# Notes for Complex Analysis - A First Course with Applications

## CHAPTER 1 Complex Numbers and the Complex Plane

• Polar form, exponential form of the complex number: $z = r(\cos\theta + i\sin\theta) = re^{i\theta} = x + iy$.
Cauchy-Riemann Equations;
It is important to be aware that a line integral is independent of the parametrization of the curve $C$, provided $C$ is given the same orientation by all sets of parametric equations defining the curve.
Cauchy's Theorem→Cauchy-Goursat Theorem; (Cauchy-Goursat Theorem for Multiply Connected Domains)
Cauchy's Integral Formula; Cauchy's Integral Formula for Derivatives
Residue; Cauchy's Residue Theorem
1.1 Complex Numbers and Their Properties

**The Imaginary Unit**

The symbol $i$ was originally used as a disguise for the embarrassing $\sqrt{-1}$. We now say that $i$ is the **imaginary unit** and define it by the property $i^2 = -1$. Using the imaginary unit, we build a general complex number out of two real numbers.

> **Definition 1.1 Complex Number**
> A complex number is any number of the form $z = a + ib$ where $a$ and $b$ are real numbers and $i$ is the imaginary unit.

**Terminology**

The notations $a + ib$ and $a + bi$ are used interchangeably. The real number $a$ in $z = a + ib$ is called the **real part** of $z$; the real number $b$ is called the **imaginary part** of $z$. The real and imaginary parts of a complex number $z$ are abbreviated $\mathrm{Re}(z)$ and $\mathrm{Im}(z)$, respectively.

> **Definition 1.2 Equality**
> Complex numbers $z_1 = a_1 + ib_1$ and $z_2 = a_2 + ib_2$ are **equal**, $z_1 = z_2$, if $a_1 = a_2$ and $b_1 = b_2$.

The totality of complex numbers or the set of complex numbers is usually denoted by the symbol $\mathbf{C}$. Because any real number $a$ can be written as $z = a + 0i$, we see that the set $\mathbf{R}$ of real numbers is a subset of $\mathbf{C}$.

**Arithmetic Operations**

Complex numbers can be added, subtracted, multiplied, and divided. If $z_1 = a_1 + ib_1$ and $z_2 = a_2 + ib_2$, these operations are defined as follows.

Addition:

$$z_1 + z_2 = (a_1 + ib_1) + (a_2 + ib_2) = (a_1 + a_2) + i(b_1 + b_2)$$

Subtraction:

$$z_1 - z_2 = (a_1 + ib_1) - (a_2 + ib_2) = (a_1 - a_2) + i(b_1 - b_2)$$

Multiplication:

$$\begin{aligned} z_1 \cdot z_2 &= (a_1 + ib_1)(a_2 + ib_2) \\ &= a_1a_2 - b_1b_2 + i(b_1a_2 + a_1b_2) \end{aligned}$$

Division:

$$\begin{aligned} \frac{z_1}{z_2} &= \frac{a_1 + ib_1}{a_2 + ib_2}, a_2 \neq 0, \text{or } b_2 \neq 0 \\ &= \frac{a_1a_2 + b_1b_2}{a_2^2 + b_2^2} + i\frac{b_1a_2 - a_1b_2}{a_2^2 + b_2^2} \end{aligned}$$

Commutative laws:

$$\begin{cases} z_1 + z_2 = z_2 + z_1 \\ z_1z_2 = z_2z_1 \end{cases}$$

Associative laws:

$$\begin{cases} z_1 + (z_2 + z_3) = (z_1 + z_2) + z_3 \\ z_1(z_2 z_3) = (z_1 z_2) z_3 \end{cases}$$

Distributive law:

$$z_1(z_2 + z_3) = z_1 z_2 + z_1 z_3$$

**Zero and Unity**

The **zero** in the complex number system is the number $0 + 0i$ and the **unity** is $1 + 0i$. The zero and unity are denoted by $0$ and $1$, respectively. The zero is the **additive identity** in the complex number system. Similarly, the unity is the **multiplicative identity** of the system.

**Conjugate**

If $z$ is a complex number, the number obtained by changing the sign of its imaginary part is called the **complex conjugate**, or simply **conjugate**, of $z$ and is denoted by the symbol $\bar{z}$. In other words, if $z = a + ib$, then its conjugate is $\bar{z} = a - ib$. From the definitions of addition and subtraction of complex numbers, it is readily shown that the conjugate of a sum and difference of two complex numbers is the sum and difference of the conjugates:

$$\overline{z_1 + z_2} = \bar{z}_1 + \bar{z}_2, \quad \overline{z_1 - z_2} = \bar{z}_1 - \bar{z}_2.$$

Moreover, we have the following three additional properties:

$$\overline{z_1 z_2} = \bar{z}_1 \bar{z}_2, \quad \overline{\left(\frac{z_1}{z_2}\right)} = \frac{\bar{z}_1}{\bar{z}_2}, \quad \bar{\bar{z}} = z.$$

Of course, the conjugate of any finite sum (product) of complex numbers is the sum (product) of the conjugates.

The definitions of addition and multiplication show that the sum and product of a complex number $z$ with its conjugate $\bar{z}$ is a real number:

$$z + \bar{z} = (a + ib) + (a - ib) = 2a$$

$$z\bar{z} = (a + ib)(a - ib) = a^2 - i^2 b^2 = a^2 + b^2$$

The difference of a complex number $z$ with its conjugate $\bar{z}$ is a pure imaginary number:

$$z - \bar{z} = (a + ib) - (a - ib) = 2ib$$

Since $a = \mathrm{Re}(z)$ and $b = \mathrm{Im}(z)$, above equations yield two useful formulas:

$$\mathrm{Re}(z) = \frac{z + \bar{z}}{2} \quad \text{and} \quad \mathrm{Im}(z) = \frac{z - \bar{z}}{2i}.$$

**Division**

To divide $z_1$ by $z_2$, multiply the numerator and denominator of $z_1/z_2$ by the conjugate of $z_2$. That is

$$\frac{z_1}{z_2} = \frac{z_1}{z_2} \cdot \frac{\bar{z}_2}{\bar{z}_2} = \frac{z_1 \bar{z}_2}{z_2 \bar{z}_2}$$

and then use the fact that $z_2 \bar{z}_2$ is the sum of the squares of the real and imaginary parts of $z_2$.

**Inverses**

In the complex number system, every number $z$ has a unique additive inverse. As in the real number system, the **additive inverse** of $z = a + ib$ is its *negative*, $-z$, where $-z = -a - ib$. For any complex number $z$, we have $z + (-z) = 0$. Similarly, every *nonzero* complex number $z$ has a **multiplicative inverse**. In symbols, for $z \neq 0$ there exists one and only one nonzero complex number $z^{-1}$ such that $zz^{-1} = 1$. The multiplicative inverse $z^{-1}$ is the same as the **reciprocal** $1/z$.

**Remarks** Comparison with Real Analysis

i. Many of the properties of the real number system $\mathbf{R}$ hold in the complex number system $\mathbf{C}$, but there are some truly remarkable differences as well. For example, the concept of *order* in the real number system does not carry over to the complex number system. In other words, we cannot compare two complex numbers $z_1 = a_1 + ib_1,\ b_1 \neq 0$, and $z_2 = a_2 + ib_2,\ b_2 \neq 0$, by means of inequalities.

ii. Some things that we take for granted as impossible in real analysis, such as $e^x = -2$ and $\sin x = 5$ when $x$ is a real variable, are perfectly correct and ordinary in complex analysis when the symbol $x$ is interpreted as a complex variable.

1.2 Complex Plane

A complex number $z = x + iy$ is uniquely determined by an *ordered pair* of real numbers $(x, y)$. The first and second entries of the ordered pairs correspond, in turn, with the real and imaginary parts of the complex number. In this manner we are able to associate a complex number $z = x + iy$ with a point $(x, y)$ in a coordinate plane.

**Complex Plane**

Because of the correspondence between a complex number $z = x + iy$ and one and only one point $(x, y)$ in a coordinate plane, we shall use the terms *complex number* and *point* interchangeably. The coordinate plane illustrated in Figure 1.1 is called the ***complex plane*** or simply the **$z$-plane**. The horizontal or $x$-axis is called the **real axis** because each point on that axis represents a real number. The vertical or $y$-axis is called the **imaginary axis** because a point on that axis represents a pure imaginary number.

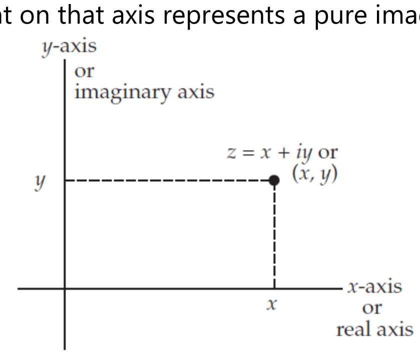



Figure 1. 1 $z$-plane

**Vectors**

A complex number $z = x + iy$ can also be viewed as a two-dimensional position **vector**, that is, a vector whose initial point is the origin and whose terminal point is the point $(x, y)$. See Figure 1.2. This vector interpretation prompts us to define the length of the vector $z$ as the distance $\sqrt{x^2 + y^2}$ from the origin to the point $(x, y)$.

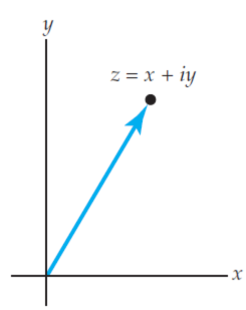



Figure 1. 2 $z$ as a position vector

> **Definition 1.3 Modulus**
> The modulus of a complex number $z = x + iy$, is the real number
> $$|z| = \sqrt{x^2 + y^2}.$$

The modulus $|z|$ of a complex number $z$ is also called the **absolute value** of $z$.

**Properties**

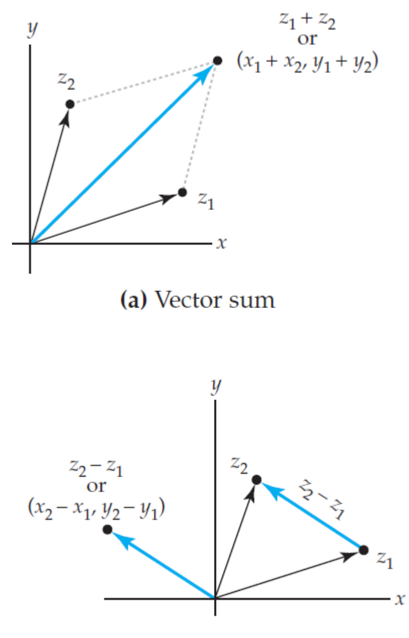
The relations

$$|z|^2 = z\bar{z} \quad \text{and} \quad |z| = \sqrt{z\bar{z}}$$

deserve to be stored in memory. The modulus of a complex number $z$ has the additional properties.

$$|z_1 z_2| = |z_1||z_2| \quad \text{and} \quad \left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}.$$

**Distance Again**




**(a)** Vector sum



**(b)** Vector difference

Figure 1. 3 Sum and difference of vectors

$$|z_2 - z_1| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}.$$

**EXAMPLE 2**: Set of Points in the Complex Plane (Page 11)

**Inequalities**

Since $|z|$ is a real number, we can compare the absolute values of two complex numbers.

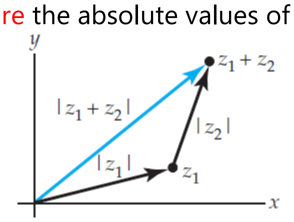



Figure 1. 4 Triangle with vector sides

$$|z_1 + z_2| \leq |z_1| + |z_2|.$$

The result above is known as the **triangle inequality**.

$$|z_1| - |z_2| \leq |z_1 + z_2|.$$
$$\big||z_1| - |z_2|\big| \leq |z_1 + z_2|.$$
$$|z_1 - z_2| \leq |z_1| + |z_2|.$$

$$||z_1| - |z_2|| \le |z_1 - z_2|.$$

**EXAMPLE 3**: An Upper Bound (Page 13)

1.3 Polar Form of Complex Numbers

Recall from calculus that a point $P$ in the plane whose rectangular coordinates are $(x, y)$ can also be described in terms of **polar coordinates**. The polar coordinate system, invented by Isaac Newton, consists of point $O$ called the **pole** and the horizontal half-line emanating from the pole called the **polar axis**. If $r$ is a directed distance from the pole to $P$ and $\theta$ is an angle of inclination (in radians) measured from the polar axis to the line $OP$, then the point can be described by the ordered pair $(r, \theta)$, called the polar coordinates of $P$. See Figure 1.5.

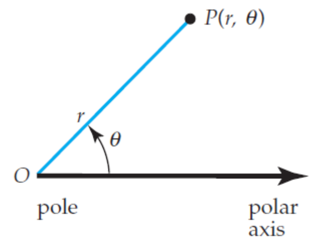



Figure 1. 5 Polar coordinates

**Polar Form**

Suppose, as shown in Figure 1.6, that a polar coordinate system is superimposed on the complex plane with the polar axis coinciding with the positive $x$-axis and the pole $O$ at the origin. Then $x, y, r$ and $\theta$ are related by $x = r\cos\theta$, $y = r\sin\theta$. These equations enable us to express a nonzero complex number $z = x + iy$ as $z = (r\cos\theta) + i(r\sin\theta)$ or

$$z = r(\cos\theta + i\sin\theta).$$

We say that the above equation is the **polar form** or **polar representation** of the complex number $z$. We shall adopt the convention that $r$ is *never negative* so that we can take $r$ to be the modulus of $z$, that is, $r = |z|$. The angle $\theta$ of inclination of the vector $z$, which *will always be measured in radians* from the positive real axis, is positive when measured counterclockwise and negative when measured clockwise. The angle $\theta$ is called an **argument** of $z$ and is denoted by $\theta = \arg(z)$. An argument $\theta$ of a complex number must satisfy the equations $\cos\theta = x/r$ and $\sin\theta = y/r$. In practice we use $\tan\theta = y/x$ to find $\theta$. However, because $\tan\theta$ is $\pi$-periodic, some care must be exercised in using the last equation. A calculator will give only angles satisfying $-\pi/2 < \tan^{-1}(y/x) < \pi/2$, that is, angles in the first and fourth quadrants. We have to choose $\theta$ consistent with the quadrant in which $z$ is located; this may require adding or subtracting $\pi$ to $\tan^{-1}(y/x)$ when appropriate.

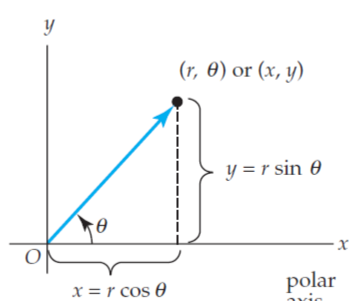



Figure 1. 6 Polar coordinates in the complex plane

Be careful using $\tan^{-1}(y/x)$.

**Principal Argument**

The symbol $\arg(z)$ actually represents a set of values, but the argument $\theta$ of a complex number that lies in the interval $-\pi < \theta \le \pi$ is called the **principal value** of $\arg(z)$ or the **principal argument** of $z$. The principal argument of $z$ is *unique* and is represented by the symbol $\mathrm{Arg}(z)$, that is,

$$-\pi < \text{Arg}(z) \leq \pi.$$

**Multiplication and Division**

The polar form of a complex number is especially convenient when multiplying or dividing two complex numbers.

Suppose

$$z_1 = r_1(\cos\theta_1 + i\sin\theta_1) \quad \text{and} \quad z_2 = r_2(\cos\theta_2 + i\sin\theta_2),$$
$$z_1z_2 = r_1r_2[\cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2)]$$
$$\frac{z_1}{z_2} = \frac{r_1}{r_2}[\cos(\theta_1-\theta_2) + i\sin(\theta_1-\theta_2)].$$

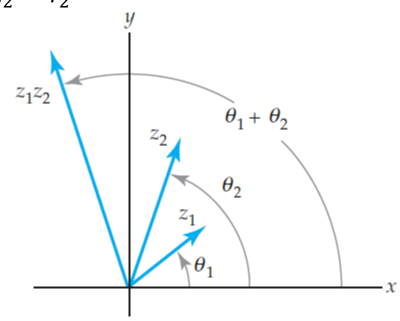



Figure 1. 7 $arg(z_1z_2) = \theta_1 + \theta_2$

$$\arg(z_1z_2) = \arg(z_1) + \arg(z_2) \quad \text{and} \quad \arg\left(\frac{z_1}{z_2}\right) = \arg(z_1) - \arg(z_2).$$

**Integer Powers of $z$**

$$z^n = r^n(\cos n\theta + i\sin n\theta).$$

**de Moivre's Formula**

When $z = (\cos\theta + i\sin\theta)$, we have $|z| = r = 1$, and so yields

$$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta.$$

**Remarks** Comparison with Real Analysis

i. Although $\arg(z_1z_2) = \arg(z_1) + \arg(z_2)$ and $\arg(z_1/z_2) = \arg(z_1) - \arg(z_2)$ is true for any arguments of $z_1$ and $z_2$, *it is not true*, in general that $\text{Arg}(z_1z_2) = \text{Arg}(z_1) + \text{Arg}(z_2)$ and $\text{Arg}(z_1/z_2) = \text{Arg}(z_1) - \text{Arg}(z_2)$.

ii. An argument can be assigned to any nonzero complex number $z$. However, for $z = 0$, $\arg(z)$ cannot be defined in any way that is meaningful.

iii. If we take $\arg(z)$ from the interval $(-\pi,\pi]$, the relationship between a complex number $z$ and its argument is single-valued; that is, every nonzero complex number has precisely one angle in $(-\pi,\pi]$. But there is nothing special about the interval $(-\pi,\pi]$; we also establish a single-valued relationship by using the interval $(0,2\pi]$ to define the principal value of the argument of $z$. For the interval $(-\pi,\pi]$, the negative real axis is analogous to a barrier that we agree not to cross; the technical name for this barrier is a **branch cut**.

iv. The "cosine $i$ sine" part of the polar form of a complex number is sometimes abbreviated cis. That is,

$$z = r(\cos\theta + i\sin\theta) = r\,\text{cis}\,\theta$$

This notation, used mainly in engineering.

1.4 Powers and Roots

Recall from algebra that $-2$ and $2$ are said to be *square roots* of the number $4$ because $(-2)^2 = 4$ and $(2)^2 = 4$. In other words, the two square roots of $4$ are distinct solutions of the equation $w^2 = 4$. In like manner we say $w = 3$ is a cube root of $27$ since $w^3 = 3^3 = 27$. This last equation points us again in the direction of complex variables since any real number has only *one real* cube root and *two complex* roots. In general, we say that a number $w$ is an $n$th root of a nonzero complex number $z$ if $w^n = z$, where $n$ is a positive integer. For example, you are urged to verify that $w_1 = \frac{1}{2}\sqrt{2} + \frac{1}{2}\sqrt{2}i$ and $w_2 = -\frac{1}{2}\sqrt{2} - \frac{1}{2}\sqrt{2}I$ are the two square roots of the complex number $z = i$ because $w_1^2 = i$ and $w_2^2 = i$.

**Roots**

Suppose $z = r(\cos\theta + i\sin\theta)$ and $w = \rho(\cos\phi + i\sin\phi)$ are polar forms of the complex numbers $z$ and $w$. Then the equation $w^n = z$ becomes

$$\rho^n(\cos n\phi + i\sin n\phi) = r(\cos\theta + i\sin\theta)$$

we define $\rho = \sqrt[n]{r}$ to be the unique **positive** $n$th root of the positive real number $r$. And the arguments $\theta$ and $\phi$ are related by $n\phi = \theta + 2k\pi$, where $k$ is an integer. Thus,

$$\phi = \frac{\theta + 2k\pi}{n}$$

As $k$ takes on the successive integer values $k = 0,1,2,\cdots,n-1$ we obtain *$n$ distinct* $n$th roots of $z$; these roots have the same modulus $\sqrt[n]{r}$ but different arguments. Notice that for $k \geq n$ we obtain the same roots because the sine and cosine are $2\pi$ periodic.

We summarize this result. The $n$ $n$th roots of a nonzero complex number $z = r(\cos\theta + i\sin\theta)$ are given by

$$w_k = \sqrt[n]{r}\left[\cos\left(\frac{\theta + 2k\pi}{n}\right) + i\sin\left(\frac{\theta + 2k\pi}{n}\right)\right],$$

where $k = 0,1,2,\cdots,n-1$.

**EXAMPLE 1**: Cube Roots of a Complex Number (Page 24)

**Principal $n$th Root**

We have pointed out that the symbol $\arg(z)$ really stands for a *set* of arguments for a complex number $z$. Stated another way, for a given complex number $z \neq 0$, $\arg(z)$ is *infinite-valued*. In like manner, $z^{1/n}$ is $n$-valued; that is, the **symbol** $z^{1/n}$ represents the **set** of $n$ $n$th roots $w_k$ of $z$. The *unique* root of a complex number $z$ (obtained by using the principal value of $\arg(z)$ with $k = 0$) is naturally referred to as the **principal $n$th root** of $w$. The choice of $\mathrm{Arg}(z)$ and $k = 0$ guarantees us that when $z$ is a positive real number $r$, the principal $n$th root is $\sqrt[n]{r}$.

Since the roots have the same modulus, the $n$ $n$th roots of a nonzero complex number $z$ lie on a circle of radius $\sqrt[n]{r}$ centered at the origin in the complex plane. Moreover, since the **difference** between the arguments of **any** two successive roots $w_k$ and $w_{k+1}$ is $2\pi/n$, the $n$ $n$th roots of $z$ are equally spaced on this circle, beginning with the root whose argument is $\theta/n$.

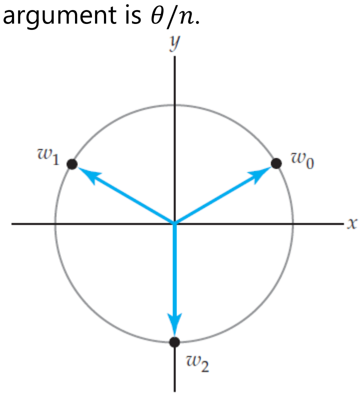



Figure 1. 8 Three cube roots of $i$

**EXAMPLE 2**: Fourth Roots of a Complex Number (Page 25)

**Remarks** Comparison with Real Analysis

i. As a consequence of $w_k = \sqrt[n]{r}\left[\cos\left(\frac{\theta+2k\pi}{n}\right) + i\sin\left(\frac{\theta+2k\pi}{n}\right)\right]$, we can say that the complex number system is *closed* under the operation of extracting roots. This means that for any $z$ in $\mathbf{C}$, $z^{1/n}$ is also in $\mathbf{C}$. The real number system does not possess a similar closure property since, if $x$ is in $\mathbf{R}$, $x^{1/n}$ is not necessarily in $\mathbf{R}$.

ii. Geometrically, the $n$ $n$th roots of a complex number $z$ can also be interpreted as the vertices of a regular polygon with $n$ sides that is inscribed within a circle of radius $\sqrt[n]{r}$ centered at the origin.

iii. When $m$ and $n$ are positive integers with no common factors, then $w_k = \sqrt[n]{r}\left[\cos\left(\frac{\theta+2k\pi}{n}\right) + i\sin\left(\frac{\theta+2k\pi}{n}\right)\right]$ enables us to define a **rational power** of $z$, that is, $z^{m/n}$. It can be shown that the set of values $\left(z^{1/n}\right)^m$ is the same as the set of values $(z^m)^{1/n}$.

1.5 Sets of Points in the Complex Plane

**Circles**

Suppose $z_0 = x_0 + iy_0$. Since $|z - z_0| = \sqrt{(x - x_0)^2 + (y - y_0)^2}$ is the distance between the points $z = x + iy$ and $z_0 = x_0 + iy_0$, the points $z = x + iy$ that satisfy the equation

$$|z - z_0| = \rho, \quad \rho > 0,$$

lie on a **circle** of radius $\rho$ centered at the point $z_0$.

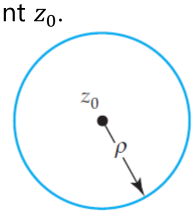



Figure 1. 9 Circle of radius $\rho$

**EXAMPLE 1**: Two Circles (Page 29)

**Disks and Neighborhoods**

The points $z$ that satisfy the inequality $|z - z_0| \le \rho$ can be either *on* the circle $|z - z_0| = \rho$ or *within* the circle. We say that the set of points defined by $|z - z_0| \le \rho$ is a **disk** of radius $\rho$ centered at $z_0$. But the points $z$ that satisfy the strict inequality $|z - z_0| < \rho$ lie within, and not on, a circle of radius $\rho$ centered at the point $z_0$. This set is called a **neighborhood** of $z_0$. Occasionally, we will need to use a neighborhood of $z_0$ that also *excludes* $z_0$. Such a neighborhood is defined by the simultaneous inequality $0 < |z - z_0| < \rho$ and is called a **deleted neighborhood** of $z_0$.

**Open Sets**

A point $z_0$ is said to be an **interior point** of a set $S$ of the complex plane if there exists some neighborhood of $z_0$ that lies **entirely** within $S$. If every point $z$ of a set $S$ is an interior point, then $S$ is said to be an **open set**. See Figure 1.10. For example, the inequality $\mathrm{Re}(z) > 1$ defines a *right half-plane*, which is an open set.

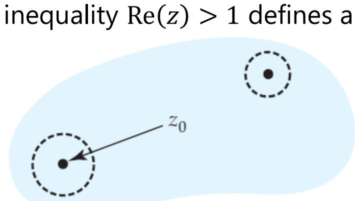



Figure 1. 10 Open set

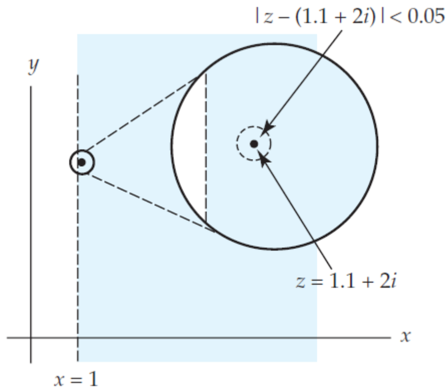


Figure 1. 11 Open set with magnified view of point near $x = 1$

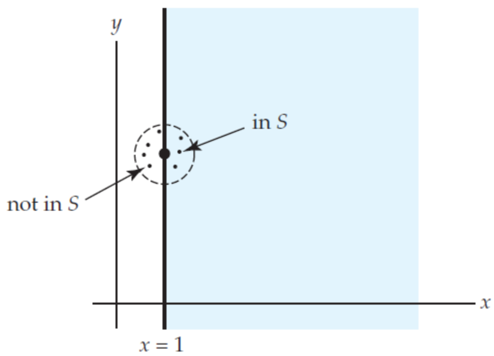


Figure 1. 12 Set $S$ not open

**EXAMPLE 2**: Some Open Sets (Page 30)
If *every* neighborhood of a point $z_0$ **of a set** $S$ contains at least one point of $S$ and at least one point not in $S$, then $z_0$ is said to be a **boundary point** of $S$. The **collection** of **boundary points** of a set $S$ is called the **boundary** of $S$. The circle $|z-i|=2$ is the boundary for both the disk $|z-i| \leq 2$ and the neighborhood $|z-i|<2$ of $z=i$. A point $z$ that is neither an interior point nor a boundary point of a set $S$ is said to be an **exterior point** of $S$; in other words, $z_0$ is an exterior point of a set $S$ if there exists some neighborhood of $z_0$ that contains no points of $S$.

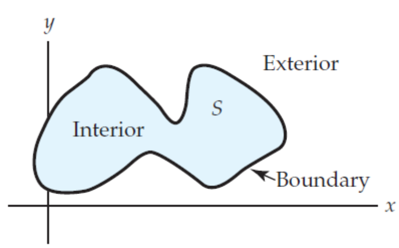


Figure 1. 13 Interior, boundary, and exterior of set $S$

**Annulus**
The set $S_1$ of points satisfying the inequality $\rho_1 < |z-z_0|$ lie exterior to the circle of radius $\rho_1$ centered at $z_0$, whereas the set $S_2$ of points satisfying $|z-z_0| < \rho_2$ lie interior to the circle of radius $\rho_2$ centered at $z_0$. Thus, if $0<\rho_1<\rho_2$, the set of points satisfying the simultaneous inequality

$$\rho_1 < |z-z_0| < \rho_2,$$

is the intersection of the sets $S_1$ and $S_2$. This intersection is an open circular ring centered at $z_0$. Figure 1.14 illustrates such a ring centered at the origin. The set defined by the above equation is called an open **circular annulus**.

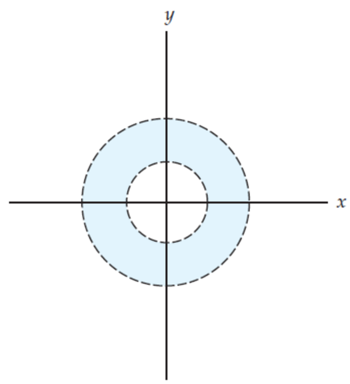



Figure 1. 14 $1 < |z| < 2$; interior of circular ring

**Domain**

If any pair of points $z_1$ and $z_2$ in a set $S$ can be connected by a polygonal line that consists of a finite number of line segments joined end to end that lies entirely in the set, then the set $S$ is said to be **connected**. See Figure 1.15. An open connected set is called a **domain**. A neighborhood of a point $z_0$ is a connected set.

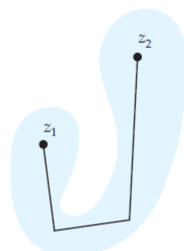



Figure 1. 15 Connected set

**Regions**

A **region** is a set of points in the complex plane with all, some, or none of its boundary points. Since an open set does not contain any boundary points, it is automatically a region. A region that contains all its boundary points is said to be **closed**. ("Closed" does not mean "not open.") The disk defined by $|z - z_0| \leq \rho$ is an example of a closed region and is referred to as a **closed disk**. A neighborhood of a point $z_0$ defined by $|z - z_0| < \rho$ is an open set or an open region and is said to be an **open disk**. If the center $z_0$ is deleted from either a closed disk or an open disk, the regions defined by $0 < |z - z_0| \leq \rho$ or $0 < |z - z_0| < \rho$ are called **punctured disks**. A *punctured open disk* is the *same* as a deleted *neighborhood* of $z_0$. A region can be neither open nor closed; the annular region defined by the inequality $1 \leq |z - 5| < 3$ contains only some of its boundary points (the points lying on the circle $|z - 5| = 1$), and so it is ***neither** open **nor** closed*. In a more general interpretation, an annulus or **annular region** may have the appearance shown in Figure 1.16.

