CMP302: Design and Analysis of Algorithms Summary

Dynamic Programming

A faster technique to solve divide-and-conquer problems, usually decreases running time from **exponential** to **linear**.

Conditions for DP solution satisfaction

1. Optimal Substructure
    - An optimal solution to the problem can be constructed from the optimal solution of its subproblems.
2. Overlapping Subproblems
    - Subproblems solutions are re-used multiple times.

Types of DP solutions

A DP solution can be one of the following two types:

1. Top-down approach **(Memoization)**
    - Start from the big problem and work your way down, decomposing the big problem to multiple smaller ones.
    - Involves recursion which usually takes more memory space than bottom-up solutions.
    - Usually easier to implement than bottom-up solutions.
2. Bottom-up approach **(Tabulation)**
    - Start from the smallest problem and work your way up to the bigger one.
    - Iterative solution (with loops).
    - Saves memory space (no recursion required).
    - Harder to implement

Top-down solution steps

Any Top-down solution contains 3 major pillars.

1. Base case
    - The recursion termination condition
2. State
    - The base parameters of the recursion method.
    - Must be able to uniquely identify all the states of the recursion.
    - The size of the state impacts the time and memory complexity of the solution.
3. Transition
    - The relation between a problem and its subproblems.
    - The way to construct the bigger problem using its smaller subproblems.

When solving any DP problem we must figure out these 3 things, usually finding out the base case is trivial, we must then think about the state or the transition (whichever is easier).

Complexity of DP

- DP complexity = $O(\text{product of state size} * \text{function work})$

Example:
State = $[1000][5000]$
Complexity = $O(1000 * 5000 * \text{function work})$

Fibonacci

Fibonacci can be implemented recursively in a **naive** way as follows:

```
def fib(n):
  if(n == 1 or n == 2)
    return 1
  else
    return fib(n-1) + fib(n-2)
```

However, this implementation is not very well as it is $O(2^n)$.

We can improve this by adding **memoization**:

```
def fib(n):
  if(n == 1 or n == 2)
    return 1

  if(n in memo)
    return memo[n]

  memo[n] = fib[n-1] + fib[n-2]
  return memo[n]
```

This improves the complexity to $O(n)$, however this takes memory space $O(n)$.

This can also be implemented with **tabulation**:

```
def fib(n):
  fib[0] = 0
  fib[1] = 1
  for i in 2:n
    fib[i] = fib[i-1] + fib[i-2]
  return fib[n]
```

Rod Cutting

Cut rod of length n to maximize revenue based on a price table.

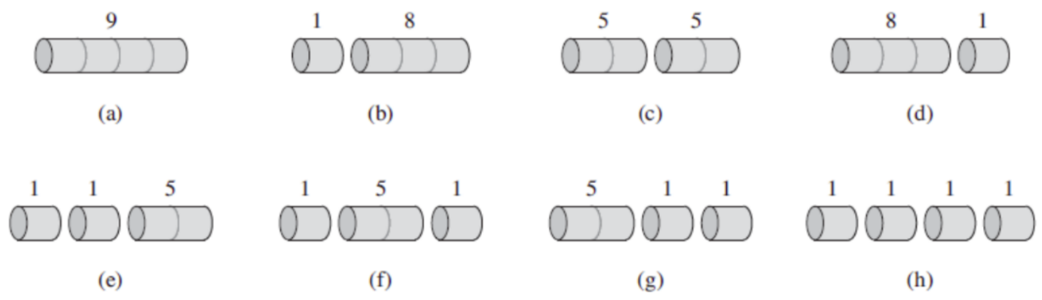
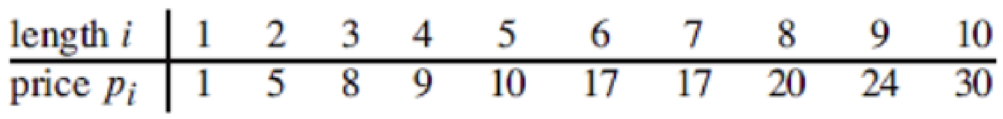


Figure 15.2 The 8 possible ways of cutting up a rod of length 4. Above each piece is the value of that piece, according to the sample price chart of Figure 15.1. The optimal strategy is part (c)—cutting the rod into two pieces of length 2—which has total value 10.



length i	1	2	3	4	5	6	7	8	9	10
price p_i	1	5	8	9	10	17	17	20	24	30

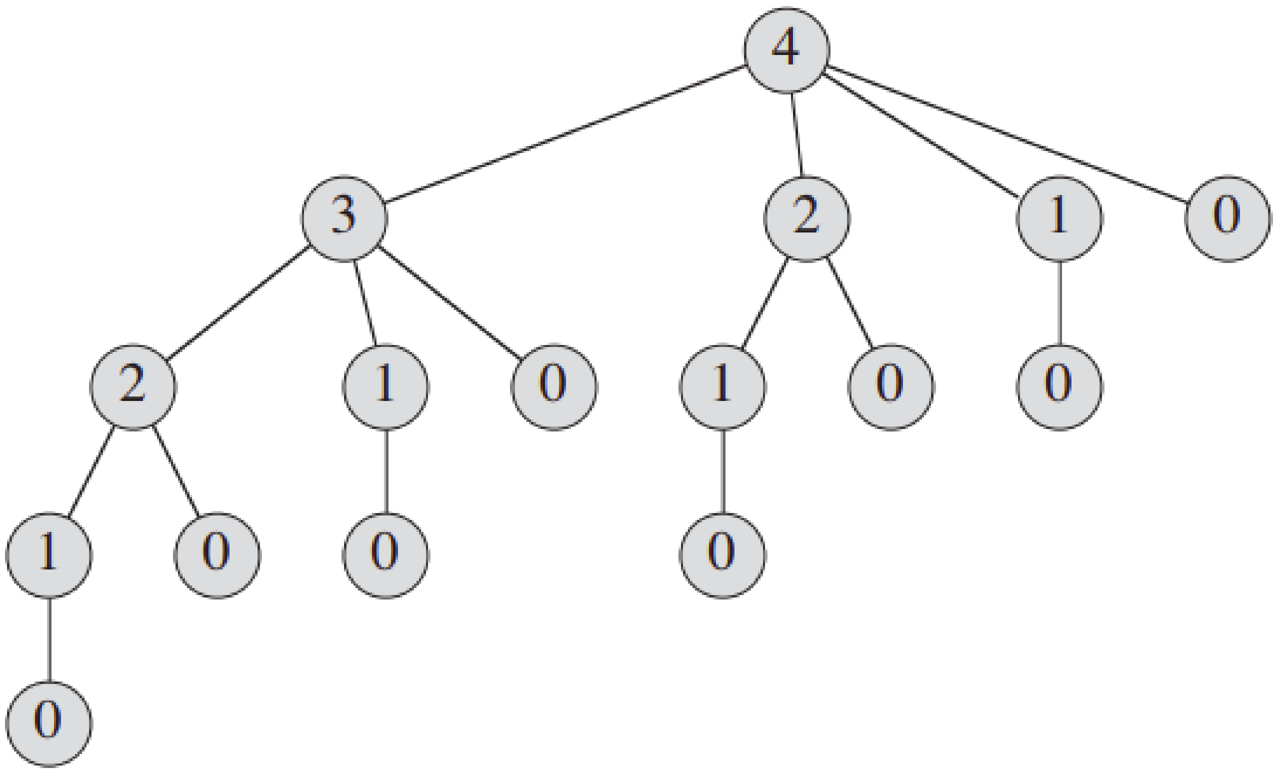
Can be solved naively by:

1. Cut a piece at the left end of the rod, and sell it.
2. Find an optimal way to cut the remainder of the rod.

```
def cut_rod(n, p): # p is array of prices of piece i
  if (n == 0)
    return 0

  q = -inf
  for i in 1:n
    q = max(q, p[i] + cut_rod(p, n - i))

  return q
```

- **Runtime:** $O(2^n)$

We can try to optimize this algorithm using dynamic programming, but first we need to check if a DP approach is applicable, by inspecting the recursion tree:



This tree has $O(2^n)$ leaves which corresponds to the solution complexity, however we can notice a lot of **overlapping subproblems**.

Implementing this with **memoization**:

```
def cut_rod(n, p):
  if (n == 0)
    return 0

  if (n in memo)
    return memo[n]

  q = -inf
  for i in 1:n
    q = max(q, p[i] + cut_rod(n - i, p))

  memo[n] = q
  return q
```

- **Runtime:** $\Theta(n^2)$

Note the similarities between the memoization solution and the naive solution, they are very similar, the memoization solution is basically the naive solution but with an extra memoization part that checks if a subproblem has been calculated before.

We can also start from smaller rods and build our answer up to the bigger rods, implementing this with **tabulation**:

```
def cut_rod(n, p):
  r[0..n] = new array
  r[0] = 0

  for j in 1:n
    q = -inf
    for i in 1:j
      q = max(q, p[i] + r[j - i])
    r[j] = q

  return r[n]
```

- **Runtime:** $\Theta(n^2)$

Explanation: $r[0..n]$ is an array containing the optimal answers for all cuts at any given iteration, for each rod length n we can define its optimal answer as $max_{i=0}^{i=n}(p[i] + r[n-i])$.
We attempt cutting a piece of length i and add it to the optimal answer of $n - i$ for all possible i.

Finding optimal cuts

We have previously calculated the maximum profit of the rod cutting problem, we have yet to find the optimal way of cutting. We can achieve that by creating a new array and for a given n store the length of the piece cut at the left of the rod.

i	0	1	2	3	4	5	6	7	8	9	10
$r[i]$	0	1	5	8	10	13	17	18	22	25	30
$s[i]$	0	1	2	3	2	2	6	1	2	3	10

```
def cut_rod_extended(n, p):
  r[0..n], s[0..n] = new array
  r[0] = 0

  for j in 1:n
    q = -inf

    for i in 1:j
      current_price = p[i] + r[j - i]
      if q < r
        s[n] = i
        q = current_price

    r[j] = q

  return r[n], s


def print_cut_rod(n, s):
  while n > 0
    print s[n]
    n = n - s[n]
```

Matrix Chain Multiplication

When multiplying matrices, the order of multiplication matters, for example, assume we have three matrices $< A_1, A_2, A_3 >$ with dimensions $10 \times 100, 100 \times 5, 5 \times 50$ respectively.

If we do the multiplication as $(A_1 A_2)A_3$, we will do $10 * 5 * 50 + 10 * 5 * 50 = 5000 + 2500 = 7500$ operations.
However if we do it as $A_1(A_2 A_3)$, this will result in $100 * 5 * 50 + 10 * 100 * 50 = 25000 + 50000 = 75000$ operations, which is 10 times faster.

