Notes for Differential Equations with Boundary-Value Problems

CHAPTER 1 Introduction to Differential Equations

Differential Equation (ordinary differential equation (ODE), partial differential equation (PDE)).

The words *differential* and *equations* suggest solving some kind of equation that contains derivatives $y', y'', \cdots$.

1.1 Definitions and Terminology

INTRODUCTION

The derivative $dy\,/\,dx$ of a function $y = \phi(x)$ is itself another function $\phi'(x)$ found by an appropriate rule.

How do you solve an equation such as $\frac{dy}{dx} = 0.2xy$ for the function $y = \phi(x)$?

A Definition

> **Definition 1.1.1 Differential Equation**
> An equation containing the derivatives of one or more unknown functions (or dependent variables), with respect to one or more independent variables, is said to be a **differential equation** (**DE**).

To talk about them, we shall classify differential equations according to **type**, **order**, and **linearity**.

CLASSIFICATION BY TYPE

If a differential equation contains only ordinary derivatives of one or more unknown functions with respect to a single independent variable, it is said to be an **ordinary differential equation** (**ODE**). An equation involving partial derivatives of one or more unknown functions of two or more independent variables is called a **partial differential equation** (**PDE**).

EXAMPLE 1: Types of Differential Equations

(a) The equations

an ODE can contain more than one unknown function

$$\frac{dy}{dx} + 5y = e^x, \quad \frac{d^2y}{dx^2} - \frac{dy}{dx} + 6y = 0, \quad \text{and} \quad \frac{dx}{dt} + \frac{dy}{dt} = 2x + y$$

are examples of ordinary differential equations.

(b) The following equations are partial differential equations:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \quad \frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2} - 2\frac{\partial u}{\partial t}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

Notice in the third equation that there are two unknown functions and two independent variables in the PDE. This means u and v must be functions of two or more independent variables.

NOTATION

Throughout this text ordinary derivatives will be written by using either the **Leibniz notation** $dy/dx, d^2y/dx^2, d^3y/dx^3, \cdots$ or the **prime notation** $y', y'', y''', \cdots$. Although less convenient to write and to typeset, the Leibniz notation has an advantage over the prime notation in that it clearly displays both the dependent and independent variables. For example, in the equation

unknown function or dependent variable

$$\frac{d^2x}{dt^2} + 16x = 0$$

independent variable

Partial derivatives are often denoted by a **subscript notation** indicating the independent variables. For example, with the subscript notation the second equation $\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2} - 2\frac{\partial u}{\partial t}$ becomes $u_{xx} = u_{tt} - 2u_t$.

CLASSIFICATION BY ORDER

The **order of a differential equation** (either ODE or PDE) is the order of the highest derivative in the equation. For example,

$$\underbrace{\frac{d^2y}{dx^2}}_{\text{second order}} + 5\underbrace{\left(\frac{dy}{dx}\right)^3}_{\text{first order}} - 4y = e^x$$

is a second-order ordinary differential equation. A first-order ordinary differential equation is sometimes written in the **differential form**

$$M(x, y)dx + N(x, y)dy = 0$$

EXAMPLE 2: Differential Form of a First-Order ODE

In symbols we can express an nth-order ordinary differential equation in one dependent variable by the general form

$$F(x, y, y', \cdots, y^{(n)}) = 0,$$

where F is a real-valued function of $n+2$ variables: $x, y, y', \cdots, y^{(n)}$. For both practical and theoretical reasons we shall also make the assumption hereafter that it is possible to solve an ordinary differential equation in the form above uniquely for the highest derivative $y^{(n)}$ in terms of the remaining $n+1$ variables. The differential equation

$$\frac{d^n y}{dx^n} = f(x, y, y', \cdots, y^{(n-1)}),$$

where f is a real-valued continuous function, is referred to as the **normal form** of $F(x, y, y', \cdots, y^{(n)}) = 0$. Thus when it suits our purposes, we shall use the normal forms

$$\frac{dy}{dx} = f(x, y) \quad \text{and} \quad \frac{d^2y}{dx^2} = f(x, y, y')$$

to represent general first- and second-order ordinary differential equations.

EXAMPLE 3: Normal Form of an ODE

CLASSIFICATION BY LINEARITY

An nth-order ordinary differential equation $F(x, y, y', \cdots, y^{(n)}) = 0$ is said to be **linear** if F is linear in $y, y', \cdots, y^{(n)}$. This means that an nth-order ODE is linear when $F(x, y, y', \cdots, y^{(n)}) = 0$ is $a_n(x)y^{(n)} + a_{n-1}(x)y^{(n-1)} + \cdots + a_1(x)y' + a_0(x)y - g(x) = 0$ or

$$a_n(x)\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_1(x)\frac{dy}{dx} + a_0(x)y = g(x).$$

Two important special cases of it above are linear first-order ($n = 1$) and linear second-order ($n = 2$) DEs:

$$a_1(x)\frac{dy}{dx} + a_0(x)y = g(x) \quad \text{and} \quad a_2(x)\frac{d^2y}{dx^2} + a_1(x)\frac{dy}{dx} + a_0(x)y = g(x).$$

In the additive combination on the left-hand side of equation $a_n(x)y^{(n)} + a_{n-1}(x)y^{(n-1)} + \cdots + a_1(x)y' + a_0(x)y = g(x)$ we see that the characteristic two properties of a linear ODE are as follows:

- The dependent variable y and all its derivatives $y', y'', \cdots, y^{(n)}$ are of the first degree, that is, the power of each term involving y is 1.
- The coefficients $a_0, a_1, \cdots, a_n$ of $y, y', \cdots, y^{(n)}$ depend at most on the independent variable x.

A **nonlinear** ordinary differential equation is simply one that is not linear. Nonlinear functions of the dependent variable or its derivatives, such as $\sin y$ or $e^{y'}$, cannot appear in a linear equation.

EXAMPLE 4: Linear and Nonlinear ODEs

(a) The equations

$$(y - x)\,dx + 4x\,dy = 0, \quad y'' - 2y' + y = 0, \quad x^3\frac{d^3y}{dx^3} + x\frac{dy}{dx} - 5y = e^x$$

are, in turn, *linear* first-, second-, and third-order ordinary differential equations.

(b) The equations

$$\underset{\substack{\text{nonlinear term:}\\ \text{coefficient depends on } y}}{(1-y)y'} + 2y = e^x, \qquad \frac{d^2y}{dx^2} + \underset{\substack{\text{nonlinear term:}\\ \text{nonlinear function of } y}}{\sin y} = 0, \qquad \text{and} \qquad \frac{d^4y}{dx^4} + \underset{\substack{\text{nonlinear term:}\\ \text{power not } 1}}{y^2} = 0$$

are examples of *nonlinear* first-, second-, and fourth-order ordinary differential equations, respectively.

Solutions

> **Definition 1.1.2 solution of an ODE**
> Any function ϕ, defined on an interval I and possessing at least n derivatives that are continuous on I, which when substituted into an nth-order ordinary differential equation reduces the equation to an identity, is said to be a **solution** of the equation on the interval.

In other words, a solution of an nth-order ordinary differential equation $F(x, y, y', \cdots, y^{(n)}) = 0$ is a function ϕ that possesses at least n derivatives and for which

$$F\left(x, \phi(x), \phi'(x), \cdots, \phi^{(n)}(x)\right) = 0$$

We say that ϕ satisfies the differential equation on I. For our purposes we shall also assume that a solution ϕ is a real-valued function.

Occasionally, it will be convenient to denote a solution by the alternative symbol $y(x)$.

INTERVAL OF DEFINITION

You cannot think solution of an ordinary differential equation without simultaneously thinking interval. The interval I in Denition 1.1.2 is variously called the **interval of definition**, the **interval of existence**, the **interval of validity**, or the **domain of the solution** and can be an open interval (a, b), a closed interval $[a, b]$, an infinite interval (a, ∞), and so on.

EXAMPLE 5: Verification of a Solution

Note, too, that each differential equation in Example 5 possesses the constant solution $y = 0$, $-\infty < x < \infty$. A solution of a differential equation that is identically zero on an interval I is said to be a **trivial solution**.

SOLUTION CURVE

The graph of a solution ϕ of an ODE is called a **solution curve**. Since ϕ is a differentiable function, it is continuous on its interval I of definition. Thus there may be a difference between the graph of the *function* ϕ and the graph of the *solution* ϕ. Put another way, the domain of the function ϕ need not be the same as the interval I of definition (or domain) of the solution ϕ.

EXAMPLE 6: Function versus Solution

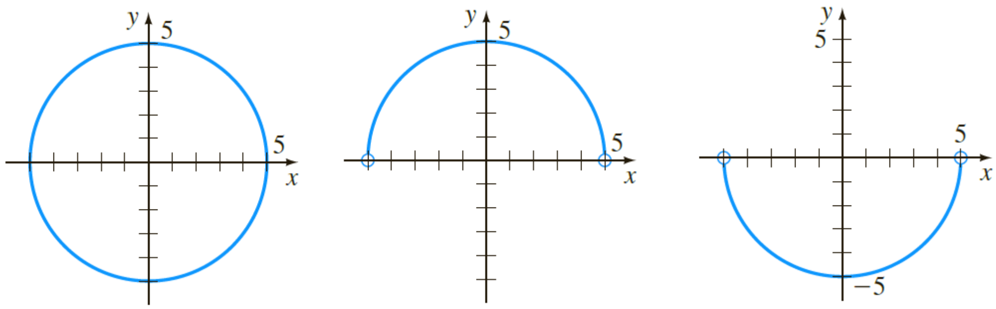
EXPLICIT AND IMPLICIT SOLUTIONS

You should be familiar with the terms *explicit functions* and *implicit functions* from your study of calculus. A solution in which the dependent variable is expressed solely in terms of the independent variable and constants is said to be an **explicit solution**.

When we get down to the business of actually solving some ordinary differential equations, you will see that methods of solution do not always lead directly to an explicit solution $y = \phi(x)$. This is particularly true when we attempt to solve nonlinear first-order differential equations. Often we have to be content with a relation or expression $G(x, y) = 0$ that defines a solution ϕ implicitly.

> **Definition 1.1.3 Implicit solution of an ODE**
> A relation $G(x, y) = 0$ is said to be an **implicit solution** of an ordinary differential equation $F(x, y, y', \cdots, y^{(n)}) = 0$ on an interval I, provided that there exists at least one function ϕ that satisfies the relation as well as the differential equation on I.

EXAMPLE 7: Verification of an Implicit Solution



(a) implicit solution $x^2 + y^2 = 25$ **(b)** explicit solution $y_1 = \sqrt{25 - x^2},\ -5 < x < 5$ **(c)** explicit solution $y_2 = -\sqrt{25 - x^2},\ -5 < x < 5$

Figure 1. 1 An implicit solution and two explicit solutions in Example 7

FAMILIES OF SOLUTIONS

The study of differential equations is similar to that of integral calculus. When evaluating an antiderivative or indefinite integral in calculus, we use a single constant c of integration. Analogously, we shall see in Chapter 2 that when solving a first-order differential equation $F(x, y, y') = 0$ we *usually* obtain a solution containing a single constant or parameter c. A solution of $F(x, y, y') = 0$ containing a constant c is a set of solutions $G(x, y, c) = 0$ called a **one-parameter family of solutions**. When solving an nth-order differential equation $F(x, y, y', \cdots, y^{(n)}) = 0$ we seek an n-parameter family of solutions $G(x, y, c_1, c_2, \cdots, c_n) = 0$. This means that a single differential equation can possess an infinite number of solutions corresponding to an unlimited number of choices for the parameter(s). A solution of a differential equation that is free of parameters is called a **particular solution**.

The parameters in a family of solutions such as $G(x, y, c_1, c_2, \cdots, c_n) = 0$ are *arbitrary* up to a point. However, it is understood that the relation should always make sense in the real number system; thus, if $c = -25$ we cannot say that $x^2 + y^2 = -25$ is an implicit solution of the differential equation.

EXAMPLE 8: Particular Solutions

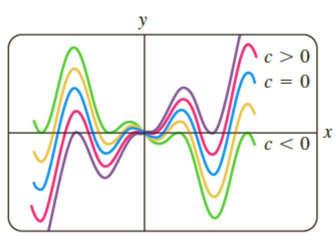


Figure 1. 2 Some solutions of DE in part (a) of Example 8

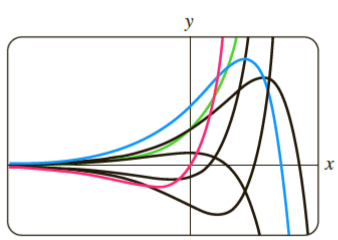
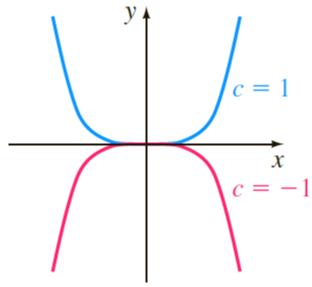


Figure 1. 3 Some solutions of DE in part (b) of Example 8

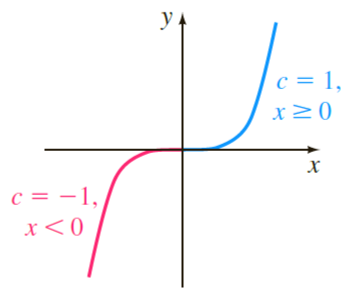
Sometimes a differential equation possesses a solution that is not a member of a family of solutions of the equation—that is, a solution that cannot be obtained by specializing any of the parameters in the family of solutions. Such an extra solution is called a **singular solution**.

EXAMPLE 9: Using Different Symbols

EXAMPLE 10: Piecewise-Defined Solution



(a) two explicit solutions



(b) piecewise-defined solution

Figure 1. 4 Some solutions of DE in Example 10

SYSTEMS OF DIFFERENTIAL EQUATIONS

Up to this point we have been discussing single differential equations containing one unknown function. But often in theory, as well as in many applications, we must deal with systems of differential equations. A **system of ordinary differential equations** is two or more equations involving the derivatives of two or more unknown functions of a single independent variable. For example, if x and y denote dependent variables and t denotes the independent variable, then a system of two first-order differential equations is given by

$$\frac{dx}{dt} = f(t, x, y)$$
$$\frac{dy}{dt} = g(t, x, y).$$

A **solution** of a system such as above equations is a pair of differentiable functions $x = \phi_1(t)$, $y = \phi_2(t)$, defined on a common interval I, that satisfy each equation of the system on this interval.

1.2 Initial-Value Problems

INTRODUCTION

We are often interested in problems in which we seek a solution $y(x)$ of a differential equation so that $y(x)$ also satisfies certain prescribed side conditions, that is, conditions that are imposed on the unknown function $y(x)$ and its derivatives at a number x_0. On some interval I containing x_0 the problem of solving an nth-order differential equation subject to n side conditions specied at x_0:

$$Solve: \quad \frac{d^n y}{dx^n} = f\left(x, y, y', \cdots, y^{(n-1)}\right)$$
$$Subject\ to: \quad y(x_0) = y_0, y'(x_0) = y_1, \cdots, y^{(n-1)}(x_0) = y_{n-1},$$

where $y_0, y_1, \cdots, y_{n-1}$ are arbitrary constants, is called an ***n*th-order initial-value problem** (**IVP**). The values of $y(x)$ and its first $n-1$ derivatives at x_0, $y(x_0) = y_0$, $y'(x_0) = y_1$, $\cdots$, $y^{(n-1)}(x_0) = y_{n-1}$ are called **initial conditions** (**IC**).

GEOMETRIC INTERPRETATION

The cases $n = 1$ and $n = 2$,

$$\begin{aligned} Solve: \quad & \frac{dy}{dx} = f(x, y) \\ Subject\ to: \quad & y(x_0) = y_0 \end{aligned}$$

and

$$\begin{aligned} Solve: \quad & \frac{d^2y}{dx^2} = f(x, y, y') \\ Subject\ to: \quad & y(x_0) = y_0, y'(x_0) = y_1 \end{aligned}$$

are examples of **first-** and **second-order** initial-value problems, respectively. These two problems are easy to interpret in geometric terms.