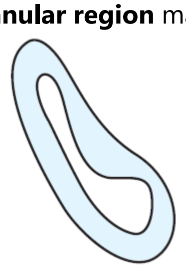


Figure 1. 16 Annular region

**Bounded Sets**

Finally, we say that a set $S$ in the complex plane is **bounded** if there exists a real number $R > 0$ such that $|z| < R$ for every $z$ in $S$. That is, $S$ is bounded if it can be completely enclosed within *some* neighborhood of the origin.

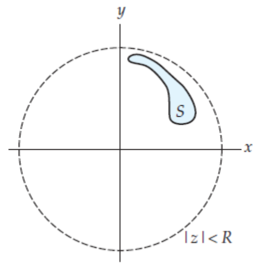



Figure 1. 17 The set $S$ is bounded since some neighborhood of the origin encloses $S$ entirely.

| |
|---|
| **Remarks** Comparison with Real Analysis<br>the **point at infinity**; **Riemann sphere**. |

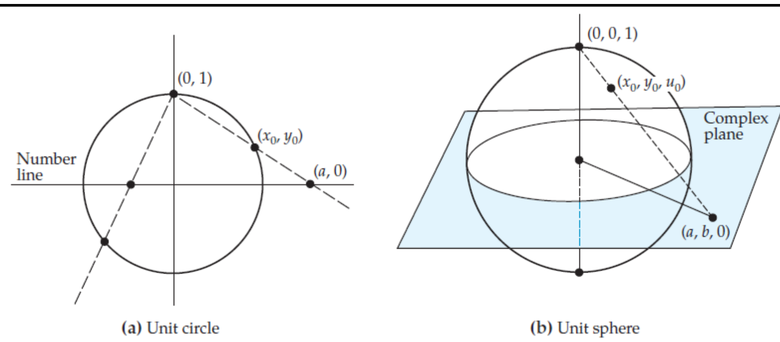



Figure 1. 18 The method of correspondence in (b) is a stereographic projection.

1.6 Applications

**Algebra**

You probably encountered complex numbers for the first time in a beginning course in algebra where you learned that roots of polynomial equations can be complex as well as real. For example, any second degree, or quadratic, polynomial equation can be solved by completing the square. In the general case, $ax^2 + bx + c = 0$, where the coefficients $a \neq 0$, $b$, and $c$ are real, completion of the square in $x$ yields the quadratic formula:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

When the discriminant $b^2 - 4ac$ is negative, the roots of the equation are complex. For example, by the above equation the two roots of $x^2 - 2x + 10 = 0$ are

$$x = \frac{-(-2) \pm \sqrt{(-2)^2 - 4(1)(10)}}{2(1)} = \frac{2 \pm \sqrt{-36}}{2}.$$

In beginning courses the imaginary unit $i$ is written $i = \sqrt{-1}$ and the assumption is made that the laws of exponents hold so that a number such as $\sqrt{-36}$ can be written $\sqrt{-36} = \sqrt{36}\sqrt{-1} = 6i$. Let us denote the two complex roots of the above equation as $z_1 = 1 + 3i$ and $z_2 = 1 - 3i$. (Note: The roots $z_1$ and $z_2$ are conjugates.)

**Quadratic Formula**

The quadratic formula is perfectly valid when the coefficients $a \neq 0$, $b$, and $c$ of a quadratic polynomial equation $ax^2 + bx + c = 0$ are complex numbers. Although the formula can be obtained in exactly the same manner as above, we choose to write the result as

$$z = \frac{-b + (b^2 - 4ac)^{1/2}}{2a}.$$

Notice that the numerator of the right-hand side of the above equation looks a little different than the traditional $-b \pm \sqrt{b^2-4ac}$. Bear in mind that when $b^2-4ac \neq 0$, the symbol $(b^2-4ac)^{1/2}$ represents the set of two square roots of the complex number $b^2-4ac$.

**EXAMPLE 1**: Using the Quadratic Formula (Note: The roots $z_1$ and $z_2$ are not conjugates in here.) (Page 37)

**Factoring a Quadratic Polynomial**

By finding all the roots of a polynomial equation we can factor the polynomial completely.

For the present, note that if $z_1$ and $z_2$ are the roots defined by $z = \frac{-b+(b^2-4ac)^{1/2}}{2a}$, then a quadratic polynomial $ax^2+bx+c=0$ factors as

$$ax^2+bx+c = a(z-z_1)(z-z_2).$$

**Differential Equations**

The first step in solving a linear second-order ordinary differential equation $ay''+by'+cy=f(x)$ with real coefficients $a$, $b$, and $c$ is to solve the associated homogeneous equation $ay''+by'+cy=0$. The latter equation possesses solutions of the form $y=e^{mx}$. To see this, we substitute $y=e^{mx}$, $y'=me^{mx}$, $y''=m^2e^{mx}$ into $ay''+by'+cy=0$:

$$ay''+by'+cy = am^2e^{mx}+bme^{mx}+ce^{mx} = e^{mx}(am^2+bm+c) = 0$$

From $e^{mx}(am^2+bm+c)=0$, we see that $y=e^{mx}$ is a solution of the homogeneous equation whenever $m$ is root of the polynomial equation $am^2+bm+c=0$. The latter quadratic equation is known as the **auxiliary equation**. Now when the coefficients of a polynomial equation are *real*, the equation cannot have just *one* complex root; that is, complex roots must always appear in conjugate pairs. Thus, if the auxiliary equation possesses complex roots $\alpha+i\beta$, $\alpha-i\beta$, $\beta>0$, then two solutions of $ay''+by'+cy=0$ are complex exponential functions $y=e^{(\alpha+i\beta)x}$ and $y=e^{(\alpha-i\beta)x}$. In order to obtain real solutions of the differential equation we use **Euler's formula**

$$e^{i\theta} = \cos\theta + i\sin\theta,$$

where $\theta$ is real. With $\theta$ replaced, in turn, by $\beta$ and $-\beta$, we can get

$$e^{(\alpha+i\beta)x} = e^{\alpha x}e^{i\beta x} = e^{\alpha x}(\cos\beta x + i\sin\beta x)$$
$$e^{(\alpha-i\beta)x} = e^{\alpha x}e^{-i\beta x} = e^{\alpha x}(\cos\beta x - i\sin\beta x).$$

**Now since** the differential equation is homogeneous, the linear combinations

$$y_1 = \frac{1}{2}\left[e^{(\alpha+i\beta)x}+e^{(\alpha-i\beta)x}\right] \quad \text{and} \quad y_2 = \frac{1}{2i}\left[e^{(\alpha+i\beta)x}-e^{(\alpha-i\beta)x}\right]$$

are also solutions. Both of the foregoing expressions are real functions

$$y_1 = e^{\alpha x}\cos\beta x \quad \text{and} \quad y_2 = e^{\alpha x}\sin\beta x$$

The general solution of the second-order differential equation is $y_1 = c_1y_1+c_2y_2$, where $c_1$ and $c_2$ are arbitrary constants.

**Exponential Form of a Complex Number**

The complex exponential $e^z$ is the complex number ***defined*** by

$$e^z = e^{x+iy} = e^x(\cos y + i\sin y)$$

The polar form of a complex number $z$, $z=r(\cos\theta+i\sin\theta)$, can now be written compactly as

$$z = re^{i\theta}.$$

This convenient form is called the **exponential form** of a complex number $z$. For example, $i = 1\left(\cos\frac{\pi}{2}+i\sin\frac{\pi}{2}\right) = e^{\pi i/2}$ and $1+i = \sqrt{2}\left(\cos\frac{\pi}{4}+i\sin\frac{\pi}{4}\right) = \sqrt{2}e^{\pi i/4}$. Also, the formula for the $n$ $n$th roots of a complex number, becomes

$$z^{1/n} = \sqrt[n]{r}e^{i(\theta+2k\pi)/n}, k=0,1,2,\cdots,n-1.$$

## CHAPTER 2 Complex Functions and Mappings

2.1 Complex Functions

In elementary calculus we studied functions whose inputs and outputs were real numbers. Such functions are called **real-valued functions of a real variable**. In this section we begin our study of functions whose inputs and outputs are complex numbers. Naturally, we call these functions **complex functions of a complex variable,** or **complex functions** for short.

**Function**

Suppose that $f$ is a function from the set $A$ to the set $B$. If $f$ assigns to the element $a$ in $A$ the element $b$ in $B$, then we say that $b$ is the **image** of $a$ under $f$, or the value of $f$ at $a$, and we write $b = f(a)$. The set $A$—the set of inputs—is called the **domain** of $f$ and the set of images in $B$—the set of outputs—is called the **range** of $f$. We denote the domain and range of a function $f$ by $\mathrm{Dom}(f)$ and $\mathrm{Range}(f)$, respectively.

**Definition 2.1 Complex Function**
A **complex function** is a function $f$ whose domain and range are subsets of the set $C$ of complex numbers.

Inputs to a complex function $f$ will typically be denoted by the variable $z$ and outputs by the variable $w = f(z)$.

**Real and Imaginary Parts of a Complex Function**

It is often helpful to express the inputs and the outputs a complex function in terms of their real and imaginary **parts**.

By setting $z = x + iy$, we can express any complex function $w = f(z)$ in terms of **two** real functions as:

$$f(z) = u(x, y) + iv(x, y).$$

The functions $u(x, y)$ and $v(x, y)$ in above are called the **real** and **imaginary** **parts** of $f$, respectively.

A complex function $w = f(z)$ can be defined by arbitrarily specifying two real functions $u(x, y)$ and $v(x, y)$, even though $w = u + iv$ may not be obtainable through familiar operations performed solely on the symbol $z$.

**Exponential Function**

**Definition 2.2 Complex Exponential Function**
The function $e^z$ defined by:

$$e^z = e^x \cos y + ie^x \sin y$$

is called the **complex exponential function**.

**Exponential Form of a Complex Number**

The exponential function **enables** us to express the polar form of a nonzero complex number $z = r(\cos\theta + i\sin\theta)$ in a **particularly convenient** and compact form:

$$z = re^{i\theta}.$$

We call it the **exponential form** of the complex number $z$.

The complex exponential function is periodic.

The complex exponential function has a pure imaginary period $2\pi i$.

**Polar Coordinates**

Given a complex function $w = f(z)$, if we replace the symbol $z$ with $r(\cos\theta + i\sin\theta)$, then we can write this function as:

$$f(z) = u(r, \theta) + iv(r, \theta).$$

We still call the real functions $u(r, \theta)$ and $v(r, \theta)$ the real and imaginary parts of $f$, respectively.

The functions $u(r, \theta)$ and $v(r, \theta)$ are not the same as the functions $u(x, y)$ and $v(x, y)$.

2.2 Complex Functions as Mappings

We use the alternative term **complex mapping** in place of "complex function" when considering the function as this correspondence between points in the $z$-plane and points in the $w$-plane.

**Mappings**

If $w = f(z)$ is a complex function, then both $z$ and $w$ lie in a complex plane. It follows that the set of all points $(z, f(z))$ lies in four-dimensional space (**two** dimensions from the input $z$ and **two** dimensions from the output $w$).

| We cannot draw the graph a complex function. |
|---|

The concept of a complex mapping provides an alternative way of giving a geometric representation of a complex function. We use the term **complex mapping** to refer to the correspondence determined by a complex function $w = f(z)$ between points in a $z$-plane and images in a $w$-plane. If the point $z_0$ in the $z$-plane corresponds to the point $w_0$ in the $w$-plane, that is, if $w_0 = f(z_0)$, then we say that $f$ **maps** $z_0$ onto $w_0$ or, equivalently, that $z_0$ is **mapped** onto $w_0$ by $f$.

> **Notation: $S'$**
> If $w = f(z)$ is a complex mapping and if $S$ is a set of points in the $z$-plane, then we call the set of images of the points in $S$ under $f$ **the image of $S$ under $f$**, and we denote this set by the symbol $S'$.

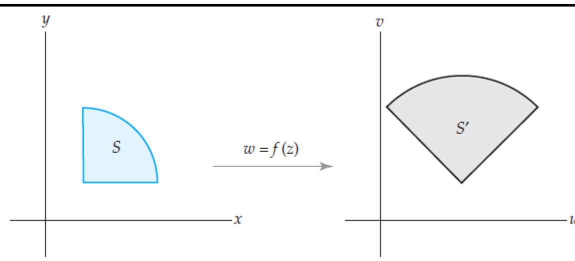



Figure 2. 1 The image of a set $S$ under a mapping $w = f(z)$

**EXAMPLE 1**: Image of a Half-Plane under $w = iz$

**EXAMPLE 2**: Image of a Line under $w = z^2$

**Parametric Curves in the Complex Plane**

> **Definition 2.3 Parametric Curves in the Complex Plane**
> If $x(t)$ and $y(t)$ are real-valued functions of a real variable $t$, then the set $C$ consisting of all points $z(t) = x(t) + iy(t)$, $a \le t \le b$, is called a **parametric curve** or a **complex parametric curve**. The complex valued function of the real variable $t$, $z(t) = x(t) + iy(t)$, is called a **parametrization** of $C$.

If $z$ is on the line containing $z_0$ and $z_1$, then there is a real number $t$ such that $z - z_0 = t(z_1 - z_0)$.

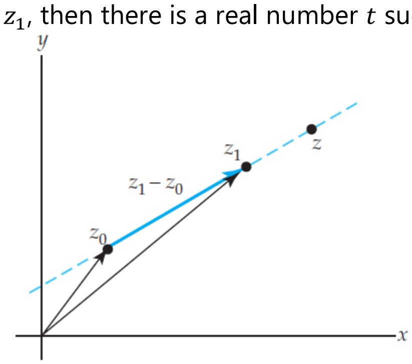



Figure 2. 2 Parametrization of a line

**Common Parametric Curves in the Complex Plane**

**Line**

A parametrization of the line containing the points $z_0$ and $z_1$ is:

$$z(t) = z_0(1 - t) + z_1 t, \quad -\infty < t < \infty.$$

**Line Segment**

A parametrization of the line segment from $z_0$ to $z_1$ is:

$$z(t) = z_0(1 - t) + z_1 t, \quad 0 \le t \le 1.$$

**Ray**

A parametrization of the ray emanating from $z_0$ and containing $z_1$ is:

$$z(t) = z_0(1 - t) + z_1 t, \quad 0 \le t < \infty.$$

**Circle**

A parametrization of the circle centered at $z_0$ with radius $r$ is:

$$z(t) = z_0 + r(\cos t + i \sin t), \quad 0 \le t \le 2\pi.$$

In exponential notation, this parametrization is:

$$z(t) = z_0 + re^{it}, \quad 0 \le t \le 2\pi.$$

**Image of a Parametric Curve under a Complex Mapping**

If $w = f(z)$ is a complex mapping **and** if $C$ is a **curve** parametrized by $z(t)$, $a \le t \le b$, then

$$w(t) = f\big(z(t)\big), \quad a \le t \le b$$

is a parametrization of the image, $C'$ of $C$ under $w = f(z)$.

In some instances, it is convenient to represent a complex mapping using a single copy of the complex plane.

**EXAMPLE 3**: Image of a Parametric Curve

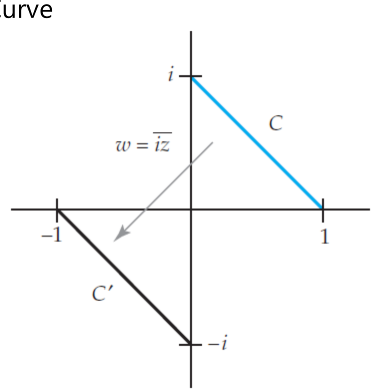



Figure 2. 3 The mapping $w = \overline{iz}$

**EXAMPLE 4**: Image of a Parametric Curve

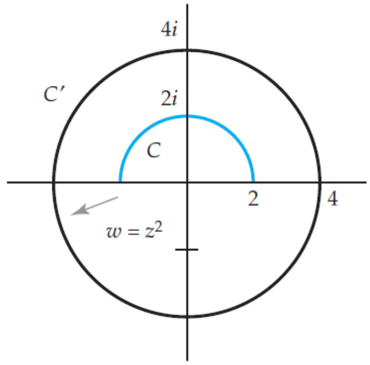



Figure 2. 4 The mapping $w = z^2$

2.3 Linear Mappings

We define a **complex linear function** to be a function of the form $f(z) = az + b$ where $a$ and $b$ are **any** complex constants. In this section, we will show that every nonconstant complex linear mapping can be described as a composition of three basic types of motions: a translation, a rotation, and a magnification. Throughout this section we use the symbols $T$, $R$, and $M$ to represent mapping by translation, rotation, and magnification, respectively.

**Translations**

A complex linear function

$$T(z) = z + b, \quad b \neq 0,$$

is called a **translation**. If we set $z = x + iy$ and $b = x_0 + iy_0$, then we obtain:

$$T(z) = (x + iy) + (x_0 + iy_0) = x + x_0 + i(y + y_0).$$

The mapping $T(z) = z + b$ is also called a ***translation*** *by* $b$.

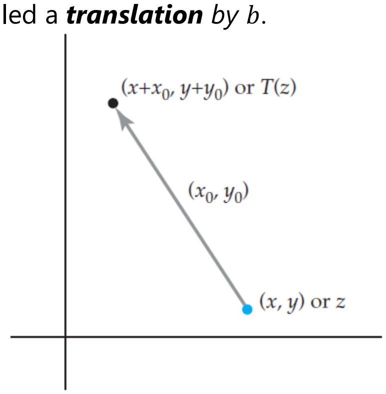



Figure 2. 5 Translation

**EXAMPLE 1**: Image of a Square under Translation

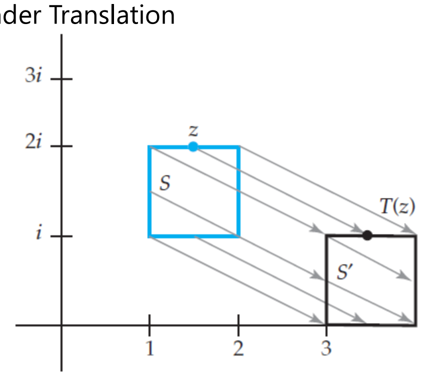



Figure 2. 6 Image of a square under translation

From our geometric description, it is clear that a translation does not change the **shape** or **size** of a figure in the complex plane. A mapping with this property is sometimes called a ***rigid motion***.

**Rotations**

A complex linear function

$$R(z) = az, \quad |a| = 1,$$

is called a **rotation**. For any nonzero complex number $\alpha$, we have that $R(z) = \frac{\alpha}{|\alpha|} z$ is a rotation.

We can write $a$ in exponential form as $a = e^{i\theta}$, then we obtain the following description of $R$:

$$R(z) = e^{i\theta} re^{i\phi} = re^{i(\theta+\phi)}.$$

The angle $\theta = \mathrm{Arg}(a)$ is called an **angle of rotation** of $R$.

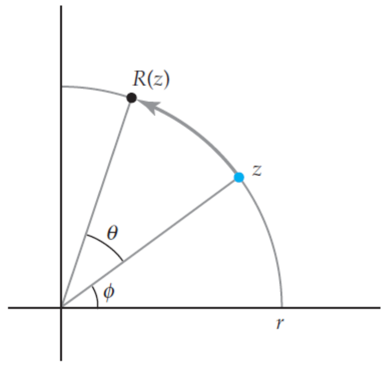



Figure 2. 7 Rotation

**EXAMPLE 2**: Image of a Line under Rotation

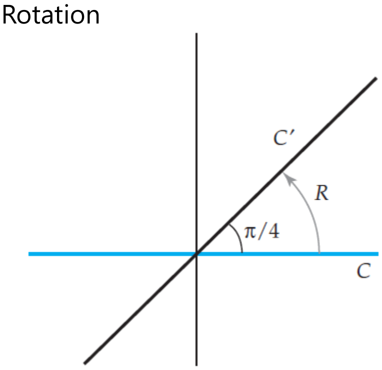



Figure 2. 8 Image of a line under rotation

**As** with translations, rotations will not change the shape or size of a figure in the complex plane.

**Magnifications**

The final type of special linear function we consider is magnification. A complex linear function

$$M(z) = az, \quad a > 0,$$

is called a **magnification**. Recall from the Remarks at the end of Section 1.1 that since there is no concept of order in the complex number system, it is implicit in the inequality $a > 0$ that the symbol $a$ represents a real number.

Using the exponential form $z = re^{i\theta}$ of $z$, we can also express the function above as:

$$M(z) = a\left(re^{i\theta}\right) = (ar)e^{i\theta}.$$

The real number $a$ is called the **magnification factor** of $M$.

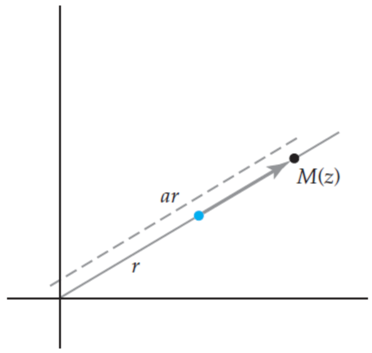



Figure 2. 9 Magnification

**EXAMPLE 3**: Image of a Circle under Magnification

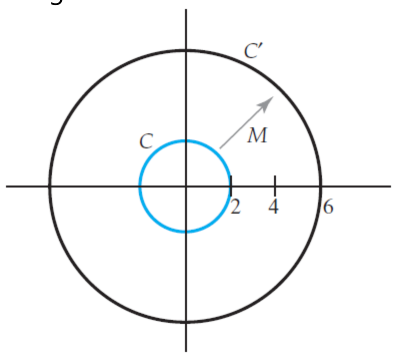



Figure 2. 10 Image of a circle under magnification

Although a magnification mapping will change the size of a figure in the complex plane, it will not change its basic shape.

**Linear Mappings**

Now suppose that $f(z) = az + b$ is a complex linear function. We assume that $a \neq 0$; otherwise, our mapping would be the **constant map** $f(z) = b$, which maps every point in the complex plane onto the single point $b$. Observe that we can express $f$ as:

$$f(z) = az + b = |a|\left(\frac{a}{|a|}z\right) + b.$$

**Image of a Point under a Linear Mapping**

Let $f(z) = az + b$ be a linear mapping with $a \neq 0$ and let $z_0$ be a point in the complex plane. If the point $w_0 = f(z_0)$ is plotted in the same copy of the complex plane as $z_0$, then $w_0$ is the point obtained by

i. rotating $z_0$ through an angle of $\mathrm{Arg}(a)$ about the origin,
ii. magnifying the result by $|a|$, and
iii. translating the result by $b$.

From above we see that **every** nonconstant complex linear mapping is a composition of *at most* one rotation, one magnification, and one translation. We emphasize the phrase “at most” in order to stress the fact that one or more of the maps involved maybe the **identity mapping** $f(z) = z$ (which maps every complex number onto itself).

A complex **linear** mapping $w = az + b$ with $a \neq 0$ can distort the **size** of $a$ figure in the complex plane, but it **cannot** alter the basic shape of the figure.

When describing a linear function as a composition of a rotation, a magnification, and a translation, keep in mind that the order of composition is important.

**EXAMPLE 4**: Image of a Rectangle under a Linear Mapping

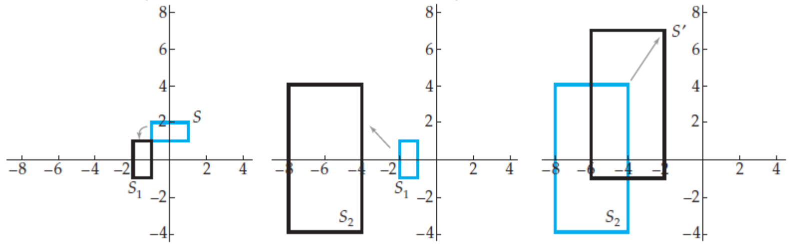


Figure 2. 11 Linear mapping of a rectangle

**EXAMPLE 5**: A Linear Mapping of a Triangle

2.4 Special Power Functions

A **complex polynomial function** is a function of the form $p(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0$ where $n$ is a positive integer and $a_n, a_{n-1}, \cdots, a_1, a_0$ are complex constants. In general, a complex polynomial mapping can be quite complicated, but in many special cases the action of the mapping is easily understood. For instance, the complex linear functions studied in Section 2.3 are complex polynomials of degree $n = 1$.

**Power Functions**

A **complex power function** is a function of the form $f(z) = z^\alpha$ where $\alpha$ is a complex constant. In this section we will restrict our attention to **special** complex power functions of the form $z^n$ and $z^{1/n}$ where $n \geq 2$ and $n$ is an integer. More complicated complex power functions such as $z^{\sqrt{2}-i}$ will be discussed in Section 4.2 following the introduction of the complex logarithmic function.

**The Power Function $z^n$**

**The Function $z^2$**

We begin by expressing this mapping in exponential notation by replacing the symbol $z$ with $re^{i\theta}$:

$$w = z^2 = \left(re^{i\theta}\right)^2 = r^2 e^{i2\theta}.$$

It is important to note that the magnification or contraction factor and the rotation angle associated to $w = f(z) = z^2$ depend on where the point $z$ is located in the complex plane.

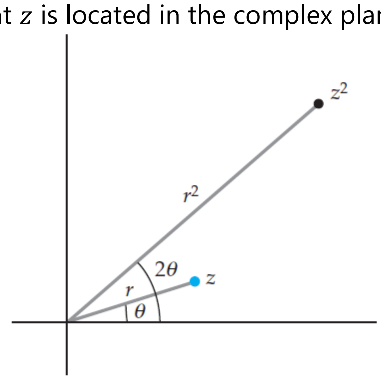



Figure 2. 12 The mapping $w = z^2$

**EXAMPLE 1**: Image of a Circular Arc under $w = z^2$

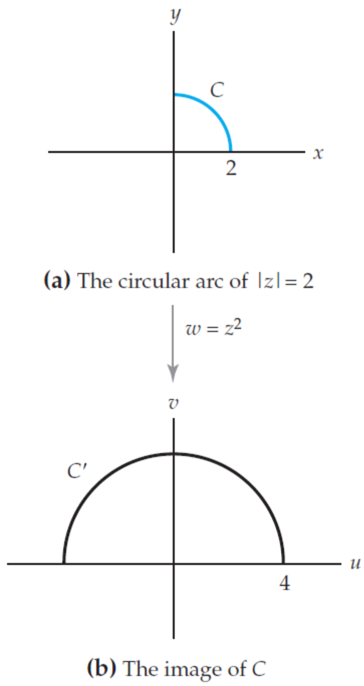


Figure 2. 13 The mapping $w = z^2$

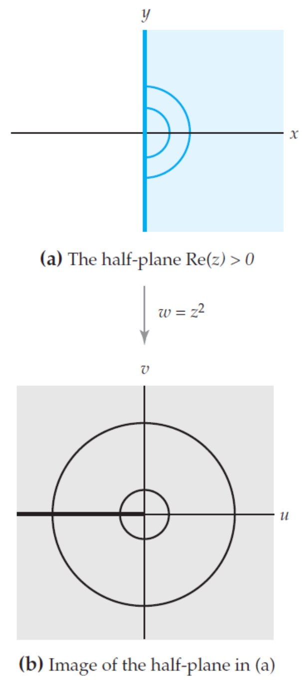


Figure 2. 14 The mapping $w = z^2$

**EXAMPLE 2**: Image of a Vertical Line under $w = z^2$

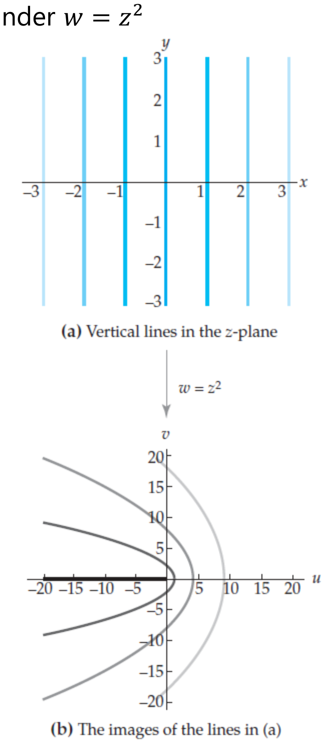
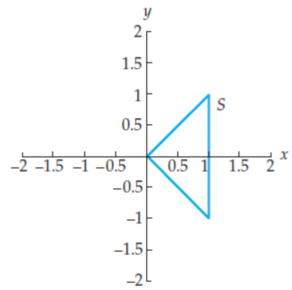
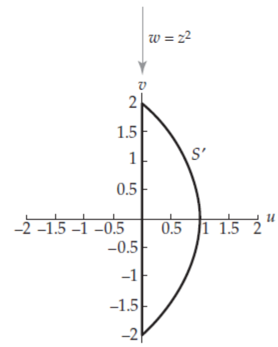


Figure 2. 15 The mapping $w = z^2$

**EXAMPLE 3**: Image of a Triangle under $w = z^2$



**(a)** A triangle in the $z$-plane



**(b)** The image of the triangle in (a)

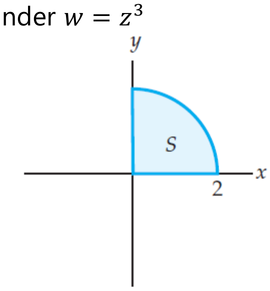
Figure 2. 16 The mapping $w = z^2$

**The Function $z^n, n > 2$**

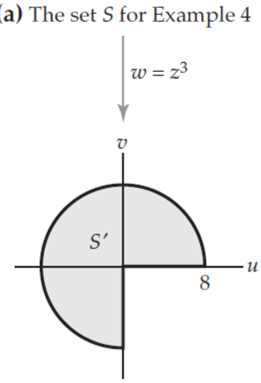
An analysis similar to that used for the mapping $w = z^2$ can be applied to the mapping $w = z^n$, $n > 2$. By replacing the symbol $z$ with $re^{i\theta}$ we obtain:

$$w = z^n = r^n e^{in\theta}.$$

**EXAMPLE 4**: Image of a Circular Wedge under $w = z^3$



**(a)** The set $S$ for Example 4



**(b)** The image of $S$

Figure 2. 17 The mapping $w = z^3$

**The Power Function $z^{1/n}$**

**Principal Square Root Function $\mathbf{z^{1/2}}$**

For $n = 2$, we have that the two square roots of $z$ are:

$$\sqrt{r}\left[\cos\left(\frac{\theta + 2k\pi}{2}\right) + i\sin\left(\frac{\theta + 2k\pi}{2}\right)\right] = \sqrt{r}e^{i(\theta+2k\pi)/2}$$

for $k = 0,1$. The formula above does not define a function because it assigns *two* complex numbers (one for $k = 0$ and one for $k = 1$) to the complex number $z$. However, by setting $\theta = \operatorname{Arg}(z)$ and $k = 0$ we can define a function that assigns to $z$ the unique principal square root. Naturally, this function is called the **principal square root function**.

