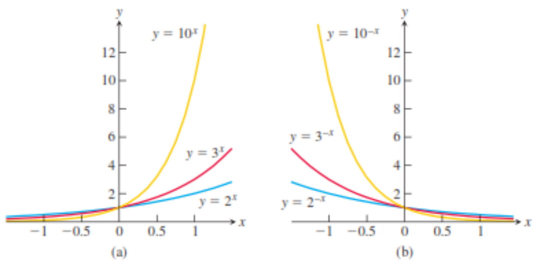



FIGURE 1.41: Graphs of exponential functions.

1. $a^0 = 1$
2. $a^{m+n} = a^m a^n$
3. $a^{m-n} = a^m / a^n$

4. $(a^m)^n = a^{mn}$

When a is the natural base i.e, $a = e \cong 2.718$ then $f(x) = e^x$ is called the natural exponential function.

**Logarithmic Functions**

**Definition 1.12.** *These are the functions $f(x) = \log_a x$, where the base $a \neq 1$ is a positive constant. They are the inverse functions of the exponential functions,*

(Figure 1.42)shows the graphs of four logarithmic functions with various bases. In each case the domain is $(0, \infty)$ and the range is $(-\infty, \infty)$.

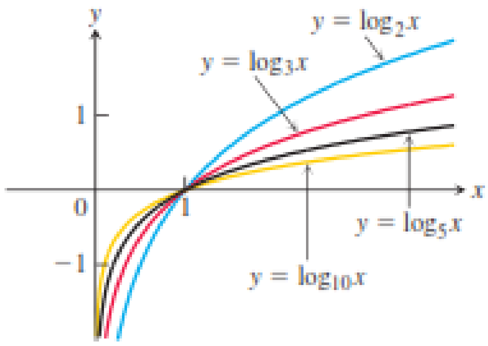



FIGURE 1.42: Graphs of four logarithmic functions.

For all real numbers $a, m, n, p, x$, and $y$, where $x > 0$, a $> 0$, a $\neq 1$.

1. If $a^x = a^y$, then $x = y$.
2. $\log_a x = y$ if and only if $x = a^y$
3. $\log_{10} x = \log x$
4. $\log_e x = \ln x$
5. $\log_a a^x = x, \;\; \ln e^x = x, \;\; a^{\log_a x} = x, \;\; e^{\ln x} = x$
6. $\log_a mn = \log_a m + \log_a n$
7. $\log_a \dfrac{m}{n} = \log_a m - \log_a n$
8. $\log_a m^p = p\,(\log_a m)$

9. $\log_a x = \dfrac{\log x}{\log a} = \dfrac{\ln x}{\ln a}, a \neq 1$

10. If $\log_a x = \log_a y$, then $x = y$.

### 1.4.6 Hyperbolic Functions

The hyperbolic functions are certain special combinations of $e^x$ and $e^{-x}$ that play a role in engineering and physics . The hyperbolic sine and cosine, often called "cinch" and "cosh," are defined as follows:

$$\sinh x = \frac{e^x - e^{-x}}{2}, \quad \cosh x = \frac{e^x + e^{-x}}{2}$$

As the terminology suggests, there are similarities between the hyperbolic and trigonometric functions. Here are some examples:

- **Parity:** The trigonometric functions and their hyperbolic analogs have the same parity. Thus, $f(x) = \sin x$ and $f(x) = \sinh x$ are both odd, and $f(x) = \cos x$ and $f(x) = \cosh x$ are both even (Figure 2 ):

$$\sinh(-x) = -\sinh x, \quad \cosh(-x) = \cosh x$$

- **Identities:** The basic trigonometric identity $\sin^2 x + \cos^2 x = 1$ has a hyperbolic analog:

$$\cosh^2 x - \sinh^2 x = 1$$

  The addition formulas satisfied by $\sin x$ and $\cos x$ also have hyperbolic analogs:

$$\sinh(x + y) = \sinh x \cosh y + \cosh x \sinh y$$
$$\cosh(x + y) = \cosh x \cosh y + \sinh x \sinh y$$

- **Hyperbola instead of the circle:** Because of the identity $\cosh^2 t - \sinh^2 t = 1$, the point $(\cosh t, \sinh t)$ lies on the hyperbola $x^2 - y^2 = 1$, just as $(\cos t, \sin t)$ lies on the unit circle $x^2 + y^2 = 1$ .

- **Other hyperbolic functions:** The hyperbolic tangent, cotangent, secant, and cosecant functions are defined like their trigonometric counterparts:

$$\tanh x = \frac{\sinh x}{\cosh x} = \frac{e^x - e^{-x}}{e^x + e^{-x}}, \quad \operatorname{sech} x = \frac{1}{\cosh x} = \frac{2}{e^x + e^{-x}}$$
$$\coth x = \frac{\cosh x}{\sinh x} = \frac{e^x + e^{-x}}{e^x - e^{-x}}, \quad \operatorname{csch} x = \frac{1}{\sinh x} = \frac{2}{e^x - e^{-x}}$$

### 1.4.7 Inverse Of Hyperbolic Functions

Just as the inverse trigonometric functions are useful in certain integrations, the inverse hyperbolic functions are useful with others. Because the hyperbolic functions are defined in terms of exponential functions, their inverses can be expressed in terms of logarithms as shown in the following . It is often more convenient to refer to $sinh^{-1}x$ than to $\ln(x+\sqrt{x^2+1})$, especially when one is working on theory and does not need to compute actual values.

**Logarithmic definitions of Inverse Hyperbolic Functions**

1. $\cosh^{-1} x = \ln\left(x+\sqrt{x^2-1}\right); \quad x \geq 1$
2. $\tanh^{-1} x = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right); \quad |x| < 1$
3. $\operatorname{sech}^{-1} x = \ln\left(\frac{1+\sqrt{1-x^2}}{x}\right); \quad 0 < x \leq 1$
4. $\sinh^{-1} x = \ln\left(x+\sqrt{x^2+1}\right)$
5. $\coth^{-1} x = \dfrac{1}{2}\ln\left(\dfrac{x+1}{x-1}\right); \quad |x| > 1$
6. $\operatorname{csch}^{-1} x = \ln\left(\frac{1}{x}+\dfrac{\sqrt{1+x^2}}{|x|}\right); \quad x \neq 0$

**Example 1.4.10.** *Prove that*

$$\sinh^{-1} x = \ln\left(x+\sqrt{x^2+1}\right)$$

*let's consider the inverses of the hyperbolic functions. We begin with the function $f(x) = sinhx$.*

$$\begin{aligned} y &= \frac{e^x - e^{-x}}{2} \\ 2y &= e^x - e^{-x} \\ 2ye^x &= e^{2x} - 1 \end{aligned}$$

*Then*

$$(e^x)^2 - 2ye^x - 1 = 0$$

*Hence*

$$\begin{aligned} e^x &= \frac{2y \pm \sqrt{4y^2+4}}{2} \\ e^x &= y \pm \sqrt{y^2+1} \\ e^x &= y + \sqrt{y^2+1} \\ x &= \ln\left(y + \sqrt{y^2+1}\right) \end{aligned}$$

*Finally, interchange the variable to find that*

$$\sinh^{-1} x = \ln\left(x + \sqrt{x^2+1}\right)$$

*In a similar manner we can derive the inverses of the other hyperbolic functions*

In such a situation, the logarithmic representation is useful. The reader is not encouraged to memorize these, but rather know they exist and know how to use them when needed.

| Function | Domain | Range | Function | Domain | Range |
|---|---|---|---|---|---|
| $\cosh x$ | $[0, \infty)$ | $[1, \infty)$ | $\cosh^{-1} x$ | $[1, \infty)$ | $[0, \infty)$ |
| $\sinh x$ | $(-\infty, \infty)$ | $(-\infty, \infty)$ | $\sinh^{-1} x$ | $(-\infty, \infty)$ | $(-\infty, \infty)$ |
| $\tanh x$ | $(-\infty, \infty)$ | $(-1, 1)$ | $\tanh^{-1} x$ | $(-1, 1)$ | $(-\infty, \infty)$ |
| $\operatorname{sech} x$ | $[0, \infty)$ | $(0, 1]$ | $\operatorname{sech}^{-1} x$ | $(0, 1]$ | $[0, \infty)$ |
| $\operatorname{csch} x$ | $(-\infty, 0) \cup (0, \infty)$ | $(-\infty, 0) \cup (0, \infty)$ | $\operatorname{csch}^{-1} x$ | $(-\infty, 0) \cup (0, \infty)$ | $(-\infty, 0) \cup (0, \infty)$ |
| $\coth x$ | $(-\infty, 0) \cup (0, \infty)$ | $(-\infty, -1) \cup (1, \infty)$ | $\coth^{-1} x$ | $(-\infty, -1) \cup (1, \infty)$ | $(-\infty, 0) \cup (0, \infty)$ |

FIGURE 1.43

## 1.5 Exercises

**1- If $a$ and $h$ are real numbers, find and simplify:**

(a) $f(a)$ (b) $f(-a)$

(c) $f(a+h)$ (d) $\frac{f(a+h)-f(a)}{h}$, provided $h \neq 0$

**For**

(1) $f(x) = x+4$ (2) $f(-a)$

(3) $\sqrt{4+x}$ (4) $f(x) = \frac{1}{x^2+1}$

**2- Find the domain of $f$, where :**

(a) $f(x) = x^3 + 4x^2 - 1$ (b) $f(x) = \sqrt{9-x^2}$

(c) $f(x) = \frac{1}{\sqrt{4-x}}$ (d) $f(x) = \frac{1}{1-\sqrt{x}}$.

**3 whether the function is even, odd, or neither. Give reasons for your answer.**

(a) $f(x) = 3$ (b) $f(x) = x^{-5}$

(c) $f(x) = x^2 + x$ (d) $x^4 + 3x^2 - 1$

(e) $f(x) = \frac{x}{2+x}$ (f) $f(x) = \frac{x}{2-1}$

**4-Write a formula for $f \circ g \circ h$**

(a) $f(x) = x+1, \quad g(x) = 3x, \quad h(x) = 4-x$

(b) $f(x) = 3x+4, \quad g(x) = 2x-1, \quad h(x) = x^2$

(c) $f(x) = \sqrt{x+1}, \quad g(x) = \frac{1}{x+4}, \quad h(x) = \frac{1}{x}$

(d) $f(x) = \frac{x+2}{3-x}, \quad g(x) = \frac{x^2}{x^2+1}, \quad h(x) = \sqrt{2-x}$.

# Chapter 2

# Limits and continuity

## 2.1 limits of Function values

Frequently when studying a function $y = f(x)$, we find ourselves interested in the function's behavior near a particular point $c$, but not at $c$. This might be the case, for instance, if $c$ is an irrational number, like $\pi$ or $\sqrt{2}$, whose values can only be approximated by "close" rational numbers at which we actually evaluate the function instead. Another situation occurs when trying to evaluate a function at $c$ leads to division by zero, which is undefined. We encountered this last circumstance when seeking the instantaneous rate of change in $y$ by considering the quotient function $\Delta y/h$ for $h$ closer and closer to zero. Here's a specific example in which we explore numerically how a function behaves near a particular point at which we cannot directly evaluate the function.

**Example 2.1.1.** *How does the function*

$$f(x) = \frac{x^2 - 1}{x - 1}$$

*behave near* $x = 1$ *?*

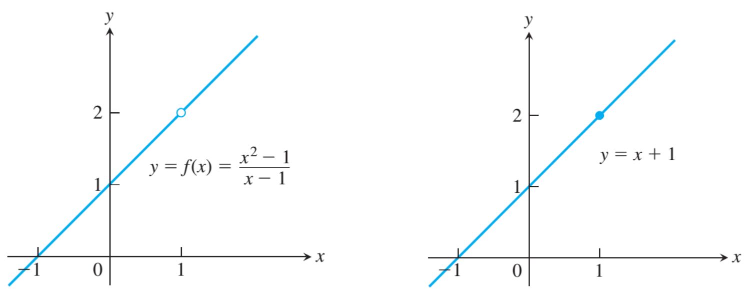



Figure 2.1

**Solution:** The given formula defines $f$ for all real numbers $x$ except $x = 1$ (we cannot divide by zero). For any $x \neq 1$, we can simplify the formula by factoring the numerator and canceling common factors:

$$f(x) = \frac{(x-1)(x+1)}{x-1} = x+1 \quad \text{for} \quad x \neq 1$$

The graph of $f$ is the line $y = x+1$ with the point $(1,2)$ removed. This removed point is shown as a "hole" in Figure 2.1. Even though $f(1)$ is not defined, it is clear that we can make the value of $f(x)$ as close as we want to 2 by choosing $x$ close enough to 1 (Table 2.1).

| $x$ | $f(x) = \frac{x^2-1}{x-1}$ |
|---|---|
| 0.9 | 1.9 |
| 1.1 | 2.1 |
| 0.99 | 1.99 |
| 1.01 | 2.01 |
| 0.999 | 1.999 |
| 1.001 | 2.001 |
| 0.999999 | 1.999999 |
| 1.000001 | 2.000001 |

TABLE 2.1 As $x$ gets closer to $1, f(x)$ gets closer to 2 .

Generalizing the idea illustrated in Example 1, suppose $f(x)$ is defined on an open interval about $c$, except possibly at $c$ itself. If $f(x)$ is arbitrarily close to the number $L$ (as close to $L$ as we like) for all $x$ sufficiently close to $c$, we say that $f$ approaches the limit $L$ as $x$ approaches $c$, and write

$$\lim_{x \to c} f(x) = L$$

which is read "the limit of $f(x)$ as $x$ approaches $c$ is $L$. " For instance, in Example 1 we would say that $f(x)$ approaches the limit 2 as $x$ approaches 1, and write

$$\lim_{x \to 1} f(x) = 2, \quad \text{or} \quad \lim_{x \to 1} \frac{x^2-1}{x-1} = 2$$

Essentially, the definition says that the values of $f(x)$ are close to the number $L$ whenever $x$ is close to $c$ (on either side of $c$ ).

### 2.1.1 The limit laws

To calculate limits of functions that are arithmetic combinations of functions having known limits, we can use several fundamental rules.

**Theorem 2.1** (Limits Laws). *If $L, M, c,$ and $k$ are real numbers and* $\lim_{x \to c} f(x) = L$ *and* $\lim_{x \to c} g(x) = M,$ *then*

1. *Sum Rule:* $\lim_{x\to c}(f(x)+g(x)) = L+M.$
2. *Difference Rule:* $\lim_{x\to c}(f(x)-g(x)) = L-M.$
3. *Constant Multiple Rule:* $\lim_{x\to c}(k\cdot f(x)) = k\cdot L.$
4. *Product Rule:* $\lim_{x\to c}(f(x)\cdot g(x)) = L\cdot M.$
5. *Quotient Rule:* $\lim_{x\to c}\frac{f(x)}{g(x)} = \frac{L}{M}, \quad M\neq 0.$
6. *Power Rule:* $\lim_{x\to c}[f(x)]^n = L^n$, *n a positive integer*
7. *Root Rule:* $\lim_{x\to c}\sqrt[n]{f(x)} = \sqrt[n]{L} = L^{1/n}$, *n a positive integer (If n is even, we assume that* $\lim_{x\to c} f(x) = L > 0$*..)*

*Moreover, If* $P(x) = a_nx^n + a_{n-1}x^{n-1} + \cdots + a_0$*, then*

$$\lim_{x\to c} P(x) = P(c) = a_nc^n + a_{n-1}c^{n-1} + \cdots + a_0.$$

*If* $P(x)$ *and* $Q(x)$ *are polynomials and* $Q(c)\neq 0$*, then*

$$\lim_{x\to c}\frac{P(x)}{Q(x)} = \frac{P(c)}{Q(c)}.$$

**Example 2.1.2.**

$$\lim_{x\to -1}\frac{x^3+4x^2-3}{x^2+5} = \frac{(-1)^3+4(-1)^2-3}{(-1)^2+5} = \frac{0}{6} = 0.$$

## 2.1.2 Eliminating Common Factors from Zero Denominators

In case of rational functions, the last case of Theorem 2.1 applies only if the denominator of the rational function is not zero at the limit point $c$. If the denominator is zero, canceling common factors in the numerator and denominator may reduce the fraction to one whose denominator is no longer zero at $c$. If this happens, we can find the limit by substitution in the simplified fraction.

**Example 2.1.3.** *Evaluate*

$$\lim_{x\to 1}\frac{x^2+x-2}{x^2-x}$$

**Solution:** We cannot substitute $x = 1$ because it makes the denominator zero. We test the numerator to see if it, too, is zero at $x = 1$. It is, so it has a factor of $(x-1)$ in common with the denominator. Canceling this common factor gives a simpler fraction with the same values as the original for $x \neq 1$

$$\frac{x^2+x-2}{x^2-x} = \frac{(x-1)(x+2)}{x(x-1)} = \frac{x+2}{x}, \qquad \text{if } x \neq 1$$

Using the simpler fraction, we find the limit of these values as $x \to 1$ by Theorem 2.1 :

$$\lim_{x\to 1} \frac{x^2+x-2}{x^2-x} = \lim_{x\to 1} \frac{x+2}{x} = \frac{1+2}{1} = 3$$

See Figure 2.2.

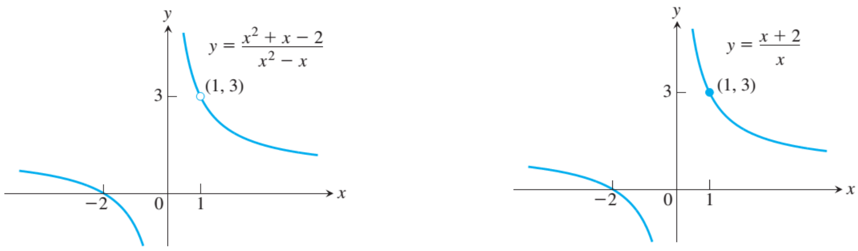



Figure 2.2

### 2.1.3 The Sandwich Theorem

The following theorem enables us to calculate a variety of limits. It is called the Sandwich Theorem because it refers to a function whose values are sandwiched between the values of two other functions $g$ and $h$ that have the same limit $L$ at a point $c$. Being trapped between the values of two functions that approach $L$, the values of must also approach $L$ (Figure 2.3).

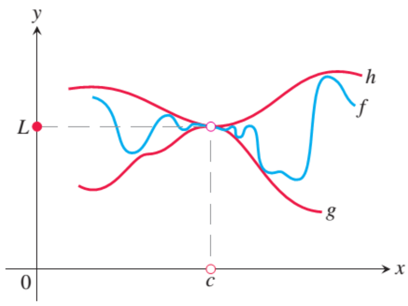



Figure 2.3

**Theorem 2.2** (The Sandwich Theorem). *Suppose that $g(x) \leq f(x) \leq h(x)$ for all $x$ in some open interval containing $c$, except possibly at $x = c$ itself. Suppose also that*

$$\lim_{x \to c} g(x) = \lim_{x \to c} h(x) = L$$

*Then* $\lim_{x \to c} f(x) = L$.

The Sandwich Theorem is also called the Squeeze Theorem or the Pinching Theorem.

**Example 2.1.4.** *Given that*

$$1 - \frac{x^2}{4} \leq u(x) \leq 1 + \frac{x^2}{2} \quad \text{for all } x \neq 0$$

*find* $\lim_{x \to 0} u(x)$, *no matter how complicated $u$ is.*

**Solution:** Since

$$\lim_{x \to 0} \left(1 - \left(x^2/4\right)\right) = 1 \quad \text{and} \quad \lim_{x \to 0} \left(1 + \left(x^2/2\right)\right) = 1$$

the Sandwich Theorem implies that $\lim_{x \to 0} u(x) = 1$ (Figure 2.4).

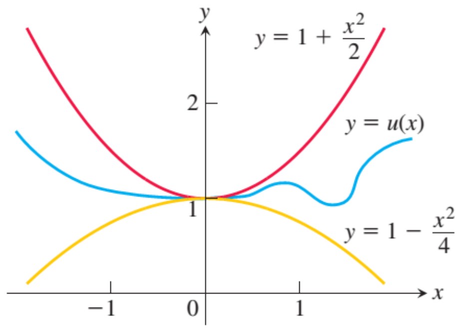



Figure 2.4

**Example 2.1.5.** *Using the Sandwich Theorem:*

*(a) If $\sqrt{5 - 2x^2} \leq f(x) \leq \sqrt{5 - x^2}$ for $-1 \leq x \leq 1$, find $\lim_{x \to 0} f(x)$.*

*(b) If $2 - x^2 \leq g(x) \leq 2\cos x$ for all $x$, find $\lim_{x \to 0} g(x)$.*

**Solution:** (a) Since

$$\lim_{x \to 0} \sqrt{5 - 2x^2} = \sqrt{5 - 2(0)^2} = \sqrt{5}$$

and

$$\lim_{x \to 0} \sqrt{5 - x^2} = \sqrt{5 - (0)^2} = \sqrt{5},$$

then by the sandwich theorem, $\lim_{x\to 0} f(x) = \sqrt{5}$.

(b) Since

$$\lim_{x\to 0}\left(2-x^2\right) = 2-0 = 2$$

and

$$\lim_{x\to 0} 2\cos x = 2(1) = 2,$$

then by the sandwich theorem, $\lim_{x\to 0} g(x) = 2$.

## 2.2 The Precise Definition of a limit

We now turn our attention to the precise definition of a limit. We replace vague phrases like "gets arbitrarily close to" in the informal definition with specific conditions that can be applied to any particular example. With a precise definition, we can avoid misunderstandings, prove the limit properties given in the preceding section, and establish many important limits.

To show that the limit of $(x)$ as $x \longrightarrow c$ equals the number $L$, we need to show that the gap between $(x)$ and $L$ can be made "as small as we choose" if $x$ is kept "close enough" to $c$. Let us see what this would require if we specified the size of the gap between $(x)$ and $L$.

**Definition 2.3.** *Let $f(x)$ be defined on an open interval about $c$, except possibly at $c$ itself. We say that the limit of $f(x)$ as $x$ approaches $c$ is the number $L$, and write*

$$\lim_{x\to c} f(x) = L$$

*if, for every number $\epsilon > 0$, there exists a corresponding number $\delta > 0$ such that for all $x$,*

$$0 < |x-c| < \delta \quad \Rightarrow \quad |f(x) - L| < \epsilon.$$

See Figure 2.5.

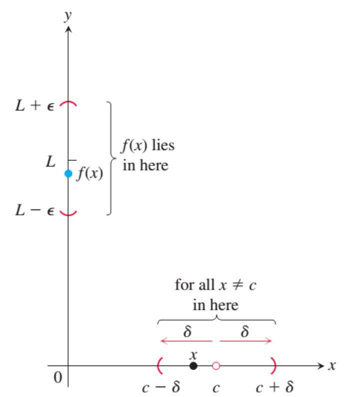


Figure 2.5

**Example 2.2.1.** *Show that*

$$\lim_{x\to 1}(5x-3)=2$$

**Solution:** Set $c=1$, $f(x)=5x-3$, and $L=2$ in the definition of limit. For any given $\epsilon>0$, we have to find a suitable $\delta>0$ so that if $x\neq 1$ and $x$ is within distance $\delta$ of $c=1$, that is, whenever

$$0<|x-1|<\delta$$

it is true that $f(x)$ is within distance $\epsilon$ of $L=2$, so

$$|f(x)-2|<\epsilon.$$

We find $\delta$ by working backward from the $\epsilon$-inequality:

$$\begin{aligned}|(5x-3)-2|=|5x-5|&<\epsilon\\ 5|x-1|&<\epsilon\\ |x-1|&<\epsilon/5\end{aligned}$$

Thus, we can take $\delta=\epsilon/5$ (Figure 2.6). If $0<|x-1|<\delta=\epsilon/5$, then

$$|(5x-3)-2|=|5x-5|=5|x-1|<5(\epsilon/5)=\epsilon$$

which proves that $\lim_{x\to 1}(5x-3)=2$ The value of $\delta=\epsilon/5$ is not the only value that will make $0<|x-1|<\delta$ imply $|5x-5|<\epsilon$. Any smaller positive $\delta$ will do as well. The definition does not ask for a "best" positive $\delta$, just one that will work.

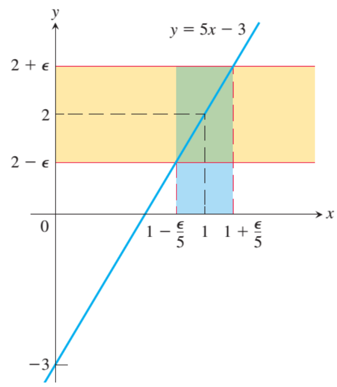



Figure 2.6

**Example 2.2.2.** *For the limit* $\lim_{x\to 5}\sqrt{x-1}=2$*, find a* $\delta>0$ *that works for* $\epsilon=1$ *That is, find a* $\delta>0$ *such that for all* $x$

$$0<|x-5|<\delta \quad\Rightarrow\quad |\sqrt{x-1}-2|<1$$

**Solution:** We organize the search into two steps:

1. Solve the inequality $|\sqrt{x-1}-2|<1$ to find an interval containing $x=5$ on which the inequality holds for all $x\neq 5$.

$$\begin{gathered}|\sqrt{x-1}-2|<1\\ -1<\sqrt{x-1}-2<1\\ 1<\sqrt{x-1}<3\\ 1<x-1<9\\ 2<x<10\end{gathered}$$

The inequality holds for all $x$ in the open interval $(2,10)$, so it holds for all $x\neq 5$ in this interval as well.

2. Find a value of $\delta>0$ to place the centered interval $5-\delta<x<5+\delta$ (centered at $x=5$ ) inside the interval $(2,10)$. The distance from 5 to the nearer endpoint of $(2,10)$ is 3. If we take $\delta=3$ or any smaller positive number, then the inequality $0<|x-5|<\delta$ will automatically place $x$ between 2 and 10 to make $|\sqrt{x-1}-2|<1$:

$$0<|x-5|<3 \quad\Rightarrow\quad |\sqrt{x-1}-2|<1.$$

**Example 2.2.3.** *Prove the limit statements in the following:*

**(a)** $\lim_{x\to 4}(9-x)=5$

**(b)** $\lim_{x\to 3}(3x-7)=2$

**(c)** $\lim_{x\to 9}\sqrt{x-5}=2$

**(d)** $\lim_{x\to 0}\sqrt{4-x}=2$

**(e)** $\lim_{x\to 1}f(x)=1 \quad$ *if* $\quad f(x)=\begin{cases}x^2, & x\neq 1\\ 2, & x=1\end{cases}$

**Solution:**

**(a)** The first step is

$$\begin{aligned}|(9-x)-5| < \epsilon &\Rightarrow -\epsilon < 4-x < \epsilon \\ &\Rightarrow -\epsilon-4 < -x < \epsilon-4 \\ &\Rightarrow \epsilon+4 > x > 4-\epsilon \\ &\Rightarrow 4-\epsilon < x < 4+\epsilon.\end{aligned}$$

The second step is

$$\begin{aligned}|x-4| < \delta &\Rightarrow -\delta < x-4 < \delta \\ &\Rightarrow -\delta+4 < x < \delta+4\end{aligned}$$

Then $-\delta+4 = -\epsilon+4 \Rightarrow \delta = \epsilon$, or $\delta+4 = \epsilon+4 \Rightarrow \delta = \epsilon$. Thus choose $\delta = \epsilon$.

**(b)** Similarly, we have

$$\begin{aligned}|(3x-7)-2| < \epsilon &\Rightarrow -\epsilon < 3x-9 < \epsilon \\ &\Rightarrow 9-\epsilon < 3x < 9+\epsilon \\ &\Rightarrow 3-\frac{\epsilon}{3} < x < 3+\frac{\epsilon}{3}.\end{aligned}$$

$$\begin{aligned}|x-3| < \delta &\Rightarrow -\delta < x-3 < \delta \\ &\Rightarrow -\delta+3 < x < \delta+3\end{aligned}$$

Then $-\delta+3 = 3-\frac{\epsilon}{3} \Rightarrow \delta = \frac{\epsilon}{3}$, or $\delta+3 = 3+\frac{\epsilon}{3} \Rightarrow \delta = \frac{\epsilon}{3}$. Thus choose $\delta = \frac{\epsilon}{3}$.

**(c)** In one hand

$$\begin{aligned}|\sqrt{x-5}-2| < \epsilon &\Rightarrow -\epsilon < \sqrt{x-5}-2 < \epsilon \\ &\Rightarrow 2-\epsilon < \sqrt{x-5} < 2+\epsilon \\ &\Rightarrow (2-\epsilon)^2 < x-5 < (2+\epsilon)^2 \\ &\Rightarrow (2-\epsilon)^2+5 < x < (2+\epsilon)^2+5.\end{aligned}$$

On the other hand

$$\begin{aligned}|x-9| < \delta &\Rightarrow -\delta < x-9 < \delta \\ &\Rightarrow -\delta+9 < x < \delta+9.\end{aligned}$$

Then $-\delta+9 = \epsilon^2-4\epsilon+9 \Rightarrow \delta = 4\epsilon-\epsilon^2$, or $\delta+9 = \epsilon^2+4\epsilon+9 \Rightarrow \delta = 4\epsilon+\epsilon^2$. Thus choose the smaller distance, $\delta = 4\epsilon-\epsilon^2$.

**(d)** We have

$$
\begin{aligned}
|\sqrt{4-x}-2| < \epsilon &\Rightarrow -\epsilon < \sqrt{4-x}-2 < \epsilon \\
&\Rightarrow 2-\epsilon < \sqrt{4-x} < 2+\epsilon \\
&\Rightarrow (2-\epsilon)^2 < 4-x < (2+\epsilon)^2 \\
&\Rightarrow -(2+\epsilon)^2 < x-4 < -(2-\epsilon)^2 \\
&\Rightarrow -(2+\epsilon)^2+4 < x < -(2-\epsilon)^2+4.
\end{aligned}
$$

Moreover

$$|x-0| < \delta \Rightarrow -\delta < x < \delta.$$

Then $-\delta = -(2+\epsilon)^2+4 = -\epsilon^2-4\epsilon \Rightarrow \delta = 4\epsilon+\epsilon^2$, or $\delta = -(2-\epsilon)^2+4 = 4\epsilon-\epsilon^2$. Thus choose the smaller distance, $\delta = 4\epsilon-\epsilon^2$

**(e)** For $x \neq 1$, we have

$$
\begin{aligned}
\left|x^2-1\right| < \epsilon &\Rightarrow -\epsilon < x^2-1 < \epsilon \\
&\Rightarrow 1-\epsilon < x^2 < 1+\epsilon \\
&\Rightarrow \sqrt{1-\epsilon} < |x| < \sqrt{1+\epsilon} \\
&\Rightarrow \sqrt{1-\epsilon} < x < \sqrt{1+\epsilon}.
\end{aligned}
$$

$$
\begin{aligned}
|x-1| < \delta &\Rightarrow -\delta < x-1 < \delta \\
&\Rightarrow -\delta+1 < x < \delta+1
\end{aligned}
$$

Then $-\delta+1 = \sqrt{1-\epsilon} \Rightarrow \delta = 1-\sqrt{1-\epsilon}$, or $\delta+1 = \sqrt{1+\epsilon} \Rightarrow \delta = \sqrt{1+\epsilon}-1$. Choose $\delta = \min\{1-\sqrt{1-\epsilon}, \sqrt{1+\epsilon}-1\}$, that is, the smaller of the two distances.

## 2.3 One-sided Limits

To have a limit $L$ as $x$ approaches $c$, a function  must be defined on both sides of $c$ and its values $(x)$ must approach $L$ as $x$ approaches $c$ from either side. That is,  must be defined in some open interval about $c$, but not necessarily at $c$. Because of this, ordinary limits are called **two-sided**.

If $f$ fails to have a two-sided limit at $c$, it may still have a one-sided limit, that is, a limit if the approach is only from one side. If the approach is from the right, the limit is a right-hand limit. From the left, it is a left-hand limit.

The function $f(x) = x/|x|$ (Figure 2.7) has limit 1 as $x$ approaches 0 from the right, and limit $-1$ as $x$ approaches 0 from the left. Since these one-sided limit values are not the same, there is no single number that $f(x)$ approaches as $x$ approaches 0. So $f(x)$ does not have a (two-sided) limit at 0 .

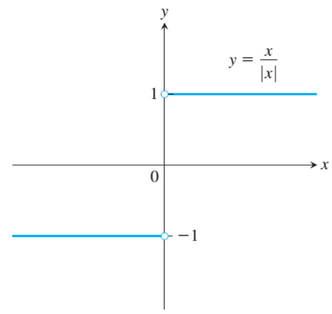



Figure 2.7

**Theorem 2.4.** *A function $f(x)$ has a limit as $x$ approaches $c$ if and only if it has left-hand and right-hand limits there and these one-sided limits are equal:*

$$\lim_{x \to c} f(x) = L \quad \Leftrightarrow \quad \lim_{x \to c^-} f(x) = L \quad \text{and} \quad \lim_{x \to c^+} f(x) = L.$$

**Example 2.3.1.** *For the function graphed in Figure 2.8,*

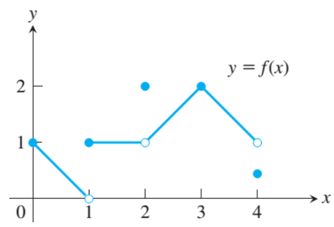



*Figure 2.8*

First we have $\lim_{x \to 0^+} f(x) = 1$. But $\lim_{x \to 0^-} f(x)$ and $\lim_{x \to 0} f(x)$ do not exist. The function is not defined to the left of $x = 0$. And at $x = 1$, $\lim_{x \to 1^-} f(x) = 0$ even though $f(1) = 1$, $\lim_{x \to 1^+} f(x) = 1$. That is, $\lim_{x \to 1} f(x)$ does not exist since the right- and left-hand limits are not equal. At $x = 2$, we have

$$\lim_{x \to 2^-} f(x) = 1, \quad \lim_{x \to 2^+} f(x) = 1, \quad \lim_{x \to 2} f(x) = 1$$

even though $f(2) = 2$. At $x = 3$, $\lim_{x \to 3^-} f(x) = \lim_{x \to 3^+} f(x) = \lim_{x \to 3} f(x) = f(3) = 2$. At $x = 4$, $\lim_{x \to 4^-} f(x) = 1$ even though $f(4) \neq 1$. And $\lim_{x \to 4^+} f(x)$, $\lim_{x \to 4} f(x)$ do not exist. The function is not defined to the right of $x = 4$.

At every other point $c$ in $[0, 4]$, $(x)$ has limit $(c)$.

### 2.3.1 Limits involving $\frac{\sin\theta}{\theta}$

We can use Sandwich Theorem to prove the following theorem.

**Theorem 2.5.** *We have*

$$\lim_{\theta\to 0}\frac{\sin\theta}{\theta}=1$$

*where $\theta$ given in radians.*

*Proof.* The plan is to show that the right-hand and left-hand limits are both 1. Then we will know that the two-sided limit is 1 as well. To show that the right-hand limit is 1 , we begin with positive values of $\theta$ less than $\pi/2$ (Figure 2.9 ).

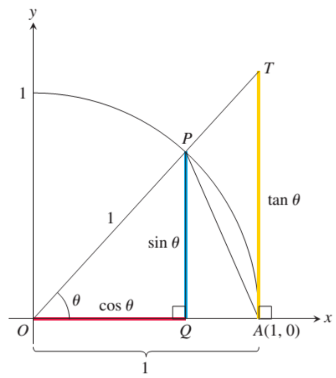



Figure 2.9

Notice that

$$\text{Area } \Delta OAP < \text{ area sector } OAP < \text{ area } \Delta OAT.$$

We can express these areas in terms of $\theta$ as follows:

$$\text{Area } \Delta OAP = \frac{1}{2}\text{ base } \times \text{ height } = \frac{1}{2}(1)(\sin\theta) = \frac{1}{2}\sin\theta.$$

$$\text{Area sector } OAP = \frac{1}{2}r^2\theta = \frac{1}{2}(1)^2\theta = \frac{\theta}{2}.$$

$$\text{Area}\Delta OAT = \frac{1}{2}\text{ base} \times \text{ height} = \frac{1}{2}(1)(\tan\theta) = \frac{1}{2}\tan\theta.$$

Thus,

$$\frac{1}{2}\sin\theta < \frac{1}{2}\theta < \frac{1}{2}\tan\theta$$

This last inequality goes the same way if we divide all three terms by the number $(1/2)\sin\theta$, which is positive, since $0<\theta<\pi/2$:

$$1 < \frac{\theta}{\sin\theta} < \frac{1}{\cos\theta}$$

Taking reciprocals reverses the inequalities:

$$1 > \frac{\sin\theta}{\theta} > \cos\theta$$

Since $\lim_{\theta\to 0^+}\cos\theta = 1$, the Sandwich Theorem gives

$$\lim_{\theta\to 0^+}\frac{\sin\theta}{\theta} = 1$$

To consider the left-hand limit, we recall that $\sin\theta$ and $\theta$ are both odd functions. Therefore, $f(\theta) = (\sin\theta)/\theta$ is an even function, with a graph symmetric about the $y$-axis. This symmetry implies that the left-hand limit at 0 exists and has the same value as the right-hand limit:

$$\lim_{\theta\to 0^-}\frac{\sin\theta}{\theta} = 1 = \lim_{\theta\to 0^+}\frac{\sin\theta}{\theta},$$

so $\lim_{\theta\to 0}(\sin\theta)/\theta = 1$. □

## Solved problems

**(a)** Show that $\lim_{h\to 0}\frac{\cos h - 1}{h} = 0$.

**Solution:** Using the half-angle formula $\cos h = 1 - 2\sin^2(h/2)$, we calculate

$$\begin{aligned} lim_{h\to 0}\frac{\cos h - 1}{h} &= \lim_{h\to 0} -\frac{2\sin^2(h/2)}{h} \\ &= -\lim_{\theta\to 0}\frac{\sin\theta}{\theta}\sin\theta \\ &= -(1)(0) = 0. \end{aligned}$$

**(b)** Show that $\lim_{x\to 0}\frac{\sin 2x}{5x} = \frac{2}{5}$.

**Solution:** We need a $2x$ in the denominator, not a $5x$. We produce it by multiplying numerator and denominator by $\frac{2}{5}$:

$$\begin{aligned} \lim_{x\to 0}\frac{\sin 2x}{5x} &= \lim_{x\to 0}\frac{(2/5)\cdot\sin 2x}{(2/5)\cdot 5x} \\ &= \frac{2}{5}\lim_{x\to 0}\frac{\sin 2x}{2x} \qquad \text{Sine rule applies with } \theta = 2x \\ &= \frac{2}{5}(1) = \frac{2}{5}. \end{aligned}$$

**(c)** Find $\lim_{t\to 0} \frac{\tan t \sec 2t}{3t}$.

**Solution:** From the definition of $\tan t$ and $\sec 2t$, we have

$$\begin{aligned}
\lim_{t\to 0} \frac{\tan t \sec 2t}{3t} &= \lim_{t\to 0} \frac{1}{3} \cdot \frac{1}{t} \cdot \frac{\sin t}{\cos t} \cdot \frac{1}{\cos 2t} \\
&= \frac{1}{3} \lim_{t\to 0} \frac{\sin t}{t} \cdot \frac{1}{\cos t} \cdot \frac{1}{\cos 2t} \\
&= \frac{1}{3}(1)(1)(1) = \frac{1}{3}.
\end{aligned}$$

## 2.4 Continuity

**Definition 2.6.** *Let $c$ be a real number on the $x$-axis. The function $f$ is continuous at* **c** *if*

$$\lim_{x\to c} f(x) = f(c)$$

*The function $f$ is right-continuous at* **c** *(or continuous from the right) if*

$$\lim_{x\to c^+} f(x) = f(c)$$

*The function $f$ is left-continuous at $c$ (or continuous from the left) if*

$$\lim_{x\to c^-} f(x) = f(c).$$

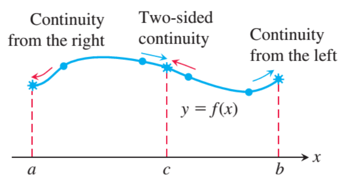



Figure 2.10

It follows that a function $f$ is continuous at an interior point $c$ of its domain if and only if it is both right-continuous and left-continuous at $c$ (Figure 2.10). We say that a function is continuous over a closed interval $[a, b]$ if it is rightcontinuous at $a$, left-continuous at $b$, and continuous at all interior points of the interval.

This definition applies to the infinite closed intervals $[a, \infty)$ and $(-\infty, b]$ as well, but only one endpoint is involved. If a function is not continuous at an interior point $c$ of its domain, we say that $f$ is discontinuous at $c$, and that $c$ is a point of discontinuity of $f$. Note that a function $f$ can be continuous, right-continuous, or left-continuous only at a point $c$ for which $f(c)$ is defined.

**Example 2.4.1.** *The function* $f(x) = \sqrt{4 - x^2}$ *is continuous over its domain* $[-2, 2]$*. It is right-continuous at* $x = -2$*, and left-continuous at* $x = 2$*.*

We summarize continuity at an interior point in the form of a test.
**Continuity Test** A function $f(x)$ is continuous at a point $x = c$ if and only if it meets the following three conditions.

**(1)** $f(c)$ exists ($c$ lies in the domain of $f$).

**(2)** $\lim_{x \to c} f(x)$ exists ($f$ has a limit as $x \to c$).

**(3)** $\lim_{x \to c} f(x) = f(c)$ (the limit equals the function value).

For one-sided continuity and continuity at an endpoint of an interval, the limits in parts 2 and 3 of the test should be replaced by the appropriate one-sided limits.

**Example 2.4.2.** *The function* $y = \lfloor x \rfloor$ *is graphed in Figure 2.11 It is discontinuous at every integer because the left-hand and right-hand limits are not equal as* $x \to n$*:*

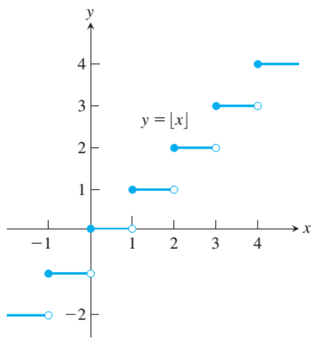



*Figure 2.11*

$$\lim_{x \to n^-} \lfloor x \rfloor = n - 1 \quad \text{and} \quad \lim_{x \to n^+} \lfloor x \rfloor = n$$

*Since* $\lfloor n \rfloor = n$*, the greatest integer function is right-continuous at every integer* $n$ *(but not left-continuous).*

*The greatest integer function is continuous at every real number other than the integers. For example,*

$$\lim_{x \to 1.5} \lfloor x \rfloor = 1 = \lfloor 1.5 \rfloor$$

*In general, if* $n - 1 < c < n$*,* $n$ *an integer, then*

$$\lim_{x \to c} \lfloor x \rfloor = n - 1 = \lfloor c \rfloor$$

**Continuous Functions:** Generally, we want to describe the continuity behavior of a function throughout its entire domain, not only at a single point. We know how to do that if the domain is a closed interval. In the same way, we define a continuous function as one that is **continuous** at every point in its domain. This is a property of the function. A function always has a specified domain, so if we change the domain, we change the function, and this may change its continuity property as well. If a function is discontinuous at one or more points of its domain, we say it is a **discontinuous function**.