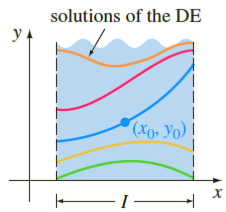


Figure 1. 5 Solution curve of first-order IVP

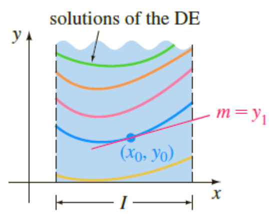
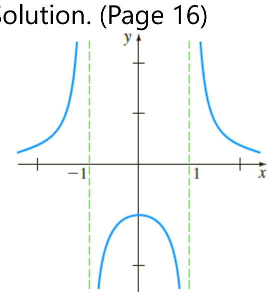
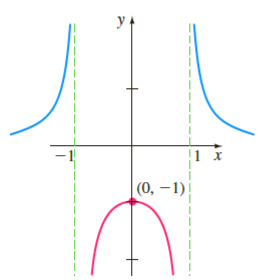


Figure 1. 6 Solution curve of second-order IVP

EXAMPLE 2: Interval I of Definition of a Solution. (Page 16)



(a) function defined for all x except $x = \pm 1$



(b) solution defined on interval containing $x = 0$

Figure 1. 7 Graphs of function and solution of IVP in Example 2

EXISTENCE AND UNIQUENESS

Two fundamental questions arise in considering an initial-value problem:

Does a solution of the problem exist? If a solution exists, is it unique?

For the first-order initial-value problem we ask:

Existence $\begin{cases} \textit{Does the differential equation } dy/dx = f\,(x,y) \textit{ possess solutions?} \\ \textit{Do any of the solution curves pass through the point } (x_0, y_0)\textit{?} \end{cases}$

Uniqueness $\begin{cases} \textit{When can we be certain that there is precisely one solution curve} \\ \textit{passing through the point } (x_0, y_0)\textit{?} \end{cases}$

Within the safe confines of a formal course in differential equations one can be fairly confident that *most* differential equations will have solutions and that solutions of initial-value problems will *probably* be unique. Real life, however, is not so idyllic.

We state here without proof a straightforward theorem that gives conditions that are sufficient to guarantee the existence and uniqueness of a solution of a first-order initial-value problem of the form $dy/dx = f(x, y)$.

Definition 1.2.1 Existence of a Unique solution

Let R be a rectangular region in the xy-plane defined by $a \leq x \leq b$, $c \leq y \leq d$ that contains the point (x_0, y_0) in its interior. If $f(x, y) = dy/dx$ and $\partial f/\partial y$ are continuous on R, then there exists some interval I_0: $(x_0 - h, x_0 + h), h > 0$, contained in $[a, b]$, and a unique function $y(x)$, defined on I_0, that is a solution of the initial-value problem $dy/dx = f(x, y)$.

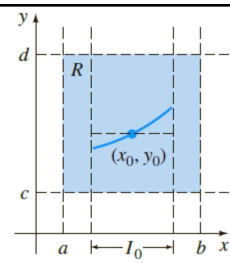


Figure 1. 8 Rectangular region R

INTERVAL OF EXISTENCE/UNIQUENESS

Suppose $y(x)$ represents a solution of the initial-value problem $dy/dx = f(x, y)$. The following three sets on the real x-axis may not be the same: the domain of the function $y(x)$, the interval I over which the solution $y(x)$ is defined or exists, and the interval I_0 of existence and uniqueness.

REMARKS

i. The conditions in Theorem 1.2.1 are sufficient but not necessary.

ii. You are encouraged to read, think about, work, and then keep in mind Problem 50 in Problem 50 in Exercises 1.2.

iii. Initial conditions are prescribed at a *single* point x_0. But we are also interested in solving differential equations that are subject to conditions specified on $y(x)$ or its derivative at *two* different points x_0 and x_1. Conditions such as

$$y(1) = 0, \quad y(5) = 0 \quad \text{or} \quad y(\pi/2) = 0, \quad y'(\pi) = 1$$

are called **boundary conditions** (**BC**). A differential equation together with boundary conditions is called a **boundary-value problem** (**BVP**). For example,

$$y'' + y = 0, \quad y'(0) = 0, \quad y'(\pi) = 0$$

is a boundary-value problem.

1.3 Differential Equations as Mathematical Models

CHAPTER 2 Introduction to Differential Equations

Before we start solving anything, you should be aware of two facts: A differential equation may have no solutions, and a differential equation may possess solutions yet there might not exist any analytical method for solving it.

2.1 Solution Curves Without a Solution

We begin our study of first-order differential equations with two ways of analyzing a DE qualitatively. Both of these ways enable us to determine, in an approximate sense, what a solution curve must look like without actually solving the equation.

DIRECTION FIELDS

SOME FUNDAMENTAL QUESTIONS

How does a solution behave near a certain point? How does a solution behave as $x \to \infty$?—can often be answered when the function f depends solely on the variable y. We begin, however, with a simple concept from calculus:

A derivative dy/dx of a differentiable function $y = y(x)$ gives slopes of tangent lines at points on its graph.

SLOPE

Because a solution $y = y(x)$ of a first-order differential equation

$$\frac{dy}{dx} = f(x, y)$$

is necessarily a differentiable function on its interval I of definition, it must also be continuous on I. Thus the corresponding solution curve on I must have no breaks and must possess a tangent line at each point $(x, y(x))$. The function f in the normal form $dy/dx = f(x, y)$ is called the **slope function** or **rate function**. The value $f(x, y)$ that the function f assigns to the point represents the slope of a line or, as we shall envision it, a line segment called a **lineal element**. For example, consider the equation $dy/dx = 0.2xy$, where $f(x, y) = 0.2xy$, at point (2,3).

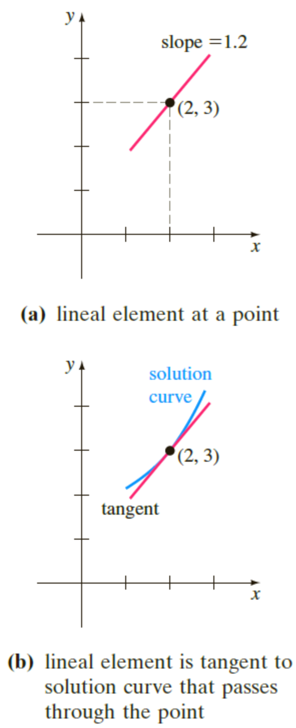


Figure 2. 1 A solution curve is tangent to lineal element at (2,3)

DIRECTION FIELD

If we systematically evaluate f over a rectangular grid of points in the xy-plane and draw a line element at

each point (x, y) of the grid with slope $f(x, y)$, then the collection of all these line elements is called a **direction field** or a **slope field** of the differential equation $dy/dx = f(x, y)$. Visually, the direction field suggests the appearance or shape of a family of solution curves of the differential equation, and consequently, it may be possible to see at a glance certain qualitative aspects of the solutions-regions in the plane.
Figure 2.2 shows a computer-generated direction field of the differential equation $dy/dx = \sin(x + y)$ over a region of the xy-plane.

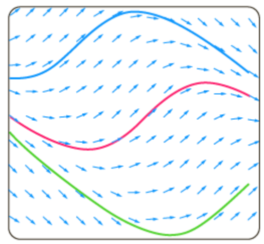
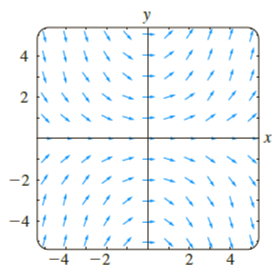
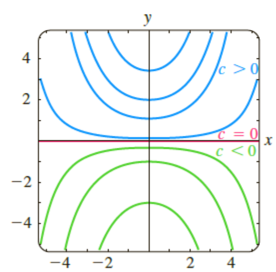


Figure 2. 2 Solution curves following flow of a direction field

EXAMPLE 1: Direction Field



(a) direction field for
$dy/dx = 0.2xy$



(b) some solution curves in the family $y = ce^{0.1x^2}$

Figure 2. 3 Direction field and solution curves in Example 1

INCREASING/DECREASING
Interpretation of the derivative dy/dx as a function that gives slope plays the key role in the construction of a direction field. Another telling property of the first derivative will be used next, namely, if $dy/dx > 0$ (or $dy/dx < 0$) for all x in an interval I, then a differentiable function $y = y(x)$ is increasing (or decreasing) on I.
AUTONOMOUS FIRST-ORDER DEs
AUTONOMOUS FIRST-ORDER DES
An ordinary differential equation in which the independent variable does not appear explicitly is said to be **autonomous**. If the symbol x denotes the independent variable, then an autonomous first-order differential equation can be written as $f(y, y')$ or in normal form as

$$\frac{dy}{dx} = f(y).$$

We shall assume throughout that the function f above and its derivative f' are continuous functions of y on some interval I. The first-order equations

$$\frac{dy}{dx} = \overset{\substack{f(y)\\ \downarrow}}{1 + y^2} \qquad \text{and} \qquad \frac{dy}{dx} = \overset{\substack{f(x,\,y)\\ \downarrow}}{0.2xy}$$

are autonomous and nonautonomous, respectively.

CRITICAL POINTS

The zeros of the function $f(y) = dy/\,dx$ are of special importance. We say that a real number c is a **critical point** of the autonomous differential equation if it is a zero of f-that is, $f(c) = 0$. A critical point is also called an **equilibrium point** or **stationary point**. Now observe that if we substitute the constant function $y(x) = c$ into $f(y)$, then both sides of the equation are zero. This means:

If c is a critical point of $f(y) = dy/\,dx$, then $y(x) = c$ is a constant solution of the autonomous differential equation.

A constant solution $y(x) = c$ of $f(y) = dy/\,dx$ is called an **equilibrium solution**; equilibria are the *only* constant solutions of $f(y) = dy/\,dx$.

2.2 Separable Equations

SOLUTION BY INTEGRATION

Consider the first-order differential equation $dy/\,dx = f(x, y)$. When f does not depend on the variable y, that is, $f(x, y) = g(x)$, the differential equation

$$\frac{dy}{dx} = g(x)$$

can be solved by **integration**. If $g(x)$ is a continuous function, then integrating both sides of it gives $y = \int g(x)\, dx = G(x) + c$, where $G(x)$ is an antiderivative (indefinite integral) of $g(x)$.

A DEFINITION

> **DEFINITION 2.2.1 Separable Equation**
> A first-order differential equation of the form
> $$\frac{dy}{dx} = g(x)h(y)$$
> is said to be **separable** or to have **separable variables**.

For example, the equations

$$\frac{dy}{dx} = y^2xe^{3x+4y} \quad \text{and} \quad \frac{dy}{dx} = y + \sin x$$

are separable and nonseparable, respectively.

$$f(x, y) = y^2xe^{3x+4y} = \overset{\substack{g(x)\\ \downarrow}}{(xe^{3x})}\overset{\substack{h(y)\\ \downarrow}}{(y^2e^{4y})},$$

Observe that by dividing by the function $h(y)$, we can write a separable equation $dy/\,dx = g(x)h(y)$ as

$$p(y)\frac{dy}{dx} = g(x),$$

where, for convenience, we have denoted $1/h(y)$ by $p(y)$. From this last form we can see immediately that it reduces to $dy/\,dx = g(x)$ when $h(y) = 1$.

Now if $y = \phi(x)$ represents a solution of the separable equation, we must have $p(\phi(x))\phi'(x) = g(x)$, and therefore

$$\int p(\phi(x))\phi'(x)\, dx = \int g(x)\, dx.$$

But $dy = \phi'(x)\, dx$, and so the above is the same as

$$\int p(y)\,dy = \int g(x)\,dx \quad \text{or} \quad H(y) = G(x) + c,$$

where $H(y)$ and $G(x)$ are antiderivatives of $p(y) = 1/h(y)$ and $g(x)$, respectively.

METHOD OF SOLUTION

Equation $H(y) = G(x) + c$ indicates the procedure for solving separable equations. A one-parameter family of solutions, usually given implicitly, is obtained by integrating both sides of $p(y)\,dy = g(x)\,dx$.

NOTE

There is no need to use two constants in the integration of a separable equation.

LOSING A SOLUTION

Some care should be exercised in separating variables, since the variable divisors could be zero at a point. Specifically, if r is a zero of the function $h(y)$, then substituting $y = r$ into $dy/\,dx = g(x)h(y)$ makes both sides zero; in other words, $y = r$ is a constant solution of the differential equation.

But after variables are separated, the left-hand side of $\frac{dy}{h(y)} = g(x)$ is undefined at r. As a consequence, $y = r$ might not show up in the family of solutions that are obtained after integration and simplification. Recall that such a solution is called a singular solution.

EXAMPLE 3: Losing a Solution

Solve $\frac{dy}{dx} = y^2 - 4$. (Page 49)

AN INTEGRAL-DEFINED FUNCTION

A solution method for a certain kind of differential equation may lead to an integral-defined function. This is especially true for separable differential equations because integration *is* the method of solution. For example, if g is continuous on some interval I containing x_0 and x, then a solution of the simple initial-value problem $dy/\,dx = g(x)$, $y(x_0) = y_0$, defined on I, is given by

$$y(x) = y_0 + \int_{x_0}^{x} g(t)dt.$$

When $\int g(t)dt$ is nonelementary—that is, cannot be expressed in terms of elementary functions—the form $y(x) = y_0 + \int_{x_0}^{x} g(t)dt$ may be the best we can do in obtaining an explicit solution of an IVP. The next example illustrates this idea.

EXAMPLE 5: An Initial-Value Problem

Solve $\frac{dy}{dx} = e^{-x^2}$, $y(3) = 5$. (Page 51)

2.3 Linear Equations

A DEFINITION

The form of a linear first-order DE reproduced here for convenience.

> **DEFINITION 2.3.1 Linear Equation**
>
> A first-order differential equation of the form
>
> $$a_1(x)\frac{dy}{dx} + a_0(x)y = g(x),$$
>
> is said to be a **linear equation** in the variable y.

STANDARD FORM

By dividing both sides of $a_1(x)\frac{dy}{dx} + a_0(x)y = g(x)$ by the lead coefficient $a_1(x)$, we obtain a more useful form, the **standard form**, of a linear equation:

$$\frac{dy}{dx} + P(x)y = f(x).$$

We seek a solution of the above equation on an interval I for which both coefficient functions P and f are continuous.

Before we examine a general procedure for solving equations of the standard form we note that in some instances of it can be solved by separation of variables. For example, you should verify that the equations

$$\frac{dy}{dx} + 2xy = 0 \quad \text{and} \quad \frac{dy}{dx} = y + 5$$

are both linear and separable, but that the linear equation

$$\frac{dy}{dx} + y = x$$

is not separable.

METHOD OF SOLUTION

The method for solving $dy/dx + P(x)y = f(x)$ hinges on a remarkable fact that the *left-hand side* of the equation can be recast into the form of the exact derivative of a product by multiplying the both sides of it by a special function $\mu(x)$. It is relatively easy to find the function $\mu(x)$ because we want

$$\frac{d}{dx}\overbrace{[\mu(x)y]}^{\text{product}} = \overbrace{\mu\frac{dy}{dx} + \frac{d\mu}{dx}y}^{\text{product rule}} = \overbrace{\mu\frac{dy}{dx} + \mu P y}^{\text{left-hand side of (2) multipled by } \mu(x)}.$$

these must be equal

The equality is true provided that

$$\frac{d\mu}{dx} = \mu P.$$

The last equation can be solved by separation of variables. Integrating

$$\frac{d\mu}{\mu} = Pdx \quad \text{and solving} \quad \ln|\mu(x)| = \int P(x)dx + c_1$$

gives $\mu(x) = c_2 e^{\int P(x)dx}$. Even though there are infinite choices of $\mu(x)$ (all constant multiples of $e^{\int P(x)dx}$), all produce the same desired result. Hence we can simplify life and choose $c_2 = 1$. The function

$$\mu(x) = e^{\int P(x)dx}$$

is called an **integrating factor** for equation $dy/dx + P(x)y = f(x)$.

Here is what we have so far:

$$e^{\int P(x)dx}\frac{dy}{dx} + P(x)e^{\int P(x)dx}y = e^{\int P(x)dx}f(x)$$
$$\frac{d}{dx}\left[e^{\int P(x)dx}y\right] = e^{\int P(x)dx}f(x)$$
$$e^{\int P(x)dx}y = \int e^{\int P(x)dx}f(x)\,dx + c$$
$$y = e^{-\int P(x)dx}\int e^{\int P(x)dx}f(x)\,dx + ce^{-\int P(x)dx}.$$

Solving a Linear First-Order Equation

i. Remember to put a linear equation into the standard form $dy/dx + P(x)y = f(x)$.
ii. From the standard form of the equation identify $P(x)$ and then find the integrating factor $e^{\int P(x)dx}$. No constant need be used in evaluating the indefinite integral $\int P(x)dx$.
iii. Multiply the both sides of the standard form equation by the integrating factor. The left-hand side of the resulting equation is automatically the derivative of the product of the integrating factor $e^{\int P(x)dx}$ and y:
iv. Integrate both sides of the last equation and solve for y.