**Definition 2.4 Principal Square Root Function**

The function $z^{1/2}$ defined by:

$$z^{1/2} = \sqrt{|z|}e^{i\operatorname{Arg}(z)/2}$$

is called the **principal square root function**.

If we set $\theta = \operatorname{Arg}(z)$ and replace $z$ with $re^{i\theta}$ above, then we obtain an alternative description of the principal square root function for $|z| > 0$:

$$z^{1/2} = \sqrt{r}e^{i\theta/2}, \quad \theta = \operatorname{Arg}(z).$$

Here we use $z^{1/2}$ to represent the *value* of the principal square root of the complex number $z$, whereas in Section 1.4 the symbol $z^{1/2}$ was used to represent the *set* of two square roots of the complex number $z$.

**Inverse Functions**

**Definition 2.5 Inverse Function**

If $f$ is a one-to-one complex function with domain $A$ and range $B$, then the **inverse function of** $f$, denoted by $f^{-1}$, is the function with domain $B$ and range $A$ defined $f^{-1}(z) = w$ if $f(w) = z$.

**EXAMPLE 6**: Inverse Function of $f(z) = z + 3i$

Solve the equation $z = f(w)$ for $w$ to find a formula for $w = f^{-1}(z)$.

**Inverse Functions of $\mathbf{z^n}$, $\mathbf{n > 2}$**

**EXAMPLE 7**: A Restricted Domain for $f(z) = z^2$

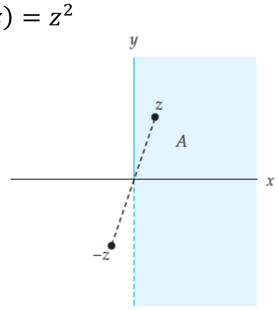



Figure 2. 18 A domain on which $f(z) = z^2$ is one-to-one

**An Inverse of $\mathbf{f(z) = z^2}$**

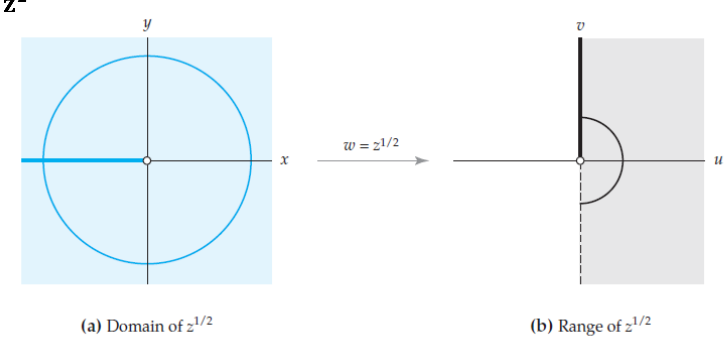



Figure 2. 19 The principal square root function $w = z^{1/2}$

**The Mapping $\mathbf{w = z^{1/2}}$**

As a mapping, the function $\mathbf{z^2}$ squares the modulus of a point and doubles its argument. Because the principal square root function $\mathrm{z}^{1/2}$ is an inverse function of $\mathrm{z}^2$, it follows that the mapping $\mathrm{w} = \mathrm{z}^{1/2}$ takes the square root of the modulus of a point and halves its principal argument. That is, if $\mathrm{w} = \mathrm{z}^{1/2}$, then we have $|w| = \sqrt{|z|}$ and $\mathrm{Arg}(w) = \frac{1}{2}\mathrm{Arg}(z)$.

**EXAMPLE 8**: Image of a Circular Sector under $\mathbf{w = z^{1/2}}$

**Principal $n$th Root Function**

We can show that the complex power function $f(z) = z^n$, $n > 2$, is one-to-one on the set defined by

$$-\frac{\pi}{n} < \arg(z) \leq \frac{\pi}{n}.$$

Analogous to the case $n = 2$, this inverse function of $z^n$ is called the **principal $n$th root function** $z^{1/n}$. The **domain** of $z^{1/n}$ is the set of all nonzero complex numbers, and the **range** of $z^{1/n}$ is the set of complex numbers $w$ satisfying $-\pi/n < \arg(w) \leq \pi/n$.

> **Definition 2.6 Principal $n$th Root Functions**
> For $n \geq 2$, the function $z^{1/n}$ defined by:
> $$z^{1/n} = \sqrt[n]{|z|}e^{i\,\mathrm{Arg}(z)/n}$$
> is called the **principal $n$th root function**.

By setting $z = re^{i\theta}$ with $\theta = \mathrm{Arg}(z)$ we can also express the principal $n$th root function as:

$$z^{1/n} = \sqrt[n]{r}e^{i\theta/n}, \quad \theta = \mathrm{Arg}(z).$$

**Multiple-Valued Functions**

> **Notation: Multiple-Valued Functions**
> When representing multiple-valued functions with functional notation, we will use uppercase letters such as $F(z) = z^{1/2}$ or $G(z) = \arg(z)$. Lowercase letters such as $f$ and $g$ will be reserved to represent functions.

**Remarks**

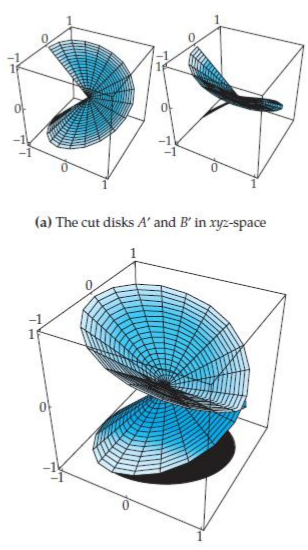



Figure 2. 20 A Riemann surface for $f(z) = z^2$

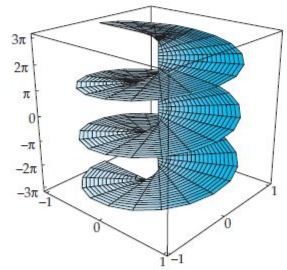


Figure 2. 21 The Riemann surface for $G(z) = arg(z)$

2.5 Reciprocal Function
Analogous to real functions, we define a **complex rational function** to be a function of the form $f(z) = p(z)/q(z)$ where both $p(z)$ and $q(z)$ are complex **polynomial** functions.

**Reciprocal Function**

The function $1/z$, whose domain is the set of all nonzero complex numbers, is called the **reciprocal function**. Given $z \neq 0$, if we set $z = re^{i\theta}$, then we obtain:

$$w = \frac{1}{z} = \frac{1}{re^{i\theta}} = \frac{1}{r}e^{-i\theta}.$$

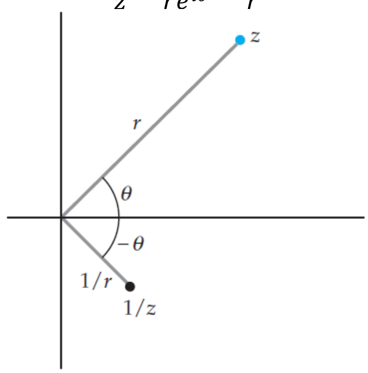


Figure 2. 22 The reciprocal mapping

As we shall see, a simple way to visualize the reciprocal function as a complex mapping is as a composition of *inversion* in the unit circle followed by *reflection* across the real axis.

**Inversion in the Unit Circle**

The function

$$g(z) = \frac{1}{r}e^{i\theta},$$

whose domain is the set of all nonzero complex numbers, is called **inversion in the unit circle**.

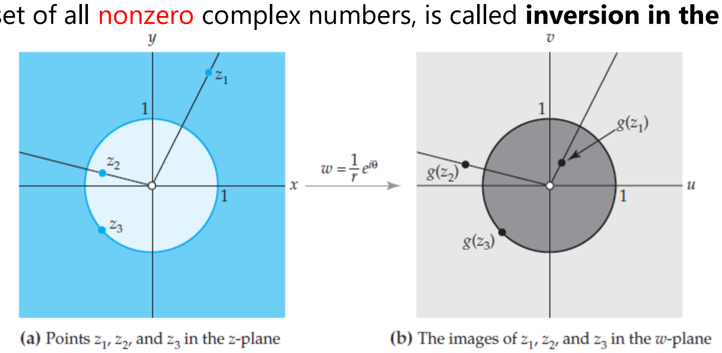


Figure 2. 23 Inversion in the unit circle

**Complex Conjugation**

The second complex mapping that is helpful for **describing** the reciprocal mapping is a reflection across the real axis. Under this mapping the image of the point $(x, y)$ is $(x, -y)$. It is easy to verify that this complex mapping is given by the function $c(z) = \bar{z}$, which we call the **complex conjugation function**. The complex conjugation function can be written as $c(z) = \bar{z} = re^{-i\theta}$.

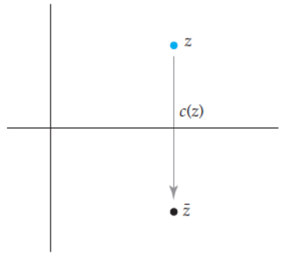



Figure 2. 24 Complex conjugation

**Reciprocal Mapping**

The reciprocal function $f(z) = \frac{1}{z}$ can be written as the composition of inversion in the unit circle and complex conjugation.

$$c(g(z)) = c\left(\frac{1}{r}e^{i\theta}\right) = \frac{1}{r}e^{-i\theta} = f(z) = 1/z.$$

This implies that, as a mapping, the reciprocal function first inverts in the unit circle, then reflects across the real axis.

**Image of a Point under the Reciprocal Mapping**

Let $z_0$ be a nonzero point in the complex plane. If the point $w_0 = f(z_0) = 1/z_0$ is plotted in the same copy of the complex plane as $z_0$, then $w_0$ is the point obtained by:

i. Inverting $z_0$ in the unit circle, then

ii. reflecting the result across the real axis.

**EXAMPLE 1**: Image of a Semicircle under $w = 1/z$

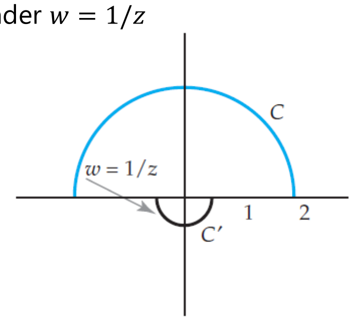



Figure 2. 25 The reciprocal mapping

**EXAMPLE 2**: Image of a Line under $w = 1/z$

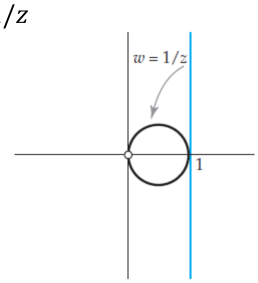


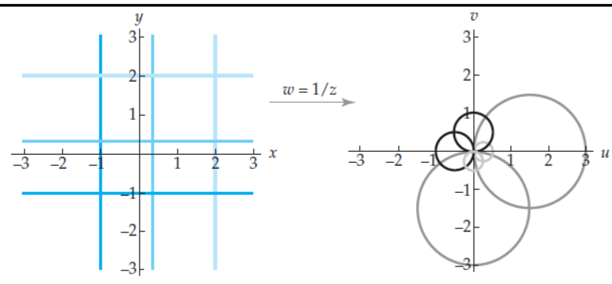

Figure 2. 26 The reciprocal mapping

In the Remarks in Section 1.5 we saw that the extended complex-number system consists of all the points in the complex plane adjoined with the ideal point ∞. In the context of mappings this set of points is commonly referred to as the **extended complex plane**.

**Definition 2.7 The Reciprocal Function on the Extended Complex Plane**
The reciprocal function on the extended complex plane is the function defined by:
$$f(z) = \begin{cases} 1/z, & \text{if } z \neq 0 \text{ or } \infty \\ \infty, & \text{if } z = 0 \\ 0, & \text{if } z = \infty. \end{cases}$$

**EXAMPLE 3**: Image of a Line under $w = 1/z$

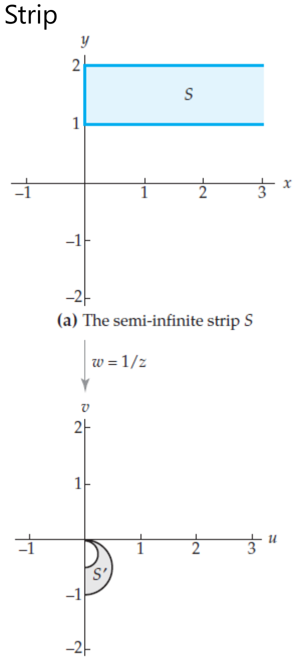
**Mapping Lines to Circles with $w = 1/z$**
The reciprocal function on the extended complex plane maps:
i. the vertical line $x = k$ with $k \neq 0$ onto the circle
$$\left|w - \frac{1}{2k}\right| = \left|\frac{1}{2k}\right|, \text{ and}$$
ii. the horizontal line $y = k$ with $k \neq 0$ onto the circle
$$\left|w + \frac{1}{2k}i\right| = \left|\frac{1}{2k}\right|.$$



**(a)** Vertical and horizontal lines **(b)** Images of the lines in (a)

Figure 2. 27 Images of vertical and horizontal lines under the reciprocal mapping

**EXAMPLE 4**: Mapping of a Semi-infinite Strip



**(a)** The semi-infinite strip $S$

**(b)** The image of $S$

Figure 2. 28 The reciprocal mapping

**Remarks**
It is easy to verify that the reciprocal function $f(z) = 1/z$ is one-to-one. Therefore, $f$ has a well-defined inverse function $f^{-1}$. We find a formula for the inverse function $f^{-1}(z)$ by solving the equation $z = f(w)$ for $w$. Clearly, this gives $f^{-1}(z) = 1/z$. This observation extends our understanding of the complex mapping $w = 1/z$. We can see that the circles $\left|w - \frac{1}{2k}\right| = \left|\frac{1}{2k}\right|$ and $\left|w + \frac{1}{2k}i\right| = \left|\frac{1}{2k}\right|$ are mapped onto the lines $x = k$ and $y = k$, respectively.

2.6 Applications
There are, however, other ways to visualize complex functions. In this section we will show that complex functions give complex representations of two-dimensional vector fields.

**Vector Fields**
In **multivariable** calculus, a **vector-valued** function of **two** real variables

$$\mathbf{F}(x, y) = \big(P(x, y), Q(x, y)\big)$$

is also called a **two-dimensional vector field**. Using the standard orthogonal unit basis vectors $\mathbf{i}$ and $\mathbf{j}$, we can also express the vector field above as:

$$\mathbf{F}(x, y) = P(x, y)\mathbf{i} + Q(x, y)\mathbf{j}.$$

For example, the function $\mathbf{F}(x, y) = (x + y)\mathbf{i} + (2xy)\mathbf{j}$ is a two-dimensional vector field, for which, say, $\mathbf{F}(1,3) = (1 + 3)\mathbf{i} + (2 \cdot 1 \cdot 3)\mathbf{j} = 4\mathbf{i} + 6\mathbf{j}$. Values of a function $\mathbf{F}$ are vectors that can be plotted as position vectors with initial point at the origin. However, in order to obtain a graphical representation of the vector field above that displays the relation between the input $(x, y)$ and the output $\mathbf{F}(x, y)$, we plot the vector $\mathbf{F}(x, y)$ with initial point $(x, y)$ and terminal point $\big(x + P(x, y), y + Q(x, y)\big)$. For example, as shown below,

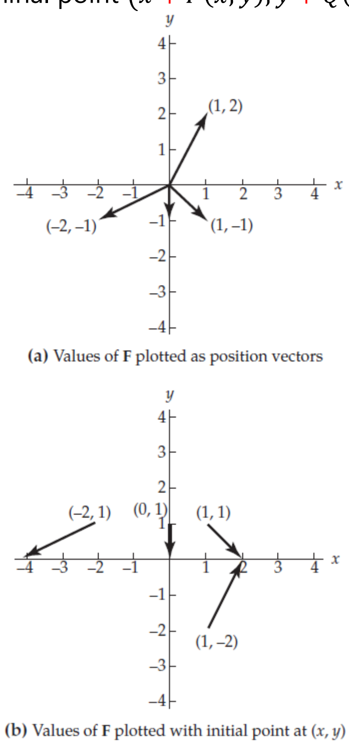



Figure 2. 29 Some vector values of the function $\boldsymbol{F}(x, y) = x\boldsymbol{i} - y\boldsymbol{j}$

**Complex Functions as Vector Fields**
There is a natural way to represent a vector field $\mathbf{F}(x, y) = P(x, y)\mathbf{i} + Q(x, y)\mathbf{j}$ with a complex function $f$. Namely, we use the functions $P$ and $Q$ as the real and imaginary parts of $f$, in which case, we say that the complex function $f(z) = P(x, y) + iQ(x, y)$ is the **complex representation** of the vector field $\mathbf{F}(x, y) =$

$P(x, y)\mathbf{i} + Q(x, y)\mathbf{j}$. Conversely, any complex function $f(z) = u(x, y) + iv(x, y)$ has an **associated** vector field $\mathbf{F}(x, y) = u(x, y)\mathbf{i} + v(x, y)\mathbf{j}$. From this point on we shall refer to both $\mathbf{F}(x, y) = P(x, y)\mathbf{i} + Q(x, y)\mathbf{j}$ and $f(x, y) = u(x, y) + iv(x, y)$ as **vector fields**.

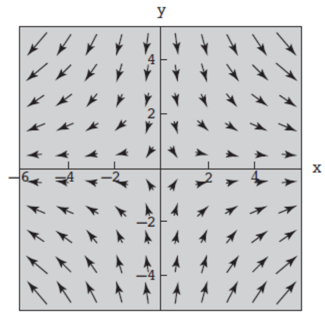


Figure 2. 30 The vector field $f(z) = \bar{z}$

When plotting a vector field $\mathbf{F}$ associated with a complex function $f$ it is helpful to note that plotting the vector $\mathbf{F}(x, y)$ with initial point $(x, y)$ is equivalent to plotting the vector representation of the complex number $f(z)$ with initial point $z$.

**EXAMPLE 1**: Plotting Vectors in a Vector Field

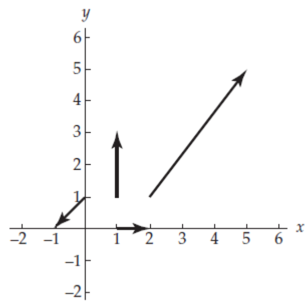


Figure 2. 31 Vectors in the vector field $f(z) = z^2$

**Use of Computers**

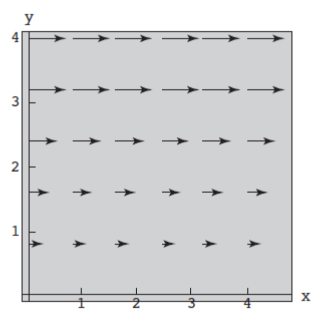


Figure 2. 32 Mathematica plot of the vector field $f(z) = iy$

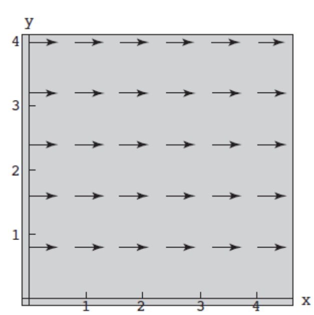


Figure 2. 33 Mathematica plot of the normalized vector field $f(z) = iy$

**Fluid Flow**

# CHAPTER 3 Analytic Functions

3.1 Limits and Continuity

In a real limit, there are two directions from which $x$ can approach $x_0$ on the real line, namely, from the left or from the right. In a complex limit, however, there are infinitely many directions from which $z$ can approach $z_0$ in the complex plane.

**Limits**

**Real Limits**

> **Limit of a Real Function $f(x)$**
> The limit of $f$ as $x$ tends $x_0$ exists and is equal to $L$ if for every $\varepsilon > 0$ there exists a $\delta > 0$ such that $|f(x) - L| < \varepsilon$ whenever $0 < |x - x_0| < \delta$.

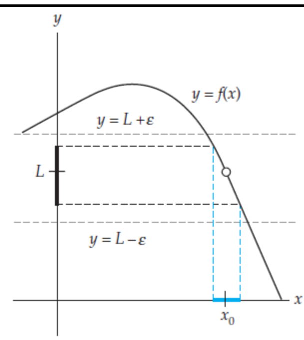



Figure 3. 1 Geometric meaning of a real limit

**Complex Limits**

> **Definition 2.8 Limit of a Complex Function**
> Suppose that a complex function $f$ is defined in a deleted neighborhood of $z_0$ and suppose that $L$ is a complex number. The **limit of $f$ as $z$ tends to $z_0$ exists and is equal to $L$**, written as $\lim_{z \to z_0} f(z) = L$, if for every $\varepsilon > 0$ there exists a $\delta > 0$ such that $|f(z) - L| < \varepsilon$ whenever $0 < |z - z_0| < \delta$.

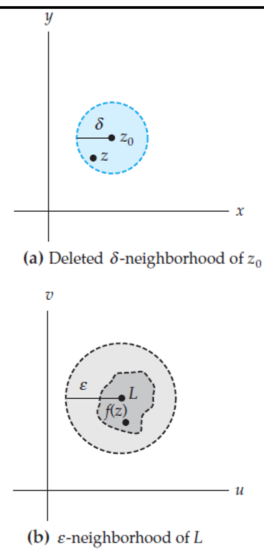



Figure 3. 2 The geometric meaning of a complex limit

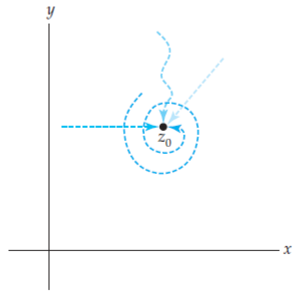



Figure 3. 3 Different ways to approach $z_0$ in a limit

**Criterion for the Nonexistence of a Limit**
If $f$ approaches two complex numbers $L_1 \neq L_2$ for two different curves or paths through $z_0$, then $\lim_{z \to z_0} f(z)$ does not exist.

**EXAMPLE 1**: A Limit That Does Not Exist
**EXAMPLE 2**: An Epsilon-Delta Proof of a Limit

**Real Multivariable Limits**

We now present a practical method for computing complex limits in Theorem 2.1.

**Limit of the Real Function $F(x, y)$**
The limit of $F$ as $(x, y)$ tends to $(x_0, y_0)$ exists and is equal to the real number $L$ if for every $\varepsilon > 0$ there exists a $\delta > 0$ such that $|F(x, y) - L| < \varepsilon$ whenever $0 < \sqrt{(x - x_0)^2 + (y - y_0)^2} < \delta$.

**Theorem 2.1 Real and Imaginary Parts of a Limit**
Suppose that $f(z) = u(x, y) + iv(x, y)$, $z_0 = x_0 + iy_0$, and $L = u_0 + iv_0$. Then $\lim_{z \to z_0} f(z) = L$ if and only if

$$\lim_{(x,y) \to (x_0,y_0)} u(x, y) = u_0 \quad \text{and} \quad \lim_{(x,y) \to (x_0,y_0)} v(x, y) = v_0.$$

Theorem 2.1 has many uses. First and foremost, it allows us to compute many complex limits by simply computing a pair of real limits. (need techniques)

**EXAMPLE 3**: Using Theorem 2.1 to Compute a Limit

**Theorem 2.2 Properties of Complex Limits**
Suppose that $f$ and $g$ are complex functions. If $\lim_{z \to z_0} f(z) = L$ and $\lim_{z \to z_0} g(z) = M$, then

i. $\lim_{z \to z_0} cf(z) = cL$, $c$ a complex constant,
ii. $\lim_{z \to z_0} \big(f(z) \pm g(z)\big) = L \pm M$,
iii. $\lim_{z \to z_0} f(z) \cdot g(z) = L \cdot M$, and
iv. $\lim_{z \to z_0} \frac{f(z)}{g(z)} = \frac{L}{M}$, provided $M \neq 0$.

**EXAMPLE 4**: Computing Limits with Theorem 2.2

**Continuity**

**Continuity of Real Functions**

**Continuity of a Real Function $f(x)$**
A function $f$ is continuous at a point $x_0$ if $\lim_{x \to x_0} f(x) = f(x_0)$.

**Continuity of Complex Functions**

| |
|---|
| **Definition 2.9 Continuity of a Complex Function**<br>A complex function $f$ is **continuous at a point** $z_0$ if<br>$$\lim_{z \to z_0} f(z) = f(z_0).$$ |

| |
|---|
| **Criteria for Continuity at a Point**<br>A complex function $f$ is continuous at a point $z_0$ if each of the following three conditions hold:<br>i. $\lim_{z \to z_0} f(z)$ exists,<br>ii. $f$ is defined at $z_0$, and<br>iii. $\lim_{z \to z_0} f(z) = f(z_0)$. |

If a complex function $f$ is not continuous at a point $z_0$ then we say that $f$ is **discontinuous** at $z_0$.

**EXAMPLE 5**: Checking Continuity at a Point

**EXAMPLE 6**: Discontinuity of Principal Square Root Function

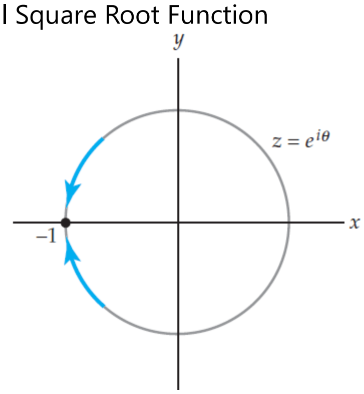



Figure 3. 4 Figure for Example 6

A complex function $f$ is **continuous on a set *S*** if $f$ is continuous at $z_0$ for each $z_0$ in $S$.

**Properties of Continuous Functions**

| |
|---|
| **Continuity of a Real Function $F(x, y)$**<br>A function $F$ is continuous at a point $(x_0, y_0)$ if<br>$$\lim_{(x,y) \to (x_0,y_0)} F(x, y) = F(x_0, y_0).$$ |

| |
|---|
| **Theorem 2.3 Real and Imaginary Parts of a Continuous Function**<br>If $f(z) = u(x, y) + iv(x, y)$ and $z_0 = x_0 + iy_0$, then the complex function $f$ is continuous at the point $z_0$ if and only if both real functions $u$ and $v$ are continuous at the point $(x_0, y_0)$. |

**EXAMPLE 7**: Checking Continuity Using Theorem 2.3

**Theorem 2.4 Properties of Continuous Functions**
If $f$ and $g$ are continuous at the point $z_0$, then the following functions are continuous at the point $z_0$:
i. $cf$, $c$ a complex constant,
ii. $f \pm g$,
iii. $f \cdot g$, and
iv. $\frac{f}{g}$ provided $g(z_0) \neq 0$.

**Theorem 2.5 Continuity of Polynomial Functions**
Polynomial functions are continuous on the **entire** complex plane $C$.

**Continuity of Rational Functions**
Rational functions are continuous on their domains.

**Bounded Functions**

**A Bounding Property**
If a complex function $f$ is continuous on a closed and bounded region $R$, then $f$ is bounded on $R$. That is, there is a real constant $M > 0$ such that $|f(z)| \leq M$ for all $z$ in $R$.

While this result assures us that a bound $M$ exists for $f$ on $R$, it offers no practical approach to find it.

**Branches**

In more rigorous terms, a **branch** of a multiple-valued function $F$ is a **function** $f_1$ that is continuous on some domain and that assigns exactly one of the multiple-values of $F$ to each point $z$ in that domain.

**Notation: Branches**
When representing branches of a multiple-valued function $F$ with functional notation, we will use lowercase letters with a numerical subscript such as $f_1$, $f_2$, and so on.

In order to obtain a branch of $F(z) = z^{1/2}$ that agrees with the principal square root function, we must restrict the domain to exclude points on the negative real axis. This gives the function

$$f_1(z) = \sqrt{r}e^{i\theta/2}, \quad -\pi < \theta < \pi.$$

**EXAMPLE 8**: A Branch of $F(z) = z^{1/2}$

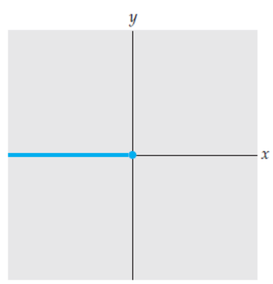
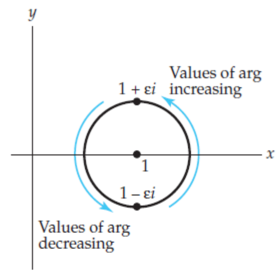



Figure 3. 5 The domain $D$ of the branch $f_1$

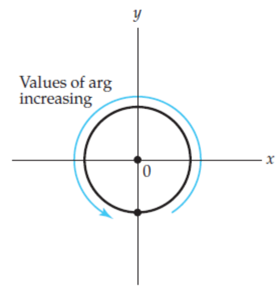
**Branch Cuts and Points**

In general, a **branch cut** for a branch $f_1$ of a multiple-valued function $F$ is a portion of a curve that is excluded from the domain of $F$ so that $f_1$ is continuous on the remaining points.

In general, a point with the property that it is on the branch cut of every branch is called a **branch point** of $F$. Alternatively, a branch point is a point $z_0$ with the following property: If we traverse any circle centered at $z_0$ with sufficiently small radius starting at a point $z_1$, then the values of any branch do not return to the value at $z_1$. (Page 127)




**(a)** $z = 1$ is not a branch point




**(b)** $z = 0$ is a branch point

Figure 3. 6 $G(z) = arg(z)$

> **Remarks** Comparison with Real Analysis
>
> $$\lim_{z\to\infty} f(z) = L \text{ if and only if } \lim_{z\to 0} f\left(\frac{1}{z}\right) = L.$$
>
> $$\lim_{z\to z_0} f(z) = \infty \text{ if and only if } \lim_{z\to z_0} \frac{1}{f(z)} = 0.$$

3.2 Differentiability and Analyticity

**The Derivative**

Suppose $z = x + iy$ and $z_0 = x_0 + iy_0$; then the change in $z_0$ is the difference $\Delta z = z - z_0$ or $\Delta z = x - x_0 + i(y - y_0) = \Delta x + i\Delta y$. If a complex function $w = f(z)$ is defined at $z$ and $z_0$, then the corresponding change in the function is the difference $\Delta w = f(z_0 + \Delta z) - f(z_0)$. The **derivative** of the function $f$ is defined in terms of a limit of the difference quotient $\Delta w/\Delta z$ as $\Delta z \to 0$.

> **Definition 3.1 Derivative of Complex Function**
>
> Suppose the complex function $f$ is defined in a neighborhood of a point $z_0$. The **derivative** of $f$ at $z_0$, denoted by $f'(z_0)$, is
>
> $$f'(z_0) = \lim_{\Delta z\to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z}$$
>
> provided this limit **exists**.

