Categorical Künneth formulas for analytic stacks

Youshua Kesting

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Abstract

In [BFN10] the authors define a class of derived stacks, called "perfect stacks" and show that for this class the categories of quasi-coherent sheaves satisfy a categorical Künneth formula. Motivated to extend their results to the theory of analytic stacks as developed by Clausen-Scholze, we investigate categorical Künneth formulas for general 6-functor formalisms. As applications we show a general Tannakian reconstruction result for analytic stacks and, following recent work of Anschütz, Le Bras and Mann [ABM24], show a p-adic version of Drinfelds lemma for certain stacks that appear conjecturally in a categorical p-adic Langlands program.

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1 Introduction

The motivating problem of this article is to understand for which class of analytic stacks the categorical Künneth formula holds. Let us briefly recall the classical Künneth formula in algebraic geometry. Let $X \to \mathrm{Spec}(k)$, $W \to \mathrm{Spec}(k)$ be two quasi-compact and separated schemes over a field k, denote by $D_{qcoh}(X)$ the derived category of X with quasi-coherent cohomology sheaves and by $R\Gamma(X, -) : D_{qcoh}(X) \to D_{qcoh}(k)$ the derived functor of global sections. Then the Künneth formula states that the canonical morphism

$$R\Gamma(X, \mathcal{O}_X) \otimes_k R\Gamma(W, \mathcal{O}_W) \to R\Gamma(X \times_{\mathrm{Spec}(k)} W, \mathcal{O}_{X \times_{\mathrm{Spec}(k)} W})$$

is an equivalence. In fact, this equivalence can even be observed on the level of the derived categories of X and W as has been investigated by Ben-Zvi, Francis and Nadler in the context of derived algebraic geometry. For $X \to Z$, $W \to Z$ any maps of perfect derived stacks and denoting by $\mathrm{QCoh}(X)$ the derived ∞-category of quasi coherent sheaves on X, they show (cf. [BFN10, Theorem 4.14]) that one obtains a categorical Künneth formula

$$\mathrm{QCoh}(X) \otimes_{\mathrm{QCoh}(Z)} \mathrm{QCoh}(W) \cong \mathrm{QCoh}(X \times_Z W). \tag{1.0.1}$$

Here the tensor product on the left hand side is the relative Lurie-tensor product in Pr^{L}, the ∞-category of presentable ∞-categories. Let us recall that a derived stack X is called perfect if its diagonal morphism $\Delta_X : X \to X \times X$ is affine, the ∞-category of quasi-coherent sheaves $\mathrm{QCoh}(X)$ is compactly generated and compact and dualisable objects coincide in $\mathrm{QCoh}(X)$ (cf. [BFN10, Proposition 3.9]). Using the formalism of higher traces, they show furthermore that the existence of such categorical Künneth formulas implies many classical statements such as versions of the Atiyah-Bott-Lefschetz and the Riemann-Roch theorem (cf. [BN13, Theorem 1.4]).

The key idea of [BFN10] to prove Equation (1.0.1) is to use that it is true for affine derived schemes, where it reduces to an abstract statement about ∞-categories of modules and to use descent of $\mathrm{QCoh}(-)$ for the general case. One difficulty arises due to the absence of a well-defined 6-functor formalism for quasi-coherent sheaves, which they circumvent by showing proper base change and the projection formula for perfect stacks directly (cf. [BFN10, Section 3.2]).

The later problem has been resolved by the recent theory of condensed mathematics and the theory of analytic stacks by Clausen-Scholze (cf. [CS22],[AS]) which provides a vast framework to study both topology and algebraic, complex and p-adic geometry. In particular, they construct a 6-functor formalism $D_{qc}(-)$ on the ∞-category AnSt of analytic stacks which satisfies a strong descent condition which they call !-descent. Working in this generality, it becomes unnatural to impose the compact generatedness of $D_{qc}(-)$. Moreover it become apparent in recent years that there are many interesting ∞-categories such as the category of nuclear modules in condensed mathematics or categories of sheaves on locally compact Hausdorff spaces which are not compactly generated but which belong to a bigger class of presentable ∞-categories called dualisable categories. The aim of this article is to show that the existence of a well-defined 6-functor formalism with strong descent properties and the passage from compact-generated categories to dualisable categories yield strong generalizations of the results in [BFN10].

More precisely, we can describe our problem as follows.

Problem 1.1. For which morphisms $X \to Z$, $W \to Z$ of analytic stacks is the natural morphism

$$D_{qc}(X) \otimes_{D_{qc}(Z)} D_{qc}(W) \to D_{qc}(X \times_Z W) \tag{1.0.2}$$

an equivalence?

Many of our arguments can be formulated for arbitrary 6-functor formalisms. Motivated by finding a replacement of the notion of "perfect stacks" for general 3- (or 6-)functor formalisms, we adopt a maximalist approach and introduce the following definition.

Definition 1.1 (Definition 3.3). Let $D : Corr(C, E) \to \mathrm{Pr}^L$ be a 6-functor formalism and $f : X \to Z$ be a map in E. We call f Künneth if for all morphisms $W \to Z$ in C the natural morphism

$$D(X) \otimes_{D(Z)} D(W) \to D(X \times_Z W) , \, A \otimes B \mapsto p_X^*(A) \otimes p_W^*(B)$$

is an equivalence, with $p_X : X \times_Z W \to X$, $p_W : X \times_Z W \to W$ the natural projection morphisms. We say D satisfies Künneth for a class of morphisms $P \subset E$ if any $f \in P$ is Künneth. We say D satisfies Künneth if it satisfies Künneth for $P = E$.

Note that for a general 6-functor formalism there may be no non-trivial Künneth morphisms at all. For our applications to analytic stacks, we will be only concerned with the case where we start with a 3-functor formalism $D_0 : Corr(C_0, E_0) \to \mathrm{Pr}^L$ which already satisfies Künneth and consider its Kan extension to a 3-functor formalism $D : Corr(C, E) \to \mathrm{Pr}^L$ on the category $C := \mathrm{Sh}_{D_0}(C_0)$ of sheaves of anima on C_0 with respect to the D_0-topology, with E an appropriate class of morphisms containing E_0 (see [HM24, Theorem 3.4.11] for the precise conditions).
As we show, the class of Künneth morphisms enjoys many permanence properties (see Lemma 3.4, Lemma 3.5, Proposition 3.21,Proposition 3.22, Definition 3.11,Proposition 3.12):

Proposition 1.2. Let $D : Corr(C, E) \to \mathrm{Pr}^L$ be a 6-functor formalism.

1. (composition) If $f : Y \to Z$ and $g : X \to Y$ are Künneth, then $f \circ g : X \to Z$ is Künneth.

2. (base change) Let $W \to Z$ be any morphism. If $f : Y \to Z$ is Künneth, then the base change $f' : Y' := Y \times_Z W \to W$ is Künneth.

3. (open immersions) If $D : Corr(C, E) \to \mathrm{Pr}_{st}^L$ is stable and $g : X \to Y$ in E is a D-open immersion, then f is Künneth.

4. (locality on source) Let $g : X \to Y$ be any morphism in E. Let $(f_i : X_i \to X)_{i \in I}$ be a universal !-cover. If $g \circ f_{[n],i_\bullet} : X_{i_0} \times_X ... \times_X X_{i_n} \to Y$ is Künneth for all $([n], i_\bullet) \in \Delta_I$, then g is Künneth.