EXAMPLE 1: Solving a Linear Equation

Solve $\frac{dy}{dx} - 3y = 0$.

Solution

The integrating factor is $e^{\int(-3)dx} = e^{-3x}$. We then multiply the given equation by this factor and recognize that

$$e^{-3x}\frac{dy}{dx} - 3e^{-3x}y = e^{-3x}\cdot 0 \quad \text{is the same as} \quad \frac{d}{dx}[e^{-3x}y] = 0.$$

Integration of the last equation,

$$\int \frac{d}{dx}[e^{-3x}y]\,dx = \int 0\,dx$$

then yields $e^{-3x}y = c$ or $y = ce^{3x}$, $-\infty < x < \infty$.

EXAMPLE 2: Solving a Linear Equation

Solve $\frac{dy}{dx} - 3y = 6$.

Solution

The integrating factor is $e^{\int(-3)dx} = e^{-3x}$.

$$e^{-3x}\frac{dy}{dx} - 3e^{-3x}y = 6e^{-3x} \quad \text{and so} \quad \frac{d}{dx}[e^{-3x}y] = 6e^{-3x}.$$

$$\int \frac{d}{dx}[e^{-3x}y]\,dx = 6\int e^{-3x}\,dx \quad \text{gives} \quad e^{-3x}y = -6\left(\frac{e^{-3x}}{3}\right) + c$$

or $y = -2 + ce^{3x}$, $-\infty < x < \infty$.

When a_1, a_0, and g in linear equation are constants, the differential equation is autonomous. In Example 2 you can verify from the normal form that -2 is a critical point and that it is unstable (a repeller).

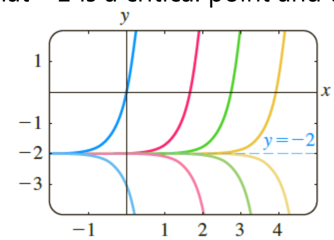


Figure 2. 4 Solution curves of DE in Example 2

GENERAL SOLUTION

Suppose again that the functions P and f in $\frac{dy}{dx} + P(x)y = f(x)$ are continuous on a common interval I. $y = e^{-\int P(x)dx}\int e^{\int P(x)dx}f(x)\,dx + ce^{-\int P(x)dx}$ is a one parameter family of solutions of equation $\frac{dy}{dx} + P(x)y = f(x)$ and *every solution* of it defined on I is a member of this family. Therefore we call $y = e^{-\int P(x)dx}\int e^{\int P(x)dx}f(x)\,dx + ce^{-\int P(x)dx}$ the **general solution** of the differential equation on the interval I. Now by writing it in the normal form $y' = F(x, y)$, we can identify $F(x, y) = -P(x)y + f(x)$ and $\partial F/\partial y = -P(x)$. From the continuity of P and f on the interval I we see that F and $\partial F/\partial y$ are also continuous on I. With Theorem 1.2.1 as our justification, we conclude that there exists one and only one solution of the first-order initial-value problem

$$\frac{dy}{dx} + P(x)y = f(x), \quad y(x_0) = y_0$$

defined on some interval I_0 containing x_0.

EXAMPLE 3: General Solution

Solve $x\frac{dy}{dx} - 4y = x^6e^x$.

Solution

Dividing by x, the standard form of the given DE is

$$\frac{dy}{dx} - \frac{4}{x}y = x^5e^x.$$

From this form we identify $P(x) = -4/x$ and $f(x) = x^5e^x$ and further observe that P and f are continuous on $(0, \infty)$. Hence the integrating factor is

we can use $\ln x$ instead of $\ln|x|$ since $x > 0$

$$e^{-4\int dx/x} = e^{-4\ln x} = e^{\ln x^{-4}} = x^{-4}.$$

Now we multiply and rewrite

$$x^{-4}\frac{dy}{dx} - 4x^{-5}y = xe^x \quad \text{as} \quad \frac{d}{dx}[x^{-4}y] = xe^x.$$

It follows from integration by parts that the general solution defined on the interval $(0, \infty)$ is $x^{-4}y = xe^x - e^x + c$ or $y = x^5e^x - x^4e^x + cx^4$.

Values of x for which $a_1(x) = 0$ are called **singular points** of the equation. Singular points are potentially troublesome. Specically, $P(x)$ (formed by dividing $a_0(x)$ by $a_1(x)$) is discontinuous at a point, the discontinuity may carry over to solutions of the differential equation.

EXAMPLE 4: General Solution

Find the general solution of $(x^2 - 9)\frac{dy}{dx} + xy = 0$.

Solution

$$\frac{dy}{dx} + \frac{x}{x^2 - 9}y = 0$$

Although $P(x)$ is continuous on $(-\infty, -3)$, $(-3,3)$, and $(3, \infty)$, we shall solve the equation on the first and third intervals. On these intervals the integrating factor is

$$e^{\int x\,dx/(x^2-9)} = e^{1/2\int 2x\,dx/(x^2-9)} = e^{1/2\ln|x^2-9|} = \sqrt{x^2 - 9}.$$

and

$$\frac{d}{dx}\left[\sqrt{x^2 - 9}y\right] = 0.$$

Integration gives $\sqrt{x^2 - 9}y = c$. Thus on either $(-\infty, -3)$ or $(3, \infty)$ the general solution of the equation is $y = \frac{c}{\sqrt{x^2-9}}$.

PIECEWISE-LINEAR DIFFERENTIAL EQUATION

When either $P(x)$ or $f(x)$ in $\frac{dy}{dx} + P(x)y = f(x)$ is a piecewise-defined function the equation is then referred to as a **piecewise-linear differential equation**.

EXAMPLE 6: An Initial-Value Problem. (Page 59)

ERROR FUNCTION

In mathematics, science, and engineering some important functions are defined in terms of nonelementary integrals. Two such special functions are the **error function** and **complementary error function**:

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}}\int_0^x e^{-t^2}dt \quad \text{and} \quad \operatorname{erfc}(x) = \frac{2}{\sqrt{\pi}}\int_x^\infty e^{-t^2}dt.$$

From the known result $\int_0^\infty e^{-t^2}dt = \sqrt{\pi}/2$ we can write $\left(2/\sqrt{\pi}\right)\int_0^\infty e^{-t^2}dt = 1$. (standard normal distribution $\left(1/\sqrt{2\pi}\right)\int_{-\infty}^\infty e^{-y^2/2}\,dy = 1 \rightarrow \int_0^\infty e^{-y^2/2}\,dy = \sqrt{2\pi}/2 \rightarrow \int_0^\infty e^{-t^2}dt = \sqrt{\pi}/2\ \left(y = \sqrt{2}t\right)$)

Using the additive interval property of definite integrals $\int_0^\infty \quad = \int_0^x \quad + \int_x^\infty \quad$ we can rewrite the last result in the alternative form

$$\frac{2}{\sqrt{\pi}}\int_0^\infty e^{-t^2}dt = \overbrace{\frac{2}{\sqrt{\pi}}\int_0^x e^{-t^2}dt}^{\operatorname{erf}(x)} + \overbrace{\frac{2}{\sqrt{\pi}}\int_x^\infty e^{-t^2}dt}^{\operatorname{erfc}(x)} = 1.$$

It is seen from above that the error function $\operatorname{erf}(x)$ and complementary error function $\operatorname{erfc}(x)$ are related by the identity

$$\operatorname{erf}(x) + \operatorname{erfc}(x) = 1$$

Because of its importance in probability, statistics, and applied partial differential equations, the error function has been extensively tabulated. Note that $\operatorname{erf}(0) = 0$ is one obvious function value.

REMARKS

i. Occasionally, a first-order differential equation is not linear in one variable but is linear in the other variable. For example, the differential equation

$$\frac{dy}{dx} = \frac{1}{x + y^2}$$

is not linear in the variable y. But its reciprocal

$$\frac{dx}{dy} = x + y^2 \quad \text{or} \quad \frac{dx}{dy} - x = y^2$$

is recognized as linear in the variable x. You should verify that the integrating factor $e^{\int (-1)\,dy} = e^{-y}$ and integration by parts yield the explicit solution $x = -y^2 - 2y - 2 + ce^y$ for the second equation. This expression is then an implicit solution of the first equation.

2.4 Exact Equations

INTRODUCTION

Although the simple first-order differential equation

$$y\,dx + x\,dy = 0$$

is both separable and linear, we can solve the equation by an alternative manner by recognizing that the expression on the left-hand side of the equality is the differential of the function $f(x, y) = xy$; that is,

$$d(xy) = ydx + xdy.$$

DIFFERENTIAL OF A FUNCTION OF TWO VARIABLES

If $z = f(x, y)$ is a function of two variables with continuous first partial derivatives in a region R of the xy −plane, then recall from calculus that its **differential** is defined to be

$$dz = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy.$$

In the special case when $f(x, y) = c$, where c is a constant, then the above gives

$$\frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy = 0.$$

In other words, given a one-parameter family of functions $f(x, y) = c$, we can generate a first-order differential equation by computing the differential of both sides of the equality. For example, if $x^2 - 5xy + y^3 = c$, then the above gives the first-order DE

$$(2x - 5y)\,dx + (-5x + 3y^2)\,dy = 0.$$

A DEFINITION

DEFINITION 2.4.1 Exact Equation

A differential expression $M(x, y)\,dx + N(x, y)\,dy$ is an **exact differential** in a region R of the xy-plane if it corresponds to the differential of some function $f(x, y)$ defined in R. A first-order differential equation of the form

$$M(x, y)\,dx + N(x, y)\,dy = 0$$

is said to be an **exact equation** if the expression on the left-hand side is an exact differential.

For example, $x^2y^3\,dx + x^3y^2\,dy = 0$ is an exact equation, because its left-hand side is an exact differential:

$$d\left(\frac{1}{3}x^3y^3\right) = x^2y^3\,dx + x^3y^2\,dy.$$

Notice that if we make the identifications $M(x, y) = x^2y^3$ and $N(x, y) = x^3y^2$, then $\partial M/\partial y = 3x^2y^2 = \partial N/\partial x$.

THEOREM 2.4.1 Criterion for an Exact Differential
Let $M(x, y)$ and $N(x, y)$ be continuous and have continuous first partial derivatives in a rectangular region R defined by $a < x < b$, $c < y < d$. Then a necessary and sufficient condition that $M(x, y)\, dx + N(x, y)\, dy$ be an exact differential is
$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}.$$

PROOF OF THE NECESSITY
For simplicity let us assume that $M(x, y)$ and $N(x, y)$ have continuous first partial derivatives for all (x, y). Now if the expression $M(x, y)\, dx + N(x, y)\, dy$ is exact, there exists some function f such that for all x in R,
$$M(x, y)\, dx + N(x, y)\, dy = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy.$$
Therefore
$$M(x, y) = \frac{\partial f}{\partial x}, \quad N(x, y) = \frac{\partial f}{\partial y},$$
and
$$\frac{\partial M}{\partial y} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial N}{\partial x}.$$
The equality of the mixed partials is a consequence of the continuity of the first partial derivatives of $M(x, y)$ and $N(x, y)$.

METHOD OF SOLUTION
Given an equation in the differential form $M(x, y)\, dx + N(x, y)\, dy = 0$, determine whether the equality $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$. If it does, then there exists a function f for which
$$\frac{\partial f}{\partial x} = M(x, y).$$
We can find f by integrating $M(x, y)$ with respect to x while holding y constant:
$$f(x, y) = \int M(x, y)\, dx + g(y),$$
where the arbitrary function $g(y)$ is the "constant" of integration. Now differentiate the above with respect to y and assume that $\partial f / \partial y = N(x, y)$:
$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y} \int M(x, y)\, dx + g'(y) = N(x, y).$$
This gives
$$g'(y) = N(x, y) - \frac{\partial}{\partial y} \int M(x, y)\, dx.$$
Finally, integrate the above with respect to y and substitute the result in $f(x, y)$ above. The implicit solution of the equation is $f(x, y) = c$.
Some observations are in order. First, it is important to realize that the expression $N(x, y) - (\partial / \partial y) \int M(x, y)\, dx$ above is independent of x, because
$$\frac{\partial}{\partial x}\left[N(x, y) - \frac{\partial}{\partial y} \int M(x, y)\, dx\right] = \frac{\partial N}{\partial x} - \frac{\partial}{\partial y}\left(\frac{\partial}{\partial x} \int M(x, y)\, dx\right) = \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = 0.$$
Second, we could just as well start the foregoing procedure with the assumption that $\partial f / \partial y = N(x, y)$. After integrating N with respect to y and then differentiating that result, we would find the analogues of $f(x, y) = \int M(x, y)\, dx + g(y)$ and $g'(y) = N(x, y) - \frac{\partial}{\partial y} \int M(x, y)\, dx$ to be, respectively,
$$f(x, y) = \int N(x, y)\, dy + h(x) \quad \text{and} \quad h'(y) = M(x, y) - \frac{\partial}{\partial x} \int N(x, y)\, dy.$$
EXAMPLE 1: Solving an Exact DE
Solve $2xy\, dx + (x^2 - 1)\, dy = 0$.
SOLUTION

With $M(x, y) = 2xy$ and $N(x, y) = x^2 - 1$ we have

$$\frac{\partial M}{\partial y} = 2x = \frac{\partial N}{\partial x}.$$

Thus the equation is exact, and so by Theorem 2.4.1 there exists a function $f(x, y)$ such that

$$\frac{\partial f}{\partial x} = 2xy \quad \text{and} \quad \frac{\partial f}{\partial y} = x^2 - 1.$$

From the first of these equations we obtain, after integrating,

$$f(x, y) = x^2 y + g(y).$$

Taking the partial derivative of the last expression with respect to y and setting the result equal to $N(x, y)$ gives

$$\frac{\partial f}{\partial y} = x^2 + g'(y) = x^2 - 1. \qquad \leftarrow N(x, y)$$

It follows that $g'(y) = -1$ and $g(y) = -y$. Hence $f(x, y) = x^2 y - y$, so the solution of the equation in implicit form is $x^2 y - y = c$. The explicit form of the solution is easily seen to be $y = c/(x^2 - 1)$ and is defined on any interval not containing either $x = 1$ or $x = -1$.

EXAMPLE 3: An Initial-Value Problem

Solve $\frac{dy}{dx} = \frac{xy^2 - \cos x \sin x}{y(1 - x^2)}$, $y(0) = 2$. (Page 68)

INTEGRATING FACTORS

It is sometimes possible to find an integrating factor $\mu(x, y)$ so that after multiplying a nonexact differential equation $M(x, y)\,dx + N(x, y)\,dy = 0$, the left-hand side of

$$\mu(x, y)M(x, y)\,dx + \mu(x, y)N(x, y)\,dy = 0$$

is an exact differential.

2.5 Solutions by Substitutions

INTRODUCTION

We usually solve a differential equation by recognizing it as a certain kind of equation (say, separable, linear, exact, and so on) and then carrying out a procedure consisting of *equation-specific mathematical steps* (say, separating variables and integrating) that yields a solution of the equation. But it is not uncommon to be stumped by a differential equation because it does not fall into one of the classes of equations that we know how to solve. The substitution procedures that are discussed in this section may be helpful in this situation.

SUBSTITUTIONS

Often the first step in solving a differential equation consists of transforming it into another differential equation by means of a **substitution**. For example, suppose we wish to transform the first-order differential equation $dy/dx = f(x, y)$ by the substitution $y = g(x, u)$, where u is regarded as a function of the variable x. If g possesses first-partial derivatives, then the Chain Rule

$$\frac{dy}{dx} = \frac{\partial g}{\partial x}\frac{dx}{dx} + \frac{\partial g}{\partial u}\frac{du}{dx} \quad \text{gives} \quad \frac{dy}{dx} = g_x(x, u) + g_u(x, u)\frac{du}{dx}.$$

If we replace $dy\,/\,dx$ by the foregoing derivative and replace y in $f(x, y)$ by $g(x, u)$, then the DE $dy/dx = f(x, y)$ becomes $g_x(x, u) + g_u(x, u)\frac{du}{dx} = f\big(x, g(x, u)\big)$, which, solved for du/dx, has the form $\frac{du}{dx} = F(x, u)$. If we can determine a solution $u = \phi(x)$ of this last equation, then a solution of the original differential equation is $y = g\big(x, \phi(x)\big)$.