If the limit above exists, then the function $f$ is said to be **differentiable** at $z_0$. Two other symbols denoting the derivative of $w = f(z)$ are $w'$ and $dw/dz$. If the latter notation is used, then the value of a derivative at a specified point $z_0$ is written $\frac{dw}{dz}|_{z=z_0}$.

**EXAMPLE 1**: Using Definition 3.1

**Rules of Differentiation**

The familiar rules of differentiation in the calculus of real variables carry over to the calculus of complex variables.

**Differentiation Rules**

Constant Rules:

$$\frac{d}{dz}c = 0 \text{ and } \frac{d}{dz}cf(z) = cf'(z)$$

Sum Rule:

$$\frac{d}{dz}[f(z) \pm g(z)] = f'(z) \pm g'(z)$$

Product Rule:

$$\frac{d}{dz}[f(z)g(z)] = f'(z)g(z) + f(z)g'(z)$$

Quotient Rule:

$$\frac{d}{dz}\left[\frac{f(z)}{g(z)}\right] = \frac{f'(z)g(z) - f(z)g'(z)}{[g(z)]^2}$$

Chain Rule:

$$\frac{d}{dz}f\big(g(z)\big) = f'\big(g(z)\big)g'(z).$$

The **power rule** for differentiation of powers of $z$ is also valid:

$$\frac{d}{dz}z^n = nz^{n-1}, \quad n \text{ an integer.}$$

the **power rule for functions**:

$$\frac{d}{dz}[g(z)]^n = n[g(z)]^{n-1}g'(z), \quad n \text{ an integer.}$$

**EXAMPLE 2**: Using the Rules of Differentiation.

For a complex function $f$ to be differentiable at a point $z_0$, we know from the preceding chapter that the limit $\lim\limits_{\Delta z \to 0} \frac{f(z_0+\Delta z)-f(z_0)}{\Delta z}$ must exist and equal the same complex number from any direction; that is, the limit **must** exist regardless how $\Delta z$ approaches $0$.

If a complex function is made up by specifying its real and imaginary parts $u$ and $v$, such as $f(z) = x + 4iy$, there is a **good chance** that it is not differentiable.

**EXAMPLE 3**: A Function That Is Nowhere Differentiable.

The basic power rule does not apply to powers of the conjugate of $z$ because, **like** the function in Example 3, the function $f(z) = \bar{z}$ is nowhere differentiable.

**Analytic Functions**

Even though the requirement of differentiability is a stringent demand, there is a class of functions that is of great importance whose members satisfy even more severe requirements. These functions are called **analytic functions**.

**Definition 3.2 Analyticity at a Point**

A complex function $w = f(z)$ is said to be **analytic at a point** $z_0$ if $f$ is differentiable at $z_0$ and at every point in some neighborhood of $z_0$.

A function $\boldsymbol{f}$ **is analytic in a domain** $\boldsymbol{D}$ if it is analytic at every point in $D$. The phrase "analytic on a domain $D$" is also used. Although we shall not use these terms in this text, a function $f$ that is analytic throughout a domain $D$ is called **holomorphic** or **regular**.

Analyticity at a point is ***not*** the same as differentiability at a point. Analyticity at a *point* is a *neighborhood property*; in other words, analyticity is a property that is defined over an **open set**.

**Entire Functions**

A function that is analytic at every point $z$ in the **complex plane** is said to be an **entire function**.

**Theorem 3.1 Polynomial and Rational Functions**

i. A polynomial function $p(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0$, where $n$ is a nonnegative integer, is an entire function.

ii. A rational function $f(z) = \frac{p(z)}{q(z)}$, where $p$ and $q$ are polynomial functions, is analytic in any domain $D$ that contains **no** point $z_0$ for which $q(z_0) = 0$.

**Singular Points**

In general, a point $z$ at which a complex function $w = f(z)$ fails to be analytic is called a **singular point** of $f$.
If the functions $f$ and $g$ are analytic in a domain $D$, it can be proved that:

**Analyticity of Sum, Product, and Quotient**

If the functions $f$ and $g$ are analytic in a domain $D$, then the sum $f(z) + g(z)$, difference $f(z) - g(z)$, and product $f(z)g(z)$ are analytic in $D$. The quotient $f(z)/g(z)$ is analytic provided $g(z) \neq 0$ in $D$.

**An Alternative Definition of $\mathbf{f'(z)}$**

Sometimes it is convenient to define the derivative of a function $f$ using an alternative form of the difference quotient $\Delta w/\Delta z$. Since $\Delta z = z - z_0$, then $z = z_0 + \Delta z$, and so

$$f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}.$$

**Theorem 3.2 Differentiability Implies Continuity**

If $f$ is differentiable at a point $z_0$ in a domain $D$, then $f$ is continuous at $z_0$.

Of course the converse of Theorem 3.2 is not true; continuity of a function $f$ at a point does not guarantee that $f$ is differentiable at the point.
As another consequence of differentiability, L'Hˆopital's rule for computing limits of the indeterminate form $0/0$, carries over to complex analysis.

**Theorem 3.3 L'Hˆopital's Rule**

Suppose $f$ and $g$ are functions that are analytic at a point $z_0$ and $f(z_0) = 0$, $g(z_0) = 0$, but $g'(z_0) \neq 0$. Then

$$\lim_{z \to z_0} \frac{f(z)}{g(z)} = \frac{f'(z_0)}{g'(z_0)}.$$

**EXAMPLE 4**: Using L'Hˆopital's Rule
3.3 Cauchy-Riemann Equations

**A Necessary Condition for Analyticity**

In the next theorem we see that if a function $f(z) = u(x, y) + iv(x, y)$ is **differentiable** at a point $z$, then the functions $u$ and $v$ **must satisfy** a pair of equations that relate their first-order partial derivatives. (Page 152)

**Theorem 3.4 Cauchy-Riemann Equations**

Suppose $f(z) = u(x, y) + iv(x, y)$ is differentiable at a point $z = x + iy$. Then at $z$ the first-order **partial derivatives** of $u$ and $v$ exist and satisfy the **Cauchy-Riemann equations**

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

**Proof**:

$$f'(z) = \lim_{\Delta z \to 0} \frac{f(z + \Delta z) - f(z)}{\Delta z}.$$

Let $\Delta z \to 0$ along a horizontal line, then $\Delta y = 0$ and $\Delta z = \Delta x$. So

$$f'(z) = \lim_{\Delta x \to 0} \frac{u(x+\Delta x, y) - u(x,y) + i[v(x+\Delta x, y) - v(x,y)]}{\Delta x}$$
$$= \lim_{\Delta x \to 0} \frac{u(x+\Delta x, y) - u(x,y)}{\Delta x} + i \lim_{\Delta x \to 0} \frac{v(x+\Delta x, y) - v(x,y)}{\Delta x}.$$

Hence,

$$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}.$$

Now let $\Delta z \to 0$ along a vertical line. With $\Delta x = 0$ and $\Delta z = i\Delta y$,

$$f'(z) = \lim_{\Delta y \to 0} \frac{u(x, y+\Delta y) - u(x,y)}{i\Delta y} + i \lim_{\Delta y \to 0} \frac{v(x, y+\Delta y) - v(x,y)}{i\Delta y}.$$

Hence,

$$f'(z) = -i\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}.$$

By equating the real and imaginary parts of $f'(z)$ above, we obtain the pair of equations in Theorem 3.4. Because Theorem 3.4 states that the Cauchy-Riemann equations hold at $z$ as a ***necessary* consequence** of $f$ being differentiable at $z$, we cannot use the theorem to help us determine where $f$ is differentiable. But it is important to realize that Theorem 3.4 can **tell** us *where* a function $f$ does not possess a derivative. If the equations in Theorem 3.4 are not satisfied at a point $z$, then $f$ **cannot** be differentiable at $z$.

**EXAMPLE 1**: Verifying Theorem 3.4

> **Criterion for Non-analyticity**
> If the Cauchy-Riemann equations are **not satisfied** at every point $z$ in a domain $D$, then the function $f(z) = u(x,y) + iv(x,y)$ cannot be analytic in $D$.

**EXAMPLE 2**: Using the Cauchy-Riemann Equations

**A Sufficient Condition for Analyticity**

By themselves, the Cauchy-Riemann equations do not ensure analyticity of a function $f(z) = u(x,y) + iv(x,y)$ at a point $z = x + iy$.

> **Theorem 3.5 Criterion for Analyticity**
> Suppose the **real** functions $u(x,y)$ and $v(x,y)$ are continuous and have continuous first-order partial derivatives in a domain $D$. If $u$ and $v$ satisfy the Cauchy-Riemann equations at all points of $D$, then the complex function $f(z) = u(x,y) + iv(x,y)$ is **analytic** in $D$.

**EXAMPLE 3**: Using Theorem 3.5

**Recall** that analyticity **implies** differentiability but not conversely. Theorem 3.5 has an analogue that gives the following criterion for differentiability.

> **Sufficient Conditions for Differentiability**
> If the real functions $u(x,y)$ and $v(x,y)$ are continuous and have continuous first-order partial derivatives in some neighborhood of a point $z$, and if $u$ and $v$ satisfy the Cauchy-Riemann equations at $z$, then the complex function $f(z) = u(x,y) + iv(x,y)$ is **differentiable** at $z$ and $f'(z)$ is given by
> $$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y}$$

**EXAMPLE 4**: A Function Differentiable on a Line

> **Theorem 3.6 Constant Functions**
> Suppose the function $f(z) = u(x,y) + iv(x,y)$ is analytic in a domain $D$.
> i. If $|f(z)|$ is constant in $D$, then so is $f(z)$.
> ii. If $f'(z) = 0$ in $D$, then $f(z) = c$ in $D$, where $c$ is a constant.

**Polar Coordinates**

Indeed, the form $f(z) = u(r,\theta) + iv(r,\theta)$ is often more convenient to use. In polar coordinates the Cauchy-Riemann equations become

$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}, \quad \frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta}.$$

**Proof**:

Let $f(z) = u(x,y) + iv(x,y)$ and $x = r\cos\theta, y = r\sin\theta$. Using chain rule.

$$\frac{\partial u}{\partial r} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial r}, \quad \frac{\partial u}{\partial \theta} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial \theta}$$
$$\frac{\partial v}{\partial r} = \frac{\partial v}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial v}{\partial y}\frac{\partial y}{\partial r}, \quad \frac{\partial v}{\partial \theta} = \frac{\partial v}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial v}{\partial y}\frac{\partial y}{\partial \theta}$$

Since

$$\frac{\partial x}{\partial r} = \cos\theta, \quad \frac{\partial x}{\partial \theta} = -r\sin\theta$$
$$\frac{\partial y}{\partial r} = \sin\theta, \quad \frac{\partial y}{\partial \theta} = r\cos\theta$$

substitute in the obtained partial derivatives,

$$u_r = \frac{\partial u}{\partial r} = \frac{\partial u}{\partial x}\cos\theta + \frac{\partial u}{\partial y}\sin\theta$$
$$u_\theta = \frac{\partial u}{\partial \theta} = -\frac{\partial u}{\partial x} r\sin\theta + \frac{\partial u}{\partial y} r\cos\theta$$
$$v_r = \frac{\partial v}{\partial r} = \frac{\partial v}{\partial x}\cos\theta + \frac{\partial v}{\partial y}\sin\theta$$
$$v_\theta = \frac{\partial v}{\partial \theta} = -\frac{\partial v}{\partial x} r\sin\theta + \frac{\partial v}{\partial y} r\cos\theta$$

Adding $u_r + \frac{-1}{r} \times v_\theta$

$$u_r + \frac{-1}{r} \times v_\theta = \frac{\partial u}{\partial x}\cos\theta + \frac{\partial u}{\partial y}\sin\theta + \frac{\partial v}{\partial x}\sin\theta - \frac{\partial v}{\partial y}\cos\theta$$

because the Cauchy-Riemann equations in Cartesian form we know that

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}, \quad \text{substitute in above and it follows that}$$
$$u_r + \frac{-1}{r} \times v_\theta = 0 \quad \text{or} \quad \frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}$$

Adding $\frac{1}{r}u_\theta + v_r$

$$\frac{1}{r}u_\theta + v_r = -\frac{\partial u}{\partial x}\sin\theta + \frac{\partial u}{\partial y}\cos\theta + \frac{\partial v}{\partial x}\cos\theta + \frac{\partial v}{\partial y}\sin\theta$$

by the Cauchy-Riemann equations,

$$\frac{1}{r}u_\theta + v_r = 0 \quad \text{or} \quad \frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta}$$

The polar version of $f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y}$ at a point $z$ whose polar coordinates are $(r,\theta)$ is then

$$f'(z) = e^{-i\theta}\left(\frac{\partial u}{\partial r} + i\frac{\partial v}{\partial r}\right) = \frac{1}{r}e^{-i\theta}\left(\frac{\partial v}{\partial \theta} - i\frac{\partial u}{\partial \theta}\right).$$

**Proof**:

Solve the two equations $u_r$ and $u_\theta$ for $u_x$ and then solve the two equations $v_r$ and $v_\theta$ for $v_x$.

First,

$$r\cos\theta\, u_r = \frac{\partial u}{\partial x} r(\cos\theta)^2 + \frac{\partial u}{\partial y} r\sin\theta\cos\theta$$
$$\sin\theta\, u_\theta = -\frac{\partial u}{\partial x} r(\sin\theta)^2 + \frac{\partial u}{\partial y} r\sin\theta\cos\theta$$

we can get $u_x$ by,

$$r\cos\theta\, u_r - \sin\theta\, u_\theta = \frac{\partial u}{\partial x} r(\cos\theta)^2 + \frac{\partial u}{\partial x} r(\sin\theta)^2 = \frac{\partial u}{\partial x} r = u_x r$$

$$u_x = \cos\theta\, u_r - \frac{1}{r}\sin\theta\, u_\theta$$

Second,

$$r\cos\theta\, v_r = \frac{\partial v}{\partial x} r(\cos\theta)^2 + \frac{\partial v}{\partial y} r\sin\theta\cos\theta$$

$$\sin\theta\, v_\theta = -\frac{\partial v}{\partial x} r(\sin\theta)^2 + \frac{\partial v}{\partial y} r\sin\theta\cos\theta$$

we can get $v_x$ by,

$$r\cos\theta\, v_r - \sin\theta\, v_\theta = \frac{\partial v}{\partial x} r(\cos\theta)^2 + \frac{\partial v}{\partial x} r(\sin\theta)^2 = \frac{\partial v}{\partial x} r = v_x r$$

$$v_x = \cos\theta\, v_r - \frac{1}{r}\sin\theta\, v_\theta$$

So

$$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \cos\theta\, u_r - \frac{1}{r}\sin\theta\, u_\theta + i\left(\cos\theta\, v_r - \frac{1}{r}\sin\theta\, v_\theta\right)$$

using the Cauchy-Riemann in polar form,

$$\begin{aligned} f'(z) &= \cos\theta\, u_r + \sin\theta\, v_r + i(\cos\theta\, v_r - \sin\theta\, u_r) \\ &= (\cos\theta - i\sin\theta)u_r + (\sin\theta + i\cos\theta)v_r \\ &= (\cos\theta - i\sin\theta)u_r + i^3(\sin\theta + i\cos\theta)iv_r \\ &= (\cos\theta - i\sin\theta)u_r + (\cos\theta - i\sin\theta)iv_r \\ &= e^{-i\theta}\left(\frac{\partial u}{\partial r} + i\frac{\partial v}{\partial r}\right) \end{aligned}$$

Again, using the Cauchy-Riemann in polar form, we can also get

$$f'(z) = \frac{1}{r}e^{-i\theta}\left(\frac{\partial v}{\partial \theta} - i\frac{\partial u}{\partial \theta}\right)$$

**Remarks** Comparison with Real Analysis
We are now in a position to show that $f(z) = e^z$ is differentiable everywhere and that this complex function shares the same derivative property as its real counterpart, that is, $f'(z) = f(z)$.

3.4 Harmonic Functions

In Section 5.5 we shall see that when a complex function $f(z) = u(x,y) + iv(x,y)$ is analytic at a point $z$, then *all* the derivatives of $f$: $f'(z)$, $f''(z)$, $f'''(z)$, $\cdots$ are also analytic at $z$. As a consequence of this remarkable fact, we can conclude that *all* partial derivatives of the real functions $u(x,y)$ and $v(x,y)$ are continuous at $z$. From the continuity of the partial derivatives we then know that the second-order mixed partial derivatives are equal. This last fact, coupled with the Cauchy-Riemann equations, will be used in this section to demonstrate that there is a connection between the real and imaginary parts of an analytic function $f(z) = u(x,y) + iv(x,y)$ and the second-order partial differential equation

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0.$$

This equation, one of the most famous in applied mathematics, is known as **Laplace's equation** in two variables. The sum $\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2}$ of the two second partial derivatives above is denoted by $\nabla^2\phi$ and is called the Laplacian of $\phi$. Laplace's equation is then abbreviated as $\nabla^2\phi = 0$.

**Harmonic Functions**

A solution $\phi(x,y)$ of Laplace's equation in a domain $D$ of the plane is given a special name.

**Definition 3.3 Harmonic Functions**
A real-valued function $\phi$ of two real variables $x$ and $y$ that has continuous first and second-order partial derivatives in a domain $D$ and satisfies Laplace's equation is said to be **harmonic** in $D$.

**Theorem 3.7 Harmonic Functions**
Suppose the complex function $f(z) = u(x, y) + iv(x, y)$ is analytic in a domain $D$. Then the functions $u(x, y)$ and $v(x, y)$ are harmonic in $D$.

**Harmonic Conjugate Functions**
Now suppose $u(x, y)$ is a given real function that is known to be harmonic in $D$. If it is possible to find another real harmonic function $v(x, y)$ so that $u$ and $v$ satisfy the Cauchy-Riemann equations throughout the domain $D$, then the function $v(x, y)$ is called a **harmonic conjugate** of $u(x, y)$. By combining the functions as $u(x, y) + iv(x, y)$ we obtain a function that is analytic in $D$.
**EXAMPLE 2**: Harmonic Conjugate (161)
3.5 Applications
**Orthogonal Families**
Suppose the function $f(z) = u(x, y) + iv(x, y)$ is analytic in some domain $D$. Then the real and imaginary parts of $f$ can be used to define two families of curves in $D$. The equations

$$u(x, y) = c_1 \quad \text{and} \quad v(x, y) = c_2,$$

where $c_1$ and $c_2$ are arbitrary real constants, are called **level curves** of $u$ and $v$, respectively. The level curves above are **orthogonal families**. Roughly, this means that each curve in one family is orthogonal to each curve in the other family. More precisely, at a point of intersection $z_0 = x_0 + iy_0$, where we shall assume that $f'(z_0) \neq 0$, the tangent line $L_1$ to the level curve $u(x, y) = u_0$ and the tangent line $L_2$ to the level curve $v(x, y) = v_0$ are perpendicular. See Figure 3.7. The numbers $u_0$ and $v_0$ are defined by evaluating $u$ and $v$ at $z_0$, that is, $c_1 = u(x_0, y_0) = u_0$ and $c_2 = v(x_0, y_0) = v_0$.
To prove that $L_1$ and $L_2$ are perpendicular at $z_0$ we demonstrate that the slope of one tangent is the negative reciprocal of the slope of the other by showing that the product of the two slopes is $-1$. We begin by differentiating $u(x, y) = u_0$ and $v(x, y) = v_0$ with respect to $x$ using the chain rule of partial differentiation:

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y}\frac{dy}{dx} = 0 \quad \text{and} \quad \frac{\partial v}{\partial x} + \frac{\partial v}{\partial y}\frac{dy}{dx} = 0.$$

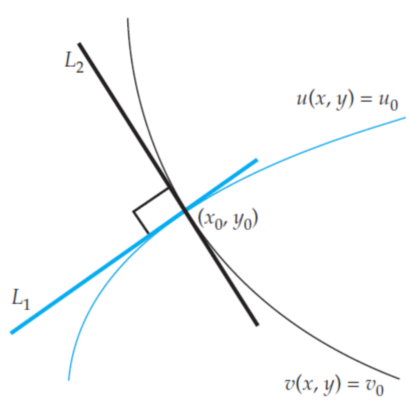



Figure 3. 7 Tangents $L_1$ and $L_2$ at point of intersection $z_0$ are perpendicular.

We then solve each of the foregoing equations for $dy/dx$:

$$\overbrace{\frac{dy}{dx}=-\frac{\partial u/\partial x}{\partial u/\partial y}}^{\text{slope of a tangent to curve } u(x,y)=u_0}, \qquad \overbrace{\frac{dy}{dx}=-\frac{\partial v/\partial x}{\partial v/\partial y}}^{\text{slope of a tangent to curve } v(x,y)=v_0},$$

At $(x_0, y_0)$ we see from above, the Cauchy-Reimann equations $u_x = v_y$, $u_y = v_x$, and from $f'(z_0) \neq 0$, that the product of the two slope functions is

$$\left(-\frac{\partial u/\partial x}{\partial u/\partial y}\right)\left(-\frac{\partial v/\partial x}{\partial v/\partial y}\right) = \left(\frac{\partial v/\partial y}{\partial v/\partial x}\right)\left(-\frac{\partial v/\partial x}{\partial v/\partial y}\right) = -1.$$

**EXAMPLE 1**: Orthogonal Families

**Gradient Vector**

In vector calculus, if $f(x, y)$ is a differentiable scalar function, then the **gradient** of $f$, written either **grad** $f$ or $\nabla f$ (the symbol $\nabla$ is called a *nabla* or *del*), is defined to be the two-dimensional vector

$$\nabla f = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j}.$$

As shown in color in Figure 3.8, the gradient vector $\nabla f(x_0, y_0)$ at a point $(x_0, y_0)$ is perpendicular to the level curve of $f(x, y)$ passing through that point, that is, to the level curve $f(x, y) = c_0$, where $c_0 = f(x_0, y_0)$. To see this, suppose that $x = g(t)$, $y = h(t)$, where $x_0 = g(t_0)$, $y_0 = h(t_0)$ are parametric equations for the curve $f(x, y) = c_0$. Then the derivative of $f\big(x(t), y(t)\big) = c_0$ with respect to $t$ is

$$\frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} = 0.$$

This last result is the dot product of $\nabla f$ with the tangent vector $r'(t) = x'(t)\mathbf{i} + y'(t)\mathbf{j}$. Specifically, at $t = t_0$, the above equation shows that if $r'(t_0) \neq 0$, then $\nabla f(x_0, y_0) \cdot r'(t_0) = 0$. This means that $\nabla f$ is perpendicular to the level curve at $(x_0, y_0)$.

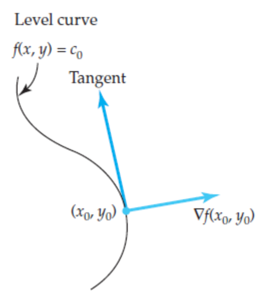



Figure 3. 8 Gradient is perpendicular to level curve at $(x_0, y_0)$

**Gradient Fields**

As discussed in Section 2.7, in complex analysis two-dimensional vector fields $\mathbf{F}(x, y) = P(x, y)\mathbf{i} + Q(x, y)\mathbf{j}$, defined in some domain $D$ of the plane, are of interest to us because $F$ can be represented equivalently as a complex function $f(z) = P(x, y) + iQ(x, y)$. Of particular importance in science are vector fields that can be written as the gradient of some scalar function $\phi$ with continuous second partial derivatives. In other words, $\mathbf{F}(x, y) = P(x, y)\mathbf{i} + Q(x, y)\mathbf{j}$ is the same as

$$\mathbf{F}(x, y) = \nabla\phi = \frac{\partial \phi}{\partial x}\mathbf{i} + \frac{\partial \phi}{\partial y}\mathbf{j},$$

where $P(x, y) = \partial\phi/\partial x$ and $Q(x, y) = \partial\phi/\partial y$. Such a vector field $\mathbf{F}$ is called a **gradient field** and $\phi$ is called a **potential function** or simply the **potential** for $\mathbf{F}$.

**Complex Potential**

In general, if a potential function $\phi(x, y)$ satisfies Laplace's equation in some domain $D$, it is harmonic, and we know from Section 3.3 that there exists a harmonic conjugate function $\psi(x, y)$ defined in $D$ so that the complex function

$$\Omega(z) = \phi(x, y) + i\psi(x, y)$$

is analytic in $D$. The function $\Omega(z)$ is called the **complex potential** corresponding to the real potential $\phi$.

**Ideal Fluid**

# CHAPTER 4 Elementary Functions

## 4.1 Exponential and Logarithmic Functions

### Complex Exponential Function

### Exponential Function and its Derivative

**Definition 4.1 Complex Exponential Function**

The function $e^z$ defined by

$$e^z = e^x \cos y + ie^x \sin y$$

is called the **complex exponential function**.

One reason why it is natural to call this function the exponential function is that it agrees with the real exponential function when $z$ is real.

**Theorem 4.1 Analyticity of $e^z$**

The exponential function $e^z$ is entire and its derivative is given by:

$$\frac{d}{dz}e^z = e^z.$$

Using the fact that the real and imaginary parts of an analytic function are harmonic conjugates, we can also show that the *only* entire function $f$ that agrees with the real exponential function $e^x$ for real input and that satisfies the differential equation $f'(z) = f(z)$ is the complex exponential function $e^z$.

### Modulus, Argument, and Conjugate

If we express the complex number $w = e^z$ in polar form:

$$w = e^x \cos y + ie^x \sin y = r(\cos\theta + i\sin\theta),$$

then

$$|e^z| = e^x$$
$$\arg(e^z) = y + 2n\pi, \quad n = 0, \pm1, \pm2, \cdots.$$

We know from calculus that $e^x > 0$ for all real $x$, and so it follows from above $|e^z| = e^x$ that $|e^z| > 0$. This implies that $e^z \neq 0$ for all complex $z$. Put another way, the point $w = 0$ is not in the range of the complex function $w = e^z$. The equation $|e^z| = e^x$ does not, however, rule out the possibility that $e^z$ is a negative real number. In fact, you should verify that if, say, $z = \pi i$, then $e^{\pi i}$ is real and $e^{\pi i} < 0$.

A formula for the conjugate of the complex exponential $e^z$ is

$$\overline{e^z} = e^{\bar{z}}.$$

### Algebraic Properties

**Theorem 4.2 Algebraic Properties of $e^z$**

If $z_1$ and $z_2$ are complex numbers, then

i. $e^0 = 1$
ii. $e^{z_1}e^{z_2} = e^{z_1+z_2}$
iii. $\frac{e^{z_1}}{e^{z_2}} = e^{z_1-z_2}$
iv. $(e^{z_1})^n = e^{nz_1}, \quad n = 0, \pm1, \pm2, \cdots.$

### Periodicity

The most striking difference between the real and complex exponential functions is the periodicity of $e^z$. Analogous to real periodic functions, we say that a complex function $f$ is periodic with period $T$ if $f(z + T) = f(z)$ for all complex $z$.

$$e^{z+2\pi i} = e^z.$$

The complex exponential function $e^z$ is periodic with a pure imaginary period $2\pi i$.

That is, for the function $f(z) = e^z$, we have $f(z + 2\pi i) = f(z)$ for all $z$.

Furthermore, if the complex exponential function $e^z$ maps the point $z$ onto the point $w$, then it also maps the points $z \pm 2\pi i$, $z \pm 4\pi i$, $z \pm 6\pi i$, and so on, onto the point $w$. Thus, the complex exponential function is not one-to-one, and all values $e^z$ are assumed in any infinite horizontal strip of width $2\pi$ in the $z$-plane. The infinite horizontal strip defined by:

$$-\infty < x < \infty, \quad -\pi < y \leq \pi,$$

Is called the **fundamental region** of the complex exponential function.

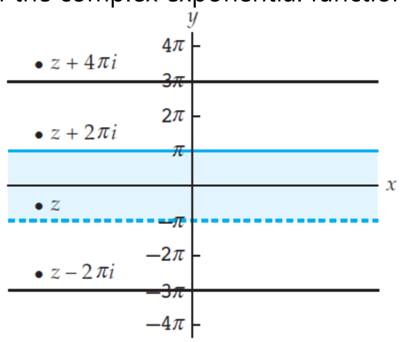



Figure 4. 1 The fundamental region of $e^z$

**The Exponential Mapping**

In order to determine the image of the fundamental region under $w = e^z$, we note that this region consists of the collection of vertical line segments $z(t) = a + it$, $-\pi < t \leq \pi$, where $a$ is an $y$ real number.

The image of the fundamental region $-\infty < x < \infty$, $-\pi < y \leq \pi$, under $w = e^z$ is the set of all complex $w$ with $w \neq 0$, or, equivalently, the set $|w| > 0$. This agrees with our observation that the point $w = 0$ is not in the range of the complex exponential function.

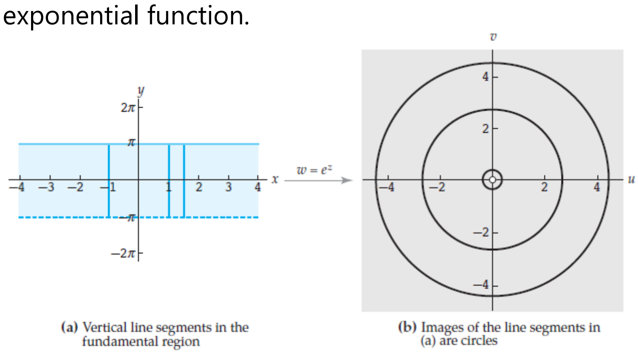
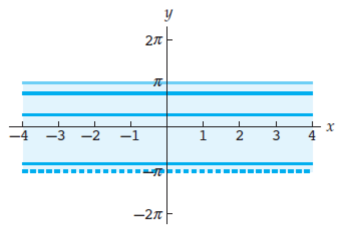



Figure 4. 2 The image of the fundamental region under $w = e^z$

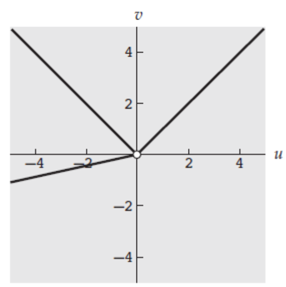
The image can also be found in the same manner by using, say, horizontal lines in the fundamental region.

$$z(t) = t + ib, -\infty < t < \infty$$



(a) Horizontal lines in the fundamental region

$w = e^z$



(b) Images of the lines in (a)

Figure 4. 3 The mapping $w = e^z$

**Exponential Mapping Properties**

i. $w = e^z$ maps the fundamental region $-\infty < x < \infty, -\pi < y \leq \pi$, onto the set $|w| > 0$.

ii. $w = e^z$ maps the vertical line segment $x = a$, $-\pi < y \leq \pi$, onto the circle $|w| = e^a$.

iii. $w = e^z$ maps the horizontal line $y = b$, $-\infty < x < \infty$, onto the ray $\arg(w) = b$.