Note: Number of operations in matrix multiplication of $(n \times m) * (m \times r)$ *is* $n * m * r$

The Matrix Chain Multiplication problem is the problem of parenthesizing the product of matrices in a way that minimizes the number of operations required for multiplication. $A_1 A_2 ... A_n$

Naive approach

Applying a naive approach to this problem simply does not work due to the large number of combinations that can be generated, the number of combinations follows the following recurrence relation:

$$P(n) = \begin{cases} 1 & \text{if } n = 1 \,, \\ \sum_{k=1}^{n-1} P(k)P(n-k) & \text{if } n \geq 2 \,. \end{cases}$$

Which is $\Omega(2^n)$.

Recursive approach

We can think about solving a general case, multiplying $A_1 A_2 ... A_n$.
We are required to find a specific k for which we split the multiplication into $(A_1 A_2 ... A_k)$ $(A_{k+1} A_{k+2} ... A_n)$ such that the cost of multiplication is minimal.
The cost of multiplication in this case is

$$Cost(A_1 A_2 ... A_k) + Cost(A_{k+1} A_{k+2} ... A_n) + (p_0 * p_k * p_n)$$

or for a more general case $(A_i A_{i+1} ... A_k)$ $(A_{k+1} A_{k+2} ... A_j)$ it is

$$Cost(A_i A_{i+1} ... A_k) + Cost(A_{k+1} A_{k+2} ... A_j) + (p_{i-1} * p_k * p_j)$$

where a matrix A_i has dimensions $p_{i-1} \times p_i$

We can then find a recurrence relation that translates the above:

$m(i, j) = m(i, k) + m(k + 1, j) + p_{i-1} p_k p_j$

with $m(i, j) = 0$ when $i = j$.

The final form of the reccurence relation:

$$m[i,j] = \begin{cases} 0 & \text{if } i = j\ , \\ \min\limits_{i \le k < j} \{m[i,k] + m[k+1,j] + p_{i-1}p_k p_j\} & \text{if } i < j\ . \end{cases}$$

This can be implemented with **recursion**:

```
def mat_chain(i, j):
  if(i == j)
    return 0

  q = inf

  for k in i:j-1
    q = min(q, mat_chain(i, k) + mat_chain(k+1, j) + (p[i - 1] * p[k] * p[j]))

  return q
```

- **Runtime is exponential**

Top-down approach

We can take the previous recursive solution and apply **memoization** to it:

```
def mat_chain(i, j):
  if(i == j)
    return 0

  if((i,j) in memo)
    return memo[i, j]

  q = inf

  for k in i:j-1
    q = min(q, mat_chain(i, k) + mat_chain(k+1, j) + (p[i - 1] * p[k] * p[j]))

  memo[i, j] = q
  return memo[i, j]
```

- **Runtime:** $O(n^3)$

Bottom-up approach

We can build up the smaller solutions by thinking backwards, computing smaller multiplications first and then bigger ones in sequence, we need to do this in a correct order to solve the problem of dependency in solutions. Visualizing the table of the DP helps:

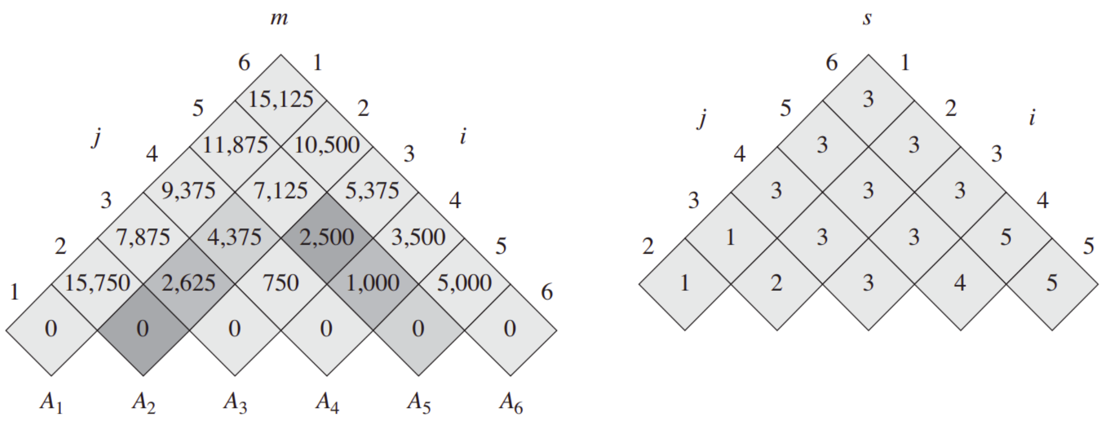


Figure 15.5 The m and s tables computed by MATRIX-CHAIN-ORDER for $n = 6$ and the following matrix dimensions:

matrix	A_1	A_2	A_3	A_4	A_5	A_6
dimension	30×35	35×15	15×5	5×10	10×20	20×25

We can notice that we need to start with chains of length 1 and work our way up to the maximum chain length. We can also compute the ideal position to parenthesis at on our way.

```
def mat_chain():
  n = p.length - 1     # p is sizes of matrices
  m[1..n][1..n], s[1..n][1..n] = new array

  for i in 1:n
    m[i, i] = 0        # chains with length = 1

  for l in 1:n         # chains with length > 1
    for i in 1:n-l+1   # i = start of chain
      j = i + l - 1    # j = end of chain
      m[i, j] = inf

      for k in i:j-1   # try cutting at k for all k
        q = m[i, k] + m[k+1, j] + p[i-1] * p[k] * p[j]
        if q < m[i, j]
          m[i, j] = q
          s[i, j] = k
  return m[1, n], s
```

- **Runtime:** $O(n^3)$

Greedy Algorithms

- A greedy algorithm always makes the best choice currently.
- Takes the locally optimal solution hoping to reach a globally optimal solution.
- Does not guarantee optimal solutions, but a greedy solution is always at least sub-optimal.
- Proving that a greedy solution is optimal or not is the hard part.
- Much faster than other solutions
- Good for optimization problems

Solution Steps

1. Interpret the optimization problem as one we make a choice and are left with one smaller subproblem to solve.
2. Prove there is an optimal solution to the original problem that makes the greedy choice.
3. Make sure that combining the optimal solution to the subproblem we have left with the greedy choice we made will leave us with the optimal solution.

Activity Selection

Find max subset of non-overlapping tasks.

i	1	2	3	4	5	6	7	8	9	10	11
s_i	1	3	0	5	3	5	6	8	8	2	12
f_i	4	5	6	7	9	9	10	11	12	14	16

A greedy solution may be as follows:

```
def activity_selection():
  sort tasks by finish time
  k = -inf
  A = []

  for task, i in tasks
    if s[i] >= k
      k = f[i]
      A.insert(task)
  return A
```

- **Runtime:** $O(nlogn)$

We sort the tasks on finish times, and count the number of non-overlapping tasks from the least finish time up to the largest finish time.

We can prove that this approach yields an optimal solution, like its dynamic programming counterpart, however, this algorithm is much faster.

Knapsack Problem

0-1 Knapsack

Thief trying to pick maximum value items while maintaining a maximum weight W, each item has a value v_i and a weight w_i.

0-1 Knapsack problem is always solved with dynamic programming as it guarantees the fastest optimal solution.

There is no optimal greedy algorithm for solving this problem, that is because when making a local choice we cannot guarantee that it is globally optimal.

Fractional Knapsack

Fractional Knapsack problem is similar to `0-1 Knapsack` but the thief can pick a fraction of the item.

It is easy to find an optimal solution to this problem greedily. Find the item with the highest $value/pound$ and take it.

```
def FractionalKnapsack(items):
  a = []
  for item in items
    a.insert({ item.v / item.w, w })

  sort_descendingly(a)

  ans = 0
  for item in a
    ans += a.value * a.weight

  return ans
```

Huffman Codes

The problem of huffman encoding is taking a set of symbols with probabilities and assign codes to each symbol, such that the entropy of the alphabet is minimal.

```
def Huffman(C):
  n = |C|
  Q = C

  for i in 1:n-1
    allocate new node z
    z.left = x = Extract_Min(Q)
    z.right = y = Extract_Min(Q)
    z.freq = x.freq + y.freq
    Q.insert(z)

  return Extract_Min(Q)
```

Graphs

Graph Representations

- Adjacency List
  - $\Theta(V+E)$ space
  - Takes less space
  - Takes more time to check if there is an edge between u and v
- Adjacency Matrix
  - $\Theta(V^2)$ space
  - Takes more space
  - Faster to check if there is an edge between u and v

Breadth First Search (BFS)

- One of the ways to traverse a graph
- Used for finding unweighted shortest path between a node and all nodes
- Expands from s to surrounding nodes

```
def bfs(G, s):
  for each vertex u in G
    u.color = white # white: unvisited, gray: current, black: visited
    u.dist = inf
    u.parent = nil

  s.color = gray
  s.dist = 0
  s.parent = nil

  Q = new queue
  Q.enqueue(s)

  while Q is not empty
    u = Q.dequeue()
    for each v in adj[u]
      if v.color is white
        v.color = gray
        v.dist = u.dist + 1
        v.parent = u
        Q.enqueue(v)

    u.color = black
```

- **Runtime:** $O(V + E)$

Depth First Search (DFS)

- One of the ways to traverse a graph
- Used as a building block for a lot of algorithms

```
def dfs(G):
  for each vertex u in G
    u.color = white
    u.parent = pi
  time = 0
  for each vertex u in G
    if u.color is white
      dfs_visit(G, u)

def dfs_visit(G, u):
  time = time + 1
  u.start = time
  u.color = gray

  for each vertex v in adj[u]
    if v.color is white
      v.parent = u
      dfs_visit(G, v)

  u.color = black
  time = time + 1
  u.finish = time
```

- **Runtime:** $O(V + E)$

Topological Sort

- Linear ordering of vertices in graph G such that if there exists edge (u, v) then u appears before v
- Must be performed on a DAG (directed acyclic graph)

```
def topo(G):
  S = new stack

  for each vertex u in G
    u.color = white

  for each vertex u in G
    if u.color is white
      topo_dfs(G, S, u)

  while S is not empty
    print(S.pop())

  return S

def topo_dfs(G, S, u):
  u.color = gray

  for each vertex v in adj[u]
    if v.color is white
      topo_dfs(G, Q, u)

  u.color = black
  S.push(u)
```

- **Runtime:** $O(V + E)$

Strongly Connected Components (SCC)

- Maximal set of vertices C such that every pair of vertices u and v are reachable from each other ($u \to v; v \to u$)

```
def scc(G):
  S = topo(G) # topological order of graph
              # order of vertices according ot finish time

  G_T = transpose(G)

  C = new list

  while S is not empty
    u = S.pop()
    if u.color is white
      c = new list
      scc_dfs(G, c, u)
      C.insert(c)

def scc_dfs(G, c, u):
  c.insert(u)
  for vertex v in adj[u]
    if v.color is white
      dfs_print(G, u)
```

Disjoint Sets Union (DSU)

- Each set has one representative (member of the same set)
- Used to identify whether two members belong to the same set or not