HOMOGENEOUS EQUATIONS

If a function f possesses the property $f(tx, ty) = t^\alpha f(x, y)$ for some real number α, then f is said to be a **homogeneous function** of degree α. For example, $f(x, y) = x^3 + y^3$ is a homogeneous function of degree 3, since

$$f(tx, ty) = (tx)^3 + (ty)^3 = t^3(x^3 + y^3) = t^3 f(x, y),$$

whereas $f(x, y) = x^3 + y^3 + 1$ is not homogeneous. A first-order DE in differential form

$$M(x,y)\,dx + N(x,y)\,dy = 0$$

is said to be **homogeneous** if both coefficient functions M and N are homogeneous functions of the *same* degree.

In addition, if M and N are homogeneous functions of degree α, we can also write

$$M(x,y) = x^{\alpha}M(1,u) \quad \text{and} \quad N(x,y) = x^{\alpha}N(1,u), \quad \text{where } u = y/x,$$

and

$$M(x,y) = y^{\alpha}M(v,1) \quad \text{and} \quad N(x,y) = y^{\alpha}N(v,1), \quad \text{where } v = x/y.$$

The above properties suggest the substitutions that can be used to solve a homogeneous differential equation. Specically, *either* of the substitutions $y = ux$ or $x = vy$, where u and v are new dependent variables, will reduce a homogeneous equation to a *separable* first-order differential equation.

To show this,

$$M(x,y)\,dx + N(x,y)\,dy = 0$$
$$x^{\alpha}M(1,u)\,dx + x^{\alpha}N(1,u)\,dy = 0 \quad \text{or} \quad M(1,u)\,dx + N(1,u)\,dy = 0, \quad y = ux$$
$$M(1,u)\,dx + N(1,u)\,[udx + xdu] = 0$$
$$[M(1,u) + uN(1,u)]\,dx + xN(1,u)du = 0$$
$$\frac{dx}{x} + \frac{N(1,u)du}{M(1,u) + uN(1,u)} = 0$$

EXAMPLE 1: Solving a Homogeneous DE

Solve $(x^2 + y^2)\,dx + (x^2 - xy)\,dy = 0$.

Solution

Inspection of $M(x,y) = x^2 + y^2$ and $N(x,y) = x^2 - xy$ shows that these coefficients are homogeneous functions of degree 2. If we let $y = ux$, then $dy = udx + xdu$, so after substituting, the given equation becomes

$$(x^2 + u^2x^2)\,dx + (x^2 - ux^2)[u\,dx + x\,du] = 0$$
$$x^2(1 + u)\,dx + x^3(1 - u)\,du = 0$$
$$\frac{1-u}{1+u}du + \frac{dx}{x} = 0$$
$$\left[-1 + \frac{2}{1+u}\right]du + \frac{dx}{x} = 0. \quad \leftarrow \text{long division}$$

After integration the last line gives

$$-u + 2\ln|1 + u| + \ln|x| = \ln|c|$$
$$-\frac{y}{x} + 2\ln\left|1 + \frac{y}{x}\right| + \ln|x| = \ln|c|. \quad \leftarrow \text{resubstituting } u = y/x$$

Using the properties of logarithms, we can write the preceding solution as

$$\ln\left|\frac{(x+y)^2}{cx}\right| = \frac{y}{x} \quad \text{or} \quad (x+y)^2 = cxe^{y/x}$$

Although either of the indicated substitutions can be used for every homogeneous differential equation, in practice we try $x = vy$ whenever the function $M(x,y)$ is simpler than $N(x,y)$. Also it could happen that after using one substitution, we may encounter integrals that are difficult or impossible to evaluate in closed form; switching substitutions may result in an easier problem.

BERNOULLI'S EQUATION

The differential equation

$$\frac{dy}{dx} + P(x)y = f(x)y^n,$$

where n is any real number, is called **Bernoulli's equation**. Note that for $n = 0$ and $n = 1$, the equation is linear. For $n \neq 0$ and $n \neq 1$ the substitution $u = y^{1-n}$ reduces any equation of the form above to a linear equation.

EXAMPLE 2: Solving a Bernoulli DE

Solve $x\frac{dy}{dx} + y = x^2y^2$.

Solution

We begin by rewriting the equation in the form $\frac{dy}{dx} + P(x)y = f(x)y^n$ by dividing by x:

$$\frac{dy}{dx} + \frac{1}{x}y = xy^2.$$

With $n = 2$ we have $u = y^{-1}$ or $y = u^{-1}$. We then substitute

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = -u^{-2}\frac{du}{dx} \quad \leftarrow \text{Chain Rule}$$

into the given equation and simplify. The result is

$$\frac{du}{dx} - \frac{1}{x}u = -x.$$

The integrating factor for this linear equation on, say, $(0, \infty)$ is

$$e^{-\int dx/x} = e^{-\ln x} = e^{\ln x^{-1}} = x^{-1}.$$

Integrating

$$\frac{d}{dx}[x^{-1}u] = -1$$

gives $x^{-1}u = -x + c$ or $u = -x^2 + cx$. Since $u = y^{-1}$, we have $y = 1/u$, so a solution of the given equation is $y = 1/(-x^2 + cx)$.

REDUCTION TO SEPARATION OF VARIABLES

A differential equation of the form

$$\frac{dy}{dx} = f(Ax + By + C)$$

can always be reduced to an equation with separable variables by means of the substitution $u = Ax + By + C$, $B \neq 0$.

EXAMPLE 3: An Initial-Value Problem

Solve $\frac{dy}{dx} = (-2x + y)^2 - 7$, $y(0) = 0$. (Page 74)

CHAPTER 7 The Laplace Transform

7.1 Definition of the Laplace Transform

INTRODUCTION

In elementary calculus you learned that differentiation and integration are ***transforms***; this means, roughly speaking, that these operations transform a function into another function. For example, the function $f(x) = x^2$ is transformed, in turn, into a linear function and a family of cubic polynomial functions by the operations of differentiation and integration:

$$\frac{d}{dx}x^2 = 2x \quad \text{and} \quad \int x^2\,dx = \frac{1}{3}x^3 + c.$$

Moreover, these two transforms possess the **linearity property** that the transform of a linear combination of functions is a linear combination of the transforms. For α and β constants

$$\frac{d}{dx}[\alpha f(x) + \beta g(x)] = \alpha f'(x) + \beta g'(x)$$

and

$$\int [\alpha f(x) + \beta g(x)]\,dx = \alpha \int f(x)\,dx + \beta \int g(x)\,dx$$

provided that each derivative and integral exists. In this section we will examine a special type of integral transform called the **Laplace transform**. In addition to possessing the linearity property the Laplace transform has many other interesting properties that make it very useful in solving linear initial-value problems.

INTEGRAL TRANSFORM

If $f(x, y)$ is a function of two variables, then a definite integral of f with respect to one of the variables leads to a function of the other variable.

A definite integral such as $\int_a^b K(s,t)f(t)dt$ transforms a function f of the variable t into a function F of the variable s. We are particularly interested in an integral transform, where the interval of integration is the unbounded interval $[0, \infty)$. If $f(t)$ is defined for $t \geq 0$, then the improper integral $\int_0^\infty K(s,t)f(t)dt$ is defined as a limit:

$$\int_0^\infty K(s,t)f(t)dt = \lim_{b\to\infty} \int_0^b K(s,t)f(t)dt.$$

If the limit above exists, then we say that the integral exists or is **convergent**; if the limit does not exist, the integral does not exist and is **divergent**. The limit above will, in general, exist for only certain values of the variable s. (We will assume throughout that s is a real variable.)

A DEFINITION

The function $K(s,t)$ above is called the **kernel** of the transform. The choice $K(s,t) = e^{-st}$ as the kernel gives us an especially important integral transform.

> **DEFINITION 7.1.1 Laplace Transform**
> Let f be a function defined for $t \geq 0$. Then the integral
> $$\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)dt$$
> is said to be the **Laplace transform** of f, provided that the integral converges.

When the defining integral above converges, the result is a function of s. In general discussion we shall use a lowercase letter to denote the function being transformed and the corresponding capital letter to denote its Laplace transform—for example,

$$\mathcal{L}\{f(t)\} = F(s), \quad \mathcal{L}\{g(t)\} = G(s), \quad \mathcal{L}\{y(t)\} = Y(s).$$

EXAMPLE 1: Applying Definition 7.1.1

Evaluate $\mathcal{L}\{1\}$.

SOLUTION

$$\mathcal{L}\{1\} = \int_0^\infty e^{-st}(1)dt = \lim_{b\to\infty}\int_0^b e^{-st}dt = \lim_{b\to\infty}\frac{-e^{-st}}{s}\Big|_0^b = \lim_{b\to\infty}\frac{-e^{-sb}+1}{s} = \frac{-e^{-st}}{s}\Big|_0^\infty = \frac{1}{s}$$

provided that $s > 0$. In other words, when $s > 0$, the exponent $-sb$ is negative, and $e^{-sb} \to 0$ as $b \to \infty$. The integral diverges for $s < 0$.

EXAMPLE 2: Applying Definition 7.1.1

Evaluate $\mathcal{L}\{t\}$.

SOLUTION

$$\mathcal{L}\{t\} = \frac{-te^{-st}}{s}\Big|_0^\infty + \frac{1}{s}\int_0^\infty e^{-st}dt = \frac{1}{s}\mathcal{L}\{1\} = \frac{1}{s}\left(\frac{1}{s}\right) = \frac{1}{s^2}. \quad (s > 0)$$

EXAMPLE 3: Applying Definition 7.1.1

Evaluate (a) $\mathcal{L}\{e^{-3t}\}$ (b) $\mathcal{L}\{e^{5t}\}$

SOLUTION

(a)

$$\mathcal{L}\{e^{-3t}\} = \int_0^\infty e^{-3t}e^{-st}dt = \int_0^\infty e^{-(s+3)t}dt = \frac{-e^{-(s+3)t}}{s+3}\Big|_0^\infty = \frac{1}{s+3}.$$

The last result is valid for $s > -3$.

(b)

$$\mathcal{L}\{e^{5t}\} = \frac{1}{s-5}. (s > 5)$$

EXAMPLE 4: Applying Definition 7.1.1

Evaluate $\mathcal{L}\{\sin 2t\}$.

SOLUTION

$$\mathcal{L}\{\sin 2t\} = \int_0^\infty e^{-st}\sin 2t\,dt = \frac{-e^{-st}\sin 2t}{s}\Big|_0^\infty + \frac{2}{s}\int_0^\infty e^{-st}\cos 2t\,dt$$

$$= \frac{2}{s}\int_0^\infty e^{-st}\cos 2t\,dt, \quad s > 0$$

$$= \frac{2}{s}\left[\overset{\lim_{t\to\infty} e^{-st}\cos 2t = 0,\ s>0}{\overset{\downarrow}{\frac{-e^{-st}\cos 2t}{s}\Big|_0^\infty}} - \frac{2}{s}\overset{\text{Laplace transform of } \sin 2t}{\overset{\downarrow}{\int_0^\infty e^{-st}\sin 2t\,dt}}\right]$$

$$= \frac{2}{s^2} - \frac{4}{s^2}\mathcal{L}\{\sin 2t\}.$$

$$\mathcal{L}\{\sin 2t\} = \frac{2}{s^2+4}, \quad s > 0.$$

$\mathcal{L}$ IS A LINEAR TRANSFORM

For a linear combination of functions we can write

$$\int_0^\infty e^{-st}[\alpha f(t) + \beta g(t)]dt = \alpha\int_0^\infty e^{-st}f(t)dt + \beta\int_0^\infty e^{-st}g(t)dt$$

whenever both integrals converge for $s > c$. Hence it follows that

$$\mathcal{L}\{\alpha f(t) + \beta g(t)\} = \alpha\mathcal{L}\{f(t)\} + \beta\mathcal{L}\{g(t)\} = \alpha F(s) + \beta G(s).$$

Because of the property given above, $\mathcal{L}$ is said to be a **linear transform**.

EXAMPLE 5: Linearity of the Laplace Transform

(a)

$$\mathcal{L}\{1 + 5t\} = \mathcal{L}\{1\} + 5\mathcal{L}\{t\} = \frac{1}{s} + \frac{5}{s^2}.$$

(b)

$$\mathcal{L}\{4e^{5t} - 10\sin 2t\} = 4\mathcal{L}\{e^{5t}\} - 10\mathcal{L}\{\sin 2t\} = \frac{4}{s-5} - \frac{20}{s^2+4}.$$

(c)

$$\mathcal{L}\{20e^{-3t} + 7t - 9\} = 20\mathcal{L}\{e^{-3t}\} + 7\mathcal{L}\{t\} - 9\mathcal{L}\{1\}$$

$$= \frac{20}{s+3} + \frac{7}{s^2} - \frac{9}{s}.$$

THEOREM 7.1.1 Transforms of Some Basic Functions

$$(a)\ \mathcal{L}\{1\} = \frac{1}{s} \quad (s > 0)$$

$(b)\ \mathcal{L}\{t^n\} = \dfrac{n!}{s^{n+1}},\quad n = 1,2,3,\cdots \quad (s > 0)$ $\qquad (c)\ \mathcal{L}\{e^{at}\} = \dfrac{1}{s-a} \quad (s > a)$

$(d)\ \mathcal{L}\{\sin kt\} = \dfrac{k}{s^2+k^2} \quad (s > 0)$ $\qquad (e)\ \mathcal{L}\{\cos kt\} = \dfrac{s}{s^2+k^2} \quad (s > 0)$

$(f)\ \mathcal{L}\{\sinh kt\} = \dfrac{k}{s^2-k^2}$ $\qquad (g)\ \mathcal{L}\{\cosh kt\} = \dfrac{s}{s^2-k^2}$

SUFFICIENT CONDITIONS FOR EXISTENCE OF $\mathcal{L}\{f(t)\}$

Sufficient conditions guaranteeing the existence of $\mathcal{L}\{f(t)\}$ are that f be piecewise continuous on $[0,\infty)$ and that f be of exponential order for $t > T$.

DEFINITION 7.1.2 Exponential Order

A function f is said to be of **exponential order** if there exist constants c, $M > 0$, and $T > 0$ such that $|f(t)| \le Me^{ct}$ for all $t > T$.

THEOREM 7.1.2 Sufficient Conditions for Existence

If f is piecewise continuous on $[0,\infty)$ and of exponential order, then $\mathcal{L}\{f(t)\}$ exists for $s > c$.

The next theorem indicates that not every arbitrary function of s is a Laplace transform of a piecewise continuous function of exponential order.

THEOREM 7.1.3 Behavior of $F(s)$ as $s \to \infty$

If f is piecewise continuous on $[0,\infty)$ and of exponential order and $F(s) = \mathcal{L}\{f(t)\}$, then $\lim_{s\to\infty} F(s) = 0$.

EXERCISES 7.1

25. Find $\mathcal{L}\{f(t)\}$: $f(t) = (t+1)^3$.

$$\mathcal{L}\{f(t)\} = \mathcal{L}\{t^3 + 3t^2 + 3t + 1\} = \frac{6}{s^4} + \frac{6}{s^3} + \frac{3}{s^2} + \frac{1}{s}$$

33. Find $\mathcal{L}\{f(t)\}$: $f(t) = \sinh kt$.

$$\mathcal{L}\{f(t)\} = \mathcal{L}\{\sinh kt\} = \mathcal{L}\left\{\frac{e^{kt} - e^{-kt}}{2}\right\} = \frac{1}{2}[\mathcal{L}\{e^{kt}\} - \mathcal{L}\{e^{-kt}\}] = \frac{k}{s^2-k^2}$$

41. We have encountered the **gamma function $\Gamma(\alpha)$** in our study. One definition of this function is given by the improper integral

$$\Gamma(\alpha) = \int_0^\infty t^{\alpha-1}e^{-t}dt, \quad \alpha > 0.$$

Use this definition to show that $\Gamma(\alpha+1) = \alpha\Gamma(\alpha)$. When $\alpha = n$ is a positive integer the last property can be used to show that $\Gamma(n+1) = n!$.

42. Use Problem 41 and the change of variable $u = st$ to obtain the generalization

$$\mathcal{L}\{t^\alpha\} = \frac{\Gamma(\alpha+1)}{s^{\alpha+1}}, \quad \alpha > -1,$$

of the result in Theorem 7.1.1(b).

Let $u = st \Longrightarrow du = sdt$, then

$$\mathcal{L}\{t^\alpha\} = \int_0^\infty e^{-st}t^\alpha dt$$

$$= \frac{1}{s}\int_0^\infty e^{-u}\left(\frac{u}{s}\right)^\alpha du$$

$$= \frac{1}{s^{\alpha+1}} \int_0^\infty e^{-u} u^\alpha du$$
$$= \frac{\Gamma(\alpha+1)}{s^{\alpha+1}}, \quad \alpha > -1.$$

50. Under what conditions is a linear function

$$f(x) = mx + b, m \neq 0, \text{a linear transform?}$$

A linear function is a linear transform if the two properties below hold:

1) $f(x+y) = f(x) + f(y)$
2) $f(a \cdot x) = a \cdot f(x)$

After looking at both properties, b must be equal to 0.