**EXAMPLE 2**: Exponential Mapping of a Grid

**Complex Logarithmic Function**

In real analysis, the natural logarithm function $\ln x$ is often defined as an inverse function of the real exponential function $e^x$.

The situation is very different in complex analysis because the complex exponential function $e^z$ is not a one-to-one function on its domain $C$. In fact, given a fixed nonzero complex number $z$, the equation $e^w = z$ has infinitely many solutions.

In general, suppose that $w = u + iv$ is a solution of $e^w = z$. Then we must have $|e^w| = |z|$ and $\arg(e^w) = \arg(z)$. Then it follows that $e^u = |z|$ and $v = \arg(z)$, or, equivalently, $u = \log_e|z|$ and $v = \arg(z)$. Therefore, given a nonzero complex number $z$ we have shown that:

$$\text{If } e^w = z, \text{ then } w = \log_e|z| + i\arg(z).$$

Because there are infinitely many arguments of $z$, the above equation gives infinitely many solutions $w$ to the equation $e^w = z$. The set of values given by the above equation defines a multiple-valued function $w = G(z)$, as described in Section 2.4, which is called the complex logarithm of $z$ and denoted by $\ln z$.

**Definition 4.2 Complex Logarithm**

The multiple-valued function $\ln z$ defined by:

$$\ln z = \log_e|z| + i\arg(z)$$

is called the **complex logarithm**.

By switching to exponential notation $z = re^{i\theta}$ above, we obtain the following alternative description of the complex logarithm:

$$\ln z = \log_e r + i(\theta + 2n\pi), \quad n = 0, \pm 1, \pm 2, \cdots.$$

**Logarithmic Identities**

**Theorem 4.3 Algebraic Properties of $\ln z$**

If $z_1$ and $z_2$ are nonzero complex numbers and $n$ is an integer, then

i. $\ln(z_1 z_2) = \ln(z_1) + \ln(z_2)$

ii. $\ln\left(\frac{z_1}{z_2}\right) = \ln(z_1) - \ln(z_2)$

iii. $\ln(z_1^n) = n \ln(z_1)$.

**Principal Value of a Complex Logarithm**

The unique value of $\ln 5$ corresponding to $n = 0$ is the same as the value of the real logarithm $\log_e 5$. In general, this value of the complex logarithm is called the **principal value of the complex logarithm** since it is found by using the principal argument $\mathrm{Arg}(z)$ in place of the argument $\arg(z)$. We denote the principal value of the logarithm by the symbol $\mathrm{Ln}\, z$.

**Definition 4.3 Principal Value of the Complex Logarithm**

The complex function $\mathrm{Ln}\, z$ defined by:

$$\mathrm{Ln}\, z = \log_e |z| + i\, \mathrm{Arg}(z)$$

is called the **principal value of the complex logarithm**.

We will use the terms logarithmic function and logarithm to refer to both the multiple-valued function $\ln z$ and the function $\mathrm{Ln}\, z$. We can see that

$$\mathrm{Ln}\, z = \log_e r + i\theta, \quad -\pi < \theta \le \pi.$$

It is important to note that the identities for the complex logarithm in Theorem 4.3 are *not* necessarily satisfied by the principal value of the complex logarithm.

**$\mathrm{Ln}\, z$ as an Inverse Function**

Because $\mathrm{Ln}\, z$ is *one* of the values of the complex logarithm $\ln z$, it follows that:

$$e^{\mathrm{Ln}\, z} = z \text{ for all } z \neq 0.$$

This suggests that the logarithmic function $\mathrm{Ln}\, z$ is an inverse function of exponential function $e^z$. The exponential function must first be restricted to a domain on which it is one-to-one in order to have a well-defined inverse function. The $e^z$ is a one-to-one function on the fundamental region $-\infty < x < \infty$, $-\pi < y \le \pi$.

We can show that:

$$\mathrm{Ln}\, e^z = z \quad \text{if } -\infty < x < \infty \text{ and } -\pi < y \le \pi.$$

**$\mathrm{Ln}\, z$ as an Inverse Function of $e^z$**

If the complex exponential function $f(z) = e^z$ is defined on the fundamental region $-\infty < x < \infty$, $-\pi < y \le \pi$, then $f$ is one-to-one and the inverse function of $f$ is the principal value of the complex logarithm $f^{-1}(z) = \mathrm{Ln}\, z$.

Bear in mind that $e^{\mathrm{Ln}\, z} = z$ holds for all nonzero complex numbers $z$, but $\mathrm{Ln}\, e^z = z$ only holds if $z$ is in the fundamental region.

**Analyticity**

The principal value of the complex logarithm $\mathrm{Ln}\, z$ is discontinuous at the point $z = 0$ since this function is not defined there. This function also turns out to be discontinuous at every point on the negative real axis. The $\mathrm{Ln}\, z$ is a continuous function on the domain

$$|z| > 0, \quad -\pi < \arg(z) < \pi,$$

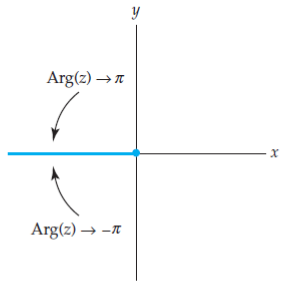



Figure 4. 4 $Ln\,z$ is discontinuous at $z = 0$ and on the negative real axis.

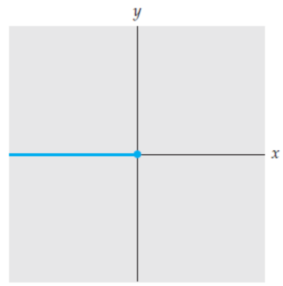



Figure 4. 5 Branch cut for $f_1$

Put another way, the function $f_1$ defined by:

$$f_1(z) = \log_e r + i\theta$$

is continuous on the domain above where $r = |z|$ and $\theta = \arg(z)$.

Using the terminology of Section 2.6, we have shown that the function $f_1$ defined by above is a branch of the multiple-valued function $F(z) = \ln z$. (Recall that branches of a multiple-valued function $F$ are denoted by $f_1$, $f_2$, and so on.) This branch is called the **principal branch of the complex logarithm**. The nonpositive real axis shown in color in Figure 4.5 is a **branch cut** for $f_1$ and the point $z = 0$ is a **branch point**.

**Theorem 4.4 Analyticity of the Principal Branch of $\ln z$**

The principal branch $f_1$ of the complex logarithm defined by $f_1(z) = \log_e r + i\theta$ is an analytic function and its derivative is given by:

$$f_1'(z) = \frac{1}{z}.$$

Because $f_1(z) = \mathrm{Ln}\, z$ for each point $z$ in the domain given in $|z| > 0$, $-\pi < \arg(z) < \pi$, it follows from Theorem 4.4 that $\mathrm{Ln}\, z$ is differentiable in this domain, and that its derivative is given by $f_1'$. That is, if $|z| > 0$ and $-\pi < \arg(z) < \pi$ then:

$$\frac{d}{dz}\mathrm{Ln}\, z = \frac{1}{z}.$$

**EXAMPLE 5**: Derivatives of Logarithmic Functions

**Logarithmic Mapping**

**Logarithmic Mapping Properties**

i. $w = \mathrm{Ln}\, z$ maps the set $|z| > 0$ onto the region $-\infty < u < \infty$, $-\pi < v \leq \pi$.

ii. $w = \mathrm{Ln}\, z$ maps the circle $|z| = r$ onto the vertical line segment $u = \log_e r$, $-\pi < v \leq \pi$.

iii. $w = \mathrm{Ln}\, z$ maps the ray $\arg(z) = \theta$ onto the horizontal line $v = \theta, -\infty < u < \infty$.

**EXAMPLE 6**: Logarithmic Mapping

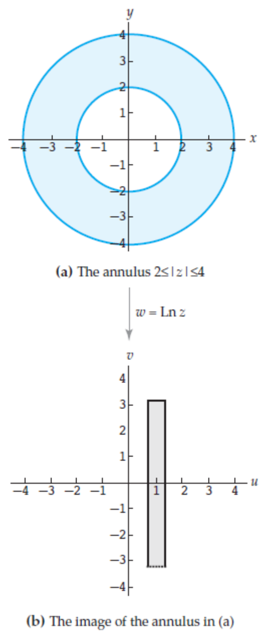



Figure 4. 6 The mapping $w = Ln\, z$

4.2 Complex Powers

**Complex Powers**

When $n$ is an integer it follows from Theorem 4.2(iv) that $z^n$ can be written as $z^n = \left(e^{\ln z}\right)^n = e^{n \ln z}$.

> **Definition 4.4 Complex Powers**
> If $\alpha$ is a complex number and $z \neq 0$, then the complex power $z^\alpha$ is defined to be:
> $$z^\alpha = e^{\alpha \ln z}.$$

When $n$ is an integer, however, the expression $z^n$ is single-valued (in agreement with fact that $z^n$ is a function when $n$ is an integer). (Page 195)

Although the previous discussion shows that $z^\alpha$ can define a single-valued function, you should bear in mind that, in general,

$$z^\alpha = e^{\alpha \ln z}$$

defines a multiple-valued function. We call the multiple-valued function given by above a **complex power function**.

**EXAMPLE 1**: Complex Powers

Complex powers defined by $z^\alpha = e^{\alpha \ln z}$ satisfy the following properties that are analogous to properties of real powers:

$$z^{\alpha_1} z^{\alpha_2} = z^{\alpha_1 + \alpha_2}, \quad \frac{z^{\alpha_1}}{z^{\alpha_2}} = z^{\alpha_1 - \alpha_2},$$
$$(z^\alpha)^n = z^{n\alpha} \quad \text{for } n = 0, \pm 1, \pm 2, \cdots.$$

Not all properties of real exponents have analogous properties for complex exponents. See the Remarks at the end of this section for an example.

**Principal Value of a Complex Power**

> **Definition 4.5 Principal Value of a Complex Powers**
> If $\alpha$ is a complex number and $z \neq 0$, then the function defined by:
> $$z^\alpha = e^{\alpha \operatorname{Ln} z}$$
> is called the **principal value of the complex power** $z^\alpha$.

**EXAMPLE 2**: Principal Value of a Complex Power

**Analyticity**

Since the function $e^{\alpha z}$ is continuous on the entire complex plane, and since the function $\operatorname{Ln} z$ is continuous on the domain $|z| > 0$, $-\pi < \arg(z) < \pi$, it follows that $z^\alpha$ is continuous on the domain $|z| > 0$, $-\pi < \arg(z) < \pi$. Using polar coordinates $r = |z|$ and $\theta = \arg(z)$ we have found that the function defined by:

$$f_1(z) = e^{\alpha(\log_e r + i\theta)}, \quad -\pi < \theta < \pi$$

is a branch of the multiple-valued function $F(z) = z^\alpha = e^{\alpha \ln z}$. This particular branch is called the **principal branch** of the complex power $z^\alpha$; its branch cut is the nonpositive real axis, and $z = 0$ is a branch point.
On the domain $|z| > 0$, $-\pi < \arg(z) < \pi$, the principal value of the complex power $z^\alpha$ is differentiable and

$$\frac{d}{dz} z^\alpha = \alpha z^{\alpha - 1}.$$

**Remarks** Comparison with Real Analysis

i. There are some properties of real powers that are not satisfied by complex powers. One example of this is that for complex powers, $(z^{\alpha_1})^{\alpha_2} \neq z^{\alpha_1 \alpha_2}$ unless $\alpha_2$ is an integer.
ii. As with complex logarithms, some properties that do hold for complex powers do not hold for principal values of complex powers. For example,
$$(z_1 z_2)^\alpha = z_1^\alpha z_2^\alpha$$
for any nonzero complex numbers $z_1$ and $z_2$.

4.3 Trigonometric and Hyperbolic Functions
Complex Trigonometric Functions
If $x$ is a real variable, then it follows from Definition 4.1 that:

$$e^{ix} = \cos x + i \sin x \quad \text{and} \quad e^{-ix} = \cos x - i \sin x.$$

By adding these equations and simplifying, we obtain an equation that relates the real cosine function with the complex exponential function:

$$\cos x = \frac{e^{ix} + e^{-ix}}{2}.$$

In a similar manner, if we subtract the two equations above, then we obtain an expression for the real sine function:

$$\sin x = \frac{e^{ix} - e^{-ix}}{2i}.$$

**Definition 4.6 Complex Sine and Cosine Functions**
The complex **sine** and **cosine** functions are defined by:

$$\sin z = \frac{e^{iz} - e^{-iz}}{2i} \quad \text{and} \quad \cos z = \frac{e^{iz} + e^{-iz}}{2}.$$

Analogous to real trigonometric functions, we next define the complex tangent, cotangent, secant, and cosecant functions using the complex sine and cosine:

$$\tan z = \frac{\sin z}{\cos z}, \quad \cot z = \frac{\cos z}{\sin z}, \quad \sec z = \frac{1}{\cos z}, \quad \text{and} \quad \csc z = \frac{1}{\sin z}.$$

These functions also agree with their real counterparts for real input.
**EXAMPLE 1**: Values of Complex Trigonometric Functions
**Identities**
Each of the results below is identical to its real analogue.

$$\sin(-z) = -\sin z \quad \cos(-z) = \cos z$$
$$\cos^2 z + \sin^2 z = 1$$
$$\sin(z_1 \pm z_2) = \sin(z_1)\cos(z_2) \pm \cos z_1 \sin z_2$$
$$\cos(z_1 \pm z_2) = \cos(z_1)\cos(z_2) \mp \sin z_1 \sin z_2$$

the double-angle formulas:

$$\sin(2z) = 2 \sin z \cos z \quad \cos(2z) = \cos^2 z - \sin^2 z$$

It is important to recognize that some properties of the real trigonometric functions are *not* satisfied by their complex counterparts. For example, $|\sin x| \leq 1$ and $|\cos x| \leq 1$ for all real $x$, but, from Example 1.

**Periodicity**

We have showed that $e^{z+2\pi i} = e^{z}$ for all complex $z$. Replacing $z$ with $iz$ in this equation we obtain $e^{iz+2\pi i} = e^{i(z+2\pi)} = e^{iz}$. Thus, $e^{iz}$ is a periodic function with real period $2\pi$. Similarly, we can show that $e^{-i(z+2\pi)} = e^{-iz}$.

$$\sin(z+2\pi) = \frac{e^{i(z+2\pi)} - e^{-i(z+2\pi)}}{2i} = \frac{e^{iz} - e^{-iz}}{2i} = \sin z.$$

In summary, we have:

$$\sin(z+2\pi) = \sin z \quad \text{and} \quad \cos(z+2\pi) = \cos z$$

for all $z$.

The periodicity of the secant and cosecant functions follows immediately from above. The identities $\sin(z+\pi) = -\sin z$ and $\cos(z+\pi) = -\cos z$ can be used to show that the complex tangent and cotangent are periodic with a real period of $\pi$.

**Trigonometric Equations**

**EXAMPLE 2**: Solving Trigonometric Equations

**Modulus**

The modulus of a complex trigonometric function can also be helpful in solving trigonometric equations.

If we replace the symbol $z$ with the symbol $x + iy$: (Page 204)

$$\sin z = \sin x \cosh y + i \cos x \sinh y.$$
$$\cos z = \cos x \cosh y + i \sin x \sinh y.$$

using the identities $\cos^2 x + \sin^2 x = 1$ and $\cosh^2 y = 1 + \sinh^2 y$:

$$|\sin z| = \sqrt{\sin^2 x + \sinh^2 y}.$$
$$|\cos z| = \sqrt{\cos^2 x + \sinh^2 y}.$$

Thus, we have shown that the complex sine and cosine functions are not bounded on the complex plane.

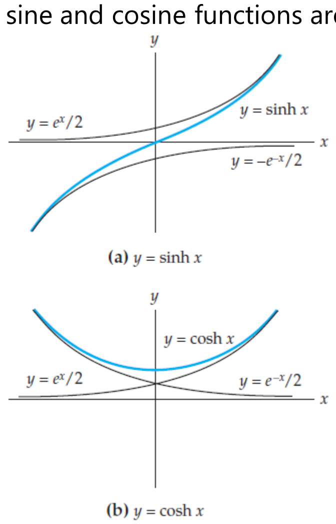



Figure 4. 7 The real hyperbolic functions

**Zeros**

In summary we have:

$$\sin z = 0 \quad \text{if and only if } z = n\pi,$$
$$\cos z = 0 \quad \text{if and only if } z = \frac{(2n+1)\pi}{2}$$

For $n = 0, \pm 1, \pm 2, \cdots$.

**Analyticity**

**Derivatives of Complex Trigonometric Functions**

$$\frac{d}{dz}\sin z = \cos z \quad \frac{d}{dz}\cos z = -\sin z$$
$$\frac{d}{dz}\tan z = \sec^2 z \quad \frac{d}{dz}\cot z = -\csc^2 z$$
$$\frac{d}{dz}\sec z = \sec z \tan z \quad \frac{d}{dz}\csc z = -\csc z \cot z$$

The sine and cosine functions are entire, but the tangent, cotangent, secant, and cosecant functions are only analytic at those points where the denominator is nonzero.

**Trigonometric Mapping**

The complex sine function is one-to-one on the domain $-\pi/2 < x < \pi/2$, $-\infty < y < \infty$.

**EXAMPLE 3**: The Mapping $w = \sin z$

See Figure 4.8. In summary, we have shown that the image of the infinite vertical strip $-\pi/2 < x < \pi/2$, $-\infty < y < \infty$, under $w = \sin z$, is the entire $w$-plane. (Page 208)

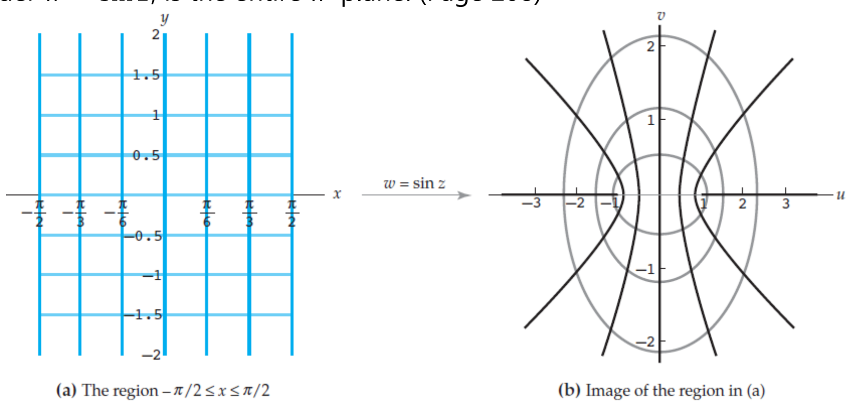



Figure 4. 8 The mapping $w = \sin z$

The mapping $w = \cos z$ can be analyzed in a similar manner, or, since $\cos z = \sin(z + \pi/2)$, we can view the mapping $w = \cos z$ as a composition of the translation $w = z + \pi/2$ and the mapping $w = \sin z$.

Complex Hyperbolic Functions

**Definition 4.7 Complex Hyperbolic Sine and Cosine**

The complex **hyperbolic sine** and **hyperbolic cosine** functions are defined by:

$$\sinh z = \frac{e^z - e^{-z}}{2} \quad \text{and} \quad \cosh z = \frac{e^z + e^{-z}}{2}.$$

The complex hyperbolic functions agree with the real hyperbolic functions for real input. Unlike the real hyperbolic functions, the complex hyperbolic functions are periodic and have infinitely many zeros.

$$\tanh z = \frac{\sinh z}{\cosh z}, \quad \coth z = \frac{\cosh z}{\sinh z}, \quad \operatorname{sech} z = \frac{1}{\cosh z}, \quad \text{and} \quad \operatorname{csch} z = \frac{1}{\sinh z}$$

Observe that the hyperbolic sine and cosine functions are entire.

**Derivatives of Complex Hyperbolic Functions**

$$\frac{d}{dz}\sinh z = \cosh z \quad \frac{d}{dz}\cosh z = \sinh z$$
$$\frac{d}{dz}\tanh z = \operatorname{sech}^2 z \quad \frac{d}{dz}\coth z = -\operatorname{csch}^2 z$$
$$\frac{d}{dz}\operatorname{sech} z = -\operatorname{sech} z \tanh z \quad \frac{d}{dz}\operatorname{csch} z = -\operatorname{csch} z \coth z$$

**Relation To Sine and Cosine**

When dealing with the *complex* trigonometric and hyperbolic functions, however, there is a simple and beautiful connection between the two.

$$\sin z = -i \sinh(iz) \quad \text{and} \quad \cos z = \cosh(iz)$$
$$\sinh z = -i \sin(iz) \quad \text{and} \quad \cosh z = \cos(iz).$$
$$\tan(iz) = \frac{\sin(iz)}{\cos(iz)} = \frac{i \sinh z}{\cosh z} = i \tanh(z).$$

We next list some of the more commonly used hyperbolic identities. Each of the results below is identical to its real analogue.

$$\sinh(-z) = -\sinh z \quad \cosh(-z) = \cosh z$$
$$\cosh^2 z - \sinh^2 z = 1$$
$$\sinh(z_1 \pm z_2) = \sinh(z_1)\cosh(z_2) \pm \cosh(z_1)\sinh(z_2)$$
$$\cosh(z_1 \pm z_2) = \cosh(z_1)\cosh(z_2) \pm \sinh(z_1)\sinh(z_2)$$

**EXAMPLE 4**: A Hyperbolic Identity

The relations between the complex trigonometric and hyperbolic functions also allow us determine the action of the hyperbolic functions as complex mappings.

4.4 Inverse Trigonometric and Hyperbolic Functions

**Inverse Sine**

**Definition 4.8 Inverse Sine**

The multiple-valued function $\sin^{-1} z$ defined by:

$$\sin^{-1} z = -i \ln\left[iz + (1-z^2)^{1/2}\right]$$

is called the **inverse sine**.

At times, we will also call the inverse sine the **arcsine** and we will denote it by $\arcsin z$.

**EXAMPLE 1**: Values of Inverse Sine

**Inverse Cosine and Tangent**

**Definition 4.9 Inverse Cosine and Inverse Tangent**

The multiple-valued function $\cos^{-1} z$ defined by:

$$\cos^{-1} z = -i \ln\left[z + i(1-z^2)^{1/2}\right]$$

is called the **inverse cosine**. The multiple-valued function $\tan^{-1} z$ defined by:

$$\tan^{-1} z = \frac{i}{2} \ln\left(\frac{i+z}{i-z}\right)$$

Is called the **inverse tangent**.

**Branches and Analyticity**

The inverse sine and inverse cosine are multiple-valued functions that can be made single-valued by specifying a single value of the square root to use for the expression $(1-z^2)^{1/2}$ and a single value of the complex logarithm used. The inverse tangent, on the other hand, can be made single-valued by just specifying a single value of $\ln z$ to use.

A branch of a multiple-valued inverse trigonometric function may be obtained by choosing a branch of the square root function and a branch of the complex logarithm.

**Derivatives of Branches $\sin^{-1} z$, $\cos^{-1} z$, and $\tan^{-1} z$**

$$\frac{d}{dz} \sin^{-1} z = \frac{1}{(1-z^2)^{1/2}}$$
$$\frac{d}{dz} \cos^{-1} z = \frac{-1}{(1-z^2)^{1/2}}$$
$$\frac{d}{dz} \tan^{-1} z = \frac{1}{1+z^2}$$

When finding the value of a derivative with $\frac{d}{dz} \sin^{-1} z$ or $\frac{d}{dz} \cos^{-1} z$, we must use the same square root as is used to define the branch.

**EXAMPLE 2**: Derivative of a Branch of Inverse Sine

**Definition 4.10 Inverse Hyperbolic Sine, Cosine, and Tangent**
The multiple-valued functions $\sinh^{-1} z$, $\cosh^{-1} z$, and $\tanh^{-1} z$, defined by:

$$\sinh^{-1} z = \ln\left[z + (z^2 + 1)^{1/2}\right],$$
$$\cosh^{-1} z = \ln\left[z + (z^2 - 1)^{1/2}\right],$$

and

$$\tanh^{-1} z = \frac{1}{2}\ln\left(\frac{1+z}{1-z}\right)$$

are called the **inverse hyperbolic sine**, the **inverse hyperbolic cosine**, and the **inverse hyperbolic tangent**, respectively.

Branches of the inverse hyperbolic functions are defined by choosing branches of the square root and complex logarithm, or, in the case of the inverse hyperbolic tangent, just choosing a branch of the complex logarithm.

**Derivatives of Branches $\sinh^{-1} z$, $\cosh^{-1} z$, and $\tanh^{-1} z$**

$$\frac{d}{dz}\sinh^{-1} z = \frac{1}{(z^2+1)^{1/2}}$$
$$\frac{d}{dz}\cosh^{-1} z = \frac{-1}{(z^2-1)^{1/2}}$$
$$\frac{d}{dz}\tanh^{-1} z = \frac{1}{1-z^2}$$

As with the inverse trigonometric functions, we should take care to be consistent in our use of branches when evaluating derivatives.

**EXAMPLE 3**: Inverse Hyperbolic Cosine

4.5 Applications

# CHAPTER 5 Integration in the Complex Plane

## 5.1 Real Integrals

**Definite Integral**

**Steps Leading to the Definition of the Definite Integral**

1. Let $f$ be a function of a single variable $x$ defined at all points in a closed interval $[a, b]$.
2. Let $P$ be a partition:
$$a = x_0 < x_1 < x_2 < \cdots < x_{n-1} < x_n = b$$
of $[a, b]$ into $n$ subintervals $[x_{k-1}, x_k]$ of length $\Delta x_k = x_k - x_{k-1}$.
3. Let $\|P\|$ be the ***norm*** of the partition $P$ of $[a, b]$, that is, the length of the longest subinterval.
4. Choose a number $x_k^*$ in each subinterval $[x_{k-1}, x_k]$ of $[a, b]$.
5. Form $n$ products $f(x_k^*)\Delta x_k$, $k = 1,2,\cdots,n$, and then sum these products:
$$\sum_{k=1}^{n} f(x_k^*)\Delta x_k.$$

**Definition 5.1 Definite Integral**

The **definite integral** of $f$ on $[a, b]$ is
$$\int_a^b f(x)\,dx = \lim_{\|P\|\to 0} \sum_{k=1}^{n} f(x_k^*)\Delta x_k.$$

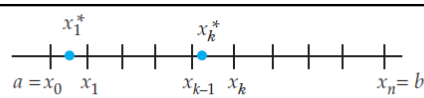



Figure 5. 1 Partition of $[a, b]$ with $x_k^*$ in each subinterval $[x_{k-1}, x_k]$

The notion of the definite integral $\int_a^b f(x)\,dx$, that is, integration of a real function $f(x)$ over an interval on the $x$-axis from $x = a$ to $x = b$ can be generalized to integration of a real multivariable function $G(x, y)$ on a curve $C$ from point $A$ to point $B$ in the Cartesian plane.

**Terminology**

Suppose a curve $C$ in the plane is parametrized by a set of equations $x = x(t)$, $y = y(t)$, $a \le t \le b$, where $x(t)$ and $y(t)$ are continuous real functions. Let the **initial** and **terminal points** of $C$, that is, $\big(x(a), y(a)\big)$ and $\big(x(b), y(b)\big)$, be denoted by the symbols $A$ and $B$, respectively. We say that:

i. $C$ is a **smooth curve** if $x'$ and $y'$ are continuous on the closed interval $[a, b]$ and not simultaneously zero on the open interval $(a, b)$.

ii. $C$ is a **piecewise smooth curve** if it consists of a finite number of smooth curves $C_1$, $C_2$,$\cdots$, $C_n$ joined end to end, that is, the terminal point of one curve $C_k$ coinciding with the initial point of the next curve $C_{k+1}$.

iii. $C$ is a **simple curve** if the curve $C$ does not cross itself except possibly at $t = a$ and $t = b$.

iv. $C$ is a **closed curve** if $A = B$.

v. $C$ is a **simple closed curve** if the curve $C$ does not cross itself and $A = B$; that is, $C$ is simple and closed.

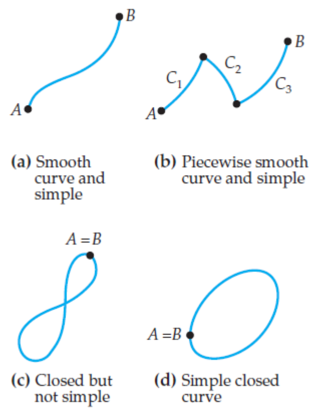



Figure 5. 2 Types of curves in the plane

### Line Integrals in the Plane

**Steps Leading to the Definition of Line Integrals**

1. Let $G$ be a function of two real variables $x$ and $y$ defined at all points on a smooth curve $C$ that lies in some region of the $xy$-plane. Let $C$ be defined by the parametrization $x = x(t)$, $y = y(t)$, $a \le t \le b$.

2. Let $P$ be a partition of the parameter interval $[a, b]$ into $n$ subintervals $[t_{k-1}, t_k]$ of length $\Delta t_k = t_k - t_{k-1}$:

$$a = t_0 < t_1 < t_2 < \cdots < t_{n-1} < t_n = b.$$

The partition $P$ induces a partition of the curve $C$ into $n$ subarcs of length $\Delta s_k$. Let the projection of each subarc onto the $x$- and $y$-axes have lengths $\Delta x_k$ and $\Delta y_k$, respectively.

3. Let $\|P\|$ be the ***norm*** of the partition $P$ of $[a, b]$, that is, the length of the longest subinterval.

4. Choose a point $(x_k^*, y_k^*)$ on each subarc of $C$.

5. Form $n$ products $G(x_k^*, y_k^*)\Delta x_k$, $G(x_k^*, y_k^*)\Delta y_k$, $G(x_k^*, y_k^*)\Delta s_k$, $k = 1, 2, \cdots, n$, and then sum these products:

$$\sum_{k=1}^{n} G(x_k^*, y_k^*)\Delta x_k, \quad \sum_{k=1}^{n} G(x_k^*, y_k^*)\Delta y_k, \quad \text{and} \quad \sum_{k=1}^{n} G(x_k^*, y_k^*)\Delta s_k.$$

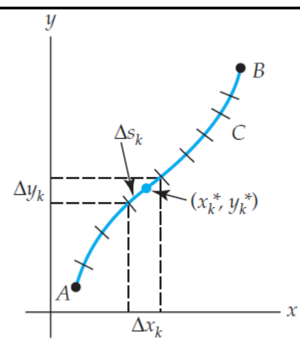



Figure 5. 3 Partition of curve $C$ into $n$ subarcs induced by a partition $P$ of the parameter interval $[a, b]$

**Definition 5.2 Line Integrals in the Plane**

The line integral of $G$ along $C$ with respect to $x$ is

$$\int_C G(x,y)\,dx = \lim_{\|P\|\to 0}\sum_{k=1}^{n} G(x_k^*, y_k^*)\Delta x_k.$$

The line integral of $G$ along $C$ with respect to $y$ is

$$\int_C G(x,y)\,dy = \lim_{\|P\|\to 0}\sum_{k=1}^{n} G(x_k^*, y_k^*)\Delta y_k.$$

The line integral of $G$ along $C$ with respect to arc length $s$ is

$$\int_C G(x,y)\,ds = \lim_{\|P\|\to 0}\sum_{k=1}^{n} G(x_k^*, y_k^*)\Delta s_k.$$

If $G$ is continuous on $C$, then the three types of line integrals above exist. The curve $C$ is referred to as the **path** of integration.