Algebraic combinations of continuous functions are continuous wherever they are defined.

**Theorem 2.7** (Properties of Continuous Functions)**.** *Functions If the functions $f$ and $g$ are continuous at $x = c$, then the following algebraic combinations are continuous at $x = c$.*

**(1)** *Sums:* $f + g$.

**(2)** *Differences:* $f - g$.

**(3)** *Constant multiples:* $k \cdot f$, *for any number* $k$.

**(4)** *Products:* $f \cdot g$.

**(5)** *Quotients:* $f/g$, *provided* $g(c) \neq 0$

**(6)** *Powers:* $f^n$, $n$ *a positive integer.*

**(7)** *Roots:* $\sqrt[n]{f}$, *provided it is defined on an open interval containing* $c$, *where* $n$ *is a positive integer.*

Using the above theorem, we the following important examples.

**Example 2.4.3.**

**(a)** *Every polynomial* $P(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0$ *is continuous because* $\lim_{x\to c} P(x) = P(c)$.

**(b)** *If* $P(x)$ *and* $Q(x)$ *are polynomials, then the rational function* $P(x)/Q(x)$ *is continuous wherever it is defined* $(Q(c) \neq 0)$.

**Example 2.4.4.** *The function* $f(x) = |x|$ *is continuous. If* $x > 0$, *we have* $f(x) = x$, *a polynomial. If* $x < 0$, *we have* $f(x) = -x$, *another polynomial. Finally, at the origin,* $\lim_{x\to 0} |x| = 0 = |0|$

The functions $y = \sin x$ and $y = \cos x$ are continuous at $x = 0$. Both functions are, in fact, continuous everywhere. It follows that all six trigonometric functions are then continuous wherever they are defined. For example, $y = \tan x$ is continuous on

$$\cdots \cup (-\pi/2, \pi/2) \cup (\pi/2, 3\pi/2) \cup \cdots$$

**Inverse Functions and Continuity**

The inverse function of any function continuous on an interval is continuous over its domain. This result is suggested by the observation that the graph of $f^{-1}$, being the reflection of the graph of $f$ across the line $y = x$, cannot have any breaks in it when the graph of $f$ has no breaks. A rigorous proof that $f^{-1}$ is continuous whenever $f$ is continuous on an interval is given in more advanced texts. It follows that the inverse trigonometric functions are all continuous over their domains.

We defined the exponential function $y = a^x$ in Chapter 1 informally by its graph. Recall that the graph was obtained from the graph of $y = a^x$ for $x$ a rational number by "filling in the holes" at the irrational points $x$, so the function $y = a^x$ was defined to be continuous over the entire real line. The inverse function $y = \log_a x$ is also continuous. In particular, the natural exponential function $y = e^x$ and the natural logarithm function $y = \ln x$ are both continuous over their domains.

**Composites** All composites of continuous functions are continuous. The idea is that if $f(x)$ is continuous at $x = c$ and $g(x)$ is continuous at $x = f(c)$, then $g \circ f$ is continuous at $x = c$ (Figure 2.12). In this case, the limit as $x \to c$ is $g(f(c))$.

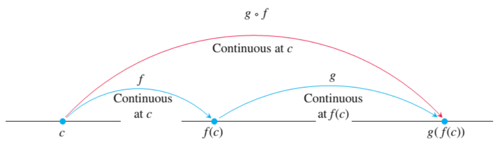



Figure 2.12

**Theorem 2.8** (Composite of Continuous Functions)**.** *If $f$ is continuous at $c$ and $g$ is continuous at $f(c)$, then the composite $g \circ f$ is continuous at $c$.*

Intuitively, Theorem 2.8 is reasonable because if $x$ is close to $c$, then $f(x)$ is close to $f(c)$, and since $g$ is continuous at $f(c)$, it follows that $g(f(x))$ is close to $g(f(c))$.

The continuity of composites holds for any finite number of functions. The only requirement is that each function be continuous where it is applied.

**Example 2.4.5.** *Show that the following functions are continuous on their natural domains.*

**(a)** $y = \sqrt{x^2 - 2x - 5}$.

***Solution:*** *The square root function is continuous on $[0, \infty)$ because it is a root of the continuous identity function $f(x) = x$ (Part 7 , Theorem 2.7 ). The given function is then the composite of the polynomial $f(x) = x^2 - 2x - 5$ with the square root function $g(t) = \sqrt{t}$, and is continuous on its natural domain.*

**(b)** $y = \frac{x^{2/3}}{1+x^4}$.

*Solution:* *The numerator is the cube root of the identity function squared; the denominator is an everywhere-positive polynomial. Therefore, the quotient is continuous.*

**(c)** $y = \left|\frac{x-2}{x^2-2}\right|$.

*Solution:* *The quotient* $(x-2)/\left(x^2-2\right)$ *is continuous for all* $x \neq \pm\sqrt{2}$, *and the function is the composition of this quotient with the continuous absolute value function.*

**(d)** $y = \left|\frac{x \sin x}{x^2+2}\right|$.

*Solution:* *Because the sine function is everywhere-continuous (Exercise 70), the numerator term* $x \sin x$ *is the product of continuous functions, and the denominator term* $x^2 + 2$ *is an everywhere-positive polynomial. The given function is the composite of a quotient of continuous functions with the continuous absolute value function.*

Theorem 2.8 is actually a consequence of a more general result, which we now state and prove.

**Theorem 2.9** (Limits of Continuous Functions)**.** *If* $g$ *is continuous at the point* $b$ *and* $\lim_{x\to c} f(x) = b$, *then*

$$\lim_{x\to c} g(f(x)) = g(b) = g\left(\lim_{x\to c} f(x)\right)$$

*Proof.* Let $\epsilon > 0$ be given. Since $g$ is continuous at $b$, there exists a number $\delta_1 > 0$ such that

$$|g(y) - g(b)| < \epsilon \quad \text{whenever} \quad 0 < |y - b| < \delta_1$$

Since $\lim_{x\to c} f(x) = b$, there exists a $\delta > 0$ such that

$$|f(x) - b| < \delta_1 \quad \text{whenever} \quad 0 < |x - c| < \delta$$

If we let $y = f(x)$, we then have that

$$|y - b| < \delta_1 \quad \text{whenever} \quad 0 < |x - c| < \delta$$

which implies from the first statement that $|g(y) - g(b)| = |g(f(x)) - g(b)| < \epsilon$ whenever $0 < |x - c| < \delta$. From the definition of limit, this proves that $\lim_{x\to c} g(f(x)) = g(b)$. □

**Example 2.4.6.** *As an application of Theorem 2.9, we have the following calculations.*

**(a)**

$$\begin{aligned}\lim_{x\to\pi/2} \cos\left(2x + \sin\left(\frac{3\pi}{2} + x\right)\right) &= \cos \lim_{x\to\pi/2}\left(2x + \sin\left(\frac{3\pi}{2} + x\right)\right) \\ &= \cos(\pi + \sin 2\pi) = \cos\pi = -1.\end{aligned}$$

**(b)**

$$\begin{aligned}\lim_{x\to 1}\sin^{-1}\left(\frac{1-x}{1-x^2}\right) &= \sin^{-1}\left(\lim_{x\to 1}\frac{1-x}{1-x^2}\right) && \textit{Arcsine is continuous}\\ &= \sin^{-1}\left(\lim_{x\to 1}\frac{1}{1+x}\right) && \textit{Cancel the factor } (1-x)\\ &= \sin^{-1}\frac{1}{2} = \frac{\pi}{6}.\end{aligned}$$

**(c)** *Since the exponential is continuous, we have*

$$\begin{aligned}\lim_{x\to 0}\sqrt{x+1}e^{\tan x} &= \lim_{x\to 0}\sqrt{x+1}\cdot\exp\left(\lim_{x\to 0}\tan x\right)\\ &= 1\cdot e^0 = 1.\end{aligned}$$

### 2.4.1 Intermediate value property

Functions that are continuous on intervals have properties that make them particularly useful in mathematics and its applications. One of these is the Intermediate Value Property. A function is said to have the Intermediate Value Property if whenever it takes on two values, it also takes on all the values in between.

We end this section by the following theorem.

**Theorem 2.10** (Intermediate Value Theorem for Continuous Functions). *If is a continuous function on a closed interval $[a,b]$, and if $y_0$ is any value between $(a)$ and $(b)$, then $y_0 = (c)$ for some $c$ in $[a,b]$.*

Theorem 2.10 says that continuous functions over finite closed intervals have the Intermediate Value Property. Geometrically, the Intermediate Value Theorem says that any horizontal line $y = y_0$ crossing the $y$-axis between the numbers $(a)$ and $(b)$ will cross the curve $y = (x)$ at least once over the interval $[a,b]$.

## 2.5 Limits involving infinity; Asymptotes of graphs

**Definition 2.11.** **(a)** *We say that $f(x)$ has the limit $L$ as $x$ approaches infinity and write*

$$\lim_{x\to\infty} f(x) = L$$

*if, for every number $\epsilon > 0$, there exists a corresponding number $M$ such that for all $x$*

$$x > M \quad \Rightarrow \quad |f(x) - L| < \epsilon$$

**(b)** *We say that $f(x)$ has the limit $L$ as $x$ approaches minus infinity and write*

$$\lim_{x \to -\infty} f(x) = L$$

*if, for every number $\epsilon > 0$, there exists a corresponding number $N$ such that for all $x$*

$$x < N \quad \Rightarrow \quad |f(x) - L| < \epsilon$$

**Example 2.5.1.** *Show that*

**(a)** $\lim_{x \to \infty} \frac{1}{x} = 0.$

**(b)** $\lim_{x \to -\infty} \frac{1}{x} = 0.$

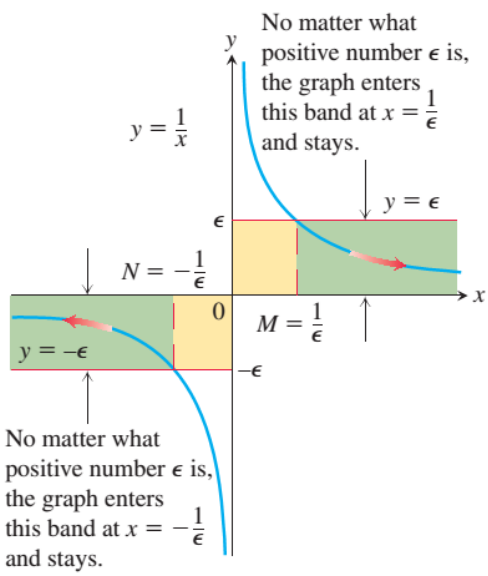



Figure 2.13

**Solution:** (a) Let $\epsilon > 0$ be given. We must find a number $M$ such that for all $x$

$$x > M \quad \Rightarrow \quad \left|\frac{1}{x} - 0\right| = \left|\frac{1}{x}\right| < \epsilon$$

The implication will hold if $M = 1/\epsilon$ or any larger positive number (Figure 2.13). This proves $\lim_{x \to \infty}(1/x) = 0$.

**Solution:** (b) Let $\epsilon > 0$ be given. We must find a number $N$ such that for all $x$

$$x < N \quad \Rightarrow \quad \left|\frac{1}{x} - 0\right| = \left|\frac{1}{x}\right| < \epsilon$$

The implication will hold if $N = -1/\epsilon$ or any number less than $-1/\epsilon$ (Figure 2.50 ). This proves $\lim_{x\to-\infty}(1/x) = 0$.

Limits at infinity have properties similar to those of finite limits.

**Theorem 2.12.** *All the Limit Laws in Theorem 1 are true when we replace* $\lim_{x\to c}$ *by* $\lim_{x\to\infty}$ *or* $\lim_{x\to-\infty}$ *. That is, the variable x may approach a finite number c or* $\pm\infty$*.*

### 2.5.1 Limits at Infinity of Rational Functions

To determine the limit of a rational function as $x \to \pm\infty$, we first divide the numerator and denominator by the highest power of $x$ in the denominator. The result then depends on the degrees of the polynomials involved.

The following examples illustrate what happens when the degree of the numerator is less than or equal to the degree of the denominator.

**Example 2.5.2.**

**(a)**

$$\begin{aligned}\lim_{x\to\infty}\frac{5x^2+8x-3}{3x^2+2} &= \lim_{x\to\infty}\frac{5+(8/x)-\left(3/x^2\right)}{3+(2/x^2)}\\ &= \frac{5+0-0}{3+0} = \frac{5}{3}\end{aligned}$$

*where we divide numerator and denominator by* $x^2$*.*

**(b)**

$$\begin{aligned}\lim_{x\to-\infty}\frac{11x+2}{2x^3-1} &= \lim_{x\to-\infty}\frac{\left(11/x^2\right)+\left(2/x^3\right)}{2-(1/x^3)}\\ &= \frac{0+0}{2-0} = 0\end{aligned}$$

*where we divide numerator and denominator by* $x^3$*.*

### 2.5.2 Horizontal Asymptotes

If the distance between the graph of a function and some fixed line approaches zero as a point on the graph moves increasingly far from the origin, we say that the graph approaches the line asymptotically and that the line is an asymptote of the graph.

Looking at $f(x) = 1/x$ (see Figure 2.14), we observe that the $x$-axis is an asymptote of the curve on the right because

$$\lim_{x\to\infty}\frac{1}{x} = 0$$

and on the left because

$$\lim_{x\to-\infty}\frac{1}{x}=0$$

We say that the $x$-axis is a horizontal asymptote of the graph of $f(x)=1/x$.

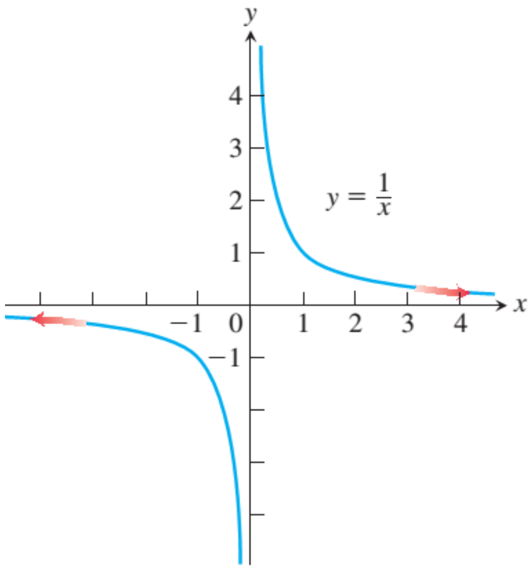



Figure 2.14

**Definition 2.13.** *A line $y=b$ is a horizontal asymptote of the graph of a function $y=f(x)$ if either*

$$\lim_{x\to\infty}f(x)=b \quad \text{or} \quad \lim_{x\to-\infty}f(x)=b.$$

The graph of the function

$$f(x)=\frac{5x^2+8x-3}{3x^2+2}$$

has the line $y=5/3$ as a horizontal asymptote on both the right and the left because

$$\lim_{x\to\infty}f(x)=\frac{5}{3} \quad \text{and} \quad \lim_{x\to-\infty}f(x)=\frac{5}{3}.$$

**Example 2.5.3.** *Find the horizontal asymptotes of the graph of*

$$f(x)=\frac{x^3-2}{|x|^3+1}$$

**Solution:** We calculate the limits as $x\to\pm\infty$.

For $x\geq 0$ :

$$\lim_{x\to\infty}\frac{x^3-2}{|x|^3+1}=\lim_{x\to\infty}\frac{x^3-2}{x^3+1}=\lim_{x\to\infty}\frac{1-(2/x^3)}{1+(1/x^3)}=1.$$

For $x < 0$ :

$$\lim_{x\to-\infty} \frac{x^3 - 2}{|x|^3 + 1} = \lim_{x\to-\infty} \frac{x^3 - 2}{(-x)^3 + 1} = \lim_{x\to-\infty} \frac{1 - (2/x^3)}{-1 + (1/x^3)} = -1.$$

The horizontal asymptotes are $y = -1$ and $y = 1$. The graph is displayed in Figure 2.15. Notice that the graph crosses the horizontal asymptote $y = -1$ for a positive value of $x$.

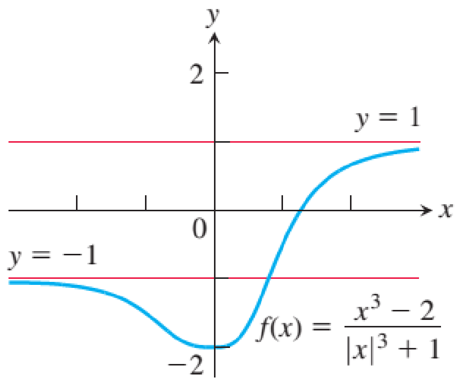



Figure 2.15

The Sandwich Theorem also holds for limits as $x \to \pm\infty$. You must be sure, though, that the function whose limit you are trying to find stays between the bounding functions at very large values of $x$ in magnitude consistent with whether $x \to \infty$ or $x \to -\infty$.

**Example 2.5.4.** *Using the Sandwich Theorem, find the horizontal asymptote of the curve*

$$y = 2 + \frac{\sin x}{x}.$$

**Solution:** We are interested in the behavior as $x \to \pm\infty$. Since

$$0 \le \left|\frac{\sin x}{x}\right| \le \left|\frac{1}{x}\right|$$

and $\lim_{x\to\pm\infty} |1/x| = 0$, we have $\lim_{x\to\pm\infty}(\sin x)/x = 0$ by the Sandwich Theorem. Hence,

$$\lim_{x\to\pm\infty} \left(2 + \frac{\sin x}{x}\right) = 2 + 0 = 2$$

and the line $y = 2$ is a horizontal asymptote of the curve on both left and right.

### 2.5.3 Oblique Asymptotes

If the degree of the numerator of a rational function is 1 greater than the degree of the denominator, the graph has an oblique or slant line asymptote. We find an equation for the asymptote by dividing numerator by denominator to express $f$ as a linear function plus a remainder that goes to zero as $x \to \pm\infty$.

**Example 2.5.5.** *Find the oblique asymptote of the graph of* $f(x) = \frac{x^2-3}{2x-4}$, *in Figure 2.16*

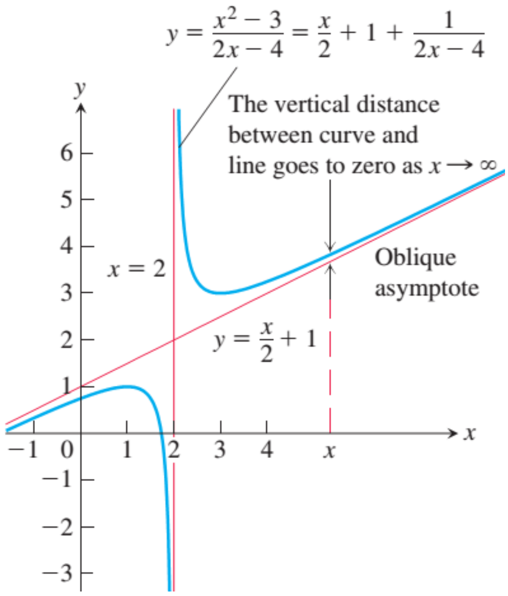



Figure 2.16

**Solution:** We are interested in the behavior as $x \to \pm\infty$. We divide $x^2 - 3$ by $2x - 4$ as follows:

$$
\begin{array}{r|l}
 & \tfrac{1}{2}x + 1. \\
\hline
2x - 4) & x^2 \qquad - 3 \\
 & -x^2 + 2x \\
\hline
 & 2x - 3 \\
 & -2x + 4 \\
\hline
 & 1
\end{array}
$$

This tells us that

$$f(x) = \frac{x^2-3}{2x-4} = \Big(\underbrace{\frac{x}{2}+1}_{\text{linear } g(x)}\Big) + \Big(\underbrace{\frac{1}{2x-4}}_{\text{remainder}}\Big)$$

As $x \to \pm\infty$, the remainder, whose magnitude gives the vertical distance between the graphs of $f$ and $g$, goes to zero, making the slanted line

$$g(x) = \frac{x}{2} + 1$$

an asymptote of the graph of $f$ (Figure 2.16). The line $y = g(x)$ is an asymptote both to the right and to the left. The next subsection will confirm that the function $f(x)$ grows arbitrarily large in absolute value as $x \to 2$ (where the denominator is zero), as shown in the graph.

## 2.6 Infinite Limits

Let us look again at the function $(x) = \frac{1}{x}$. As $x \to 0^+$, the values of  grow without bound, eventually reaching and surpassing every positive real number. That is, given any positive real number $B$, however large, the values of  become larger still (Figure 2.17).

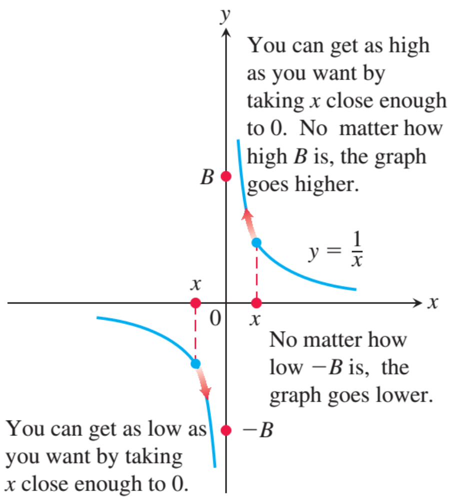



Figure 2.17

Thus, $f$ has no limit as $x \to 0^+$.It is nevertheless convenient to describe the behavior of $f$ by saying that $f(x)$ approaches $\infty$ as $x \to 0^+$. We write

$$\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} \frac{1}{x} = \infty$$

In writing this equation, we are not saying that the limit exists. Nor are we saying that there is a real number $\infty$, for there is no such number. Rather, we are saying that $\lim_{x \to 0^+}(1/x)$ does not exist because $1/x$ becomes arbitrarily large and positive as $x \to 0^+$.

As $x \to 0^-$, the values of $f(x) = 1/x$ become arbitrarily large and negative. Given any negative real number $-B$, the values of $f$ eventually lie below $-B$. (See Figure 2.17.) We write

$$\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} \frac{1}{x} = -\infty$$

Again, we are not saying that the limit exists and equals the number $-\infty$. There *is* no real number $-\infty$. We are describing the behavior of a function whose limit as $x \to 0^-$does not exist because its values become arbitrarily large and negative.

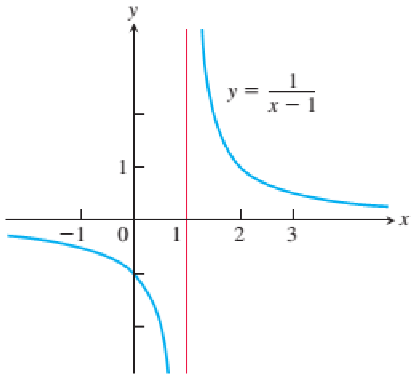



Figure 2.18

**Example 2.6.1.** *Find* $\lim_{x\to1^+} \frac{1}{x-1}$ *and* $\lim_{x\to1^-} \frac{1}{x-1}$.

**Geometric Solution:** The graph of $y = 1/(x-1)$ is the graph of $y = 1/x$ shifted 1 unit to the right (Figure 2.18). Therefore, $y = 1/(x-1)$ behaves near 1 exactly the way $y = 1/x$ behaves near 0:

$$\lim_{x\to1^+} \frac{1}{x-1} = \infty \quad \text{and} \quad \lim_{x\to1^-} \frac{1}{x-1} = -\infty.$$

**Analytic Solution:** Think about the number $x - 1$ and its reciprocal. As $x \to 1^+$, we have $(x-1) \to 0^+$and $1/(x-1) \to \infty$. As $x \to 1^-$, we have $(x-1) \to 0^-$and $1/(x-1) \to -\infty$.

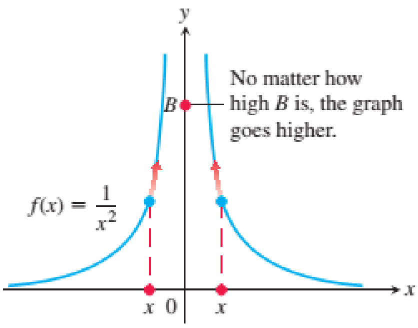



Figure 2.19

**Example 2.6.2.** *Discuss the behavior of*

$$f(x) = \frac{1}{x^2} \quad \textit{as} \quad x \to 0.$$

**Solution:** As $x$ approaches zero from either side, the values of $1/x^2$ are positive and become arbitrarily large (Figure 2.19). This means that

$$\lim_{x\to 0} f(x) = \lim_{x\to 0} \frac{1}{x^2} = \infty$$

The function $y = 1/x$ shows no consistent behavior as $x \to 0$. We have $1/x \to \infty$ if $x \to 0^+$, but $1/x \to -\infty$ if $x \to 0^-$. All we can say about $\lim_{x\to 0}(1/x)$ is that it does not exist. The function $y = 1/x^2$ is different. Its values approach infinity as $x$ approaches zero from either side, so we can say that $\lim_{x\to 0}\left(1/x^2\right) = \infty$.

**More Examples:** These examples illustrate that rational functions can behave in various ways near zeros of the denominator.

**(a)** $\lim_{x\to 2} \frac{(x-2)^2}{x^2-4} = \lim_{x\to 2} \frac{(x-2)^2}{(x-2)(x+2)} = \lim_{x\to 2} \frac{x-2}{x+2} = 0$

**(b)** $\lim_{x\to 2} \frac{(x-2)^2}{x^2-4} = \lim_{x\to 2} \frac{(x-2)^2}{(x-2)(x+2)} = \lim_{x\to 2} \frac{x-2}{x+2} = 0$

**(c)** $\lim_{x\to 2^+} \frac{x-3}{x^2-4} = \lim_{x\to 2^+} \frac{x-3}{(x-2)(x+2)} = -\infty$. The values are negative for $x > 2$, $x$ near 2.

**(d)** $\lim_{x\to 2^-} \frac{x-3}{x^2-4} = \lim_{x\to 2^-} \frac{x-3}{(x-2)(x+2)} = \infty$. The values are positive for $x < 2$, $x$ near 2.

**(e)** $\lim_{x\to 2} \frac{x-3}{x^2-4} = \lim_{x\to 2} \frac{x-3}{(x-2)(x+2)}$ does not exist. See parts (c) and (d).

**(f)** $\lim_{x\to 2} \frac{2-x}{(x-2)^3} = \lim_{x\to 2} \frac{-(x-2)}{(x-2)^3} = \lim_{x\to 2} \frac{-1}{(x-2)^2} = -\infty$.

In parts (a) and (b) the effect of the zero in the denominator at $x = 2$ is canceled because the numerator is zero there also. Thus a finite limit exists. This is not true in part (f), where cancellation still leaves a zero factor in the denominator

**Example 2.6.3.** *Find* $\lim_{x\to -\infty} \frac{2x^5-6x^4+1}{3x^2+x-7}$.

**Solution:** We are asked to find the limit of a rational function as $x \to -\infty$, so we divide the numerator and denominator by $x^2$, the highest power of $x$ in the denominator:

$$\begin{aligned}
\lim_{x\to -\infty} \frac{2x^5 - 6x^4 + 1}{3x^2 + x - 7} &= \lim_{x\to -\infty} \frac{2x^3 - 6x^2 + x^{-2}}{3 + x^{-1} - 7x^{-2}} \\
&= \lim_{x\to -\infty} \frac{2x^2(x-3) + x^{-2}}{3 + x^{-1} - 7x^{-2}} \\
&= -\infty, \quad x^{-n} \to 0, x - 3 \to -\infty
\end{aligned}$$

because the numerator tends to $-\infty$ while the denominator approaches 3 as $x \to -\infty$.

### 2.6.1 Vertical Asymptotes

Notice that the distance between a point on the graph of $f(x) = 1/x$ and the $y$-axis approaches zero as the point moves vertically along the graph and away from the origin (Figure 2.20). The function $f(x) = 1/x$ is unbounded as $x$ approaches 0 because

$$\lim_{x\to 0^+} \frac{1}{x} = \infty \quad \text{and} \quad \lim_{x\to 0^-} \frac{1}{x} = -\infty$$

We say that the line $x = 0$ (the $y$-axis) is a vertical asymptote of the graph of $f(x) = 1/x$ Observe that the denominator is zero at $x = 0$ and the function is undefined there.

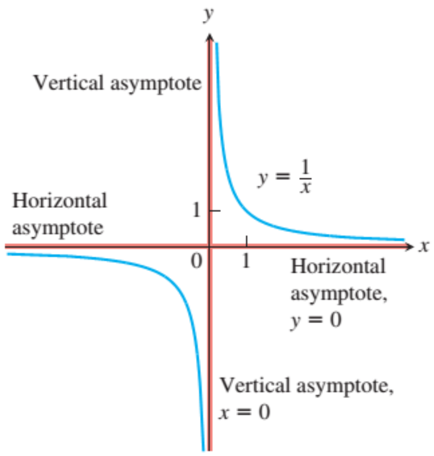



Figure 2.20

**Definition 2.14.** *A line $x = a$ is a vertical asymptote of the graph of a function $y = f(x)$ if either*

$$\lim_{x\to a^+} f(x) = \pm\infty \quad \textit{or} \quad \lim_{x\to a^-} f(x) = \pm\infty.$$

**Example 2.6.4.** *Find the horizontal and vertical asymptotes of the curve*

$$y = \frac{x+3}{x+2}.$$

**Solution:** We are interested in the behavior as $x \to \pm\infty$ and the behavior as $x \to -2$, where the denominator is zero.

The asymptotes are quickly revealed if we recast the rational function as a polynomial with a remainder, by dividing $(x + 2)$ into $(x + 3)$:

$$\begin{array}{rl} & \phantom{-x}\ \ 1. \\ x+2 & \enclose{longdiv}{\ x+3} \\ & \underline{-x-2} \\ & \phantom{-x-}\ 1 \end{array}$$

As $x \to \pm\infty$, the curve approaches the horizontal asymptote $y = 1$; as $x \to -2$, the curve approaches the vertical asymptote $x = -2$. We see that the curve in question is the graph of $f(x) = 1/x$ shifted 1 unit up and 2 units left (Figure 2.21). The asymptotes, instead of being the coordinate axes, are now the lines $y = 1$ and $x = -2$.

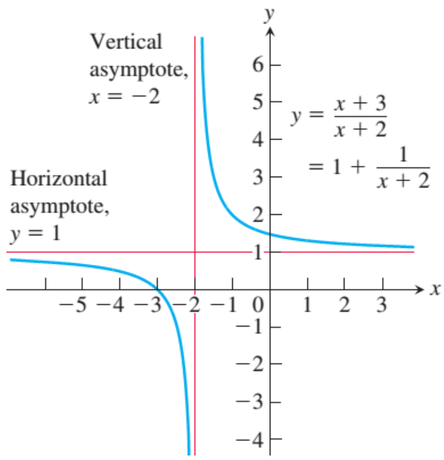



Figure 2.21

**Example 2.6.5.** *Find the horizontal and vertical asymptotes of the graph of*

$$f(x) = -\frac{8}{x^2 - 4}.$$

**Solution:** We are interested in the behavior as $x \to \pm\infty$ and as $x \to \pm 2$, where the denominator is zero. Notice that $f$ is an even function of $x$, so its graph is symmetric with respect to the $y$-axis.

(a) The behavior as $x \to \pm\infty$. Since $\lim_{x\to\infty} f(x) = 0$, the line $y = 0$ is a horizontal asymptote of the graph to the right. By symmetry it is an asymptote to the left as well (Figure 2.22). Notice that the curve approaches the $x$-axis from only the negative side (or from below). Also, $f(0) = 2$.

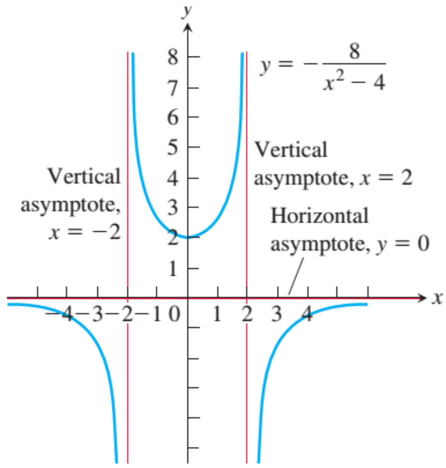



Figure 2.22

(b) The behavior as $x \to \pm 2$. Since

$$\lim_{x \to 2^+} f(x) = -\infty \quad \text{and} \quad \lim_{x \to 2^-} f(x) = \infty$$

the line $x = 2$ is a vertical asymptote both from the right and from the left. By symmetry, the line $x = -2$ is also a vertical asymptote. There are no other asymptotes because $f$ has a finite limit at all other points.

**Example 2.6.6.** *The graph of the natural logarithm function has the y-axis (the line $x = 0$ ) as a vertical asymptote.*

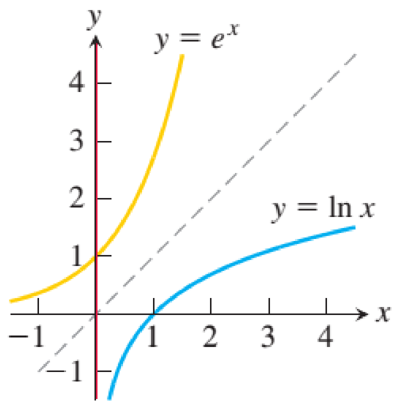



*Figure 2.23*

*We see this from the graph sketched in Figure 2.23 (which is the reflection of the graph of the natural exponential function across the line $y = x$ ) and the fact that the x-axis is a horizontal asymptote of $y = e^x$. Thus,*

$$\lim_{x \to 0^+} \ln x = -\infty$$

*The same result is true for $y = \log_a x$ whenever $a > 1$.*

**Example 2.6.7.** *The curves*

$$y = \sec x = \frac{1}{\cos x} \quad \textit{and} \quad y = \tan x = \frac{\sin x}{\cos x}$$

*both have vertical asymptotes at odd-integer multiples of* $\pi/2$, *where* $\cos x = 0$ *(Figure 2.24).*

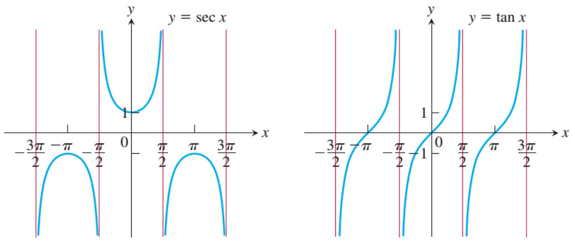



Figure 2.24

## 2.7 Exercises

### Limits

1. Find the following limits:

   (a) $\lim_{x\to\pi} \sin(x - \sin x)$.

   (b) $\lim_{t\to 0} \sin\left(\frac{\pi}{2}\cos(\tan t)\right)$.

   (c) $\lim_{y\to 1} \sec\left(y\sec^2 y - \tan^2 y - 1\right)$.

   (d) $\lim_{x\to 0} \tan\left(\frac{\pi}{4}\cos\left(\sin x^{1/3}\right)\right)$.

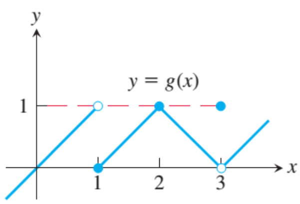
   (e) $\lim_{t\to 0} \cos\left(\frac{\pi}{\sqrt{19-3\sec 2t}}\right)$.

   (f) $\lim_{x\to\pi/6} \sqrt{\csc^2 x + 5\sqrt{3}\tan x}$.

   (g) $\lim_{x\to 0^+} \sin\left(\frac{\pi}{2}e^{\sqrt{x}}\right)$.

   (h) $\lim_{x\to 1} \cos^{-1}(\ln\sqrt{x})$.

2. For the function $g(x)$ graphed here, find the following limits or explain why they do not exist.



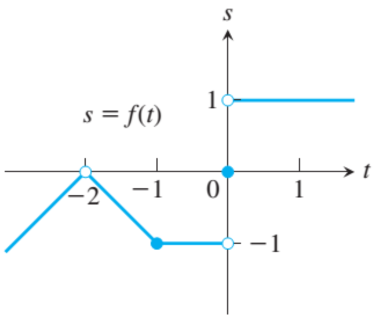

   (a) $\lim_{x\to 1} g(x)$

   (b) $\lim_{x\to 2} g(x)$

   (c) $\lim_{x\to 3} g(x)$

   (d) $\lim_{x\to 2.5} g(x)$.

3. For the function $f(t)$ graphed here, find the following limits or explain why they do not exist.




(a) $\lim_{t\to -2} f(t)$

(b) $\lim_{t\to -1} f(t)$

(c) $\lim_{t\to 0} f(t)$

(d) $\lim_{t\to -0.5} f(t)$.

4. Calculate the following limits:

(a) $\lim_{x\to 2} \frac{2x+5}{11-x^3}$

(b) $\lim_{s\to 2/3}(8-3s)(2s-1)$.

(c) $\lim_{x\to -1/2} 4x(3x+4)^2$

(d) $\lim_{y\to 2} \frac{y+2}{y^2+5y+6}$.

(e) $\lim_{y\to -3}(5-y)^{4/3}$

(f) $\lim_{z\to 4} \sqrt{z^2-10}$.

(g) $\lim_{h\to 0} \frac{3}{\sqrt{3h+1}+1}$

(h) $\lim_{h\to 0} \frac{\sqrt{5h+4}-2}{h}$.

(i) $\lim_{x\to 5} \frac{x-5}{x^2-25}$

(j) $\lim_{x\to -3} \frac{x+3}{x^2+4x+3}$.

(k) $\lim_{x\to -5} \frac{x^2+3x-10}{x+5}$

(l) $\lim_{x\to 2} \frac{x^2-7x+10}{x-2}$.

(m) $\lim_{t\to 1} \frac{t^2+t-2}{t^2-1}$

(n) $\lim_{t\to -1} \frac{t^2+3t+2}{t^2-t-2}$.

(o) $\lim_{x\to -2} \frac{-2x-4}{x^3+2x^2}$

(p) $\lim_{y\to 0} \frac{5y^3+8y^2}{3y^4-16y^2}$.

(q) $\lim_{u\to 1} \frac{u^4-1}{u^3-1}$

(r) $\lim_{v\to 2} \frac{v^3-8}{v^4-16}$.

(s) $\lim_{x\to 9} \frac{\sqrt{x}-3}{x-9}$

(t) $\lim_{x\to 4} \frac{4x-x^2}{2-\sqrt{x}}$.

(u) $\lim_{x\to 1} \frac{x-1}{\sqrt{x+3}-2}$

(v) $\lim_{x\to -1} \frac{\sqrt{x^2+8}-3}{x+1}$.

(w) $\lim_{x\to 2} \frac{\sqrt{x^2+12}-4}{x-2}$

(x) $\lim_{x\to -2} \frac{x+2}{\sqrt{x^2+5}-3}$.

(y) $\lim_{x\to -3} \frac{2-\sqrt{x^2-5}}{x+3}$

(z) $\lim_{x\to 4} \frac{4-x}{5-\sqrt{x^2+9}}$.

5. Calculate the following limits:

(a) $\lim_{x\to -0.5^-} \sqrt{\frac{x+2}{x+1}}$

(b) $\lim_{x\to 1^+} \sqrt{\frac{x-1}{x+2}}$.

(c) $\lim_{x\to -2^+} \left(\frac{x}{x+1}\right)\left(\frac{2x+5}{x^2+x}\right)$

(d) $\lim_{x\to 1^-} \left(\frac{1}{x+1}\right)\left(\frac{x+6}{x}\right)\left(\frac{3-x}{7}\right)$.

(e) $\lim_{h\to 0^+} \frac{\sqrt{h^2+4h+5}-\sqrt{5}}{h}$

(f) $\lim_{x\to 0} \frac{x^2-x+\sin x}{2x}$.

(g) $\lim_{x\to 0} \frac{x+x\cos x}{\sin x\cos x}$

(h) $\lim_{\theta\to 0} \frac{1-\cos\theta}{\sin 2\theta}$.

(i) $\lim_{x\to 0} \frac{x-x\cos x}{\sin^2 3x}$

(j) $\lim_{t\to 0} \frac{\sin(1-\cos t)}{1-\cos t}$.

(k) $\lim_{h\to 0} \frac{\sin(\sin h)}{\sin h}$

(l) $\lim_{\theta\to 0} \frac{\sin\theta}{\sin 2\theta}$.

(m) $\lim_{x\to 0} \frac{\sin 5x}{\sin 4x}$

(n) $\lim_{\theta\to 0} \theta\cos\theta$.

(o) $\lim_{\theta\to 0} \sin\theta\cot 2\theta$

(p) $\lim_{x\to 0} \frac{\tan 3x}{\sin 8x}$.

6. Later we will prove that for any number $x$, $\lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n = e^x$. Use this fact to find the following limits:

(a) $\lim_{x\to 0}(1-x)^{\frac{1}{x}}$

(b) $\lim_{x\to\infty}\left(\frac{x^2+1}{x^2-1}\right)^{x^2}$.

(c) $\lim_{x\to 1} x^{\frac{1}{1-x}}$

(d) $\lim_{x\to 0}(1-3x)^{\frac{1}{x}}$.

(e) $\lim_{x\to\infty}\left(1+\frac{2}{x}\right)^{-x}$.

(f) $\lim_{x\to\infty}(1+\tan x)^{\cot x}$

(g) $\lim_{x\to\infty}\left(\frac{x+1}{x-1}\right)^{x}$.

## Continuity

1. At what points are the following functions continuous?

(a) $y = \frac{1}{x-2} - 3x$

(b) $y = \frac{1}{(x+2)^2} + 4$.

(c) $y = \frac{x+1}{x^2-4x+3}$

(d) $y = \frac{1}{|x|+1} - \frac{x^2}{2}$.

(e) $y = |x-1| + \sin x$

(f) $y = \frac{x+3}{x^2-3x-10}$.

(g) $y = \frac{\cos x}{x}$

(h) $y = \frac{x+2}{\cos x}$.

(i) $y = \csc 2x$

(j) $y = \tan\frac{\pi x}{2}$.

(k) $y = \frac{x\tan x}{x^2+1}$

(l) $y = \frac{\sqrt{x^4+1}}{1+\sin^2 x}$.