54. If $\mathcal{L}\{f(t)\} = F(s)$ and $a > 0$ is a constant, show that

$$\mathcal{L}\{f(at)\} = \frac{1}{a} F\left(\frac{s}{a}\right).$$

This result is known as the **change of scale theorem**.

Let $u = at \Longrightarrow du = adt$ and $t = \frac{u}{a}$,

$$\mathcal{L}\{f(at)\} = \int_0^\infty e^{-st} f(at) dt$$
$$= \frac{1}{a} \int_0^\infty e^{-\frac{s}{a}u} f(u) du$$
$$= \frac{1}{a} F\left(\frac{s}{a}\right).$$

7.2 Inverse Transforms and Transforms of Derivatives

INVERSE TRANSFORMS

THE INVERSE PROBLEM

If $F(s)$ represents the Laplace transform of a function $f(t)$, that is, $\mathcal{L}\{f(t)\} = F(s)$, we then say $f(t)$ is the inverse Laplace transform of $F(s)$ and write $f(t) = \mathcal{L}^{-1}\{F(s)\}$.

Transform	Inverse Transform
$\mathscr{L}\{1\} = \frac{1}{s}$	$1 = \mathscr{L}^{-1}\left\{\frac{1}{s}\right\}$
$\mathscr{L}\{t\} = \frac{1}{s^2}$	$t = \mathscr{L}^{-1}\left\{\frac{1}{s^2}\right\}$
$\mathscr{L}\{e^{-3t}\} = \frac{1}{s+3}$	$e^{-3t} = \mathscr{L}^{-1}\left\{\frac{1}{s+3}\right\}$

The idea is simply this: Suppose $F(s) = \frac{-2s+6}{s^2+4}$ is a Laplace transform; find a function $f(t)$ such that $\mathcal{L}\{f(t)\} = F(s)$.

THEOREM 7.1.1 Transforms of Some Basic Functions

$$(a)\ 1 = \mathcal{L}^{-1}\left\{\frac{1}{s}\right\} \quad (s > 0)$$

$$(b)\ t^n = \mathcal{L}^{-1}\left\{\frac{n!}{s^{n+1}}\right\}, \quad n = 1,2,3,\cdots \quad (s > 0) \qquad (c)\ e^{at} = \mathcal{L}^{-1}\left\{\frac{1}{s-a}\right\} \quad (s > a)$$

$$(d)\ \sin kt = \mathcal{L}^{-1}\left\{\frac{k}{s^2+k^2}\right\} \quad (s > 0) \qquad (e)\ \cos kt = \mathcal{L}^{-1}\left\{\frac{s}{s^2+k^2}\right\} \quad (s > 0)$$

$$(f)\ \sinh kt = \mathcal{L}^{-1}\left\{\frac{k}{s^2-k^2}\right\} \qquad (g)\ \cosh kt = \mathcal{L}^{-1}\left\{\frac{s}{s^2-k^2}\right\}$$

In evaluating inverse transforms, it often happens that a function of s under consideration does not match exactly the form of a Laplace transform $F(s)$ given in a table. It may be necessary to "fix up" the function of s by multiplying and dividing by an appropriate constant.

EXAMPLE 1: Applying Theorem 7.2.1

Evaluate (a) $\mathcal{L}^{-1}\left\{\frac{1}{s^5}\right\}$ (b) $\mathcal{L}^{-1}\left\{\frac{1}{s^2+7}\right\}$.

SOLUTION

$$\mathcal{L}^{-1}\left\{\frac{1}{s^5}\right\} = \frac{1}{4!}\mathcal{L}^{-1}\left\{\frac{4!}{s^5}\right\} = \frac{1}{24}t^4.$$

$$\mathcal{L}^{-1}\left\{\frac{1}{s^2+7}\right\} = \frac{1}{\sqrt{7}}\mathcal{L}^{-1}\left\{\frac{\sqrt{7}}{s^2+7}\right\} = \frac{1}{\sqrt{7}}\sin\sqrt{7}t.$$

$\mathcal{L}^{-1}$ IS A LINEAR TRANSFORM

The inverse Laplace transform is also a linear transform; that is, for constants α and β

$$\mathcal{L}^{-1}\{\alpha F(s) + \beta G(s)\} = \alpha\mathcal{L}^{-1}\{F(s)\} + \beta\mathcal{L}^{-1}\{G(s)\},$$

where F and G are the transforms of some functions f and g.

EXAMPLE 2: **Termwise** Division and Linearity

Evaluate $\mathcal{L}^{-1}\left\{\frac{-2s+6}{s^2+4}\right\}$.

SOLUTION

$$\mathscr{L}^{-1}\left\{\frac{-2s+6}{s^2+4}\right\} \overset{\text{termwise division}}{=} \mathscr{L}^{-1}\left\{\frac{-2s}{s^2+4} + \frac{6}{s^2+4}\right\} \overset{\text{linearity and fixing up constants}}{=} -2\,\mathscr{L}^{-1}\left\{\frac{s}{s^2+4}\right\} + \frac{6}{2}\mathscr{L}^{-1}\left\{\frac{2}{s^2+4}\right\}$$

$$= -2\cos 2t + 3\sin 2t. \quad \leftarrow \text{parts (e) and (d) of Theorem 7.2.1 with } k = 2$$

PARTIAL FRACTIONS

Partial fractions play an important role in finding inverse Laplace transforms.

EXAMPLE 3: Partial Fractions: Distinct Linear Factors. (Page 288)

TRANSFORMS OF DERIVATIVES

As was pointed out in the introduction to this chapter, our immediate goal is to use the Laplace transform to solve differential equations. To that end we need to evaluate quantities such as $\mathcal{L}\{dy/dt\}$ and $\mathcal{L}\{d^2y/dt^2\}$. For example, if f' is continuous for $t \geq 0$, then integration by parts gives

$$\mathcal{L}\{f'(t)\} = \int_0^\infty e^{-st}f'(t)dt = e^{-st}f(t)\Big|_0^\infty + s\int_0^\infty e^{-st}f(t)dt$$
$$= -f(0) + s\mathcal{L}\{f(t)\}$$

or $\mathcal{L}\{f'(t)\} = sF(s) - f(0)$.

Here we have assumed that $e^{-st}f(t) \to 0$ as $t \to \infty$. Similarly, with the aid of above equation,

$$\mathcal{L}\{f''(t)\} = \int_0^\infty e^{-st}f''(t)dt = e^{-st}f'(t)\Big|_0^\infty + s\int_0^\infty e^{-st}f'(t)dt$$
$$= -f'(0) + s\mathcal{L}\{f'(t)\}$$
$$= s[sF(s) - f(0)] - f'(0) \quad \leftarrow \text{from } \mathcal{L}\{f'(t)\}$$

or $\mathcal{L}\{f''(t)\} = s^2F(s) - sf(0) - f'(0)$.

In like manner it can be shown that

$$\mathcal{L}\{f'''(t)\} = s^3F(s) - s^2f(0) - sf'(0) - f''(0).$$

The recursive nature of the Laplace transform of the derivatives of a function f should be apparent from the results above.

> **THEOREM 7.2.2 Transform of a Derivative**
>
> If $f, f', \cdots, f^{(n-1)}$ are continuous on $[0,\infty)$ and are of exponential order and if $f^{(n)}(t)$ is piecewise continuous on $[0,\infty)$, then
>
> $$\mathcal{L}\{f^{(n)}(t)\} = s^nF(s) - s^{n-1}f(0) - s^{n-2}f'(0) - \cdots - f^{(n-1)}(0),$$
>
> where $F(s) = \mathcal{L}\{f(t)\}$.

SOLVING LINEAR ODES

It is apparent from the general result given in Theorem 7.2.2 that $\mathcal{L}\{d^ny/dt^n\}$ depends on $Y(s) = \mathcal{L}\{y(t)\}$ and the $n-1$ derivatives of $y(t)$ evaluated at $t = 0$. This property makes the Laplace transform ideally suited for solving linear initial-value problems in which the differential equation has *constant coefficients*.

Such a differential equation is simply a linear combination of terms $y, y', y'', \cdots, y^{(n)}$:

$$a_n \frac{d^n y}{dt^n} + a_{n-1} \frac{d^{n-1} y}{dt^{n-1}} + \cdots + a_0 y = g(t),$$
$$y(0) = y_0, y'(0) = y_1, \cdots, y^{(n-1)}(0) = y_{n-1},$$

where the $a_i, i = 0,1, \cdots, n$ and $y_0, y_1, \cdots, y_{n-1}$ are constants. By the linearity property the Laplace transform of this linear combination is a linear combination of Laplace transforms:

$$a_n \mathcal{L}\left\{\frac{d^n y}{dt^n}\right\} + a_{n-1} \mathcal{L}\left\{\frac{d^{n-1} y}{dt^{n-1}}\right\} + \cdots + a_0 \mathcal{L}\{y\} = \mathcal{L}\{g(t)\}.$$

From Theorem 7.2.2, the above equation becomes

$$a_n\left[s^n Y(s) - s^{n-1} y(0) - \cdots - y^{(n-1)}(0)\right] + a_{n-1}\left[s^{n-1} Y(s) - s^{n-2} y(0) - \cdots - y^{(n-2)}(0)\right] + \cdots + a_0 Y(s) = G(s),$$

where $\mathcal{L}\{y(t)\} = Y(s)$ and $\mathcal{L}\{g(t)\} = G(s)$. In other words,

The Laplace transform of a linear differential equation with constant coefficients becomes an algebraic equation in $Y(s)$.

If we solve the general transformed equation above for the symbol $Y(s)$, we first obtain $P(s)Y(s) = Q(s) + G(s)$ and then write

$$Y(s) = \frac{Q(s)}{P(s)} + \frac{G(s)}{P(s)},$$

where $P(s) = a_n s^n + a_{n-1} s^{n-1} + \cdots + a_0$, $Q(s)$ is a polynomial in s of degree less than or equal to $n-1$ consisting of the various products of the coefficients a_i, $i = 1, \cdots, n$ and the prescribed initial conditions $y_0, y_1, \cdots, y_{n-1}$, and $G(s)$ is the Laplace transform of $g(t)$.

Finally, the solution $y(t)$ of the original initial-value problem is $y(t) = \mathcal{L}^{-1}\{Y(s)\}$, where the inverse transform is done term by term.

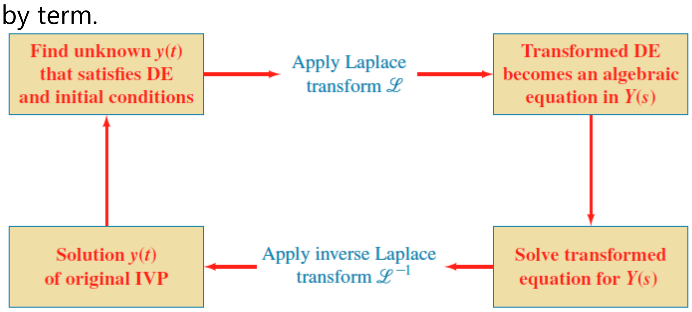


Figure 7. 1 Steps in solving an IVP by the Laplace transform

EXAMPLE 4: Solving a First-Order IVP (Page 290)

Use the Laplace transform to solve the initial-value problem

$$\frac{dy}{dt} + 3y = 13\sin 2t, \quad y(0) = 6.$$

EXAMPLE 5: Solving a Second-Order IVP (Page 291)

Solve $y'' - 3y' + 2y = e^{-4t}$, $y(0) = 1$, $y'(0) = 5$.

REMARKS

i. The inverse Laplace transform of a function $F(s)$ may not be unique; in other words, it is possible that $\mathcal{L}\{f_1(t)\} = \mathcal{L}\{f_2(t)\}$ and yet $f_1 \neq f_2$. For our purposes this is not anything to be concerned about. If f_1 and f_2 are piecewise continuous on $[0, \infty)$ and of exponential order, then f_1 and f_2 are *essentially* the same. However, if f_1 and f_2 are continuous on $[0, \infty)$ and $\mathcal{L}\{f_1(t)\} = \mathcal{L}\{f_2(t)\}$, then $f_1 = f_2$ on the interval.

ii. This remark is for those of you who will be required to do partial fraction decompositions by hand. There is another way of determining the coefficients in a partial fraction decomposition in the special case when $\mathcal{L}\{f(t)\} = F(s)$ is a rational function of s and the denominator of F is a product of *distinct* linear factors.

This special technique for determining coefficients is naturally known as the **cover-up method**. (Page 292)

7.3 Operational Properties I

TRANSLATION ON THE s-AXIS

A TRANSLATION

In general, if we know the Laplace transform of a function f, $\mathcal{L}\{f(t)\} = F(s)$, it is possible to compute the Laplace transform of an exponential multiple of f, that is, $\mathcal{L}\{e^{at}f(t)\}$, with no additional effort other than *translating*, or *shifting*, the transform $F(s)$ to $F(s-a)$. This result is known as the **first translation theorem** or **first shifting theorem**.

THEOREM 7.3.1 First Translation Theorem

If $\mathcal{L}\{f(t)\} = F(s)$ and a is any real number, then

$$\mathcal{L}\{e^{at}f(t)\} = F(s-a).$$

Proof

The proof is immediate, since by Definition 7.1.1

$$\mathcal{L}\{e^{at}f(t)\} = \int_0^\infty e^{-st}e^{at}f(t)dt = \int_0^\infty e^{-(s-a)t}f(t)dt = F(s-a).$$

If we consider s a real variable, then the graph of $F(s-a)$ is the graph of $F(s)$ shifted on the s-axis by the amount $|a|$. If $a > 0$, the graph of $F(s)$ is shifted a units to the right, whereas if $a < 0$, the graph is shifted $|a|$ units to the left.

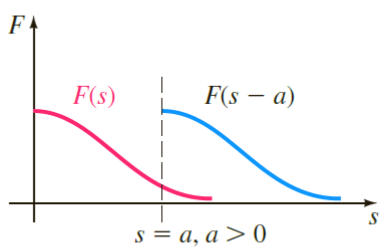


Figure 7. 2 Shift on s-axis

For emphasis it is sometimes useful to use the symbolism

$$\mathcal{L}\{e^{at}f(t)\} = \mathcal{L}\{f(t)\}|_{s\to s-a},$$

where $s \to s-a$ means that in the Laplace transform $F(s)$ of $f(t)$ we replace the symbol s wherever it appears by $s-a$.

INVERSE FORM OF THEOREM 7.3.1

To compute the inverse of $F(s-a)$, we must recognize $F(s)$, find $f(t)$ by taking the inverse Laplace transform of $F(s)$, and then multiply $f(t)$ by the exponential function e^{at}. This procedure can be summarized symbolically in the following manner:

$$\mathcal{L}^{-1}\{F(s-a)\} = \mathcal{L}^{-1}\{F(s)|_{s\to s-a}\} = e^{at}f(t),$$

where $f(t) = \mathcal{L}^{-1}\{F(s)\}$.

EXAMPLE 2: Partial Fractions: Repeated Linear Factors (Page 296)

TRANSLATION ON THE t-AXIS

UNIT STEP FUNCTION

In engineering, one frequently encounters functions that are either "off" or "on."

It is convenient, then, to define a special function that is the number 0 (off) up to a certain time then the number 1 (on) after that time. This function is called the **unit step function** or the **Heaviside function**.

> **Definition 7.3.1 Unit Step Function**
> The **unit step function** $\mathcal{U}(t-a)$ is defined to be
> $$\mathcal{U}(t-a) = \begin{cases} 0, & 0 \le t < a \\ 1, & t \ge a. \end{cases}$$

Notice that we define $\mathcal{U}(t-a)$ only on the nonnegative t-axis, since this is all that we are concerned with in the study of the Laplace transform. In a broader sense $\mathcal{U}(t-a) = 0$ for $t < a$. The graph of $\mathcal{U}(t-a)$ is given in Figure 7.3. In the case when $a = 0$, we take $\mathcal{U}(t) = 1$ for $t \ge 0$.

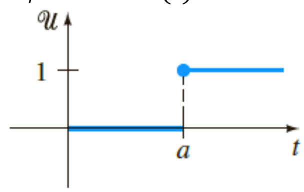


Figure 7. 3 Graph of unit step function

When a function f defined for $t \ge 0$ is multiplied by $\mathcal{U}(t-a)$, the unit step function "turns off " a portion of the graph of that function.