**Method of Evaluation–$C$ Defined Parametrically**

The ***basic*** idea is to convert a line integral to a definite integral in a single variable.

If $C$ is smooth curve parametrized by $x = x(t),\ \ y = y(t),\ a \le t \le b$, then replace $x$ and $y$ in the integral by the functions $x(t)$ and $y(t)$, and the appropriate differential $dx$, $dy$, or $ds$ by

$$x'(t)dt,\quad y'(t)dt,\quad \text{or}\quad \sqrt{[x'(t)]^2 + [y'(t)]^2}dt.$$

$$\int_C G(x,y)\,dx = \int_a^b G\big(x(t), y(t)\big)x'(t)dt,$$

$$\int_C G(x,y)\,dy = \int_a^b G\big(x(t), y(t)\big)y'(t)dt,$$

$$\int_C G(x,y)\,ds = \int_a^b G\big(x(t), y(t)\big)\sqrt{[x'(t)]^2 + [y'(t)]^2}dt.$$

**EXAMPLE 1**: $C$ Defined Parametrically

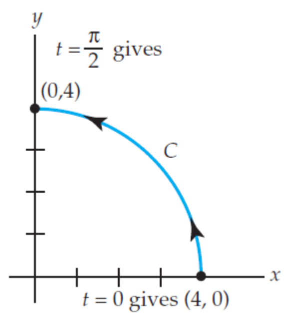



Figure 5. 4 Path $C$ of integration

**Method of Evaluation–$C$ Defined by a Function**

If the path of integration $C$ is the graph of an explicit function $y = f(x)$, $a \le x \le b$, then we can use $x$ as a parameter.

$$\int_C G(x,y)\,dx = \int_a^b G\big(x, f(x)\big)dx,$$

$$\int_C G(x,y)\,dy = \int_a^b G\big(x, f(x)\big)f'(x)dx,$$

$$\int_C G(x,y)\,ds = \int_a^b G\big(x, f(x)\big)\sqrt{1 + [f'(x)]^2}dx.$$

A line integral along a piecewise smooth curve $C$ is defined as the sum of the integrals over the various smooth curves whose union comprises $C$. For example, to evaluate $\int_C G(x,y)\,ds$ when $C$ is composed of two smooth curves $C_1$ and $C_2$, we begin by writing

$$\int_C G(x,y)\,ds = \int_{C_1} G(x,y)\,ds + \int_{C_2} G(x,y)\,ds.$$

**Notation**

In many applications, line integrals appear as a sum $\int_C P(x,y)\,dx + \int_C Q(x,y)\,dy$. It is common practice to write this sum as one integral without parentheses as

$$\int_C P(x,y)\,dx + Q(x,y)\,dy \quad \text{or simply} \quad \int_C P\,dx + Q\,dy.$$

A line integral along a closed curve $C$ is usually denoted by

$$\oint_C P\,dx + Q\,dy.$$

**EXAMPLE 2**: $C$ Defined by an Explicit Function

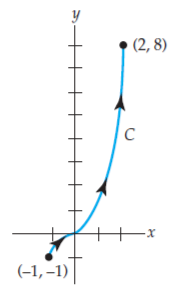



Figure 5. 5 Graph of $y = x^3$ on the interval $-1 \le x \le 2$

**EXAMPLE 3**: $C$ is a Closed Curve
**EXAMPLE 4**: $C$ is a Closed Curve

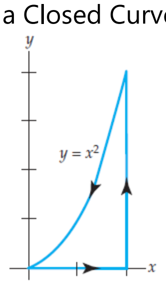



Figure 5. 6 Piecewise smooth path of integration

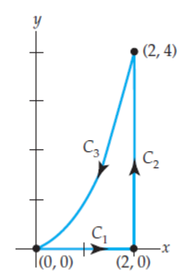



Figure 5. 7 $C$ consists of the union of $C_1$, $C_2$, and $C_3$.

**Orientation of a Curve**

$$\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx.$$

If $C$ is not a closed curve, then we say the **positive direction** on $C$, or that $C$ has **positive orientation**, if we traverse $C$ from its initial point $A$ to its terminal point $B$.

If $C$ is traversed in the sense opposite to that of the positive orientation, then $C$ is said to have **negative orientation**. If $C$ has an orientation (positive or negative), then the **opposite curve**, the curve with the opposite orientation, will be denoted by the symbol $-C$.

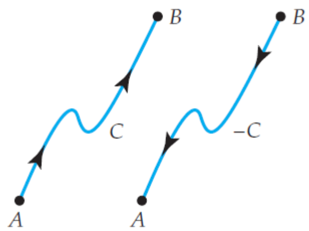


Figure 5. 8 Curve $C$ and its opposite $-C$

$$\int_{-C} P\,dx + Q\,dy = -\int_{C} P\,dx + Q\,dy,$$

or, equivalently

$$\int_{-C} P\,dx + Q\,dy + \int_{C} P\,dx + Q\,dy = 0.$$

It is important to be aware that a line integral is independent of the parametrization of the curve $C$, provided $C$ is given the same orientation by all sets of parametric equations defining the curve.

5.2 Complex Integrals

**Curves Revisited**

We can describe the points $z$ on $C$ by means of a complex-valued function of a real variable $t$ called a parametrization of $C$:

$$z(t) = x(t) + iy(t), \quad a \le t \le b.$$

The point $z(a) = x(a) + iy(a)$ or $A = \big(x(a), y(a)\big)$ is called the **initial point** of $C$ and $z(b) = x(b) + iy(b)$ or $B = \big(x(b), y(b)\big)$ is its **terminal point**.

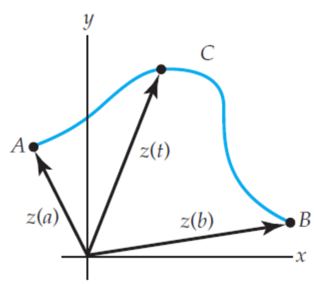


Figure 5. 9 $z(t) = x(t) + iy(t)$ as a position vector

**Contours**

Suppose the derivative of $z(t) = x(t) + iy(t), a \le t \le b$ is $z'(t) = x'(t) + iy'(t)$. We say a curve $C$ in the complex plane is smooth if $z'(t)$ is continuous and never zero in the interval $a \le t \le b$.

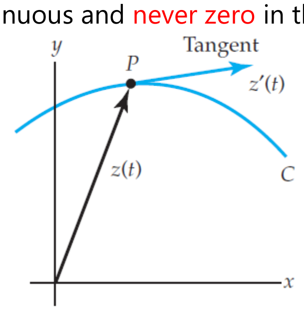


Figure 5. 10

A **piecewise smooth curve** $C$ has a continuously turning tangent, except possibly at the points where the component smooth curves $C_1, C_2, \cdots, C_n$ are joined together. A curve $C$ in the complex plane is said to be a **simple** if $z(t_1) \ne z(t_2)$ for $t_1 \ne t_2$, except possibly for $t = a$ and $t = b$. $C$ is a **closed curve** if $z(a) = z(b)$. $C$

is a **simple closed curve** if $z(t_1) \neq z(t_2)$ for $t_1 \neq t_2$ and $z(a) = z(b)$. In complex analysis, a piecewise smooth curve $C$ is called a **contour** or **path**.

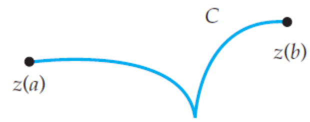
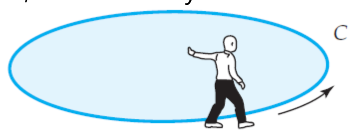
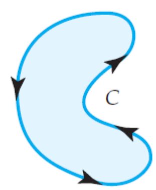



Figure 5. 11 Curve $C$ is not smooth since it has a cusp.

We define the **positive direction** on a contour $C$ to be the direction on the curve corresponding to increasing values of the parameter $t$. It is also said that the curve $C$ has **positive orientation**. In the case of a simple closed curve $C$, the positive direction roughly corresponds to the counterclockwise direction or the direction that a person must walk on $C$ in order to keep the interior of $C$ to the left. The **negative direction** on a contour $C$ is the direction opposite the positive direction. If $C$ has an orientation, the **opposite curve**, that is, a curve with opposite orientation, is denoted by $-C$.




**(a)** Positive direction




**(b)** Positive direction

Figure 5. 12 Interior of each curve is to the left.

**Complex Integral**

An integral of a function $f$ of a complex variable $z$ that is defined on a contour $C$ is denoted by $\int_C f(z)dz$ and is called a **complex integral**.

**Steps Leading to the Definition of the Complex Integrals**

1. Let $f$ be a function of a complex variable $z$ defined at all points on a smooth curve $C$ that lies in some region of the complex plane. Let $C$ be defined by the parametrization $z(t) = x(t) + iy(t)$, $a \le t \le b$.

2. Let $P$ be a partition of the parameter interval $[a, b]$ into $n$ subintervals $[t_{k-1}, t_k]$ of length $\Delta t_k = t_k - t_{k-1}$:

$$a = t_0 < t_1 < t_2 < \cdots < t_{n-1} < t_n = b.$$

The partition $P$ induces a partition of the curve $C$ into $n$ subarcs whose initial and terminal points are the pairs of numbers

$$\begin{array}{ll} z_0 = x(t_0) + iy(t_0), & z_1 = x(t_1) + iy(t_1), \\ z_1 = x(t_1) + iy(t_1), & z_2 = x(t_2) + iy(t_2), \\ \quad\vdots & \quad\vdots \\ z_{n-1} = x(t_{n-1}) + iy(t_{n-1}), & z_n = x(t_n) + iy(t_n). \end{array}$$

Let $\Delta z_k = z_k - z_{k-1}$, $k = 1,2,\cdots,n$.

3. Let $\|P\|$ be the ***norm*** of the partition $P$ of $[a, b]$, that is, the length of the longest subinterval.

4. Choose a point $z_k^* = x_k^* + iy_k^*$ on each subarc of $C$.

5. Form $n$ products $f(z_k^*)\Delta z_k$, $k = 1,2,\cdots,n$, and then sum these products:

$$\sum_{k=1}^{n} f(z_k^*)\Delta z_k.$$

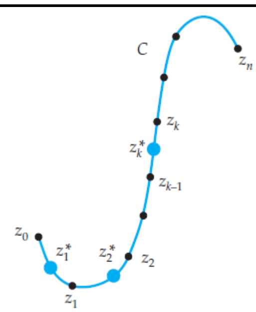



Figure 5. 13 Partition of curve $C$ into $n$ subarcs is induced by a partition $P$ of the parameter interval $[a, b]$

**Definition 5.3 Complex Integral**

The **complex integral** of $f$ on $C$ is

$$\int_C f(z)\,dz = \lim_{\|P\|\to 0} \sum_{k=1}^{n} f(z_k^*)\Delta z_k.$$

If the limit above exists, then $f$ is said to be **integrable** on $C$. The limit exists whenever if $f$ is continuous at all points on $C$ and $C$ is either smooth or piecewise smooth.

We will use the notation $\oint_C f(z)dz$ to represent a complex integral around a *positively oriented* closed curve $C$. When it is important to distinguish the direction of integration around a closed curve, we will employ the notations

$$\ointctrclockwise_C f(z)dz \quad \text{and} \quad \ointclockwise_C f(z)dz$$

to denote integration in the positive and negative directions, respectively.

We shall from now on refer to a complex integral $\oint_C f(z)dz$ by its more common name, **contour integral**.

**Complex-Valued Function of a Real Variable**

If $f_1$ and $f_2$ are real-valued functions of a real variable $t$ continuous on a common interval $a \le t \le b$, then we *define* the integral of the complex-valued function $f(t) = f_1(t) + if_2(t)$ on $a \le t \le b$ in terms of the definite integrals of the real and imaginary parts of $f$:

$$\int_a^b f(t)dt = \int_a^b f_1(t)dt + i\int_a^b f_2(t)dt.$$

The continuity of $f_1$ and $f_2$ on $[a, b]$ guarantees that both $\int_a^b f_1(t)dt$ and $\int_a^b f_2(t)dt$ exist.

If $f(t) = f_1(t) + if_2(t)$ and $g(t) = g_1(t) + ig_2(t)$ are complex-valued functions of a real variable $t$ continuous on an interval $a \leq t \leq b$, then

$$\int_a^b kf(t)dt = k\int_a^b f(t)dt\,, \quad k \text{ a complex constant},$$

$$\int_a^b \big(f(t) + g(t)\big)dt = \int_a^b f(t)dt + \int_a^b g(t)dt,$$

$$\int_a^b f(t)dt = \int_a^c f(t)dt + \int_c^b f(t)dt,$$

$$\int_b^a f(t)dt = -\int_a^b f(t)dt.$$

we choose to assume that the real number $c$ is in the interval $[a, b]$.

**Evaluation of Contour Integrals**

**Theorem 5.1 Evaluation of a Contour Integral**

If $f$ is continuous on a smooth curve $C$ given by the parametrization $z(t) = x(t) + iy(t)$, $a \leq t \leq b$, then

$$\int_C f(z)\,dz = \int_a^b f\big(z(t)\big)z'(t)\,dt.$$

Suppose $z(t) = x(t) + iy(t)$ and $z'(t) = x'(t) + iy'(t)$. Then the integrand $f\big(z(t)\big)z'(t)$ is a complex-valued function of a real variable $t$.

**EXAMPLE 1**: Evaluating a Contour Integral

**EXAMPLE 2**: Evaluating a Contour Integral

In general, if $x$ and $y$ are related by means of a continuous real function $y = f(t)$, then the corresponding curve $C$ in the complex plane can be parametrized by $z(x) = x + if(x)$. Equivalently, we can let $x = t$ so that a set of parametric equations for $C$ is $x = t$, $y = f(t)$.

**Theorem 5.2 Properties of Contour Integrals**

Suppose the functions $f$ and $g$ are continuous in a domain $D$, and $C$ is a smooth curve lying entirely in $D$. Then

i. $\int_C kf(z)dz = k\int_C f(z)dz\,, k$ a complex constant.

ii. $\int_C [f(z) + g(z)]dz = \int_C f(z)dz + \int_C g(z)dz$.

iii. $\int_C f(z)dz = \int_{C_1} f(z)dz + \int_{C_2} f(z)dz$, where $C$ consists of the smooth curves $C_1$ and $C_2$ joined end to end.

iv. $\int_{-C} f(z)dz = -\int_C f(z)dz$, where $-C$ denotes the curve having the opposite orientation of $C$.

**EXAMPLE 3**: $C$ Is a Piecewise Smooth Curve

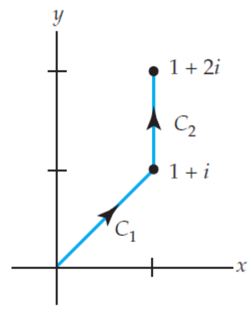



Figure 5. 14 Contour $C$ is piecewise smooth.

In the next theorem we use the fact that the length of a plane curve is $L = \int_a^b \sqrt{[x'(t)]^2 + [y'(t)]^2}dt$. But if $z'(t) = x'(t) + iy'(t)$, then $|z'(t)| = \sqrt{[x'(t)]^2 + [y'(t)]^2}$ and, consequently, $L = \int_a^b |z'(t)|dt$.

**Theorem 5.3 A Bounding Theorem**

If $f$ is continuous on a smooth curve $C$ and if $|f(t)| \leq M$ for all $z$ on $C$, then $\left|\int_C f(z)dz\right| \leq ML$, where $L$ is the length of $C$.

Theorem 5.3 is used often in the theory of complex integration and is sometimes referred to as the ***ML*-inequality**.

**EXAMPLE 4**: A Bound for a Contour Integral

5.3 Cauchy-Goursat Theorem

In this section we shall concentrate on contour integrals, where the contour $C$ is a simple closed curve with a positive (counterclockwise) orientation. Specifically, we shall see that when $f$ is analytic in a special kind of domain $D$, the value of the contour integral $\oint_C f(z)dz$ is the same for *any* simple closed curve $C$ that lies entirely within $D$. This theorem, called the **Cauchy-Goursat theorem**, is one of *the* fundamental results in complex analysis.

**Simply and Multiply Connected Domains**

We say that a domain $D$ is **simply connected** if every simple closed contour $C$ lying entirely in $D$ can be shrunk to a point without leaving $D$. Expressed yet another way, a simply connected domain has no "holes" in it.

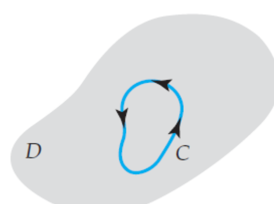



Figure 5. 15 Simply connected domain $D$

A domain that is not simply connected is called a **multiply connected domain**; that is, a multiply connected domain has "holes" in it. We call a domain with one "hole" **doubly connected**, a domain with two "holes" **triply connected**, and so on. The open disk defined by $|z| < 2$ is a simply connected domain; the open circular annulus defined by $1 < |z| < 2$ is a doubly connected domain.

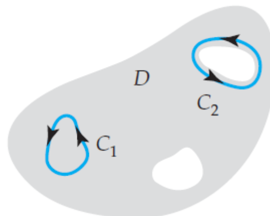


Figure 5. 16 Multiply connected domain $D$

**Cauchy's Theorem**

**Cauchy's Theorem**
Suppose that a function $f$ is analytic in a simply connected domain $D$ and that $f'$ is continuous in $D$. Then for every simple closed contour $C$ in $D$, $\oint_C f(z)dz = 0$.

**Cauchy's Proof of the above theorem**: The proof of this theorem is an immediate consequence of Green's theorem in the plane and the Cauchy-Riemann equations. Recall from calculus that if $C$ is a positively oriented, piecewise smooth, simple closed curve forming the boundary of a region $R$ within $D$, and if the real-valued functions $P(x, y)$ and $Q(x, y)$ along with their first-order partial derivatives are continuous on a domain that contains $C$ and $R$, then

$$\oint_C Pdx + Qdy = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA.$$

Green's theorem expresses a real line integral as a double integral.
We assumed that $f'$ is continuous throughout the domain $D$. As a consequence, the real and imaginary parts of $f(z) = u + iv$ and their first partial derivatives are continuous throughout $D$. We write $\oint_C f(z)dz$ in terms of real line integrals and apply Green's theorem to each line integral:

$$\oint_C f(z)dz = \oint_C f(z)(dx + idy) = \oint_C u(x, y)dx - v(x, y)dy + i\oint_C v(x, y)dx + u(x, y)dy$$
$$= \iint_R \left(-\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) dA + i\iint_R \left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right) dA.$$

Because $f$ is analytic in $D$, the real functions $u$ and $v$ satisfy the Cauchy-Riemann equations, $\partial u/\partial x = \partial v/\partial y$ and $\partial u/\partial y = -\partial v/\partial x$, at every point in $D$. Using the Cauchy-Riemann equations to replace $\partial u/\partial y$ and $\partial u/\partial x$ above shows that

$$\oint_C f(z)dz = \iint_R \left(-\frac{\partial v}{\partial x} + \frac{\partial v}{\partial x}\right) dA + i\iint_R \left(\frac{\partial v}{\partial y} - \frac{\partial v}{\partial y}\right) dA$$
$$= \iint_R (0)dA + i\iint_R (0)dA = 0.$$

This completes the proof.

**Theorem 5.4 Cauchy-Goursat Theorem**
Suppose that a function $f$ is analytic in a simply connected domain $D$. Then for every simple closed contour $C$ in $D$, $\oint_C f(z)dz = 0$.

Since the interior of a simple closed contour is a simply connected domain, the Cauchy-Goursat theorem can be stated in the slightly more practical manner:

If $f$ is analytic at all points within and on a simple closed contour $C$, then $\oint_C f(z)dz = 0$.

**EXAMPLE 1**: Applying the Cauchy-Goursat Theorem

The point of Example 1 is that $\oint_C e^z dz = 0$ for *any* simple closed contour in the complex plane. Indeed, it follows that for any simple closed contour $C$ and any entire function $f$, such as $f(z) = \sin z$, $f(z) = \cos z$, and $p(z) = a_n z^n + a_{n-1}z^{n-1} + \cdots + a_1 z + a_0$, $n = 0,1,2,\cdots$, that

$$\oint_C \sin z\, dz = 0, \quad \oint_C \cos z\, dz = 0, \quad \oint_C p(z)dz = 0,$$

And so on.

**EXAMPLE 2**: Applying the Cauchy-Goursat Theorem

**Multiply Connected Domains**

If $f$ is analytic in a multiply connected domain $D$ then we cannot conclude that $\oint_C f(z)dz = 0$ for every simple closed contour $C$ in $D$.

By introducing the crosscut $AB$ shown in Figure 5.17, the region bounded between the curves is now simply connected.

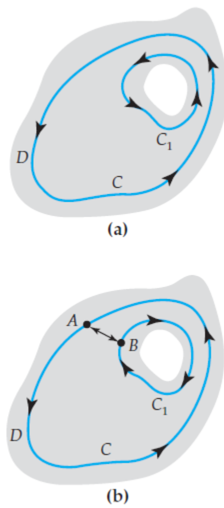



Figure 5. 17 Doubly connected domain $D$

$$\oint_C f(z)dz + \int_{AB} f(z)dz + \int_{-AB} f(z)dz + \oint_{C_1} f(z)dz = 0$$

or

$$\oint_C f(z)dz = \oint_{C_1} f(z)dz.$$

The last result is sometimes called the **principle of deformation of contours** since we can think of the contour $C_1$ as a continuous deformation of the contour $C$. Under this deformation of contours, the value of the integral does not change. In other words, the above equation allows us to evaluate an integral over a complicated simple closed contour $C$ by replacing $C$ with a contour $C_1$ that is more convenient.

**EXAMPLE 3**: Applying Deformation of Contours

Evaluate $\oint_C \frac{dz}{z-i}$, where $C$ is the contour shown in black in Figure 5.18.

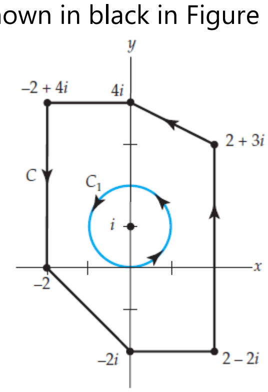



Figure 5. 18 We use the simpler contour $C_1$ in Example 3.

**Solution**:
Note that $f(z) = \frac{1}{z-i}$ is analytic on the multiply connected domain consisting of the complex plane excluding the point $z = i$. We choose the more convenient circular contour $C_1$ drawn in color in the figure. By taking the radius of the circle to be $r = 1$, we are guaranteed that $C_1$ lies within $C$. In other words, $C_1$ is the circle $|z - i| = 1$, which can be parametrized by $z = i + e^{it}$, $0 \leq t \leq 2\pi$. From $z - i = e^{it}$ and $dz = ie^{it}dt$ we obtain

$$\oint_C \frac{dz}{z-i} = \oint_{C_1} \frac{dz}{z-i} = \int_0^{2\pi} \frac{ie^{it}}{e^{it}} dt = i\int_0^{2\pi} dt = 2\pi i.$$

The result obtained in Example 3 can be generalized. It can be shown that if $z_0$ is any constant complex number interior to *any* simple closed contour $C$, then for $n$ an integer we have

$$\oint_C \frac{dz}{(z-z_0)^n} = \begin{cases} 2\pi i & n = 1 \\ 0, & n \neq 1. \end{cases}$$

The fact that the integral above is zero when $n \neq 1$ follows only partially from the Cauchy-Goursat theorem. When $n$ is zero or a negative integer, $1/(z - z_0)^n$ is a *polynomial* and therefore entire. Theorem 5.4 and the discussion following Example 1 then indicate that $\oint_C dz/(z - z_0)^n = 0$. It is left as an exercise to show that the integral is still zero when $n$ is a positive integer different from 1. (when $n \geq 2$ and $C_1 : z - z_0 = re^{it}$ lies within $C$, $\oint_C dz/(z - z_0)^n = \oint_C (ire^{it}dt)/(r^n e^{nit}) = \oint_C (idt)/\left(r^{n-1}e^{(n-1)it}\right) = -\frac{1}{r^{n-1}(n-1)} e^{-(n-1)it}\Big|_0^{2\pi} = 0$)

**EXAMPLE 4**: Applying Formula above

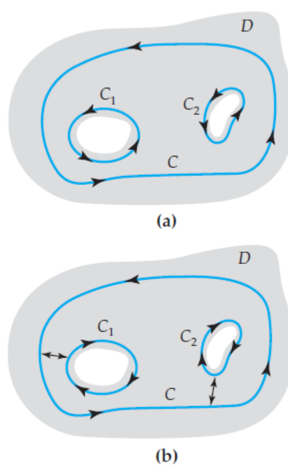



Figure 5. 19 Triply connected domain $D$

> **Theorem 5.5 Cauchy-Goursat Theorem for Multiply Connected Domains**
> Suppose $C$, $C_1$, $\cdots$, $C_n$ are simple closed curves with a positive orientation such that $C_1$, $C_2$, $\cdots$, $C_n$ are interior to $C$ but the regions interior to each $C_k$, $k = 1,2, \cdots, n$, have no points in common. If $f$ is analytic on each contour and at each point interior to $C$ but exterior to all the $C_k$, $k = 1,2, \cdots, n$, then
> $$\oint_C f(z)dz = \sum_{k=1}^{n} \oint_{C_k} f(z)dz.$$

**EXAMPLE 5**: Applying Theorem 5.5

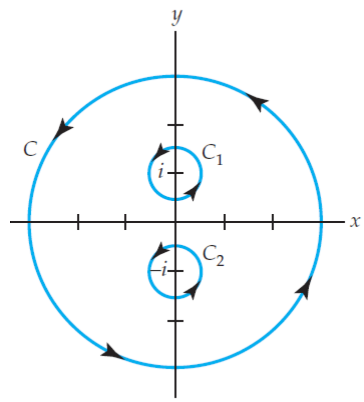



Figure 5. 20 Contour for Example 5

**Remarks**
Throughout the foregoing discussion we assumed that $C$ was a simple closed contour, in other words, $C$ did not intersect itself. Although we shall not give the proof, it can be shown that the Cauchy-Goursat theorem is **valid** for any closed contour $C$ in a simply connected domain $D$. As shown in Figure 5.21, the contour $C$ is closed but not simple. Nevertheless, if $f$ is analytic in $D$, then $\oint_C f(z)dz = 0$.

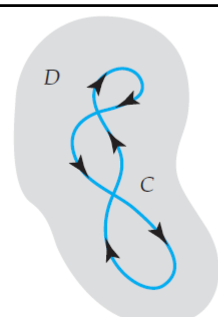



Figure 5. 21 Contour $C$ is closed but not simple.

5.4 Independence of Path

**Path Independence**

The definition of **path independence** for a contour integral $\int_C f(z)dz$ is essentially the same as for a real line integral $\int_C P\,dx + Q\,dy$.

**Definition 5.4 Independence of the Path**
Let $z_0$ and $z_1$ be points in a domain $D$. A contour integral $\int_C f(z)dz$ is said to be **independent of the path** if its value is the same for all contours $C$ in $D$ with initial point $z_0$ and terminal point $z_1$.

At the end of the preceding section we noted that the Cauchy-Goursat theorem also holds for closed contours, not just simple closed contours, in a simply connected domain $D$.

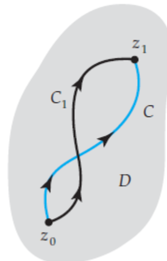



Figure 5. 22 If $f$ is analytic in $D$, integrals on $C$ and $C_1$ are equal.

$$\int_C f(z)dz + \int_{-C_1} f(z)dz = 0.$$

which is equivalent to

$$\int_C f(z)dz = \int_{C_1} f(z)dz.$$

The result above is also an example of the principle of deformation of contours.

**Theorem 5.6 Analyticity Implies Path Independence**
Suppose that a function $f$ is analytic in a simply connected domain $D$ and $C$ is any contour in $D$. Then $\int_C f(z)dz$ is independent of the path $C$.

**EXAMPLE 1**: Choosing a Different Path

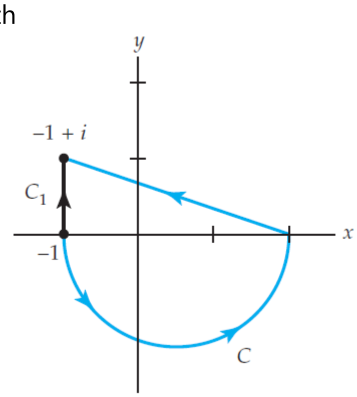



Figure 5. 23 Contour for Example 1

A contour integral $\int_C f(z)dz$ that is independent of the path $C$ is usually written $\int_{z_0}^{z_1} f(z)dz$, where $z_0$ and $z_1$ are the initial and terminal points of $C$.

There is an easier way to evaluate the contour integral in Example 1, but before proceeding we need another definition.

**Definition 5.5 Antiderivative**
Suppose that a function $f$ is continuous on a domain $D$. If there exists a function $F$ such that $F'(z) = f(z)$ for each $z$ in $D$, then $F$ is called an **antiderivative** of $f$.

As in calculus of a real variable, the most general antiderivative, or **indefinite integral**, of a function $f(z)$ is written $\int f(z)dz = F(z) + C$, where $F'(z) = f(z)$ and $C$ is some complex constant.

**Theorem 5.7 Fundamental Theorem for Contour Integrals**
Suppose that a function $f$ is **continuous** on a domain $D$ and $F$ is an antiderivative of $f$ in $D$. Then for any contour $C$ in $D$ with initial point $z_0$ and terminal point $z_1$,

$$\int_C f(z)dz = F(z_1) - F(z_0).$$

**EXAMPLE 2**: Applying Theorem 5.7
**EXAMPLE 3**: Applying Theorem 5.7
**Some Conclusions**
We can draw several immediate conclusions from Theorem 5.7. First, observe that if the contour $C$ is closed, then $z_0 = z_1$ and, consequently,

$$\oint_C f(z)dz = 0.$$

If a continuous function $f$ has an antiderivative $F$ in $D$, then $\int_C f(z)dz$ is independent of the path.