(m) $y = \sqrt{2x+3}$

(n) $y = \sqrt[4]{3x-1}$.

(o) $y = (2x-1)^{1/3}$

(p) $y = (2-x)^{1/5}$.

(q) $g(x) = \begin{cases} \frac{x^2-x-6}{x-3}, & x \neq 3 \\ 5, & x = 3 \end{cases}$

(r) $f(x) = \begin{cases} \frac{x^3-8}{x^2-4}, & x \neq 2, x \neq -2 \\ 3, & x = 2 \\ 4, & x = -2 \end{cases}$.

2. For what value of $a$ is
$$f(x) = \begin{cases} x^2 - 1, & x < 3 \\ 2ax, & x \geq 3 \end{cases}$$
continuous at every $x$?

3. For what value of $b$ is
$$g(x) = \begin{cases} x, & x < -2 \\ bx^2, & x \geq -2 \end{cases}$$
continuous at every $x$?

4. For what values of $a$ is
$$f(x) = \begin{cases} a^2x - 2a, & x \geq 2 \\ 12, & x < 2 \end{cases}$$
continuous at every $x$?

## Limits Involving Infinity; Asymptotes of Graphs

1. Find the following limits:

(a) $\lim_{x\to-\infty} \left(\frac{1-x^3}{x^2+7x}\right)^5$

(b) $\lim_{x\to\infty} \sqrt{\frac{x^2-5x}{x^3+x-2}}$.

(c) $\lim_{x\to\infty} \frac{2\sqrt{x}+x^{-1}}{3x-7}$

(d) $\lim_{x\to\infty} \frac{2+\sqrt{x}}{2-\sqrt{x}}$.

(e) $\lim_{x\to-\infty} \frac{\sqrt[3]{x}-\sqrt[5]{x}}{\sqrt[3]{x}+\sqrt[5]{x}}$

(f) $\lim_{x\to\infty} \frac{x^{-1}+x^{-4}}{x^{-2}-x^{-3}}$.

(g) $\lim_{x\to\infty} \frac{2x^{5/3}-x^{1/3}+7}{x^{8/5}+3x+\sqrt{x}}$

(h) $\lim_{x\to-\infty} \frac{\sqrt[3]{x}-5x+3}{2x+x^{2/3}-4}$.

(i) $\lim_{x\to\infty} \frac{\sqrt{x^2+1}}{x+1}$

(j) $\lim_{x\to-\infty} \frac{\sqrt{x^2+1}}{x+1}$.

(k) $\lim_{x\to\infty} \frac{x-3}{\sqrt{4x^2+25}}$

(l) $\lim_{x\to-\infty} \frac{4-3x^3}{\sqrt{x^6+9}}$.

(m) $\lim_{x\to\infty} \frac{2x+3}{5x+7}$

(n) $\lim_{x\to-\infty} \frac{7x^3}{x^3-3x^2+6x}$.

(o) $\lim_{x\to\infty} \frac{3x^7+5x^2-1}{6x^3-7x+3}$

(p) $\lim_{x\to-\infty} \frac{3x+7}{x^2-2}$.

2. Find the limits:

(a) $\lim_{x\to0^+} \frac{1}{3x}$

(b) $\lim_{x\to0^-} \frac{5}{2x}$.

(c) $\lim_{x\to2^-} \frac{3}{x-2}$

(d) $\lim_{x\to3^+} \frac{1}{x-3}$.

(e) $\lim_{x\to-8^+} \frac{2x}{x+8}$

(f) $\lim_{x\to-5^-} \frac{3x}{2x+10}$.

(g) $\lim_{x\to7} \frac{4}{(x-7)^2}$

(h) $\lim_{x\to 0} \frac{-1}{x^2(x+1)}$.

(i) $\lim_{x\to(\pi/2)^-} \tan x$

(j) $\lim_{x\to(-\pi/2)^+} \sec x$.

(k) $\lim_{\theta\to 0^-}(1+\csc\theta)$

(l) $\lim_{\theta\to 0}(2-\cot\theta)$.

3. Use limits to determine the equations for all vertical asymptotes.

   (a) $y = \frac{x^2+4}{x-3}$.

   (b) $f(x) = \frac{x^2-x-2}{x^2-2x+1}$.

   (c) $y = \frac{x^2+x-6}{x^2+2x-8}$.

4. Use limits to determine the equations for all horizontal asymptotes.

   (a) $y = \frac{1-x^2}{x^2+1}$

   (b) $f(x) = \frac{\sqrt{x}+4}{\sqrt{x+4}}$

   (c) $g(x) = \frac{\sqrt{x^2+4}}{x}$

   (d) $y = \sqrt{\frac{x^2+9}{9x^2+1}}$

# Chapter 3

# Differentiation

Differential calculus is the study of the derivative, and differentiation is the process of computing derivatives. What is a derivative? There are three equally important answers:A derivative is a rate of change, it is the slope of a tangent line, and (more formally), it is the limit of a difference quotient, as we will explain shortly. In this chapter, we explore all three facets of the derivative and develop the basic rules of differentiation. When you master these techniques, you will possess one of the most useful and flexible tools that mathematics has to offer.

## 3.1 Definition of the Derivative

We begin with two questions: What is the precise definition of a tangent line? And how can we compute its slope?

The secant line through distinct points $P = (a, f(a))$ and $Q = (x, f(x))$ on the graph of a function $f$ has slope [Figure 6.1 ]

$$\frac{\Delta f}{\Delta x} = \frac{f(x) - f(a)}{x - a}$$

where

$$\Delta f = f(x) - f(a) \quad \text{and} \quad \Delta x = x - a$$

The expression $\dfrac{f(x) - f(a)}{x - a}$ is called the difference quotient. Based on this intuition, we define the derivative $f'(a)$ (which is read " $f$ prime of $a$ ") at a point $x = a$ as the limit

$$\underbrace{f'(a)}_{\substack{\text{Slope of the} \\ \text{tangent line}}} = \underbrace{\lim_{x \to a} \frac{f(x) - f(a)}{x - a}}_{\text{Limit of slopes of secant lines}}$$

There is another way of writing the difference quotient using a new variable $h$ :

$$h = x - a$$

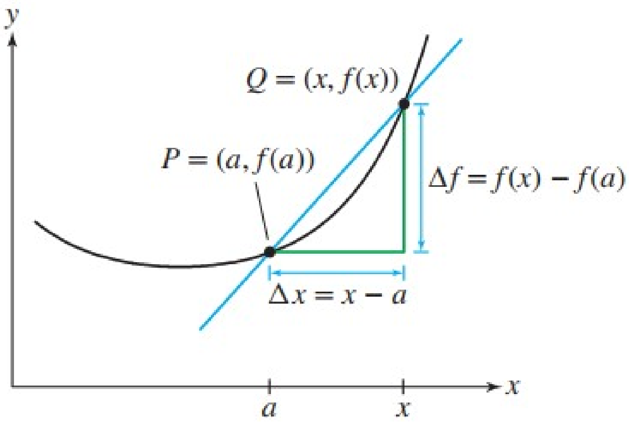



FIGURE 3.1

We have $x = a + h$ and, for $x \neq a$ (Figure 3.2 ),

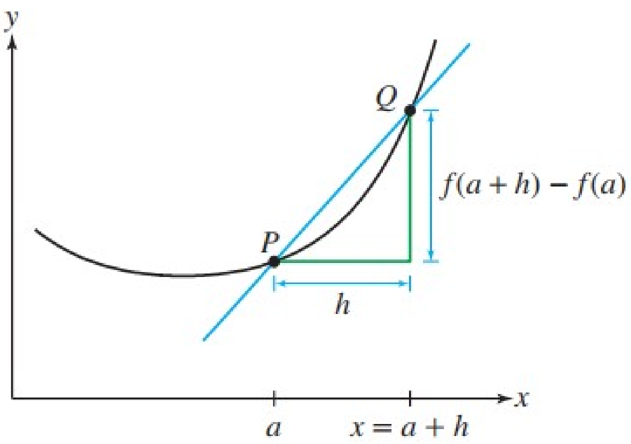



FIGURE 3.2

$$\frac{f(x) - f(a)}{x - a} = \frac{f(a+h) - f(a)}{h}$$

The variable $h$ approaches 0 as $x \to a$, so we can rewrite the derivative as

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

Each way of writing the derivative is useful. The version using $h$ is often more convenient in computations.

**Definition 3.1.** *The Derivative The derivative of $f$ at a point $a$ is the limit of the difference quotients (if it exists):*

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

*When the limit exists, we say that $f$ is differentiable at $a$. An equivalent definition of the derivative at a point $a$ is*

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

We now investigate the derivative as a function derived from $f$ by considering the limit at each point $x$ in the domain of $f$

**Definition 3.2.** *The derivative of the function $f(x)$ with respect to the variable $x$ is the function $f'$ whose value at $x$ is*

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

*provided the limit exists.*

**Example 3.1.1.** *Differentiate*

$$f(x) = \frac{x}{x-1}$$

***Solution*** *We use the definition of derivative, which requires us to calculate $f(x+h)$ and then subtract $f(x)$ to obtain the numerator in the difference quotient. We have*

$$f(x) = \frac{x}{x-1} \quad \textit{and} \quad f(x+h) = \frac{(x+h)}{(x+h)-1}$$

*Then*

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} \\ &= \lim_{h\to 0} \frac{\frac{x+h}{x+h-1} - \frac{x}{x-1}}{h} \\ &= \lim_{h\to 0} \frac{1}{h} \cdot \frac{(x+h)(x-1) - x(x+h-1)}{(x+h-1)(x-1)} \\ &= \lim_{h\to 0} \frac{1}{h} \cdot \frac{-h}{(x+h-1)(x-1)} \\ &= \lim_{h\to 0} \frac{-1}{(x+h-1)(x-1)} \\ &= \frac{-1}{(x-1)^2}. \end{aligned}$$

**Example 3.1.2.** *Find the derivative of* $f(x) = \sqrt{x}$ *for* $x > 0$.

***Solution***

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} \\ &= \lim_{h\to 0} \frac{\sqrt{x+h}-\sqrt{x}}{h} \\ &= \lim_{h\to 0} \frac{\sqrt{x+h}-\sqrt{x}}{h} \frac{\sqrt{x+h}+\sqrt{x}}{\sqrt{x+h}+\sqrt{x}} \\ &= \lim_{h\to 0} \frac{x+h-x}{h(\sqrt{x+h}+\sqrt{x})} \\ &= \lim_{h\to 0} \frac{1}{(\sqrt{x+h}+\sqrt{x})} \\ &= \frac{1}{2\sqrt{x}} \end{aligned}$$

**Example 3.1.3.** *Compute* $f'(3)$, *where* $f(x) = x^2 - 8x$.

***Solution*** *Since*

$$
\begin{aligned}
f'(3) &= \lim_{h\to 0} \frac{f(3+h)-f(3)}{h} \\
&= \frac{\big((3+h)^2 - 8(3+h)\big) - \big(3^2 - 8(3)\big)}{h} \\
&= \frac{\big((9+6h+h^2) - (24+8h)\big) - (9-24)}{h} \\
&= \frac{h^2 - 2h}{h} \\
&= \lim_{h\to 0}(h-2) = -2
\end{aligned}
$$

## 3.2 Differentiation Rules

This section introduces several rules that allow us to differentiate constant functions, power functions, polynomials, rational functions, and certain combinations of them, simply and directly, without having to take limits each time.

### 3.2.1 Powers, Multiples, Sums, and Differences

A simple rule of differentiation is that the derivative of every constant function is zero.

If $f$ has the constant value $f(x) = c$, then

$$\frac{df}{dx} = \frac{d}{dx}(c) = 0$$

*Proof.* We apply the definition of the derivative to $f(x) = c$, the function whose outputs have the constant value $c$ . At every value of $x$, we find that

$$
\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} \\
&= \lim_{h\to 0} \frac{c-c}{h} = \lim_{h\to 0} 0 = 0
\end{aligned}
$$

□

If $n$ is any real number, then

$$\frac{d}{dx}x^n = nx^{n-1}$$

for all $x$ where the powers $x^n$ and $x^{n-1}$ are defined.

**Example 3.2.1.** *Differentiate the following powers of $x$.*

$$(a)\ x^3 \quad (b) x^{2/3} \quad (c)\ x^{\sqrt{2}} \quad (d)\ \frac{1}{x^4} \quad (e) x^{-4/3} \quad (f) \sqrt{x^{2+\pi}}$$

***Solution:***

- (a) $\dfrac{d}{dx}\left(x^3\right) = 3x^{3-1} = 3x^2$
- (b) $\dfrac{d}{dx}\left(x^{2/3}\right) = \dfrac{2}{3}x^{(2/3)-1} = \dfrac{2}{3}x^{-1/3}$
- (c) $\dfrac{d}{dx}\left(x^{\sqrt{2}}\right) = \sqrt{2}x^{\sqrt{2}-1}$
- (d) $\dfrac{d}{dx}\left(\frac{1}{x^4}\right) = \dfrac{d}{dx}\left(x^{-4}\right) = -4x^{-4-1} = -4x^{-5} = -\dfrac{4}{x^5}$
- (e) $\dfrac{d}{dx}\left(x^{-4/3}\right) = -\dfrac{4}{3}x^{-(4/3)-1} = -\dfrac{4}{3}x^{-7/3}$
- (f) $\dfrac{d}{dx}\left(\sqrt{x^{2+\pi}}\right) = \dfrac{d}{dx}\left(x^{1+(\pi/2)}\right) = \left(1+\dfrac{\pi}{2}\right)x^{1+(\pi/2)-1} = \dfrac{1}{2}(2+\pi)\sqrt{x^\pi}$

The next rule says that when a differentiable function is multiplied by a constant, its derivative is multiplied by the same constant.

If $u$ and $v$ are differentiable functions of $x$, then their sum $u+v$ is differentiable at every point where $u$ and $v$ are both differentiable. At such points,

$$\frac{d}{dx}(u+v) = \frac{du}{dx} + \frac{dv}{dx}$$

*Proof.* We apply the definition of the derivative to $f(x) = u(x) + v(x)$ :

$$\begin{aligned}
\frac{d}{dx}[u(x)+v(x)] &= \lim_{h\to 0} \frac{[u(x+h)+v(x+h)] - [u(x)+v(x)]}{h} \\
&= \lim_{h\to 0}\left[\frac{u(x+h)-u(x)}{h} + \frac{v(x+h)-v(x)}{h}\right] \\
&= \lim_{h\to 0}\frac{u(x+h)-u(x)}{h} + \lim_{h\to 0}\frac{v(x+h)-v(x)}{h} = \frac{du}{dx} + \frac{dv}{dx}.
\end{aligned}$$

□

**Example 3.2.2.** *Find the derivative of the polynomial*

$$y = x^3 + \frac{4}{3}x^2 - 5x + 1$$

***Solution***

$$\begin{aligned}\frac{dy}{dx} &= \frac{d}{dx}x^3 + \frac{d}{dx}\left(\frac{4}{3}x^2\right) - \frac{d}{dx}(5x) + \frac{d}{dx}(1)\\ &= 3x^2 + \frac{4}{3}\cdot 2x - 5 + 0\\ &= 3x^2 + \frac{8}{3}x - 5\end{aligned}$$

### 3.2.2 Products and Quotients

While the derivative of the sum of two functions is the sum of their derivatives, the derivative of the product of two functions is not the product of their derivatives. For instance,

$$\frac{d}{dx}(x\cdot x) = \frac{d}{dx}\left(x^2\right) = 2x, \quad \text{while} \quad \frac{d}{dx}(x)\cdot\frac{d}{dx}(x) = 1\cdot 1 = 1$$

The derivative of a product of two functions is the sum of two products, as we now explain.

| |
|---|
| If $u$ and $v$ are differentiable at $x$, then so is their product $uv$, and $$\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$$ |

*Proof.* **Proof of the Derivative Product Rule**

$$\frac{d}{dx}(uv) = \lim_{h\to 0}\frac{u(x+h)v(x+h) - u(x)v(x)}{h}$$

To change this fraction into an equivalent one that contains difference quotients for the derivatives of $u$ and $v$, we subtract and add $u(x+h)v(x)$ in the numerator:

$$\begin{aligned}\frac{d}{dx}(uv) &= \lim_{h\to 0}\frac{u(x+h)v(x+h) - u(x+h)v(x) + u(x+h)v(x) - u(x)v(x)}{h}\\ &= \lim_{h\to 0}\left[u(x+h)\frac{v(x+h)-v(x)}{h} + v(x)\frac{u(x+h)-u(x)}{h}\right]\\ &= \lim_{h\to 0}u(x+h)\cdot\lim_{h\to 0}\frac{v(x+h)-v(x)}{h} + v(x)\cdot\lim_{h\to 0}\frac{u(x+h)-u(x)}{h}\end{aligned}$$

As $h$ approaches zero, $u(x+h)$ approaches $u(x)$ because $u$, being differentiable at $x$, is continuous at $x$. The two fractions approach the values of $dv/dx$ at $x$ and $du/dx$ at $x$. Therefore,

$$\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$$

□

**Example 3.2.3.** *Find the derivative of*

$$y = \left(x^2+1\right)\left(x^3+3\right)$$

***Solution*** *From the Product Rule with $u = x^2+1$ and $v = x^3+3$, we find*

$$\begin{aligned}\frac{d}{dx}\left[\left(x^2+1\right)\left(x^3+3\right)\right] &= \left(x^2+1\right)\left(3x^2\right)+\left(x^3+3\right)(2x)\\ &= 3x^4+3x^2+2x^4+6x\\ &= 5x^4+3x^2+6x\end{aligned}$$

The derivative of the quotient of two functions is given by the Quotient Rule.

If $u$ and $v$ are differentiable at $x$ and if $v(x) \neq 0$, then the quotient $u/v$ is differentiable at $x$, and

$$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

In function notation,

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{g(x)f'(x) - f(x)g'(x)}{g^2(x)}$$

*Proof.* **Proof of the Derivative Quotient Rule**

$$\begin{aligned}\frac{d}{dx}\left(\frac{u}{v}\right) &= \lim_{h\to 0}\frac{\frac{u(x+h)}{v(x+h)} - \frac{u(x)}{v(x)}}{h}\\ &= \lim_{h\to 0}\frac{v(x)u(x+h)-u(x)v(x+h)}{hv(x+h)v(x)}\end{aligned}$$

To change the last fraction into an equivalent one that contains the difference quotients for the derivatives of $u$ and $v$, we subtract and add $v(x)u(x)$ in the numerator. We then get

$$\begin{aligned}\frac{d}{dx}\left(\frac{u}{v}\right) &= \lim_{h\to 0}\frac{v(x)u(x+h)-v(x)u(x)+v(x)u(x)-u(x)v(x+h)}{hv(x+h)v(x)}\\ &= \lim_{h\to 0}\frac{v(x)\frac{u(x+h)-u(x)}{h} - u(x)\frac{v(x+h)-v(x)}{h}}{v(x+h)v(x)}\\ &= \frac{v(x)\lim_{h\to 0}\frac{u(x+h)-u(x)}{h} - u(x)\lim_{h\to 0}\frac{v(x+h)-v(x)}{h}}{\lim_{h\to 0}v(x+h)v(x)}\\ &= \frac{v(x)\frac{du}{dx} - u(x)\frac{dv}{dx}}{v^2(x)}\end{aligned}$$

□

**Example 3.2.4.** *Find the derivative of*

$$y = \frac{(x-1)\left(x^2-2x\right)}{x^4}$$

***Solution*** *Using the Quotient Rule here will result in a complicated expression with many terms. Instead, use some algebra to simplify the expression. First expand the numerator and divide by $x^4$ :*

$$y = \frac{(x-1)\left(x^2-2x\right)}{x^4} = \frac{x^3-3x^2+2x}{x^4} = x^{-1}-3x^{-2}+2x^{-3}$$

*Then use the Sum and Power Rules:*

$$\begin{aligned}\frac{dy}{dx} &= -x^{-2}-3(-2)x^{-3}+2(-3)x^{-4}\\ &= -\frac{1}{x^2}+\frac{6}{x^3}-\frac{6}{x^4}\end{aligned}$$

## 3.3 Derivatives of Trigonometric Functions

Many phenomena of nature are approximately periodic (electromagnetic fields, heart rhythms, tides, weather). The derivatives of sines and cosines play a key role in describing periodic changes. This section shows how to differentiate the six basic trigonometric functions.

To calculate the derivative of $f(x) = sinx$, since

$$\sin(x+h) = \sin x \cos h + \cos x \sin h$$

Then

$$\begin{aligned}f'(x) &= \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{h\to 0}\frac{\sin(x+h)-\sin x}{h}\\ &= \lim_{h\to 0}\frac{(\sin x\cos h+\cos x\sin h)-\sin x}{h}\\ &= \lim_{h\to 0}\frac{\sin x(\cos h-1)+\cos x\sin h}{h}\\ &= \lim_{h\to 0}\left(\sin x\cdot\frac{\cos h-1}{h}\right)+\lim_{h\to 0}\left(\cos x\cdot\frac{\sin h}{h}\right)\\ &= \sin x\cdot\lim_{h\to 0}\frac{\cos h-1}{h}+\cos x\cdot\lim_{h\to 0}\frac{\sin h}{h}\\ &= \sin x\cdot 0+\cos x\cdot 1\\ &= \cos x\end{aligned}$$

**Example 3.3.1.** *Differentiate*

$$y = x^2 \sin x$$

. ***Solution:*** *Using the Product Rule and Formula , we have*

$$\begin{aligned}\frac{dy}{dx} &= x^2 \frac{d}{dx}(\sin x) + \sin x \frac{d}{dx}\left(x^2\right) \\ &= x^2 \cos x + 2x \sin x\end{aligned}$$

**Example 3.3.2.** *Differentiate*

$$y = \frac{\sin x}{x}$$

***Solution:*** *Using the Quotient Rule and Formula , we have*

$$\begin{aligned}\frac{dy}{dx} &= \frac{x \cdot \frac{d}{dx}(\sin x) - \sin x \cdot 1}{x^2} \\ &= \frac{x \cos x - \sin x}{x^2}\end{aligned}$$

To find the derivative of $f(x) = \cos(x)$, since

$$\cos(x+h) = \cos x \cos h - \sin x \sin h$$

we can compute the limit of the difference quotient:

$$\begin{aligned}\frac{d}{dx}(\cos x) &= \lim_{h\to 0} \frac{\cos(x+h) - \cos x}{h} \\ &= \lim_{h\to 0} \frac{(\cos x \cos h - \sin x \sin h) - \cos x}{h} \\ &= \lim_{h\to 0} \frac{\cos x(\cos h - 1) - \sin x \sin h}{h} \\ &= \lim_{h\to 0} \cos x \cdot \frac{\cos h - 1}{h} - \lim_{h\to 0} \sin x \cdot \frac{\sin h}{h} \\ &= \cos x \cdot \lim_{h\to 0} \frac{\cos h - 1}{h} - \sin x \cdot \lim_{h\to 0} \frac{\sin h}{h} \\ &= \cos x \cdot 0 - \sin x \cdot 1 \\ &= -\sin x.\end{aligned}$$

**Example 3.3.3.** *Differentiate*

$$y = \sin x \cos x$$

***Solution:***

$$\begin{aligned}
\frac{dy}{dx} &= \sin x \frac{d}{dx}(\cos x) + \cos x \frac{d}{dx}(\sin x) \\
&= \sin x(-\sin x) + \cos x(\cos x) \\
&= \cos^2 x - \sin^2 x
\end{aligned}$$

**Example 3.3.4.** *Differentiate*

$$y = \frac{\cos x}{1 - \sin x}$$

***Solution:***

$$\begin{aligned}
\frac{dy}{dx} &= \frac{(1-\sin x)\frac{d}{dx}(\cos x) - \cos x \frac{d}{dx}(1 - \sin x)}{(1-\sin x)^2} \\
&= \frac{(1-\sin x)(-\sin x) - \cos x(0 - \cos x)}{(1-\sin x)^2} \\
&= \frac{1 - \sin x}{(1-\sin x)^2} \\
&= \frac{1}{1 - \sin x}
\end{aligned}$$

**Example 3.3.5.** *Verify the formula*

$$\frac{d}{dx}(\tan x) = \sec^2 x$$

***Solution:*** *Use the Quotient Rule and the identity* $\cos^2 x + \sin^2 x = 1$ *:*

$$\begin{aligned}
\frac{d}{dx}(\tan x) &= \left(\frac{\sin x}{\cos x}\right)' \\
&= \frac{\cos x \cdot (\sin x)' - \sin x \cdot (\cos x)'}{\cos^2 x} \\
&= \frac{\cos x \cos x - \sin x(-\sin x)}{\cos^2 x} \\
&= \frac{\cos^2 x + \sin^2 x}{\cos^2 x} = \frac{1}{\cos^2 x} = \sec^2 x
\end{aligned}$$

**Example 3.3.6.** *Verify the formula*

$$\frac{d}{dx}(\sec x) = \sec x \tan x$$

***Solution:*** *Since*

$$\sec x = \frac{1}{\cos x}$$

*Then*

$$\begin{aligned}
\frac{d}{dx}(\sec x) &= \frac{d}{dx}\frac{1}{\cos x} \\
&= \frac{\frac{d}{dx}(1)\cdot(\cos x) - \frac{d}{dx}\cos x\cdot(1)}{(\cos x)^2} \\
&= \frac{\sin x}{(\cos x)(\cos x)} \\
&= \left(\frac{\sin x}{\cos x}\right)\left(\frac{1}{\cos x}\right) \\
&= \tan x \sec x.
\end{aligned}$$

**Example 3.3.7.** *Differentiate*

$$f(x) = \frac{\sec x}{1+\tan x}$$

***Solution:*** *The Quotient Rule gives*

$$\begin{aligned}
f'(x) &= \frac{(1+\tan x)\frac{d}{dx}(\sec x) - \sec x\frac{d}{dx}(1+\tan x)}{(1+\tan x)^2} \\
&= \frac{(1+\tan x)\sec x\tan x - \sec x\cdot\sec^2 x}{(1+\tan x)^2} \\
&= \frac{\sec x\left(\tan x + \tan^2 x - \sec^2 x\right)}{(1+\tan x)^2} \\
&= \frac{\sec x(\tan x - 1)}{(1+\tan x)^2}
\end{aligned}$$

We collect all the differentiation formulas for trigonometric functions in the following table. Remember that they are valid only when $x$ is measured in radians.

**Derivatives of Trigonometric Functions**

$$\begin{aligned}
\frac{d}{dx}(\sin x) &= \cos x & \frac{d}{dx}(\csc x) &= -\csc x\cot x \\
\frac{d}{dx}(\cos x) &= -\sin x & \frac{d}{dx}(\sec x) &= \sec x\tan x \\
\frac{d}{dx}(\tan x) &= \sec^2 x & \frac{d}{dx}(\cot x) &= -\csc^2 x
\end{aligned}$$

## 3.4 The Chain Rule

The Chain Rule is used to differentiate composite functions such as $y = \cos\left(x^3\right)$ and $y = \sqrt{x^4+1}$

Recall that a composite function is obtained by "plugging" one function into another. The composite of $f$ and $g$, denoted $f \circ g$, is defined by

$$(f \circ g)(x) = f(g(x))$$

For convenience, we call $f$ the outside function and $g$ the inside function. Often, we write the composite function as $f(u)$, where $u = g(x)$. For example, $y = \cos\left(x^3\right)$ is the function $y = \cos u$, where $u = x^3$

**Chain Rule** If $f$ and $g$ are differentiable, then the composite function $(f \circ g)(x) = f(g(x))$ is differentiable and

$$(f(g(x)))' = f'(g(x))g'(x)$$

**Example 3.4.1.** *Calculate the derivative of*

$$y = \cos\left(x^3\right)$$

***Solution*** *As noted above,* $y = \cos\left(x^3\right)$ *is a composite* $f(g(x))$*, where*

$$\begin{aligned} f(u) &= \cos u, & u &= & g(x) &= x^3 \\ f'(u) &= -\sin u, & g'(x) &= 3x^2 \end{aligned}$$

*Since* $u = x^3, f'(g(x)) = f'(u) = f'\left(x^3\right) = -\sin\left(x^3\right)$. *So, by the Chain Rule*

$$\begin{aligned} \frac{d}{dx}\cos\left(x^3\right) &= \underbrace{-\sin\left(x^3\right)}_{f'(g(x))}\underbrace{\left(3x^2\right)}_{g'(x)} \\ &= -3x^2\sin\left(x^3\right) \end{aligned}$$

**Example 3.4.2.** *Calculate the derivative of*

$$y = \sqrt{x^4+1}$$

***Solution*** *The function* $y = \sqrt{x^4+1}$ *is a composite* $f(g(x))$*, where*

$$\begin{aligned} f(u) &= u^{1/2}, & u = g(x) &= x^4+1 \\ f'(u) &= \frac{1}{2}u^{-1/2}, & g'(x) &= 4x^3 \end{aligned}$$

*Note that* $f'(g(x)) = \frac{1}{2}\left(x^4+1\right)^{-1/2}$*, so by the Chain Rule,*

$$\begin{aligned} \frac{d}{dx}\sqrt{x^4+1} &= \underbrace{\frac{1}{2}\left(x^4+1\right)^{-1/2}}_{f'(g(x))}\underbrace{\left(4x^3\right)}_{g'(x)} \\ &= \frac{4x^3}{2\sqrt{x^4+1}} \end{aligned}$$

**Example 3.4.3.** *Calculate* $\frac{dy}{dx}$ *for*

$$y = \tan\left(\frac{x}{x+1}\right)$$

***Solution*** *The outside function is* $f(u) = \tan u$*. Because* $f'(u) = \sec^2 u$*, the Chain Rule gives us* $\frac{d}{dx}\tan\left(\frac{x}{x+1}\right) = \sec^2\left(\frac{x}{x+1}\right) \underbrace{\frac{d}{dx}\left(\frac{x}{x+1}\right)}_{\text{Derivative of inside function}}$ *Now, by the Quotient Rule,*

$$\frac{d}{dx}\left(\frac{x}{x+1}\right) = \frac{(x+1)\frac{d}{dx}x - x\frac{d}{dx}(x+1)}{(x+1)^2} = \frac{1}{(x+1)^2}$$

*We obtain*

$$\begin{aligned} \frac{d}{dx}\tan\left(\frac{x}{x+1}\right) &= \sec^2\left(\frac{x}{x+1}\right)\frac{1}{(x+1)^2} \\ &= \frac{\sec^2\left(\frac{x}{x+1}\right)}{(x+1)^2} \end{aligned}$$

It is instructive to write the Chain Rule in Leibniz notation. Let

$$y = f(u) = f(g(x))$$

Then, by the Chain Rule,

$$\frac{dy}{dx} = f'(u)g'(x) = \frac{df}{du}\frac{du}{dx}$$

or

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$$

We now discuss some important special cases of the Chain Rule.

**General Power and Exponential Rules:** If $g$ is differentiable, then

$$\frac{d}{dx}(g(x))^n = n(g(x))^{n-1}g'(x)$$

for any number $n$)

**Example 3.4.4.** *Find the derivatives of*

- *(a)* $y = \left(x^2+7x+2\right)^{-1/3}$.
- *(b)* $y = \sec^4 t$.

***Solution:*** *Apply* $\frac{d}{dx}g(x)^n = ng(x)^{n-1}g'(x)$ .

- *(a)*

$$\begin{aligned}\frac{d}{dx}\left(x^2+7x+2\right)^{-1/3} &= -\frac{1}{3}\left(x^2+7x+2\right)^{-4/3}\frac{d}{dx}\left(x^2+7x+2\right)\\ &= -\frac{1}{3}\left(x^2+7x+2\right)^{-4/3}(2x+7)\end{aligned}$$

- *(b)*

$$\begin{aligned}\frac{d}{dt}\sec^4 t &= 4\sec^3 t\frac{d}{dt}\sec t\\ &= 4\sec^3 t(\sec t\tan t)\\ &= 4\sec^4 t\tan t\end{aligned}$$

**Example 3.4.5.** *Calculate*

$$\frac{d}{dx}\sqrt{1+\sqrt{x^2+1}}.$$

***Solution:*** *First apply the Chain Rule with inside function* $u = 1+\sqrt{x^2+1}$ *:*

$$\frac{d}{dx}\left(1+\left(x^2+1\right)^{1/2}\right)^{1/2} = \frac{1}{2}\left(1+\left(x^2+1\right)^{1/2}\right)^{-1/2}\frac{d}{dx}\left(1+\left(x^2+1\right)^{1/2}\right)$$

*Then apply the Chain Rule again to the remaining derivative:*

$$\begin{aligned}\frac{d}{dx}\left(1+\left(x^2+1\right)^{1/2}\right)^{1/2} &= \frac{1}{2}\left(1+\left(x^2+1\right)^{1/2}\right)^{-1/2}\left(\frac{1}{2}\left(x^2+1\right)^{-1/2}(2x)\right)\\ &= \frac{1}{2}x\left(x^2+1\right)^{-1/2}\left(1+\left(x^2+1\right)^{1/2}\right)^{-1/2}\end{aligned}$$

**Example 3.4.6.** *Find* $f'(x)$ *if*

$$f(x) = \frac{1}{\sqrt[3]{x^2+x+1}}$$

***Solution:*** *First rewrite* $f : f(x) = \left(x^2+x+1\right)^{-1/3}$ *Thus*

$$\begin{aligned} f'(x) &= -\frac{1}{3}\left(x^2+x+1\right)^{-4/3}\frac{d}{dx}\left(x^2+x+1\right) \\ &= -\frac{1}{3}\left(x^2+x+1\right)^{-4/3}(2x+1) \end{aligned}$$

**Example 3.4.7.** *Find the derivative of the function*

$$g(t) = \left(\frac{t-2}{2t+1}\right)^9$$

***Solution:****Combining the Power Rule, Chain Rule, and Quotient Rule, we get*

$$\begin{aligned} g'(t) &= 9\left(\frac{t-2}{2t+1}\right)^8 \frac{d}{dt}\left(\frac{t-2}{2t+1}\right) \\ &= 9\left(\frac{t-2}{2t+1}\right)^8 \frac{(2t+1)\cdot 1 - 2(t-2)}{(2t+1)^2} = \frac{45(t-2)^8}{(2t+1)^{10}} \end{aligned}$$

**Example 3.4.8.** *Differentiate*

$$y = (2x+1)^5\left(x^3-x+1\right)^4$$

***Solution:*** *In this example we must use the Product Rule before using the Chain Rule:*

$$\begin{aligned} \frac{dy}{dx} &= (2x+1)^5\frac{d}{dx}\left(x^3-x+1\right)^4 + \left(x^3-x+1\right)^4\frac{d}{dx}(2x+1)^5 \\ &= (2x+1)^5\cdot 4\left(x^3-x+1\right)^3\frac{d}{dx}\left(x^3-x+1\right) \\ &\quad + \left(x^3-x+1\right)^4\cdot 5(2x+1)^4\frac{d}{dx}(2x+1) \\ &= 4(2x+1)^5\left(x^3-x+1\right)^3\left(3x^2-1\right) + 5\left(x^3-x+1\right)^4(2x+1)^4\cdot 2 \\ &= 2(2x+1)^4\left(x^3-x+1\right)^3\left(17x^3+6x^2-9x+3\right) \end{aligned}$$

## 3.5 Derivatives of Inverse Trigonometric Functions

In fact, if $f$ is any one-to-one differentiable function, it can be proved that its inverse function $f^{-1}$ is also differentiable, except where its tangents are vertical. This is plausible because the graph of a differentiable function has no corner or kink and so if we reflect it about $y = x$, the graph of its inverse function also has no corner or kink.

Recall the definition of the arcsine function:

$$y = \sin^{-1} x \quad \text{means} \quad \sin y = x \quad \text{and} \quad -\frac{\pi}{2} \leqslant y \leqslant \frac{\pi}{2}$$

Differentiating $\sin y = x$ implicitly with respect to $x$, we obtain

$$\cos y \frac{dy}{dx} = 1 \quad \text{or} \quad \frac{dy}{dx} = \frac{1}{\cos y}$$

Now $\cos y \geqslant 0$, since $-\pi/2 \leqslant y \leqslant \pi/2$, so

$$\cos y = \sqrt{1 - \sin^2 y} = \sqrt{1 - x^2}$$

Therefore

$$\frac{dy}{dx} = \frac{1}{\cos y} = \frac{1}{\sqrt{1 - x^2}}$$

Hence

$$\frac{d}{dx}\left(\sin^{-1} x\right) = \frac{1}{\sqrt{1 - x^2}}$$

The formula for the derivative of the arctangent function is derived in a similar way. If $y = \tan^{-1} x$, then $\tan y = x$. Differentiating this latter equation implicitly with respect to $x$, we have

$$\begin{aligned} \sec^2 y \frac{dy}{dx} &= 1 \\ \frac{dy}{dx} &= \frac{1}{\sec^2 y} = \frac{1}{1 + \tan^2 y} = \frac{1}{1 + x^2} \\ \frac{d}{dx}\left(\tan^{-1} x\right) &= \frac{1}{1 + x^2} \end{aligned}$$

The inverse trigonometric functions that occur most frequently are the ones that we have just discussed. The derivatives of the remaining four are given in the following table. The proofs of the formulas are left as exercises.

**Derivatives of Inverse Trigonometric Functions**

$$\begin{aligned} \frac{d}{dx}\left(\sin^{-1} x\right) &= \frac{1}{\sqrt{1 - x^2}} & \frac{d}{dx}\left(\csc^{-1} x\right) &= -\frac{1}{x\sqrt{x^2 - 1}} \\ \frac{d}{dx}\left(\cos^{-1} x\right) &= -\frac{1}{\sqrt{1 - x^2}} & \frac{d}{dx}\left(\sec^{-1} x\right) &= \frac{1}{x\sqrt{x^2 - 1}} \\ \frac{d}{dx}\left(\tan^{-1} x\right) &= \frac{1}{1 + x^2} & \frac{d}{dx}\left(\cot^{-1} x\right) &= -\frac{1}{1 + x^2} \end{aligned}$$

**Example 3.5.1.** *Differentiate*

- *(a)* $y = \frac{1}{\sin^{-1} x}$ .
- *(b)* $f(x) = x \tan^{-1} \sqrt{x}$.

***Solution:***

- *(a)*

$$\begin{aligned}\frac{dy}{dx} &= \frac{d}{dx}\left(\sin^{-1} x\right)^{-1} = -\left(\sin^{-1} x\right)^{-2} \frac{d}{dx}\left(\sin^{-1} x\right) \\ &= -\frac{1}{\left(\sin^{-1} x\right)^{2} \sqrt{1-x^2}}\end{aligned}$$

- *(b)*

$$\begin{aligned}f'(x) &= x\frac{1}{1+(\sqrt{x})^2}\left(\frac{1}{2}x^{-1/2}\right) + \arctan\sqrt{x} \\ &= \frac{\sqrt{x}}{2(1+x)} + \arctan\sqrt{x}\end{aligned}$$

**Example 3.5.2.** *Differentiate*

$$y = \cot^{-1}\left(\frac{1}{x^2}\right)$$

$$\begin{aligned}y' &= \left(\cot^{-1}\frac{1}{x^2}\right)' \\ &= -\frac{1}{1+\left(\frac{1}{x^2}\right)^2} \cdot \left(\frac{1}{x^2}\right)' \\ &= -\frac{1}{1+\frac{1}{x^4}} \cdot \left(-2x^{-3}\right) \\ &= \frac{2x^4}{\left(x^4+1\right)x^3} \\ &= \frac{2x}{1+x^4}\end{aligned}$$

**Example 3.5.3.** *Find the derivative of each of the following function:*

$$f(x) = \cos^{-1}\left(x^3 + x\right)$$

$$
\begin{aligned}
f'(x) &= \frac{d}{dx}\left[\cos^{-1}\left(x^3+x\right)\right] \\
&= -\frac{1}{\sqrt{1-(x^3+x)^2}}\cdot\left(3x^2+1\right) \\
&= -\frac{3x^2+1}{\sqrt{1-(x^3+x)^2}}
\end{aligned}
$$

## 3.6 Derivative of the Natural Exponential Function

Thus the exponential function $f(x) = e^x$ has the property that it is its own derivative. The geometrical significance of this fact is that the slope of a tangent line to the curve $y = e^x$ is equal to the $y$-coordinate of the point .

$$\frac{d}{dx}\left(e^x\right) = e^x$$

**Example 3.6.1.** *Differentiate the function*

$$y = e^{\tan x}.$$

***Solution:*** *To use the Chain Rule, we let $u = \tan x$. Then we have $y = e^u$, so*

$$
\begin{aligned}
\frac{dy}{dx} &= \frac{dy}{du}\frac{du}{dx} \\
&= e^u\frac{du}{dx} \\
&= e^{\tan x}\sec^2 x
\end{aligned}
$$

In general , we have

$$\frac{d}{dx}\left(e^u\right) = e^u\frac{du}{dx}$$

**Example 3.6.2.** *Find $y'$ if*

$$y = e^{-4x}\sin 5x$$

***Solution:*** *Using Product Rule, we have*

$$
\begin{aligned}
y' &= e^{-4x}(\cos 5x)(5) + (\sin 5x)e^{-4x}(-4) \\
&= e^{-4x}(5\cos 5x - 4\sin 5x)
\end{aligned}
$$

## 3.7 Derivative Of Logarithmic Functions

We begin by establishing that $f(x) = \ln x$ is differentiable for $x > 0$ by using the derivative definition to find its derivative. To obtain this derivative, we need the fact that $\ln x$ is continuous for $x > 0$. Since $e^x$ is continuous , we know that $\ln x$ is continuous for $x > 0$ . We will also need the limit

$$\lim_{v\to 0}(1+v)^{1/v} = e$$

Using the definition of the derivative, we obtain

$$\begin{aligned}
\frac{d}{dx}[\ln x] &= \lim_{h\to 0}\frac{\ln(x+h)-\ln x}{h} \\
&= \lim_{h\to 0}\frac{1}{h}\ln\left(\frac{x+h}{x}\right) \\
&= \lim_{h\to 0}\frac{1}{h}\ln\left(1+\frac{h}{x}\right) \\
&= \lim_{v\to 0}\frac{1}{vx}\ln(1+v) \\
&= \frac{1}{x}\lim_{v\to 0}\frac{1}{v}\ln(1+v) \\
&= \frac{1}{x}\lim_{v\to 0}\ln(1+v)^{1/v} \\
&= \frac{1}{x}\ln\left[\lim_{v\to 0}(1+v)^{1/v}\right] \\
&= \frac{1}{x}\ln e \\
&= \frac{1}{x}
\end{aligned}$$

Thus,

$$\frac{d}{dx}[\ln x] = \frac{1}{x}, \quad x > 0$$

A derivative formula for the general logarithmic function $\log_b x$ can be obtained the following from

$$\frac{d}{dx}\left[\log_b x\right] = \frac{d}{dx}\left[\frac{\ln x}{\ln b}\right] = \frac{1}{\ln b}\frac{d}{dx}[\ln x]$$

It follows from this that

$$\frac{d}{dx}\left[\log_b x\right] = \frac{1}{x\ln b}, \quad x > 0$$

**Example 3.7.1.** *Find* $\frac{d}{dx}\left[\ln\left(x^2+1\right)\right]$ ***Solution:*** *Let* $u = x^2+1$ *we obtain*

$$\begin{aligned}\frac{d}{dx}\left[\ln\left(x^2+1\right)\right] &= \frac{1}{x^2+1}\cdot\frac{d}{dx}\left[x^2+1\right] = \frac{1}{x^2+1}\cdot 2x\\ &= \frac{2x}{x^2+1}\end{aligned}$$

**Example 3.7.2.** *Find*

$$\frac{d}{dx}\left[\ln\left(\frac{x^2\sin x}{\sqrt{1+x}}\right)\right]$$

***Solution:***

$$\begin{aligned}\frac{d}{dx}\left[\ln\left(\frac{x^2\sin x}{\sqrt{1+x}}\right)\right] &= \frac{d}{dx}\left[2\ln x+\ln(\sin x)-\frac{1}{2}\ln(1+x)\right]\\ &= \frac{2}{x}+\frac{\cos x}{\sin x}-\frac{1}{2(1+x)}\\ &= \frac{2}{x}+\cot x-\frac{1}{2+2x}\end{aligned}$$

We now consider a technique called **logarithmic differentiation** that is useful for differentiating functions that are composed of products, quotients, and powers.