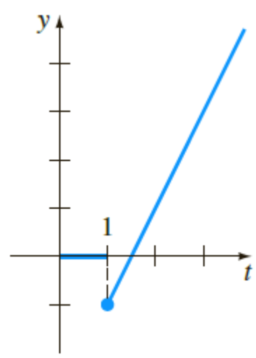


Figure 7. 4 Function is $f(t) = (2t-3)\mathcal{U}(t-1)$

The unit step function can also be used to write piecewise-defined functions in a compact form.

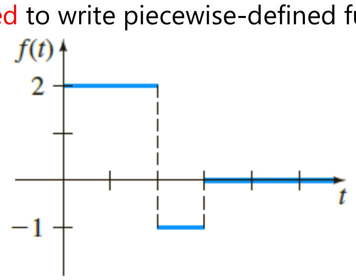


Figure 7. 5 Function is $f(t) = 2 - 3\mathcal{U}(t-2) + \mathcal{U}(t-3)$

Also, a general piecewise-defined function of the type
$$f(\mathrm{t}) = \begin{cases} g(t), & 0 \le t < a \\ h(t), & t \ge a \end{cases}$$

is the same as

$$f(t) = g(t) - g(t)\mathcal{U}(t-a) + h(t)\mathcal{U}(t-a).$$

Similarly, a function of the type

$$f(\mathrm{t}) = \begin{cases} 0, & 0 \le t < a \\ g(t), & a \le t < b \\ 0, & t \ge b \end{cases}$$

can be written

$$f(\mathrm{t}) = g(t)[\mathcal{U}(t-a) - \mathcal{U}(t-b)].$$

Consider a general function $y = f(t)$ defined for $t \ge 0$. The piecewise-defined function

$$f(t-a)\mathcal{U}(t-a) = \begin{cases} 0, & 0 \le t < a \\ f(t-a), & t \ge a \end{cases}$$

plays a significant role in the discussion that follows. As shown in Figure 7.6, for $a > 0$ the graph of the function $y = f(t-a)\mathcal{U}(t-a)$ coincides with the graph of $y = f(t-a)$ for $t \ge a$ (which is the *entire* graph of $y = f(t)$, $t \ge a$ shifted a units to the right on the t-axis), but is identically zero for $0 \le t < a$.

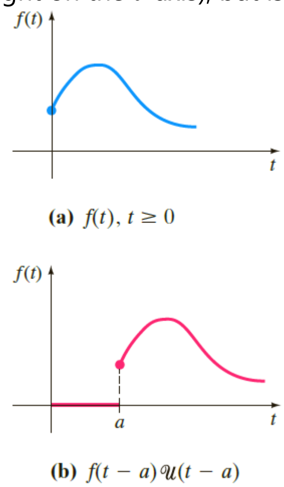


Figure 7. 6 Shift on t-axis

As a consequence of the next theorem we see that whenever $F(s)$ is multiplied by an exponential function e^{-as}, $a > 0$, the inverse transform of the product $e^{-as}F(s)$ is the function f shifted along the t-axis in the manner illustrated in Figure 7.6(b). This result, presented next in its direct transform version, is called the **second translation theorem** or **second shifting theorem**.

THEOREM 7.3.2 Second Translation Theorem

If $F(s) = \mathcal{L}\{f(t)\}$ and $a > 0$, then

$$\mathcal{L}\{f(t-a)\mathcal{U}(t-a)\} = e^{-as}F(s).$$

Proof

By the additive interval property of integrals,

$$\int_0^\infty e^{-st} f(t-a)\mathcal{U}(t-a)dt$$

can be written as two integrals:

$$\mathcal{L}\{f(t-a)\,\mathcal{U}(t-a)\} = \int_0^a e^{-st}f(t-a)\,\underbrace{\mathcal{U}(t-a)}_{\substack{\text{zero for}\\ 0 \le t < a}}\,dt + \int_a^\infty e^{-st}f(t-a)\,\underbrace{\mathcal{U}(t-a)}_{\substack{\text{one for}\\ t \ge a}}\,dt = \int_a^\infty e^{-st}f(t-a)\,dt.$$

Now if we let $v = t - a$, $dv = dt$ in the last integral, then

$$\mathcal{L}\{f(t-a)\mathcal{U}(t-a)\} = \int_0^\infty e^{-s(v+a)} f(v)dv = e^{-as}\int_0^\infty e^{-sv} f(v)dv = e^{-as}\mathcal{L}\{f(t)\}.$$

We often wish to find the Laplace transform of just a unit step function.

If we identify $f(t) = 1$, then $f(t-a) = 1$, $F(s) = \mathcal{L}\{1\} = 1/s$, and so

$$\mathcal{L}\{\mathcal{U}(t-a)\} = \frac{e^{-as}}{s}.$$

INVERSE FORM OF THEOREM 7.3.2

If $f(t) = \mathcal{L}^{-1}\{F(s)\}$, the inverse form of Theorem 7.3.2, $a > 0$, is

$$\mathcal{L}^{-1}\{e^{-as}F(s)\} = f(t-a)\mathcal{U}(t-a).$$

ALTERNATIVE FORM OF THEOREM 7.3.2

We are frequently confronted with the problem of finding the Laplace transform of a product of a function g and a unit step function $\mathcal{U}(t-a)$ where the function g lacks the precise shifted form $f(t-a)$ in Theorem 7.3.2.

Using Definition 7.1.1, the definition of $\mathcal{U}(t-a)$, and the substitution $u = t - a$, we obtain

$$\mathcal{L}\{g(t)\mathcal{U}(t-a)\} = \int_a^\infty e^{-st}g(t)dt = \int_0^\infty e^{-s(u+a)}g(u+a)du.$$

That is,

$$\mathcal{L}\{g(t)\mathcal{U}(t-a)\} = e^{-as}\mathcal{L}\{g(t+a)\}.$$

EXAMPLE 9: An Initial-Value Problem (Page 301)

Solve $y' + y = f(t)$, $y(0) = 5$, where $f(t) = \begin{cases} 0, & 0 \le t < \pi \\ 3\cos t, & t \ge \pi \end{cases}$.

REMARKS

Outside the discussion of the Laplace transform, the unit step function is defined on the interval $(-\infty, \infty)$, that is,

$$\mathcal{U}(t-a) = \begin{cases} 0, & t < a \\ 1, & t \ge a \end{cases}.$$

The **boxcar function** is denoted by

$$\prod(t) = \mathcal{U}(t-a) - \mathcal{U}(t-b).$$

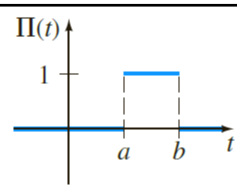


Figure 7. 7 Boxcar function

7.4 Operational Properties II

INTRODUCTION

In this section we develop several more operational properties of the Laplace transform.

DERIVATIVES OF A TRANSFORM

MULTIPLYING A FUNCTION BY t^n

The Laplace transform of the product of a function $f(t)$ with t can be found by differentiating the Laplace transform of $f(t)$. To motivate this result, let us assume that $F(s) = \mathcal{L}(f(t))$ exists and that it is possible to interchange the order of differentiation and integration.

$$\frac{d}{ds}F(s) = \frac{d}{ds}\int_0^\infty e^{-st}f(t)dt = \int_0^\infty \frac{\partial}{\partial s}[e^{-st}f(t)]dt = -\int_0^\infty e^{-st}tf(t)dt = -\mathcal{L}\{tf(t)\};$$

that is,

$$\mathcal{L}\{tf(t)\} = -\frac{d}{ds}\mathcal{L}\{f(t)\}.$$

$$\mathcal{L}\{t^2f(t)\} = \mathcal{L}\{t \cdot tf(t)\} = -\frac{d}{ds}\mathcal{L}\{tf(t)\} = -\frac{d}{ds}\left(-\frac{d}{ds}\mathcal{L}\{f(t)\}\right) = \frac{d^2}{ds^2}\mathcal{L}\{f(t)\}.$$

THEOREM 7.4.1 Derivatives of Transforms
If $F(s) = \mathcal{L}\{f(t)\}$ and $n = 1,2,3,\cdots$, then
$$\mathcal{L}\{t^n f(t)\} = (-1)^n \frac{d^n}{ds^n} F(s).$$

EXAMPLE 1: Using Theorem 7.4.1 (Page 307)
NOTE To find transforms of functions $t^n e^{at}$ we can use either Theorem 7.3.1 or Theorem 7.4.1.
TRANSFORMS OF INTEGRALS
CONVOLUTION
If functions f and g are piecewise continuous on the interval $[0, \infty)$, then the **convolution** of f and g, denoted by the symbol $f * g$, is a function defined by the integral
$$f * g = \int_0^t f(\tau) g(t - \tau) d\tau.$$
Because we are integrating it with respect to the variable τ (the lower case Greek letter *tau*), the convolution $f * g$ is a function of t. To emphasize this fact, it is also written $(f * g)(t)$. As the notation $f * g$ suggests, the convolution is often interpreted as a *generalized product* of two functions f and g.
EXAMPLE 3: Convolution of Two Functions (Page 308)
It is left as an exercise to show that
$$\int_0^t f(\tau) g(t - \tau) d\tau = \int_0^t f(t - \tau) g(\tau) d\tau,$$
that is, $f * g = g * f$. In other words, the convolution of two functions is commutative.
CONVOLUTION THEOREM
We have seen that if f and g are both piecewise continuous for $t \geq 0$, then the Laplace transform of a sum $f + g$ is the sum of the individual Laplace transforms. While it is *not* true that the Laplace transform of the product fg is the product of the Laplace transforms, we see in the next theorem—called the **convolution theorem**—that the Laplace transform of the generalized product $f * g$ is the product of the Laplace transforms of f and g.

THEOREM 7.4.2 Convolution Theorem
If $f(t)$ and $g(t)$ are piecewise continuous on $[0, \infty)$ and of exponential order, then
$$\mathcal{L}\{f * g\} = \mathcal{L}\{f(t)\}\mathcal{L}\{g(t)\} = F(s)G(s).$$

Proof
$$F(s) = \mathcal{L}\{f(t)\} = \int_0^\infty e^{-s\tau} f(\tau) d\tau$$
$$G(s) = \mathcal{L}\{g(t)\} = \int_0^\infty e^{-s\beta} g(\beta) d\beta.$$
$$\begin{aligned} F(s)G(s) &= \left(\int_0^\infty e^{-s\tau} f(\tau) d\tau\right)\left(\int_0^\infty e^{-s\beta} g(\beta) d\beta\right) \\ &= \int_0^\infty \int_0^\infty e^{-s(\tau+\beta)} f(\tau) g(\beta) d\tau d\beta \\ &= \int_0^\infty f(\tau) d\tau \int_0^\infty e^{-s(\tau+\beta)} g(\beta) d\beta. \end{aligned}$$
Holding τ fixed, we let $t = \tau + \beta$, $dt = d\beta$, so that
$$F(s)G(s) = \int_0^\infty f(\tau) d\tau \int_\tau^\infty e^{-st} g(t - \tau) dt.$$
In the $t\tau$-plane we are integrating over the shaded region in Figure 7.8. Since f and g are piecewise continuous on $[0, \infty)$ and of exponential order, it is possible to interchange the order of integration:

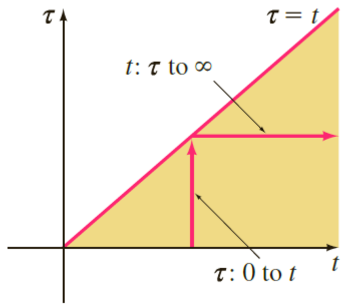


Figure 7. 8 Changing order of integration from t first to τ first

$$F(s)G(s) = \int_0^\infty e^{-st} dt \int_0^t f(\tau)g(t-\tau)d\tau = \int_0^\infty e^{-st} \left\{ \int_0^t f(\tau)g(t-\tau)d\tau \right\} dt = \mathcal{L}\{f * g\}.$$

Theorem 7.4.2 shows that we can find the Laplace transform of the convolution $f * g$ of two functions without actually evaluating the definite integral $\int_0^t f(\tau)g(t-\tau)d\tau$.

INVERSE FORM OF THEOREM 7.4.2

The convolution theorem is sometimes useful in finding the inverse Laplace transform of the product of two Laplace transforms. From Theorem 7.4.2 we have

$$\mathcal{L}^{-1}\{F(s)G(s)\} = f * g.$$

TRANSFORM OF AN INTEGRAL

When $g(t) = 1$ and $\mathcal{L}\{g(t)\} = G(s) = 1/s$, the convolution theorem implies that the Laplace transform of the integral of f is

$$\mathcal{L}\left\{ \int_0^t f(\tau)d\tau \right\} = \frac{F(s)}{s}.$$

The inverse form of it,

$$\int_0^t f(\tau)d\tau = \mathcal{L}^{-1}\left\{ \frac{F(s)}{s} \right\},$$

can be used in lieu of partial fractions when s^n is a factor of the denominator and $f(t) = \mathcal{L}^{-1}\{F(s)\}$ is easy to integrate. For example, we know for $f(t) = \sin t$ that $F(s) = 1/(s^2+1)$,

$$\mathcal{L}^{-1}\left\{ \frac{1}{s(s^2+1)} \right\} = \mathcal{L}^{-1}\left\{ \frac{1/(s^2+1)}{s} \right\} = \int_0^t \sin\tau\, d\tau = 1 - \cos t$$

$$\mathcal{L}^{-1}\left\{ \frac{1}{s^2(s^2+1)} \right\} = \mathcal{L}^{-1}\left\{ \frac{1/s(s^2+1)}{s} \right\} = \int_0^t (1-\cos\tau)d\tau = t - \sin t$$

$$\mathcal{L}^{-1}\left\{ \frac{1}{s^3(s^2+1)} \right\} = \mathcal{L}^{-1}\left\{ \frac{1/s^2(s^2+1)}{s} \right\} = \int_0^t (\tau - \sin\tau)d\tau = \frac{1}{2}t^2 - 1 + \cos t$$

and so on.

EXERCISES

7. $\mathcal{L}\{te^{2t}\sin 6t\}$

$$F(s) = \mathcal{L}\{e^{2t}\sin 6t\} = \frac{6}{(s-2)^2+36}$$

$$\rightarrow \frac{d}{ds}\left(\frac{6}{(s-2)^2+36} \right) = \frac{24-12s}{((s-2)^2+36)^2}$$

$$(-1)\left(\frac{24-12s}{((s-2)^2+36)^2} \right) = -\frac{24-12s}{((s-2)^2+36)^2}.$$

CHAPTER 11 Fourier Series

11.1 Orthogonal Functions

INTRODUCTION

The concepts of geometric vectors in two and three dimensions, orthogonal or perpendicular vectors, and the inner product of two vectors have been generalized. It is perfectly routine in mathematics to think of a function as a vector.

INNER PRODUCT

Recall that if $\mathbf{u}$ and $\mathbf{v}$ are two vectors in R^3 or 3-space, then the inner product $(\mathbf{u},\mathbf{v})$ (in calculus this is called the dot product and written as $\mathbf{u}\cdot\mathbf{v}$) possesses the following properties:

i. $(\mathbf{u},\mathbf{v}) = (\mathbf{v},\mathbf{u})$,
ii. $(k\mathbf{u},\mathbf{v}) = k(\mathbf{u},\mathbf{v}), k$ a scalar
iii. $(\mathbf{u},\mathbf{u}) = 0$ if $\mathbf{u} = \mathbf{0}$ and $(\mathbf{u},\mathbf{u}) > 0$ if $\mathbf{u} \neq \mathbf{0}$,
iv. $(\mathbf{u}+\mathbf{v},\mathbf{w}) = (\mathbf{u},\mathbf{w}) + (\mathbf{v},\mathbf{w})$.

We expect that any generalization of the inner product concept should have these same properties.
Suppose that f_1 and f_2 are functions defined on an interval $[a,b]$. Since a *definite integral* on $[a,b]$ of the product $f_1(x)f_2(x)$ possesses the foregoing properties (i)–(iv) of an inner product whenever the integral exists, we are prompted to make the following definition.

THEOREM 11.1.1 Inner Product of Functions
The **inner product** of two functions f_1 and f_2 on an interval $[a,b]$ is the number

$$(f_1,f_2) = \int_a^b f_1(x)f_2(x)\,dx$$

ORTHOGONAL FUNCTIONS

Motivated by the fact that two geometric vectors $\mathbf{u}$ and $\mathbf{v}$ are orthogonal whenever their inner product is zero, we define **orthogonal functions** in a similar manner.

