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Fibonacci Numbers

The Fibonacci sequence is defined as follows:

$$F_0 = 0, F_1 = 1, F_n = F_{n-1} + F_{n-2}$$

The first elements of the sequence (OEIS A000045) are:

$$0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, \ldots$$

Properties

Fibonacci numbers possess a lot of interesting properties. Here are a few of them:

- Cassini's identity:

$$F_{n-1} F_{n+1} - F_n^2 = (-1)^n$$

- The "addition" rule:

$$F_{n+k} = F_k F_{n+1} + F_{k-1} F_n$$

- Applying the previous identity to the case $k = n$, we get:

$$F_{2n} = F_n (F_{n+1} + F_{n-1})$$

- From this we can prove by induction that for any positive integer k, F_{nk} is multiple of F_n.
- The inverse is also true: if F_m is multiple of F_n, then m is multiple of n.
- GCD identity:

$$GCD(F_m, F_n) = F_{GCD(m, n)}$$

- Fibonacci numbers are the worst possible inputs for Euclidean algorithm (see Lame's theorem in Euclidean algorithm)

Fibonacci Coding

We can use the sequence to encode positive integers into binary code words. According to Zeckendorf's theorem, any natural number n can be uniquely represented as a sum of Fibonacci numbers:

$$N = F_{k_1} + F_{k_2} + \ldots + F_{k_r}$$

such that $k_1 \geq k_2 + 2,\ k_2 \geq k_3 + 2,\ \ \ldots,\ k_r \geq 2$ (i.e.: the representation cannot use two consecutive Fibonacci numbers).

It follows that any number can be uniquely encoded in the Fibonacci coding. And we can describe this representation with binary codes $d_0 d_1 d_2 \dots d_s 1$, where d_i is 1 if F_{i+2} is used in the representation. The code will be appended by a 1 to indicate the end of the code word. Notice that this is the only occurrence where two consecutive 1-bits appear.

$$\begin{eqnarray}
1 &=& 1 &=& F_2 &=& (11)_F \\
2 &=& 2 &=& F_3 &=& (011)_F \\
6 &=& 5 + 1 &=& F_5 + F_2 &=& (10011)_F \\
8 &=& 8 &=& F_6 &=& (000011)_F \\
9 &=& 8 + 1 &=& F_6 + F_2 &=& (100011)_F \\
19 &=& 13 + 5 + 1 &=& F_7 + F_5 + F_2 &=& (1001011)_F
\end{eqnarray}$$

The encoding of an integer n can be done with a simple greedy algorithm:

1. Iterate through the Fibonacci numbers from the largest to the smallest until you find one less than or equal to n.
2. Suppose this number was F_i. Subtract F_i from n and put a 1 in the $i-2$ position of the code word (indexing from 0 from the leftmost to the rightmost bit).
3. Repeat until there is no remainder.
4. Add a final 1 to the codeword to indicate its end.

To decode a code word, first remove the final 1. Then, if the i-th bit is set (indexing from 0 from the leftmost to the rightmost bit), sum F_{i+2} to the number.

Formulas for the n^{th} Fibonacci number

Closed-form expression

$$\begin{pmatrix} F_{2k+1} & F_{2k} \\ F_{2k} & F_{2k-1} \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^{2k} = \begin{pmatrix} F_{k+1} & F_{k} \\ F_{k} & F_{k-1} \end{pmatrix}^{2}$$

we can find these simpler equations:

$$\begin{aligned} F_{2k+1} &= F_{k+1}^2 + F_k^2 \\ F_{2k} &= F_k(F_{k+1} + F_{k-1}) = F_k(2F_{k+1} - F_k) \end{aligned}.$$

Thus using above two equations Fibonacci numbers can be calculated easily by the following code:

```
pair<int, int> fib (int n) {
    if (n == 0)
        return {0, 1};

    auto p = fib(n >> 1);
    int c = p.first * (2 * p.second - p.first);
    int d = p.first * p.first + p.second * p.second;
    if (n & 1)
        return {d, c + d};
    else
        return {c, d};
}
```

The above code returns F_n and F_{n+1} as a pair.

Periodicity modulo p

Consider the Fibonacci sequence modulo p. We will prove the sequence is periodic.

Let us prove this by contradiction. Consider the first $p^2 + 1$ pairs of Fibonacci numbers taken modulo p:

$$(F_0,\ F_1),\ (F_1,\ F_2),\ \ldots,\ (F_{p^2},\ F_{p^2+1})$$

There can only be p different remainders modulo p, and at most p^2 different pairs of remainders, so there are at least two identical pairs among them. This is sufficient to prove the sequence is periodic, as a Fibonacci number is only determined by its two predecessors. Hence if two pairs of consecutive numbers repeat, that would also mean the numbers after the pair will repeat in the same fashion.

We now choose two pairs of identical remainders with the smallest indices in the sequence. Let the pairs be $(F_a,\ F_{a+1})$ and $(F_b,\ F_{b+1})$. We will prove that $a = 0$. If this was false, there would be two previous pairs $(F_{a-1},\ F_a)$ and $(F_{b-1},\ F_b)$, which, by the property of Fibonacci numbers, would also be equal. However, this contradicts the fact that we had chosen pairs with the smallest indices, completing our proof that there is no pre-period (i.e the numbers are periodic starting from F_0).