**Example 3.7.3.** *Find the derivative of*

$$y = \frac{x^2\sqrt[3]{7x-14}}{(1+x^2)^4}$$

***Solution:*** *Take the natural logarithm of both sides and then use its properties, we can write*

$$\ln y = 2\ln x + \frac{1}{3}\ln(7x-14) - 4\ln\left(1+x^2\right)$$

*Differentiating both sides with respect to x yields*

$$\frac{1}{y}\frac{dy}{dx} = \frac{2}{x}+\frac{7/3}{7x-14}-\frac{8x}{1+x^2}$$

*Thus, on solving for* $dy/dx$ *we obtain*

$$\frac{dy}{dx} = \frac{x^2\sqrt[3]{7x-14}}{(1+x^2)^4}\left[\frac{2}{x}+\frac{1}{3x-6}-\frac{8x}{1+x^2}\right]$$

**Example 3.7.4.** *Find*

$$\frac{d}{dx}\ln\frac{x+1}{\sqrt{x-2}}$$

***Solution:***

$$\begin{aligned}\frac{d}{dx}\ln\frac{x+1}{\sqrt{x+2}} &= \frac{d}{dx}\left[\ln(x+1)-\frac{1}{2}\ln(x-2)\right]\\ &= \frac{1}{x+1}-\frac{1}{2}\left(\frac{1}{x-2}\right)\end{aligned}$$

**Steps in Logarithmic Differentiation**

1. Take natural logarithms of both sides of an equation $y = f(x)$ and use the properties of logarithms to simplify.
2. Differentiate implicitly with respect to $x$.
3. Solve the resulting equation for $y'$.

**Example 3.7.5.** *Differentiate*

$$y = x^{\sqrt{x}}$$

***Solution:****Since both the base and the exponent are variable, we use logarithmic differentiation:*

$$\begin{aligned}\ln y &= \ln x^{\sqrt{x}} = \sqrt{x}\ln x\\ \frac{y'}{y} &= \sqrt{x}\cdot\frac{1}{x}+(\ln x)\frac{1}{2\sqrt{x}}\\ y' &= y\left(\frac{1}{\sqrt{x}}+\frac{\ln x}{2\sqrt{x}}\right) = x^{\sqrt{x}}\left(\frac{2+\ln x}{2\sqrt{x}}\right)\end{aligned}$$

**Example 3.7.6.** *Differentiate*

$$y = x^x + (\sin x)^{\cos x}$$

***Solution:*** *Consider*

$$y = y_1 + y_2$$

*where* $y_1 = x^x, \quad y_2 = (\sin x)^{\cos x}$ *Since*

$$\begin{aligned}\ln y_1 &= \ln x^x\\ &= x\ln x\\ \frac{1}{y_1}y_1' &= x\frac{1}{x}+\ln(x)\\ y_1' &= x^x[1+\ln(x)]\end{aligned}$$

*and*

$$\begin{aligned}
\ln y_2 &= \ln(\sin x)^{\cos x} \\
&= (\cos x)\ln(\sin x) \\
\frac{1}{y_2}y_2' &= \frac{\cos^2 x}{\sin x} - \sin x \ln(\sin x) \\
y_2' &= (\sin x)^{\cos x}[\cot x \csc x - \sin x \ln(\sin x)]
\end{aligned}$$

*Then*

$$\begin{aligned}
y' &= y_1' + y_2' \\
&= x^x[1+\ln(x)] + (\sin x)^{\cos x}[\cot x \csc x - \sin x \ln(\sin x)]
\end{aligned}$$

## 3.8 Derivatives of Hyperbolic Functions

The derivatives of the hyperbolic functions are easily computed. For example,

$$\frac{d}{dx}(\sinh x) = \frac{d}{dx}\left(\frac{e^x - e^{-x}}{2}\right) = \frac{e^x + e^{-x}}{2} = \cosh x$$

We list the differentiation formulas for the hyperbolic functions as in the following table . The remaining proofs are left as exercises. Note the analogy with the differentiation formulas for trigonometric functions, but beware that the signs are different in some cases.

**Derivatives of Hyperbolic Functions**

$$\begin{aligned}
\frac{d}{dx}(\sinh x) &= \cosh x & \frac{d}{dx}(\operatorname{csch} x) &= -\operatorname{csch} x \coth x \\
\frac{d}{dx}(\cosh x) &= \sinh x & \frac{d}{dx}(\operatorname{sech} x) &= -\operatorname{sech} x \tanh x \\
\frac{d}{dx}(\tanh x) &= \operatorname{sech}^2 x & \frac{d}{dx}(\coth x) &= -\operatorname{csch}^2 x
\end{aligned}$$

**Example 3.8.1.** *Differentiate*

$$y = \cosh\sqrt{x}$$

***Solution:***

$$
\begin{aligned}
\frac{dy}{dx} &= \frac{d}{dx}(\cosh\sqrt{x}) \\
&= \sinh\sqrt{x}\cdot\frac{d}{dx}\sqrt{x} \\
&= \frac{\sinh\sqrt{x}}{2\sqrt{x}}
\end{aligned}
$$

**Example 3.8.2.** *Differentiate*

$$y = \cosh\sqrt{x}$$

***Solution:***

$$
\begin{aligned}
\frac{dy}{dx} &= \frac{d}{dx}\left[\cosh\left(x^3\right)\right] \\
&= \sinh\left(x^3\right)\cdot\frac{d}{dx}\left[x^3\right] \\
&= 3x^2\sinh\left(x^3\right)
\end{aligned}
$$

**Example 3.8.3.** *Differentiate*

$$y = \ln(\tanh x)$$

***Solution:***

$$
\begin{aligned}
\frac{dy}{dx} &= \frac{d}{dx}[\ln(\tanh x)] \\
&= \frac{1}{\tanh x}\cdot\frac{d}{dx}[\tanh x] \\
&= \frac{\operatorname{sech}^2 x}{\tanh x}
\end{aligned}
$$

## 3.9 Derivatives Of Inverse Hyperbolic Functions

Formulas for the derivatives of the inverse hyperbolic functions can be obtained as the following

$$\begin{aligned}
\frac{d}{dx}\left[\sinh^{-1} x\right] &= \frac{d}{dx}\left[\ln\left(x+\sqrt{x^2+1}\right)\right] \\
&= \frac{1}{x+\sqrt{x^2+1}}\left(1+\frac{x}{\sqrt{x^2+1}}\right) \\
&= \frac{\sqrt{x^2+1}+x}{\left(x+\sqrt{x^2+1}\right)\left(\sqrt{x^2+1}\right)} \\
&= \frac{1}{\sqrt{x^2+1}}
\end{aligned}$$

The following table list the generalized derivative formulas and corresponding formulas for the inverse hyperbolic functions. Some of the proofs appear as exercises.

$$\boxed{\begin{aligned}
\frac{d}{dx}\left(\sinh^{-1} u\right) &= \frac{1}{\sqrt{1+u^2}}\frac{du}{dx} \\
\frac{d}{dx}\left(\coth^{-1} u\right) &= \frac{1}{1-u^2}\frac{du}{dx}, \quad |u|>1 \\
\frac{d}{dx}\left(\cosh^{-1} u\right) &= \frac{1}{\sqrt{u^2-1}}\frac{du}{dx}, \quad u>1 \\
\frac{d}{dx}\left(\operatorname{sech}^{-1} u\right) &= -\frac{1}{u\sqrt{1-u^2}}\frac{du}{dx}, \quad 0<u<1 \\
\frac{d}{dx}\left(\tanh^{-1} u\right) &= \frac{1}{1-u^2}\frac{du}{dx}, \quad |u|<1 \\
\frac{d}{dx}\left(\operatorname{csch}^{-1} u\right) &= -\frac{1}{|u|\sqrt{1+u^2}}\frac{du}{dx}, \quad u\neq 0
\end{aligned}}$$

**Example 3.9.1.** *Verify*

$$\frac{d}{dx}\coth x = -\operatorname{csch}^2 x$$

***Solution:*** *By the Quotient Rule and the identity* $\cosh^2 x - \sinh^2 x = 1$,

$$\begin{aligned}
\frac{d}{dx}\coth x &= \left(\frac{\cosh x}{\sinh x}\right)' \\
&= \frac{(\sinh x)(\cosh x)' - (\cosh x)(\sinh x)'}{\sinh^2 x} \\
&= \frac{\sinh^2 x - \cosh^2 x}{\sinh^2 x} = \frac{-1}{\sinh^2 x} \\
&= -\operatorname{csch}^2 x
\end{aligned}$$

**Example 3.9.2.** *Calculate:*

- *(a)* $\dfrac{d}{dx}\cosh\left(3x^2+1\right)$
- *(b)* $\dfrac{d}{dx}\sinh x \tanh x$.

***Solution:***

- *(a) By the Chain Rule,*

$$\frac{d}{dx}\cosh\left(3x^2+1\right) = 6x\sinh\left(3x^2+1\right)$$

.

- *(b) By the Product Rule,*

$$\begin{aligned}
\frac{d}{dx}(\sinh x \tanh x) &= \sinh x \operatorname{sech}^2 x + \tanh x \cosh x \\
&= \operatorname{sech} x \tanh x + \sinh x
\end{aligned}$$

**Example 3.9.3.** *Verify the formula*

$$\frac{d}{dx}\tanh^{-1} x = \frac{1}{1-x^2}$$

***Solution*** *Recall that if* $g$ *is the inverse of* $f$, *then* $g'(x) = 1/f'(g(x))$. *Applying this to* $f(x) = \tanh x$, *and using the formula* $(\tanh x)' = \operatorname{sech}^2 x$, *we have*

$$\frac{d}{dx}\tanh^{-1} x = \frac{1}{\operatorname{sech}^2\left(\tanh^{-1} x\right)}$$

*Now let* $t = \tanh^{-1} x$. *Then*

$$\begin{aligned} \cosh^2 t - \sinh^2 t &= 1 && \textit{(basic identity)} \\ 1 - \tanh^2 t &= \operatorname{sech}^2 t && \textit{(divide by } \cosh^2 t\textit{)} \\ 1 - x^2 &= \operatorname{sech}^2\left(\tanh^{-1} x\right) && \textit{(because } x = \tanh t\textit{)} \end{aligned}$$

*Hence*

$$\frac{d}{dx}\tanh^{-1} x = \frac{1}{\operatorname{sech}^2\left(\tanh^{-1} x\right)} = \frac{1}{1-x^2}$$

## 3.10 Implicit Differentiation

Most of the functions we have dealt with so far have been described by an equation of the form $y = f(x)$ that expresses $y$ explicitly in terms of the variable $x$. We have learned rules for differentiating functions defined in this way. Another situation occurs when we encounter equations like

$$x^3 + y^3 - 9xy = 0, \quad y^2 - x = 0, \quad \text{or} \quad x^2 + y^2 - 25 = 0$$

These equations define an implicit relation between the variables $x$ and $y$. In some cases we may be able to solve such an equation for $y$ as an explicit function (or even several functions) of $x$. When we cannot put an equation $F(x, y) = 0$ in the form $y = f(x)$ to differentiate it in the usual way, we may still be able to find $dy/dx$ by implicit differentiation. This section describes the technique.

**Implicit Differentiation**

1. Differentiate both sides of the equation with respect to $x$, treating $y$ as a differentiable function of $x$.
2. Collect the terms with $dy/dx$ on one side of the equation and solve for $dy/dx$.

**Example 3.10.1.** *Find* $dy/dx$ *if*

$$y^2 = x^2 + \sin xy$$

***Solution:*** *We differentiate the equation implicitly.*

$$\begin{aligned}
y^2 &= x^2 + \sin xy \\
\frac{d}{dx}\left(y^2\right) &= \frac{d}{dx}\left(x^2\right) + \frac{d}{dx}(\sin xy) \\
2y\frac{dy}{dx} &= 2x + (\cos xy)\frac{d}{dx}(xy) \\
2y\frac{dy}{dx} &= 2x + (\cos xy)\left(y + x\frac{dy}{dx}\right) \\
2y\frac{dy}{dx} - (\cos xy)\left(x\frac{dy}{dx}\right) &= 2x + (\cos xy)y \\
(2y - x\cos xy)\frac{dy}{dx} &= 2x + y\cos xy \\
\frac{dy}{dx} &= \frac{2x + y\cos xy}{2y - x\cos xy}
\end{aligned}$$

**Example 3.10.2.** *Find $d^2y/dx^2$ if*

$$2x^3 - 3y^2 = 8$$

***Solution:*** *To start, we differentiate both sides of the equation with respect to $x$ in order to find $y' = dy/dx$*

$$\begin{aligned}
\frac{d}{dx}\left(2x^3 - 3y^2\right) &= \frac{d}{dx}(8) \\
6x^2 - 6yy' &= 0 \\
y' &= \frac{x^2}{y}, \quad \textit{when } y \neq 0
\end{aligned}$$

*We now apply the Quotient Rule to find $y''$.*

$$\begin{aligned}
y'' &= \frac{d}{dx}\left(\frac{x^2}{y}\right) \\
&= \frac{2xy - x^2y'}{y^2} \\
&= \frac{2x}{y} - \frac{x^2}{y^2}\cdot y'
\end{aligned}$$

*Finally, we substitute $y' = x^2/y$ to express $y''$ in terms of $x$ and $y$.*

$$\begin{aligned}
y'' &= \frac{2x}{y} - \frac{x^2}{y^2}\left(\frac{x^2}{y}\right) \\
&= \frac{2x}{y} - \frac{x^4}{y^3}, \quad \textit{when } y \neq 0
\end{aligned}$$

## 3.11 Exercises

1. Compute $f'(a)$ using the limit definition of $f$ at $x = a$

   (a) $f(x) = x^2 - x, \quad a = 1$

   (b) $f(x) = 5 - 3x, \quad a = 2$

   (c) $f(x) = x^{-1}, \quad a = 4$

   (d) $f(x) = x^3, \quad a = -2$

2. Compute $dy/dx$ using the limit definition.

   (a) $y = 4 - x^2$

   (b) $y = \sqrt{2x+1}$

   (c) $y = \frac{1}{2-x}$

   (d) $y = \frac{1}{(x-1)^2}$

3. Compute the derivative.

   - $y = 3x^5 - 7x^2 + 4$
   - $y = 4x^{-3/2}$
   - $y = t^{-7.3}$
   - $y = 4x^2 - x^{-2}$
   - $y = \frac{x+1}{x^2+1}$
   - $y = \frac{3t-2}{4t-9}$
   - $y = \left(x^4 - 9x\right)^6$
   - $y = \left(3t^2 + 20t^{-3}\right)^6$
   - $y = \left(2 + 9x^2\right)^{3/2}$
   - $y = (x+1)^3(x+4)^4$
   - $y = \frac{z}{\sqrt{1-z}}$
   - $y = \left(1 + \dfrac{1}{x}\right)^3$
   - $y = \frac{x^4+\sqrt{x}}{x^2}$
   - $y = \frac{1}{(1-x)\sqrt{2-x}}$
   - $y = \sqrt{x + \sqrt{x + \sqrt{x}}}$
   - $h(z) = \left(z + (z+1)^{1/2}\right)^{-3/2}$
   - $y = \tan\left(t^{-3}\right)$
   - $y = 4\cos(2 - 3x)$
   - $y = \sin(2x)\cos^2 x$
   - $y = \sin\left(\frac{4}{\theta}\right)$
   - $y = \frac{t}{1+\sec t}$
   - $y = z\csc(9z+1)$
   - $f(x) = x^5 + 5^x$
   - $g(x) = x\sin\left(2^x\right)$
   - $G(x) = 4^{c/x}$
   - $F(t) = 3^{\cos 2t}$
   - $y = (\cos x)^x$
   - $y = (\sin x)^{\ln x}$
   - $y = (\tan x)^{1/x}$
   - $y = (\ln x)^{\cos x}$
   - $y = \left(\tan^{-1} x\right)^2$
   - $g(x) = \arccos\sqrt{x}$
   - $y = \sin^{-1}(2x+1)$
   - $R(t) = \arcsin(1/t)$

- $y = x\sin^{-1}x + \sqrt{1-x^2}$
- $F(x) = x\sec^{-1}\left(x^3\right)$
- $y = \cos^{-1}\left(\sin^{-1}t\right)$
- $y = \arctan\sqrt{\dfrac{1-x}{1+x}}$

4. Use implicit differentiation to find $dy/dx$ .

   (a) $x^2y + xy^2 = 6$
   (b) $x^3 + y^3 = 18xy$
   (c) $2xy + y^2 = x + y$
   (d) $x^3 - xy + y^3 = 1$
   (e) $x^2(x-y)^2 = x^2 - y^2$
   (f) $(3xy+7)^2 = 6y$
   (g) $y^2 = \frac{x-1}{x+1}$
   (h) $x^3 = \frac{2x-y}{x+3y}$
   (i) $x = \sec y$
   (j) $xy = \cot(xy)$
   (k) $x + \tan(xy) = 0$
   (l) $x^4 + \sin y = x^3y^2$
   (m) $y\sin\left(\frac{1}{y}\right) = 1 - xy$
   (n) $x\cos(2x+3y) = y\sin x$

5. Find $y'$ if $e^{x/y} = x - y$.

6. Find an equation of the tangent line to the curve $xe^y + ye^x = 1$ at the point $(0,1)$

7. Show that the function $y = e^x + e^{-x/2}$ satisfies the differential equation $2y'' - y' - y = 0$

8. Show that the function $y = Ae^{-x} + Bxe^{-x}$ satisfies the differential equation $y'' + 2y' + y = 0$

9. For what values of $r$ does the function $y = e^{rx}$ satisfy the differential equation $y'' + 6y' + 8y = 0$?

10. Find the values of $\lambda$ for which $y = e^{\lambda x}$ satisfies the equation $y + y' = y''$.

11. If $f(x) = e^{2x}$, find a formula for $f^{(n)}(x)$.

12. Find the thousandth derivative of $f(x) = xe^{-x}$.

13. Find $y'$ if $\tan^{-1}\left(x^2y\right) = x + xy^2$.

14. Prove that $x^2y'' + xy' + y = 0$ , where $y = \sin(\ln(x)) + \cos(\ln(x))$.

15. Prove that $x^4y'' + y = 0$ , where $y = x\sin\left(\dfrac{1}{x}\right)$.

# Chapter 4

# Applications of Derivatives

In the previous chapter we focused almost exclusively on the computation of derivatives. In this chapter will focus on applications of derivatives. It is important to always remember that we didn't spend a whole chapter talking about computing derivatives just to be talking about them. There are many very important applications to derivatives.

The two main applications that we'll be looking at in this chapter are using derivatives to determine information about graphs of functions and optimization problems. These will not be the only applications however. We will be revisiting limits and taking a look at an application of derivatives that will allow us to compute limits that we haven't been able to compute previously.

## 4.1 Tangent and Normal Lines

The derivative of a function at a point is the slope of the tangent line at this point. The normal line is defined as the line that is perpendicular to the tangent line at the point of tangency. Therefore, we have:

1. The equation of the tangent line for the curve $y = f(x)$ at the point $(x_1, y_1)$ is given by

$$\frac{y - y_1}{x - x_1} = m, \quad \text{where } m = f'(x_1) := f'(x)\big|_{x=x_1}$$

2. The equation of the normal line for the curve $y = f(x)$ at the point $x = a$ is given by

$$\frac{y - y_1}{x - x_1} = -\frac{1}{m}$$

**Example 4.1.1.** *Find the equation of the tangent and normal lines to the graph of $f(x) = \sqrt{x^2 + 3}$ at the point $(-1, 2)$*

***Solution:***

$$
\begin{aligned}
f(x) &= \left(x^2+3\right)^{1/2} \\
f'(x) &= \frac{1}{2}\left(x^2+3\right)^{-1/2} \cdot (2x) \\
f'(x) &= \frac{x}{\sqrt{x^2+3}}
\end{aligned}
$$

*At the point $(-1,2), f'(-1) = -1/2$ and the equation of the line is*

$$
\begin{aligned}
y - y_1 &= m\,(x - x_1) \\
y - 2 &= -\frac{1}{2}(x+1) \\
2y - 4 &= -x - 1 \\
x + 2y &= 3
\end{aligned}
$$

*Similarly, the equation of the normal line at the point $(-1,2)$ is*

$$
\begin{aligned}
\frac{y-2}{x+1} &= 2 \\
y - 2 &= 2(x+1) \\
y - 2x - 4 &= 0
\end{aligned}
$$

**Example 4.1.2.** *Find the equation of the tangent and normal lines to the graph of $y = x^2$ at the point $(-1,1)$*

***Solution:*** *By differentiating, we have*

$$
\begin{aligned}
f(x) &= x^2 \\
f(x) &= 2x \\
m &= f'(-1) = -2
\end{aligned}
$$

*Hence, the equation of the tangent line at the point $(-1,1)$ is*

$$
\begin{aligned}
y - 1 &= -2(x+1) \\
y + 2x + 1 &= 0
\end{aligned}
$$

*Similarly, the equation of the normal line at the point $(-1,1)$ is*

$$
\begin{aligned}
y - 1 &= \frac{1}{2}(x+1) \\
2y - x - 3 &= 0
\end{aligned}
$$

## 4.2 Maclaurin and Taylor series

The Maclaurin series expansion for $y = f(x)$ (as a power in $x$ ) is defined by.

$$f(x) = f(0) + \frac{x}{1!}f'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f''(0) + \frac{x^4}{4!}f^{(4)}(0) + \ldots + \frac{x^n}{n!}f^{(n)}(0) + \ldots$$

**Example 4.2.1.** *Find the Maclaurin series for $f(x) = e^x$ , and hence for $f(x) = \sinh(x)$.*
***Solution:*** *We have*

$$\begin{aligned}
f(x) &= e^x & f(0) &= 1\\
f'(x) &= e^x & f'(0) &= 1\\
f''(x) &= e^x & f''(0) &= 1\\
f'''(x) &= e^x & f'''(0) &= 1\\
&\vdots & &\\
f^{(n)}(x) &= e^x & f^{(n)}(0) &= 1
\end{aligned}$$

*Now, using the Maclaurin expansion, we get*

$$\begin{aligned}
e^x &= 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \ldots\ldots\\
e^{-x} &= 1 - \frac{x}{1!} + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} - \frac{x^5}{5!} + \ldots\ldots
\end{aligned}$$

*Hence*

$$\sinh x = \frac{e^x - e^{-x}}{2} = \frac{x}{1!} + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \frac{x^9}{9!} + \ldots\ldots$$

**Example 4.2.2.** *Find the Maclaurin series for $f(x) = \sin x$ and hence for$f(x) = \cos x$ .*
***Solution:*** *We know that the Maclaurin series expansion for $y = f(x)$ is defined by.Now, we have*

$$\begin{aligned}
f(x) &= \sin x & f(0) &= 0\\
f'(x) &= \cos x & f'(0) &= 1\\
f''(x) &= -\sin x & f''(0) &= 0\\
f'''(x) &= -\cos x & f''(0) &= -1\\
f^{(4)}(x) &= \sin x & f^{(4)}(0) &= 0\\
f^{(5)}(x) &= \cos x & f^{(5)}(0) &= 1
\end{aligned}$$

*Then*

$$\begin{aligned}\sin x &= \frac{x}{1!} - \frac{x^3}{3!} + \frac{x^3}{5!} - \frac{x'}{7!}\ldots\ldots\\ \cos x &= \frac{d}{dx}[\sin x]\\ &= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{5!} + \ldots\ldots\end{aligned}$$

**Remark 4.1.** *The Maclaurin expansions for* $\cosh x, \cos x, \ln(1+x), \ln(1-x), \tan^{-1} x$ *and more are readily found using similar application. However note that a power series expression for* $\ln x$ *is not possible since* $\ln 0$ *is not defined, and neither are derivatives of* $\ln x$ *at* $x = 0$

**Definition 4.2.** *The Talyor series expansion for* $y = f(x)$ *at* $x = 0$ *(as a power in terms of* $(x-a)$ *is defined by:*

$$f(x) = f(a) + \frac{x-a}{1!}f'(a) + \frac{(x-a)^2}{2!}f''(a) + \frac{(x-a)^3}{3!}f''(a) + \frac{(x-a)^4}{4!}f^{(4)}(a) + \ldots.. + \frac{(x-a)^n}{n!}f^{(n)}(a) + \ldots$$

*OR*

$$f(x+a) = f(a) + \frac{x}{1!}f'(a) + \frac{x^2}{2!}f''(a) + \frac{x^3}{3!}f''(a) + \frac{x^4}{4!}f^{(4)}(a) + \ldots. + \frac{x^n}{n!}f^{(n)}(a) + \ldots$$

This means that Maclaurin series expansion is the Taylor expansion at $x = 0$.

**Example 4.2.3.** *Find the expansion of* $\cos(a+x)$*, and hence the expansion for* $\cos(x)$ *Solution: We have*

$$\begin{aligned}f(x) &= \cos x & f(a) &= \cos a\\ f'(x) &= -\sin x & f(a) &= -\sin a\\ f''(x) &= -\cos x & f''(a) &= -\cos a\\ f''(x) &= \sin x & f'''(a) &= \sin a\end{aligned}$$

*The pattern now established*

$$\cos(a+x) = \cos a - (\sin a)x - (\cos a)\frac{x^2}{2!} + (\sin a)\frac{x^3}{3!} + \ldots$$

*Note that putting* $a = 0$ *in this expansion*

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots.$$

**Example 4.2.4.** *Find the expansion (the binomial expansion.) for* $(a+x)^n$ . ***Solution:*** *We have*

$$\begin{aligned} f(x) &= x^n & f(a) &= a^n \\ f'(x) &= nx^{n-1} & f'(a) &= na^{n-1} \\ f''(x) &= n(n-1)x^{n-2} & f''(a) &= n(n-1)a^{n-2} \\ &\vdots \end{aligned}$$

*Then*

$$(a+x)^n = a^n + na^{n-1} + n(n-1)a^{n-2}\frac{x^2}{2!} + \dots$$

*readily recognizable as the binomial expansion.*

**Remark 4.3.** *The series can be used for approximation. For example in the expansion for* $\cos(a+x)$*, aforegoing, putting* $a = \frac{\pi}{3}$ $(60^\circ)$ *and* $x = \frac{\pi}{180}$ $(1^\circ)$ *gives an approximate value for* $\cos 61^\circ$*. The more terms taken, the better the approximation. So*

$$\cos 61^\circ = \cos\frac{\pi}{3} - \left(\sin\frac{\pi}{3}\right)\left(\frac{\pi}{180}\right) - \left(\cos\frac{\pi}{3}\right)\frac{\left(\frac{\pi}{180}\right)^2}{2!} + \left(\sin\frac{\pi}{3}\right)\frac{\left(\frac{\pi}{180}\right)^3}{3!} + \dots$$

**Example 4.2.5.** *Find expansion for* $\ln x$ *in terms of powers of* $(x-a)$.

***Solution:*** *One can be found using this first form of Taylor expansion. With*

$$\begin{aligned} f(x) &= \ln x, & f(a) &= \ln a \\ f'(x) &= \frac{1}{x} & f'(a) &= \frac{1}{a} \\ f''(x) &= -\frac{1}{x^2} & f''(a) &= -\frac{1}{a^2} \\ f'''(x) &= \frac{2}{x^3} & f'''(a) &= \frac{2}{a^3} \end{aligned}$$

*Then*

$$\ln x = \ln a + \frac{1}{a}(x-a) - \frac{1}{a^2}\frac{(x-a)^2}{2!} + \frac{2}{a^3}\frac{(x-a)^3}{3!} + \dots$$

**Remark 4.4.** *Approximations when* $x$ *is close to* $a$ *such that* $(x-a)$ *is small, and consequently successive powers of* $(x-a)$ $\left[\mathrm{eg}(x-a)^2, (x-a)^3\right]$ *become smaller and smaller approximate values can be found*

**Example 4.2.6.** *In the a foregoing an approximate value of* $\ln(1.01)$ *can be found by letting* $x = 1.01, a = 1$ *and* $(x-a) = 0.01$

***Solution :***

$$
\begin{aligned}
\ln 1.01 &= \ln 1 + \frac{1}{1}(0.01) - \frac{1}{1^2}\frac{(0.01)^2}{2!} + \frac{2}{1^3}\frac{(0.01)^3}{3!} + \dots \\
&= 0 + 0.01 - 0.00005 + 0.000000333. \\
&= \underline{\underline{0.0099503}} \quad \textit{Compare with calculator}
\end{aligned}
$$

**Other approximations:**

1. ***A*** linear approximation for $f(x)$ when $x$ is close to $a$ can be found by ignoring $(x-a)^2$ and higher powers, for then $f(x) \approx f(a) + (x-a)f'(a)$
2. ***A*** quadratic approximation for $f(x)$ when $\underline{x}$ is close to $a$ can be found by ignoring $(x-a)^3$ and higher powers, for then $f(x) \approx f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2!}f''(a) + \dots$

**Example 4.2.7.** *Find the linear approximation for* $\tan x$ *when* $x$ *is close to* $\frac{\pi}{4}$
***Solution :*** *Since*

$$
\begin{aligned}
f(x) &= \tan x & f\left(\frac{\pi}{4}\right) &= \tan\frac{\pi}{4} = 1 \\
f'(x) &= \sec^2 x & f'\left(\frac{\pi}{4}\right) &= \sec^2\frac{\pi}{4} = 2
\end{aligned}
$$

*Then*

$$
\begin{aligned}
f(x) &\approx 1 + \left(x - \frac{\pi}{4}\right)2 \\
&= f(x) \approx 1 - \frac{\pi}{2} + 2x
\end{aligned}
$$

## 4.3 Critical points, Increasing and Decreasing Functions, relative maximum and relative minimum points:

The derivative of a function may be used to determine whether the function is increasing or decreasing on any intervals in its domain.

**Definition 4.5.** *Let* $y = f(x)$ *be a differentiable function. Then, we have*

1. *A point* $c$ *is called critical point of* $f(x)$ *if* $f'(c) = 0$.
2. *The function is said to be increasing on* $I$ *if* $f'(x) > 0$ *at each point in an interval* $I$.
3. *The function is said to be decreasing on* $I$ *if* $f'(x) < 0$ *at each point in an interval* $I$.

**Remark 4.6.** *We use the first derivative test Or the second derivative test to check that a critical point $x = c$ of a function $y = f(x)$ is relative maximum or relative minimum or neither nor (See below)*

**First Derivative Test:** Suppose that $x = c$ is a critical point of $f(x)$ then,

1. If $f'(x) > 0$ to the left of $x = c$ and $f'(x) < 0$ to the right of $x = c$ then $x = c$ is a relative maximum.
2. If $f'(x) < 0$ to the left of $x = c$ and $f'(x) > 0$ to the right of $x = c$ then $x = c$ is a relative minimum.
3. If $f'(x)$ is the same sign on both sides of $x = c$ then $x = c$ is neither a relative maximum nor a relative minimum.

**Second Derivative Test:** Suppose that $x = c$ is a critical point of $f'(c)$ such that $f'(c) = 0$ and that $f''(x)$ is continuous in a region around $x = c$. Then,

1. If $f'(c) < 0$ then $x = c$ is a relative maximum.
2. If $f'(c) > 0$ then $x = c$ is a relative minimum.
3. If $f'(c) = 0$ then $x = c$ can be a relative maximum, relative minimum or neither.

**Example 4.3.1.** *Find the critical points of*

$$f(x) = x^3 - 9x^2 + 24x - 10$$

***Solution:*** *The function $f$ is differentiable everywhere , so the critical points are the solutions of $f'(x) = 0$*

$$\begin{aligned} f'(x) = 3x^2 - 18x + 24 &= 3\left(x^2 - 6x + 8\right) \\ &= 3(x-2)(x-4) = 0 \end{aligned}$$

*The critical points are the roots $c = 2$ and $c = 4$*

**Example 4.3.2.** *Find the critical points of*

$$h(t) = 10te^{3-1^2}$$

***Solution:*** *Here's the derivative for this function.*

$$\mathrm{h}'(\mathrm{t}) = 10\mathrm{e}^{3-1^2} + 10te^{3-1^2}(-2\mathrm{t}) = 10\mathrm{e}^{3-1^2} - 20\mathrm{t}^2\mathrm{e}^{3-1^2}$$

*Now, this looks unpleasant, however with a little factoring we can clean things up a little as follows,*

$$h'(t) = 10e^{3-1^2}\left(1 - 2t^2\right)$$

*This function will exist everywhere and so no critical points will come from that. Determining where this is zero is easier than it looks. We know that exponentials are never zero and so the only way the derivative will be zero is if,*

$$\begin{aligned} 1 - 2t^2 &= 0 \\ 1 &= 2t^2 \\ \frac{1}{2} &= t^2 \end{aligned}$$

*We will have two critical points for this function.*

$$t = \pm\frac{1}{\sqrt{2}}$$

**Steps to determining increasing or decreasing intervals and relative maximum (minimum) points**

1. Find the domain of $f(x)$ and calculate the first derivative to finding all its critical points .

2. Test all intervals in the domain of the function to the left and to the right of these critical points to determine if the derivative ispositive or negative. (If $f'(x) > 0$, then $f$ is increasing on the interval, and if $f'(x) < 0$, then $f$ is decreasing on the interval). positive or negative. (If $f'(x) > 0$, then $f$ is increasing on the interval, and if $f'(x) < 0$, then $f$ is decreasing on the interval). 3) Check that a critical point $x = c$ of a function $y = f(x)$ is relative maximum or relative minimum or neither nor using the first derivative test (or the second derivative test) . Check that a critical point $x = c$ of a function $y = f(x)$ is relative maximum or relative minimum or neither nor using the first derivative test (or the second derivative test) .

**Example 4.3.3.** *For $f(x) = x^4 - 8x^2$ determine all intervals where $f$ is increasing or decreasing.*

***Solution:*** *The domain of $f(x)$ is all real numbers, and its derivatives given by*

$$f'(x) = 4x^3 - 16x = 4x(x-2)(x+2) = 0$$

*So its critical points occur at $x = -2, 0,$ and $2$ . Testing all intervals to the left and right of these values for $f'(x) = 4x^3 - 16x$, you find that*

$$\begin{aligned} f'(x) &< 0 \text{ on } (-\infty, -2) \\ f'(x) &> 0 \text{ on } (-2, 0) \\ f'(x) &< 0 \text{ on } (0, 2) \\ f'(x) &> 0 \text{ on } (2, +\infty) \end{aligned}$$

*Hence, $f$ is increasing on $(-2, 0)$ and $(2, +\infty)$ and decreasing on $(-\infty, -2)$ and $(0, 2)$.*

**Example 4.3.4.** *For $f(x) = \sin x + \cos x$ on $[0, 2\pi]$, determine all intervals where $f$ is increasing or decreasing.*

***Solution:*** *The domain of $f(x)$ is restricted to the closed interval $[0, 2\pi]$, and its critical points occur at $\pi/4$ and $5\pi/4$. Testing all intervals to the left and right of these values for $f'(x) = \cos x - \sin x$, you find that*

$$\begin{aligned} f'(x) &> 0 \text{ on } \left[0, \frac{\pi}{4}\right) \\ f'(x) &< 0 \text{ on } \left(\frac{\pi}{4}, \frac{5\pi}{4}\right) \\ f'(x) &> 0 \text{ on } \left(\frac{5\pi}{4}, 2\pi\right] \end{aligned}$$

*Hence, $f$ is increasing on $[0, \pi/4](5\pi/4, 2\pi)$ and decreasing on $(\pi/4, 5\pi/4)$.*

**Example 4.3.5.** *Find d classify all the critical points of the following function. Give the intervals where the function is increasing and decreasing.*

$$g(t) = t\sqrt[3]{t^2 - 4}$$

***Solution:*** *First we'll need the derivative so we can get our hands on the critical points. Note as well that we'll do some simplification on the derivative to help us find the critical points.*

$$\begin{aligned} g'(t) &= \left(t^2 - 4\right)^{\frac{1}{3}} + \frac{2}{3}t^2\left(t^2 - 4\right)^{-\frac{2}{3}} \\ &= \left(t^2 - 4\right)^{\frac{1}{3}} + \frac{2t^2}{3\left(t^2 - 4\right)^{\frac{2}{3}}} \\ &= \frac{3\left(t^2 - 4\right) + 2t^2}{3\left(t^2 - 4\right)^{\frac{2}{3}}} \\ &= \frac{5t^2 - 12}{3\left(t^2 - 4\right)^{\frac{2}{3}}} \end{aligned}$$

*So, it looks like we'll have four critical points here. They are,*

$$t = \pm 2 \quad \textit{The derivative doesn't exist here.}$$
$$t = \pm\sqrt{\frac{12}{5}} = \pm 1.549 \quad \textit{The derivative is zero here.}$$

*Finding the intervals of increasing and decreasing will also give the classification of the critical points so let's get those first. Here is a number line with the critical points graphed and test points. So, it looks like we've got the following intervals of increasing and decreasing.*

- *Increase :* $-\infty < x < -\sqrt{\frac{12}{5}}$ *and* $\sqrt{\frac{12}{5}} < x < \infty$
- *Decrease :* $-\sqrt{\frac{12}{5}} < \mathrm{x} < \sqrt{\frac{12}{5}}$

*From this it looks like* $t = -2$ *and* $t = 2$ *are neither relative minimum or relative maximums since the function is increasing on both side of them. On the other hand,* $t = -\sqrt{\dfrac{1}{5}}$ *is a relative maximum and* $t = \sqrt{\dfrac{12}{5}}$ *is a relative minimum.*

## 4.4 Concave Up, Concave Down, Points of Inflection:

We have seen previously that the sign of the derivative provides us with information about where a function (and its graph) is increasing, decreasing or stationary. We now look at the "direction of bending" of a graph, i.e. whether the graph is "concave up" or "concave down".

**Definition 4.7.** *Let* $y = f(x)$ *be a differentiable function. A graph of* $y = f(x)$ *is said to be*

1. *concave up at a point if the tangent line to the graph at that point lies below the graph in the vicinity of the point (where* $f''(x) > 0$)
2. *concave down at a point if the tangent line lies above the graph in the vicinity of the point (where* $f''(x) < 0$).
3. *A point where the concavity changes (from up to down or down to up) is called a point of inflection (POI) (where* $f''(x) = 0$ *). Note that the tangent line to a graph at a point of inflection must cross the graph at that point.*

**Steps to determining the concavity of the function's curve at any point**

1. Calculate the second derivative.
2. Substitute the value of $x$.
3. If $f''(x) > 0$, the graph is concave upward at that value of $x$.
4. If $f''(x) = 0$, the graph may have a point of inflection at that value of $x$. To check, consider the value of $f''(x)$ at values of $x$ to either side of the point of interest.
5. If $f''(x) < 0$, the graph is concave downward at that value of $x$.

**Example 4.4.1.** *Find the points of inflection and the intervals on which*

$$f(x) = 3x^5 - 5x^4 + 1$$

*is concave up and concave down.*

***Solution:*** *The first derivative is* $f'(x) = 15x^4 - 20x^3$ *and*

$$f''(x) = 60x^3 - 60x^2 = 60x^2(x-1)$$

*The zeros of* $f''(x) = 60x^2(x-1)$ *are* $x = 0$ *and* $x = 1$. *They divide the x-axis into three intervals:* $(-\infty, 0)$, $(0, 1)$, *and* $(1, \infty)$. *We determine the sign of* $f''(x)$ *and the concavity of* $f$ *by computing "test values" within each interval (Figure 7):*

| *Interval* | *Test Value* | *Sign of* $f''(x)$ | *Behavior of* $f(x)$ |
|---|---|---|---|
| $(-\infty, 0)$ | $f''(-1) = -120$ | $-$ | *Concave down* |
| $(0, 1)$ | $f''\left(\frac{1}{2}\right) = -\frac{15}{2}$ | $-$ | *Concave down* |
| $(1, \infty)$ | $f''(2) = 240$ | $+$ | *Concave up* |

*We can read off the points of inflection from this table:*

- $c = 0$ : *no point of inflection, because* $f''(x)$ *does not change sign at* 0.
- $c = 1$ : *point of inflection, because* $f''(x)$ *changes sign at 1 .*

**Example 4.4.2.** *Find the point of inflection for*

$$f(x) = x^4$$

***Solution:*** *Consider $f(x) = x^4$. Solving $f''(x) = 12x^2 = 0$ yields $x = 0$. At values of $x < 0$, the second derivative is positive. At values of $x > 0$, the second derivative is positive. $(0,0)$ is a local minimum ( is not inflection).*

## 4.5 L'Hospital's Rule

What happens when we try to evaluate $\lim_{x\to 1} \dfrac{\ln x}{x-1}$?