THEOREM 11.1.2 Orthogonal Functions
Two functions f_1 and f_2 are **orthogonal** on an interval $[a,b]$ if

$$(f_1,f_2) = \int_a^b f_1(x)f_2(x)\,dx = 0.$$

Unlike in vector analysis, in which the word *orthogonal* is a synonym for *perpendicular*, in this present context the term *orthogonal* and condition $(f_1,f_2) = 0$ have no geometric significance. Note that the zero function is orthogonal to every function.

ORTHOGONAL SETS

We are primarily interested in infinite sets of orthogonal functions that are all defined on the same interval $[a,b]$.

THEOREM 11.1.3 Orthogonal Set
A set of real-valued functions $\{\phi_0(x),\phi_1(x),\phi_2(x),\cdots\}$ is said to be **orthogonal** on an interval $[a,b]$ if

$$(\phi_m,\phi_n) = \int_a^b \phi_m(x)\phi_n(x)\,dx = 0,\quad m \neq n.$$

ORTHONORMAL SETS

The norm, or length $\|\mathbf{u}\|$, of a vector $\mathbf{u}$ can be expressed in terms of the inner product. The expression $(\mathbf{u},\mathbf{u}) = \|\mathbf{u}\|^2$ is called the square norm, and so the norm is $\|\mathbf{u}\| = \sqrt{(\mathbf{u},\mathbf{u})}$. Similarly, the **square norm** of a function ϕ_n is $\|\phi_n\|^2 = (\phi_n,\phi_n)$, and so the **norm**, or its generalized length, is $\|\phi_n\| = \sqrt{(\phi_n,\phi_n)}$. In other words, the square norm and norm of a function ϕ_n in an orthogonal set $\{\phi_n(x)\}$ are, respectively,

$$\|\phi_n\|^2 = \int_a^b \phi_n^2(x)\,dx \quad \text{and} \quad \|\phi_n\| = \sqrt{\int_a^b \phi_n^2(x)\,dx}.$$

If $\{\phi_n(x)\}$ is an orthogonal set of functions on the interval $[a, b]$ with the additional property that $\|\phi_n\| = 1$ for $n = 0,1,2,\cdots$, then $\{\phi_n(x)\}$ is said to be an **orthonormal set** on the interval.

EXAMPLE 2: Orthogonal Set of Functions

Show that the set $\{1, \cos x, \cos 2x, \cdots\}$ is orthogonal on the interval $[-\pi, \pi]$.

SOLUTION

If we make the identification $\phi_0(x) = 1$ and $\phi_n(x) = \cos nx$, we must then show that $\int_{-\pi}^{\pi} \phi_0(x)\phi_n(x)dx = 0$, $n \neq 0$, and $\int_{-\pi}^{\pi} \phi_m(x)\phi_n(x)dx = 0$, $m \neq n$. We have, in the first case,

$$(\phi_0, \phi_n) = \int_{-\pi}^{\pi} \phi_0(x)\phi_n(x)dx = \int_{-\pi}^{\pi} \cos nx\,dx$$

$$= \frac{1}{n}\sin nx \Big|_{-\pi}^{\pi} = \frac{1}{n}[\sin n\pi - \sin(-n\pi)] = 0, \quad n \neq 0,$$

and, in the second,

$$(\phi_m, \phi_n) = \int_{-\pi}^{\pi} \phi_m(x)\phi_n(x)dx$$

$$= \int_{-\pi}^{\pi} \cos mx \cos nx\,dx$$

$$= \frac{1}{2}\int_{-\pi}^{\pi} [\cos(m+n)x + \cos(m-n)x]dx \quad \leftarrow \text{trigonometric identity}$$

$$= \frac{1}{2}\left[\frac{\sin(m+n)x}{m+n} + \frac{\sin(m-n)x}{m-n}\right]\Big|_{-\pi}^{\pi} = 0, \quad m \neq n.$$

EXAMPLE 3: Norms

Find the norm of each function in the orthogonal set given in Example 2.

SOLUTION

For $\phi_0(x) = 1$ we have,

$$\|\phi_0(x)\|^2 = \int_{-\pi}^{\pi} dx = 2\pi,$$

so $\|\phi_0(x)\| = \sqrt{2\pi}$. For $\phi_n(x) = \cos nx$, $n > 0$, it follows that

$$\|\phi_n(x)\|^2 = \int_{-\pi}^{\pi} \cos^2 nx\,dx = \frac{1}{2}\int_{-\pi}^{\pi} [1 + \cos 2nx]dx = \pi.$$

Thus for $n > 0$, $\|\phi_n(x)\| = \sqrt{\pi}$.

NORMALIZATION

Any orthogonal set of nonzero functions $\{\phi_n(x)\}$, $n = 0,1,2,\cdots$ can be made into an orthonormal set by **normalizing** each function in the set, that is, by dividing each function by its norm.

EXAMPLE 4: Orthonormal Set

In Example 2 we proved that the set

$$\{1, \cos x, \cos 2x, \cdots\}$$

is orthogonal on the interval $[-\pi, \pi]$. In Example 3, we then saw that the norms of the functions in the foregoing set are

$$\|\phi_0(x)\| = \|1\| = \sqrt{2\pi} \quad \text{and} \quad \|\phi_n(x)\| = \|\cos nx\| = \sqrt{\pi}, n = 1,2,\cdots$$

By dividing each function by its norm we obtain the set

$$\left\{\frac{1}{\sqrt{2\pi}}, \frac{\cos x}{\sqrt{\pi}}, \frac{\cos 2x}{\sqrt{\pi}}, \cdots\right\}$$

which is orthonormal on the interval $[-\pi, \pi]$.

VECTOR ANALOGY

In the introduction to this section, we stated that our purpose for studying orthogonal functions is to be able to expand a function in terms of an infinite set $\{\phi_n(x)\}$ of orthogonal functions. To motivate this concept we shall make one more analogy between vectors and functions. (Page 427)

ORTHOGONAL SERIES EXPANSION

Suppose $\{\phi_n(x)\}$ is an infinite orthogonal set of functions on an interval $[a,b]$. We ask: If $y=f(x)$ is a function defined on the interval $[a,b]$, is it possible to determine a set of coefficients c_n, $n=0,1,2,\cdots$, for which

$$f(x)=c_0\phi_0(x)+c_1\phi_1(x)+\cdots+c_n\phi_n(x)+\cdots?$$

As in the foregoing discussion on finding components of a vector we can find the desired coefficients c_n by using the inner product. Multiplying the above equation by $\phi_m(x)$ and integrating over the interval $[a,b]$ gives

$$\int_a^b f(x)\phi_m(x)\,dx=c_0\int_a^b \phi_0(x)\phi_m(x)\,dx+c_1\int_a^b \phi_1(x)\phi_m(x)\,dx+\cdots+c_n\int_a^b \phi_n(x)\phi_m(x)\,dx+\cdots$$
$$=c_0(\phi_0,\phi_m)+c_1(\phi_1,\phi_m)+\cdots+c_n(\phi_n,\phi_m)+\cdots.$$

By orthogonality each term on the right-hand side of the last equation is zero *except* when $m=n$. In this case we have

$$\int_a^b f(x)\phi_n(x)\,dx=c_n\int_a^b \phi_n^2(x)\,dx.$$

It follows that the required coefficients c_n are given by

$$c_n=\frac{\int_a^b f(x)\phi_n(x)\,dx}{\int_a^b \phi_n^2(x)\,dx},\quad n=0,1,2,\cdots.$$

In other words,

$$f(x)=\sum_{n=0}^{\infty}c_n\phi_n(x),$$

where

$$c_n=\frac{\int_a^b f(x)\phi_n(x)\,dx}{\|\phi_n(x)\|^2}.$$

With inner product notation, $f(x)$ becomes

$$f(x)=\sum_{n=0}^{\infty}\frac{(f,\phi_n)}{\|\phi_n(x)\|^2}\phi_n(x).$$

Thus the above is seen to be the function analogue of the vector result given in VECTOR ANALOGY.

> **THEOREM 11.1.4 Orthogonal Set/Weight Function**
> A set of real-valued functions $\{\phi_0(x),\phi_1(x),\phi_2(x),\cdots\}$ is said to be **orthogonal with respect to a weight function** $w(x)$ on an interval $[a,b]$ if
> $$\int_a^b w(x)\phi_m(x)\phi_n(x)\,dx=0,\quad m\neq n.$$

The usual assumption is that $w(x)>0$ on the interval of orthogonality $[a,b]$. The set $\{1,\cos x,\cos 2x,\cdots\}$ in Example 2 is orthogonal with respect to the weight function $w(x)=1$ on the interval $[-\pi,\pi]$.

If $\{\phi_n(x)\}$ is orthogonal with respect to a weight function $w(x)$ on the interval $[a,b]$, then multiplying $f(x)=c_0\phi_0(x)+c_1\phi_1(x)+\cdots+c_n\phi_n(x)+\cdots$ by $w(x)\phi_n(x)$ and integrating yields

$$c_n=\frac{\int_a^b f(x)w(x)\phi_n(x)\,dx}{\|\phi_n(x)\|^2},$$

where

$$\|\phi_n(x)\|^2=\int_a^b w(x)\phi_n^2(x)\,dx.$$

The series $f(x)=\sum_{n=0}^{\infty}c_n\phi_n(x)$ with coefficients c_n given by either $c_n=\frac{\int_a^b f(x)\phi_n(x)\,dx}{\|\phi_n(x)\|^2}$ or $c_n=$

$\frac{\int_a^b f(x)w(x)\phi_n(x)\,dx}{\|\phi_n(x)\|^2}$ is said to be an **orthogonal series expansion** of f or a **generalized Fourier series**.

COMPLETE SETS

The procedure outlined for determining the coefficients c_n in $c_n = \frac{\int_a^b f(x)\phi_n(x)\,dx}{\|\phi_n(x)\|^2}$ was *formal*; that is, fundamental questions about whether or not an orthogonal series expansion of a function f such as $f(x) = \sum_{n=0}^{\infty} c_n\phi_n(x)$ actually converges to the function were ignored. It turns out that for some specific orthogonal sets such series expansions do indeed converge to the function. (Page 429)

EXERCISES

12. Show that the given set of functions is orthogonal on the indicated interval. Find the norm of each function in the set.

$$\left\{1, \cos\frac{n\pi}{p}x, \sin\frac{m\pi}{p}x\right\},\ n = 1,2,3,\cdots,\ m = 1,2,3,\cdots;\ [-p,p]$$

Orthogonality:

$$\left(1, \cos\frac{n\pi}{p}x\right) = \int_{-p}^{p} \cos\frac{n\pi}{p}x\,dx$$
$$= 0.$$
$$\left(1, \sin\frac{m\pi}{p}x\right) = \int_{-p}^{p} \sin\frac{m\pi}{p}x\,dx$$
$$= 0.$$
$$\left(\cos\frac{n\pi}{p}x, \sin\frac{m\pi}{p}x\right) = \int_{-p}^{p} \cos\frac{n\pi}{p}x\sin\frac{m\pi}{p}x\,dx$$
$$= \frac{1}{2}\int_{-p}^{p}\left[\sin\frac{(m+n)\pi}{p}x + \sin\frac{(m-n)\pi}{p}x\right]dx$$
$$= 0.$$
$$\left(\cos\frac{n\pi}{p}x, \cos\frac{m\pi}{p}x\right) = \int_{-p}^{p} \cos\frac{n\pi}{p}x\cos\frac{m\pi}{p}x\,dx \quad \text{where } n \neq m$$
$$= \frac{1}{2}\int_{-p}^{p}\left[\cos\frac{(n+m)\pi}{p}x + \cos\frac{(n-m)\pi}{p}x\right]dx$$
$$= 0.$$
$$\left(\sin\frac{n\pi}{p}x, \sin\frac{m\pi}{p}x\right) = \int_{-p}^{p} \sin\frac{n\pi}{p}x\sin\frac{m\pi}{p}x\,dx \quad \text{where } n \neq m$$
$$= \frac{1}{2}\int_{-p}^{p}\left[\cos\frac{(n-m)\pi}{p}x - \cos\frac{(n+m)\pi}{p}x\right]dx$$
$$= 0.$$

Given the above results, the set of functions are orthogonal on the interval $[-p,p]$.

Norm:

$$\|1\| = \sqrt{\int_{-p}^{p} dx}$$
$$= \sqrt{2p}.$$
$$\left\|\cos\frac{n\pi}{p}x\right\| = \sqrt{\int_{-p}^{p} \cos^2\frac{n\pi}{p}x\,dx}$$
$$= \sqrt{p}.$$
$$\left\|\sin\frac{n\pi}{p}x\right\| = \sqrt{\int_{-p}^{p} \sin^2\frac{n\pi}{p}x\,dx}$$
$$= \sqrt{p}.$$

17. Let $\{\phi_n(x)\}$ be an orthogonal set of functions on $[a,b]$. Show that $\|\phi_m(x) + \phi_n(x)\|^2 = \|\phi_m(x)\|^2 +$

$\|\phi_n(x)\|^2$, $m \neq n$.

$$\|\phi_m(x)+\phi_n(x)\|^2 = \int_a^b \big(\phi_m(x)+\phi_n(x)\big)^2 dx$$
$$= \int_a^b \big(\phi_m^2(x)+\phi_n^2(x)+2\phi_m(x)\phi_n(x)\big)dx$$
$$= \int_a^b \phi_m^2(x)dx + \int_a^b \phi_n^2(x)dx + 2\int_a^b \phi_m(x)\phi_n(x)dx$$
$$= \|\phi_m(x)\|^2 + \|\phi_n(x)\|^2.$$

26. An orthogonal set can be constructed out of any linearly independent set $\{f_0(x), f_1(x), f_2(x), \cdots\}$ of real-valued functions continuous on an interval $[a,b]$ using the **Gram-Schmidt orthogonalization process**. With the inner product $(f_n, \phi_n) = \int_a^b f_n(x)\phi_n(x)dx$, define the functions in the set $B' = \{\phi_0(x),\ \phi_1(x),\ \phi_2(x), \cdots\}$ to be

$$\phi_0(x) = f_0(x)$$
$$\phi_1(x) = f_1(x) - \frac{(f_1,\phi_0)}{\|\phi_0\|^2}\phi_0(x)$$
$$\phi_2(x) = f_2(x) - \frac{(f_2,\phi_0)}{\|\phi_0\|^2}\phi_0(x) - \frac{(f_2,\phi_1)}{\|\phi_1\|^2}\phi_1(x)$$
$$\vdots$$

and so on.

(a) Write out $\phi_3(x)$ in the set B'.

$$\phi_3(x) = f_3(x) - \frac{(f_3,\phi_0)}{\|\phi_0\|^2}\phi_0(x) - \frac{(f_3,\phi_1)}{\|\phi_1\|^2}\phi_1(x) - \frac{(f_3,\phi_2)}{\|\phi_2\|^2}\phi_2(x)$$

(b) By construction, the set $B' = \{\phi_0(x), \phi_1(x), \phi_2(x), \cdots\}$ is orthogonal on $[a,b]$. Demonstrate that $\phi_0(x)$, $\phi_1(x)$, and $\phi_2(x)$ are mutually orthogonal.

$$(\phi_0,\phi_1) = \int_a^b \phi_0(x)\phi_1(x)dx = \int_a^b \phi_0(x)\left[f_1(x) - \frac{(f_1,\phi_0)}{\|\phi_0\|^2}\phi_0(x)\right]dx = 0$$
$$(\phi_0,\phi_2) = 0$$
$$(\phi_1,\phi_2) = 0.$$

11.2 Fourier Series

The orthogonal set of trigonometric functions

$$\left\{1, \cos\frac{\pi}{p}x, \cos\frac{2\pi}{p}x, \cos\frac{3\pi}{p}x, \cdots, \sin\frac{\pi}{p}x, \sin\frac{2\pi}{p}x, \sin\frac{3\pi}{p}x, \cdots\right\}$$

will be of particular importance later on in the solution of certain kinds of boundary-value problems involving linear partial differential equations. The set above is orthogonal on the interval $[-p,p]$.