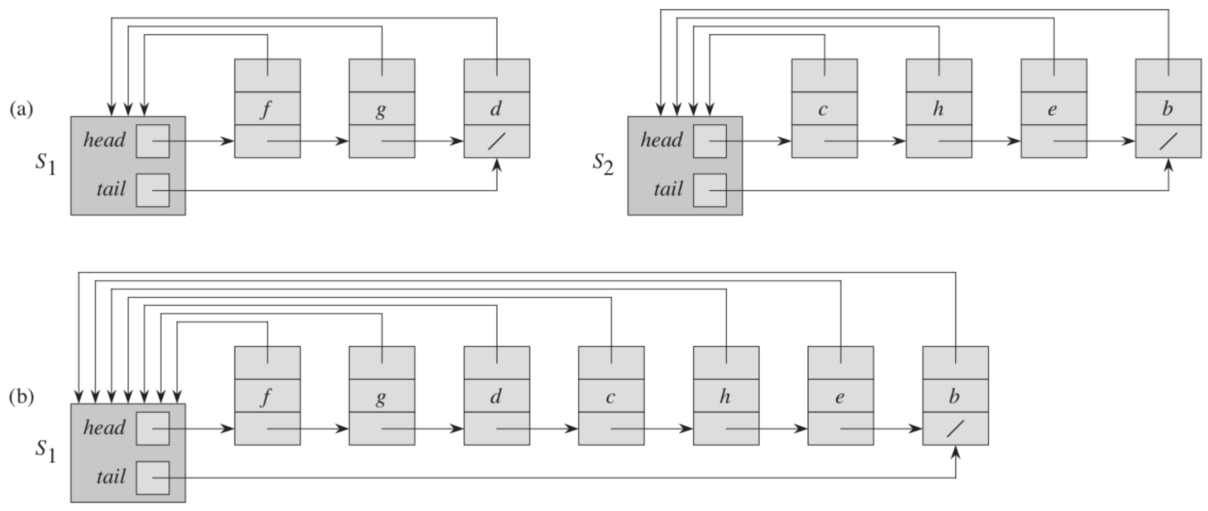
DSU Operations

1. $make\_set(x)$: creates a new set with only x as a member.
2. $union(x, y)$: unites sets containing x and y.
3. $find\_set(x)$: returns the representative of the set containing x.

DSU Representations

Linked List representation

- Each set is a separate linked list



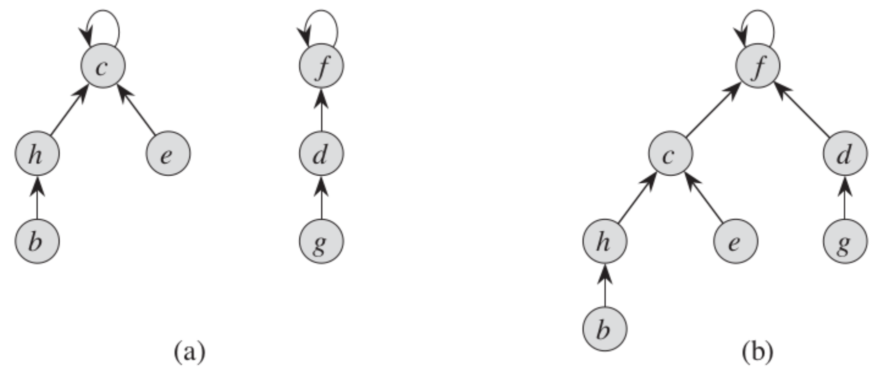
Operations:

1. $make\_set(x)$ creates a new list with x in it. $O(1)$
2. $union(x, y)$ just join two ends of the linked lists, however we need to loop through elements of one list and update their pointers one by one. $O(n)$
3. $find\_set(x)$ returns the pointer to the set object the element is pointing at. $O(1)$

- To create components for a graph with n connected vertices $\rightarrow \Theta(n^2)$
  - Weight Union: can try to append shorter list to the longer one $\rightarrow O(m + nlgn)$ for m DSU operations n of which are $make\_set$ operations.

Tree representation

- Each set is a separate tree



Operations:

1. $make\_set(x)$ creates a new tree with x in it. $O(1)$
2. $union(x, y)$ one tree's root points to the other tree's root. $O(n)$
3. $find\_set(x)$ traverse the tree up. $O(n)$

- If tree representation has worse complexity than linked list, why use it at all? heuristics!:
    1. Union By Rank: union the smaller set to the larger set. $O(mlgn)$
    2. Path Compression: while traversing in $find\_set(x)$ can update the parent of bottom childs to the root of the tree, significantly decreasing tree height.
        - Union by rank + Path compression $\rightarrow O(m\ \alpha(n))$

DSU Applications

Graph Connected Components

- DSU can be used to find the connected components of a graph.
- Done by adding all vertices with edges between them to the same set.

```
def connect_components(G):
  for each vertex v in G
    make_set(v)
  for each edge (u, v) in G
    if find_set(u) != find_set(v)
      union(u, v)

def same_component(u, v):
  if find_set(u) == find_set(v)
    return true
  return false
```

Minimum Spanning Tree (MST)

- Problem of transforming a graph into a tree with minimum summation of edge weights. $min[w(T) = \sum_{(u,v) \epsilon T} w(u,v)]$
- Used in reducing cost of components
    - i.e. wiring electronics components, building city roads, etc..
- Approach the problem in a greedy way.

Kruskal's algorithm

- Main idea is to sort the edges in a non-decreasing order based on their weights.
- Pick edges with minimum weights to connect all vertices.

```
def kruskal(G):
  A = new set

  for each vertex in v
    make_set(v)

  sort G.edges in non decreasing order according to weights

  for each edge e == (u, v) in non decreasing order of weights
    if find_set(u) != find_set(v)
      A.insert(e)
      union(u, v)

  return A
```

- **Runtime:** $O(E\ logE) = O(E\ logV^2) = O(E\ logV)$

Prim's algorithm

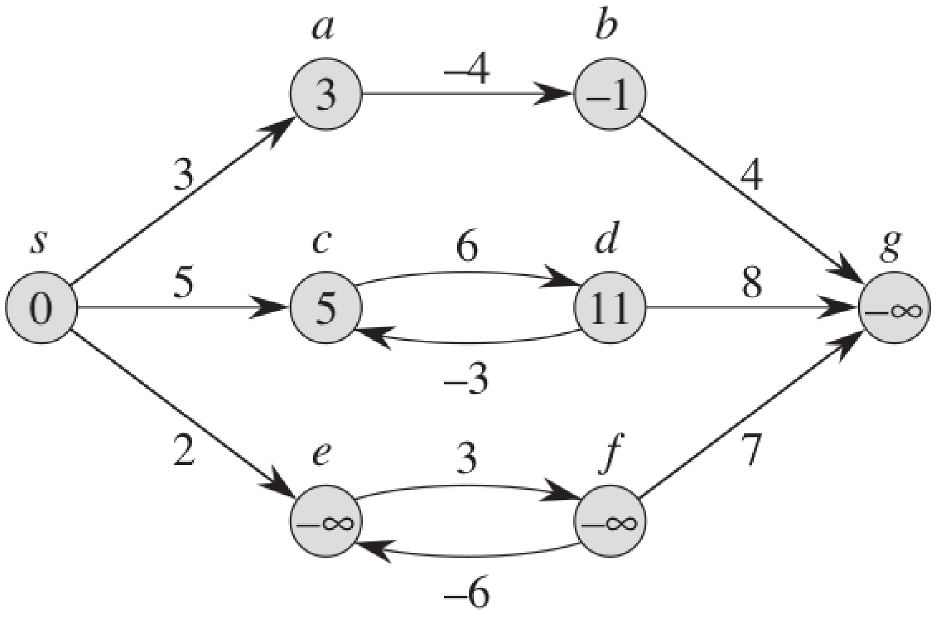
- Similar idea to Kruskal but works on vertices instead of edges.
- Start with a one node tree and check the minimum edge connected to the tree, take it and expand.

```
def prim(G, r): # r is root
  for each vertex u in G
    u.key = inf
    u.par = nil

  r.key = 0

  Q = new min_heap

  while Q is not empty
    u = Q.extract_min
    for each vertex v in adj[u]
      if v in Q and w(u, v) < v.key
        v.par = u
        v.key = w(u, v) # also updates heap
```

- **Runtime:** $O(V\ logV + E\ logV) = O(E\ logV)$
  - Using fibonacci heap: $O(E + V\ logV)$

Shortest Paths

All shortest paths algorithms utilize the optimal substructure of the shortest paths problem, that is: The path required to go from vertex v_0 to v_k can be decomposed to a summation of different shortest subpaths.

- Types of shortest paths problems:
  - Single-source shortest path problem
    - Can be solved directly using defined algorithms.
  - Single-pair shortest path problem
    - Can be solved by finding all shortest paths and then taking the pair we want only.
  - Single-destination shortest path problem
    - Reverse edges of graph and solve a single-source problem using the destination as the source.
  - All-source shortest paths
    - Use specialized algorithms instead of doing n iterations of single-source runs.
- Negative edge weights
  - Some algorithms allow them, some do not.
- How algorithms handle different cycles:
  - 

  - Negative-weight cycles (summation of cycle is negative):
    - Either prohibited or detected
  - Positive-weight cycles (summation of cycle is positive):
    - Theoretically not possible
  - Zero-weight cycles:
    - Eliminated from the path

Single Source Shortest Paths

Bellman Ford

- Single source, all destination
- Allows negative edges
- Does not allow negative cycles but it detects them.

1. Iterate over all edges (u, v) and relax them.
    - If the order of iteration is correct, we will have the shortest path now, however we cannot guarantee that the order of iteration is correct.
2. Repeat d times where d is the length of the longest path between the source and the destination.
    - Repetition is required to guarantee that we will ignore the order of iteration.

```
def bellman_ford(G, s):
  # initialization
  for each vertex v in G
    v.dist = inf
    v.parent = nil
  s.d = 0

  # length of longest possible path
  for i in 1:|G.V|-1
    for each edge e == (u, v) in G
      if u.dist + e.weight < v.dist
        v.dist = u.dist + e.weight
        v.par = u

  # check for negative cycles
  for each edge e == (u, v) in G
    if v.dist > u.dist + e.weight
      return false

  return true
```

- **Runtime:** $O(VE)$

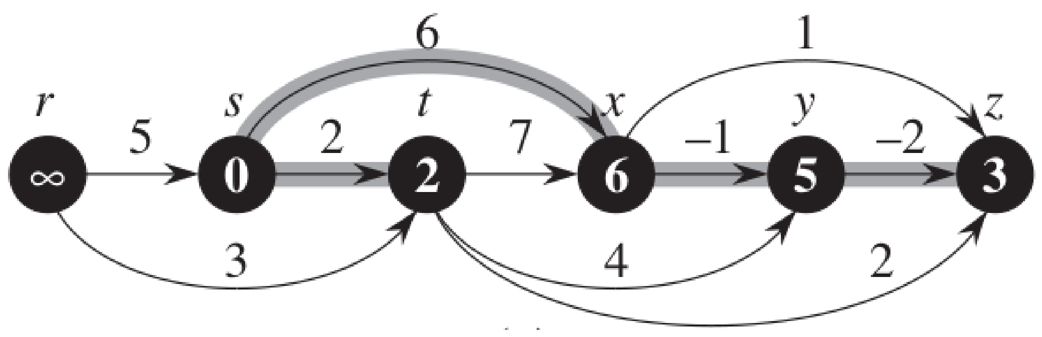
Directed Acyclic Graph (DAG) shortest-path

Since it is a DAG, we can topologically sort the graph and find the correct ordering.