5. (locality on target) $f : X \to Z$ be any morphism in E and $(g_i : Z_i \to Z)_{i \in I}$ be a universal !-cover. Denote by $f_{([n],i_\bullet)} : X_{([n],i_\bullet)} := X \times_Z Z_{[n],i_\bullet} \to Z_{[n],i_\bullet}$ the base change of f by $g_{[n],i_\bullet} : Z_{[n],i_\bullet} \to Z$. If $f_{[n],i_\bullet}$ is Künneth for all $([n], i_\bullet) \in \Delta_I$, then f is Künneth.

Using these properties we show our main theorem (Theorem 3.26, see Proposition 3.23 for the notation).

Theorem 1.3. Let $D : Corr(C, E) \to \mathrm{Pr}^L$ be a sheafy 6-functor formalism on a subcanonical site C. Let $S \in C$ and assume that $D_S : ((C_E)_{/S})^{op} \to \mathrm{Mod}_{D(S)}(\mathrm{Pr}^L)$ is symmetric monoidal. Then the extension $\tilde{D}_S : ((\mathrm{Shv}(C)_{\tilde{E}})_{/S})^{op} \to \mathrm{Mod}_{D(S)}(\mathrm{Pr}^L)$ is symmetric monoidal. The same is true if $S \in \mathrm{Shv}(C)$ has a representable diagonal Δ_S and a !-cover $U \to S$ with $U \in C$ and $D : Corr(C, E) \to \mathrm{Pr}^L$ is Künneth.

To formulate our first application, we introduce the following terminology (cf. Definition 3.31).

Definition 1.4. Let $D : Corr(C, E) \to CAlg(Pr^L)$ be a 6-functor formalism and $f : X \to S$ in C. We call f Tannakian if the functor

$$\text{Hom}_{C_{/S}}(Y, X) \cong \text{Fun}_{D(S)}^{L, \otimes}(D(X), D(Y))$$

is an equivalence for all $Y \in C_{/S}$. We call D Tannakian if all morphisms in C are Tannakian.

As a consequence of Theorem 1.3, we prove the following Tannakian reconstruction result:

Theorem 1.5. Theorem 3.33 Let C be a subcanonical site and $D : Corr(C, E) \to CAlg(Pr^L)$ be a Tannakian sheafy 6-functor formalism satisfying Künneth. Consider its extension $D : Corr(\text{Shv}(C), \tilde{E}) \to CAlg(Pr^L)$. Let $f : X \to S$ be a morphism in $\tilde{E}$ and assume that there is a !-cover $g : S' \to S$ with $S' \in C$ and $g \in E_0$. Then f is Tannakian.

We apply these results to the 6-functor formalism $D : Corr(\text{AnRing}, E) \to Pr^L$ on the category AnRing of analytic rings with E the class of !-able maps (see Section 4 for a construction and the definition of (D, AnRing, E)). We show that $D : Corr(\text{AnRing}, E) \to Pr^L$ is Künneth and Tannakian (Proposition 4.11) and obtain thus a partial answer to Problem 1.1, using Theorem 1.3.

Corollary 1.5.1. Let $S \in \text{AnSt} \coloneqq \text{Shv}(\text{AnRing}))$ be an analytic stack with representable diagonal Δ_S, and assume S admits a !-cover $U \to S$. Then any morphism of analytic stacks $X \to S$ in $\tilde{E}$ is Künneth and Tannakian.

Our interest in categorical Künneth formulas stems from finding an analogue of Drinfeld's lemma in the context of the p-adic local Langlands correspondence. Let us briefly recall the Drinfeld lemma in the ℓ-adic setting (cf. [FS21, Chapter IV.7]). Consider $E/\mathbb{Q}_p$ a finite extension, $k = \bar{\mathbb{F}}_q$ an algebraic closure of its residue field and $\Lambda = \bar{\mathbb{Q}}_\ell$. For Y a small v-stack, consider the full subcategory $D_{dlb}(Y, \Lambda) \subset D_{et}(Y, \Lambda)$ of dualisable objects in the category of étale sheaves with coefficients in Λ on Y. There is a small v-stack called Div_k^1 whose significance comes from the fact that it "geometrises" smooth Λ-representations of the Weil group W_E of E (cf. [FS21, Theorem V.1.1]). Let W be any small v-stack, the simplest instance of the Drinfeld lemma can then be phrased as an equivalence of categories

$$D_{dlb}(\text{Div}_k^1, \Lambda) \otimes_{D_{dlb}(\text{Spd}(k), \Lambda)} D_{dlb}(W, \Lambda) \cong D_{dlb}(\text{Div}_k^1 \times_{\text{Spd}(k)} W, \Lambda)$$

(cf. [FS21, Proposition 4.7.3]). Note that the tensor product on the left hand side is the one induced from Pr^L on the category of idempotent complete stable ∞-categories.

In the p-adic setting there is currently no conjecture along the lines of [FS21, Conjecture I.10.2]. Motivated by understanding the properties of pro-étale $\mathbb{Q}_p$-cohomology on rigid spaces, Anschütz, Le Bras and Mann [ABM24] constructed recently a 6-functor formalism $S \to D_{(0,\infty)}(S) \in Pr^L$ on the category vStack of small v-stacks, a variant of which might be a suitable category of coefficients for a p-adic Langlands correspondence. To support this claim, they introduce a small v-stack Div_E^1 and show that it geometrises (φ, Γ)-modules on the Robba ring $\mathcal{R}_E$, where $\Gamma = \text{Gal}(E^{\text{cycl}}/E)$. More precisely they show ([ABM24, Proposition 6.3.15]) that the category of (φ, Γ)-modules on the $\mathcal{R}_E$ is contained fully faithfully in the category of dualisable objects in $D_{(0,\infty)}(\text{Div}_E^1)$. A natural question from the context of a p-adic Langlands correspondence is thus whether the Drinfeld lemma holds in this setting, that is whether for W any small v-stack we have an equivalence

$$D_{(0,\infty)}(\text{Div}_E^1) \otimes_{D_{(0,\infty)}(\text{Spd}(\mathbb{F}_q))} D_{(0,\infty)}(W) \cong D_{(0,\infty)}(\text{Div}_E^1 \times_{\text{Spd}(\mathbb{F}_q)} W).$$

On $S = \mathrm{Spa}(R, R^+)$ an affinoid perfectoid space which admits a morphism of finite trg.dim to a totally disconnected perfectoid space, the category $D_{(0,\infty)}(S) \cong D_{\hat{\square}}(Y_{(0,\infty),S})$ is given by (cf. [ABM24, Theorem 1.2.1]) the category of (modified) solid quasi-coherent sheaves on the analytic adic space

$$Y_{(0,\infty),S} := \mathrm{Spa}(W(R^+))\backslash V([\pi]p).$$

Here $W(-)$ denotes the p-typical Witt vectors and $\pi \in R^+$ a pseudo-uniformiser.