If $f$ is continuous and $\int_C f(z)dz$ is independent of the path $C$ in a domain $D$, then $f$ has an antiderivative everywhere in $D$.

**Theorem 5.8 Existence of an Antiderivative**
Suppose that a function $f$ is analytic in a simply connected domain $D$. Then $f$ has an antiderivative in $D$; that is, there exists a function $F$ such that $F'(z) = f(z)$ for all $z$ in $D$.

**EXAMPLE 4**: Using the Logarithmic Function
**EXAMPLE 5**: Using an Antiderivative of $z^{-1/2}$

**Remarks** Comparison with Real Analysis

i. Suppose $f$ and $g$ are analytic in a simply connected domain $D$. Then
$$\int f(z)g'(z)dz = f(z)g(z) - \int g(z)f'(z)dz.$$
In addition, if $z_0$ and $z_1$ are the initial and terminal points of a contour $C$ lying entirely in $D$, then
$$\int_{z_0}^{z_1} f(z)g'(z)dz = f(z)g(z)\big|_{z_0}^{z_1} - \int_{z_0}^{z_1} g(z)f'(z)dz$$

ii. There is no such complex counterpart for **mean-value theorem** in calculus.

5.5 Cauchy's Integral Formulas and Their Consequences
Cauchy's Two Integral Formulas
**First Formula**

**Theorem 5.9 Cauchy's Integral Formula**
Suppose that $f$ is analytic in a simply connected domain $D$ and $C$ is any simple closed contour lying entirely within $D$. Then for any point $z_0$ within $C$,

$$f(z_0) = \frac{1}{2\pi i}\oint_C \frac{f(z)}{z - z_0}dz.$$

Cauchy's integral formula can be used to evaluate contour integrals. Since we often work problems without a simply connected domain explicitly defined, a more practical restatement of Theorem 5.9 is:

If $f$ is analytic at all points within and on a simple closed contour $C$, and $z_0$ is any point interior to $C$, then $f(z_0) = \frac{1}{2\pi i} \oint_C \frac{f(z)}{z-z_0} dz$.

**EXAMPLE 1**: Using Cauchy's Integral Formula
**EXAMPLE 2**: Using Cauchy's Integral Formula
**Second Formula**

**Theorem 5.10 Cauchy's Integral Formula for Derivatives**
Suppose that $f$ is analytic in a simply connected domain $D$ and $C$ is any simple closed contour lying entirely within $D$. Then for any point $z_0$ within $C$,

$$f^{(n)}(z_0) = \frac{n!}{2\pi i} \oint_C \frac{f(z)}{(z-z_0)^{n+1}} dz.$$

**EXAMPLE 3**: Using Cauchy's Integral Formula for Derivatives
**EXAMPLE 4**: Using Cauchy's Integral Formula for Derivatives

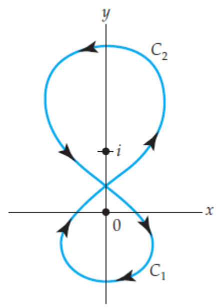



Figure 5. 24 Contour for Example 4

**Some Consequences of the Integral Formulas**
An immediate and important corollary to Theorem 5.10 is summarized next.

**Theorem 5.11 Derivative of an Analytic Function Is Analytic**
Suppose that $f$ is analytic in a simply connected domain $D$, Then $f$ possesses derivatives of all orders at every point $z$ in $D$. The derivatives $f'$, $f''$, $f'''$, $\cdots$ are analytic functions in $D$.

**Cauchy's Inequality**

**Theorem 5.12 Cauchy's Inequality**
Suppose that $f$ is analytic in a simply connected domain $D$ and $C$ is a circle defined by $|z - z_0| = r$ that lies entirely in $D$. If $|f(z)| \leq M$ for all points $z$ on $C$, then

$$\left|f^{(n)}(z_0)\right| \leq \frac{n!\, M}{r^n}.$$

Note that above if $n = 0$, then $M \geq |f(z_0)|$ for *any* circle $C$ centered at $z_0$ as long as $C$ lies within $D$. In other words, an upper bound $M$ of $|f(z)|$ on $C$ cannot be smaller than $|f(z_0)|$.

**Liouville's Theorem**

**Theorem 5.13 Liouville's Theorem**
The only bounded entire functions are constants.

**Fundamental Theorem of Algebra**

**Theorem 5.14 Fundamental Theorem of Algebra**
If $p(z)$ is a nonconstant polynomial, then the equation $p(z) = 0$ has at least one root.

**Theorem 5.15 Morera's Theorem**
If $f$ is continuous in a simply connected domain $D$ and if $\oint_C f(z)dz = 0$ for every closed contour $C$ in $D$, then $f$ is analytic in $D$.

**Theorem 5.16 Maximum Modulus Theorem**
Suppose that $f$ is analytic and nonconstant on a closed region $R$ bounded by a simple closed curve $C$. Then the modulus $|f(z)|$ attains its maximum on $C$.

If the stipulation that $f(z) \neq 0$ for all $z$ in $R$ is added to the hypotheses of Theorem 5.16, then the modulus $|f(z)|$ also attains its *minimum* on $C$.

**EXAMPLE 5**: Maximum Modulus

5.6 Applications

## CHAPTER 6 Series and Residues

6.1 Sequences and Series

**Sequences**

A **sequence** $\{z_n\}$ is a function whose domain is the set of positive integers and whose range is a subset of the complex numbers $C$.

For example, the sequence $\{1+i^n\}$ is

$$\begin{array}{ccccc} 1+i, & 0, & 1-i, & 2, & 1+i, \cdots \\ \uparrow & \uparrow & \uparrow & \uparrow & \uparrow \\ n=1, & n=2, & n=3, & n=4, & n=5, \cdots \end{array}$$

If $\lim_{n\to\infty} z_n = L$, we say the sequence $\{z_n\}$ is **convergent**. In other words, $\{z_n\}$ converges to the number $L$ if for each positive real number $\varepsilon$ an $N$ can be found such that $|z_n - L| < \varepsilon$ whenever $n > N$.

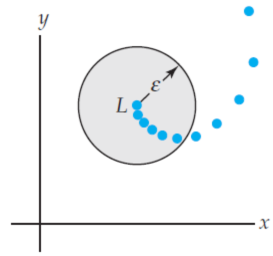



Figure 6. 1 If $\{z_n\}$ converges to $L$, all but a finite number of terms are in every $\varepsilon$-neighborhood of $L$.

A sequence that is not convergent is said to be **divergent**

**EXAMPLE 1**: A Convergent Sequence

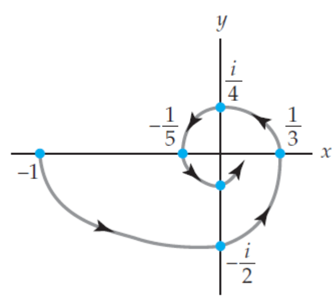



Figure 6. 2 The terms of the sequence $\{i^{n+1}/n\}$ spiral in toward 0.

> **Theorem 6.1 Criterion for Convergence**
> A sequence $\{z_n\}$ converges to a complex number $L = a + ib$ if and only if $\text{Re}(z_n)$ converges to $\text{Re}(L) = a$ and $\text{Im}(z_n)$ converges to $\text{Im}(L) = b$.

**EXAMPLE 2**: Illustrating Theorem 6.1

**Series**

An **infinite series** or **series** of complex numbers

$$\sum_{k=1}^{\infty} z_k = z_1 + z_2 + z_3 + \cdots + z_n + \cdots$$

is **convergent** if the sequence of partial sums $\{S_n\}$, where

$$S_n = z_1 + z_2 + z_3 + \cdots + z_n$$

converges. If $S_n \to L$ as $n \to \infty$, we say that the series converges to $L$ or that the **sum** of the series is $L$.

**Geometric Series**

A **geometric series** is any series of the form

$$\sum_{k=1}^{\infty} az^{k-1} = a + az + az^2 + \cdots + az^{n-1} + \cdots.$$

For the above equation, the $n$th term of the sequence of partial sums is

$$S_n = a + az + az^2 + \cdots + az^{n-1}.$$

When an infinite series is a geometric series, it is always possible to find a formula for $S_n$.

$$zS_n = az + az^2 + az^3 + \cdots + az^n,$$

$$S_n - zS_n = (a + az + az^2 + \cdots + az^{n-1}) - (az + az^2 + az^3 + \cdots + az^n)$$

$$= a - az^n$$

or $(1-z)S_n = a(1-z^n)$. Solving the last equation for $S_n$ gives us

$$S_n = \frac{a(1-z^n)}{1-z}.$$

Now $z^n \to 0$ as $n \to \infty$ whenever $|z| < 1$, and so $S_n \to a/(1-z)$. In other words, for $|z| < 1$ the sum of a geometric series is $a/(1-z)$:

$$\frac{a}{1-z} = a + az + az^2 + \cdots + az^{n-1} + \cdots.$$

A geometric series diverges when $|z| \geq 1$.

**Special Geometric Series**

If we set $a = 1$, the above equality is

$$\frac{1}{1-z} = 1 + z + z^2 + z^3 + \cdots.$$

whenever $|z| < 1$. If we then replace the symbol $z$ by $-z$, we get a similar result

$$\frac{1}{1+z} = 1 - z + z^2 - z^3 + \cdots.$$

the equality above is valid for $|z| < 1$ since $|-z| = |z|$.

**EXAMPLE 3**: Convergent Geometric Series

We turn now to some important theorems about convergence and divergence of an infinite series.

**Theorem 6.2 A Necessary Condition for Convergence**
If $\sum_{k=1}^{\infty} z_k$ converges, then $\lim_{n\to\infty} z_n = 0$.

**A Test for Divergence**

**Theorem 6.3 The $n$th Term Test for Divergence**
If $\lim_{n\to\infty} z_n \neq 0$, then $\sum_{k=1}^{\infty} z_k$ diverges.

**Definition 6.1 Absolute and Conditional Convergence**
An infinite series $\sum_{k=1}^{\infty} z_k$ is said to be **absolutely convergent** if $\sum_{k=1}^{\infty} |z_k|$ converges. An infinite series $\sum_{k=1}^{\infty} z_k$ is said to be **conditionally convergent** if it converges but $\sum_{k=1}^{\infty} |z_k|$ diverges.

In elementary calculus a real series of the form $\sum_{k=1}^{\infty} \frac{1}{k^p}$ is called a ***p*-series** and converges for $p > 1$ and diverges for $p \leq 1$.

**EXAMPLE 4**: Absolute Convergence

As in real calculus:

Absolute convergence implies convergence.

**Tests for Convergence**

**Theorem 6.4 Ratio Test**
Suppose $\sum_{k=1}^{\infty} z_k$ is a series of nonzero complex terms such that
$$\lim_{n\to\infty}\left|\frac{z_{n+1}}{z_n}\right| = L.$$
i. If $L < 1$, then the series converges absolutely.
ii. If $L > 1$ or $L = \infty$, then the series diverges.
iii. If $L = 1$, the test is inconclusive.

**Theorem 6.5 Root Test**
Suppose $\sum_{k=1}^{\infty} z_k$ is a series of complex terms such that
$$\lim_{n\to\infty}\sqrt[n]{|z_n|} = L.$$
i. If $L < 1$, then the series converges absolutely.
ii. If $L > 1$ or $L = \infty$, then the series diverges.
iii. If $L = 1$, the test is inconclusive.

We are interested primarily in applying the tests in Theorems 6.4 and 6.5 to power series.

**Power Series**

The notion of a power series is important in the study of analytic functions. An infinite series of the form
$$\sum_{k=0}^{\infty} a_k(z-z_0)^k = a_0 + a_1(z-z_0) + a_2(z-z_0)^2 + \cdots,$$
where the coefficients $a_k$ are complex constants, is called a **power series** in $z - z_0$. The power series above is said to be **centered** at $z_0$; the complex point $z_0$ is referred to as the **center** of the series.

**Circle of Convergence**

Every complex power series has a **radius of convergence**. Analogous to the concept of an interval of convergence for real power series, a complex power series has a **circle of convergence**, which is the circle centered at $z_0$ of largest radius $R > 0$ for which converges at every point *within* the circle $|z - z_0| = R$. A power series converges absolutely at all points $z$ within its circle of convergence, that is, for all $z$ satisfying $|z - z_0| < R$, and diverges at all points $z$ exterior to the circle, that is, for all $z$ satisfying $|z - z_0| > R$. The radius of convergence can be:

i. $R = 0$ (in which case the complex power series converges only at its center $z = z_0$),
ii. $R$ a finite positive number (in which case the complex power series converges at all interior points of the circle $|z - z_0| = R$), or
iii. $R = \infty$ (in which case the complex power series converges for all $z$).

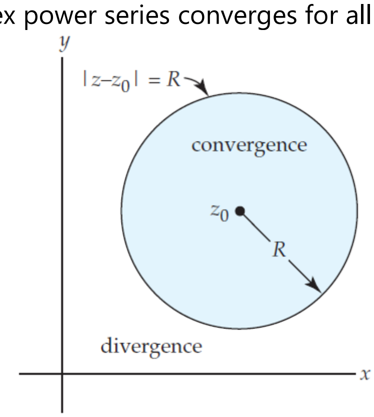



Figure 6. 3 No general statement concerning convergence at points on the circle $|z - z_0| = R$ can be made.

**EXAMPLE 5**: Circle of Convergence

It should be clear from Theorem 6.4 and Example 5 that for a power series $\sum_{k=0}^{\infty} a_k(z-z_0)^k$, the limit from ratio test depends only on the coefficients $a_k$. Thus, if

i. $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = L \neq 0$, the radius of convergence is $R = \frac{1}{L}$;
ii. $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = 0$, the radius of convergence is $R = \infty$;
iii. $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \infty$, the radius of convergence is $R = 0$.

Similar conclusions can be made for the root test by utilizing

$$\lim_{n\to\infty}\sqrt[n]{|a_n|}.$$

For example, if $\lim_{n\to\infty}\sqrt[n]{|a_n|} = L \neq 0$, then $R = 1/L$.

**EXAMPLE 6**: Radius of Convergence

**EXAMPLE 7**: Radius of Convergence

**The Arithmetic of Power Series**

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6.2 Taylor Series

**Differentiation and Integration of Power Series**

**Theorem 6.6 Continuity**
A power series $\sum_{k=0}^{\infty} a_k(z-z_0)^k$ represents a continuous function $f$ within its circle of convergence $|z-z_0| = R$.

**Theorem 6.7 Term-by-Term Differentiation**
A power series $\sum_{k=0}^{\infty} a_k(z-z_0)^k$ can be differentiated term by term within its circle of convergence $|z-z_0| = R$.

Differentiating a power series term-by-term gives,

$$\frac{d}{dz}\sum_{k=0}^{\infty} a_k(z-z_0)^k = \sum_{k=0}^{\infty} a_k \frac{d}{dz}(z-z_0)^k = \sum_{k=1}^{\infty} a_k k(z-z_0)^{k-1}.$$

Note that the summation index in the last series starts with $k = 1$ because the term corresponding to $k = 0$ is zero.

It follows as a corollary to Theorem 6.7 that *a power series defines an infinitely differentiable function* within its circle of convergence and each differentiated series has the same radius of convergence $R$ as the original power series.

**Theorem 6.8 Term-by-Term Integration**
A power series $\sum_{k=0}^{\infty} a_k(z-z_0)^k$ can be integrated term-by-term within its circle of convergence $|z-z_0| = R$, for every contour $C$ lying entirely within the circle of convergence.

**Taylor Series**

Suppose a power series represents a function $f$ within $|z-z_0| = R$, that is

$$f(z) = \sum_{k=0}^{\infty} a_k(z-z_0)^k = a_0 + a_1(z-z_0) + a_2(z-z_0)^2 + a_3(z-z_0)^3 + \cdots.$$

It follows from Theorem 6.7 that the derivatives of $f$ are the series

$$f'(z) = \sum_{k=1}^{\infty} a_k k(z-z_0)^{k-1} = a_1 + 2a_2(z-z_0) + 3a_3(z-z_0)^2 + \cdots,$$

$$f''(z) = \sum_{k=2}^{\infty} a_k k(k-1)(z-z_0)^{k-2} = 2\cdot 1a_2 + 3\cdot 2a_3(z-z_0) + \cdots,$$

$$f'''(z) = \sum_{k=3}^{\infty} a_k k(k-1)(k-2)(z-z_0)^{k-3} = 3 \cdot 2 \cdot 1 a_3 + \cdots,$$

and so on.

We conclude that *a power series represents an analytic function* within its circle of convergence.

There is a relationship between the coefficients $a_k$ above and the derivatives of $f$.

$$f(z_0) = a_0, \quad f'(z_0) = 1!\, a_1, \quad f''(z_0) = 2!\, a_2, \quad \text{and} \quad f'''(z_0) = 3!\, a_3,$$

In general, $f^{(n)}(z_0) = n!\, a_n$, or

$$a_n = \frac{f^{(n)}(z_0)}{n!}, \quad n \geq 0.$$

Substituting the above equation into $f(z) = \sum_{k=0}^{\infty} a_k (z-z_0)^k$ yields

$$f(z) = \sum_{k=0}^{\infty} \frac{f^{(k)}(z_0)}{k!} (z-z_0)^k.$$

This series is called the **Taylor series** for $f$ centered at $z_0$. A Taylor series with center $z_0 = 0$,

$$f(z) = \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} (z)^k,$$

is referred to as a **Maclaurin series**.

We have just seen that a power series with a nonzero radius $R$ of convergence represents an analytic function. On the other hand we ask: If we are given a function $f$ that is analytic in some domain $D$, can we represent it by a power series of the form $f(z) = \sum_{k=0}^{\infty} \frac{f^{(k)}(z_0)}{k!} (z-z_0)^k$ or $f(z) = \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} (z)^k$?

**Theorem 6.9 Taylor's Theorem**

Let $f$ be analytic within a domain $D$ and let $z_0$ be a point in $D$. Then $f$ has the series representation

$$f(z) = \sum_{k=0}^{\infty} \frac{f^{(k)}(z_0)}{k!} (z-z_0)^k$$

valid for the largest circle $C$ with center at $z_0$ and radius $R$ that lies entirely within $D$.

We can **find** the radius of convergence of a Taylor series in exactly the same manner illustrated in Examples 5–7 of the preceding section. However, we can simplify matters even further by noting that the radius of convergence $R$ is the distance from the center $z_0$ of the series to the *nearest* ***isolated singularity*** of $f$.

An isolated singularity is a point at which $f$ fails to be analytic but is, nonetheless, analytic at all other points throughout some neighborhood of the point.

**Some Important Maclaurin Series**

$$e^z = 1 + \frac{z}{1!} + \frac{z^2}{2!} + \cdots = \sum_{k=0}^{\infty} \frac{z^k}{k!}$$

$$\sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots = \sum_{k=0}^{\infty} (-1)^k \frac{z^{2k+1}}{(2k+1)!}$$

$$\cos z = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \cdots = \sum_{k=0}^{\infty} (-1)^k \frac{z^{2k}}{(2k)!}$$

**EXAMPLE 1**: Radius of Convergence

Stated in another way, the power series expansion of a function, with center $z_0$, is *unique*. On a practical level this means that a power series expansion of an analytic function $f$ centered at $z_0$, irrespective of the method used to obtain it, is *the* Taylor series expansion of the function.

**EXAMPLE 2**: Maclaurin Series

**EXAMPLE 3**: Taylor Series

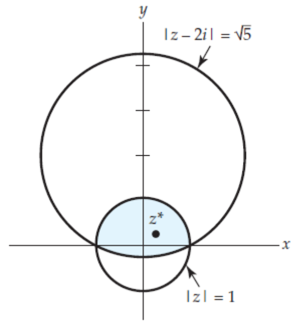


Figure 6. 4 Series $\frac{1}{1-z} = \sum_{k=0}^{\infty} z^k$ and $\frac{1}{1-z} = \sum_{k=0}^{\infty} \frac{1}{1-2i}(z-2i)^k$ both converge in the shaded region.

6.3 Laurent Series

If a complex function $f$ fails to be analytic at a point $z = z_0$, then this point is said to be a **singularity** or **singular point** of the function.

**Isolated Singularities**

Suppose that $z = z_0$ is a singularity of a complex function $f$. The point $z = z_0$ is said to be an **isolated singularity** of the function $f$ if there exists some deleted neighborhood, or punctured open disk, $0 < |z - z_0| < R$ of $z_0$ throughout which $f$ is analytic.

We say that a singular point $z = z_0$ of a function $f$ is **nonisolated** if every neighborhood of $z_0$ contains at least one singularity of $f$ other than $z_0$. For example, the branch point $z = 0$ is a nonisolated singularity of $\operatorname{Ln} z$ since every neighborhood of $z = 0$ contains points on the negative real axis.

**A New Kind of Series**

If $z = z_0$ is a singularity of a function $f$, then certainly $f$ cannot be expanded in a power series with $z_0$ as its center. However, about an isolated singularity $z = z_0$, it is possible to represent $f$ by a series involving both negative and nonnegative integer powers of $z - z_0$; that is,

$$f(z) = \cdots + \frac{a_{-2}}{(z-z_0)^2} + \frac{a_{-1}}{z-z_0} + a_0 + a_1(z-z_0) + a_2(z-z_0)^2 + \cdots.$$

As a *very* simple example let us consider the function $f(z) = 1/(z-1)$.

$$f(z) = \cdots + \frac{0}{(z-1)^2} + \frac{1}{z-1} + 0 + 0 \cdot (z-1) + 0 \cdot (z-1)^2 + \cdots$$

The series representation above is valid for $0 < |z-1| < \infty$.

Using summation notation, we can write the new kind of series as the sum of two series

$$f(z) = \sum_{k=1}^{\infty} a_{-k}(z-z_0)^{-k} + \sum_{k=1}^{\infty} a_k(z-z_0)^k.$$

The two series on the right-hand side above are given special names. The part with negative powers of $z - z_0$, that is,

$$\sum_{k=1}^{\infty} a_{-k}(z-z_0)^{-k} = \sum_{k=1}^{\infty} \frac{a_{-k}}{(z-z_0)^k}$$

is called the **principal part** of the new kind of series and will converge for $|1/(z-z_0)| < r^*$ or equivalently for $|z-z_0| > 1/r^* = r$.

The part consisting of the nonnegative powers of $z - z_0$,

$$\sum_{k=1}^{\infty} a_k(z-z_0)^k,$$

is called the **analytic part** of the new kind of series and will converge for $|z-z_0| < R$. Hence, the sum of the principal part and the analytic part converges when $z$ satisfies both $|z-z_0| > r$ *and* $|z-z_0| < R$, that is, when $z$ is a point in an annular domain defined by $r < |z-z_0| < R$.

By summing over negative and nonnegative integers, the new kind of series can be written compactly as

$$f(z) = \sum_{k=-\infty}^{\infty} a_k (z - z_0)^k.$$

**EXAMPLE 1**: Series of the Form Given in the new kind of series

A series representation of a function $f$ that has the form given in the new kind of series, is called a **Laurent series** or a **Laurent expansion** of $f$ about $z_0$ on the annulus $r < |z - z_0| < R$.

**Theorem 6.10 Laurent's Theorem**

Let $f$ be analytic within the annular domain $D$ defined by $r < |z - z_0| < R$. Then $f$ has the series representation

$$f(z) = \sum_{k=-\infty}^{\infty} a_k (z - z_0)^k$$

valid for $r < |z - z_0| < R$. The coefficients $a_k$ are given by

$$a_k = \frac{1}{2\pi i} \oint_C \frac{f(s)}{(s - z_0)^{k+1}} ds, \quad k = 0, \pm 1, \pm 2, \cdots,$$

where $C$ is a simple closed curve that lies entirely within $D$ and has $z_0$ in its interior.

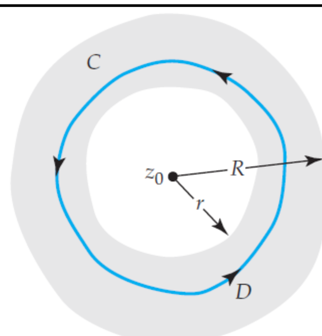


Figure 6. 5 Contour for Theorem 6.10

In the case when $a_{-k} = 0$ for $k = 1,2,3,\cdots$, the principal part is zero and the Laurent series reduces to a Taylor series. Thus, a Laurent expansion can be considered as a generalization of a Taylor series.

The annular domain in Theorem 6.10 defined by $r < |z - z_0| < R$ need not have the "ring" shape illustrated in Figure 6.5. Here are some other possible annular domains:

$$(i) r = 0, R \text{ finite}, \quad (ii) r \neq 0, R = \infty, \quad \text{and} \quad (iii) r = 0, R = \infty.$$

The integral formula in Theorem 6.10 for the coefficients of a Laurent series are rarely used in actual practice.

**EXAMPLE 2**: Four Laurent Expansions

6.4 Zeros and Poles

**Classification of Isolated Singular Points**

An isolated singular point $z = z_0$ of a complex function $f$ is given a classification depending on whether the principal part of its Laurent expansion contains zero, a finite number, or an infinite number of terms.

If the principal part is zero, that is, *all* the coefficients $a_{-k}$ are zero, then $z = z_0$ is called a removable singularity.

If the principal part contains a finite number of nonzero terms, then $z = z_0$ is called a **pole**. If, in this case, the last nonzero coefficient is $a_{-n}$, $n \geq 1$, then we say that $z = z_0$ is a **pole of order** $n$. If $z = z_0$ is pole of order 1, then the principal part contains exactly one term with coefficient $a_{-1}$. A pole of order 1 is commonly called a **simple pole**.

If the principal part contains an infinitely many nonzero terms, then $z = z_0$ is called an **essential singularity**.

| $z = z_0$ | **Laurent Series for $0 < \lvert z - z_0 \rvert < R$** |
|---|---|
| Removable singularity | $a_0 + a_1(z - z_0) + a_2(z - z_0)^2 + \cdots$ |
| Pole of order $n$ | $\frac{a_{-n}}{(z - z_0)^n} + \frac{a_{-(n-1)}}{(z - z_0)^{n-1}} + \cdots + \frac{a_{-1}}{z - z_0} + a_0 + a_1(z - z_0) + \cdots$ |
| Simple pole | $\frac{a_{-1}}{z - z_0} + a_0 + a_1(z - z_0) + a_2(z - z_0)^2 + \cdots$ |
| Essential singularity | $\cdots + \frac{a_{-2}}{(z - z_0)^2} + \frac{a_{-1}}{z - z_0} + a_0 + a_1(z - z_0) + a_2(z - z_0)^2 + \cdots$ |

Figure 6. 6 Forms of Laurent series

**EXAMPLE 1**: Removable Singularity

If a function $f$ has a removable singularity at the point $z = z_0$, then we can always supply an appropriate definition for the value of $f(z_0)$ so that $f$ becomes analytic at $z = z_0$.

**EXAMPLE 2**: Poles and Essential Singularity

**Zeros**

Recall, a number $z_0$ is **zero** of a function $f$ if $f(z_0) = 0$. We say that an analytic function $f$ has a **zero of order** $n$ at $z = z_0$ if

$$\overbrace{f(z_0) = 0,\quad f'(z_0) = 0,\quad f''(z_0) = 0,\quad \ldots,\quad f^{(n-1)}(z_0) = 0}^{z_0 \text{ is a zero of } f \text{ and of its first } n-1 \text{ derivatives}},\quad \text{but}\quad f^{(n)}(z_0) \neq 0.$$

A zero of order $n$ is also referred to as a **zero of multiplicity** $n$. For example, for $f(z) = (z - 5)^3$ we see that $f(5) = 0$, $f'(5) = 0$, $f''(5) = 0$, but $f'''(5) = 6 \neq 0$. Thus $f$ has a zero of order (or multiplicity) 3 at $z_0 = 5$. A zero of order 1 is called a **simple zero**.

**Theorem 6.11 Zero of Order $n$**

A function $f$ that is analytic in some disk $|z - z_0| < R$ has a zero of order $n$ at $z = z_0$ if and only if $f$ can be written

$$f(z) = (z - z_0)^n \phi(z),$$

where $\phi$ is analytic at $z = z_0$ and $\phi(z_0) \neq 0$.

**EXAMPLE 3**: Order of a Zero

**Poles**

We can characterize a pole of order $n$ in a manner analogous to zeros.

**Theorem 6.12 Pole of Order $n$**

A function $f$ that is analytic in a punctured disk $0 < |z - z_0| < R$ has a pole of order $n$ at $z = z_0$ if and only if $f$ can be written

$$f(z) = \frac{\phi(z)}{(z - z_0)^n},$$

where $\phi$ is analytic at $z = z_0$ and $\phi(z_0) \neq 0$.

**Zeros Again**

**Theorem 6.13 Pole of Order $n$**

If the functions $g$ and $h$ are analytic at $z = z_0$ and $h$ has a zero of order $n$ at $z = z_0$ and $g(z_0) \neq 0$, then the function $f(z) = g(z)/h(z)$ has a pole of order $n$ at $z = z_0$.

**EXAMPLE 4**: Order of Poles

6.5 Residues and Residue Theorem

We saw in the last section that if a complex function $f$ has an isolated singularity at a point $z_0$, then $f$ has a Laurent series representation

$$f(z) = \sum_{k=-\infty}^{\infty} a_k(z - z_0)^k = \cdots + \frac{a_{-2}}{(z - z_0)^2} + \frac{a_{-1}}{z - z_0} + a_0 + a_1(z - z_0) + a_2(z - z_0)^2 + \cdots,$$

which converges for all $z$ near $z_0$. More precisely, the representation is valid in some deleted neighborhood of $z_0$ or punctured open disk $0 < |z - z_0| < R$.

**Residue**

The coefficient $a_{-1}$ of $1/(z - z_0)$ in the Laurent series given above is called the **residue** of the function $f$ at the isolated singularity $z_0$. We shall use the notation

$$a_{-1} = \mathrm{Res}(f(z), z_0)$$

to denote the residue of $f$ at $z_0$.