A TRIGONOMETRIC SERIES

Suppose that f is a function defined on the interval $(-p,p)$ and can be expanded in an orthogonal series consisting of the trigonometric functions in the orthogonal set above; that is,

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n\cos\frac{n\pi}{p}x + b_n\sin\frac{n\pi}{p}x\right).$$

Note that we have chosen to write the coefficient of 1 in the set as $\frac{a_0}{2}$ rather than a_0. This is for convenience only; the formula of a_n will then reduce to a_0 for $n = 0$.

$$\int_{-p}^{p} f(x)\cdot 1dx = \frac{a_0}{2}\int_{-p}^{p} dx = pa_0.$$

Solving for a_0 yields

$$a_0 = \frac{1}{p}\int_{-p}^{p} f(x)dx.$$

$$\int_{-p}^{p} f(x)\cdot\cos\frac{m\pi}{p}x\,dx = a_n p,$$

$$a_n = \frac{1}{p}\int_{-p}^{p} f(x)\cos\frac{n\pi}{p}x\,dx.$$
$$\int_{-p}^{p} f(x)\cdot\sin\frac{m\pi}{p}x\,dx = b_n p,$$
$$b_n = \frac{1}{p}\int_{-p}^{p} f(x)\sin\frac{n\pi}{p}x\,dx.$$

The trigonometric series $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n\cos\frac{n\pi}{p}x + b_n\sin\frac{n\pi}{p}x\right)$ with coefficients a_0, a_n, and b_n defined by $a_0 = \frac{1}{p}\int_{-p}^{p} f(x)dx$, $a_n = \frac{1}{p}\int_{-p}^{p} f(x)\cos\frac{n\pi}{p}x\,dx$ and $b_n = \frac{1}{p}\int_{-p}^{p} f(x)\sin\frac{n\pi}{p}x\,dx$, respectively, is said to be the **Fourier series** of f.

The formulas above for coefficients a_0, a_n, and b_n in a Fourier series are known as the **Euler formulas**.

DEFINITION 11.2.1 Fourier Series

The **Fourier series** of a function f defined on the interval $(-p, p)$ is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n\cos\frac{n\pi x}{p} + b_n\sin\frac{n\pi x}{p}\right)$$

where

$$a_0 = \frac{1}{p}\int_{-p}^{p} f(x)dx$$
$$a_n = \frac{1}{p}\int_{-p}^{p} f(x)\cos\frac{n\pi x}{p}dx$$
$$b_n = \frac{1}{p}\int_{-p}^{p} f(x)\sin\frac{n\pi x}{p}dx.$$

CONVERGENCE OF A FOURIER SERIES

In the absence of any stated conditions that guarantee the validity of the steps leading to the coefficients a_0, a_n, and b_n, the equality sign in $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n\cos\frac{n\pi x}{p} + b_n\sin\frac{n\pi x}{p}\right)$ should not be taken in a strict or literal sense.

In view of the fact that most functions in applications are of the type that guarantee convergence of the series, we shall use the equality symbol. Is it possible for a series to converge at a number x in the interval $(-p, p)$, and yet not be equal to $f(x)$? The answer is an emphatic *Yes*.

PIECEWISE-CONTINUOUS FUNCTIONS

We use the symbols $f(x^+)$ and $f(x^-)$ to denote the one-sided limits

$$f(x^+) = \lim_{\substack{h\to 0\\ h>0}} f(x+h), \quad f(x^-) = \lim_{\substack{h\to 0\\ h>0}} f(x-h),$$

called, respectively, the **right-** and **left-hand limits** of f at x. A function f is said to be **piecewise continuous** on a closed interval $[a, b]$ if there are

- a finite number of points $x_1 < x_2 < \cdots < x_n$ in $[a, b]$ at which f has a finite (or jump) discontinuity, and
- f is continuous on each open interval (x_k, x_{k+1}).

As a consequence of this definition, the one-sided limits $f(x^+)$ and $f(x^-)$ must exist at every x satisfying $a < x < b$. The limits $f(a^+)$ and $f(b^-)$ must also exist but it is not required that f be continuous or even defined at either a or b.

THEOREM 11.2.1 Conditions for Convergence
Let f and f' be piecewise continuous on the interval $[-p, p]$. Then for all x in the interval $(-p, p)$, the Fourier series of f converges to $f(x)$ at a point continuity. At a point of discontinuity the Fourier series converges to the average

$$\frac{f(x^+) + f(x^-)}{2},$$

where $f(x^+)$ and $f(x^-)$ are the right- and left-hand limits of f at x, respectively.

EXAMPLE 1: Expansion in a Fourier Series
Expand

$$f(x) = \begin{cases} 0, & -\pi < x < 0 \\ \pi - x, & 0 \le x < \pi \end{cases}$$

in a Fourier series.

SOLUTION
The graph of f is given in Figure 11.1. With $p = \pi$ we have

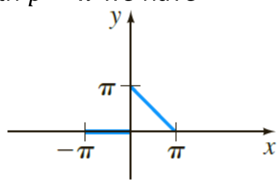


Figure 11. 1 Piecewise-continuous function f in Example 1

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx = \frac{\pi}{2}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\, dx = \frac{1}{\pi}\left[\int_{-\pi}^{0} 0 dx + \int_{0}^{\pi} (\pi - x)\cos nx\, dx\right]$$

$$= \frac{1}{\pi}\left[(\pi - x)\frac{\sin nx}{n}\Big|_0^{\pi} + \frac{1}{n}\int_0^{\pi} \sin nx\, dx\right] \quad \leftarrow \text{integration by parts}$$

$$= -\frac{1}{n\pi}\frac{\cos nx}{n}\Big|_0^{\pi} = \frac{1 - (-1)^n}{n^2\pi},$$

where we have used $\cos nx = (-1)^n$. In like manner we find

$$b_n = \frac{1}{\pi}\int_0^{\pi} (\pi - x)\sin nx\, dx = \frac{1}{n}.$$

Therefore

$$f(x) = \frac{\pi}{4} + \sum_{n=1}^{\infty}\left\{\frac{1 - (-1)^n}{n^2\pi}\cos nx + \frac{1}{n}\sin nx\right\}.$$

PERIODIC EXTENSION
Observe that each of the functions in the basic set $\left\{1, \cos\frac{\pi}{p}x, \cos\frac{2\pi}{p}x, \cdots, \sin\frac{\pi}{p}x, \sin\frac{2\pi}{p}x, \cdots\right\}$ has a different fundamental period – namely, $2p/n$, $n \ge 1$- but since a positive integer multiple of a period is also a period, we see that all the functions have in common the period $2p$. Hence the right-hand side of $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi}{p}x + b_n \sin\frac{n\pi}{p}x\right)$ is $2p$-periodic; indeed, $2p$ is the **fundamental period** of the sum. We conclude that a Fourier series not only represents the function on the interval $(-p, p)$ but also gives the **periodic extension** of f outside this interval. ($f(x + 2p) = f(x)$)
When f is piecewise continuous and the right- and left-hand derivatives exist at $x = -p$ and $x = p$, respectively, then the series $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi x}{p} + b_n \sin\frac{n\pi x}{p}\right)$ converges to the average

$$\frac{f(p^+) + f(-p^+)}{2}$$

at these endpoints and to this value extended periodically to $\pm 3p, \pm 5p, \pm 7p$, and so on.

SEQUENCE OF PARTIAL SUMS

It is interesting to see how the sequence of partial sums $\{S_N(x)\}$ of a Fourier series approximates a function. (Page 435)

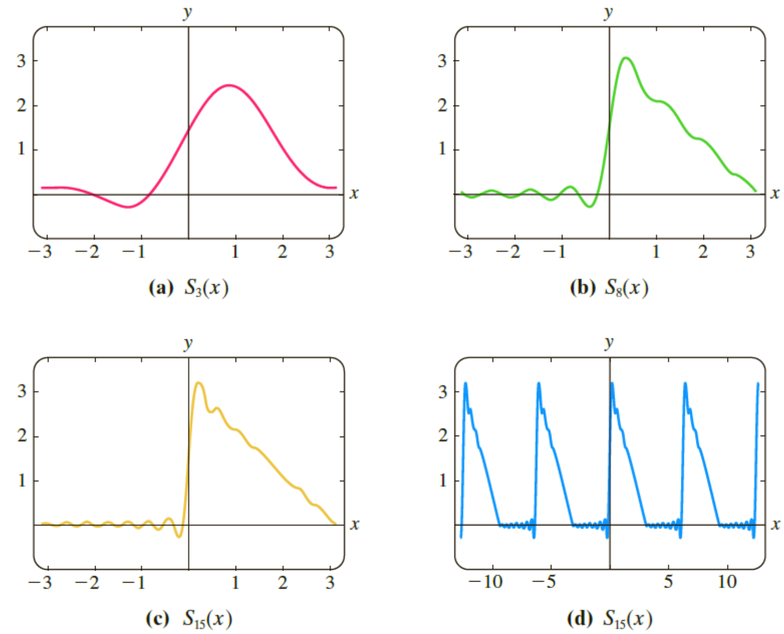


Figure 11. 2 Partial sums of Fourier series in Example 1

EXERCISES

Find the Fourier series of f on the given interval. Give the number to which the Fourier series converges at a point of discontinuity of f.

5. $f(x) = \begin{cases} 0, & -\pi < x < 0 \\ x^2, & 0 \le x < \pi \end{cases}$

$$a_0 = \frac{\pi^2}{3}$$

$$a_n = \frac{2}{n^2}(-1)^n$$

$$b_n = \frac{n^2\pi^2(-1)^{n+1} + 2(-1)^n - 2}{\pi n^3}$$

$$f(x) = \frac{\pi^2}{6} + \sum_{n=1}^{\infty}\left[\frac{2}{n^2}(-1)^n \cos nx + \frac{n^2\pi^2(-1)^{n+1} + 2(-1)^n - 2}{\pi n^3}\sin nx\right]$$

The function is continuous on the interval $(-\pi, \pi)$, and it converges at the end points of the interval to the average $\frac{\pi^2}{2}$.

9. $f(x) = \begin{cases} 0, & -\pi < x < 0 \\ \sin x, & 0 \le x < \pi \end{cases}$

$$a_0 = \frac{2}{\pi}$$

$$a_n = \frac{(-1)^n + 1}{\pi(1 - n^2)} \quad \text{For } n \neq 1$$

$$a_1 = 0$$

$$b_n = \frac{1}{2\pi}\left(\frac{\sin(1-n)x}{1-n} - \frac{\sin(1+n)x}{1+n}\right)\Big|_0^{\pi} = 0 \quad \text{For } n \neq 1$$

$$b_1 = \frac{1}{2}$$

$$f(x)=\frac{1}{\pi}+\frac{\sin x}{2}+\sum_{n=2}^{\infty}\left[\frac{(-1)^n+1}{\pi(1-n^2)}\cos nx\right]$$

The function is continuous on the interval $(-\pi,\pi)$, and it converges at the end points of the interval to the average 0.

19. Use the result of Problem 5 to show that

$$\frac{\pi^2}{6}=1+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\cdots$$

and

$$\frac{\pi^2}{12}=1-\frac{1}{2^2}+\frac{1}{3^2}-\frac{1}{4^2}+\cdots.$$

$$f(\pi)=\frac{\pi^2}{2}=\frac{\pi^2}{6}+\sum_{n=1}^{\infty}\frac{2}{n^2}$$

$$\frac{\pi^2}{6}=1+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\cdots.$$

$$f(0)=0=\frac{\pi^2}{6}+\sum_{n=1}^{\infty}\left[\frac{2}{n^2}(-1)^n\right]$$

$$0=\frac{\pi^2}{6}+2\left(-1+\frac{1}{2^2}+\frac{(-1)}{3^2}+\frac{1}{4^2}+\cdots\right)$$

$$\frac{\pi^2}{12}=1-\frac{1}{2^2}+\frac{1}{3^2}-\frac{1}{4^2}+\cdots.$$

Parseval's Theorem

If a function f has a Fourier series given by

$$f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}\left(a_n\cos\frac{n\pi x}{p}+b_n\sin\frac{n\pi x}{p}\right)$$

Then

$$\int_{-p}^{p}[f(x)]^2dx=\frac{a_0^2}{4}\int_{-p}^{p}dx+\sum_{n=1}^{\infty}\left(a_n^2\int_{-p}^{p}\left[\cos\frac{n\pi x}{p}\right]^2dx+b_n^2\int_{-p}^{p}\left[\sin\frac{n\pi x}{p}\right]^2dx\right)$$

$$=\frac{a_0^2}{2}p+\sum_{n=1}^{\infty}(a_n^2p+b_n^2p).$$

so

$$\frac{1}{p}\int_{-p}^{p}[f(x)]^2dx=\frac{a_0^2}{2}+\sum_{n=1}^{\infty}(a_n^2+b_n^2).$$

11.3 Fourier Cosine and Sine Series

INTRODUCTION

The effort that is expended in evaluation of the definite integrals that define the coefficients the a_0, a_n, and b_n in the expansion of a function f in a Fourier series is reduced significantly when f is either an even or an odd function.

Recall that a function f is said to be

even if $f(-x)=f(x)$ and **odd** if $f(-x)=-f(x)$.

EVEN AND ODD FUNCTIONS

$$f(x)=x^{\overset{\text{even integer}}{\downarrow}2}\text{ is even}\quad\text{since } f(-x)=(-x)^2=x^2=f(x)$$

$$f(x)=x^{\overset{\text{odd integer}}{\downarrow}3}\text{ is odd}\quad\text{since } f(-x)=(-x)^3=-x^3=-f(x).$$

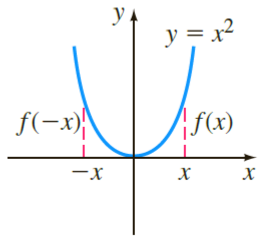


Figure 11. 3 Even function; graph symmetric with respect to y-axis

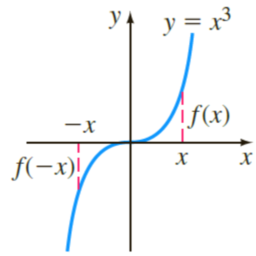


Figure 11. 4 Odd function; graph symmetric with respect to origin

The trigonometric cosine and sine functions are even and odd functions, respectively, since $\cos(-x) = \cos(x)$ and $\sin(-x) = -\sin(x)$. The exponential functions $f(x) = e^x$ and $f(x) = e^{-x}$ are neither odd nor even.

PROPERTIES

THEOREM 11.3.1 Properties of Even/Odd Functions
(a) The product of two even functions is even.
(b) The product of two odd functions is even.
(c) The product of an even function and an odd function is odd.
(d) The sum (difference) of two even functions is even.
(e) The sum (difference) of two odd functions is odd.
(f) If f is even, then $\int_{-a}^{a} f(x)dx = 2\int_0^a f(x)dx$.
(g) If f is odd, then $\int_{-a}^{a} f(x)dx = 0$.

COSINE AND SINE SERIES

If f is an even function on $(-p, p)$,

$$a_0 = \frac{1}{p}\int_{-p}^{p} f(x)\,dx = \frac{2}{p}\int_0^p f(x)\,dx$$

$$a_n = \frac{1}{p}\underbrace{\int_{-p}^{p} f(x)\cos\frac{n\pi}{p}x\,dx}_{\text{even}} = \frac{2}{p}\int_0^p f(x)\cos\frac{n\pi}{p}x\,dx$$

$$b_n = \frac{1}{p}\underbrace{\int_{-p}^{p} f(x)\sin\frac{n\pi}{p}x\,dx}_{\text{odd}} = 0.$$

Similarly, when f is odd on the interval $(-p, p)$,

$$a_n = 0, n = 0,1,2,\cdots, \quad b_n = \frac{2}{p}\int_0^p f(x)\sin\frac{n\pi}{p}x\,dx.$$

DEFINITION 11.3.1 Fourier Cosine and Sine Series

i. The Fourier series of an even function f defined on the interval $(-p, p)$ is the **cosine series**

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi}{p}x,$$

where

$$a_0 = \frac{2}{p}\int_0^p f(x)dx$$

$$a_n = \frac{2}{p}\int_0^p f(x)\cos\frac{n\pi}{p}x\,dx.$$

ii. The Fourier series of an odd function f defined on the interval $(-p, p)$ is the **sine series**

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi}{p}x,$$

where

$$b_n = \frac{2}{p}\int_0^p f(x)\sin\frac{n\pi}{p}x\,dx.$$

Because the term $\sin(n\pi x/p)$ is 0 at $x = -p$, $x = 0$, and $x = p$, a sine series converges to 0 at those points regardless of whether f is defined at these points.

EXAMPLE 1: Expansion in a Sine Series (Page 438)

CHAPTER 14 Integral Transforms

14.1 Error Function

14.2 Laplace Transform

14.3 Fourier Integral

14.4 Fourier Transforms

TRANSFORM PAIRS

FOURIER TRANSFORM PAIRS

DEFINITION 11.3.1 Fourier Cosine and Sine Series

i. Fourier transform:

$$\mathcal{F}\{f(x)\} = \int_{-\infty}^{\infty} f(x)e^{i\alpha x}dx = F(\alpha)$$

Inverse Fourier transform:

$$\mathcal{F}^{-1}\{F(\alpha)\} = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\alpha)e^{-i\alpha x}d\alpha = f(x)$$