In order to avoid dealing with modified solid sheaves, we will consider the analytic stack $\mathrm{Div}_E^{1,\mathrm{ct}} := Y_{(0,\infty),\mathrm{Div}_E^1}$ and the category $D(Y_{(0,\infty),\mathrm{Div}_E^1})$ instead of $D_{(0,\infty)}(\mathrm{Div}_E^1)$. By v-descent one can evaluate $D_{(0,\infty)}(\mathrm{Div}_E^1)$ by taking a v-cover $S_\bullet \to \mathrm{Div}_E^1$, giving an equivalence $D_{(0,\infty)}(\mathrm{Div}_E^1) \cong \lim_{[n]\in\Delta} D_{\hat{\square}}(Y_{(0,\infty),S_n})$. This identifies, up to the modification of the solid structure, with the descent datum given by the cover $Y_{(0,\infty),),S_\bullet} \to \mathrm{Div}_E^{1,\mathrm{ct}}$ for $D_{qc}(-)$. One technical reason we prefer working with analytic stacks directly is the existence of !-descent for $D_{qc}(-)$. Another advantage of working directly on the level of analytic stacks is that one can "geometrise" various kinds of p-adic representations within a single category. For example, for G a p-adic Lie group, one has the analytic stacks G, G^{la} given by sending a compact open $U \subset G$ to the algebras $C(U, \mathbb{Q}_p)$, $C^{\mathrm{la}}(U, \mathbb{Q}_p)$ of continuous functions or locally analytic respectively. Consequently, the categories $D_{qc}(\mathrm{AnSpec}(\mathbb{Q}_p)/G^?)$, $? \in \{\mathrm{ct}, \mathrm{la}\}$ correspond to certain continuous (respectively locally analytic) $\mathbb{Q}_p$-representations of G. We will use this freedom to consider also a locally analytic version of the stack $\mathrm{Div}_E^{1,\mathrm{ct}}$ which we denote by $\mathrm{Div}_E^{1,\mathrm{la}}$ (cf. Corollary 4.12.2). By Corollary 1.5.1 the categorical Künneth formula holds for all of them.

Corollary 1.5.2. (p-adic Drinfeld lemma) Let $? \in \{\mathrm{ct}, \mathrm{la}\}$. Then the morphism $\mathrm{Div}_{\mathbb{Q}_p}^{1,?} \to \mathrm{AnSpec}(\mathbb{Q}_p)$ is Künneth. In other words, for any analytic stack $W \to \mathrm{AnSpec}(\mathbb{Q}_p)$ we have equivalences

$$D_{qc}(\mathrm{Div}_{\mathbb{Q}_p}^{1,?}) \otimes_{D_{qc}(\mathbb{Q}_p)} D_{qc}(W) \cong D_{qc}(\mathrm{Div}_{\mathbb{Q}_p}^{1,?} \times_{\mathrm{AnSpec}(\mathbb{Q}_p)} W).$$

The significance of Drinfeld's lemma in the ℓ-adic setting of Fargues-Scholze lies in the construction of the spectral action (cf. [FS21, chapter VI, X]). We hope that the above theorem will be useful to construct a spectral action in a p-adic (locally analytic) setting.

After this preprint was in its final form, Montagnani and Pavia informed us that they also independently obtained similar results to ours following similar ideas.

Outline

In Section 2 we will collect some facts on presentable categories and the categories of modules over a presentably symmetric monoidal category. We will be brief and introduce just enough notation and definitions to discuss the characterisation of dualisable objects in $\mathrm{Mod}_E(\mathrm{Pr}^{\mathrm{L}})$ for $E \in \mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L}})$ a commutative algebra in Pr^{L} due to Ramzi (cf.[Ram24]). We refer to [Ram24, Chapter 1] for a more detailed discussion.

The main results of this paper are contained in Section 3. In **??** and Section 3.2 we introduce the notion of Künneth morphisms for arbitrary lax symmetric monoidal functors $D : C^{op} \to \mathrm{Pr}^{\mathrm{L}}$ for a geometric setup (C, E) and show first closure properties of Künneth maps. In Section 3.3 we recall the well known fact that for a 6-functor formalism which satisfies Künneth, any morphism $X \to Y$ makes $D(X) \in \mathrm{Mod}_{D(Y)}(\mathrm{Pr}^{\mathrm{L}})$ into a dualisable module. In Section 3.4 we prove properties of Künneth morphisms under descent. Finally, in Section 4 and Section 4.3 we recall the 6-functor

formalism of analytic stacks and give examples of Künneth morphisms in this context. Our main reference for 6-functor formalisms is [HM24].

Acknowledgments

1.1 Notations and conventions

- We denote by Pr^L the (very large) ∞-category with objects presentable ∞-categories and morphisms colimit preserving functors. $\widehat{\mathrm{Cat}_\infty}$ denotes the ∞-category of large ∞-categories. We consider Pr^L as a symmetric monoidal ∞-category with respect to the Lurie-tensor product.

- We denote by Ani the ∞-category of spaces/anima and by Sp the ∞-category of spectra.

- For C a symmetric monoidal ∞-category we denote by $\mathrm{CAlg}(C)$ the ∞-category of commutative algebra objects in C and by $\mathrm{cCAlg}(C) \coloneqq \mathrm{CAlg}(C^{op})^{op}$ the ∞-category of cocommutative coalgebra objects in C. For $A \in \mathrm{CAlg}(C), B \in \mathrm{cCAlg}(C)$ we denote by $\mathrm{Mod}_A(C)$ the ∞-category of A-modules and by $\mathrm{coMod}_B(C)$ the ∞-category of B-comodules in C. We denote by $\mathrm{Pr}_{st}^L \coloneqq \mathrm{Mod}_{\mathrm{Sp}}(\mathrm{Pr}^L)$ the ∞-category of stable presentable ∞-categories.

- For $M, N \in \mathrm{Pr}^L$ we denote by $\mathrm{Fun}^L(M, N)$ the ∞-category of functors in Pr^L. Similarly, for $E \in \mathrm{CAlg}(\mathrm{Pr}^L)$, $M, N \in \mathrm{Mod}_E(\mathrm{Pr}^L)$ we denote by $\mathrm{Fun}_E^L(M, N)$ the ∞-category of functors in $\mathrm{Mod}_E(\mathrm{Pr}^L)$. We denote the symmetric monoidal tensor product on $\mathrm{Mod}_E(\mathrm{Pr}^L)$ induced by the Lurie tensor product on Pr^L by $(-) \otimes_E (-)$.

- For $E \in \mathrm{CAlg}(\mathrm{Pr}^L)$ we denote by $\mathrm{Cat}_{\infty E}$ the category of E-enriched ∞-categories. For $M \in \mathrm{Cat}_{\infty E}$ an E-enriched category, we denote by $P_E(M)$ the category of E-enriched presheaves [HM24, Definition C.2.8].

- By "category" we will from now on mean ∞-category. We will refer to categories in which every inner horn has a unique filler as 1-categories.

2 Dualisable modules

In this chapter we will recall the notion of presentable dualisable modules. Following [Ram24, Chapter 1], we introduce the notion of atomically generated categories in order to state an alternative characterisation of dualisability due to Ramzi [Ram24, Proposition 1.40]. We will barely give any proofs and refer to [Ram24, Chapter 1] for a more thourough treatment.

2.1 Dualisable presentable categories

Let us recall the notion of a dualisable object in a symmetric monoidal category.

Definition 2.1. Let C be a symmetric monoidal category. An object $P \in C$ is called dualisable if there are an object $P^\vee$, called the dual of P and morphisms

$$ev : P \otimes P^\vee \to 1, \; co : 1 \to P \otimes P^\vee$$

called the evaluation and coevaluation maps such there are homotopy coherent diagrams

$$
\begin{array}{ccc}
P \xrightarrow{co \otimes id_P} P \otimes P^\vee \otimes P & \qquad & P^\vee \xrightarrow{id_{P^\vee} \otimes co} P^\vee \otimes P \otimes P^\vee \\
 \searrow_{id_P} \quad \downarrow^{id_P \otimes ev} & & \searrow_{id_{P^\vee}} \quad \downarrow^{ev \otimes id_{P^\vee}} \\
 \quad P & & \quad P^\vee
\end{array}
$$

We denote the full subcategory of C spanned by the dualisable objects by $C^{dlb} \subset C$.

In the case of $C = \mathrm{Pr}^L_{st}$, there is the following description of dualisable objects due to Lurie.