- Single source, all destination
- Allows negative edges
- Negative cycles will not exist since it is a DAG.

1. Topologically sort the graph
2. Relax edges in the topological order

This is similar to Bellman Ford except that it knows the correct ordering of vertices



```
def DAG_shortest(G, s):
  topological_sort(G)

  # initialization
  for each vertex v in G
    v.dist = inf
    v.parent = nil
  s.d = 0

  # iterate over topologically sorted graph and relax all edges
  for each vertex u in topologically ordered graph G
    for each edge e == (u, v) in adj[u]
        if u.dist + e.weight < v.dist
          v.dist = u.dist + e.weight
          v.par = u
```

- **Runtime:** $O(V + E)$

Djikstra

- Single source, multiple destination.
- Does not allow negative edges.
- Does not allow negative cycles but allows positive ones.

BFS is a special case of Djikstra when edges have same weight

```
def djikstra(G, s):
  Q = new min heap sorting vertices by weights
  Q.insert(s)
  while Q is not empty
    u = Q.pop()
    for each edge e == (u, v) in adj[u]
      if u.dist + e.weight < v.dist
        v.dist = u.dist + e.weight # updates min heap
```

- **Runtime:**
  - Min-heap: $O(E\ logV)$
  - Fibonacci heap: $O(V\ logV + E)$
  - Array: $O(V^2)$

All-pairs shortest paths

In all-pairs shortest paths we take an input $G(V, E)$ with weighted edges and we are required to compute an output $n \times n$ matrix W of shortest paths $\delta(u, v)$ for all u, v.

We can run single source algorithms for all vertices v to find all shortest paths between all pairs of vertices.

Let's see how the complexity will be:

- Bellman Ford: $O(V^2E)$ or $O(V^4)$ if the graph is dense
- Djikstra:
  - Min-heap: $O(VE\ logV)$
  - Fibonacci heap: $O(V^2\ logV + VE)$
  - Array: $O(V^3)$

Really costly, we can do better.

Dynamic Programming shortest paths

Thinking in a bottom-up way we can:

1. Compute shortest path (SP) for all pairs with length one
2. Use the previously computed SP of length one to expand and compute length two.
3. Repeat for longer lengths.

This way is $O(n^4)$, we can optimize a bit as we can notice that there is an analogy between **fast exponentiation** and this algorithm.

Since
$L^{(1)} = L^{(0)}.W = W$
$L^{(2)} = L^{(1)}.W = W^2$
...
$L^{(n)} = L^{(n-1)}.W = W^n$

we can compute this in $O(n^3 logn)$ instead.

```
def extend_shortest_path(G, L, W):
  n = L.rows
  let L' = new matrix[n][n]
  for i in 1:n
    for j in 1:n
      L[i][j] = inf
      for k in 1:n
        L[i][j] = min(L[i][j], L[i][k] + W[k][j])
  return L

def all_pairs_shortest_path(G, W):
  n = W.rows
  L = new list of n*n matrices
  L[1] = W
  m = 1

  while m < n-1
    L[2*m] = extend_shortest_path(L[m], L[m])
    m = 2*m

  return L[m]
```

- **Runtime:** $\Theta(n^3 logn) = \Theta(V^3 logV)$

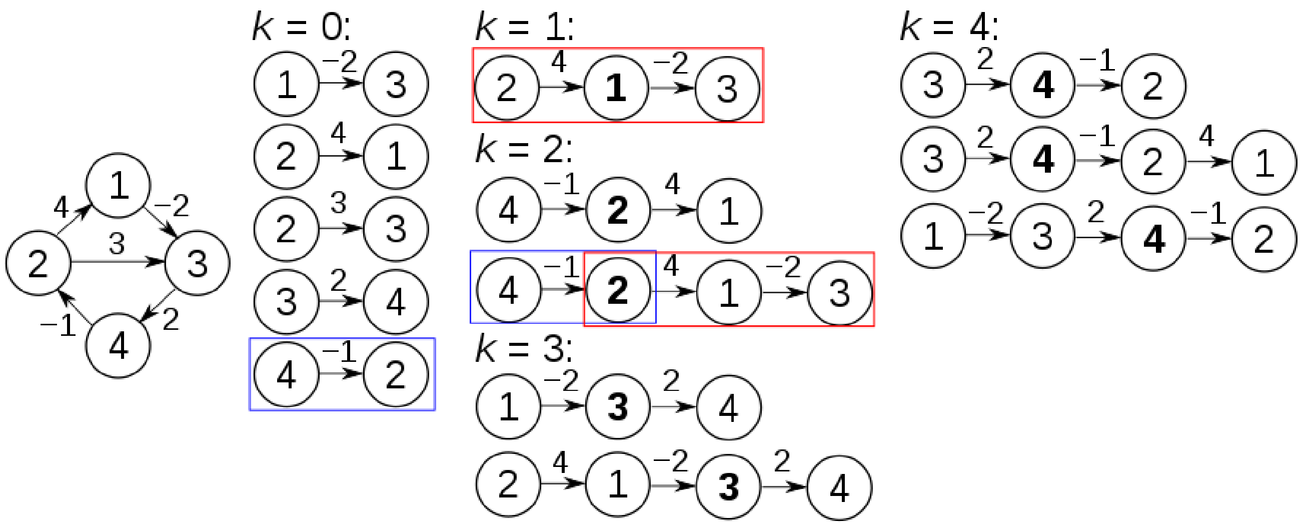
Transforming the same logic to a top-down way we notice:

- Shortest path from $i \to j$ can be written as $SP(i \to k) + SP(k \to j)$
- Divide and conquer approach
  - $l_{ij}^{(m)} = min_{i \leq k \leq n}\{l_{ik}^{m-1} + w_{kj}\}$

Floyd-Warshall algorithm

- Uses the same approach as the dynamic programming approach but on vertices rather than edges.
- Negative edges allowed
- Negative cycles are not allowed

$$d_{ij}^{(k)} = \begin{cases} w_{ij} & \text{if } k = 0 \,, \\ \min\left(d_{ij}^{(k-1)}, d_{ik}^{(k-1)} + d_{kj}^{(k-1)}\right) & \text{if } k \geq 1 \,. \end{cases}$$



```
def floyd(W):
  n = W.rows
  D = W

  for k in 1:n
    for j in 1:n
      for i in 1:n
        D[i][j] = min(D[i][j], D[i][k] + D[k][j])
```

- **Runtime:** $\Theta(n^3) = \Theta(V^3)$

Implementation can include a parent matrix to know the exact path.

Applications on all-pairs shortest paths

Transitive closure

Given a graph $G(V, E)$ we need to know if there exists an edge between all vertex pairs u, v.

1. Assign weights 1 to each edge in E.
2. Run Floyd-Warshall.
3. If $d_{ij} < n$ then i and j are connected, otherwise $d_{ij} = \infty$ therefore not connected.

```
def transitive_closure(G):
  n = |G.V|
  T = new n*n matrix

  # build transitive closure graph
  for i in 1:n
    for j in 1:n
      if i == j or (i, j) exists in G.E
        T[i][j] = 1
      else
        T[i][j] = 0

  for k in 1:n
    for i in 1:n
      for j in 1:n
        T[i][j] = T[i][j] or (T[i][k] and T[k][j])
```

Johnson's algorithm

- Used in sparse graphs $(E \ll V^2)$

It goes as follows

- If no negative edges
  - Apply Djikstra with fibonacci heap in $O(V^2 logV + VE)$
  - Better than Floyd-Warshall
- Else if no negative-weight cycles
  - Compute new set of non-negative edge weights that allow us to use same method
  - New edge weights $\hat{w}$ must satisfy
    - $\hat{w}$ is nonnegative
    - Path p is a shortest path from u to v in w if and only if it is also a shortest path in $\hat{w}$

So we conclude that we need to use Djikstra but we need to solve the problem of negative weights, to do that we change the edges from w to $\hat{w}$ using the following mapping

$$\hat{w}(u,v) = w(u,v) + h(u) - h(v)$$

$h(x)$ is a weight assigned to each vertex.

- How to calculate the weights $h(x)$ of all nodes?
    1. Add a new vertex u to the graph
    2. Connect u to all other vertices using edges of weight 0
    3. Run Bellman Ford on the modified graph
    4. The distance from vertex u to a vertex v is $h(v)$: the weight that is supposed to be added to node v
    5. Compute $\hat{w}$ using the computed weights and run djikstra as supposed.
    6. Subtract the values $h(u), h(v)$ added previously.

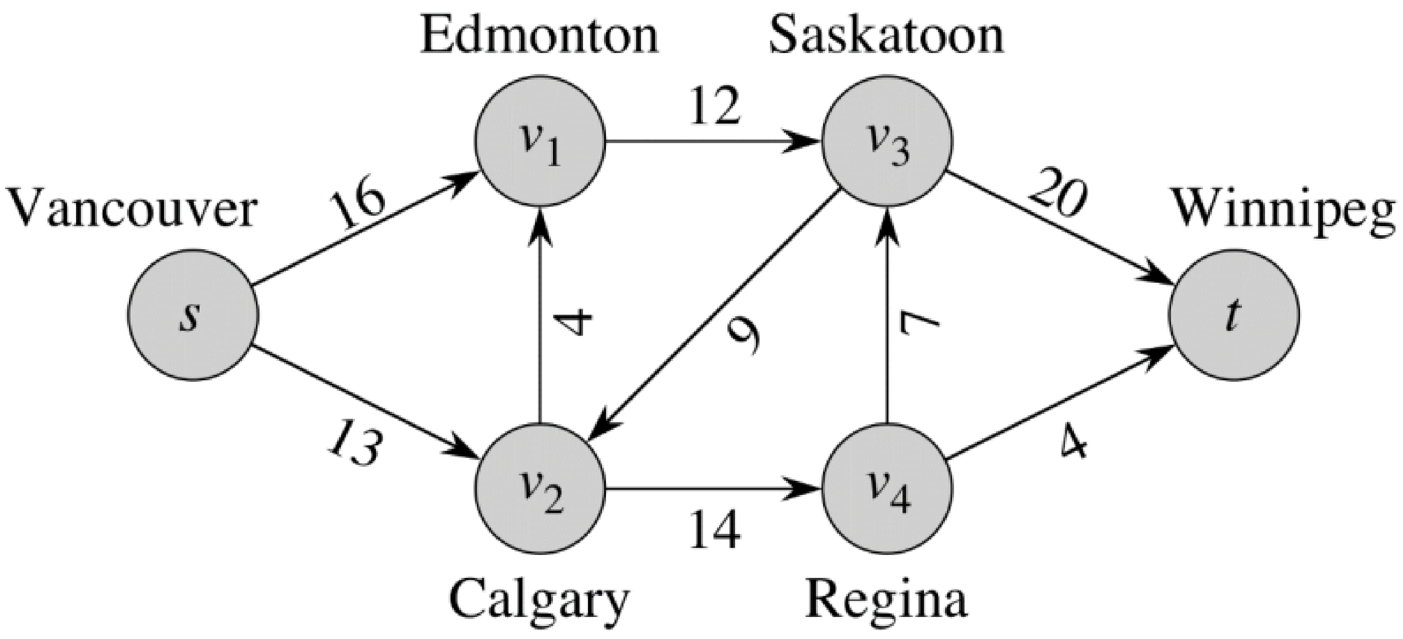
```
def johnson(G):
  G' = new graph
  G' = G
  insert new vertex s in G'
  for all vertices v in G
    insert new edge e == (s,v) with weight 0 to graph G'

  if bellman_ford(G', s) == false
    throw("Negative edges")
  else
    for each vertex v in G'
      v.hval = D[s][v] # D is the weights calculated by Bellman-Ford

    for each edge e == (u, v) in G'
      e.weight = e.weight + u.hval - v.hval

    D = new n*n matrix

    for each vertex u in G
      d = djikstra(G', u) # weights computed by djikstra

      for each vertex v in G
        D[u][v] = d[u][v] - (u.hval - v.hval) # assign the djikstra weight and remove the hval added

  return D
```