- Can't do the limits of the numerator and denominator separately. Why?
- No common terms to cancel

The limit may or may not exist and is called an indeterminate form of type $\infty/\infty$. There is also an indeterminate form of type $0/0$. Ex: $\lim_{x\to 0} \frac{x}{x^2}$

**L'Hospital's Rule:** Suppose that we have one of the following cases,

$$\lim_{x\to a} \frac{\mathrm{f(x)}}{\mathrm{g(x)}} = \frac{0}{0} \quad \text{OR} \quad \lim_{x\to 2} \frac{\mathrm{f(x)}}{\mathrm{g(x)}} = \frac{\pm\infty}{\pm\infty}$$

where a can be any real number, infinity or negative infinity. In these cases we have,

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$$

**Example 4.5.1.** *Evaluate the following limits*

1. $\lim_{x\to 0} \dfrac{\sin 5x}{3x}$
2. $\lim_{x\to 0} \dfrac{\ln(1+x)}{x}$

1. $\lim_{x\to 0} \dfrac{\sin 5x}{3x} = \frac{0}{0}$ *Hence, using L'Hospital's rule, we get*

$$\lim_{x\to 0} \frac{\sin 5x}{3x} = \text{Lim}_{x\to 0} \frac{5\cos 5x}{3} = \frac{5}{3}$$

2. $\lim_{x\to 0} \dfrac{\ln(1+x)}{x} = \frac{0}{0}$ *Hence, using L'Hospital's rule, we obtain*

$$\lim_{x\to 0} \frac{\ln(1+x)}{x} = \text{Lim}_{x\to 0} \frac{1}{1+x} = 1$$

**Example 4.5.2.** *Evaluate the following limits*

1. $\lim_{x\to 0} \dfrac{e^x - e^{-x} - 2x}{x - \sin x}$
2. $\lim_{x\to 0} \dfrac{\sqrt{1+\sin x} - \sqrt{1-\sin x}}{x}$
3. $\lim_{x\to 0}(\csc x - \cot x)$

*Using L'Hospital's Rule for three times, we get*

1.

$$\begin{aligned}
\lim_{x\to 0} \frac{e^x - e^{-x} - 2x}{x - \sin x} &= \lim_{x\to 0} \frac{e^x + e^{-x} - 2}{1 - \cos x} \\
&= \lim_{x\to 0} \frac{e^x - e^{-x}}{\sin x} \\
&= \lim_{x\to 0} \frac{e^x - e^{-x}}{\sin x} \\
&= \lim_{x\to 0} \frac{e^x + e^{-x}}{\cos x} \\
&= \frac{2}{1} = 2
\end{aligned}$$

2.

$$\begin{aligned}
\lim_{x\to 0} \frac{\sqrt{1+\sin x} - \sqrt{1-\sin x}}{x} &= \frac{1-1}{0} \\
&= \frac{0}{0}
\end{aligned}$$

*Hence, using L'Hospital's rule, we get*

$$\lim_{x\to 0} \frac{\sqrt{1+\sin x} - \sqrt{1-\sin x}}{x} = \frac{0}{0}$$

*Hence, using L'Hospital's rule, we get*

$$\begin{aligned}
\lim_{x\to 0} \frac{\sqrt{1+\sin x} - \sqrt{1-\sin x}}{x} &= \lim_{x\to 0} \frac{\frac{\cos x}{2\sqrt{1+\sin x}} + \frac{\cos x}{2\sqrt{1-\sin x}}}{1} \\
&= \frac{1}{2} + \frac{1}{2} = 1
\end{aligned}$$

*3.*

$$\begin{aligned}
\lim_{x\to 0}(\csc x - \cot x) &= \lim_{x\to 0}\left(\frac{1}{\sin x} - \frac{\cos x}{\sin x}\right) \\
&= \lim_{x\to 0}\left(\frac{1-\cos x}{\sin x}\right) = \frac{1-1}{0} = \frac{0}{0} \\
&= \lim_{x\to 0}\left(\frac{1-\cos x}{\sin x}\right) = \lim_{x\to 0}\left(\frac{\sin x}{\cos x}\right) = 0
\end{aligned}$$

L'Hospital's Rule works great on the two indeterminate forms $0/0$ and $\pm\infty \pm \infty$. However, there are many more indeterminate forms $\left\{0\cdot\mp\infty, \infty-\infty, 0^0, \infty^0, 1^\infty\right\}$. Let's take a look at some of those and see how we deal with those kinds of indeterminate forms.

**Example 4.5.3.** *Evaluate the following limit*

$$\lim_{x\to 0^+}(x\ln x)$$

***Solution:*** *Since,*

$$\lim_{x\to 0^+}(x\ln x) = 0\cdot(-\infty)$$

*Now, in the limit, we get the indeterminate form $(0)(-\infty)$. L'Hospital's Rule won't work on products, it only works on quotients. However, we can turn this into a fraction if we rewrite things a little.*

$$\lim_{x\to 0^-} x\ln x = \lim_{x\to 0^*}\frac{\ln x}{1/x}$$

*The function is the same, just rewritten, and the limit is now in the form $-\infty/\infty$ and we can now use L'Hospital's Rule.*

$$\lim_{x\to 0^+} \mathrm{x}\ln x = \lim_{x\to 1^\circ}\frac{\ln x}{1/x} = \lim_{x\to 0^-}\frac{1/x}{-1/x^2}$$

*Now, this is a mess, but it cleans up nicely.*

$$\lim_{x\to 0^\circ} x\ln x = \lim_{x\to 0^\circ}\frac{1/x}{-1/x^2} = \lim_{x\to 0^-}(-x) = 0$$

**Example 4.5.4.**

$$\lim_{x\to 0^+}(1+\sin 4x)^{\cot x}$$

***Solution:*** *Since* $\lim_{x\to 0^+}(1+\sin 4x)^{\cot x} = 1^\infty$ *(indeterminate power), putting*

$$y = (1+\sin 4x)^{\cot x} \Rightarrow \lim_{x\to 0^+} y?$$

*Taking the natural logarthim* ln*, we have*

$$\ln y = \cot \ln(1 + \sin 4x) = \frac{\ln(1 + \sin 4x)}{\tan x}$$
$$\lim_{x \to 0^+} \ln y = \lim_{x \to 0^+} \frac{\ln(1 + \sin 4x)}{\tan x} = \frac{0}{0}$$

*Now, we can use L'Hospital rules*

$$\begin{aligned} \ln \lim_{x \to 0^+} y &= \lim_{x \to 0^+} \frac{4 \cos 4x / (1 + \sin 4x)}{\sec^2 x} \\ &= \lim_{x \to 0^+} \frac{4 \cos 4x}{\sec^2 x (1 + \sin 4x)} \\ &= \lim_{x \to 0^+} \frac{4 \cos 4x \cos^2 x}{(1 + \sin 4x)} = 1 \end{aligned}$$

*Hence, we obtain*

$$\lim_{x \to 0^+} y = e$$

## 4.6 EXERCISES

1. Find the equation of the tangent and normal lines to following curves at the corresponding points:
   (a) $x^2 + y^2 - 4x + 6y - 3 = 0; \quad (2. - 7)$
   (b) $x^2 - y^2 = 9 \quad ; (3\sqrt{2}, 3)$
   (c) $y^2 = x - 1 \quad ; (5, 2)$
   (d) $x = 2y^2 - y + 1; (4, -1)$
   (e) $x = \ln t \quad ; \quad y = 2t^2 \quad ; \quad (0, 2)$
2. Find the equation of the tangent $y = x^3 - 6x + 2$ and parallel to the line $y = 6x - 2$
3. Prove the two tangent lines for $x^2 - 4y + 4 = 0$ at $(3/2, 0)$ are perpendicular.
4. Determine the increasing and decreasing intervals for the following functions:
   (a) $y = 2x^2 - 4x + 5$
   (b) $y = x^2 - 4x + 1$
   (c) $y = x^3 - 3x^2 + 5$
   (d) $y = 2x^3 - 3x^2 + 1$
   (e) $y = x^4 - 2x^2 + 1$
   (f) $y = x^4 - 4x^2 + 5$
5. Find the maximum and minimum extremes for the following functions:
   (a) $y = x^2 - 2x + 3$
   (b) $y = x^4 - 8x^2 + 2$
   (c) $y = -x^4 + 2x^3$
   (d) $y = x^3 - 9x^2 + 15x + 3$
6. Evaluate the following limits;
   (a) $\lim_{x \to 0} \dfrac{1 - \cos x}{x^2}$
   (b) $\lim_{x \to \pi/2} \dfrac{2 \cos x}{2x - \pi}$
   (c) $\lim_{x \to 0} \dfrac{\cos^{-1}(1 + x)}{\ln(1 + x)}$
   (d) $\lim_{x \to 0} \dfrac{x - \tan x}{x - \sin x}$
   (e) $\lim_{x \to \infty} \dfrac{\ln x}{\sqrt[3]{x}}$
   (f) $\lim_{x \to 0} \dfrac{\tan x - x}{x^3}$
7. Determine the point(s) on $y = x^2 + 1$ that are closest to $(0, 2)$.
8. Determine the area of the largest rectangle that can be inscribed in a circle of radius 4.
9. For the following function find the inflection points and use the second derivative test, if possible, to classify the critical points. Also, determine the intervals of increase, decrease and the intervals of concave up, concave down and sketch the graph of the functions:

(a) $y = x(6-x)^{\frac{2}{3}}$

(b) $y = x^3 + 3x^2 - 9x - 20$

10. Use the second derivative test to classify the critical points of the function $y = 3x^5 - 5x^3 + 3$

# Chapter 5

# INTEGRATION

## 5.1 Antidifferentiation

How can a known rate of inflation be used to determine future prices? What is the velocity of an object moving along a straight line with known acceleration? How can knowing the rate at which a population is changing be used to predict future population levels? In all these situations, the derivative (rate of change) of a quantity is known and the quantity itself is required. Here is the terminology we will use in connection with obtaining a function from its derivative.

**Antidifferentiation** A function $F(x)$ is said to be an antiderivative of $f(x)$ if

$$F'(x) = f(x)$$

for every $x$ in the domain of $f(x)$. The process of finding antiderivatives is called antidifferentiation or indefinite integration.

Later in this section, you will learn techniques you can use to find antiderivatives. Once you have found what you believe to be an antiderivative of a function, you can always check your answer by differentiating. You should get the original function back. Here is an example

**Example 5.1.1.** *Verify that* $F(x) = \frac{1}{3}x^3 + 5x + 2$ *is an antiderivative of* $f(x) = x^2 + 5$
***Solution:*** $F(x)$ *is an antiderivative of* $f(x)$ *if and only if* $F'(x) = f(x)$. *Differentiate* $F$ *and you will find that*

$$\begin{aligned} F'(x) &= \frac{1}{3}\left(3x^2\right) + 5 \\ &= x^2 + 5 = f(x) \end{aligned}$$

*as required.*

A function has more than one antiderivative. For example, one antiderivative of the function $f(x) = 3x^2$ is $F(x) = x^3$, since

$$F'(x) = 3x^2 = f(x)$$

but so are $x^3 + 12$ and $x^3 - 5$ and $x^3 + \pi$, since

$$\frac{d}{dx}\left(x^3 + 12\right) = 3x^2 \quad \frac{d}{dx}\left(x^3 - 5\right) = 3x^2 \quad \frac{d}{dx}\left(x^3 + \pi\right) = 3x^2$$

In general, if $F$ is one antiderivative of $f$, then so is any function of the form $G(x) = F(x) + C$, for constant $C$ since

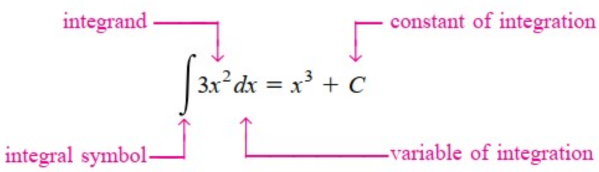
$$\begin{aligned} G'(x) &= [F(x) + C]' \\ &= F'(x) + C' \\ &= F'(x) + 0 \\ &= f(x) \end{aligned}$$

**Fundamental Property of Antiderivatives** If $F(x)$ is an antiderivative of the continuous function $f(x)$, then any other antiderivative of $f(x)$ has the form $G(x) = F(x) + C$ for some constant $C$

You have just seen that if $F(x)$ is one antiderivative of the continuous function $f(x)$ then all such antiderivatives may be characterized by $F(x) + C$ for constant $C$. The family of all antiderivatives of $f(x)$ is written

$$\int f(x)dx = F(x) + C$$

In the context of the indefinite integral $\int f(x)dx = F(x) + C$, the integral symbol is $\int_0^0 f$, the function $f(x)$ is called the integrand, $C$ is the constant of integration, and $dx$ is a differential that specifies $x$ as the variable of integration. These features are displayed in this diagram for the indefinite integral of $f(x) = 3x^2$



For any differentiable function $F$, we have

$$\int F'(x)dx = F(x) + C$$

since by definition, $F(x)$ is an antiderivative of $F'(x)$. Equivalently,

$$\int \frac{dF}{dx}dx = F(x) + C$$

**Rules for Integrating Common Functions**

- The constant rule: $\int k dx = kx + C$ for constant $k$
- The power rule: $\int x^n dx = \frac{x^{n+1}}{n+1} + C$ for all $n \neq -1$
- The logarithmic rule: $\int \frac{1}{x}dx = \ln|x| + C \quad$ for all $x \neq 0$
- The exponential rule: $\int e^{kx}dx = \frac{1}{k}e^{kx} + C \quad$ for constant $k \neq 0$

**Example 5.1.2.** *Evaluate*

$$\int \left(2x^5 + 8x^3 - 3x^2 + 5\right) dx$$

***Solution:***

$$\begin{aligned}
\int \left(2x^5 + 8x^3 - 3x^2 + 5\right) dx &= 2\int x^5 dx + 8\int x^3 dx - 3\int x^2 dx + \int 5dx \\
&= 2\left(\frac{x^6}{6}\right) + 8\left(\frac{x^4}{4}\right) - 3\left(\frac{x^3}{3}\right) + 5x + C \\
&= \frac{1}{3}x^6 + 2x^4 - x^3 + 5x + C
\end{aligned}$$

**Example 5.1.3.** *Evaluate*

$$\int \left(\frac{x^3 + 2x - 7}{x}\right) dx$$

***Solution:***

$$\begin{aligned}
\int \left(\frac{x^3 + 2x - 7}{x}\right) dx &= \int \left(x^2 + 2 - \frac{7}{x}\right) dx \\
&= \frac{1}{3}x^3 + 2x - 7\ln|x| + C
\end{aligned}$$

**Example 5.1.4.** *Evaluate*

$$\int \left(3e^{-5t} + \sqrt{t}\right) dt$$

***Solution:***

$$\begin{aligned}
\int \left(3e^{-5t} + \sqrt{t}\right) dt &= \int \left(3e^{-5t} + t^{1/2}\right) dt \\
&= 3\left(\frac{1}{-5}e^{-5t}\right) + \frac{1}{3/2}t^{3/2} + C \\
&= -\frac{3}{5}e^{-5t} + \frac{2}{3}t^{3/2} + C
\end{aligned}$$

**Example 5.1.5.** *Evaluate*

$$\int \frac{dx}{1+e^x}$$

***Solution:***

$$\begin{aligned}
\int \frac{dx}{1+e^x} &= \int \frac{dx}{1+e^x}\frac{e^{-x}}{e^{-x}} \\
&= \int \frac{e^{-x}dx}{e^{-x}+1} \\
&= -\ln|1+e^{-x}| + C
\end{aligned}$$

**Example 5.1.6.** *Evaluate*

$$\int \frac{xdx}{1+x^2}$$

***Solution:***

$$\begin{aligned}
\int \frac{xdx}{1+x^2} &= \frac{1}{2}\int \frac{2xdx}{1+x^2} \\
&= \frac{1}{2}\ln|1+x^2| + C
\end{aligned}$$

## 5.2 Basic Trigonometric Integrals

**Basic Trigonometric Integrals**

- $\displaystyle\int \sin kx dx = \frac{-1}{k}\cos x + C$
- $\displaystyle\int \cos kx dx = \frac{1}{k}\sin x + C$
- $\displaystyle\int \sec^2 kx dx = \frac{1}{k}\tan x + C$
- $\displaystyle\int \csc^2 kx dx = \frac{-1}{k}\cot x + C$
- $\displaystyle\int \sec kx \tan kx dx = \frac{1}{k}\sec x + C$
- $\displaystyle\int \csc kx \cot kx dx = \frac{-1}{k}\csc x + C$

**Example 5.2.1.** *Evaluate*

$$\int \tan x dx$$

***Solution:***

$$\begin{aligned}\int \tan x dx &= \int \frac{\sin x}{\cos x} dx \\ &= -\int \frac{-\sin x}{\cos x} dx \\ &= -\ln \cos x + C \\ &= \ln|\sec x| + C\end{aligned}$$

**Example 5.2.2.** *Evaluate*

$$\int \cot x dx$$

***Solution:***

$$\begin{aligned}\int \cot x dx &= \int \frac{\cos x}{\sin x} dx \\ &= \ln \sin x + C\end{aligned}$$

**Example 5.2.3.** *Evaluate*

$$\int \sec x dx$$

***Solution:***

$$\begin{aligned}
\int \sec x dx &= \int \sec x \frac{\sec x + \tan x}{\sec x + \tan x} dx \\
&= \int \frac{\sec^2 x + \sec x \tan x}{\sec x + \tan x} dx \\
&= \ln|\sec x + \tan x| + C
\end{aligned}$$

**Example 5.2.4.** *Evaluate*

$$\int \csc x dx$$

***Solution:***

$$\begin{aligned}
\int \csc x dx &= \int \csc x \frac{\csc x + \cot x}{\csc x + \cot x} dx \\
&= \int \frac{\csc^2 x + \csc x \cot x}{\csc x + \cot x} dx \\
&= -\ln|\csc x + \cot x| + C
\end{aligned}$$

**Example 5.2.5.** *Evaluate*

$$\int (\sin 8t + 20 \cos 9t) dt$$

***Solution:***

$$\begin{aligned}
\int (\sin 8t + 20 \cos 9t) dt &= \int \sin 8t dt + 20 \int \cos 9t dt \\
&= -\frac{1}{8} \cos 8t + \frac{20}{9} \sin 9t + C
\end{aligned}$$

## 5.3 Substitution Method

Integration (antidifferentiation) is generally more difficult than differentiation. There are no sure-fire methods, and many antiderivatives cannot be expressed in terms of elementary functions. However, there are a few important general techniques. One such technique is the Substitution Method, which uses the Chain Rule "in reverse."

Consider the integral $\int 2x \cos\left(x^2\right) dx$. We can evaluate it if we remember the Chain Rule calculation

$$\frac{d}{dx} \sin\left(x^2\right) = 2x \cos\left(x^2\right)$$

This tells us that $\sin\left(x^2\right)$ is an antiderivative of $2x\cos\left(x^2\right)$, and therefore,

$$\int \underbrace{2x}_{\substack{\text{Derivative of}\\ \text{inside function}}} \cos(\underbrace{x^2}_{\substack{\text{Inside}\\ \text{function}}})dx = \sin\left(x^2\right) + C$$

A similar Chain Rule calculation shows that

$$\int \underbrace{\left(1+3x^2\right)}_{\substack{\text{Derivative of}\\ \text{inside function}}} \cos(\underbrace{x+x^3}_{\substack{\text{Inside}\\ \text{function}}})dx = \sin\left(x+x^3\right) + C$$

In both cases, the integrand is the product of a composite function and the derivative of the inside function. The Chain Rule does not help if the derivative of the inside function is missing. For instance, we cannot use the Chain Rule to compute $\int \cos\left(x+x^3\right) dx$ because the factor $\left(1+3x^2\right)$ does not appear.

In general, if $F'(u) = f(u)$ then by the Chain Rule,

$$\frac{d}{dx}F(u(x)) = F'(u(x))u'(x) = f(u(x))u'(x)$$

This translates into the following integration formula.

**The Substitution Method** If $F'(x) = f(x)$, then

$$\int f(u(x))u'(x)dx = F(u(x)) + C$$

**Example 5.3.1.** *Evaluate*

$$\int 3x^2 \sin\left(x^3\right) dx$$

***Solution:*** *Let $u = x^3$, then $du = 3x^2dx$, hence*

$$\begin{aligned}\int 3x^2 \sin\left(x^3\right) dx &= \int \sin(u)\, du\\ &= -\cos u + C\\ &= -\cos\left(x^3\right) + C\end{aligned}$$

**Example 5.3.2.** *Evaluate*

$$\int x\left(x^2+9\right)^5 dx$$

***Solution:*** *Let $u = x^2 + 9$, then $du = 2xdx$, hence*

$$\begin{aligned}\int x\left(x^2+9\right)^5 dx &= \frac{1}{2}\int u^5 du \\ &= \frac{1}{12}u^6 + C \\ &= \frac{1}{12}\left(x^2+9\right)^6 + C\end{aligned}$$

**Example 5.3.3.** *Evaluate*

$$\int \frac{\left(x^2+2x\right)dx}{\left(x^3+3x^2+12\right)^6}$$

***Solution:*** *Let $u = x^3 + 3x^2 + 12$, then $du = (3x^2 + 6x)dx$, hence*

$$\begin{aligned}\int \frac{\left(x^2+2x\right)dx}{\left(x^3+3x^2+12\right)^6} &= \int \left(x^3+3x^2+12\right)^{-6}\left(x^2+2x\right)dx \\ &= \frac{1}{3}\int u^{-6} du \\ &= \left(\frac{1}{3}\right)\left(\frac{u^{-5}}{-5}\right) + C \\ &= -\frac{1}{15}\left(x^3+3x^2+12\right)^{-5} + C\end{aligned}$$

**Example 5.3.4.** *Evaluate*

$$\int x\sqrt{5x+1}dx$$

***Solution:*** *Let $u = 5x + 1\theta + 5$, then $du = 5dx$, hence*

$$\begin{aligned}\int x\sqrt{5x+1}dx &= \int \left(\frac{1}{5}(u-1)\right)\frac{1}{5}\sqrt{u}du \\ &= \frac{1}{25}\int \left(u^{3/2} - u^{1/2}\right) du \\ &= \frac{1}{25}\left(\frac{2}{5}u^{5/2} - \frac{2}{3}u^{3/2}\right) + C \\ &= \frac{2}{125}(5x+1)^{5/2} - \frac{2}{75}(5x+1)^{3/2} + C\end{aligned}$$

**Example 5.3.5.** *Evaluate*

$$\int \frac{1}{x}\sin^6(\ln x)\cos(\ln x)dx$$

***Solution:*** *Let* $u = \ln x$ *, then* $du = \frac{1}{x}dx$*, hence*

$$\int \frac{1}{x}\sin^6(\ln x)\cos(\ln x)dx = \int \sin^6 u \cos u du$$
$$= \frac{1}{7}\sin^7(u) + c \qquad = \frac{1}{7}\sin^7(\ln x) + c$$

**Example 5.3.6.** *Evaluate*

$$\int \frac{[1 + (1/t)]^5}{t^2}dt$$

***Solution:*** *Let* $u = \frac{1}{t}$ *, then* $du = \frac{-1}{t^2}dt$*, hence*

$$\begin{aligned}\int \frac{[1 + (1/t)]^5}{t^2}dt &= -\int u^5 du \\ &= -\frac{u^6}{6} + c \\ &= -\frac{[1 + (1/t)]^6}{6} + c\end{aligned}$$

**Example 5.3.7.** *Evaluate*

$$\int \frac{3x + 6}{\sqrt{2x^2 + 8x + 3}}dx$$

***Solution:*** *Let* $u = 2x^2 + 8x + 3$ *, then* $du = (4x + 8)dx$*, hence*

$$\begin{aligned}\int \frac{3x + 6}{\sqrt{2x^2 + 8x + 3}}dx &= \int \frac{1}{\sqrt{2x^2 + 8x + 3}}[(3x + 6)dx] \\ &= \int \frac{1}{\sqrt{u}\left(\frac{3}{4}du\right)} = \frac{3}{4}\int u^{-1/2}du \\ &= \frac{3}{4}\left(\frac{u^{1/2}}{1/2}\right) + C = \frac{3}{2}\sqrt{u} + c \\ &= \frac{3}{2}\sqrt{2x^2 + 8x + 3} + C\end{aligned}$$

**Example 5.3.8.** *Evaluate*

$$\int \frac{(\ln x)^2}{x}dx$$

***Solution:*** *Let $u = \ln x$ , then $du = \frac{1}{x}dx$, hence*

$$\begin{aligned}
\int \frac{(\ln x)^2}{x}dx &= \int (\ln x)^2 \left(\frac{1}{x}dx\right) \\
&= \int u^2 du = \frac{1}{3}u^3 + C \\
&= \frac{1}{3}(\ln x)^3 + C
\end{aligned}$$

**Example 5.3.9.** *Evaluate*

$$\int \frac{x}{x-1}dx$$

***Solution:*** *Let $u = x - 1$ , then $du = dx$, hence*

$$\begin{aligned}
\int \frac{x}{x-1}dx &= \int \frac{u+1}{u}du \\
&= \int \left[1 + \frac{1}{u}\right] du \\
&= u + \ln|u| + C \\
&= x - 1 + \ln|x-1| + C
\end{aligned}$$

**Example 5.3.10.** *Evaluate*

$$\int x^3 e^{x^4+2}dx$$

***Solution:*** *Let $u = x^4 + 2$ , then $du = 4x^3dx$, hence*

$$\begin{aligned}
\int x^3 e^{x^4+2}dx &= \int e^{x^4+2} \left(x^3 dx\right) \\
&= \int e^{n\left(\frac{1}{4}du\right)} \\
&= \frac{1}{4}e^u + C \\
&= \frac{1}{4}e^{x^4+2} + C
\end{aligned}$$

**Example 5.3.11.** *Evaluate*

$$\int e^{5x+2}dx$$

***Solution:*** *Let $u = 5x + 2$ , then $du = 5dx$, hence*

$$\begin{aligned}\int e^{5x+2}dx &= \int e^{5x}e^{2}dx\\ &= e^{2}\int e^{5x}dx\\ &= e^{2}\left[\frac{e^{5x}}{5}\right]+c\\ &= \frac{1}{5}e^{5x+2}+c\end{aligned}$$

## 5.4 Integrals Resulting in Inverse Trigonometric Functions

Let us begin this last section of the chapter with the three formulas. Along with these formulas, we use substitution to evaluate the integrals. We prove the formula for the inverse sine integral.

**The following integration formulas yield inverse trigonometric functions:**

1. $\displaystyle\int \frac{du}{\sqrt{a^2-u^2}} = \sin^{-1}\frac{u}{a} + C$
2. $\displaystyle\int \frac{du}{a^2+u^2} = \frac{1}{a}\tan^{-1}\frac{u}{a} + C$
3. $\displaystyle\int \frac{du}{u\sqrt{u^2-a^2}} = \frac{1}{a}\sec^{-1}\frac{u}{a} + C$

**Example 5.4.1.** *Evaluate*

$$\int \frac{dx}{\sqrt{4-9x^2}}$$

***Solution:*** *Let $u = 3x$ and $du = 3dx$ , then*

$$\begin{aligned}\int \frac{dx}{\sqrt{4-9x^2}} &= \frac{1}{3}\int \frac{du}{\sqrt{4-u^2}}\\ &= \frac{1}{3}\sin^{-1}\left(\frac{u}{2}\right)+C\\ &= \frac{1}{3}\sin^{-1}\left(\frac{3x}{2}\right)+C\end{aligned}$$

**Example 5.4.2.** *Evaluate*

$$\int \frac{1}{1+4x^2}dx$$

***Solution:*** *Let $u = 2x$ and $du = 2dx$ , then*

$$\begin{aligned}
&= \frac{1}{2}\int \frac{1}{1+u^2}du \\
&= \frac{1}{2}\tan^{-1} u + C \\
&= \frac{1}{2}\tan^{-1}(2x) + C
\end{aligned}$$

**Example 5.4.3.** *Evaluate*

$$\int \frac{1}{1+4x^2}dx$$

***Solution:*** *Let $u = 2x$ and $du = 2dx$ , then*

$$\begin{aligned}
\int \frac{dx}{x\sqrt{4x^2-1}} &= \int \frac{\frac{1}{2}du}{\frac{1}{2}u\sqrt{u^2-1}} \\
&= \int \frac{du}{u\sqrt{u^2-1}} \\
&= \sec^{-1}(2x) + C
\end{aligned}$$

**Example 5.4.4.** *Evaluate*

$$\int \frac{x+\sin^{-1} x}{\sqrt{1-x^2}}dx$$

***Solution:***

$$\begin{aligned}
\int \frac{x+\sin^{-1} x}{\sqrt{1-x^2}}dx &= \int \frac{\sin^{-1} x}{\sqrt{1-x^2}}dx - \int \frac{-xdx}{\sqrt{1-x^2}} \\
&= \int \sin^{-1} xd\sin^{-1} x - \frac{1}{2}\int \frac{-2xdx}{\sqrt{1-x^2}} \\
&= \frac{1}{2}\left(\sin^{-1} x\right)^2 - \sqrt{1-x^2} + c
\end{aligned}$$

**Example 5.4.5.** *Evaluate*

$$\int \frac{1}{x\left(1+\ln^2 x\right)}dx$$

***Solution:*** *Let* $u = \ln x$ *, then* $du = \frac{1}{x}dx$*, hence*

$$\begin{aligned}\int \frac{1}{x\left(1+\ln^2 x\right)}dx &= \int \frac{d(\ln x)}{\left(1+\ln^2 x\right)}\\ &= \int \frac{du}{(1+u^2)}\\ &= \tan^{-1} u + c = \tan^{-1}(\ln x) + c\end{aligned}$$

**Example 5.4.6.** *Evaluate*

$$\int \frac{x^2 dx}{\sqrt{1-x^6}}$$

***Solution:*** *Let* $u = x^3$ *, then* $du = 3x^2 dx$*, hence*

$$\begin{aligned}\int \frac{x^2 dx}{\sqrt{1-x^6}} &= \frac{1}{3}\int \frac{du}{\sqrt{1-u^2}}\\ &= \frac{1}{3}\sin^{-1} u + c = \frac{1}{3}\sin^{-1} x^3 + c\end{aligned}$$

**Example 5.4.7.** *Evaluate*

$$\int \frac{2x^2+3}{x\sqrt{9x^2-4}}dx$$

***Solution:***

$$\begin{aligned}\int \frac{2x^2+3}{x\sqrt{9x^2-4}}dx &= \int \left[\frac{2x^2}{x\sqrt{9x^2-4}} + \frac{3}{x\sqrt{9x^2-4}}\right] dx\\ &= \int \frac{2x}{\sqrt{9x^2-4}}dx + \int \frac{3}{x\sqrt{9x^2-4}}dx\end{aligned}$$

To evaluate $\int \frac{2x}{\sqrt{9x^2-4}}dx$, Let $u = 9x^2 - 4$ and $du = 18x dx$, then

$$\begin{aligned}\int \frac{2x}{\sqrt{9x^2-4}}dx &= \frac{1}{9}\int \frac{du}{\sqrt{u}}\\ &= \frac{1}{9}\cdot 2\sqrt{u} + c\\ &= \frac{2\sqrt{9x^2-4}}{9} + c\end{aligned}$$

To evaluate $\int \frac{3}{x\sqrt{9x^2-4}}dx$, Let $u = 3x$ and $du = 3dx$, then

$$\begin{aligned}\int \frac{3dx}{x\sqrt{9x^2-4}} &= 3\int \frac{du}{u\sqrt{u^2-4}}\\ &= 3\sec^{-1}\left|\frac{u}{2}\right| + c\\ &= 3\sec^{-1}\left|\frac{3x}{2}\right| + c\end{aligned}$$

Hence

$$\int \frac{2x^2+3}{x\sqrt{9x^2-4}}dx = \frac{2\sqrt{9x^2-4}}{9} + c + 3\sec^{-1}\left|\frac{3x}{2}\right| + c$$

## 5.5 Integrals Involving Hyperbolic and Inverse Hyperbolic Functions

**Hyperbolic Integral Formulas**

1. $\displaystyle\int \sinh kx dx = \frac{1}{k}\cosh kx + C$
2. $\displaystyle\int \cosh kx dx = \frac{1}{k}\sinh kx + C$
3. $\displaystyle\int \operatorname{sech}^2 kx dx = \frac{1}{k}\tanh kx + C$
4. $\displaystyle\int \operatorname{csch}^2 kx dx = \frac{-1}{k}\coth kx + C$
5. $\displaystyle\int \operatorname{sech} kx \tanh x dx = \frac{-1}{k}\operatorname{sech} kx + C$
6. $\displaystyle\int \operatorname{csch} kx \coth x dx = \frac{-1}{k}\operatorname{csch} kx + C$

**Example 5.5.1.** *Evaluate*

$$\int x\cosh\left(x^2\right)dx$$

***Solution***: *Let* $u = x^2$ *and* $du = 2xdx$, *then*

$$\begin{aligned}\int x\cosh\left(x^2\right)dx &= \frac{1}{2}\int \cosh u du\\ &= \frac{1}{2}\sinh u + C\\ &= \frac{1}{2}\sinh\left(x^2\right) + C\end{aligned}$$

**Example 5.5.2.** *Evaluate*

$$\int \cosh(3x)dx$$

***Solution***:

$$\begin{aligned}\int \cosh(3x)dx &= \frac{1}{3}\int 3\cdot\cosh(3x)dx\\ &= \frac{1}{3}\sinh(3x) + C\end{aligned}$$

**Example 5.5.3.** *Evaluate*

$$\int x\sinh\left(x^2+1\right)dx$$

***Solution***: *Let* $u = x^2 + 1$ *and* $du = 2xdx$, *then*

$$\begin{aligned}\int x\sinh\left(x^2+1\right)dx &= \frac{1}{2}\int \sinh u du\\ &= \frac{1}{2}\cosh u + C\\ &= \frac{1}{2}\cosh\left(x^2+1\right) + C\end{aligned}$$

**Example 5.5.4.** *Evaluate*

$$\int \tanh x \operatorname{sech}^2 x dx$$

***Solution***: *Let* $u = \tanh x$, $du = \operatorname{sech}^2 x dx$, *then*

$$\begin{aligned}\int \tanh x\operatorname{sech}^2 x dx &= \int u du\\ &= \frac{u^2}{2} + C\\ &= \frac{1}{2}\tanh^2 x + C\end{aligned}$$

**Example 5.5.5.** *Evaluate*

$$\int \frac{\cosh x}{3\sinh x + 4}dx$$

***Solution**: Let $u = 3\sinh x + 4$, $du = 3\cosh x dx$, then*

$$\begin{aligned}\int \frac{\cosh x}{3\sinh x + 4} dx &= \frac{1}{3}\int \frac{1}{u} du \\ &= \frac{1}{3}\ln|u| + C \\ &= \frac{1}{3}\ln|3\sinh x + 4| + C\end{aligned}$$

**Integrals Involving Inverse Hyperbolic Functions**

1. $\displaystyle\int \frac{dx}{\sqrt{x^2+1}} = \sinh^{-1} x + C$
2. $\displaystyle\int \frac{dx}{\sqrt{x^2-1}} = \cosh^{-1} x + C \quad (\text{ for } x > 1)$
3. $\displaystyle\int \frac{dx}{1-x^2} = \tanh^{-1} x + C \quad (\text{ for } |x| < 1)$
4. $\displaystyle\int \frac{dx}{1-x^2} = \coth^{-1} x + C \quad (\text{ for } |x| > 1)$
5. $\displaystyle\int \frac{dx}{x\sqrt{1-x^2}} = -\operatorname{sech}^{-1} x + C \quad (\text{ for } 0 < x < 1)$
6. $\displaystyle\int \frac{dx}{|x|\sqrt{1+x^2}} = -\operatorname{csch}^{-1} x + C \quad (\text{ for } x \neq 0)$

**Example 5.5.6.** *Evaluate*

$$\int \frac{1}{\sqrt{16+25x^2}} dx$$

***Solution**:*

$$\begin{aligned}\int \frac{1}{\sqrt{16+25x^2}} dx &= \int \frac{1}{\sqrt{16\left(1+\frac{25}{16}x^2\right)}} dx \\ &= \int \frac{1}{4\sqrt{1+\left(\frac{5x}{4}\right)^2}} dx \\ &= \frac{4}{5}\int \frac{\frac{5}{4}}{\sqrt{1+\left(\frac{5x}{4}\right)^2}} dx \\ &= \frac{1}{5}\sinh^{-1}\left(\frac{5x}{4}\right) + C\end{aligned}$$

**Example 5.5.7.** *Evaluate*

$$\int \frac{\tanh^{-1} x}{x^2 - 1} dx$$

***Solution***: *Let* $u = \tanh^{-1} x$ *and* $du = \frac{1}{x^2-1} dx$

$$\begin{aligned}
\int \frac{\tanh^{-1} x}{x^2 - 1} dx &= \int -u du \\
&= \frac{-u^2}{2} + C \\
&= \frac{-\left(\tanh^{-1} x\right)^2}{2} + C
\end{aligned}$$

**Example 5.5.8.** *Evaluate*

$$\int \frac{1}{x\sqrt{x^2 + 16}} dx$$

***Solution***: *Let* $u =$ *and* $du =$

$$\begin{aligned}
\int \frac{1}{x\sqrt{x^2 + 16}} dx &= \int \frac{1}{x\sqrt{16\left(\frac{x^2}{16} + 1\right)}} dx \\
&= \int \frac{1}{4x\sqrt{\left(\left(\frac{x}{4}\right)^2 + 1\right)}} dx \\
&= \int \frac{\frac{1}{4}}{4 \cdot \frac{x}{4}\sqrt{\left(\left(\frac{x}{4}\right)^2 + 1\right)}} dx \\
&= \frac{1}{4}\left[-\operatorname{csch}^{-1}\left(\frac{x}{4}\right)\right]
\end{aligned}$$

**Some Useful Relations**

1. $\sinh^{-1} x = \ln(x + \sqrt{x^2 + 1})$
2. $\cosh^{-1} x = \ln(x + \sqrt{x^2 - 1})$
3. $\tanh^{-1} x = \dfrac{1}{2} \ln \frac{1+x}{1-x}$
4. $\operatorname{sech}^{-1} x = \ln \left( \dfrac{1 + \sqrt{1 - x^2}}{x} \right)$
5. $\operatorname{csch}^{-1} x = \ln \left( \dfrac{1}{x} + \dfrac{\sqrt{1 + x^2}}{|x|} \right)$
6. $\coth^{-1} x = \frac{1}{2} \ln \dfrac{x+1}{x-1}$

## 5.6 Exercises

**Evaluate the following integral**

1- $\int e^{2x} dx$ 2- $\int 2^x dx$ 3- $\frac{2}{x} dx$

4- $\int \frac{\ln x}{x} dx$ 5- $\int \frac{dx}{x(\ln x)^2}$ 6- $\int \frac{dx}{x \ln x}$

7- $\int \frac{\ln(\sin x)}{\tan x} dx$ 8- $\int \frac{\cos x - x \sin x}{x \cos x} dx$ 9- $\int \ln(\cos x) \tan x dx$

10- $\int e^{\ln x} \frac{dx}{x}$ 11- $\int \frac{e^{\ln(1-t)}}{1-t} dt$ 12- $\int e^{\tan x} \sec^2 x dx$

13- $\int \frac{x \sin\left(x^2\right)}{\cos\left(x^2\right)} dx$ 14- $\int \sec^2(3x+2) dx$ 15- $\int \frac{4x^3}{\left(x^4+1\right)^2} dx$

16- $\int \frac{(1+\sqrt{x})^{1/3}}{\sqrt{x}} dx$ 17- $\int \frac{1}{x^2} \cos^2\left(\frac{1}{x}\right) dx$ 18- $\int \frac{1}{\sqrt{5s+4}} ds$

19- $\int \tan^7 \frac{x}{2} \sec^2 \frac{x}{2} dx$ 20- $\int \frac{\cos\sqrt{\theta}}{\sqrt{\theta} \sin^2 \sqrt{\theta}} d\theta$ 21- $\int \frac{t \tan^{-1}\left(t^2\right)}{1+t^4} dt$

22- $\int \frac{e^t \cos^{-1}\left(e^t\right)}{\sqrt{1-e^{2t}}} dt$ 23- $\int \frac{2e^{-2x}}{\sqrt{1-e^{-4x}}} dx$ 24- $\int \frac{(\sin x + x \cos x)}{1 + x^2 \sin^2 x} dx$

25- $\int \frac{1}{x + x \ln^2 x} dx$ 26- $\int \frac{dt}{t\sqrt{1-\ln^2 t}}$ 27- $\int \frac{e^t}{1+e^{2t}} dt$

# Chapter 6

# TECHNIQUES OF INTEGRATION

In this chapter, we study some additional techniques, including some ways of approximating definite integrals when normal techniques do not work.

## 6.1 Integration by Parts

By now we have a fairly thorough procedure for how to evaluate many basic integrals. However, although we can integrate $\int x \sin\left(x^2\right) dx$ by using the substitution, $u = x^2$, something as simple looking as $\int x \sin x dx$ defies us. Many students want to know whether there is a product rule for integration. There isn't, but there is a technique based on the product rule for differentiation that allows us to exchange one integral for another. We call this technique integration by parts.

If, $h(x) = f(x)g(x)$, then by using the product rule, we obtain $h'(x) = f'(x)g(x) + g'(x)f(x)$. Although at first it may seem counterproductive, let's now integrate both sides of this equation:

$$\int h'(x)dx = \int \left(g(x)f'(x) + f(x)g'(x)\right) dx$$

This gives us

$$h(x) = f(x)g(x) = \int g(x)f'(x)dx + \int f(x)g'(x)dx$$

Now we solve for $\int f(x)g'(x)dx$ :

$$\int f(x)g'(x)dx = f(x)g(x) - \int g(x)f'(x)dx$$

By making the substitutions $u = f(x)$ and $v = g(x)$, which in turn make $du = f'(x)dx$ and $dv = g'(x)dx$, we have the more compact form

$$\int udv = uv - \int vdu$$

**Theorem 6.1.** *Let $u = f(x)$ and $v = g(x)$ be functions with continuous derivatives. Then, the integration-by-parts formula for the integral involving these two functions is:*

$$\int udv = uv - \int vdu$$

The advantage of using the integration-by-parts formula is that we can use it to exchange one integral for another, possibly easier, integral. The following example illustrates its use.