Theorem 2.2. [Lur18, Proposition D.7.3.1] Let $C \in \mathrm{Pr}^L_{st}$ be a stable presentable category. Then the following assertions are equivalent:

1. C is dualisable in Pr^L_{st}.

2. C is a retract of a compactly generated stable category.

3. There is an adjunction in Pr^L_{st}

$$C \underset{G}{\overset{F}{\rightleftarrows}} D$$

   with D a compactly generated stable category such that F is fully faithful and G commutes with all colimits.

2.2 Dualisable modules in presentable categories

For the rest of this chapter let $E \in \mathrm{CAlg}(\mathrm{Pr}^L)$ be a commutative algebra object in Pr^L. In the following subsection, we will be studying the category $\mathrm{Mod}_E(\mathrm{Pr}^L)$ and dualisable objects in it. We will sometimes refer to an object of $\mathrm{Mod}_E(\mathrm{Pr}^L)$ as a presentable E-module.

As a first step to understand dualisable objects in $\mathrm{Mod}_E(\mathrm{Pr}^L)$, we want to obtain an analogue of the characterisation given in Theorem 2.2, generalising from the case $E = \mathrm{Sp}$ to a general $E \in \mathrm{CAlg}(\mathrm{Pr}^L)$. To achieve that we need to generalise the notion of compactly generated categories to incorporate this additional module structure.

Definition 2.3. [Ram24, Definition 1.9] Let $f : M \to N$ be a functor in $\mathrm{Mod}_E(\mathrm{Pr}^L)$ with right adjoint f_R. We call f an internal left adjoint if f_R commutes with colimits and is E-linear, that is if the canonical projection map

$$x \otimes f_R(n) \to f_R(x \otimes n)$$

is an equivalence for all $n \in N$ and all $x \in E$.

We denote by $\mathrm{Mod}(E)^{dlb}$ the (non full) subcategory of $\mathrm{Mod}_E(\mathrm{Pr}^L)$ spanned by dualisable objects and 1-morphisms given by internal left adjoints.

Example 2.1. [Ram24, Example 1.16] Let $C \in \mathrm{Pr}^L$ and $A \to B$ be a map in $\mathrm{CAlg}(C)$, then the functor given by base change of algebras $\mathrm{Mod}_A(C) \to \mathrm{Mod}_B(C)$ is an internal left adjoint.

The following notion generalises the notion of compact objects in Pr^L.

Definition 2.4. [Ram24, Definition 1.22] Let $M \in \mathrm{Mod}_E(\mathrm{Pr}^L)$. An object $x \in M$ is called E-atomic (or just atomic, if E is clear from the context) if the functor $x \otimes (-) : E \to M$ is an internal left adjoint.

We see that if $E = \mathrm{Sp}$ is the category of spectra the Sp-atomic objects in $\mathrm{Mod}_{\mathrm{Sp}}(\mathrm{Pr}^L) \cong \mathrm{Pr}^L_{st}$ are exactly the compact objects. Following [Ram24], we now introduce the following generalization of compactly generated categories.

Definition 2.5. [Ram24, Definition 1.27] An E-module $M \in \mathrm{Mod}_E(\mathrm{Pr}^L)$ is called E-atomically generated if the smallest full sub-E-module of M closed under colimits and containing the atomics of M is M itself.

In the following we will make use of the notion of E-enriched categories, our main reference is [Hei23] and [HM24, Appendix C]. For $M \in \mathrm{Cat}_{\infty E}$ an E-enriched category, we denote by $\widehat{\mathrm{Cat}_{\infty E}}$ the category of large E-enriched categories. By [Hei23, Theorem 1.2] there is a natural functor

$$\mu : \mathrm{Mod}_E(\mathrm{Pr}^L) \to \widehat{\mathrm{Cat}_{\infty E}}$$

which is fully faithful on hom categories.

Lemma 2.6. [Ram24, Observation 1.28] An E-module $M \in \mathrm{Mod}_E(\mathrm{Pr}^L)$ is E-atomically generated if and only if it is equivalent to a category of E-enriched presheafs $P_E(M_0)$ for M_0 some E-enriched category.

E-atomically generated categories are thus generalisations of compactly generated categories. One crucial step in the proof of Theorem 2.2 is to show that compactly generated categories are dualisable in Pr^L_{st}. The following Lemma due to Berman [Ber20, Theorem 1.7] shows that the analogous assertion is true for E-atomically generated modules in $\mathrm{Mod}_E(\mathrm{Pr}^L)$.

Lemma 2.7. [Ram24, Proposition 1.40] Any E-atomically generated module $M \in \mathrm{Mod}_E(\mathrm{Pr}^L)$ is dualisable in $\mathrm{Mod}_E(\mathrm{Pr}^L)$.

We now have all the necessary concepts to obtain a characterisation of dualisable objects in $\mathrm{Mod}_E(\mathrm{Pr}^L)$.

Theorem 2.8. [Ram24, Theorem 1.49] Let $E \in \mathrm{CAlg}(\mathrm{Pr}^L)$ and $C \in \mathrm{Mod}_E(\mathrm{Pr}^L)$ be a presentable E-module. Then the following assertions are equivalent:

1. C is dualisable in $\mathrm{Mod}_E(\mathrm{Pr}^{\mathrm{L}})$.

2. C is a retract of an E-atomically generated category.

3. There is an internally left adjoint fully faithful embedding $j : C \hookrightarrow D$ for some E-atomically generated category D.

2.3   Extension and restriction of modules

We end this chapter with a few lemmas on the behaviour of dualisable modules under extensions and restrictions of modules.

Lemma 2.9. Any morphism $g : V \to E$ in $\mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L}})$ induces a functor $G_L \coloneqq E \otimes_V (-) :$ $\mathrm{Mod}_V(\mathrm{Pr}^{\mathrm{L}}) \to \mathrm{Mod}_E(\mathrm{Pr}^{\mathrm{L}})$ which is symmetric monoidal and commutes with small colimits.

Proof. The existence and symmetric monoidality of the functor G_L as a functor in $\widehat{\mathrm{Cat}_\infty}$ is [Lur17, Remark 4.5.3.2]. We note that the cocartesian fibration $\mathrm{Mod}(\mathrm{Pr}^{\mathrm{L}}) \to \mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L}}) \times N(Fin_*)$ is presentable, in the sense that its classifying functor $\mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L}}) \times N(Fin_*) \to \widehat{\mathrm{Cat}_\infty}$ factorises over $\mathrm{Pr}^{\mathrm{L}} \subset \widehat{\mathrm{Cat}_\infty}$. The fact that G_L commutes with small colimits follows since $\otimes : \mathrm{Pr}^{\mathrm{L}} \times \mathrm{Pr}^{\mathrm{L}} \to \mathrm{Pr}^{\mathrm{L}}$ commutes with small colimits in each variable separately. $\square$

Lemma 2.10. Let $g : V \to E$ be a morphism in $\mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L}})$. Assume E is dualisable in $\mathrm{Mod}_V(\mathrm{Pr}^{\mathrm{L}})$. Then $G_L : \mathrm{Mod}_V(\mathrm{Pr}^{\mathrm{L}}) \to \mathrm{Mod}_E(\mathrm{Pr}^{\mathrm{L}})$ commutes with limits.