Maximum Flow Problem

Maximum flow problem applies to a graph $G(V, E)$ called a **flow network** which has

- One vertex labeled as a source
  - This node has an indegree of zero
- One vertex labeled as a sink
  - This node has an outdegree of zero
- An edge (u, v) represents a pipe which has a maximum flow capacity of $c(u, v)$
- Each edge (u, v) has nonnegative capacity $c(u, v) > 0$
- If there exists an edge (u, v) then there is no edge in reverse direction (v, u)
- No self loops

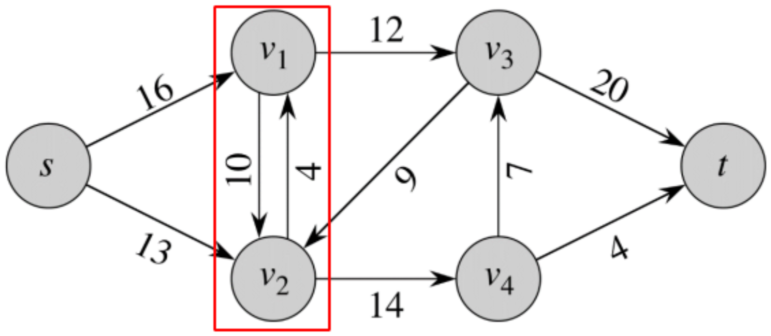
The problem is to find the maximum flow that goes from source s to sink t.

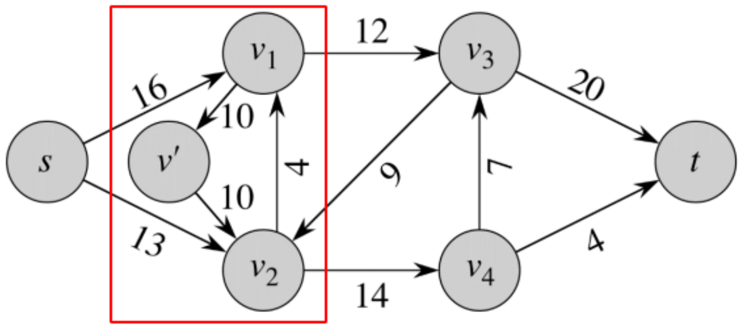


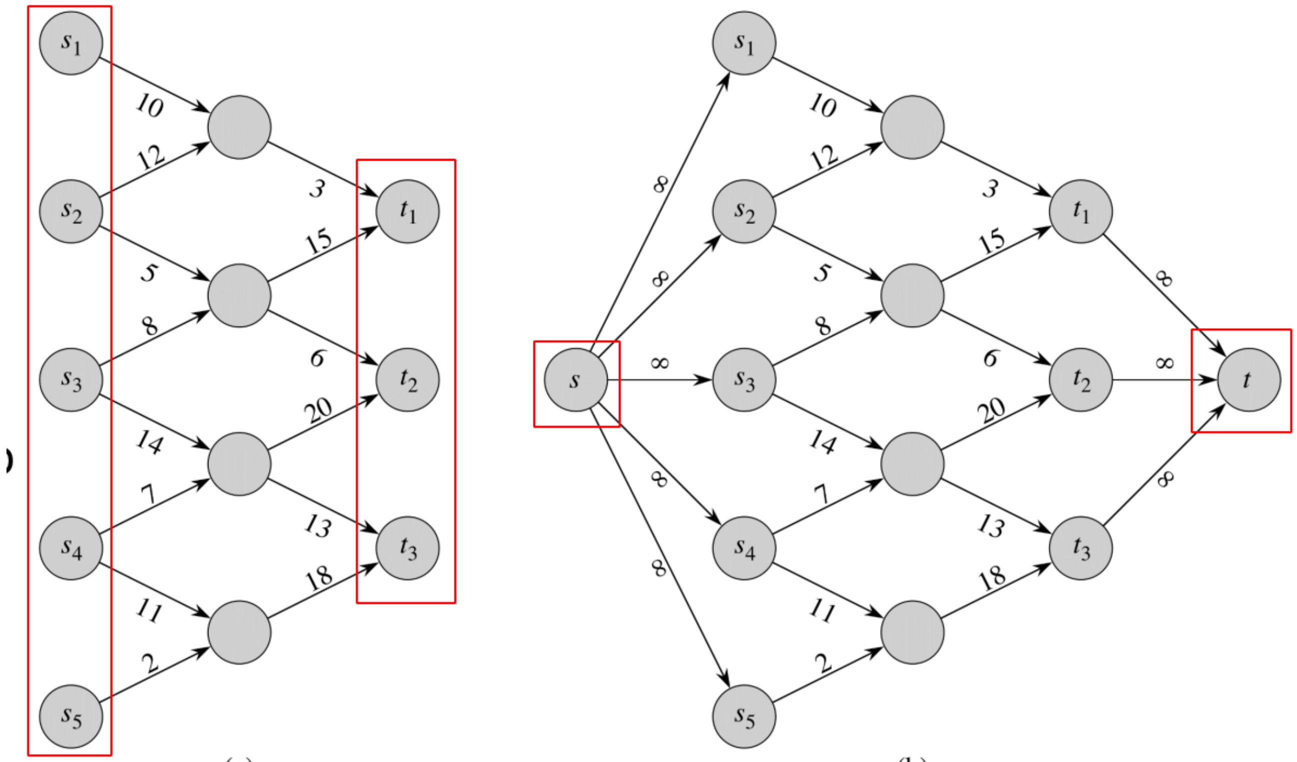
Flow is a real valued function $f : V \times V \rightarrow \mathbb{R}$ that satisfies:

- Capacity constraint
  - for all $u, v \,\epsilon\, E$, $0 \leq f(u, v) \leq c(u, v)$
- Flow constraint
  - for all $u, v \,\epsilon\, V - \{s, t\}$, for each u
    - $\sum_{v \epsilon V} f(v, u) = \sum_{v \epsilon V} f(u, v)$

Modelling different graphs to a flow network

1. Antiparallel edges
   - (u, v) and (v, u) are antiparallel edges
     - 

   - Split one of the edges to a new vertex
     - 

2. Multiple sources/sinks
   - Add a new vertices as source and sink and connect them to all source nodes and sink nodes with edges of capacity ∞
     - 

(a) (b)

Flow Networks definitions

Residual networks

- Residual network G_f is similar to flow networks, except it allows for reversed edges
- Whenever we change flow of an edge (u, v) in the flow network
  - Subtract the flow of that edge $f(u, v)$ from the capacity $c(u, v)$ in the residual network
  - Add a new edge (v, u) with the flow $f(u, v)$

Augmenting paths

- Augmenting path p is a simple path (all vertices on path are distinct) from s to t in the residual network.

Cuts of flow networks

- A cut (S, T) of a flow network $G(V, E)$ is a partition V into two distinct sets S and T
- **Net flow** $f(S, T)$ across the cut (S, T) is difference of flow between nodes in T and nodes in S
- **Capacity of cut** (S, T) is the summation of capacities on the edges which we cut at.
- A **minimum cut** of a network is the cut with the minimum capacity.

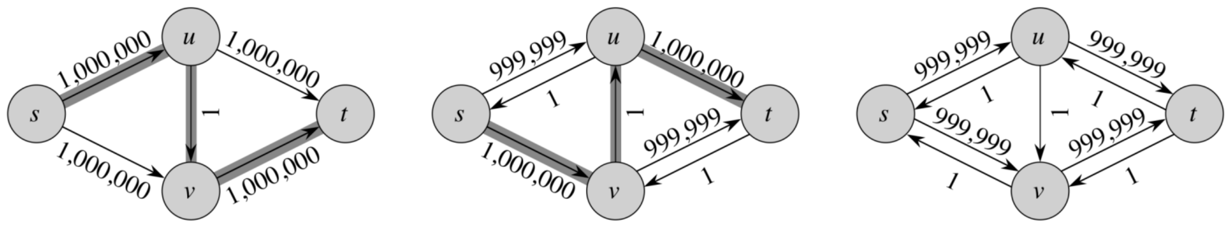
Max-flow min-cut

- The value of the max-flow f of a network is equal to the minimum cut of the same network.
- The residual network G_f contains no augmenting paths.

Ford-Fulkerson algorithm

1. Iterate over all paths possible in residual network Gf
2. While there is a path, find the minimum capacity p_flow along that path
3. Iterate over all edges on that path
    - If the edge is a forward edge, add p_flow to its flow
    - If the edge is a backward edge, subtract p_flow from its flow

```
def ford_fulkerson(G, s, t):
  for each edge e == (u, v) in G:
    e.flow = 0

  Gf = G # residual network

  while there exists a path p from s to t in Gf
    p_flow = min(capacity of edges on path p)
    for each edge e == (u, v) in p
      if (u, v) in G.E # it is a forward edge
        (u, v).flow = (u, v).flow + p_flow
      else # reverse edge
        (v, u).flow = (v, u).flow - p_flow
```

- **Runtime:** $O(E|f^*|)$, where f^* is the maximum flow.
    - Runtime is really slow for networks with high flow!
        - 

        - Results in 1000000 iterations.
        - Can use BFS to ignore the low weighted edge.