**EXAMPLE 1**: Residues

**Theorem 6.14 Residue at a Simple Pole**

If $f$ has a simple pole at $z = z_0$, then

$$\mathrm{Res}(f(z), z_0) = \lim_{z \to z_0} (z - z_0) f(z).$$

**Theorem 6.15 Residue at a Pole of Order $n$**

If $f$ has a pole of order $n$ at $z = z_0$, then

$$\mathrm{Res}(f(z), z_0) = \frac{1}{(n-1)!} \lim_{z \to z_0} \frac{d^{n-1}}{dz^{n-1}} (z - z_0)^n f(z).$$

**EXAMPLE 2**: Residue at a Pole

When $f$ is not a rational function, calculating residues by means of Theorem 6.14 or Theorem 6.15 can sometimes be tedious. It is possible to devise alternative residue formulas. In particular, suppose a function $f$ can be written as a quotient $f(z) = g(z)/h(z)$, where $g$ and $h$ are analytic at $z = z_0$. If $g(z_0) \neq 0$ and if the function $h$ has a zero of order 1 at $z_0$, then $f$ has a simple pole at $z = z_0$ and

$$\mathrm{Res}(f(z), z_0) = \frac{g(z_0)}{h'(z_0)}.$$

**EXAMPLE 3**: Using above equation to Compute Residues

**Residue Theorem**

We come now to the reason why the residue concept is important.

**Theorem 6.16 Cauchy's Residue Theorem**

Let $D$ be a simply connected domain and $C$ a simple closed contour lying entirely within $D$. If a function $f$ is analytic on and within $C$, except at a finite number of isolated singular points $z_1, z_2, \cdots, z_n$ within $C$, then

$$\oint_C f(z)dz = 2\pi i \sum_{k=1}^{n} \mathrm{Res}(f(z), z_k).$$

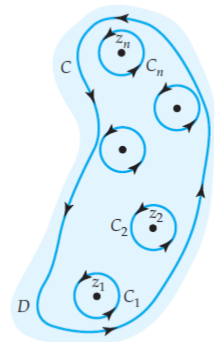

Figure 6. 7 $n$ singular points within contour $C$

**EXAMPLE 4**: Evaluation by the Residue Theorem

6.6 Some Consequences of the Residue Theorem

**Evaluation of Real Trigonometric Integrals**

**Integrals of the Form $\int_0^{2\pi} \mathbf{F}(\cos\boldsymbol{\theta}, \sin\boldsymbol{\theta})\mathbf{d}\boldsymbol{\theta}$**

The basic idea here is to convert a real trigonometric integral of form above into a complex integral, where the contour $C$ is the unit circle $|z| = 1$ centered at the origin.

We can write

$$z = e^{i\theta}, 0 \le \theta \le 2\pi$$

$$dz = ie^{i\theta}d\theta, \quad \cos\boldsymbol{\theta} = \frac{e^{i\theta} + e^{-i\theta}}{\mathbf{2}}, \quad \sin\boldsymbol{\theta} = \frac{e^{i\theta} - e^{-i\theta}}{\mathbf{2}\boldsymbol{i}}.$$

since $dz = ie^{i\theta}d\theta = izd\theta$ and $z^{-1} = 1/z = e^{-i\theta}$,

$$d\theta = \frac{dz}{iz}, \quad \cos\boldsymbol{\theta} = \frac{\mathbf{1}}{\mathbf{2}}(\boldsymbol{z} + \boldsymbol{z}^{-\mathbf{1}}), \quad \sin\boldsymbol{\theta} = \frac{\mathbf{1}}{\mathbf{2}\boldsymbol{i}}(\boldsymbol{z} - \boldsymbol{z}^{-\mathbf{1}}).$$

then

$$\int_0^{2\pi} \mathrm{F}(\cos\theta, \sin\theta)\mathrm{d}\theta = \oint_C \mathrm{F}\left(\frac{\mathbf{1}}{\mathbf{2}}(\boldsymbol{z} + \boldsymbol{z}^{-\mathbf{1}}), \frac{\mathbf{1}}{\mathbf{2}\boldsymbol{i}}(\boldsymbol{z} - \boldsymbol{z}^{-\mathbf{1}})\right)\frac{dz}{iz},$$

where $C$ is the unit circle $|z| = 1$.

**EXAMPLE 1**: A Real Trigonometric Integral

**Evaluation of Real Improper Integrals**

**Integrals of the Form $\int_{-\infty}^{\infty} \mathbf{f}(\mathbf{x})\mathbf{dx}$**

Suppose $y = f(x)$ is a real function that is defined and continuous on the interval $[0, \infty)$. In elementary calculus the improper integral $I_1 = \int_0^{\infty} f(x)dx$ is defined as the limit

$$I_1 = \int_0^{\infty} f(x)dx = \lim_{R\to\infty}\int_0^R f(x)dx.$$

If the limit exists, the integral $I_1$ is said to be **convergent**; otherwise, it is **divergent**. The improper integral $I_2 = \int_{-\infty}^{0} f(x)dx$ is defined similarly:

$$I_2 = \int_{-\infty}^{0} f(x)dx = \lim_{R\to\infty}\int_{-R}^{0} f(x)dx.$$

Finally, if $f$ is continuous on $(-\infty, \infty)$, then $\int_{-\infty}^{\infty} f(x)dx$ is defined to be

$$\int_{-\infty}^{\infty} f(x)dx = \int_{-\infty}^{0} f(x)dx + \int_0^{\infty} f(x)dx = I_1 + I_2,$$

provided ***both*** integrals $I_1$ and $I_2$ are convergent. If either one, $I_1$ or $I_2$, is divergent, then $\int_{-\infty}^{\infty} f(x)dx$ is divergent. It is important to remember that the right-hand side of the above equation is ***not*** the same as

$$\lim_{R\to\infty}\left[\int_{-R}^{0} f(x)dx + \int_0^R f(x)dx\right] = \lim_{R\to\infty}\int_{-R}^{R} f(x)dx.$$

*But*, in the event that we know (a priori) that an improper integral $\int_{-\infty}^{\infty} f(x)dx$ converges, we can then evaluate it by means of the single limiting process given above:

$$\int_{-\infty}^{\infty} f(x)dx = \lim_{R\to\infty} \int_{-R}^{R} f(x)dx.$$

On the other hand, the symmetric limit above may exist even though the improper integral $\int_{-\infty}^{\infty} f(x)dx$ is divergent.

The symmetric limit above, if it exists, is called the **Cauchy principal value** (P.V.) of the integral and is written

$$\text{P.V.} \int_{-\infty}^{\infty} f(x)dx = \lim_{R\to\infty} \int_{-R}^{R} f(x)dx.$$

**Cauchy Principal Value**

When an integral of form $\int_{-\infty}^{\infty} f(x)dx$ converges, its Cauchy principal value is the same as the value of the integral. If the integral diverges, it may still possess a Cauchy principal value.

One final point about the Cauchy principal value: Suppose $f(x)$ is continuous on $(-\infty, \infty)$ and is an *even* function, that is, $f(-x) = f(x)$. Then its graph is symmetric with respect to the $y$-axis and as a consequence

$$\int_{-R}^{0} f(x)dx = \int_{0}^{R} f(x)dx$$

and

$$\int_{-R}^{R} f(x)dx = \int_{-R}^{0} f(x)dx + \int_{0}^{R} f(x)dx = 2\int_{0}^{R} f(x)dx.$$

From above we **conclude** that if the Cauchy principal value exists, then both $\int_{0}^{\infty} f(x)dx$ and $\int_{-\infty}^{\infty} f(x)dx$ converge. The values of the integrals are

$$\int_{0}^{\infty} f(x)dx = \frac{1}{2}\text{P.V.} \int_{-\infty}^{\infty} f(x)dx \quad \text{and} \quad \int_{-\infty}^{\infty} f(x)dx = \text{P.V.} \int_{-\infty}^{\infty} f(x)dx$$

To evaluate an integral $\int_{-\infty}^{\infty} f(x)dx$, where the rational function $f(x) = p(x)/q(x)$ is continuous on $(-\infty, \infty)$, by residue theory we replace $x$ by the complex variable $z$ and integrate the complex function $f$ over a closed contour $C$ that consists of the interval $[-R, R]$ on the real axis and a semicircle $C_R$ of radius large enough to enclose all the poles of $f(x) = p(x)/q(x)$ in the upper half-plane $\text{Im}(z) > 0$. See Figure 6.11.

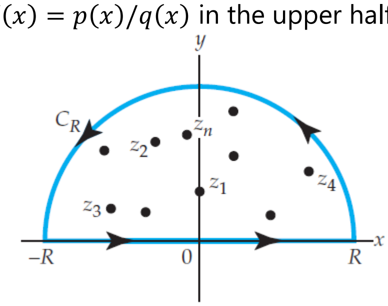


Figure 6. 8 Semicircular contour

By Theorem 6.16 of Section 6.5 we have

$$\oint_C f(z)dz = \int_{C_R} f(z)dz + \int_{-R}^{R} f(x)dx = 2\pi i \sum_{k=1}^{n} \text{Res}(f(z), z_k),$$

where $z_k$, $k = 1,2,\cdots,n$ denotes poles in the upper half-plane. If we can show that the integral $\int_{C_R} f(z)dz \to 0$ as $R \to \infty$, then we have

$$\text{P.V.} \int_{-\infty}^{\infty} f(x)dx = \lim_{R\to\infty} \int_{-R}^{R} f(x)dx = 2\pi i \sum_{k=1}^{n} \text{Res}(f(z), z_k).$$

**EXAMPLE 2**: Cauchy P.V. of an Improper Integral

It is often tedious to have to show that the contour integral along $C_R$ approaches zero as $R \to \infty$. Sufficient conditions under which this behavior is always true are summarized in the next theorem.

> **Theorem 6.17 Behavior of Integral as $R \to \infty$**
> Suppose $f(z) = \frac{p(z)}{q(z)}$ is a rational function, where the degree of $p(z)$ is $n$ and the degree of $q(z)$ is $m \geq n+2$. If $C_R$ is a semicircular contour $z = Re^{i\theta}$, $0 \leq \theta \leq \pi$, then $\int_{C_R} f(z)dz \to 0$ as $R \to \infty$.

**EXAMPLE 3**: Cauchy P.V. of an Improper Integral

**Integrals of the Form $\int_{-\infty}^{\infty} \mathrm{f(x)} \cos \alpha x \, \mathrm{dx}$ and $\int_{-\infty}^{\infty} \mathrm{f(x)} \sin \alpha x \, \mathrm{dx}$**

Because improper integrals of the form $\int_{-\infty}^{\infty} \mathrm{f(x)} \sin \alpha x \, \mathrm{dx}$ are encountered in applications of Fourier analysis, they often are referred to as **Fourier integrals**. Fourier integrals appear as the real and imaginary parts in the improper integral $\int_{-\infty}^{\infty} \mathrm{f(x)} e^{i\alpha x} \mathrm{dx}$. In view of Euler's formula $e^{i\alpha x} = \cos \alpha x + i \sin \alpha x$, where $\alpha$ is a positive real number, we can write

$$\int_{-\infty}^{\infty} \mathrm{f(x)} e^{i\alpha x} \mathrm{dx} = \int_{-\infty}^{\infty} \mathrm{f(x)} \cos \alpha x \, \mathrm{dx} + i \int_{-\infty}^{\infty} \mathrm{f(x)} \sin \alpha x \, \mathrm{dx}$$

whenever both integrals on the right-hand side converge.

> **Theorem 6.18 Behavior of Integral as $R \to \infty$**
> Suppose $f(z) = \frac{p(z)}{q(z)}$ is a rational function, where the degree of $p(z)$ is $n$ and the degree of $q(z)$ is $m \geq n+1$. If $C_R$ is a semicircular contour $z = Re^{i\theta}$, $0 \leq \theta \leq \pi$, and $\alpha > 0$, then $\int_{C_R} f(z) e^{i\alpha z} dz \to 0$ as $R \to \infty$.

**EXAMPLE 4**: Using Symmetry

**Indented Contours**

In the situation where $f$ has poles on the real axis, we must modify the procedure illustrated in Examples 2–4. For example, to evaluate $\int_{-\infty}^{\infty} \mathrm{f(x)dx}$ by residues when f(x) has a pole at $z = c$, where $c$ is a real number, we use an **indented contour** as illustrated in Figure 6.13. The symbol $C_r$ denotes a semicircular contour centered at $z = c$ and oriented in the *positive* direction.

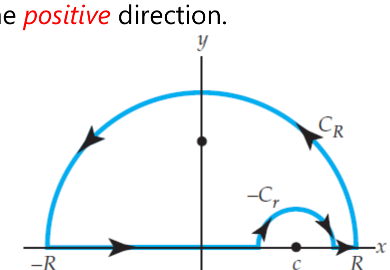


Figure 6. 9 Indented contour

> **Theorem 6.19 Behavior of Integral as $r \to 0$**
> Suppose $f$ has a simple pole $z = c$ on the real axis. If $C_r$ is the contour defined by $z = c + re^{i\theta}$, $0 \leq \theta \leq \pi$, then
> $$\lim_{r \to 0} \int_{C_r} f(z)dz = \pi i \operatorname{Res}(f(z), c).$$

**EXAMPLE 5**: Using an Indented Contour

Integration along a Branch Cut

**Branch Point at $\mathbf{z = 0}$**

In the next discussion we examine integrals of the form $\int_0^{\infty} f(x)dx$, where the integrand $f(x)$ is algebraic. But similar to Example 5, these integrals require a special type of contour because when $f(x)$ is converted to a complex function, the resulting integrand $f(z)$ has, in addition to poles, a nonisolated singularity at $z = 0$.

**EXAMPLE 6**: Integration along a Branch Cut

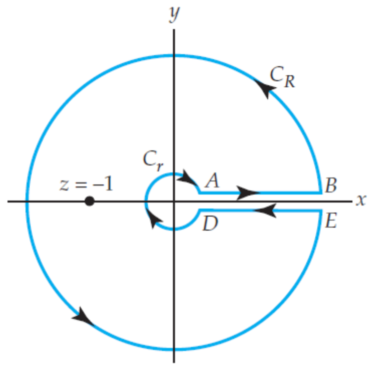


Figure 6. 10 Contour for Example 6

The Argument Principle and Rouch´e's Theorem

**Argument Principle**

In the first theorem we need to count the number of zeros and poles of a function $f$ that are located within a simple closed contour $C$; in this counting we include the order or multiplicity of each zero and pole. For example, if

$$f(z) = \frac{(z-1)(z-9)^4(z+i)^2}{(z^2-2z+2)^2(z-i)^6(z+6i)^7}$$

and $C$ is taken to be the circle $|z| = 2$. Therefore, the number $N_0$ of zeros inside $C$ is taken to be $N_0 = 1 + 2 = 3$ ($z = 1$ (a simple zero) and $z = -i$ (a zero of order or multiplicity 2)). The number $N_p$ of poles inside $C$ is taken to be $N_p = 2 + 2 + 6 = 10$ ( $z = 1 - i$ (pole of order 2) , $z = 1 + i$ (pole of order 2) , and $z = i$ (pole of order 6)).

**Theorem 6.20 Argument Principle**

Let $C$ be a simple closed contour lying entirely within a domain $D$. Suppose $f$ is analytic in $D$ except at a finite number of poles inside $C$, and that $f(z) \neq 0$ on $C$. Then

$$\frac{1}{2\pi i}\oint_C \frac{f'(z)}{f(z)}dz = N_0 - N_p,$$

where $N_0$ is the total number of zeros of $f$ inside $C$ and $N_p$ is the total number of poles of $f$ inside $C$. In determining $N_0$ and $N_p$, zeros and poles are counted according to their order or multiplicities.

To illustrate Theorem 6.20, using the example above:

$$\oint_C \frac{f'(z)}{f(z)}\,dz = \overbrace{[2\pi i(1) + 2\pi i(2)]}^{\substack{\text{contribution} \\ \text{of zeros of } f}} + \overbrace{[2\pi i(-2) + 2\pi i(-2) + 2\pi i(-6)]}^{\substack{\text{contribution} \\ \text{of poles of } f}} = -14\pi i.$$

**Why the Name?**

Why is Theorem 6.20 called the *argument principle*?

In point of fact there is a relation between the number $N_0 - N_p$ in Theorem 6.20 and $\arg(f(z))$. More precisely,

$$N_0 - N_p = \frac{1}{2\pi}\left[\text{change in } \arg(f(z)) \text{ as } z \text{ traverses } C \text{ once in the positive direction}\right].$$

**Rouch´e's Theorem**

The next result follows as a consequence of the argument principle. The theorem is helpful in determining the number of zeros of an analytic function.

> **Theorem 6.21 Rouché's Theorem**
> Let $C$ be a simple closed contour lying entirely within a domain $D$. Suppose $f$ and $g$ are analytic in $D$. If the strict inequality $|f(z) - g(z)| < |f(z)|$ holds for all $z$ on $C$, then $f$ and $g$ have the same number of zeros (counted according to their order or multiplicities) inside $C$.

**EXAMPLE 7**: Location of Zeros

Summing Infinite Series

**Using $\cot \pi z$**

In some specialized circumstances, the residues at the simple poles of the trigonometric function $\cot \pi z$ enable us to find the sum of an infinite series.

In Section 4.3 we saw that the zeros of $\sin z$ were the real numbers $z = k\pi$, $k = 0, \pm 1, \pm 2, \cdots$. Thus the function $\cot \pi z$ has simple poles at the zeros of $\sin \pi z$, which are $\pi z = k\pi$ or $z = k$, $k = 0, \pm 1, \pm 2, \cdots$. If a polynomial function $p(z)$ has $(i)$ real coefficients, $(ii)$ degree $n \geq 2$, and $(iii)$ no *integer* zeros, then the function

$$f(z) = \frac{\pi \cot \pi z}{p(z)}$$

has an infinite number of simple poles $z = 0, \pm 1, \pm 2, \cdots$ from $\cot \pi z$ and a finite number of poles $z_{p_1}, z_{p_2}, \cdots$, $z_{p_r}$ from the zeros of $p(z)$. The closed rectangular contour $C$ shown in Figure 6.11 has vertices $\left(n + \frac{1}{2}\right) + ni$, $-\left(n + \frac{1}{2}\right) + ni$, $-\left(n + \frac{1}{2}\right) - ni$, and $\left(n + \frac{1}{2}\right) - ni$, where $n$ is taken large enough so that $C$ encloses the simple poles $z = 0, \pm 1, \pm 2, \cdots, \pm n$ and *all* of the poles $z_{p_1}, z_{p_2}, \cdots, z_{p_r}$.

We can get our desired result: (Page 368)

$$\sum_{k=-\infty}^{\infty} \frac{1}{p(k)} = -\sum_{j=1}^{r} \operatorname{Res}\left(\frac{\pi \cot \pi z}{p(z)}, z_{p_j}\right).$$

**Using $\csc \pi z$**

There exist several more summation formulas similar to above.

For

$$f(z) = \frac{\pi \csc \pi z}{p(z)}$$

We have that

$$\sum_{k=-\infty}^{\infty} \frac{(-1)^k}{p(k)} = -\sum_{j=1}^{r} \operatorname{Res}\left(\frac{\pi \csc \pi z}{p(z)}, z_{p_j}\right).$$

**EXAMPLE 8**: Summing an Infinite Series

6.7 Applications

In other courses in mathematics or engineering you may have used the **Laplace transform** of a real function $f$ defined for $t \geq 0$,

$$\mathcal{L}\{f(t)\} = \int_0^{\infty} e^{-st} f(t) dt.$$

When the integral above converges, the result is a function of $s$. It is common practice to emphasize the relationship between a function and its transform by using a lowercase letter to denote the function and the corresponding uppercase letter to denote its Laplace transform, for example $\mathcal{L}\{f(t)\} = F(s)$, $\mathcal{L}\{y(t)\} = Y(s)$, and so on.

In the application of the Laplace transform we face two problems:

i. The *direct* problem: Given a function $f(t)$ satisfying certain conditions, find its Laplace transform.
ii. The *inverse* problem: Find the function $f(t)$ that has a given transform $F(s)$.

The function $F(s)$ is called the **inverse Laplace transform** and is denoted by $\mathcal{L}^{-1}\{F(s)\}$.

The Laplace transform is an invaluable aid in solving solve certain kinds of applied problems involving differential equations. In these problems we deal with the transform $Y(s)$ of an unknown function $y(t)$. The determination of $y(t)$ requires the computation of $\mathcal{L}^{-1}\{Y(s)\}$. In the case when $Y(s)$ is a rational function of $s$, you may recall employing partial fractions, operational properties, or tables to compute this inverse.

**Integral Transforms**

Suppose $f(x, y)$ is a real-valued function of two real variables. Then a definite integral of $f$ with respect to one of the variables leads to a function of the other variable. For example, if we hold $y$ constant, integration with respect to the real variable $x$ gives $\int_1^2 4xy^2 dx = 6y^2$. Thus a definite integral such as $F(\alpha) = \int_a^b f(x)K(\alpha, x)dx$ transforms a function $f$ of the variable $x$ into a function $F$ of the variable $\alpha$. We say that

$$F(\alpha) = \int_a^b f(x)K(\alpha, x)dx$$

is an **integral transform** of the function $f$. Integral transforms appear in **transform pairs**. This means that the original function $f$ can be recovered by another integral transform

$$f(x) = \int_c^d F(\alpha)H(\alpha, x)d\alpha,$$

called the **inverse transform**. The function $K(\alpha, x)$ in equation $F(\alpha)$ above and the function $H(\alpha, x)$ in equation $f(x)$ above are called the **kernels** of their respective transforms. We note that if $\alpha$ represents a complex variable, then the definite integral $f(x) = \int_c^d F(\alpha)H(\alpha, x)d\alpha$ is replaced by a contour integral.

**The Laplace Transform**

Suppose now in $F(\alpha) = \int_a^b f(x)K(\alpha, x)dx$ that the symbol $\alpha$ is replaced by the symbol $s$, and that $f$ represents a represents a real function (On occasion $f(t)$ could be a complex-valued function of a real variable $t$.) that is defined on the unbounded interval $[0, \infty)$. Then $F(s)$ is an improper integral and is defined as the limit

$$\int_0^\infty K(s, t)f(t)dt = \lim_{b \to \infty} \int_0^b K(s, t)f(t)dt.$$

If the limit above exists, we say that the integral exists or is convergent; if the limit does not exist the integral does not exist and is said to be divergent. The choice $K(s, t) = e^{-st}$, where $s$ is a complex variable, for the kernel above gives the Laplace transform $\mathcal{L}\{f(t)\}$ defined previously. The integral that defines the Laplace transform may not converge for certain kinds of functions $f$. For example, neither $\mathcal{L}\{e^{t^2}\}$ nor $\mathcal{L}\{1/t\}$ exist. Also, the limit above will exist for only certain values of the variable $s$.

**EXAMPLE 1**: Existence of a Laplace Transform

**Existence of $\mathcal{L}\{f(t)\}$**

Conditions that are *sufficient* to guarantee the existence of $\mathcal{L}\{f(t)\}$ are that $f$ be piecewise continuous on $[0, \infty)$ and that $f$ be of exponential order.

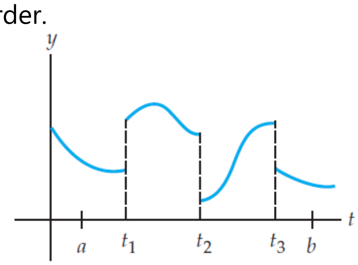


Figure 6. 11 Piecewise continuity on $[0, \infty)$

A function $f$ is said to be **exponential order $c$** if there exist constants $c$, $M > 0$, and $T > 0$ so that $|f(t)| \leq Me^{ct}$, for $t > T$. The condition $|f(t)| \leq Me^{ct}$ for $t > T$ states that the graph of $f$ on the interval $(T, \infty)$ does not grow faster than the graph of the exponential function $Me^{ct}$. See Figure 6.12.

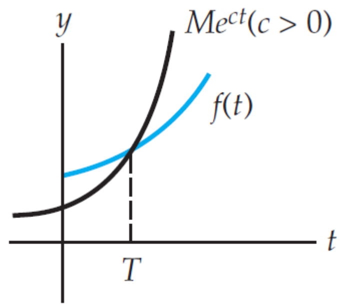


Figure 6. 12 Exponential order

Alternatively, $e^{-ct}|f(t)|$ is bounded; that is, $e^{-ct}|f(t)| \leq M$ for $t > T$. As can be seen in Figure 6.13, the function $f(t) = \cos t$, $t \geq 0$ is of exponential order $c = 0$ for $t > 0$. Indeed, it follows that all bounded functions are necessarily of exponential order.

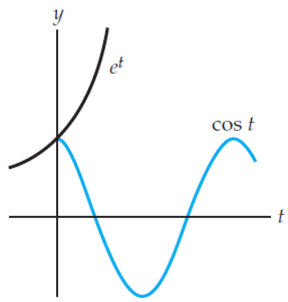


Figure 6. 13 $f(t) = cos\, t$ is of exponential order $c = 0$.

**Theorem 6.22 Sufficient Conditions for Existence**
Suppose $f$ is piecewise continuous on $[0, \infty)$ and of exponential order $c$ for $t > T$. Then $\mathcal{L}\{f(t)\}$ exists for $\mathrm{Re}(s) > c$.

**Theorem 6.23 Analyticity of the Laplace Transform**
Suppose $f$ is piecewise continuous on $[0, \infty)$ and of exponential order $c$ for $t > 0$. Then the Laplace transform of $f$,

$$F(s) = \int_0^\infty e^{-st} f(t)dt$$

is an analytic function in the right half-plane defined by $\mathrm{Re}(s) > c$.

**The Inverse Laplace Transform**

Although Theorem 6.23 indicates that the complex function $F(s)$ is analytic to the right of the line $x = c$ in the complex plane, $F(s)$ will, in general, have singularities to the left of that line.

**Theorem 6.24 Inverse Laplace Transform**
If $f$ and $f'$ are piecewise continuous on $[0, \infty)$ and $f$ is of exponential order $c$ for $t \geq 0$, and $F(s)$ is a Laplace transform, then the **inverse Laplace transform** $\mathcal{L}^{-1}\{F(s)\}$ is

$$f(t) = \mathcal{L}^{-1}\{F(s)\} = \frac{1}{2\pi i} \lim_{R\to\infty} \int_{\gamma - iR}^{\gamma + iR} e^{st} F(s) ds,$$

where $\gamma > c$.

The limit above, which defines a principal value of the integral, is usually written as

$$f(t) = \mathcal{L}^{-1}\{F(s)\} = \frac{1}{2\pi i} \int_{\gamma - i\infty}^{\gamma + i\infty} e^{st} F(s) ds,$$

where the limits of integration indicate that the integration is along the infinitely long vertical-line contour $\mathrm{Re}(s) = x = \gamma$. Here $\gamma$ is a positive real constant greater than $c$ and greater than all the real parts of the singularities in the left half-plane. The integral above is called a **Bromwich contour integral**. We can see that the kernel of the inverse transform is $H(\alpha, t) = e^{st}/2\pi i$.

The fact that $F(s)$ has singularities $s_1, s_2, \cdots, s_n$ to the left of the line $x = \gamma$ makes it possible for us to evaluate $f(t) = \mathcal{L}^{-1}\{F(s)\}$ by using an appropriate closed contour encircling the singularities. A closed contour $C$ that is commonly used consists of a semicircle $C_R$ of radius $R$ centered at $(\gamma, 0)$ and a vertical line segment $L_R$ parallel to the $y$-axis passing through the point $(\gamma, 0)$ and extending from $y = \gamma - iR$ to $y = \gamma + iR$. See Figure 6.14. We take the radius $R$ of the semicircle to be larger than the largest number in set of moduli of the singularities $\{|s_1|, |s_2|, \cdots, |s_n|\}$, that is, large enough so that all the singularities lie within the semicircular region. With the contour $C$ chosen in this manner, $f(t) = \mathcal{L}^{-1}\{F(s)\}$ can often be evaluated using Cauchy's residue theorem. If we allow the radius $R$ of the semicircle to approach $\infty$, the vertical part of the contour approaches the infinite vertical line that is the contour in Bromwich contour integral above.

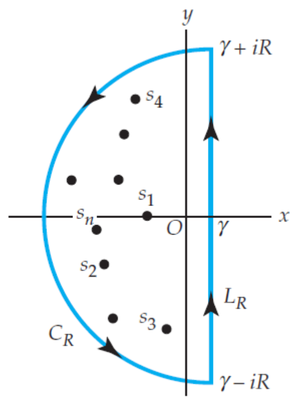


Figure 6. 14 Possible contour that could be used to evaluate $f(t) = \mathcal{L}^{-1}\{F(s)\}$

**Theorem 6.25 Inverse Laplace Transform**

Suppose $F(s)$ is a Laplace transform that has a finite number of poles $s_1, s_2, \cdots, s_n$ to the left of the vertical line $\mathrm{Re}(s) = \gamma$ and that $C$ is the contour illustrated in Figure 6.14. If $sF(s)$ is bounded on $C_R$ as $R \to \infty$, then

$$\mathcal{L}^{-1}\{F(s)\} = \sum_{k=1}^{n} \mathrm{Res}(e^{st}F(s), s_k).$$

**EXAMPLE 2**: Inverse Laplace Transform

The Laplace transform utilizes only the values of a function $f(t)$ for $t > 0$, and so $f$ is often taken to be $0$ for $t < 0$. This is no major handicap because the functions we deal with in applications are for the most part defined only for $t > 0$. Although we shall not delve into details, the inversion integral can be derived from a result known as the Fourier integral formula. In that analysis it is shown that

$$\frac{1}{2\pi i}\int_{\gamma - i\infty}^{\gamma + i\infty} e^{st}F(s)ds = \begin{cases} f(t), & t > 0 \\ 0, & t < 0 \end{cases}.$$

**EXAMPLE 3**: Inverse Laplace Transform

In the study of the Laplace transform the **unit step function**,

$$\mathcal{U}(t-a) = \begin{cases} 1, & t \geq a \\ 0, & t < a \end{cases}$$

proves to be extremely useful when working with piecewise continuous functions.

**Fourier Transform**

Suppose now that $f(x)$ is a real function defined on the interval $(-\infty, \infty)$. Another important transform pair is the **Fourier transform**

$$\mathcal{F}\{f(x)\} = \int_{-\infty}^{\infty} f(x)e^{i\alpha x}dx = F(\alpha)$$

and the **inverse Fourier transform**

$$\mathcal{F}^{-1}\{F(\alpha)\} = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\alpha)e^{-i\alpha x}d\alpha = f(x).$$

we see that the kernel of the Fourier transform is $K(\alpha, x) = e^{i\alpha x}$, whereas the kernel of the inverse transform is $H(\alpha, x) = e^{-i\alpha x}/2\pi$. Above we assume that $\alpha$ is a real variable. Also, observe that in contrast to the Inverse Laplace Transform, the inverse transform is not a contour integral.

**EXAMPLE 4**: Fourier Transform

**EXAMPLE 5**: Inverse Fourier Transform