Proof. Let $F : I \to \mathrm{Mod}_V(\mathrm{Pr}^{\mathrm{L}})$ be a limit diagram, then we have equivalences

$$
\begin{aligned}
E \otimes_V \lim_n F(n) &\cong \mathrm{Fun}_V^L(V, E \otimes_V \lim_n F(n)) \\
&\cong \mathrm{Fun}_V^L(E^\vee, \lim_n F(n)) \\
&\cong \lim_n \mathrm{Fun}_V^L(E^\vee, F(n)) \\
&\cong \lim_n E \otimes_V F(n).
\end{aligned}
$$

$\square$

Lemma 2.11. Let $g : V \to E$ be a morphism in $\mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L}})$. The right adjoint G_R of G_L preserves dualisable objects if and only if E is dualisable in $\mathrm{Mod}_V(\mathrm{Pr}^{\mathrm{L}})$. It reflects dualisable objects if E is dualisable as a $E \otimes_V E$-module.

Proof. The claim that G_R preserves dualisable objects if and only if E is dualisable in $\mathrm{Mod}_V(\mathrm{Pr}^{\mathrm{L}})$ follows by [Lur17, Proposition 4.6.4.4. (8)]. The second part follows by [Lur17, Proposition 4.6.4.12 (6)]. $\square$

3 Künneth formulas for stacks

We introduce the notion of Künneth morphisms for general 3-functor formalisms and discuss its relation to the dualisable presentable modules discussed in the previous chapter. For a 6-functor formalism $D : Corr(C, E) \to \mathrm{Pr}^{\mathrm{L}}$, a morphism $f : X \to Z$ in E will be called Künneth if a general categorical Künneth formula holds, by which we mean that for all $W \to Z$ in C the natural morphism

$$m_{X,W} : D(X) \otimes_{D(Z)} D(W) \to D(X \times_Z W)$$

is an equivalence. We show that Künneth morphisms are stable under composition and base change (Lemma 3.4, Lemma 3.5) and in certain cases, !-local on the source and target (Proposition 3.21, Proposition 3.22). Finally, we recall the 6-functor formalisms of analytic stacks and give examples of Künneth morphisms in this context, giving a generalisation of the discussion in [BFN10, Chapter 3-4] in the context of quasi-coherent sheaves on perfect stacks. We refer to [HM24, Chapter 1-4] for a reference on general 6-functor formalisms.

3.1 Künneth morphisms

A very powerful tool to construct 3-functor formalisms is the construction result of Heyer-Mann [HM24, Proposition 3.3.3]. We start by recalling the following definition.

Definition 3.1. [HM24, Definition 2.1.1, Remark 2.1.2] A geometric setup is a tuple (C, E) of a category C together with a homotopy class of edges E in C such that the following conditions are satisfied.

1. E contains all isomorphisms.

2. E is stable under compositions and under pullback along edges in C.

3. If $f : X \to Y$ lies in E, then so does its diagonal $\Delta_f : X \to X \times_Y X$.

For the construction result of Heyer-Mann, assume we are given a geometric setup (C, E), where C is a category with finite limits, and a lax symmetric monoidal functor

$$D : C^{op} \to \mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L}}), \ X \mapsto D(X)$$
$$(f : Y \to X) \mapsto D(f) := f^* : D(X) \to D(Y)$$

where we consider C with its cartesian monoidal structure. We will furthermore assume that the class E decomposes into a suitable class $P \subset E$ of "proper maps" and a class $I \subset E$ of "local isomorphisms, such that any $f \in E$ can be written as a composition of a "local isomorphism" and a "proper map". We now want to extend the functor D to a 3-functor formalism

$$D : Corr(C, E) \to \mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L}})$$

such that for any $f \in P$ the functor $f_!$ is a right adjoint of f^* and for any $f \in I$ the functor $f_!$ is a left adjoint to f^*. The construction theorem of Mann states that such an extension to a 3-functor formalism exists given a certain amount of data (cf. [HM24, Proposition 3.3.3] for the conditions).

In view of this construction theorem, we will begin by discussing categorical Künneth formulas quite abstractly. Let C be a category with finite limits and consider C with its cartesian monoidal structure. Let $D : C^{op} \to \mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L}})$ be a lax symmetric monoidal functor. Let us recall how to define the morphisms $m_{X,W}$ in this context. Let $X, W \in C$ be two objects and $p_X : X \times_Z W \to X$ and $p_W : X \times_Z W \to W$ the canonical projections. Then the functors p_X^* and p_W^* are $D(Z)$-linear and combine to a $D(Z)$-bilinear functor

$$D(X) \times D(W) \to D(X \times_Z W), \quad (A, B) \mapsto p_X^*(A) \otimes p_W^*(B).$$

By the universal property of the Lurie-tensor product, this functor extends to a $D(Z)$-linear functor which we denote by

$$m_{X,W} : D(X) \otimes_{D(Z)} D(W) \to D(X \times_Z W).$$

Lemma 3.2. Let (C, E) be a geometric setup, and consider C^{op} as a symmetric monoidal category with its coCartesian symmetric monoidal structure and $D : C^{op} \to \mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L}})$ be a lax symmetric monoidal functor. Let $X \to Z, W \to Z$ and $f : Y \to W$ be any maps in C. Then the following diagram commutes

$$
\begin{array}{ccc}
D(X) \otimes_{D(Z)} D(W) & \xrightarrow{id_{D(X)} \otimes f^*} & D(X) \otimes_{D(Z)} D(Y) \\
{\scriptstyle m_{X,W}} \downarrow & & \downarrow {\scriptstyle m_{X,Y}} \\
D(X \times_Z W) & \xrightarrow{(id_X \times f)^*} & D(X \times_Z Y)
\end{array}
$$

Proof. This follows from the lax monoidality of $D : C^{op} \to \mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L}})$. $\qquad\square$

We introduce the following definition.

Definition 3.3. Let (C, E) be a geometric setup, and consider C^{op} as a symmetric monoidal category with its coCartesian symmetric monoidal structure.

1. Let $D : C^{op} \to \mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L}})$ be a lax symmetric monoidal functor. Let $f : X \to Z$ be a map in E. We call f Künneth if for all objects $W \in C_{/Z}$ the natural morphism

$$m_{X,Y} : D(X) \otimes_{D(Z)} D(W) \to D(X \times_Z W)$$

   is an isomorphism. We say that D satisfies Künneth for a class $P \subset E$ if any morphism $f \in P$ is Künneth. We say D satisfies Künneth if it satisfies Künneth for $P = E$.

2. If $D : Corr(C, E) \to \mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L}})$ is a 3-functor formalism and $f : X \to Y$ in E, we call f Künneth if it is Künneth for the functor $C^{op} \to Corr(C, E) \to \mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L}})$.

3.2 Monoidal properties of Künneth morphisms

In this subsection we consider Künneth morphisms for $D : C^{op} \to \mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L}})$ a lax symmetric monoidal functor for (C, E) a geometric setup. We show certain closure properties of Künneth morphisms such as stability under composition and base change and give first examples of Künneth morphisms.

Lemma 3.4. Let $D : C^{op} \to \mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L}})$ be a lax symmetric monoidal functor, $f : X \to Y$, $g : Y \to Z$ two morphisms. If f and g are Künneth, then $g \circ f$ is Künneth.

Proof. Let $W \to Z$ be any morphism in C. Since f is Künneth and using the isomorphism $X \times_Z W \cong X \times_Y (Y \times_Z W)$ we obtain an equivalence

$$m_{X, Y \times_Z W} : D(X) \otimes_{D(Y)} D(Y \times_Z W) \cong D(X \times_Z W).$$

Using that g is Künneth, we obtain equivalences

$$D(X) \otimes_{D(Y)} D(Y \times_Z W) \cong D(X) \otimes_{D(Z)} D(W).$$

By functoriality of pullbacks and Lemma 3.2 we see that the resulting equivalence $D(X) \otimes_{D(Z)} D(W) \to D(X \times_Z W)$ is given by $m_{X,W}$ which proves that $g \circ f$ is Künneth. $\square$

Lemma 3.5. Let $D : C^{op} \to \mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L}})$ be a lax symmetric monoidal functor, $f : X \to Z$, $g : Y \to Z$ two morphisms. If f is Künneth, then the base change $f' : X' := X \times_Z Y \to Y$ of f along g is Künneth.