**Example 6.1.1.** *Evaluate*

$$\int x \sin x dx$$

*Let*

$$\begin{aligned} u &= x & dv &= \sin x dx \\ du &= dx & v &= -\cos x \end{aligned}$$

*Then,*

$$\begin{aligned} \int x \sin x dx &= (x)(-\cos x) - \int (-\cos x)(1dx) \\ &= -x \cos x + \int \cos x dx \\ &= -x \cos x + \sin x + C \end{aligned}$$

**Example 6.1.2.** *Evaluate*

$$\int xe^x dx$$

*Let*

$$\begin{aligned} u &= x & dv &= e^x dx \\ du &= dx & v &= e^x \end{aligned}$$

*Then,*

$$\begin{aligned}\int xe^x dx &= xe^x - \int e^x dx \\ &= xe^x - e^x + C\end{aligned}$$

**Example 6.1.3.** *Evaluate*

$$\int x^2 e^x dx$$

*Let*

$$\begin{aligned} u &= x^2 & dv &= e^x dx \\ du &= 2x dx & v &= e^x \end{aligned}$$

*Then,*

$$\begin{aligned}\int x^2 e^x dx &= x^2 \cdot e^x - \int 2x \cdot e^x dx \\ &= x^2 e^x - 2 \int xe^x dx\end{aligned}$$

*Let*

$$\begin{aligned} u &= x & dv &= e^x dx \\ du &= dx & v &= e^x \end{aligned}$$

*Then,*

$$\begin{aligned}\int x^2 e^x dx &= x^2 e^x - 2\left[xe^x - \int 1 \cdot e^x dx\right] \\ &= x^2 e^x - 2\left[xe^x - e^x\right] + C \\ &= x^2 e^x - 2xe^x - 2e^x + C \\ &= e^x \left(x^2 - 2x - 2\right) + C\end{aligned}$$

**Example 6.1.4.** *Evaluate*

$$\int \ln x dx$$

*Let*

$$u = \ln x \qquad dv = dx$$
$$du = \frac{1}{x}dx \qquad v = x$$

*Then,*

$$\int \ln x dx = x \ln x - \int x \frac{1}{x} dx$$
$$= x \ln x - x + C$$

**Example 6.1.5.** *Evaluate*

$$\int x \ln x dx$$

*Let*

$$u = \ln x \qquad dv = x dx$$
$$du = \frac{1}{x}dx \qquad v = \frac{1}{2}x^2$$

*Then,*

$$\int x \ln x dx = \frac{1}{2}x^2 \ln x - \frac{1}{2}\int \frac{x^2}{x} dx$$
$$= \frac{1}{2}x^2 \ln x - \frac{1}{4}x^2 + C$$

**Example 6.1.6.** *Evaluate*

$$\int \frac{\ln x}{x^3} dx$$

*Let*

$$u = \ln x \qquad dv = x^{-3}dx$$
$$du = \frac{1}{x}dx \qquad v = \frac{-1}{2}x^{-2}$$

*Then,*

$$\begin{aligned}
\int \frac{\ln x}{x^3}dx &= \int x^{-3}\ln x dx \\
&= (\ln x)\left(-\frac{1}{2}x^{-2}\right) - \int \left(-\frac{1}{2}x^{-2}\right)\left(\frac{1}{x}dx\right) \\
&= -\frac{1}{2}x^{-2}\ln x + \int \frac{1}{2}x^{-3}dx \\
&= -\frac{1}{2}x^{-2}\ln x - \frac{1}{4}x^{-2} + C \\
&= -\frac{1}{2x^2}\ln x - \frac{1}{4x^2} + C
\end{aligned}$$

**Example 6.1.7.** *Evaluate*

$$\int \sin 2xe^x dx$$

*Let*

$$\begin{aligned}
u &= \sin 2x & dv &= e^x dx \\
du &= \cos 2x dx & v &= e^x
\end{aligned}$$

*Then,*

$$\begin{aligned}
\int \sin 2xe^x dx &= (\sin 2x)\left(e^x\right) - \int e^x 2\cos 2x dx \\
&= e^x \sin 2x - 2\int \cos 2xe^x dx
\end{aligned}$$

*For the integral part again , let*

$$\begin{aligned}
u &= \cos 2x & dv &= e^x dx \\
du &= -\sin 2x dx & v &= e^x
\end{aligned}$$

*Then,*

$$\begin{aligned}
\int \sin 2xe^x dx &= e^x \sin 2x - 2\left(e^x \cos 2x + 2\int e^x \sin 2x dx\right) \\
\int \sin 2xe^x dx &= e^x \sin 2x - 2e^x \cos 2x - 4\int e^x \sin 2x dx \\
5\int \sin 2xe^x dx &= e^x \sin 2x - 2e^x \cos 2x \\
\int \sin 2xe^x dx &= \frac{1}{5}e^x(\sin 2x - 2\cos 2x) + C
\end{aligned}$$

**Example 6.1.8.** *Evaluate*

$$\int \sin(\ln x)dx$$

*Let*

$$u = \sin(\ln x) \qquad dv = dx$$
$$du = \frac{1}{x}\cos(\ln x)dx \qquad v = x$$

*Then,*

$$\int \sin(\ln x)dx = x\sin(\ln x) - \int \cos(\ln x)dx$$

*For the integral part again , let*

$$u = \cos(\ln x) \qquad dv = dx$$
$$du = \frac{-1}{x}\sin(\ln x)dx \qquad v = x$$

*Then,*

$$\begin{aligned}
\int \sin(\ln x)dx &= x\sin(\ln x) - \left(x\cos(\ln x) - \int -\sin(\ln x)dx\right) \\
&= x\sin(\ln x) - x\cos(\ln x) - \int \sin(\ln x)dx \\
2\int \sin(\ln x)dx &= x\sin(\ln x) - x\cos(\ln x) \\
\int \sin(\ln x)dx &= \frac{1}{2}\left[x\sin(\ln x) - x\cos(\ln x)\right]
\end{aligned}$$

**Example 6.1.9.** *Evaluate*

$$\int \sin^{-1} x dx$$

*Let*

$$u = \sin^{-1} x \qquad dv = dx$$
$$du = \frac{1}{\sqrt{1-x^2}}dx \qquad v = x$$

*Then,*

$$\begin{aligned}
\int \sin^{-1} x dx &= \sin^{-1} x \cdot x - \int \frac{1}{\sqrt{1-x^2}} \cdot x dx \\
&= x \sin^{-1} x - \int -\frac{-2x}{2\sqrt{1-x^2}} dx \\
&= x \sin^{-1} x + \sqrt{1-x^2} + C
\end{aligned}$$

**Example 6.1.10.** *Evaluate*

$$\int \tan^{-1} x dx$$

*Let*

$$\begin{aligned}
u &= \tan^{-1} x & dv &= dx \\
du &= \frac{1}{1+x^2} dx & v &= x
\end{aligned}$$

*Then,*

$$\begin{aligned}
\int \tan^{-1} x dx &= x \tan^{-1} x - \int \frac{x}{1+x^2} dx \\
&= x \tan^{-1} x - \frac{1}{2} \int \frac{2x}{1+x^2} dx \\
&= x \tan^{-1} x - \frac{1}{2} \ln\left(1+x^2\right) + C
\end{aligned}$$

**Example 6.1.11.** *Evaluate*

$$\int x \tan^{-1} x dx$$

*Let*

$$\begin{aligned}
u &= \tan^{-1} x & dv &= x dx \\
du &= \frac{1}{1+x^2} dx & v &= \frac{1}{2} x^2
\end{aligned}$$

*Then,*

$$\begin{aligned}\int x\tan^{-1} x dx &= \frac{1}{2}x^2\tan^{-1} x - \frac{1}{2}\int \frac{x^2}{1+x^2}dx \\ &= \frac{x^2\tan^{-1} x}{2} - \frac{1}{2}\int \left(1 - \frac{1}{1+x^2}\right) dx \\ &= \frac{x^2\tan^{-1} x}{2} - \frac{1}{2}x + \frac{1}{2}\tan^{-1} x + C\end{aligned}$$

**Example 6.1.12.** *Evaluate*

$$\int \sec^3 x dx$$

*Since*

$$\int \sec^3 x dx \int \sec x \sec^2 x dx$$

*Let*

$$u = \sec x \qquad dv = \sec^2 x dx$$

$$du = \sec x \tan x dx \qquad v = \tan x$$

*Then,*

$$\begin{aligned}\int \sec^3 x dx &= \sec x \tan x - \int \sec x \tan^2 x dx \\ &= \sec x \tan x - \int \sec x(\sec^2 x - 1)dx \\ &= \sec x \tan x - \int \sec^3 x dx + \int \sec x dx \\ 2\int \sec^3 x dx &= \sec x \tan x + \int \sec x dx \\ &= \sec x \tan x + \ln|\sec x + \tan x| \\ \int \sec^3 x dx &= \frac{1}{2}[\sec x \tan x + \ln|\sec x + \tan x|]\end{aligned}$$

**Example 6.1.13.** *Prove that*

$$\int x^n e^x dx = x^n e^x - n\int x^{n-1}e^x dx$$

*Let*

$$u = x^n \qquad dv = e^x dx$$
$$du = nx^{n-1} dx \qquad v = e^x$$

*Then,*

$$\int x^n e^x dx = x^n e^x - n \int x^{n-1} e^x dx$$

**Example 6.1.14.** *Prove that*

$$\int (\ln x)^k dx = x(\ln x)^k - k \int (\ln x)^{k-1} dx$$

*and use it to evaluate*

$$\int (\ln x)^3 dx$$

*Let*

$$u = (\ln x)^k \qquad dv = dx$$
$$du = \frac{k}{x} (\ln x)^{k-1} dx \qquad v = x$$

*Then,*

$$\int (\ln x)^k dx = x(\ln x)^k - k \int (\ln x)^{k-1} dx$$

*To evaluate*

$$\int (\ln x)^3 dx$$

*Here $k = 3$, then*

$$\begin{aligned}
\int (\ln x)^3 dx &= x(\ln x)^3 - 3\int (\ln x)^2 dx \\
&= x(\ln x)^3 - 3\left(x(\ln x)^2 - 2\int (\ln x) dx\right) \\
&= x(\ln x)^3 - 3x(\ln x)^2 + 6\int (\ln x) dx \\
&= x(\ln x)^3 - 3x(\ln x)^2 + 6\left(x\ln x - \int (\ln x)^0 dx\right) \\
&= x(\ln x)^3 - 3x(\ln x)^2 + 6x\ln x - 6x + C
\end{aligned}$$

### 6.1.1 Exercises

**Evaluate the following integral**

| | | | | | |
|---|---|---|---|---|---|
| 1- | $\int x^3 e^{2x} dx$ | 2- | $\int x^3 \ln(x) dx$ | 3- | $\int t \ln 2t dt$ |
| 4- | $\int t \ln t^2 dt$ | 5- | $\int \frac{\ln x}{x^2} dx$ | 6- | $\int x^3 e^{x^2} dx$ |
| 7- | $\int x^2 \sin x dx$ | 8- | $\int e^{-x} \sin x dx$ | 9- | $\int \sec^{-1} x dx$ |
| 10- | $\int x 5^x dx$ | 11- | $\int 3^x \cos x dx$ | 12- | $\int \sinh^{-1} x dx$ |
| 13- | $\int \frac{\ln(\ln x) \ln x dx}{x}$ | 14- | $\int \sin(\ln x) dx$ | 15- | $\int \cos x \ln(\sin x) dx$ |
| 16- | $\int \frac{\ln(\ln x) dx}{x}$ | 17- | $\int \frac{(\ln x)^2 dx}{x^2}$ | 18- | $\int \frac{x \tan^{-1} x dx}{\sqrt{1+x^2}}$ |
| 19- | $\int x \sec^2 x dx$ | 20- | $\int x \sin x \cos x dx$ | 21- | $\int x \sin^{-1} x dx$ |
| 22- | $\int \ln(x^2+1) dx$ | 23- | $\int \ln(x+\sqrt{x^2+1}) dx$ | 24- | $\int \left(\sin^{-1} x\right)^2 dx$ |
| 25- | $\int \cos x \cosh x dx$ | 26- | $\int e^{3x} \cos 4x dx$ | 27- | $\int x^2 \tan^{-1} x dx$ |

## 6.2 Trigonometric Integrals

In this section we look at how to integrate a variety of products of trigonometric functions. These integrals are called . They are an important part of the integration technique called trigonometric substitution, which is featured in Trigonometric Substitution. This technique allows us to convert algebraic expressions that we may not be able to integrate into expressions involving trigonometric functions, which we may be able to integrate using the techniques described in this section. In addition, these types of integrals appear frequently when we study polar, cylindrical, and spherical coordinate systems later. Let's begin our study with products of $\sin x$ and $\cos x$

### 6.2.1 Integrating Products and Powers of $\sin x$ and $\cos x$

A key idea behind the strategy used to integrate combinations of products and powers of $\sin x$ and $\cos x$ involves rewriting these expressions as sums and differences of the form $\int \sin^j x \cos x dx$ or $\int \cos^j x \sin x dx$. After rewriting these integrals, we evaluate them using $u$ -substitution. Before defore describing the general process in detail, let's take a look at the following examples.

**Example 6.2.1.** *Evaluate*

$$\int \cos^3 x \sin x dx$$

*Use u-substitution and let* $u = \cos x$ *In this case* $du = \sin x dx$*, Thus,*

$$\begin{aligned} \int \cos^3 x \sin x dx &= -\int u^3 du \\ &= -\frac{1}{4}u^4 + C \\ &= -\frac{1}{4}\cos^4 x + C \end{aligned}$$

Evaluate

$$\int \sin^4 x \cos x dx$$

**Example 6.2.2.** *A Preliminary Example: Integrating* $\int \cos^j x \sin^k x dx$ *Where k is odd*

*Evaluate*

$$\int \cos^2 x \sin^3 x dx$$

*To convert this integral to integrals of the form* $\int \cos^j x \sin x dx$*, rewrite* $\sin^3 x = \sin^2 x \sin x$ *and make the substitution* $\sin^2 x = 1 - \cos^2 x$*. Let* $u = \cos x$ *, then* $du = -\sin x dx$*, Hence ,*

$$\begin{aligned}
\int \cos^2 x \sin^3 x dx &= \int \cos^2 x \left(1 - \cos^2 x\right) \sin x dx \\
&= -\int u^2 \left(1 - u^2\right) du \\
&= \int \left(u^4 - u^2\right) du \\
&= \frac{1}{5}u^5 - \frac{1}{3}u^3 + C \\
&= \frac{1}{5}\cos^5 x - \frac{1}{3}\cos^3 x + C
\end{aligned}$$

Evaluate

$$\int \sin^2 x \cos^3 x dx$$

In the next example, we see the strategy that must be applied when there are only even powers of $\sin x$ and $\cos x$ For integrals of this type, the identities

$$\sin^2 x = \frac{1}{2} - \frac{1}{2}\cos(2x) = \frac{1 - \cos(2x)}{2}$$

and

$$\cos^2 x = \frac{1}{2} + \frac{1}{2}\cos(2x) = \frac{1 + \cos(2x)}{2}$$

are invaluable. These identities are sometimes known as power-reducing identities and they may be derived from the double-angle identity $\cos(2x) = \cos^2 x - \sin^2 x$ and the Pythagorean identity $\cos^2 x + \sin^2 x = 1$

**Example 6.2.3.** *Evaluate*

$$\int \sin^2 x dx$$

*To evaluate this integral, let's use the trigonometric identity* $\sin^2 x = \frac{1}{2}(1 - \cos(2x))$*, Then*

$$\begin{aligned}
\int \sin^2 x dx &= \int \left(\frac{1}{2} - \frac{1}{2}\cos(2x)\right) dx \\
&= \frac{1}{2}x - \frac{1}{4}\sin(2x) + C
\end{aligned}$$

**Example 6.2.4.** *Evaluate*

$$\int \cos^2 x dx$$

*To evaluate this integral, let's use the trigonometric identity* $\cos^2 x = \frac{1}{2}(1 + \cos(2x))$*, Then*

$$\begin{aligned}\int \cos^2 x dx &= \int \left(\frac{1}{2} + \frac{1}{2}\cos(2x)\right) dx \\ &= \frac{1}{2}x + \frac{1}{4}\sin(2x) + C\end{aligned}$$

The general process for integrating products of powers of $\sin x$ and $\cos x$ is summarized in the following set of guidelines.

**Problem-Solving Strategy: Integrating Products and Powers of** $\sin x$ **and** $\cos x$

To integrate $\int \cos^j x \sin^k x dx$ use the following strategies:

1. If $k$ is odd, rewrite $\sin^k x = \sin^k -1x \sin x$ and use the identity $\sin^2 x = 1 - \cos^2 x$ to rewrite $\sin^{k-1} x$ in terms of $\cos x$. Integrate using the substitution $u = \cos x$. This substitution makes $du = -\sin x dx$

2. If $j$ is odd, rewrite $\cos^j x = \cos^{j-1} x \cos x$ and use the identity $\cos^2 x = 1 - \sin^2 x$ to rewrite $\cos^{j-1} x$ in terms of $\sin x$. Integrate using the substitution $u = \sin x$. This substitution makes $du = \cos x dx$. (Note: If both $j$ and $k$ are odd, either strategy 1 or strategy 2 may be used.)

3. If both $j$ and $k$ are even, use $\sin^2 x = (1/2) - (1/2)\cos(2x)$ and $\cos^2 x = (1/2) + (1/2)\cos(2x)$. After applying these formulas, simplify and reapply strategies 1 through 3 as appropriate.

**Example 6.2.5.** *Evaluate*

$$\int \cos^8 x \sin^5 dx$$

*Since the power on* $\sin x$ *is odd, use strategy 1. Thus,*

$$\begin{aligned}
\int \cos^8 x \sin^5 x dx &= \int \cos^8 x \sin^4 x \sin x dx \\
&= \int \cos^8 x \left(\sin^2 x\right)^2 \sin x dx \\
&= \int \cos^8 x \left(1-\cos^2 x\right)^2 \sin x dx \quad \textit{Let } u = \cos x \textit{ and } du = -\sin x dx \\
&= \int u^8 \left(1-u^2\right)^2 (-du) \\
&= \int \left(-u^8 + 2u^{10} - u^{12}\right) du \\
&= -\frac{1}{9}u^9 + \frac{2}{11}u^{11} - \frac{1}{13}u^{13} + C \\
&= -\frac{1}{9}\cos^9 x + \frac{2}{11}\cos^{11} x - \frac{1}{13}\cos^{13} x + C^2
\end{aligned}$$

**Example 6.2.6.** *Evaluate*

$$\int \sin^4 x dx$$

*since the power on* $\sin x$ *is even* $(k = 4)$ *and the power on* $\cos x$ *is even* $(j = 0)$, *we must use strategy* 3. *Thus,*

$$\begin{aligned}
\int \sin^4 x dx &= \int \left(\sin^2 x\right)^2 dx \\
&= \int \left(\frac{1}{2} - \frac{1}{2}\cos(2x)\right)^2 dx \\
&= \int \left(\frac{1}{4} - \frac{1}{2}\cos(2x) + \frac{1}{4}\cos^2(2x)\right) dx \\
&= \int \left(\frac{1}{4} - \frac{1}{2}\cos(2x) + \frac{1}{4}\left(\frac{1}{2} + \frac{1}{2}\cos(4x)\right)\right) dx \\
&= \int \left(\frac{3}{8} - \frac{1}{2}\cos(2x) + \frac{1}{8}\cos(4x)\right) dx \\
&= \frac{3}{8}x - \frac{1}{4}\sin(2x) + \frac{1}{32}\sin(4x) + C
\end{aligned}$$

Evaluate

$$\int \cos^3 x dx, \qquad \int \cos^2(3x) dx$$

In some areas of physics, such as quantum mechanics, signal processing, and the computation of Fourier series, it is often necessary to integrate products that include $\sin(ax)$, $\sin(bx)$, $\cos(ax)$ and $\cos(bx)$ These integrals are evaluated by applying trigonometric identities, as outlined in the following rule.

**Integrating Products of Sines and Cosines of Different Angles**

$$\sin(ax)\sin(bx) = \frac{1}{2}\left[\cos((a-b)x) - \cos((a+b)x)\right] \tag{6.1}$$

$$\sin(ax)\cos(bx) = \frac{1}{2}\left[\sin((a-b)x) + \sin((a+b)x)\right] \tag{6.2}$$

$$\cos(ax)\cos(bx) = \frac{1}{2}\left[\cos((a-b)x) + \cos((a+b)x)\right] \tag{6.3}$$

These formulas may be derived from the sum-of-angle formulas for sine and cosine.

**Example 6.2.7.** *Evaluate*

$$\int \sin(5x)\cos(3x)dx$$

*Apply the identity* $\sin(5x)\cos(3x) = \frac{1}{2}[\sin(2x) - \cos(8x)]$ *Thus,*

$$\begin{aligned}\int \sin(5x)\cos(3x)dx &= \frac{1}{2}\int [\sin(2x) - \cos(8x)]dx \\ &= -\frac{1}{4}\cos(2x) - \frac{1}{16}\sin(8x) + C\end{aligned}$$

**Example 6.2.8.** *Evaluate*

$$\int \cos(6x)\cos(5x)dx$$

*Apply the identity* $\cos(6x)\cos(5x) = \frac{1}{2}[\cos(x) + \cos(11x)]$ *Thus,*

$$\begin{aligned}\int \cos(6x)\cos(5x)dx &= \frac{1}{2}\int [\cos(x) + \cos(11x)]dx \\ &= \frac{1}{2}\sin(x) + \frac{1}{22}\sin(11x) + C\end{aligned}$$

### 6.2.2 Integrating Products and Powers of $\tan x$ and $\sec x$

Before discussing the integration of products and powers of $\tan x$ and $\sec x$ it is useful to recall the integrals involving $\tan x$ and $\sec x$ we have already learned:

1. $\displaystyle\int \sec^2 x dx = \tan x + C$

2. $\displaystyle\int \sec x \tan x dx = \sec x + C$

3. $\displaystyle\int \tan x dx = \ln|\sec x| + C$

4. $\displaystyle\int \sec x dx = \ln|\sec x + \tan x| + C$

For most integrals of products and powers of $\tan x$ and sec $x$, we rewrite the expression we wish to integrate as the sum or difference of integrals of the form $\int \tan^j x \sec^2 x dx$ or $\int \sec^j x \tan x dx$. As we see in the following example, we can evaluate these new integrals by using $u$ -substitution.

**Example 6.2.9.** *Evaluate*

$$\int \sec^5 x \tan x dx$$

*Start by rewriting* $\sec^5 x \tan x$ *as* $\sec^4 x \sec x \tan x$ ,

$$\begin{aligned}\int \sec^5 x \tan x dx &= \int \sec^4 x \sec x \tan x dx \quad \text{Let } u = \sec x \rightarrow du = \sec x \tan x dx \\ &= \int u^4 du \\ &= \frac{1}{5}u^5 + C \\ &= \frac{1}{5}\sec^5 x + C\end{aligned}$$

We now take a look at the various strategies for integrating products and powers of $\tan x$ and $\sec x$

**Problem-Solving Strategy: Integrating $\int \tan^k x \sec^j x dx$**

To integrate $\int \tan^k x \sec^j x dx$ use the following strategies:

1. If $j$ is even and $j \geq 2$, rewrite $\sec^j x = \sec^{j-2} x \sec^2 x$ and use $\sec^2 x = \tan^2 x + 1$ to rewrite sec ${}^{j-2}x$ in terms of $\tan x$. Let $u = \tan x$ and $du = \sec^2 x$

2. If $k$ is odd and $j \geq 1$, rewrite tan ${}^k x \sec^j x = \tan^{k-1} x \sec^{j-1} x \sec x \tan x$ and use $\tan^2 x = \sec^2 x - 1$ to rewrite $\tan^{k-1} x$ in terms of $\sec x$. Let $u = \sec x$ and $du = \sec x \tan x dx$. (Note: If $j$ is even and $k$ is odd then either strategy 1 or strategy 2 may be used.)

3. If $k$ is odd where $k \geq 3$ and $j = 0$, rewrite $\tan^k x = \tan^{k-2} x \tan^2 x = \tan^{k-2} x \left(\sec^2 x - 1\right) = \tan^{k-2} x \sec^2 x - \tan^{k-2} x$. It may be necessary to repeat this process on the $\tan^{k-2} x$ term.

4. If $k$ is even and $j$ is odd, then use $\tan^2 x = \sec^2 x - 1$ to express tan $x$ in terms of sec $x$. Use integration by parts to integrate odd powers of $\sec x$.

**Example 6.2.10.** *Integrating* $\int \tan^k x \sec^j x dx$ *when j is Even*
*Evaluate*

$$\int \tan^6 x \sec^4 x dx$$

*Since the power on sec x is even, rewrite* $\sec^4 x = \sec^2 x \sec^2 x$ *and use* $\sec^2 x = \tan^2 x + 1$ *to rewrite the first* $\sec^2 x$ *in terms of* $\tan x$*. Thus,*

$$\begin{aligned}
\int \tan^6 x \sec^4 x dx &= \int \tan^6 x \left(\tan^2 x + 1\right) \sec^2 x dx \quad \textit{Let } u = \tan x \textit{ and } du = \sec^2 x \\
&= \int u^6 \left(u^2 + 1\right) du \\
&= \int \left(u^8 + u^6\right) du \\
&= \frac{1}{9}u^9 + \frac{1}{7}u^7 + C \\
&= \frac{1}{9}\tan^9 x + \frac{1}{7}\tan^7 x + C
\end{aligned}$$

**Example 6.2.11.** *Integrating* $\int \tan^k x \sec^j x dx$ *when k is odd*
*Evaluate*

$$\int \tan^5 x \sec^3 x dx$$

*Rewrite* $\tan^5 x \sec^3 x$ *as* $\tan^4 x \sec^2 x \tan x \sec x$*. Thus,*

$$\begin{aligned}
\int \tan^5 x \sec^3 x dx &= \int \left(\tan^2 x\right)^2 \sec^2 x \sec x \tan x dx \\
&= \int \left(\sec^2 x - 1\right)^2 \sec^2 x \sec x \tan x dx \quad \textit{Let } u = \sec x \textit{ and } du = \sec x \tan x dx \\
&= \int \left(u^2 - 1\right)^2 u^2 du \\
&= \int \left(u^6 - 2u^4 + u^2\right) du \\
&= \frac{1}{7}u^7 - \frac{2}{5}u^5 + \frac{1}{3}u^3 + C \\
&= \frac{1}{7}\sec^7 x - \frac{2}{5}\sec^5 x + \frac{1}{3}\sec^3 x + C
\end{aligned}$$

**Example 6.2.12.** *Integrating* $\int \tan^k x dx$ *when k is odd and* $k \geq 3$
*Evaluate*

$$\int \tan^3 x dx$$

*Begin by rewriting* $\tan^3 x = \tan x \tan^2 x = \tan x \left(\sec^2 x - 1\right) = \tan x \sec^2 x - \tan x$. *Thus,*

$$\begin{aligned}
\int \tan^3 x dx &= \int \Big(\tan x \sec^2 x - \tan x\Big)\, dx \\
&= \int \tan x \sec^2 x dx - \int \tan x dx \\
&= \frac{1}{2}\tan^2 x - \ln|\sec x| + C
\end{aligned}$$

### 6.2.3 Exercises

**Evaluate the following integral**

| | | | | | |
|---|---|---|---|---|---|
| 1- | $\int \sin^3 x dx$ | 2- | $\int \cos^3 x dx$ | 3- | $\int \sin x \cos x dx$ |
| 4- | $\int \cos^5 x dx$ | 5- | $\int \sin^5 x \cos^2 x dx$ | 6- | $\int \sin^3 x \cos^3 x dx$ |
| 7- | $\int \tan^3 x \sec x dx$ | 8- | $\int \tan^2 x \sec x dx$ | 9- | $\int \tan^2 x \sec^4 x dx$ |
| 10- | $\int \tan^8 x \sec^2 x dx$ | 11- | $\int \cot^3 x dx$ | 12- | $\int \cot^5 x \csc^2 x dx$ |
| 13- | $\int \frac{\cos^5 x}{\sin^3 x} dx$ | 14- | $\int \frac{\sin^7 x}{\cos^4 x} dx$ | 15- | $\int \sin 2x \cos 2x dx$ |
| 16- | $\int \cos 4x \cos 6x dx$ | 17- | $\int \frac{\tan^3(\ln t)}{t} dt$ | 18- | $\int t \cos^3\left(t^2\right) dt$ |

19- **Prove the formula** $\int \sec^m x dx = \dfrac{\tan x \sec^{m-2} x}{m-1} + \dfrac{m-2}{m-1}\int \sec^{m-2} x dx$

20- **Prove the formula** $\int \tan^m x dx = \dfrac{\tan^{m-1} x}{m-1} - \int \tan^{m-2} x dx$

| | | | | | |
|---|---|---|---|---|---|
| 21- | $\int \cot^5 x \csc^5 x dx$ | 22- | $\int \tan^6 x \sec^4 x dx$ | 23- | $\int \tan^3 \theta \sec^3 \theta d\theta$ |

**Hint : Use Ex-19 and Ex-20 to solve Ex-21,22,23**

## 6.3 Trigonometric Substitution

In this section, we explore integrals containing expressions of the form $\sqrt{a^2 - x^2}$, $\sqrt{a^2 + x^2}$, and $\sqrt{x^2 - a^2}$, where the values of $a$ are positive. We have already encountered and evaluated integrals containing some expressions of this type, but many y still remain inaccessible. The technique of trigonometric substitution comes in very handy when evaluating these integrals. This technique substitution to rewrite these integrals as trigonometric integrals.

### 6.3.1 Integrals Involving $\sqrt{a^2 - x^2}$

To evaluate integrals involving $\sqrt{a^2 - x^2}$ we make the substitution $x = a\sin\theta$ and $dx = a\cos\theta d\theta$ by making the substitution $x = a\sin\theta$ we are able to convert an integral involving a radical into an integral involving trigonometric functions. After we evaluate the integral, we can convert the solution back to an expression involving $x$ . Since $\sin\theta = \frac{x}{a}$ we can draw the reference triangle in Figure 6.1 to assist in expressing the values of $\cos\theta$ and $\sin\theta$ the remaining trigonometric functions in terms of $x$

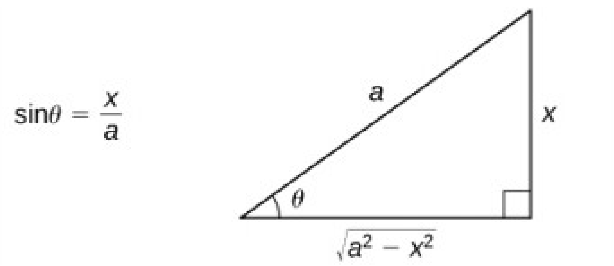



FIGURE 6.1: A reference triangle can help express the trigonometric functions evaluated at $\theta$ in terms of $x$

The essential part of this discussion is summarized in the following problem-solving strategy.

**Problem-Solving Strategy: Integrating Expressions Involving $\sqrt{a^2 - x^2}$**

1. It is a good idea to make sure the integral cannot be evaluated easily in another way. For example, although this method can be applied to integrals of the form $\int \frac{1}{\sqrt{a^2 - x^2}} dx$, $\int \frac{x}{\sqrt{a^2 - x^2}} dx$, and $\int x\sqrt{a^2 - x^2} dx$, they can be integrated directly either by formula or by a simple $u$ -substitution.

2. Make the substitution $x = a \sin\theta$ and $dx = a \cos\theta d\theta$. Note: This substitution yields $\sqrt{a^2 - x^2} = a \cos\theta$

3. Simplify the expression.

4. Evaluate the integral using techniques from the section on trigonometric integrals.

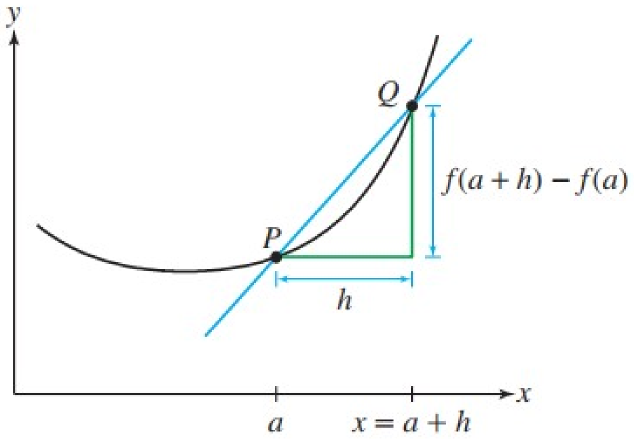
5. Use the reference triangle from Figure 6.1 to rewrite the result in terms of $x$ You may also need to use some trigonometric identities and the relationship $\theta = \sin^{-1}\left(\frac{x}{a}\right)$

The following example demonstrates the application of this problem-solving strategy.

**Example 6.3.1.** *Evaluate*

$$\int \sqrt{9 - x^2} dx$$

*Begin by making the substitutions* $x = 3 \sin\theta$ *and* $dx = 3 \cos\theta d\theta$*. since* $\sin\theta = \frac{x}{3}$*, we can construct the reference triangle shown in the following figure.*

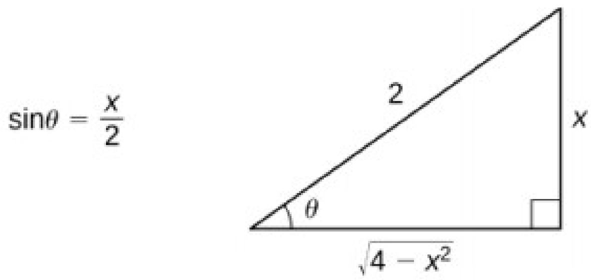



$$
\begin{aligned}
\int \sqrt{9-x^2}dx &= \int \sqrt{9-(3\sin\theta)^2}3\cos\theta d\theta \\
&= \int \sqrt{9\left(1-\sin^2\theta\right)}3\cos\theta d\theta \\
&= \int \sqrt{9\cos^2\theta}3\cos\theta d\theta \\
&= \int 9\cos^2\theta d\theta \\
&= \int 9\left(\frac{1}{2}+\frac{1}{2}\cos(2\theta)\right)d\theta \\
&= \frac{9}{2}\theta+\frac{9}{4}\sin(2\theta)+C \\
&= \frac{9}{2}\theta+\frac{9}{4}(2\sin\theta\cos\theta)+C \\
&= \frac{9}{2}\sin^{-1}\left(\frac{x}{3}\right)+\frac{9}{2}\cdot\frac{x}{3}\cdot\frac{\sqrt{9-x^2}}{3}+C \\
&= \frac{9}{2}\sin^{-1}\left(\frac{x}{3}\right)+\frac{x\sqrt{9-x^2}}{2}+C
\end{aligned}
$$

**Example 6.3.2.** *Evaluate*

$$\int \frac{\sqrt{4-x^2}}{x}dx$$

*Begin by making the substitutions* $x = 2\sin\theta$ *and* $dx = 2\cos\theta d\theta$*. since* $\sin\theta = \frac{x}{2}$*, we can construct the reference triangle shown in the following figure.*

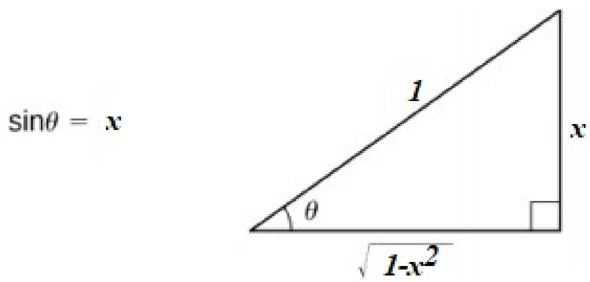



$$\begin{aligned}\int \frac{\sqrt{4-x^2}}{x}dx &= \int \frac{\sqrt{4-(2\sin\theta)^2}}{2\sin\theta}2\cos\theta d\theta \\ &= \int \frac{2\cos^2\theta}{\sin\theta}d\theta \\ &= \int \frac{2\left(1-\sin^2\theta\right)}{\sin\theta}d\theta \\ &= 2\ln|\csc\theta-\cot\theta|+2\cos\theta+C \\ &= 2\ln\left|\frac{2}{x}-\frac{\sqrt{4-x}}{x}\right|+\sqrt{4-x^2}+C\end{aligned}$$

**Example 6.3.3.** *Evaluate*

$$\int x^3\sqrt{1-x^2}dx$$

*Begin by making the substitutions* $x=\sin\theta$ *and* $dx=\cos\theta d\theta$. *since* $\sin\theta=x$, *we can construct the reference triangle shown in the following figure.*



$$\begin{aligned}\int x^3\sqrt{1-x^2}dx &= \int \sin^3\theta\cos^2\theta d\theta \\ &= \int \left(1-\cos^2\theta\right)\cos^2\theta\sin\theta d\theta \\ &= \int \left(u^4-u^2\right)du \\ &= \frac{1}{5}\cos^5\theta - \frac{1}{3}\cos^3\theta + C \\ &= \frac{1}{5}\left(1-x^2\right)^{5/2} - \frac{1}{3}\left(1-x^2\right)^{3/2} + C\end{aligned}$$

### 6.3.2 Integrating Expressions Involving $\sqrt{a^2+x^2}$

For integrals containing $\sqrt{a^2+x^2}$ let's first consider the domain of this expression. Since $\sqrt{a^2+x^2}$ is defined for all real values of we restrict our choice to those trigonometric functions that have a range of all real numbers. Thus, our choice is restricted to selecting either $x = a\tan\theta$ or $x = a\cot x$

**Problem-Solving Strategy: Integrating Expressions Involving $\sqrt{a^2+x^2}$**

1. Check to see whether the integral can be evaluated easily by using another method. In some cases, it is more convenient to use an alternative method.
2. Substitute $x = a\tan\theta$ and $dx = a\sec^2\theta d\theta$. This substitution yields $\sqrt{a^2+x^2} = \sqrt{a^2+(a\tan\theta)^2} = \sqrt{a^2\left(1+\tan^2\theta\right)} = \sqrt{a^2\sec^2\theta} = |a\sec\theta$. (since $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$ and $\sec\theta > 0$ over this interval, $|a\sec\theta| = a\sec\theta$.).
3. Simplify the expression.
4. Evaluate the integral using techniques from the section on trigonometric integrals.
5. Use the reference triangle from Figure 6.2 to rewrite the result in terms of $x$. You may also need to use some trigonometric identities and the relationship $\theta = \tan^{-1}\left(\frac{x}{a}\right)$. (Note: The reference triangle is based on the assumption that $x > 0$; however, the trigonometric ratios produced from the reference triangle are the same as the ratios for which $x \leq 0$.)

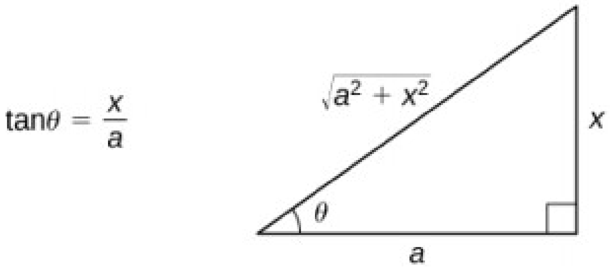
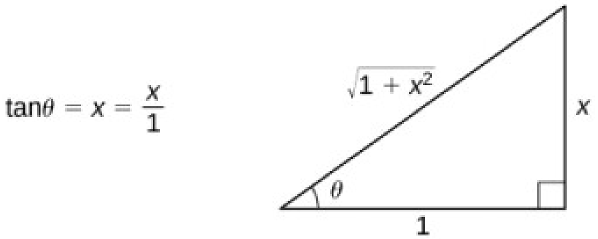



FIGURE 6.2: A reference triangle can help express the trigonometric functions evaluated at $\theta$ in terms of $x$

**Example 6.3.4.** *Evaluate*

$$\int \frac{1}{\sqrt{1+x^2}}dx$$

*Begin with the substitution* $x = \tan\theta$ *and* $dx = \sec^2\theta d\theta$. *since tan* $\theta = x$, *draw the reference triangle in the following figure.*

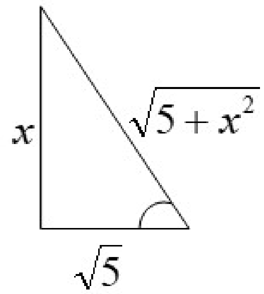



$$\begin{aligned}\int \frac{dx}{\sqrt{1+x^2}} &= \int \frac{\sec^2\theta}{\sec\theta}d\theta \\ &= \int \sec\theta d\theta \\ &= \ln|\sec\theta + \tan\theta| + C \\ &= \ln|\sqrt{1+x^2} + x| + C\end{aligned}$$

**Example 6.3.5.** *Evaluate*

$$\int \frac{1}{x^2\sqrt{5+x^2}}dx$$

*Begin with the substitution* $x = \sqrt{5}\tan\theta$ *and* $dx = \sqrt{5}\sec^2\theta d\theta$. *since tan* $\theta = x$, *draw the reference triangle in the following figure.*




$$
\begin{aligned}
\int \frac{1}{x^2\sqrt{5+x^2}}dx &= \int \frac{\sqrt{5}\sec^2\theta}{5\sqrt{5}\tan^2\theta\sec\theta}d\theta \\
&= \frac{1}{5}\int \sec\theta\cot^2\theta d\theta \\
&= \frac{1}{5}\int \frac{1}{\cos\theta}\frac{\cos^2\theta}{\sin^2\theta}d\theta \\
&= \frac{1}{5}\int \cos\theta\sin^{-2}\theta d\theta \\
&= \frac{-1}{5}\csc\theta + C \\
&= -\frac{\sqrt{5+x^2}}{5x} + c
\end{aligned}
$$

### 6.3.3 Integrating Expressions Involving $\sqrt{a^2+x^2}$

The domain of the expression $\sqrt{x^2-a^2}$ is $(-\infty,-a] \cup [a,+\infty)$. Thus, either $x < -a$ or $x > a$. Hence, $\frac{x}{a} \leq -1$ or $\frac{x}{a} \geq 1$. since these intervals correspond to the range of sec $\theta$ on the set $\left[0,\frac{\pi}{2}\right) \cup \left(\frac{\pi}{2},\pi\right]$, it makes sense to use the substitution $\sec\theta = \frac{x}{a}$ or, equivalently, $x = a\sec\theta$, where $0 \leq \theta < \frac{\pi}{2}$ or $\frac{\pi}{2} < \theta \leq \pi$. The corresponding substitution for $dx$ is $dx = a\sec\theta\tan\theta d\theta$. The procedure for using this substitution is outlined in the following problem-solving strategy.

**Problem-Solving Strategy: Integrating Expressions Involving $\sqrt{x^2 - a^2}$**

1. Check to see whether the integral cannot be evaluated using another method. If so, we may wish to consider applying an alternative technique.

2. Substitute $x = a\sec\theta$ and $dx = a\sec\theta\tan\theta d\theta$. This substitution yields

   $$\sqrt{x^2 - a^2} = \sqrt{(a\sec\theta)^2 - a^2} = \sqrt{a^2\left(\sec^2\theta + 1\right)} = \sqrt{a^2\tan^2\theta} = |a\tan\theta|$$

   For $x \geq a, \quad |a\tan\theta| = a\tan\theta$ and for $x \leq -a, \quad |a\tan\theta| = -a\tan\theta$

3. Simplify the expression.

4. Evaluate the integral using techniques from the section on trigonometric integrals.

5. Use the reference triangles from Figure 6.3 to rewrite the result in terms of $x$. You may also need to use some trigonometric identities and the relationship $\theta = \sec^{-1}\left(\frac{x}{a}\right)$. (Note: We need both reference triangles, since the values of some of the trigonometric ratios are different depending on whether $x > a$ or $x < -a$.)