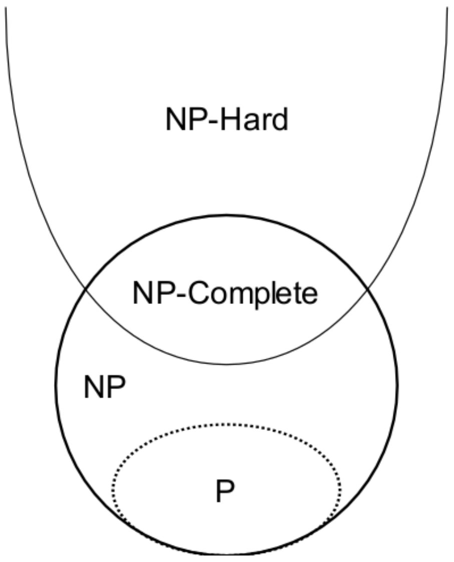
Edmonds-Karp algorithm

- Uses BFS for augmenting path (shortest path)
- Total number of flow augmentation is $O(VE)$,
- **Runtime:** $O(VE^2)$

NP Completeness

Complexity classes

- P: Problems that can be solved in **polynomial** time.
- NP (Non-determinstic Polynomial time): Problems which their solutions can be **verified** in **polynomial** time.
    - The problem solution itself can be polynomial or non-polynomial
- NP-hard: Problems which are **at least as hard as** the hardest problems in NP.
- NP-complete: Decision problems which contain **the hardest problems** in NP.
    - Hard problems, probably non-polynomial, but their solutions can still be verified in polynomill time.
    - To prove a problem is NP-complete, we need to prove that is NP and NP-hard
        - Can achieve proof of NP by checking if validation of a solution is polynomial.
        - Can achieve proof of NP-hard by reduction of a hard problem to the unknown one.



Problems in NP-complete class are interesting, because if we proved that $P = NP$, we will solve a wide array of problems.

P vs NP

Very similar problems but P vs NP:

- Shortest (P) vs Longest (NP) simple paths
- Euler tour (P) vs Hamiltonian cycle (NP)
- 2-CNF satisfiability (P) vs 3-CNF satisfiability (NP)

Decision vs Optimization

- Optimization problem: solution achieves **min/max**.
- Decision problem: solution is **yes** or **no**.

If an optimization problem is easy, the related decision problem is easy as well.
If the decision problem is hard, the optimization problem is hard as well.

Reduction

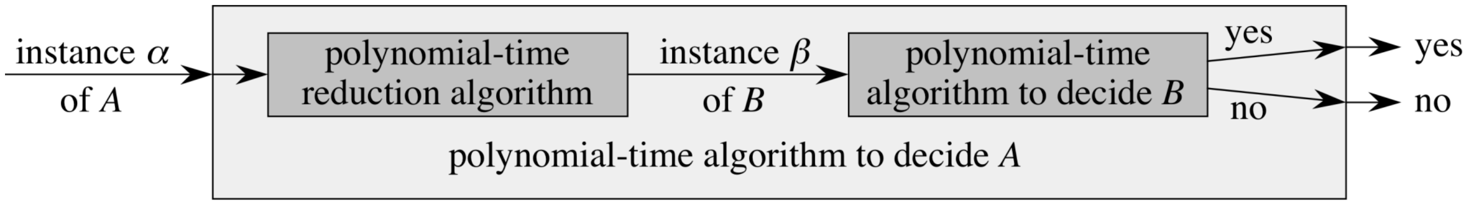
Given two problems X and A, and if we know that X is hard.

If we can find a reduction (mapping) from X to A that can be done **in polynomial time**:

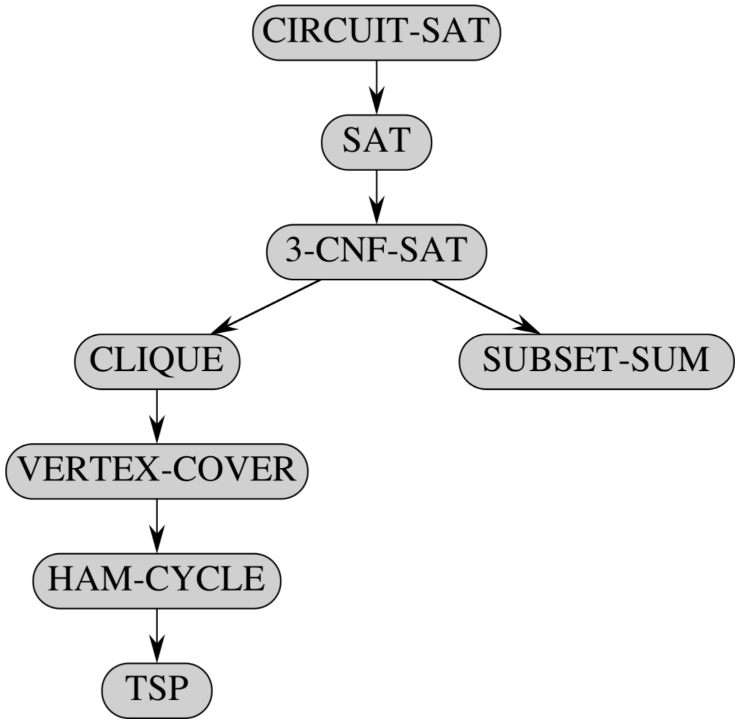
- A must be as hard as X.
- If we can solve A, then we can solve X in the same time as A.

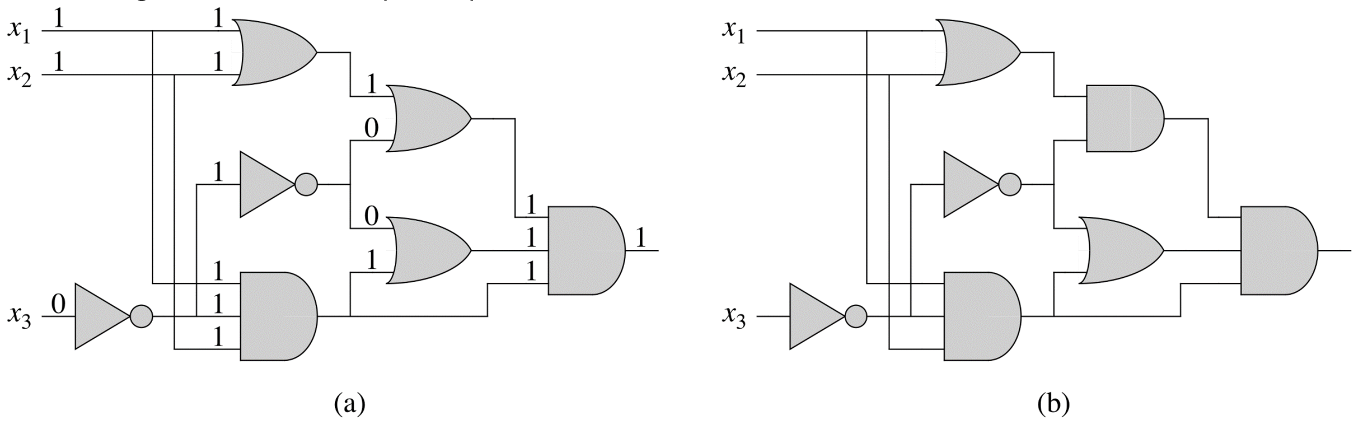
So we can deduce that solving one NP problem will lead us to solving all other NP problems through reductions.

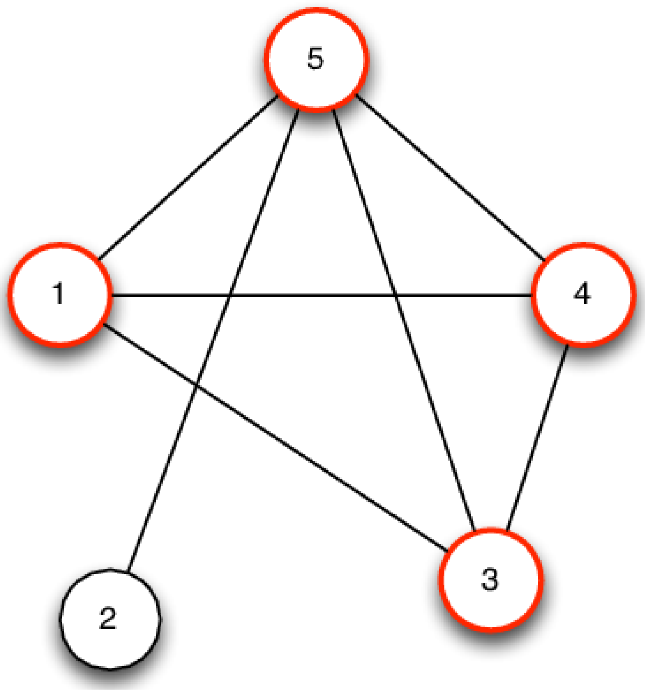
*Note: We always reduce a **hard problem** into an **unknown one**, not the other way around.*

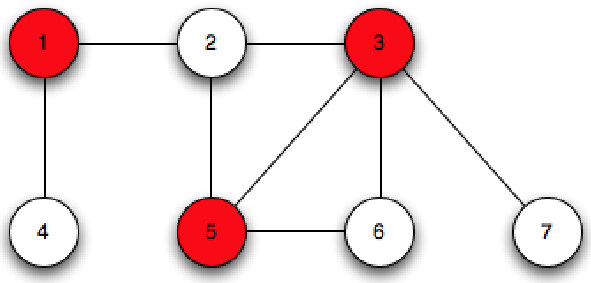

instance α of A
polynomial-time reduction algorithm
instance β of B
polynomial-time algorithm to decide B
yes
no
yes
no
polynomial-time algorithm to decide A

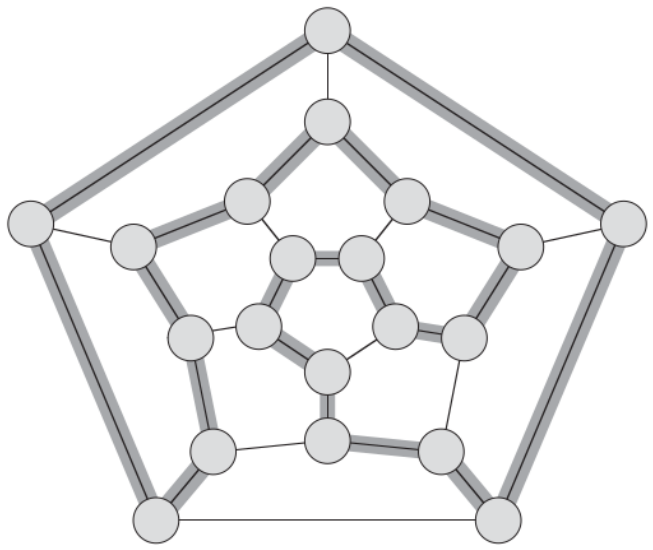
Reducible problems

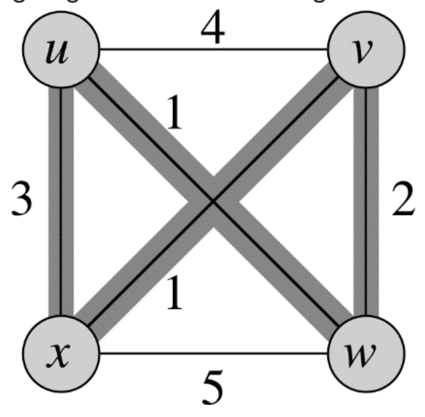


1. Circuit-SAT
    - Given a logic circuit, find an input sequence that satisfies that circuit
    - 

2. SAT
    - Given a boolean expression, find an input sequence that satisfies that expression
    - $\phi = ((x_1 \to x_2) \vee \neg((\neg x_1 \leftrightarrow x_3) \vee x_4)) \wedge \neg x_2$
3. 3-CNF-SAT
    - Given a trinomial POS (product of sums) expression, find an input sequence that satisfies that expression.
    - $\phi = (x_1 \vee \neg x_1 \vee \neg x_2) \wedge (x_3 \vee x_2 \vee x_4) \wedge (\neg x_1 \vee \neg x_3 \vee \neg x_4)$
4. Subset-sum
    - Given a positive numbers set, take a subset of that set to evaluate to a sum t.
        - $S = \{1, 2, 7, 14, 49\},\ t = 57$
5. Clique
    - Given a graph $G(V, E)$, find the maximal subgraph $G'(V, E)$ such that all nodes in G' are pairwise connected.
    - 