Proof. Let $W \to Y$ be an arbitrary morphism. We have a chain of equivalences

$$\begin{aligned} D(X') \otimes_{D(Y)} D(W) &\cong (D(X) \otimes_{D(Z)} D(Y)) \otimes_{D(Y)} D(W) \\ &\cong D(X) \otimes_{D(Z)} D(W) \\ &\cong D(X' \times_Y W) \end{aligned}$$

Here we used that f is Künneth in the first line, the natural isomorphism in the second line and the natural isomorphism $X' \times_Y W \cong X \times_Z W$ together with the fact that f is Künneth in the last line. $\square$

Lemma 3.6. Let $D : C^{op} \to \mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L}})$ be a lax symmetric monoidal functor, $f : X \to Y$, $g : Y \to Z$ two morphisms. Assume that g and the diagonal $\Delta_g : Y \to Y \times_Z Y$ are Künneth. If $g \circ f$ is Künneth, then f is Künneth.

Proof. We write f as the composition $X \stackrel{(id,f)}{\to} X \times_Z Y \stackrel{pr_2}{\to} Y$. By Lemma 3.4, it suffices to show that (id, f) and pr_2 are Künneth. The morphism (id, f) is given by the base change of $(f, id) : X \times_Z Y \to Y \times_Z Y$ along the diagonal $\Delta_g : Y \to Y \times_Z Y$ an is thus Künneth by Lemma 3.5. Similarly, the morphism pr_2 is the base change of $g \circ f$ along g and is thus Künneth by Lemma 3.5. $\square$

Lemma 3.7. Let $D : C^{op} \to \mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L}})$ be a lax symmetric monoidal functor. If $f : X_1 \to Z$, $g : X_2 \to Z$ are Künneth, then $X_1 \times_Z X_2 \to Z$ is Künneth.

Proof. This follows from Lemma 3.5 and Lemma 3.4. $\square$

We will now give first examples of Künneth morphisms. We recall the following definition

Definition 3.8. [CS22, Lemma 6.4, Proposition 6.5] Let $F : \mathcal{D} \to \mathcal{C}$ be a map in $\mathrm{CAlg}(\mathrm{Pr}^L_{st})$.

1. We call F a categorical open immersion, if F has a fully faithful left adjoint $F_L : \mathcal{C} \to \mathcal{D}$ which satisfies the projection formula, i.e. for any $M \in \mathcal{C}$ and any $N \in \mathcal{D}$ the natural map

$$F_L(M \otimes F(N)) \to F_L(M) \otimes N$$

is an equivalence.

Proposition 3.9. [CS22, Proposition 6.5] Let $F : \mathcal{D} \to \mathcal{C}$ be a map in $\mathrm{CAlg}(\mathrm{Pr}^L_{st})$. Then F is a categorical open immersion if and only if there is a (necessarily unique) idempotent algebra $A \in \mathcal{D}$ such that $F(A) \cong 0$ and the induced map $\mathcal{D}/\mathrm{Mod}_A \mathcal{D} \to \mathcal{C}$ is an equivalence.

Proof. Assume that F is a categorical open immersion. We claim that $A := \mathrm{cofib}(F_L(1) \to 1) \in \mathcal{D}$ is an idempotent algebra. Indeed, note that $F(A) \cong \mathrm{cofib}(F(F_L(1)) \to 1) \cong 0$ since F_L is fully faithful. We thus have equivalences

$$A \otimes A \cong \mathrm{cofib}(F_L(1) \otimes A \to A)$$
$$\cong \mathrm{cofib}(F_L(F(A)) \to A)$$
$$\cong \mathrm{cofib}(0 \to A) \cong A$$

where the first equivalence follows since cofibers commute with tensor products and the second line follows by the projection formula and the last line follows since $F(A) \cong 0$. Thus $A \in \mathcal{D}$ is an idempotent algebra and it suffices to show that

$$\ker(F) \cong \mathrm{Mod}_A \mathcal{D}$$

as full subcategories of $\mathcal{D}$. If $M \in \ker(F)$, then $F_L(1) \otimes M \cong F_L(F(M)) \cong 0$ by the projection formula, so $A \otimes M \cong M$ and thus M lies in $\mathrm{Mod}_A \mathcal{D}$. If $M \in \mathrm{Mod}_A \mathcal{D}$ we have $A \otimes M \cong M$ and thus $F(A \otimes M) \cong F(A) \otimes F(M) \cong 0$ since $F(A) = 0$. Conversely, assume that there exists an idempotent algebra $A \in \mathcal{D}$ such that $F : \mathcal{D} \to \mathcal{D}/\mathrm{Mod}_A \mathcal{D} \cong \mathcal{C}$. Then $F(-) \cong \mathrm{fib}(1 \to A) \otimes (-)$ and F_L is the inclusion, so the projection formula is trivially satisfied. $\qquad\square$

Lemma 3.10. [CS22, Lemma 6.4, Corollary 6.6]

1. Let $F : \mathcal{D} \to \mathcal{C}$, $G : \mathcal{C} \to \mathcal{B}$ be categorical open immersions, then $G \circ F$ is a categorical open immersion.

2. Let $F : \mathcal{D} \to \mathcal{C}$ be a categorical open immersion, $G : \mathcal{D} \to \mathcal{B}$ be a functor in $\mathrm{CAlg}(\mathrm{Pr}^L_{st})$. Then the pushout $\tilde{F} : \mathcal{B} \to \mathcal{C} \otimes_{\mathcal{D}} \mathcal{B}$ is again a categorical open immersion.

3. Let $F : \mathcal{D} \to \mathcal{C}$, $G : \mathcal{C} \to \mathcal{B}$ be maps in $\mathrm{CAlg}(\mathrm{Pr}^L_{st})$. If $G \circ F$ is a categorical open immersion, then G is a categorical open immersion.

4. Let $G : \mathcal{D} \to \mathcal{B}$ be a map in $\mathrm{CAlg}(\mathrm{Pr}^L_{st})$ and consider the pushout diagram in $\mathrm{CAlg}(\mathrm{Pr}^L_{st})$.

$$
\begin{array}{ccc}
\mathcal{B} & \xrightarrow{\tilde{F}} & \mathcal{B} \otimes_{\mathcal{D}} \mathcal{C} \\
\uparrow{\scriptstyle G} & & \uparrow{\scriptstyle \tilde{G}} \\
\mathcal{D} & \xrightarrow{F} & \mathcal{C}
\end{array}
$$

If F is a categorical open immersion, the diagram above is left adjointable.

Definition 3.11. $D : C^{op} \to \mathrm{CAlg}(\mathrm{Pr}^L_{st})$ be a lax symmetric monoidal functor and $f : X \to Y$ any morphism in E. We call f a D-closed (respectively D-open) immersion if $D(f) := f^* : D(Y) \to D(X)$ is a categorical closed (respectively open) immersion in the sense of Definition 3.8.

We will now see that in the context of a 3-functor formalisms, these notions yield our first examples of Künneth morphisms.

Proposition 3.12. $D : Corr(C, E) \to \mathrm{CAlg}(\mathrm{Pr}_{st}^L)$ be a 3-functor formalism and $f : X \to Y$ any morphism in E. Let f be a D-open immersion, then f is Künneth.