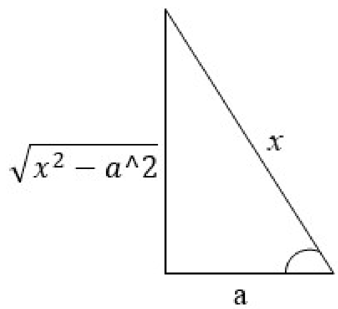
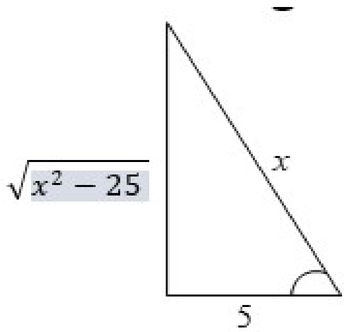



FIGURE 6.3: A reference triangle can help express the trigonometric functions evaluated at $\theta$ in terms of $x$

**Example 6.3.6.** *Evaluate*

$$\int \frac{1}{x^2\sqrt{x^2-25}}dx$$

*Begin with the substitution* $x = 5\sec\theta$ *and* $dx = 5\sec\theta\tan\theta d\theta$*. draw the reference triangle in the following figure.*




$$\begin{aligned}
\int \frac{1}{x^2\sqrt{x^2-25}}dx &= \int \frac{5\sec\theta\tan\theta d\theta}{25\sec^2\theta\sqrt{25\sec^2\theta-25}} \\
&= \int \frac{5\sec\theta\tan\theta d\theta}{125\sec^2\theta\tan\theta} \\
&= \int \frac{d\theta}{25\sec\theta} \\
&= \frac{1}{25}\int \cos\theta d\theta \\
&= \frac{1}{25}\sin\theta + c \\
&= \frac{\sqrt{x^2-25}}{25x} + c
\end{aligned}$$

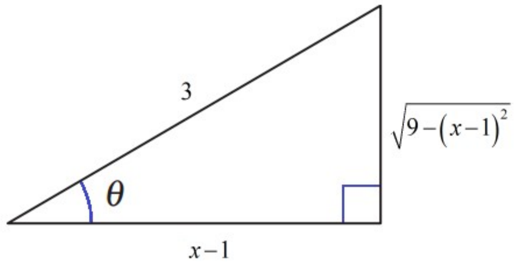
**Example 6.3.7.** *Evaluate*

$$\int \frac{1}{\sqrt{8-x^2+2x-1+1}}dx$$

*Complete the square , we get*

$$\begin{aligned}
\int \frac{1}{\sqrt{8-x^2+2x-1+1}}dx &= \int \frac{1}{\sqrt{9-(x^2-2x+1)}}dx \\
&= \int \frac{1}{\sqrt{9-(x-1)^2}}dx
\end{aligned}$$

*Begin with the substitution* $x-1 = 3\sin\theta$ *and* $dx = 3\cos\theta d\theta$*. draw the reference triangle in the following figure.*




$$\begin{aligned}
\int \frac{1}{\sqrt{9-(x-1)^2}}dx &= \int \frac{3\cos\theta d\theta}{\sqrt{9-9\sin^2\theta}} \\
&= \int \frac{3\cos\theta d\theta}{3\cos\theta} \\
&= \int d\theta \\
&= \theta + c \\
&= \sin^{-1}\left(\frac{x-1}{3}\right) + c
\end{aligned}$$

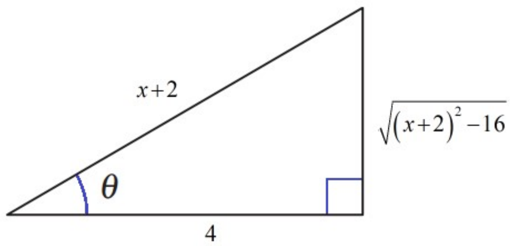
**Example 6.3.8.** *Evaluate*

$$\int \frac{1}{\sqrt{(x^2+4x+4)-16}}dx$$

*Complete the square , we get*

$$\int \frac{1}{\sqrt{(x^2+4x+4)-16}}dx = \int \frac{1}{\sqrt{(x+2)^2-16}}dx$$

*Begin with the substitution $x+2 = 4\sec\theta$ and $dx = 4\sec\theta\tan\theta d\theta$. draw the reference triangle in the following figure.*



$$\begin{aligned}
\int \frac{1}{\sqrt{(x+2)^2-16}}dx &= \int \frac{4\sec\theta\tan\theta d\theta}{\sqrt{16\sec^2\theta-16}} \\
&= \int \frac{4\sec\theta\tan\theta d\theta}{4\tan\theta} \\
&= \int \sec\theta d\theta \\
&= \ln|\sec\theta+\tan\theta|+C \\
&= \ln\left|\frac{x+2}{4}+\frac{\sqrt{(x+2)^2-16}}{4}\right|+C
\end{aligned}$$

### 6.3.4 Exercises

**Evaluate the following integrals**

1- $\int \frac{x^2dx}{\sqrt{9-x^2}}$ 2- $\int \frac{dt}{(16-t^2)^{3/2}}$ 3- $\int \frac{dx}{x\sqrt{x^2+16}}$

4- $\int \sqrt{12+4t^2}dt$ 5- $\int \frac{dx}{\sqrt{x^2-9}}$ 6- $\int \frac{dt}{t^2\sqrt{t^2-25}}$

7- $\int \frac{dy}{y^2\sqrt{5-y^2}}$ 8- $\int x^3\sqrt{9-x^2}dx$ 9- $\int \frac{dx}{\sqrt{25x^2+2}}$

10- $\int \frac{dt}{(9t^2+4)^2}$ 11- $\int \frac{dz}{z^3\sqrt{z^2-4}}$ 12- $\int \frac{dy}{\sqrt{y^2-9}}$

13- $\int \frac{x^2dx}{(6x^2-49)^{1/2}}$ 14- $\int \frac{dx}{(x^2-4)^2}$ 15- $\int \frac{x^2dx}{(x^2-1)^{3/2}}$

16- $\int \frac{x^2dx}{(x^2+1)^{3/2}}$ 17- $\int \frac{dx}{\sqrt{x+6x^2}}$ 18- $\int \frac{dx}{\sqrt{12x-x^2}}$

19- $\int \sqrt{x^2-4x+3}dx$ 20- $\int \frac{dx}{(x^2+6x+6)^2}$

21- $\int \sec^{-1} xdx$ 22- $\int \frac{\sin^{-1} x}{x^2}dx$ 24- $\int \ln\left(x^2+1\right) dx$

## 6.4 PARTIAL FRACTIONS

In algebra, one learns to combine two or more fractions into a single fraction by finding a common denominator. For example,

$$\frac{2}{x-4}+\frac{3}{x+1}=\frac{2(x+1)+3(x-4)}{(x-4)(x+1)}=\frac{5x-10}{x^2-3x-4}$$

However, for purposes of integration, the left side of the previous equation is preferable to the right side since each of the terms is easy to integrate:

$$\int\frac{5x-10}{x^2-3x-4}dx=\int\frac{2}{x-4}dx+\int\frac{3}{x+1}dx=2\ln|x-4|+3\ln|x+1|+C$$

To illustrate how this can be done, we begin by noting that on the left side the numerators are constants and the denominators are the factors of the denominator on the right side.
In this section, we examine the method of partial fraction decomposition, which allows us to decompose rational functions into sums of simpler, more easily intagrated rational functions. Using this method, we can rewrite an expression such as: $\frac{3x}{x^2-x-2}$ as an expression as $\frac{1}{x+1}+\frac{2}{x-2}$.
The key to the method of partial fraction decomposition is being able to anticipate the form that the decomposition of a rational function will take. As we shall see, this form is both predictable and highly dependent on the factorization of the denominator of the rational function. It is also extremely important to keep in mind that partial fraction decomposition can be applied to a rational function $\frac{P(x)}{Q(x)}$ only if $\deg(P(x)) < \deg(Q(x))$. In the case when $\deg(P(x)) \geq \deg(Q(x)) \geq \deg(Q(x))$, we must first perform long division to rewrite the quotient $\frac{P(x)}{Q(x)}$ in the form $A(x)+\frac{R(x)}{Q(x)}$, where $\deg(R(x)) < \deg(Q(x))$ We then do a partial fraction decomposition on $\frac{R(x)}{Q(x)}$. The following example, although not requiring partial fraction decomposition, illustrates our approach to integrals of rational functions of the form $\int\frac{P(x)}{Q(x)}dx$, where $\deg(P(x)) \geq \deg(Q(x))$

Now that we are beginning to get the idea of how the technique of partial fraction decomposition works, let's outline the basic method in the following problem-solving strategy

**Problem-Solving Strategy: Partial Fraction Decomposition**

To decompose the rational function $\dfrac{P(x)}{Q(x)}$ use the following steps:

1. Make sure that degree( $P(x)) <$ degree$(Q(x))$. If not, perform long division of polynomials.

2. Factor $Q(x)$ into the product of linear and irreducible quadratic factors. An irreducible quadratic is a quadratic that has no real zeros.

3. Assuming that $\deg(P(x)) < \deg(Q(x))$, the factors of $Q(x)$ determine the form of the decomposition of $P(x)/Q(x)$.

   - If $Q(x)$ can be factored as $(a_1x+b_1)(a_2x+b_2)\dots(a_nx+b_n)$, where each linear factor is distinct,then it is possible to find constants $A_1, A_2, \dots A_n$ satisfying

   $$\frac{P(x)}{Q(x)} = \frac{A_1}{a_1x+b_1} + \frac{A_2}{a_2x+b_2} + \cdots + \frac{A_n}{a_nx+b_n}$$

   - If $Q(x)$ contains the repeated linear factor $(ax+b)^n$, then the decomposition must contain

   $$\frac{A_1}{ax+b} + \frac{A_2}{(ax+b)^2} + \cdots + \frac{A_n}{(ax+b)^n}$$

   - For each irreducible quadratic factor $ax^2+bx+c$ that $Q(x)$ contains, the decomposition must include

   $$\frac{Ax+B}{ax^2+bx+c}$$

   - For each repeated irreducible quadratic factor $\left(ax^2+bx+c\right)^n$, the decomposition must include

   $$\frac{A_1x+B_1}{ax^2+bx+c} + \frac{A_2x+B_2}{\left(ax^2+bx+c\right)^2} + \cdots + \frac{A_nx+B_n}{\left(ax^2+bx+c\right)^n}$$

   - After the appropriate decomposition is determined, solve for the constants.

   - Last, rewrite the integral in its decomposed form and evaluate it using previously developed techniques or integration formulas.

**Example 6.4.1.** *Evaluate*

$$\int \frac{x^2+3x+5}{x+1}dx$$

*Since* $\deg\left(x^2+3x+5\right) \geq \deg(x+1)$, *we perform long division to obtain*

$$\frac{x^2+3x+5}{x+1} = x+2+\frac{3}{x+1}$$

*Then*

$$\begin{aligned}\int \frac{x^2+3x+5}{x+1}dx &= \int \left(x+2+\frac{3}{x+1}\right)dx \\ &= \frac{1}{2}x^2+2x+3\ln|x+1|+C\end{aligned}$$

**Example 6.4.2.** *Evaluate*

$$\int \frac{3x+2}{x^3-x^2-2x}dx$$

*Since* $\deg(3x+2) < \deg\left(x^3-x^2-2x\right)$, *we begin by factoring the denominator of* $\frac{3x+2}{x^3-x^2-2x}$. *We can see that* $x^3-x^2-2x = x(x-2)(x+1)$. *Thus, there constants* $A, \quad B$, *and* $C$ *satisfying*

$$\frac{3x+2}{x(x-2)(x+1)} = \frac{A}{x}+\frac{B}{x-2}+\frac{C}{x+1}$$

*We must now find these constants. To do so, we begin by getting a common denominator on the right. Thus,*

$$\frac{3x+2}{x(x-2)(x+1)} = \frac{A(x-2)(x+1)+Bx(x+1)+Cx(x-2)}{x(x-2)(x+1)}$$

*Now, we set the numerators equal to each other, obtaining*

$$3x+2 = A(x-2)(x+1)+Bx(x+1)+Cx(x-2)$$

*Then* $A=-1, \quad B=4/3, \quad C=-1/3$, *hence*

$$\begin{aligned}\int \frac{3x+2}{x^3-x^2-2x}dx &= \int \left(-\frac{1}{x}+\frac{4}{3}\cdot\frac{1}{(x-2)}-\frac{1}{3}\frac{1}{(x+1)}\right)dx \\ &= -\ln|x|+\frac{4}{3}\ln|x-2|-\frac{1}{3}\ln|x+1|+C\end{aligned}$$

**Example 6.4.3.** *Evaluate*

$$\int \frac{x^2+3x+1}{x^2-4}dx$$

*since degree* $\left(x^2+3x+1\right) \geq$ *degree* $\left(x^2-4\right)$ *, we must perform long division of polynomials. This results in*

$$\frac{x^2+3x+1}{x^2-4} = 1 + \frac{3x+5}{x^2-4}$$

*Next, we perform partial fraction decomposition on* $\frac{3x+5}{x^2-4} = \frac{3x+5}{(x+2)(x-2)}$. *We have*

$$\frac{3x+5}{(x-2)(x+2)} = \frac{A}{x-2} + \frac{B}{x+2}$$

*Then*

$$x+5 = A(x+2) + B(x-2)$$

*Solving for* $A$ *and* $B$ *using either method, we obtain* $A = 11/4$ *and* $B = 1/4$

$$\begin{aligned}\int \frac{x^2+3x+1}{x^2-4}dx &= \int \left(1 + \frac{11}{4}\cdot\frac{1}{x-2} + \frac{1}{4}\cdot\frac{1}{x+2}\right)dx \\ &= x + \frac{11}{4}\ln|x-2| + \frac{1}{4}\ln|x+2| + C\end{aligned}$$

**Example 6.4.4.** *Evaluate*

$$\int \frac{\cos x}{\sin^2 x - \sin x}dx$$

*Let's begin by letting* $u = \sin x$. *Consequently,* $du = \cos x dx$. *After making these substitutions, we have*

$$\int \frac{\cos x}{\sin^2 x - \sin x}dx = \int \frac{du}{u^2-u} = \int \frac{du}{u(u-1)}$$

*Applying partial fraction decomposition to* $1/u(u-1)$ *gives* $\dfrac{1}{u(u-1)} = -\dfrac{1}{u} + \dfrac{1}{u-1}$

$$\begin{aligned}\int \frac{\cos x}{\sin^2 x - \sin x}dx &= -\ln|u| + \ln|u-1| + C \\ &= -\ln|\sin x| + \ln|\sin x - 1| + C\end{aligned}$$

**Example 6.4.5.** *Evaluate*

$$\int \frac{x-2}{(2x-1)^2(x-1)}dx$$

*We have degree(* $x-2) <$ *degree* $\left((2x-1)^2(x-1)\right)$ *, so we can proceed with the decomposition. since* $(2x-1)^2$ *is a repeated linear factor, include* $\dfrac{A}{2x-1} + \dfrac{B}{(2x-1)^2}$ *in the decomposition. Thus,*

$$\frac{x-2}{(2x-1)^2(x-1)} = \frac{A}{2x-1} + \frac{B}{(2x-1)^2} + \frac{C}{x-1}$$

*After getting a common denominator and equating the numerators, we have*

$$x - 2 = A(2x-1)(x-1) + B(x-1) + C(2x-1)^2$$

*We then use the method of equating coefficients to find the values of* $A, \quad B,$ *and* $C$.

$$x - 2 = (2A + 4C)x^2 + (-3A + B - 4C)x + (A - B + C)$$

*Equating coefficients yields* $2A + 4C = 0, \quad -3A + B - 4C = 1,$ *and* $A - B + C = -2$. *Solving this system yields* $A = 2, \quad B = 3,$ *and* $C = -1$

$$\begin{aligned}\int \frac{x-2}{(2x-1)^2(x-1)}dx &= \int \left(\frac{2}{2x-1} + \frac{3}{(2x-1)^2} - \frac{1}{x-1}\right)dx \\ &= \ln|2x-1| - \frac{3}{2(2x-1)} - \ln|x-1| + C\end{aligned}$$

**Example 6.4.6.** *Evaluate*

$$\int \frac{2x-3}{x^3+x}dx$$

*since* $\deg(2x-3) < \deg\left(x^3+x\right)$, *factor the denominator and proceed with partial fraction decomposition. since* $x^3 + x = x\left(x^2+1\right)$ *contains the irreducible quadratic factor* $x^2+1$, *include* $\frac{Ax+B}{x^2+1}$ *as part of the decomposition, along with* $\frac{C}{x}$ *for the linear term* $x$. *Thus, the decomposition has the form*

$$\frac{2x-3}{x\left(x^2+1\right)} = \frac{Ax+B}{x^2+1} + \frac{C}{x}$$

*After getting a common denominator and equating the numerators, we obtain the equation*

$$2x - 3 = (Ax+B)x + C\left(x^2+1\right)$$

*Solving for* $A, B,$ *and* $C$, *we get* $A = 3, \quad B = 2,$ *and* $C = -3$

$$\begin{aligned}\int \frac{2x-3}{x^3+x}dx &= \int \left(\frac{3x+2}{x^2+1} - \frac{3}{x}\right)dx \\ &= 3\int \frac{x}{x^2+1}dx + 2\int \frac{1}{x^2+1}dx - 3\int \frac{1}{x}dx \\ &= \frac{3}{2}\ln\left|x^2+1\right| + 2\tan^{-1}x - 3\ln|x| + C\end{aligned}$$

**Example 6.4.7.** *Evaluate*

$$\int \frac{dx}{x^3-8}$$

*We can start by factoring* $x^3-8=(x-2)\left(x^2+2x+4\right)$. *We see that the quadratic factor* $x^2+2x+4$ *is irreducible since* $2^2-4(1)(4)=-12<0$. *Using the decomposition described in the problem-solving strategy, we get*

$$\frac{1}{(x-2)\left(x^2+2x+4\right)}=\frac{A}{x-2}+\frac{Bx+C}{x^2+2x+4}$$

*After obtaining a common denominator and equating the numerators, this becomes*

$$1=A\left(x^2+2x+4\right)+(Bx+C)(x-2)$$

*Applying either method, we get* $A=\frac{1}{12}, B=-\frac{1}{12}$, *and* $C=-\frac{1}{3}$

$$\int \frac{dx}{x^3-8}=\frac{1}{12}\int \frac{1}{x-2}dx-\frac{1}{12}\int \frac{x+4}{x^2+2x+4}dx$$

*We can see that* $\displaystyle\int \frac{1}{x-2}dx=\ln|x-2|+C$, *but* $\displaystyle\int \frac{x+4}{x^2+2x+4}dx$ *requires a bit more effort. Let's begin by completing the square on* $x^2+2x+4$ *to obtain*

$$x^2+2x+4=(x+1)^2+3$$

*By letting* $u=x+1$ *and consequently* $du=dx$, *we see that*

$$\begin{aligned}
\int \frac{x+4}{x^2+2x+4}dx &= \int \frac{x+4}{(x+1)^2+3}dx \\
&= \int \frac{u+3}{u^2+3}du \\
&= \int \frac{u}{u^2+3}du+\int \frac{3}{u^2+3}du \\
&= \frac{1}{2}\ln\left|u^2+3\right|+\frac{3}{\sqrt{3}}\tan^{-1}\frac{u}{\sqrt{3}}+C \\
&= \frac{1}{2}\ln\left|x^2+2x+4\right|+\sqrt{3}\tan^{-1}\left(\frac{x+1}{\sqrt{3}}\right)+C
\end{aligned}$$

### 6.4.1 Exercises

**Evaluate the following integrals**

| | | |
|---|---|---|
| 1- $\displaystyle\int \frac{\left(x^3+2x^2+1\right)dx}{x+2}$ | 2- $\displaystyle\int \frac{\left(x^2+2\right)dx}{x+3}$ | 3- $\displaystyle\int \frac{xdx}{3x-4}$ |
| 4- $\displaystyle\int \frac{\left(x^3+1\right)dx}{x^2+1}$ | 5- $\displaystyle\int \frac{dx}{(x-2)(x-4)}$ | 6- $\displaystyle\int \frac{(x+3)dx}{x+4}$ |
| 7- $\displaystyle\int \frac{dx}{(x-2)(x-3)(x+2)}$ | 8- $\displaystyle\int \frac{(2x-1)dx}{x^2-5x+6}$ | 9- $\displaystyle\int \frac{dx}{x(2x+1)}$ |
| 10- $\displaystyle\int \frac{\left(x^2+11x\right)dx}{(x-1)(x+1)^2}$ | 11- $\displaystyle\int \frac{3dx}{(x+1)\left(x^2+x\right)}$ | 12- $\displaystyle\int \frac{x^2dx}{x^2+9}$ |
| 13- $\displaystyle\int \frac{\left(4x^2-21x\right)dx}{(x-3)^2(2x+3)}$ | 14- $\displaystyle\int \frac{8dx}{x(x+2)^3}$ | 15- $\displaystyle\int \frac{4x^2-20}{(2x+5)^3}dx$ |
| 16- $\displaystyle\int \frac{\left(x^2-x+1\right)dx}{x^2+x}$ | 17- $\displaystyle\int \frac{xdx}{x^4+1}$ | 18- $\displaystyle\int \frac{e^xdx}{e^{2x}-e^x}$ |
| 19- $\displaystyle\int \frac{\sec^2\theta d\theta}{\tan^2\theta-1}$ | 20- $\displaystyle\int \frac{\sqrt{x}dx}{x-1}$ | 21- $\displaystyle\int \frac{dx}{x^{1/2}-x^{1/3}}$ |
| 22- $\displaystyle\int \frac{dx}{x^{5/4}-4x^{3/4}}$ | 23- $\displaystyle\int \frac{\cos ydy}{\sin^2y+\sin y-6}$ | 24- $\displaystyle\int \frac{1}{x^4+x}dx$ |

# Chapter 7

# Definite Integrals and Application

A great achievement of classical geometry was obtaining formulas for the areas and volumes of triangles, spheres, and cones. In this chapter we develop a method to calculate the areas and volumes of very general shapes. This method, called integration, is a way to calculate much more than areas and volumes. The definite integral is the key tool in calculus for defining and calculating many important quantities, such as areas, volumes, lengths of curved paths, probabilities, averages, energy consumption, the weights of various objects, and the forces against a dam's floodgates, just to mention a few

## 7.1 Area and estimating with Finite Sums

The basis for formulating definite integrals is the construction of appropriate approximations by finite sums.

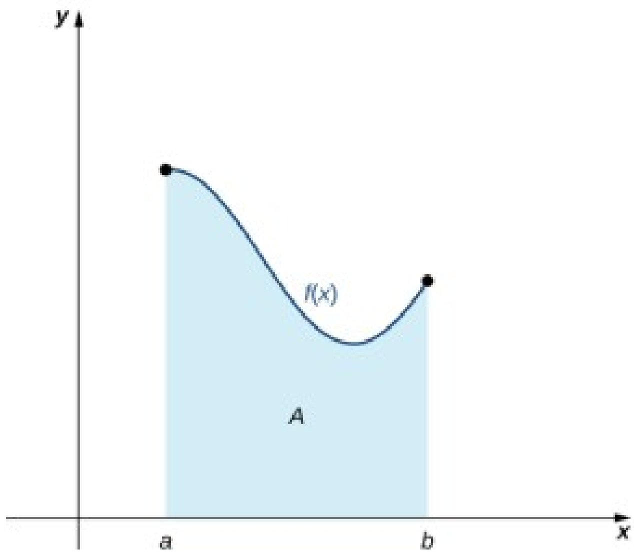
**Sums and Powers of Integers**

1. $\sum_{i=1}^{n} i = 1 + 2 + \cdots + n = \dfrac{n(n+1)}{2}$

2. $\sum_{i=1}^{n} i^2 = 1^2 + 2^2 + \cdots + n^2 = \dfrac{n(n+1)(2n+1)}{6}$

3. $\sum_{i=1}^{n} i^3 = 1^3 + 2^3 + \cdots + n^3 = \dfrac{n^2(n+1)^2}{4}$

Now that we have the necessary notation, we return to the problem at hand: approximating the area under a curve. Let $f(x)$ be a continuous, nonnegative function defined on the closed interval $[a, b]$. We want to approximate the area $A$ bounded by $f(x)$ above, the $x$ -axis below, the line $x = a$ on the left, and the line $x = b$ on the right . How do we approximate the area under this curve? The approach is a geometric one. By dividing a region into many small shapes that have lenown area formulas, we can sum these areas and obtain a reasonable




estimate of the true area. We begin by dividing the interval $[a, b]$ into $n$ subintervals of equal width, $\frac{b-a}{n}$. We do this by selecting equally spaced points $x_0, x_1, x_2, \ldots, x_n$ with $x_0 = a, x_n = b$, and

$$x_i - x_{i-1} = \frac{b-a}{n}, \quad i = 1, 2, 3, \ldots, n$$

We denote the width of each subinterval with the notation $\Delta x$, so $\Delta x = \frac{b-a}{n}$ and

$$x_i = x_0 + i\Delta x$$

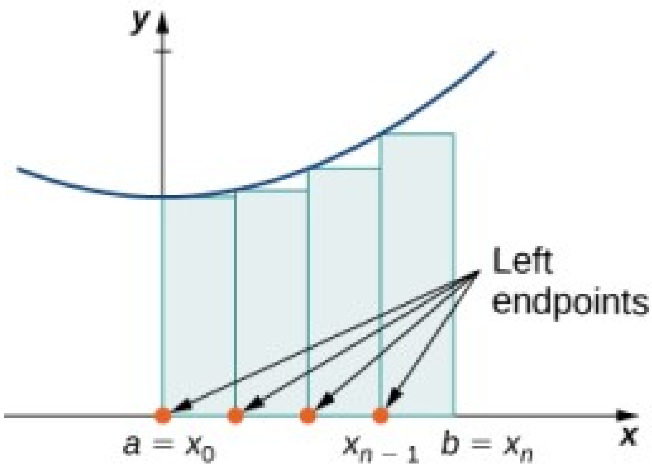
for $i = 1, 2, 3, \ldots, n$. This notion of dividing an interval $[a, b]$ into subintervals by selecting points from within the interval is used quite often in approximating the area under a curve, so let's define some relevant terminology.

> A set of points $P = \{x_i\}$ for $i = 0, 1, 2, \ldots, n$ with $a = x_0 < x_1 < x_2 < \cdots < x_n = b$, which divides the interval $[a, b]$ into subintervals of the form $[x_0, x_1], [x_1, x_2], \ldots, [x_{n-1}, x_n]$ is called a partition of $[a, b]$. If the subintervals all have the same width, the set of points forms a regular partition of the interval $[a, b]$

We can use this regular partition as the basis of a method for estimating the area under the curve. We next examine two methods: the left-endpoint approximation and the right-endpoint approximation.

**Rule: Left-Endpoint Approximation**
On each subinterval $[x_{i-1}, x_i]$ (for $i = 1, 2, 3, \ldots, n$), construct a rectangle with width $\Delta x$ and height equal to $f(x_{i-1})$, which is the function value at the left endpoint of the subinterval. Then the area of this rectangle is $f(x_{i-1})\Delta x$. Adding the areas of all these rectangles, we get an approximate for $A$. We use the notation $L_n$ to denote that this is a left-endpoint approximation of $A$ using $n$ subintervals.

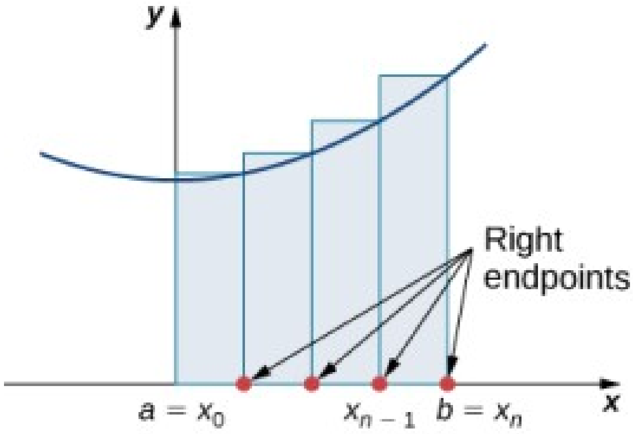
$$\begin{aligned} A \approx L_n &= f(x_0)\Delta x + f(x_1)\Delta x + \cdots + f(x_{n-1})\Delta x \\ &= \sum_{i=1}^{n} f(x_{i-1})\Delta x \end{aligned}$$




**Rule: Right-Endpoint Approximation**
Construct a rectangle on each subinterval $[x_i - 1, x_i]$, only this time the height of the rectangle is determined by the function value $f(x_i)$ at the right endpoint of the subinterval. Then, the area of each rectangle is $f(x_i)\Delta x$ and the approximation for $A$ is given by

$$\begin{aligned} A \approx R_n &= f(x_1)\Delta x + f(x_2)\Delta x + \cdots + f(x_n)\Delta x \\ &= \sum_{i=1}^{n} f(x_i)\Delta x \end{aligned}$$

The notation $R_n$ indicates this is a right-endpoint approximation for $A$ in the following figure

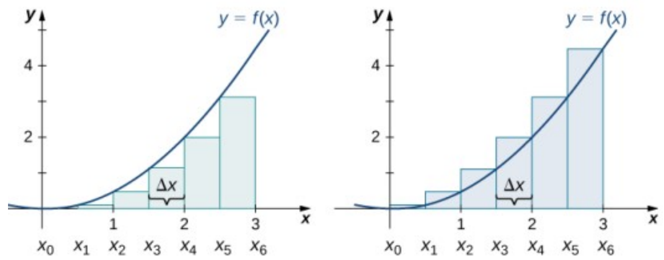



**Example 7.1.1.** *Find area under* $f(x) = \dfrac{x^2}{2}$ *on* $[0,3]$ *with* $n = 6$

***Solution:****we divide the region represented by the interval* $[0,3]$ *into six subintervals, each of width* 0.5. *Thus,* $\Delta x = 0.5$. *We then form six rectangles by drawing vertical lines perpendicular to* $x_{i-1}$, *the left endpoint of each subinterval.*

*We determine the height of each rectangle by calculating* $f(x_{i-1})$ *for* $i = 1,2,3,4,5,6$. *The intervals are* $[0,0.5], [0.5,1], [1,1.5], [1.5,2], [2,2.5], [2.5,3]$. *We find the area of each rectangle by multiplying the height by the width. Then, the sum of the rectangular areas approximates the area between* $f(x)$ *and the* $x$ *-axis. When the left endpoints are used to calculate height, we have a left-endpoint approximation.*

$$
\begin{aligned}
A \approx L_6 &= \sum_{i=1}^{6} f(x_{i-1})\,\Delta x \\
&= f(x_0)\,\Delta x + f(x_1)\,\Delta x + f(x_2)\,\Delta x + f(x_3)\,\Delta x + f(x_4)\,\Delta x + f(x_5)\,\Delta x \\
&= f(0)0.5 + f(0.5)0.5 + f(1)0.5 + f(1.5)0.5 + f(2)0.5 + f(2.5)0.5 \\
&= (0)0.5 + (0.125)0.5 + (0.5)0.5 + (1.125)0.5 + (2)0.5 + (3.125)0.5 \\
&= 0 + 0.0625 + 0.25 + 0.5625 + 1 + 1.5625 \\
&= 3.4375
\end{aligned}
$$




*This is a right-endpoint approximation of the area under $f(x)$.*

$$\begin{aligned}
A \approx R_6 &= \sum_{i=1}^{6} f(x_i)\,\Delta x \\
&= f(x_1)\,\Delta x + f(x_2)\,\Delta x + f(x_3)\,\Delta x + f(x_4)\,\Delta x + f(x_5)\,\Delta x + f(x_6)\,\Delta x \\
&= f(0.5)0.5 + f(1)0.5 + f(1.5)0.5 + f(2)0.5 + f(2.5)0.5 + f(3)0.5 \\
&= (0.125)0.5 + (0.5)0.5 + (1.125)0.5 + (2)0.5 + (3.125)0.5 + (4.5)0.5 \\
&= 0.0625 + 0.25 + 0.5625 + 1 + 1.5625 + 2.25 \\
&= 5.6875
\end{aligned}$$

**Example 7.1.2.** *Use both left-endpoint and right-endpoint approximations to approximate the area under the curve of $f(x) = x^2$ on the interval $[0,2]$; use $n = 4$.*

***The solution is up to the student***

**Example 7.1.3.** *Sketch left-endpoint and right-endpoint approximations for $f(x) = \dfrac{1}{x}$ on $[1,2]$; use $n = 4$. Approximate the area using both methods*

***The solution is up to the student***

### 7.1.1 Forming Riemann Sums

So far we have been using rectangles to approximate the area under a curve. The heights of these rectangles have been determined by evaluating the function at either the right or left endpoints of the subinterval $[x_{i-1}, x_i]$. In reality, there is no reason to restrict evaluation of the function to one of these two points only. We could evaluate the function at any point $c_i$ in the subinterval $[x_{i-1}, x_i]$, and use $f\left(x_i^*\right)$ as the height of our rectangle. This gives us an estimate for the area of the form

$$A \approx \sum_{i=1}^{n} f\left(x_i^*\right)\Delta x$$

A sum of this form is called a Riemann sum, named for the 19th-century mathematician Bernhard Riemann, who developed the idea.

Let $f(x)$ be defined on a closed interval $[a, b]$ and let $P$ be a regular partition of $[a, b]$. Let $\Delta x$ be the width of each subinterval $[x_{i-1}, x_i]$ and for each $i$, let $x_i^*$ be any point in $[x_{i-1}, x_i]$ . A Riemann sum is defined for $f(x)$ as

$$\sum_{i=1}^{n} f\left(x_i^*\right) \Delta x$$

Recall that with the left- and right-endpoint approximations, the estimates seem to get better and better as n get larger and larger. The same thing happens with Riemann sums. Riemann sums give better approximations for larger values of n. We are now ready to define the area under a curve in terms of Riemann sums.

Let $f(x)$ be a continuous, nonnegative function on an interval $[a, b]$, and let $\sum_{i=1}^{n} f\left(x_i^*\right) \Delta x$ be a Riemann sum for $f(x)$. Then, the area under the curve $y = f(x)$ on $[a, b]$ is given by

$$A = \lim_{n \to \infty} \sum_{i=1}^{n} f\left(x_i^*\right) \Delta x$$

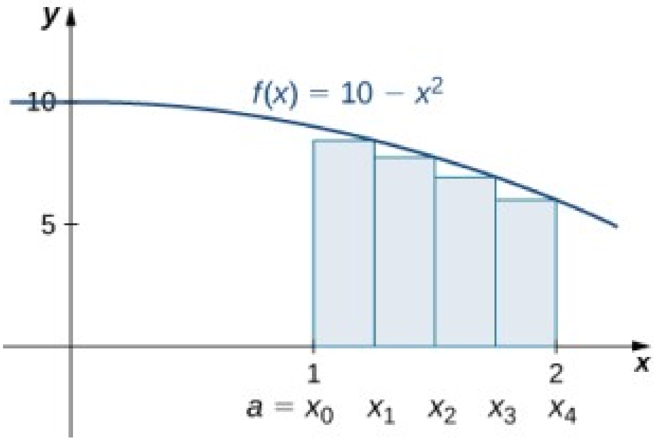
We look at some examples shortly. But, before we do, let's take a moment and talk about some specific choices for $\{x \neq\}$ Although any choice for $\left\{x_i^*\right\}$ gives us an estimate of the area under the curve, we don't necessarily know whether that estimate is too high (overestimate) or too low (underestimate). If it is important to know whether our estimate is high or low, we can select our value for $\left\{x_i^*\right\}$ to guarantee one result or the other.

If we want an overestimate, for example, we can choose $\left\{x_i^*\right\}$ such that for $i = 1, 2, 3, \ldots, n, f\left(x_i^*\right) \geq f(x)$ for all $x \in [x_{i-1}, x_i]$ . In other words, we choose $\left\{x_i^*\right\}$ so that for $i = 1, 2, 3, \ldots, n, f\left(x_i^*\right)$ is the maximum function value on the interval $[x_{i-1}, x_i]$ . If we select $\left\{x_i^*\right\}$ in this way, then the Riemann sum $\sum_{i=1}^{n} f\left(x_i^*\right) \Delta x$ is called an **upper sum**.

Similarly, if we want an underestimate, we can choose $\left\{x_i^*\right\}$ so that for $i = 1, 2, 3, \ldots, n, f\left(x_i^*\right)$ is the minimum function value on the interval $[x_{i-1}, x_i]$ . In this case, the associated Riemann sum is called a **lower sum**. Note that if $f(x)$ is either increasing or decreasing throughout the interval $[a, b]$, then the maximum and minimum values of the function occur at the endpoints of the subintervals, so the upper and lower sums are just the same as the left-and right-endpoint approximations.

**Example 7.1.4.** *Find a lower sum for $f(x) = 10 - x^2$ on $[1, 2]$; let $n = 4$ subintervals.*

***Solution:*** *With* $n = 4$ *over the interval* $[1, 2]$, $\Delta x = \frac{1}{4}$. *We can list intervals as* $[1, 1.25], [1.25, 1.5][1.5, 1.7$ *Because the function is decreasing over the interval* $[1, 2]$, *the following figure shows that a lower sum is obtained by using the right endpoints. The Riemann sum is*




$$\begin{aligned}\sum_{k=1}^{4}\left(10-x^2\right)(0.25) &= 0.25\left[10-(1.25)^2+10-(1.5)^2+10-(1.75)^2+10-(2)^2\right]\\ &= 0.25[8.4375+7.75+6.9375+6]\\ &= 7.28\end{aligned}$$

## 7.2 The Definite Integral

In the preceding section we defined the area under a curve in terms of Riemann sums:

$$A = \lim_{n\to\infty}\sum_{i=1}^{n} f\left(x_i^*\right)\Delta x$$

However, this definition came with restrictions. We required $f(x)$ to be continuous and nonnegative. Unfortunately, realworld problems don't always meet these restrictions. In this section, we look at how to apply the concept of the area under the curve to a broader set of functions through the use of the definite integral.

If $f(x)$ is a function defined on an interval $[a,b]$, the definite integral of $f$ from $a$ to $b$ is given by

$$\int_a^b f(x)dx = \lim_{n\to\infty}\sum_{i=1}^{n} f\left(x_i^*\right)\Delta x$$

provided the limit exists. If this limit exists, the function $f(x)$ is said to be integrable on $[a,b]$, or is an integrable function.