6. Vertex-cover
    - Given a graph $G(V, E)$, find **minimal** subset of vertices V' such that all edges in G are covered (connected) with vertices of V'
    - 

7. Hamiltonian cycle
    - Given a graph $G(V, E)$, check if there exists a cycle in the graph that contains all vertices V
    - 

8. Travelling Salesman Problem (TSP)
    - Given a graph $G(V, E)$, find the minimum cost to traverse a graph passing by all the vertices and then going back to the starting vertex.
    - 

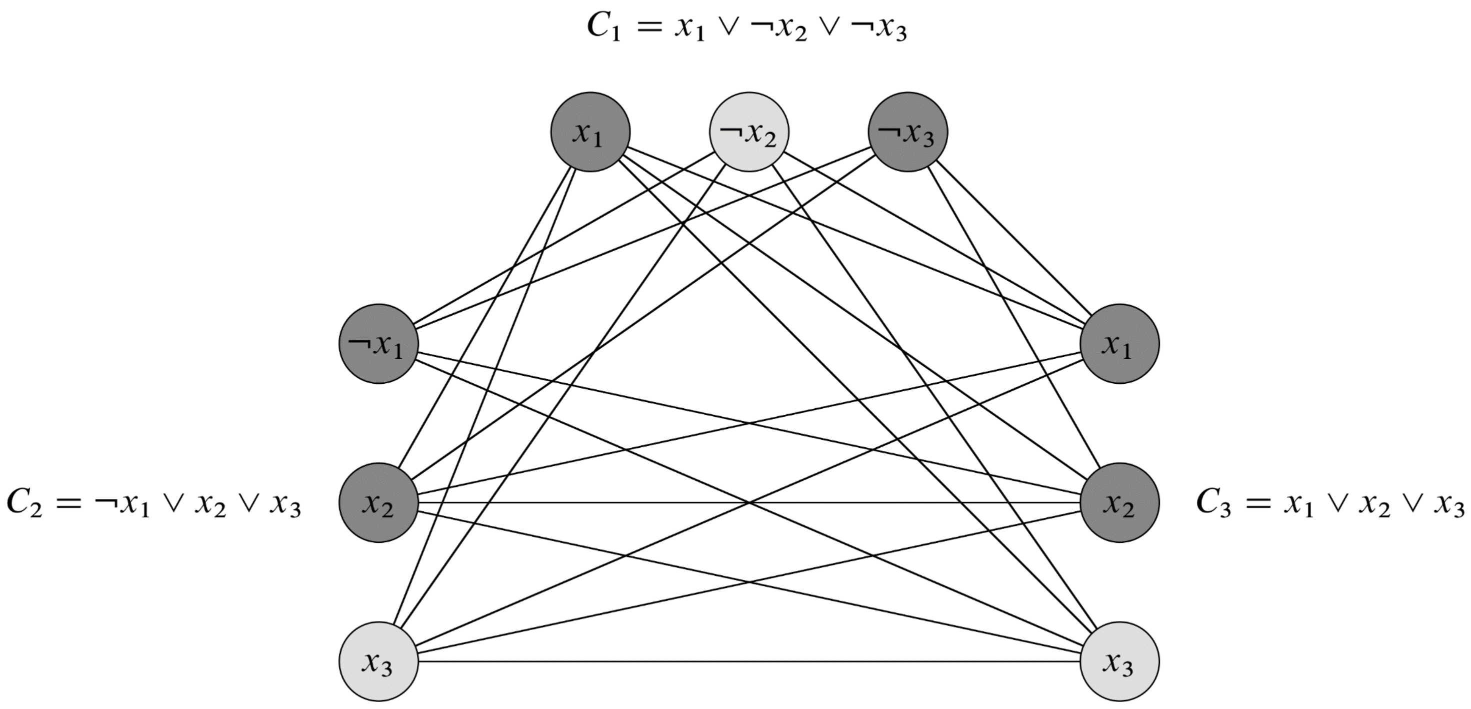
3-CNF-SAT to Clique

- We can reduce 3-CNF-SAT to Clique, this will prove that Clique problem is NP-Complete because we know that 3-CNF-SAT is NP-Complete.
- We can deduce that Clique problem is as hard as 3-CNF-SAT, because we can find 3-CNF-SAT inside clique.

Reduction:

1. Convert each clause in the 3-CNF-SAT to a set of 3 vertices.
2. Each vertex represents a variable in that clause.
3. Connect each vertex u of a clause to all other vertices in other clauses that are not $\neg u$.
4. Solve the clique problem on that graph.
5. The vertices included in the clique are the inputs that satisfy the expression.

The expression $(x_1 \vee \neg x_2 \vee \neg x_3) \wedge (\neg x_1 \vee x_2 \vee x_3) \wedge (x_1 \vee x_2 \vee x_3)$ *corresponds to the following graph*

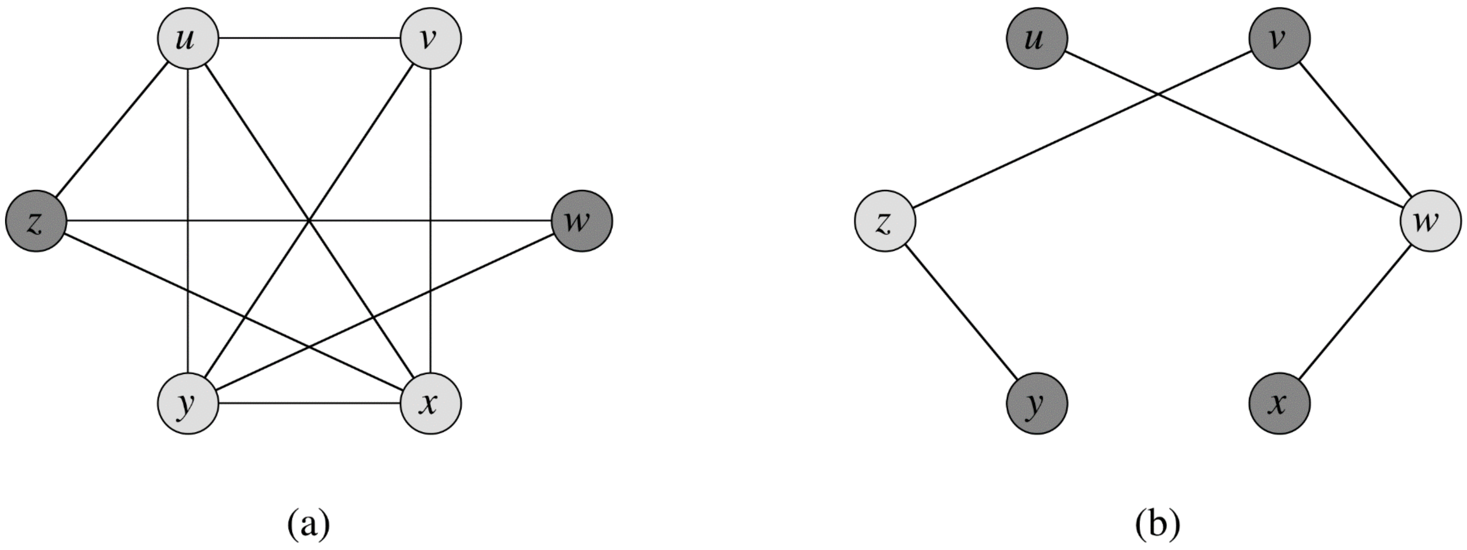


$C_1 = x_1 \vee \neg x_2 \vee \neg x_3$
x_1
$\neg x_2$
$\neg x_3$
$\neg x_1$
x_2
x_3
$C_2 = \neg x_1 \vee x_2 \vee x_3$
x_1
x_2
x_3
$C_3 = x_1 \vee x_2 \vee x_3$

Clique to Vertex-cover

Reduction:

1. Given a graph $G(V, E)$, find graph $G'(V, E)$ such that if edge (u, v) exists in G, it will not exist in G', and vice versa.
2. Find vertex-cover of $G'(V, E)$ to get set of vertices V' that is the solution to the vertex-cover problem.
3. The clique of $G(V, E)$ is $V - V'$, that is all the vertices in G except the ones in V'



The vertex cover of graph (b) is $\{w, z\}$ so the clique of graph (a) is $\{u, v, w, x, y, z\} - \{w, z\} = \{u, v, x, y\}$

Hamiltonian Cycle to TSP

Reduction:

1. Given an unweighted graph $G(V, E)$, find weighted graph $G'(V, E)$ such that:
   - If edge (u, v) exists in G, it exists in G' with weight 0
   - Else, it exists in G' with weight 1.
2. Solve TSP on graph $G'(V, E)$
3. If the solution of TSP on that graph is 0, then there exists a hamiltonian cycle and the cycle itself is the same of the TSP path.

$$c(i, j) = \begin{cases} 0 & \text{if } (i, j) \in E \ , \\ 1 & \text{if } (i, j) \notin E \ . \end{cases}$$

3-CNF-SAT to Subset Sum

Reduction:

1. Create a table with $(\text{number of variables} + \text{number of clauses}) * 2$ rows and $\text{number of variables} + \text{number of clauses}$ columns
2. For each variable x_i in the 3-CNF-SAT:
    1. create a column for x_i.
    2. create two rows v_i and v_i'.
3. For each clause c_i in the 3-CNF-SAT:
    1. create a column for c_i.
    2. create two rows s_i and s_i'.
4. Fill cells of (v_i, x_i) and (v_i', x_i) with 1's, and the rest is 0's
5. Fill cells of (v_i, c_i) with
    - 1's if x_i appears in c_i
    - 0's if x_i does not appear in c_i
6. Fill cells of v_i' with inverse of v_i
    - 0 if v_i is 1
    - 1 if v_i is 0
7. Fill cell (s_i, c_i) with 1, and 0's for rest
8. Fill cell (s_i', c_i) with 2, and 0's for rest
9. Looking at each row as a 10-base number, try to find subset-sum of these numbers that can sum up to $t = 1111..4444..$ where there are number of variables $|X|$ 1's and number of clauses $|C|$ 4's
10. The subset-sum of those numbers (rows) is the input solution to the 3-CNF-SAT

		x_1	x_2	x_3	C_1	C_2	C_3	C_4
v_1	=	1	0	0	1	0	0	1
v_1'	=	1	0	0	0	1	1	0
v_2	=	0	1	0	0	0	0	1
v_2'	=	0	1	0	1	1	1	0
v_3	=	0	0	1	0	0	1	1
v_3'	=	0	0	1	1	1	0	0
s_1	=	0	0	0	1	0	0	0
s_1'	=	0	0	0	2	0	0	0
s_2	=	0	0	0	0	1	0	0
s_2'	=	0	0	0	0	2	0	0
s_3	=	0	0	0	0	0	1	0
s_3'	=	0	0	0	0	0	2	0
s_4	=	0	0	0	0	0	0	1
s_4'	=	0	0	0	0	0	0	2
t	=	1	1	1	4	4	4	4

Table corresponding to

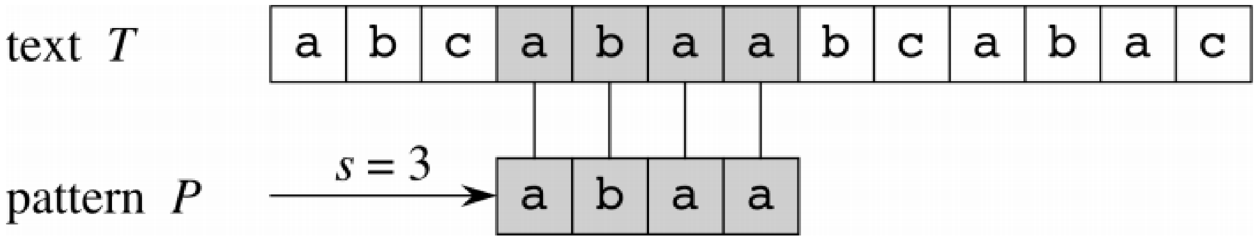
$$\phi = C_1 \wedge C_2 \wedge C_3 \wedge C_4 = (x_1 \vee \neg x_2 \vee \neg x_3) \wedge (\neg x_1 \vee \neg x_2 \vee \neg x_3) \wedge (\neg x_1 \vee \neg x_2 \vee x_3) \wedge (x_1 \vee x_2 \vee x_3)$$

Try to find subset-sum from the set
$\{1001001, 1000110, 100001, 101110, 10011, 11100, 1000, 2000, 100, 200, 10, 20, 1, 2\}$ *with* $t = 1114444$

String Matching

Find all occurrences of a string (pattern) p inside another string (text) t.

Can also be seen as the shift s that makes p match t



Naive Approach

- Iterate over all possible overlaps and check if there is a match

```
def naive_matcher(P, T):
  n = T.length
  m = P.length

  for i in 0:(n-m)
    if P[1..m] == T[i+1..i+m]
      # match found
```

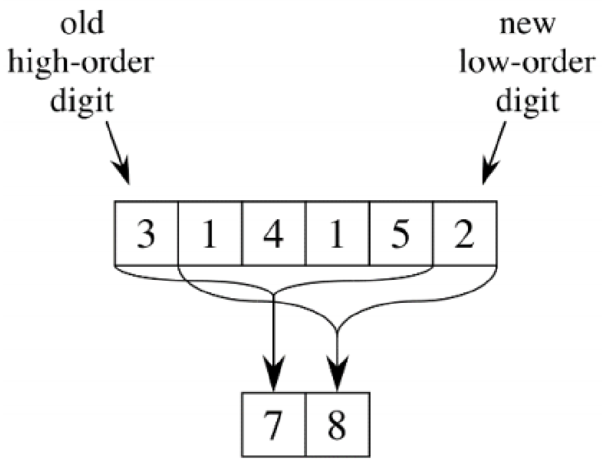
- **Runtime:** $O((n-m+1)m)$

Rabin-Karp

- Compare hashes instead of comparing strings
- This is more efficient because we can compute a new hash using previous hash.

$$hash(i+1) = (hash(i) - T[i+1] * c_1) * c_2 + T[i+m+1] \ mod \ c_3$$

This relation corresponds to taking old hash, removing the most significant character, adding the new character, then taking mod.

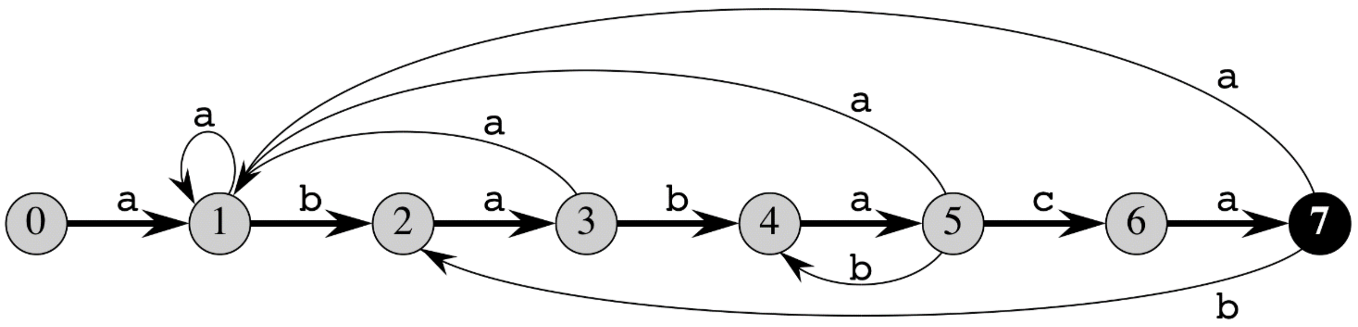


```
def rabin_karp_matcher(P, T, d, q): # d and q are hash parameters
  n = T.length
  m = P.length

  p = 0 # hash of P
  t = 0 # hash of T

  for i in 1:m
    p = (d * p + P[i]) mod q # compute hash of pattern P
    t = (d * p + T[i]) mod q # compute hash of first part of T

  for i in 0:(n-m)
    if p == t # hashes match
      if P[1..m] == T[i+1..t+m] # check if collision or not
        # pattern occurs
    if i < n - m
      t = (d * (t - T[i+1] * h) + T[i+m+1]) mod q # update hash of T
```

- **Runtime:**
  - Preprocessing: $\Theta(m)$
  - Matching:
    - Average case $O(n+m)$
    - Worst case $O((n-m+1)m)$
      - Happens if hashes always match.

Finite Automata

- Build a finite automata for the pattern P and then traverse it on the string.
- Must build the automata table correctly.
    - Find the longest string that is a suffix of $P_q a$
        - Start from state 0 and calculate where transitions should go
        - For each state q, and on adding a letter a
            - Let $k = q + 1$
            - Check if P_k is suffix of $P_q a$
                - If true, then $D(q, a) = k$
                - If false, decrement k and try again.



(a)

	input			
state	a	b	c	P
0	1	0	0	a
1	1	2	0	b
2	3	0	0	a
3	1	4	0	b
4	5	0	0	a
5	1	4	6	c
6	7	0	0	a
7	1	2	0	

(b)

i	—	1	2	3	4	5	6	7	8	9	10	11
$T[i]$	—	a	b	a	b	a	b	a	c	a	b	a
state $\phi(T_i)$	0	1	2	3	4	5	4	5	6	7	2	3

(c)

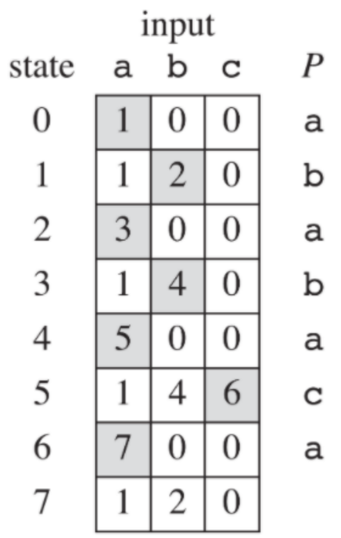
```
def compute_automata_table(P):
  m = P.length

  D = new matrix m * alphabet_size

  for q in 0:m
    for each character c in alphabet
      k = min(m + 1, q + 2) # to avoid overflow
      do
        k = k - 1
      until P[1..k] is suffix of P[1..q] + c

      D[q,a] = k

  return D

def finite_automata_matcher(T, D, m):
  n = T.length
  q = 0
  for i in 1:n
    q = D[q, T[i]]  # new state
    if q == m       # if we reached last state, then there is a match
      # match occurs
```

Runtime:

- Preprocessing: $O(m|\Sigma|)$
- Matching: $\Theta(n)$

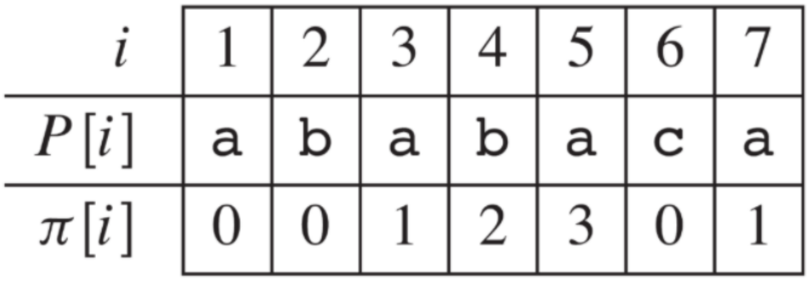
KMP

- Main idea is to optimize FA's preprocessing and squeeze the table to only one column.
- Check only transitions of mismatch
- The new transition table π is computed as follows:
  - For each state $\pi[i]$, find the length of the longest suffix in π that is also a prefix in π



	input			
state	a	b	c	P
0	1	0	0	a
1	1	2	0	b
2	3	0	0	a
3	1	4	0	b
4	5	0	0	a
5	1	4	6	c
6	7	0	0	a
7	1	2	0	

⇒



i	1	2	3	4	5	6	7
$P[i]$	a	b	a	b	a	c	a
$\pi[i]$	0	0	1	2	3	0	1

```
def compute_KMP_table(P):
  m = P.length
  pi[1..m] = new array
  pi[1] = 0

  k = 0

  for q in 2:m
    while k > 0 and P[k+1] != P[q]
      # while the character does not match next character
      # keep backtracking
      k = pi[k]
    if P[k+1] == P[q]
      k = k + 1
    pi[q] = k

  return pi

def KMP_matcher(T, P):
  n = T.length
  m = P.length
  pi = compute_KMP_table(P)

  q = 0
  for i in 1:n
    while q > 0 and P[q + 1] != T[i]
      q = pi[q]

    if P[q+1] == T[i]
      q = q + 1

    if q == m
      # match occurs
      q = pi[q]
```

- **Runtime:**
  - Preprocessing: $\Theta(m)$
  - Matching: $\Theta(n)$