Proof. Let $f : X \to Y$ be a D-open immersion and $g : W \to Y$ any morphism in C. Let $I \in D(Y)$ be the idempotent algebra associated with f via Lemma 4.4. We have a localisation sequence

$$\mathrm{Mod}_I D(Y) \hookrightarrow D(Y) \xrightarrow{f^*} D(X).$$

Now the Lurie-tensor product in $\mathrm{Mod}_{D(Y)}(\mathrm{Pr}^L)$ preserves localizations (cf.[Ram24, Corollary 1.46]), thus after applying $(-) \otimes_{D(Y)} D(W)$ we obtain a localization sequence

$$
\begin{array}{ccccc}
\mathrm{Mod}_I D(Y) \otimes_{D(Y)} D(W) & \longrightarrow & D(Y) \otimes_{D(Y)} D(W) & \xrightarrow{f^* \otimes id_{D(W)}} & D(X) \otimes_{D(Y)} D(W) \\
\wr \downarrow & & \downarrow \wr \, m_{Y,W} & & \downarrow m_{X,W} \\
\mathrm{Mod}_{g^*(I)} D(W) & \longrightarrow & D(W) & \xrightarrow{\tilde{f}^*} & D(X \times_Y W)
\end{array}
$$

Note that $g^*(I)$ is idempotent so the lower left functor is again fully faithful. The right square commutes by Lemma 3.2. We need to show $\ker(\tilde{f}^*) \cong \mathrm{Mod}_{g^*(I)} D(W)$ as full subcategories of $D(W)$. The inclusion $\mathrm{Mod}_{g^*(I)} D(W) \subset \ker(\tilde{f}^*)$ is obvious by the commutativity of the diagram. Now let $M \in \ker(\tilde{f}^*)$, we want to show that $g^*(I) \otimes M \cong M$. This follows by the following calculation

$$
\begin{aligned}
g^*(I) \otimes M &\cong \mathrm{cofib}(g^* f_!(1) \otimes M \to M) \\
&\cong \mathrm{cofib}(\tilde{f}_!(1) \otimes M \to M) \\
&\cong \mathrm{cofib}(\tilde{f}_!(\tilde{f}^*(M)) \to M) \\
&\cong \mathrm{cofib}(0 \to M) \cong M.
\end{aligned}
$$

Thus we conclude that $\tilde{f}$ is a D-open immersion and since $m_{Y,W}$ and the left vertical morphism are isomorphisms also $m_{X,W}$ is an isomorphism. $\qquad\square$

Corollary 3.12.1. $D : Corr(C, E) \to \mathrm{CAlg}(\mathrm{Pr}^L)$ be a 3-functor formalism and $f : X \to Y$ a D-open immersion, $g : Z \to Y$ any morphism in C. Then the base change $\tilde{f} : X \times_Y Z \to Z$ is again a D-open immersion.
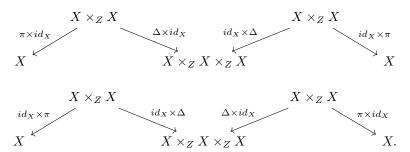
Proof. This follows by Proposition 3.12 together with point 2 in Lemma 3.10. $\qquad\square$

3.3 Dualisable modules in 6-functor formalisms

Now let $D : Corr(C, E) \to \mathrm{Pr}^L$ be a 6-functor formalism satisfying Künneth and $f : X \to Z$ a map in E. We want to see that $D(X)$ is then automatically dualisable in $\mathrm{Mod}_{D(Z)} \mathrm{Pr}^L$ via the map f^*.

Lemma 3.13. Let (C, E) be a geometric setup, $\pi : X \to Z$ a map in E and $\Delta : X \to X \times_Z X$ the diagonal map. Then the following two compositions in $Corr(C, E)$ represent the identity. In

particular, X is self-dual in $\mathrm{Corr}(C, E)$.

$$
\begin{array}{ccccc}
& X \times_Z X & & X \times_Z X & \\
{\scriptstyle \pi \times id_X} \swarrow & & \searrow {\scriptstyle \Delta \times id_X} \quad {\scriptstyle id_X \times \Delta} \swarrow & & \searrow {\scriptstyle id_X \times \pi} \\
X & & X \times_Z X \times_Z X & & X
\end{array}
$$

$$
\begin{array}{ccccc}
& X \times_Z X & & X \times_Z X & \\
{\scriptstyle id_X \times \pi} \swarrow & & \searrow {\scriptstyle id_X \times \Delta} \quad {\scriptstyle \Delta \times id_X} \swarrow & & \searrow {\scriptstyle \pi \times id_X} \\
X & & X \times_Z X \times_Z X & & X.
\end{array}
$$

Proof. The first composition is given by

$$
\begin{array}{ccc}
& X & \\
{\scriptstyle \Delta} \swarrow & & \searrow {\scriptstyle \Delta} \\
X \times_Z X & & X \times_Z X \\
\end{array}
$$

As $(\pi \times id_X) \circ \Delta = id_X = (id_X \times \pi) \circ \Delta$ this proves the claim. The claim for the second composition follows by the same argument. $\qquad\square$

Remark 3.14. Let $D : Corr(C, E) \to \mathrm{Pr}^{\mathrm{L}}$ be a 6-functor formalism, $Y, W, Z \in C$ with morphisms $\pi_Y : Y \to Z$ and $\pi_W : W \to Z$ and $f : Y \to W$ a map in E such that $\pi_Y = \pi_W \circ f$. Then the functors $f_!$ and f^* are $D(Z)$-linear with respect to the $D(Z)$-module structures on $D(Y)$ and $D(W)$ induced by the functors π_Y^* and π_W^*, respectively (cf. [6Fun, Lecture 3, Remark 3.13]).

Corollary 3.14.1. Let $D : Corr(C, E) \to \mathrm{Pr}^{\mathrm{L}}$ be a 6-functor formalism, $\pi : X \to Z$ a map in E and $\Delta : X \to X \times_Z X$ the diagonal map. Then we have isomorphisms of functors

$$
(id_X \times \pi)_! (id_X \times \Delta)^* (\Delta \times id_X)_! (\pi \times id_X)^* \cong id_{D(X)} \quad , \quad (\pi \times id_X)_! (\Delta \times id_X)^* (id_X \times \Delta)_! (id_X \times \pi)^* \cong id_{D(X)}.
$$

Proof. Apply the functor $D(-)$ to the diagrams of Lemma 3.13 and use that they represent the identity morphisms in $\mathrm{Corr}(C, E)$. $\qquad\square$

For $\pi : X \to Z$ our fixed morphism, we introduce the notation $m := m_{X,X}$, $m_{12} := m_{X, X \times_Z X}$, $m_{21} := m_{X \times_Z X, X}$. As the next two propositions show, in order to deduce that $D(X)$ is dualisable as a $D(Z)$-module, it is sufficient for the map m to be an equivalence.

Proposition 3.15. Let $D : Corr(C, E) \to \mathrm{Pr}^{\mathrm{L}}$ be a 6-functor formalism and let $\pi : X \to Z$ be a morphism in E. If the functor

$$
m : D(X) \otimes_{D(Z)} D(X) \to D(X \times_Z X)
$$

is an equivalence, then $D(X)$ is self-dual as a $D(Z)$-module.