**Example 7.2.1.** *Use the definition of the definite integral to evaluate* $\int_0^2 x^2dx$. *Use a right-endpoint approximation to generate the Riemann sum.*

***Solution:*** *We first want to set up a Riemann sum. Based on the limits of integration, we have* $a = 0$ *and* $b = 2$. *For* $i = 0, 1, 2, \ldots, n$, *let* $P = \{x_i\}$ *be a regular partition of* $[0, 2]$. *Then*

$$\Delta x = \frac{b-a}{n} = \frac{2}{n}$$

*since we are using a right-endpoint approximation to generate Riemann sums, for each* $i$, *we need to calculate the function value at the right endpoint of the interval* $[x_{i-1}, x_i]$. *The right endpoint of the interval is* $x_i$, *and since P is a regular partition,*

$$x_i = x_0 + i\Delta x = 0 + i\left[\frac{2}{n}\right] = \frac{2i}{n}$$

*Thus, the function value at the right endpoint of the interval is*

$$f(x_i) = x_i^2 = \left(\frac{2i}{n}\right)^2 = \frac{4i^2}{n^2}$$

*Then the Riemann sum takes the form*

$$\begin{aligned}
\sum_{i=1}^{n} f(x_i)\,\Delta x &= \sum_{i=1}^{n} \left(\frac{4i^2}{n^2}\right)\frac{2}{n} = \sum_{i=1}^{n} \frac{8i^2}{n^3} \\
&= \frac{8}{n^3}\sum_{i=1}^{n} i^2 \\
&= \frac{8}{n^3}\left[\frac{n(n+1)(2n+1)}{6}\right] \\
&= \frac{8}{n^3}\left[\frac{2n^3+3n^2+n}{6}\right] \\
&= \frac{16n^3+24n^2+n}{6n^3} \\
&= \frac{8}{3} + \frac{4}{n} + \frac{1}{6n^2}
\end{aligned}$$

*Now, to calculate the definite integral, we need to take the limit as $n \to \infty$. We get*

$$\begin{aligned}
\int_0^2 x^2 dx &= \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\,\Delta x \\
&= \lim_{n\to\infty} \left(\frac{8}{3} + \frac{4}{n} + \frac{1}{6n^2}\right) \\
&= \lim_{n\to\infty} \left(\frac{8}{3}\right) + \lim_{n\to\infty} \left(\frac{4}{n}\right) + \lim_{n\to\infty} \left(\frac{1}{6n^2}\right) \\
&= \frac{8}{3} + 0 + 0 = \frac{8}{3}
\end{aligned}$$

The definite integral is often called, more simply, the integral of f over $[a, b]$. The process of computing integrals is called integration and $f(x)$ is called the integrand. The endpoints a and b of $[a, b]$ are called the limits of integration. Finally, we remark that any variable may be used as a variable of integration (this is a "dummy" variable). Thus, the following three integrals all denote the same quantity:

$$\int_a^b \sin x dx, \quad \int_a^b \sin t dt, \quad \int_a^b \sin u du$$

## 7.3 Interpretation of the Definite Integral as Signed Area

When $f(x) \geq 0$, the definite integral defines the area under the graph. To interpret the integral when $f(x)$ takes on both positive and negative values, we define the notion of signed area, where regions below the $x$ -axis are considered to have "negative area" that is,

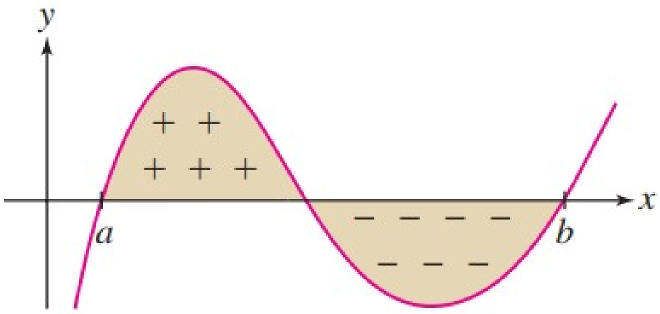



FIGURE 7.1: Signed area is the area above the $x-$axis minus the area below the $x-$axis

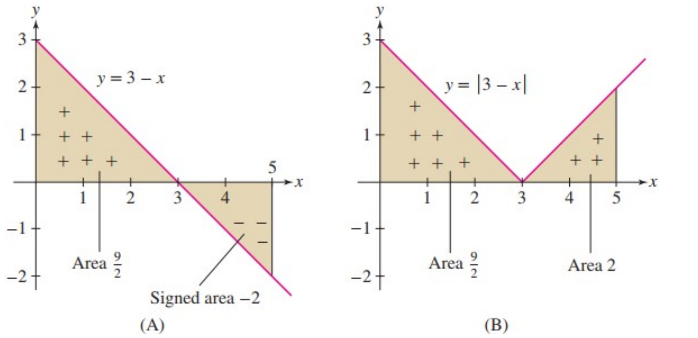
| |
|---|
| Signed area of a region $=$ ( area above $x$ -axis) $-$ (area below $x$ -axis) |

**Example 7.3.1.** *Calculate*

$$\int_0^5 (3-x)dx \quad \text{and} \quad \int_0^5 |3-x|dx$$

*The region between $y=3-x$ and the $x$ -axis consists of two triangles of areas $\frac{9}{2}$ and $2$[ Figure (A)]. However, the second triangle lies below the $x$ -axis, so it has signed area $-2$. In the graph of $y=|3-x|$, both triangles lie above the $x$ -axis [Figure (B)] Therefore,*

$$\int_0^5 (3-x)dx = \frac{9}{2} - 2 = \frac{5}{2} \quad \int_0^5 |3-x|dx = \frac{9}{2} + 2 = \frac{13}{2}$$



**Properties of definite integrals:**

1. $\int_a^b (f(x)+g(x))dx = \int_a^b f(x)dx + \int_a^b g(x)dx$
2. $\int_a^b Cf(x)dx = C\int_a^b f(x)dx$ for any constant $C$
3. $\int_b^a f(x)dx = -\int_a^b f(x)dx$
4. $\int_a^a f(x)dx = 0$
5. $\int_a^b f(x)dx + \int_b^c f(x)dx = \int_a^c f(x)dx$ for all $a, b, c$
6. Comparison Theorem: If $f(x) \leq g(x)$ on $[a, b]$, then
$$\int_a^b f(x)dx \leq \int_a^b g(x)dx$$
7. $\int_{-a}^a f(x)dx = \begin{cases} 2\int_0^a f(x)dx, & \text{if } f(x) \text{ even} \\ 0, & \text{if } f(x) \text{ odd.} \end{cases}$

**Example 7.3.2.** *Evaluate*

$$\int_1^4 \left(\frac{1}{x} - x^2\right) dx$$

***Solution:***

$$\begin{aligned}\int_1^4 \left(\frac{1}{x} - x^2\right) dx &= \left(\ln|x| - \frac{1}{3}x^3\right)\Big|_1^4 \\ &= \left[\ln 4 - \frac{1}{3}(4)^3\right] - \left[\ln 1 - \frac{1}{3}(1)^3\right] \\ &= \ln 4 - 21 \approx -19.6137\end{aligned}$$

**Example 7.3.3.** *Let $f(x)$ and $g(x)$ be functions that are continuous on the interval $-2 \leq x \leq 5$ and that satisfy*

$$\int_{-2}^5 f(x)dx = 3 \quad \int_{-2}^5 g(x)dx = -4 \quad \int_3^5 f(x)dx = 7$$

*Use this information to evaluate each of these definite integrals:*

$$a. \int_{-2}^{5} [2f(x) - 3g(x)]dx \quad b. \int_{-2}^{3} f(x)dx$$

***Solution:***
*(a)By combining the difference rule and constant multiple rule and substituting the given information, we find that*

$$\begin{aligned}
\int_{-2}^{5} [2f(x) - 3g(x)]dx &= \int_{-2}^{5} 2f(x)dx - \int_{-2}^{5} 3g(x)dx \\
&= 2\int_{-2}^{5} f(x)dx - 3\int_{-2}^{5} g(x)dx \\
&= 2(3) - 3(-4) = 18
\end{aligned}$$

*(b) According to the subdivision rule*

$$\int_{-2}^{5} f(x)dx = \int_{-2}^{3} f(x)dx + \int_{3}^{5} f(x)dx$$

*Solving this equation for the required integral* $\int_{-2}^{3} f(x)dx$ *and substituting the given information, we obtain*

$$\begin{aligned}
\int_{-2}^{3} f(x)dx &= \int_{-2}^{5} f(x)dx - \int_{3}^{5} f(x)dx \\
&= 3 - 7 = -4
\end{aligned}$$

**Example 7.3.4.** *Evaluate*

$$\int_{1/4}^{2} \left(\frac{\ln x}{x}\right) dx$$

***Solution:***
*use the substitution* $u = \ln x$ *and* $du = \frac{dx}{x}$ *to transform the limits of integration:*

$$\begin{aligned}
&\text{when } x = \frac{1}{4}, \text{ then } u = \ln\frac{1}{4} \\
&\text{when } x = 2, \text{ then } u = \ln 2
\end{aligned}$$

*Substituting, we find*

$$\begin{aligned}
\int_{1/4}^{2} \frac{\ln x}{x} dx &= \int_{\ln 1/4}^{\ln 2} u du = \frac{1}{2}u^2 \Big|_{\ln 1/4}^{\ln 2} \\
&= \frac{1}{2}(\ln 2)^2 - \frac{1}{2}\left(\ln \frac{1}{4}\right)^2 \approx -0.721
\end{aligned}$$

**Example 7.3.5.** *Prove the inequality*

$$\int_1^4 \frac{1}{x^2} dx \leq \int_1^4 \frac{1}{x} dx$$

***Solution:***
*If $x \geq 1$, then $x^2 \geq x$, and $x^{-2} \leq x^{-1}$ . Therefore, the inequality follows from the Comparison Theorem, applied with $g(x) = x^{-2}$ and $f(x) = x^{-1}$*

### 7.3.1 Exercises

**Use properties of the integral to calculate the integrals**

1- $\displaystyle\int_0^4 (6t-3)dt$ 2- $\displaystyle\int_{-3}^{2} (4x+7)dx$

3- $\displaystyle\int_0^9 x^2dx$ 4- $\int_{-3}^{3} \left(9x-4x^2\right) dx$

5- $\displaystyle\int_{-a}^{1} \left(x^2+x\right) dx$ 6- $\displaystyle\int_{-3}^{1} \left(7t^2+t+1\right) dt$

**Calculate the integral, assuming that** $\displaystyle\int_0^5 f(x)dx = 5,\quad \int_0^5 g(x)dx = 12$

7- $\displaystyle\int_0^5 (f(x)+g(x))dx$ 8- $\displaystyle\int_0^5 \left(2f(x)-\frac{1}{3}g(x)\right) dx$

9- `$\displaystyle\int_{5}^{0}g(x)dx` 10- $\displaystyle\int_0^5 (f(x)-x)dx$

**Express each integral as a single integral**

11- $\displaystyle\int_0^3 f(x)dx + \int_3^7 f(x)dx$ 12- $\displaystyle\int_2^9 f(x)dx - \int_4^9 f(x)dx$

13- $\displaystyle\int_2^9 f(x)dx - \int_2^5 f(x)dx$ 14- $\displaystyle\int_7^3 f(x)dx + \int_3^9 f(x)dx$

**Calculate the integral, assuming that $f$ is integrable and**

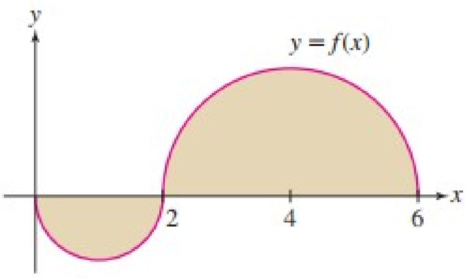
$\displaystyle\int_1^b f(x)dx = 1-b^{-1}$ for all $b>0$

15- $\displaystyle\int_1^5 f(x)dx$ 16- $\displaystyle\int_3^5 f(x)dx$

17- $\displaystyle\int_1^6 (3f(x)-4)dx$ 18- $\displaystyle\int_{1/2}^{1} f(x)dx$

19- Evaluate: (a) $\displaystyle\int_0^2 f(x)dx$ \quad (b) $\displaystyle\int_0^6 f(x)dx$

20- Evaluate: $(a)$ $\displaystyle\int_1^4 f(x)dx$ \quad (b) $\displaystyle\int_1^6 |f(x)|dx$ Evaluate:

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## 7.4 Area of a Region between Two Curves

**Regions Defined with Respect to $x$**

Let $f(x)$ and $g(x)$ be continuous functions over an interval $[a,b]$ such that $f(x) \geq g(x)$ on $[a,b]$. We want to find the area between the graphs of the functions, as shown in the following figure. As we did before, we are going to partition the interval on the $x$ -axis and approximate

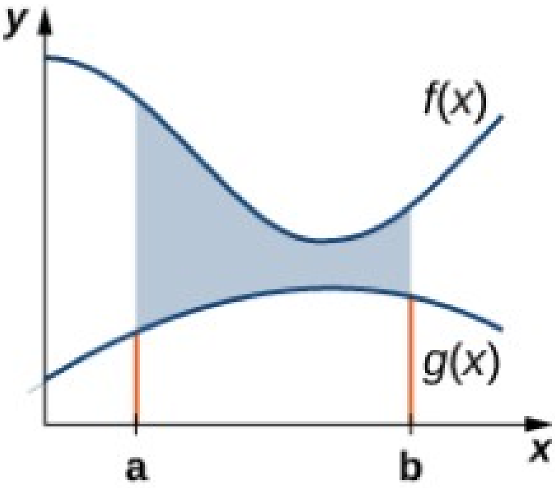



FIGURE 7.2: The area between the graphs of two functions, $f(x)$ and $g(x)$, on the interval $[a, b]$

the area between the graphs of the functions with rectangles. So, for $i = 0, 1, 2, \ldots, n$, let $P = \{x_i\}$ be a regular partition of $[a, b]$. Then, for $i = 1, 2, \ldots, n$, choose a point $x_i^* \in [x_{i-1}, x_i]$, and on each interval $[x_{i-1}, x_i]$ construct a rectangle that extends vertically from $g(x_i^*)$ to $f(x_i^*)$. The following figure (a) shows the rectangles when $x_i^*$ is selected to be the left endpoint of the . interval and $n = 10$. The following figure (b) shows a representative rectangle in detail. The

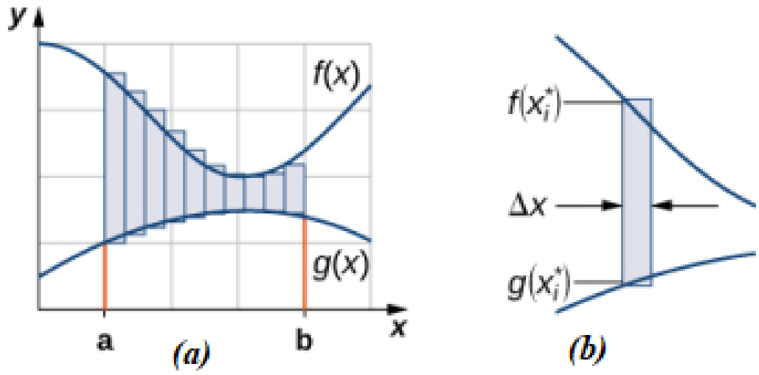



FIGURE 7.3

height of each individual rectangle is $f(x_i^*) - g(x_i^*)$ and the width of each rectangle is $\Delta x$. Adding the areas of all the rectangles, we see that the area between the curves is approximated by

$$A \approx \sum_{i=1}^{n} [f(x_i^*) - g(x_i^*)] \Delta x$$

This is a Riemann sum, so we take the limit as $n \to \infty$ and we get

$$A = \lim_{n\to\infty} \sum_{i=1}^{n} [f(x_i^*) - g(x_i^*)]\,\Delta x = \int_a^b [f(x) - g(x)]dx$$

These findings are summarized in the following theorem.

**Theorem 7.1.** *Let $f(x)$ and $g(x)$ be continuous functions such that $f(x) \geq g(x)$ over an interval $[a, b]$. Let $R$ denote the region bounded above by the graph of $f(x)$, below by the graph of $g(x)$, and on the left and right by the lines $x = a$ and $x = b$, respectively. Then, the area of $R$ is given by*

$$A = \int_a^b [f(x) - g(x)]dx$$

**Regions Defined with Respect to $y$**

Let $u(y)$ and $v(y)$ be continuous functions over an interval $[c, d]$ such that $u(y) \geq v(y)$ for all $y \in [c, d]$. We want to find the area between the graphs of the functions, as shown in the following figure. This time, we are going to partition the interval on the $y$ -axis and use

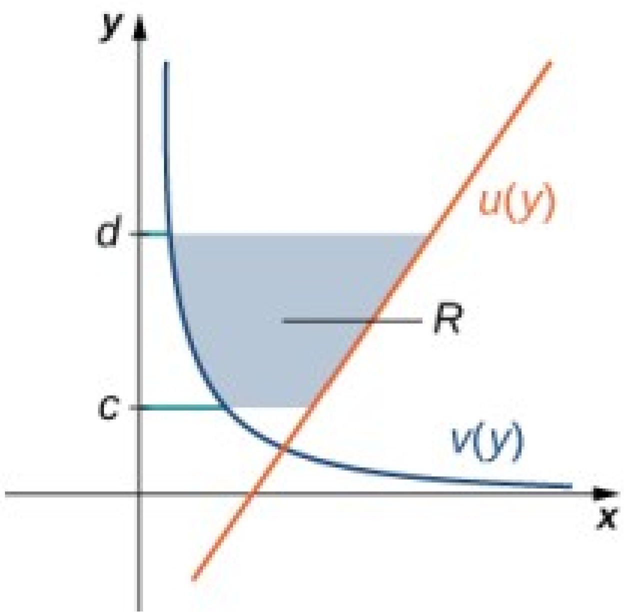



FIGURE 7.4

horizontal rectangles to approximate the area between the functions. So, for $i = 0, 1, 2, \ldots, n$, let $Q = \{y_i\}$ be a regular partition of $[c, d]$. Then, for $i = 1, 2, \ldots, n$, choose a point $y_i^* \in$

$[y_{i-1}, y_i]$ , then over each interval $[y_{i-1}, y_i]$ construct a rectangle that extends horizontally from $v(y_i^*)$ to $u(y_i^*)$. The following figure (a) shows the rectangles when $y_i^*$ is selected to be the lower endpoint of the interval and $n = 10$ The following figure (b) shows a representative rectangle in detail.

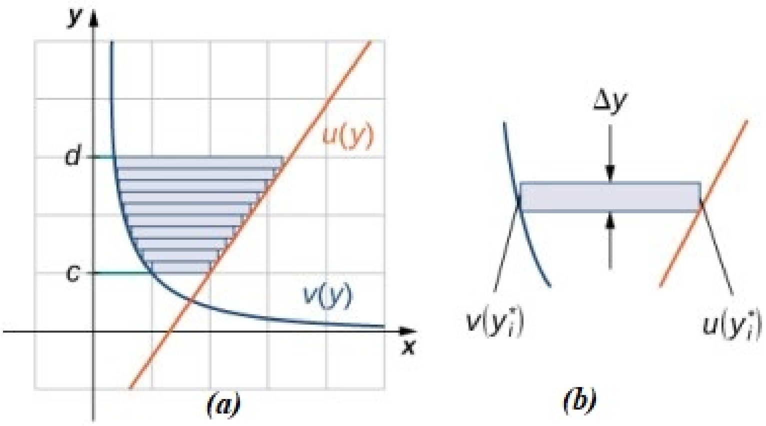



FIGURE 7.5

The height of each individual rectangle is $\Delta y$ and the width of each rectangle is $u(y_i^*) - v(y_i^*)$ . Therefore, the area between the curves is approximately

$$A \approx \sum_{i=1}^{n} [u(y_i^*) - v(y_i^*)] \Delta y$$

This is a Riemann sum, so we take the limit as $n \to \infty$, obtaining

$$A = \lim_{n\to\infty} \sum_{i=1}^{n} [u(y_i^*) - v(y_i^*)] \Delta y = \int_c^d [u(y) - v(y)]dy$$

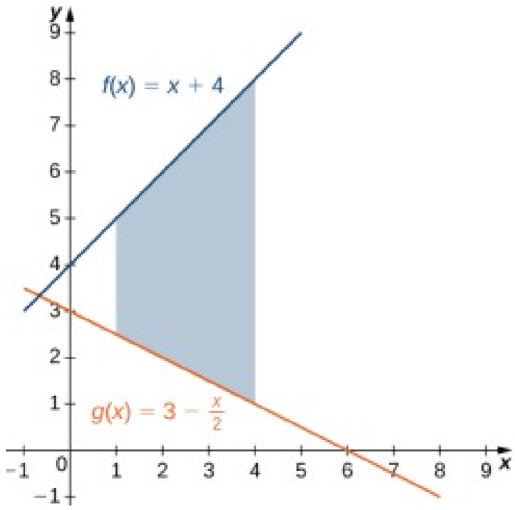
These findings are summarized in the following theorem.

**Theorem 7.2.** *Let $u(y)$ and $v(y)$ be continuous functions such that $u(y) \geq v(y)$ for all $y \in [c, d]$. Let $R$ denote the region bounded on the right by the graph of $u(y)$, on the left by the graph of $v(y)$, and above and below by the lines $y = d$ and $y = c$, respectively. Then, the area of $R$ is given by*

$$A = \int_c^d [u(y) - v(y)]dy$$

**Example 7.4.1.** *If $R$ is the region bounded above by the graph of the function $f(x) = x + 4$ and by the graph of the function $g(x) = 3 - \frac{x}{2}$ over the interval $[1, 4]$, find the area of $R$*

***Solution:***
*The region is depicted in the following figure. The area given by*




$$
\begin{aligned}
A &= \int_a^b [f(x) - g(x)]dx \\
&= \int_1^4 \left[(x+4) - \left(3 - \frac{x}{2}\right)\right] dx = \int_1^4 \left[\frac{3x}{2} + 1\right] dx \\
&= \left[\frac{3x^2}{4} + x\right]\Bigg|_1^4 = \left(16 - \frac{7}{4}\right) = \frac{57}{4}
\end{aligned}
$$

**Example 7.4.2.** *If R is the region bounded above by the graph of the function $f(x) = 9 - (x/2)^2$ and below by the graph of the function $g(x) = 6 - x$, find the area of region R*

***Solution:****The region is depicted in the following figure. We first need to compute where the graphs of the functions intersect. Setting f (x) = g(x), we get*

$$
\begin{aligned}
f(x) &= g(x) \\
9 - \left(\frac{x}{2}\right)^2 &= 6 - x \\
9 - \frac{x^2}{4} &= 6 - x \\
36 - x^2 &= 24 - 4x \\
x^2 - 4x - 12 &= 0 \\
(x-6)(x+2) &= 0
\end{aligned}
$$

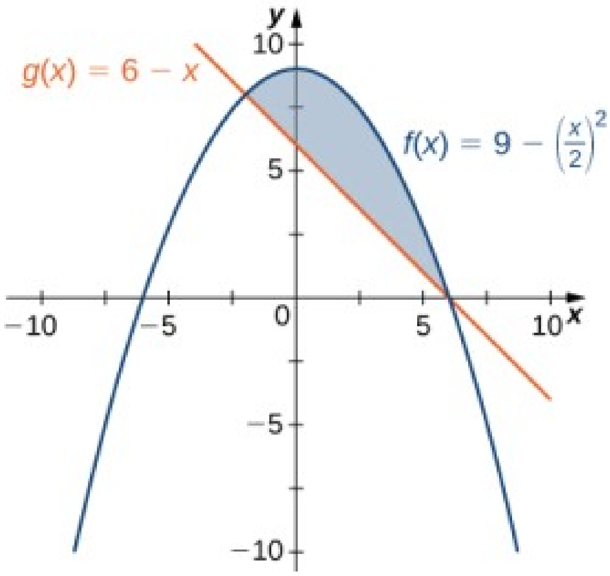



FIGURE 7.6

*The graphs of the functions intersect when* $x = 6$ *or* $x = -2$*, so we want to integrate from* $-2$ *to* $6$*. since* $f(x) \geq g(x)$ *for* $-2 \leq x \leq 6$*, we obtain The area given by*

$$\begin{aligned} A &= \int_a^b [f(x) - g(x)]dx \\ &= \int_{-2}^{6} \left[9 - \left(\frac{x}{2}\right)^2 - (6-x)\right] dx = \int_{-2}^{6} \left[3 - \frac{x^2}{4} + x\right] dx \\ &= \left[3x - \frac{x^3}{12} + \frac{x^2}{2}\right]\Bigg|_{-2}^{6} = \frac{64}{3} \end{aligned}$$

**Example 7.4.3.** *If R is the region between the graphs of the functions* $f(x) = \sin x$ *and* $g(x) = \cos x$ *over the interval* $[0, \pi]$ *find the area of region R.*

***Solution:****The region is depicted in the following figure. The graphs of the functions intersect at* $x = \pi/4$*. For* $x \in [0, \pi/4]$, $\quad \cos x \geq \sin x$*, so*

$$[f(x) - g(x)] = |\sin x - \cos x| = \cos x - \sin x$$

*On the other hand, for* $x \in [\pi/4, \pi]$, $\quad \sin x \geq \cos x$*, so*

$$|f(x) - g(x)| = |\sin x - \cos x| = \sin x - \cos x$$

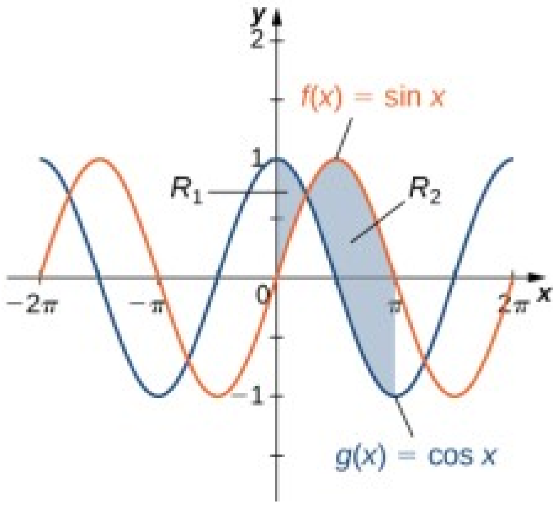



FIGURE 7.7

*Then*

$$\begin{aligned} A &= \int_a^b |f(x) - g(x)| dx \\ &= \int_0^{\pi} |\sin x - \cos x| dx \\ &= \int_0^{x/4} (\cos x - \sin x) dx + \int_{\pi/4}^{\pi} (\sin x - \cos x) dx \\ &= [\sin x + \cos x]|_0^{x/4} + [-\cos x - \sin x]|_{x/4}^{x} \\ &= (\sqrt{2} - 1) + (1 + \sqrt{2}) = 2\sqrt{2} \end{aligned}$$

**Example 7.4.4.** *Find the area of the region R enclosed by the curves* $y = x^3$ *and* $y = x^2$

***Solution:*** *To find the points where the curves intersect, solve the equations simultaneously as follows:*

$$\begin{aligned} x^3 &= x^2 \\ x^3 - x^2 &= 0 \\ x^2(x-1) &= 0 \\ x &= 0, 1 \end{aligned}$$

*The corresponding points* $(0,0)$ *and* $(1,1)$ *are the only points of intersection. The region R enclosed by the two curves is bounded above by* $y = x^2$ *and below by* $y = x^3$*, over the interval* $0 \leq x \leq 1$*. The*

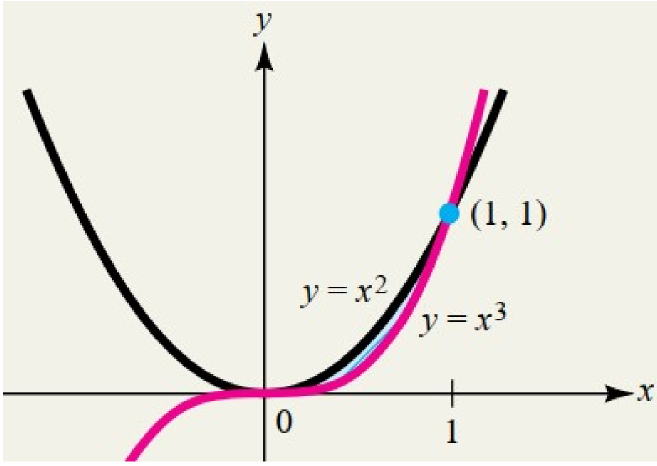



FIGURE 7.8

*area of this region is given by the integral*

$$\begin{aligned} A &= \int_0^1 \left(x^2 - x^3\right) dx \\ &= \frac{1}{3}x^3 - \frac{1}{4}x^4 \bigg|_0^1 \\ &= \left[\frac{1}{3}(1)^3 - \frac{1}{4}(1)^4\right] - \left[\frac{1}{3}(0)^3 - \frac{1}{4}(0)^4\right] \\ &= \frac{1}{12} \end{aligned}$$

**Example 7.4.5.** *Find the area of the region R is the region bounded by the curves $y = x^2$ $y = -x^2$, and the line $x = 1$*

***Solution:*** *To find the points where the curves intersect, solve the equations simultaneously as follows:*

$$\begin{aligned} x^2 &= -x^2 \\ 2x^2 &= 0 \\ x &= 0 \end{aligned}$$

*the intersection points are $(0,0)$, and $f(x) \geq g(x)$ on $[0,1]$ The area of this region is given by the*

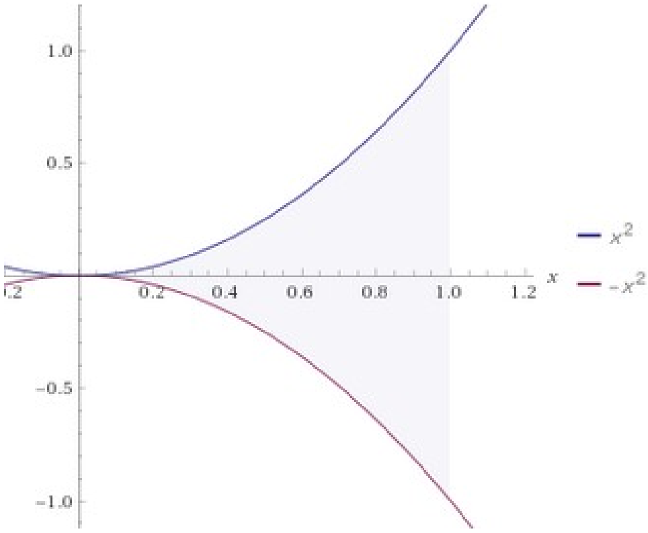



FIGURE 7.9

*integral*

$$\begin{aligned} Area &= \int_0^1 [f(x) - g(x)]dx \\ &= \int_0^1 \left[x^2 + x^2\right] dx \\ &= \frac{2}{3}x^3\Big|_0^1 \\ &= \frac{2}{3} \end{aligned}$$

**Example 7.4.6.** *Find the area of the region $R$ is the region bounded by the curves $f(x) = x^2$, $g(x) = 2 - x$ and $x-$axis*

***Solution:*** *The intersection points are $(0,0)$, $(1,1)$ and $(2,1)$ Over the interval $[0,1]$, the region is bounded above by $f(x) = x^2$ and below by the $x$ -axis, so we have*

$$\begin{aligned} A_1 &= \int_0^1 x^2 dx \\ &= \frac{x^3}{3}\Big|_0^1 \\ &= \frac{1}{3} \end{aligned}$$

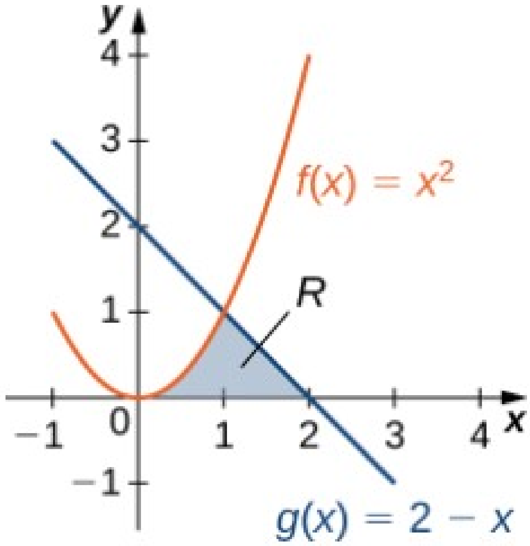



FIGURE 7.10

*Over the interval* $[1,2]$, *the region is bounded above by* $g(x) = 2 - x$ *and below by the* $x$ *-axis, so we have*

$$A_2 = \int_1^2 (2-x)dx = \left[2x - \frac{x^2}{2}\right]\Big|_1^2 = \frac{1}{2}$$

*Adding these areas together, we obtain*

$$A = A_1 + A_2 = \frac{1}{3} + \frac{1}{2} = \frac{5}{6}$$

**Remark 7.3.** *It is possible to solve the previous example by integrating with respect to y*

**Solution:** We must first express the graphs as functions of $y$. As we saw at the beginning of this section, the curve on the left can be represented by the function $x = v(y) = \sqrt{y}$, and the curve on the right can be represented by the function $x = u(y) = 2 - y$ .

Now we have to determine the limits of integration. The region is bounded below by the $x$ -axis, so the lower limit of integration is $y = 0$. The upper limit of integration is determined by the point where the two graphs intersect, which is the point $(1,1)$, so the upper limit of integration is $y = 1$. Thus, we have $[c, d] = [0, 1]$. Calculating the area of the region, we get

$$\begin{aligned} A &= \int_c^d [u(y) - v(y)dy \\ &= \int_0^1 [(2-y) - \sqrt{y}]dy \\ &= \left[2y - \frac{y^2}{2} - \frac{2}{3}y^{3/2}\right]\Big|_0^1 \\ &= \frac{5}{6} \end{aligned}$$

**Example 7.4.7.** *Find the area of the region R is the region bounded by the curves $y = 4 - x^2, \quad y = x^2 - 4$*

***Solution:*** *We will find where the curve intersect and it is calculated when $f(x) = g(x)$.*

$$\begin{aligned} f(x) &= g(x) \\ 4 - x^2 &= x^2 - 4 \\ 2\left(x^2 - 4\right) &= 0 \\ (x+2)(x-2) &= 0 \end{aligned}$$

*Hence $x = 2$ and $x = -2$ are point of intersection. First we must determine which curve is on the top. We get that $f(x)$ lies on the top of $g(x)$ implies $f(x) \geq g(x)$ on $[-2, 2]$. The area given by*

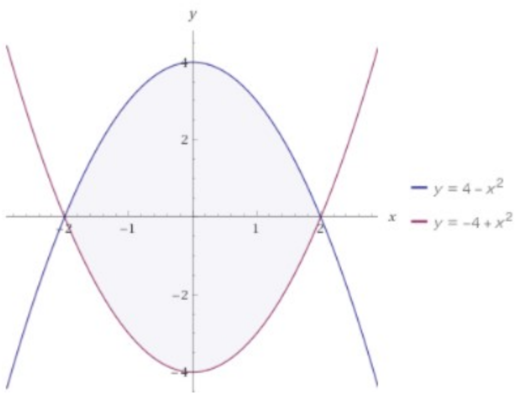



FIGURE 7.11

$$\begin{aligned}
A &= \int_{-2}^{2} (f(x) - g(x))dx \\
&= \int_{-2}^{2} \left( \left(4 - x^2\right) - \left(x^2 - 4\right) \right) dx \\
&= \int_{-2}^{2} \left( \left(4 - x^2\right) - \left(x^2 - 4\right) \right) dx \\
&= \int_{-2}^{2} \left(8 - 2x^2\right) dx \\
&= 2 \int_{0}^{2} \left(8 - 2x^2\right) dx \\
&= 2 \left[ 8x - \frac{2}{3}x^3 \right]_0^2 \\
&= \frac{64}{3}
\end{aligned}$$

**Example 7.4.8.** *Find the area of the region R is the region bounded by* $y = 8 - 3x, \quad y = 6 - x, \quad y = 2$ ***Solution:*** *We solve this example by integration with respect to* $y$*,*

$$\begin{aligned}
f(y) &= g(y) \\
\frac{1}{3}(8 - y) &= 6 - y \\
2y &= 10
\end{aligned}$$

*then intersection point is* $(1, 5)$ *and* $6 - y \geq \frac{1}{3}(8 - y)$ *for* $2 \leq y \leq 5$*. Hence area given by*

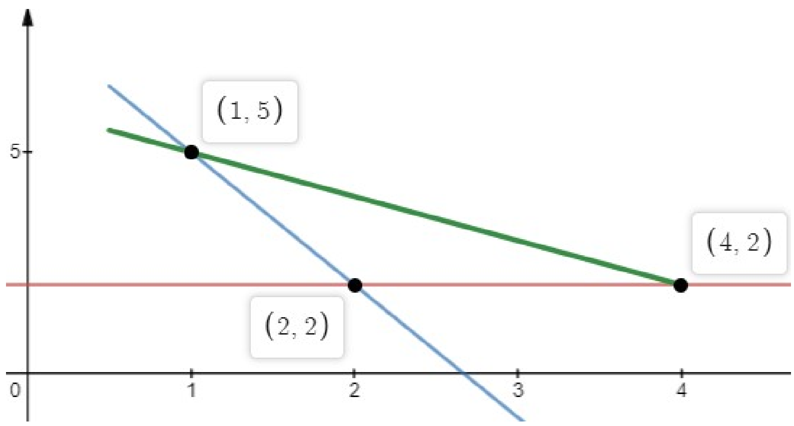



FIGURE 7.12

$$
\begin{aligned}
A &= \int_2^5 \left[6 - y - \frac{8}{3} + \frac{1}{3}y\right] dy \\
&= \int_2^5 \left[\frac{10}{3} - \frac{2}{3}y\right] dx \\
&= \left[\frac{10}{3}y - \frac{1}{3}y^2\right]_2^5 \\
&= \left[\frac{50}{3} - \frac{25}{3} - \frac{20}{3} + \frac{4}{3}\right] \\
&= 3
\end{aligned}
$$

**Example 7.4.9.** *Find the area of the region R is the region bounded by R is the region between the curve $y = x^3$ and the line $y = 9x$, for $x \geq 0$.*

***Solution:*** *since we have two curves $f(x) = 9x$, and $g(x) = x^3$, then*

$$
\begin{aligned}
x^3 &= 9x \\
x^3 - 9x &= 0 \\
x(x-3)(x+3) &= 0
\end{aligned}
$$

*The intersection points are $(0,0), (3,27), (-3,-27)$, we reject $x = -3$ and*

$$f(x) \geq g(x) \text{ on } [0,3]$$

*The area given by*

$$
\begin{aligned}
Area &= \int_0^3 [f(x) - g(x)]dx \\
&= \int_0^3 \left[9x - x^3\right] dx \\
&= \frac{9}{2}x^2 - \frac{1}{4}x^4\Big|_0^3 \\
&= \frac{81}{2} - \frac{81}{4} \\
&= \frac{81}{4}
\end{aligned}
$$

### 7.4.1 Exercises

For the following exercises, graph the equations and shade the area of the region between the curves. Determine its area by integrating over the $x-$axis.

1. $y = x^2$ and $y = -x^2 + 18x$

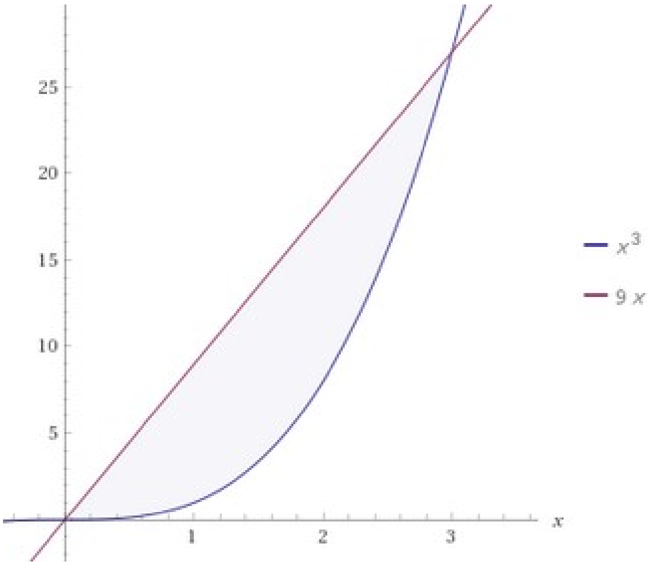



FIGURE 7.13

2. $y = \frac{1}{x}, y = \frac{1}{x^2}$, and $x = 3$

3. $y = \cos x$ and $y = \cos^2 x$ on $x = [-\pi, \pi]$

4. $y = e^x, y = e^{2x-1}$, and $x = 0$

5. $y = e^x, y = e^{-x}, x = -1$ and $x = 1$

For the following exercises, graph the equations and shade the area of the region between the curves. If necessary, break the region into sub-regions to determine its entire area.

1. $y = \sin(\pi x), y = 2x$, and $x > 0$

2. $y = 12 - x, y = \sqrt{x}$, and $y = 1$

3. $y = \sin x$ and $y = \cos x$ over $x = [-\pi, \pi]$

4. $y = x^3$ and $y = x^2 - 2x$ over $x = [-1, 1]$

5. $y = x^2 + 9$ and $y = 10 + 2x$ over $x = [-1, 3]$

6. $y = x^3 + 3x$ and $y = 4x$

For the following exercises, graph the equations and shade the area of the region between the curves. Determine its area by integrating over the $y-$axis

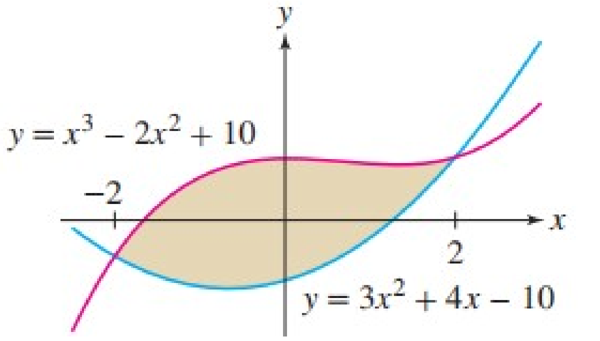
1. $x = y^3$ and $x = 3y - 2$

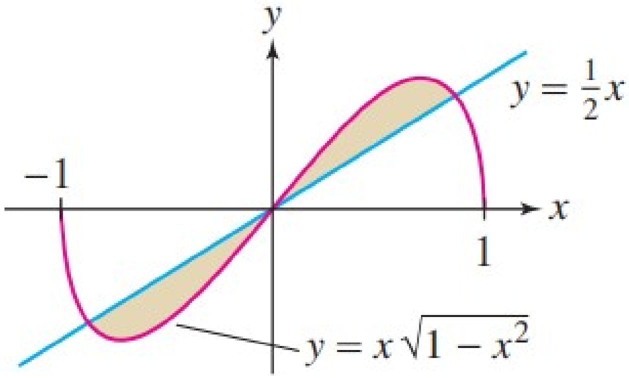
2. $x = 2y$ and $x = y^3 - y$

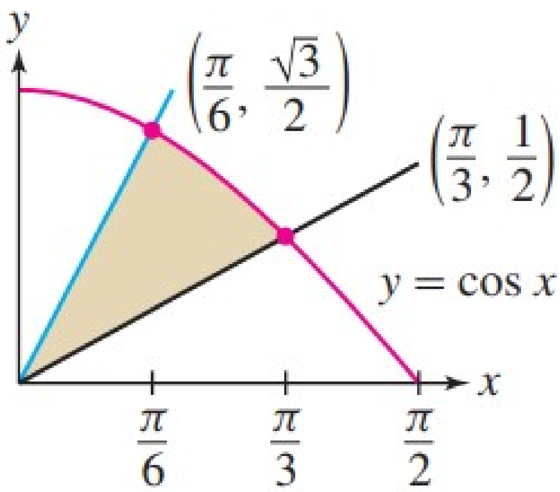
3. $x = -3 + y^2$ and $x = y - y^2$

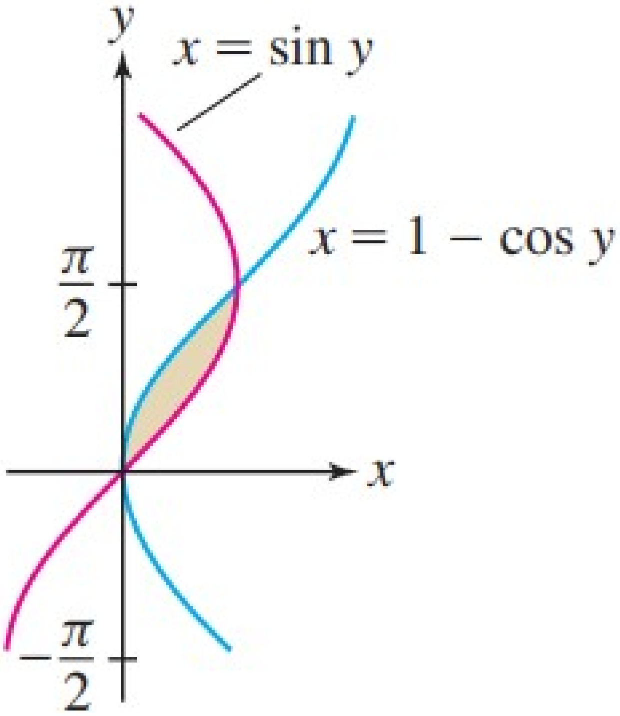
4. $y^2 = x$ and $x = y + 2$

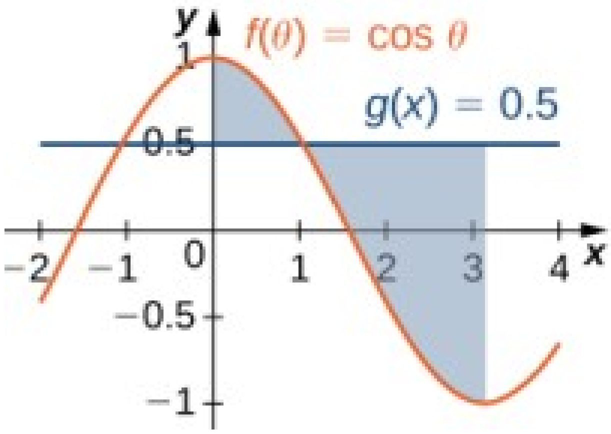
Find the area of the shaded region in Figure

1. 


2. 


3. 

4. 


5. 

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