Proof. We introduce the functors

$$t := \pi_! \Delta^* : D(X \times_Z X) \to D(Z)$$
$$u := \Delta_! \pi^* : D(Z) \to D(X \times_Z X)$$

which are $D(Z)$-linear by Remark 3.14. Using the $D(Z)$-linear equivalence m, we can define the evaluation ev and coevaluation co by the diagrams

In order to see that $D(X)$ is dualizable as a $D(Z)$-module, we need to verify that

$$(ev \otimes id_{D(X)}) \circ (id_{D(X)} \otimes co) \cong id_{D(X)}.$$
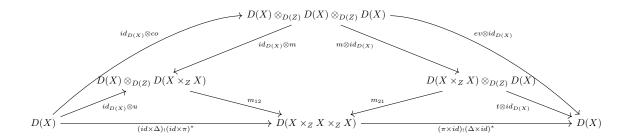
To see this, consider the following diagram:

Figure 1:

Note that we suppressed the equivalence $m_{X,Z} : D(X) \otimes_{D(Z)} D(Z) \cong D(X)$ in the lower left corner and the equivalence $m_{Z,X} : D(Z) \otimes_{D(Z)} D(X) \cong D(X)$ in the lower right corner of Figure 1 above. First observe that the outer triangles involving ev and co commute by definition of ev and co. Next we show the commutativity of the lower triangles, which follows by the naturality of m: Indeed, let $f : Y \to W$ be any morphism, then it follows from Lemma 3.2 that the following diagram commutes

$$
\begin{array}{ccc}
D(Y) \otimes_{D(Z)} D(X) & \overset{f_! \otimes id_{D(X)}}{\longrightarrow} & D(W) \otimes_{D(Z)} D(X) \\
{\scriptstyle m_{Y,X}} \downarrow & & \downarrow {\scriptstyle m_{W,X}} \\
D(Y \times_Z X) & \xrightarrow{(f \times id_X)_!} & D(W \times_Z X).
\end{array}
\tag{3.15.1}
$$

Combining Lemma 3.2 and Equation (3.15.1), we see that each of the two lower triangles in Figure 1 commute. Finally, one easily verifies that the central square commutes by definition of the morphisms $m_{i,j}$. Thus we infer that

$$(ev \otimes id_{D(X)}) \circ (id_{D(X)} \otimes co) \cong (\pi \times id)_! (\Delta \times id)^* (id \times \Delta)_! (id \times \pi)^*.$$

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But this is isomorphic to the identity by Corollary 3.14.1. The other identity

$$(id_{D(X)} \otimes ev) \circ (co \otimes id_{D(X)}) \cong id_{D(X)}$$

follows by an analogous argument. This shows that $D(X) \in \mathrm{Mod}_{D(Z)}(\mathrm{Pr}^{\mathrm{L}})$ is self-dual. $\qquad\square$

Corollary 3.15.1. Let $f : X \to Z$ be Künneth. Then $D(X) \in \mathrm{Mod}_{D(Z)}(\mathrm{Pr}^{\mathrm{L}})$ is dualisable.

Proof. Since f is Künneth we have in particular that the morphism $m_{X,X} : D(X) \otimes_{D(Z)} D(X) \to D(X \times_Z X)$ is an equivalence. The result then follows by Proposition 3.15. $\qquad\square$

Corollary 3.15.2. Let $D : Corr(C, E) \to \mathrm{Pr}^{\mathrm{L}}$ be a 6-functor formalism and let $X \to Z$ be a morphism in C. Assume that the functor $m : D(X) \otimes_{D(Z)} D(X) \to D(X \times_Z X)$ is an equivalence. Then we have an equivalence

$$\mathrm{Fun}^L_{D(Z)}(D(X), D(X)) \cong D(X \times_Z X).$$

Proof. We claim that there are equivalences

$$\mathrm{Fun}^L_{D(Z)}(D(X), D(X)) \cong \mathrm{Fun}^L_{D(Z)}(D(Z), D(X) \otimes_{D(Z)} D(X)) \cong D(X) \otimes_{D(Z)} D(X).$$

Indeed, by Proposition 3.15, we know that $D(X)$ is self dual as a $D(Z)$ module which shows the first equivalence. The second equivalence is natural. $\qquad\square$

3.4 Descent properties of Künneth morphisms

In this subsection we will be interested in the interaction of the notion of Künneth morphisms with descent of the 6-functor formalism $D : Corr(C, E) \to \mathrm{Pr}^{\mathrm{L}}$. We denote by

$$D^* : C^{op} \to \mathrm{Pr}^{\mathrm{L}}$$

the functor given by $D^*(X) = D(X)$ for $X \in C$ and $D^*(f) = f^*$ for morphisms f in C and

$$D^! : C^{op}_E \to \mathrm{Cat}_\infty$$

the functor given by $D^!(X) = D(X)$ for $X \in C$ and $D^!(f) = f^!$ for morphisms f in E (c.f [HM24, Definition 3.1.4 and 3.2.3]). In the following we use the notion of sieves as defined in [HM24, Definition A.4.1].

Definition 3.16. Let C, D be categories and assume that all limits exist in D, let $\mathcal{U} \subset C_{/U}$ a sieve on an object $U \in C$ and $F : C^{op} \to D$ a functor. We say F descends along $\mathcal{U}$ if the natural map

$$F(U) \to \lim_{V \in \mathcal{U}^{op}} F(V)$$

is an isomorphism. We say F descends universally along $\mathcal{U}$ if it descends along every pullback of $\mathcal{U}$.

Let us recall the definition of a universal !- and ∗-covers.

Definition 3.17. [HM24, Definition 3.4.6] Let $D : Corr(C, E) \to \mathrm{Pr}^{\mathrm{L}}$ be a 6-functor formalism.

1. We say that a sieve $\mathcal{U} \subset (C_E)_{/U}$ is a small !-cover if it is generated by a small family of maps $(U_i \to U)_i$ and the functor $D^!$ descends along $\mathcal{U}$. We say $\mathcal{U}$ is a universal !-cover if for every map $V \to U$ in C the family $(U_i \times_U V \to V)_i$ generates a small !-cover.

2. We say that a sieve $\mathcal{U}$ in C is a (universal) $*$-cover if D^* descends (universally) along $\mathcal{U}$.

3. The D-topology is the site on C where covers are canonical covers which have universal !-and $*$-descent.

For our purposes the following two classes of examples of !-covers will be particularly important.

Proposition 3.18. Let $D : Corr(C, E) \to \mathrm{Pr}_{st}^L$ be a stable 6-functor formalism, $f : X \to Y$ a morphism in E. Assume either of the following conditions are satisfied.

1. (smooth cover) The object 1_X is f-smooth and the functor f^* is conservative.

2. (descendable cover) The object 1_X is f-proper and $f_*(1_X) \in \mathrm{CAlg}(D(Y))$ is descendable.

Then the arrow $f : X \to Y$ is a universal !- and a universal $*$-cover.

Proof. This is [6Fun, Proposition 6.18, Proposition 6.19]. $\square$

We will call a morphism $f : X \to Y$ in E a smooth (respectively descendable) !-cover if it satisfies condition 1 (respectively condition 2) of Proposition 3.18. As the following proposition shows, for $f : X \to Y$ a descendable or smooth !-cover, the category $D(Y)$ can be made more explicit.

Proposition 3.19. [Cam24, Proposition 3.1.27] Let $D : Corr(C, E) \to \mathrm{Pr}_{st}^L$ be a stable 6-functor formalism and $f : X \to Y$ be a morphism in E.

1. If f is a descendable !-cover we have
$$D(Y) \cong \mathrm{coMod}_{f^* f_*} D(X).$$

2. If f is a smooth !-cover we have
$$D(Y) \cong \mathrm{Mod}_{f^! f_!} D(X).$$

Definition 3.20. [HM24, Definition A.4.5] $D : Corr(C, E) \to \mathrm{Pr}^L$ be a 6-functor formalism and $(f_i : X_i \to X)_{i \in I}$ be a universal !-cover of $X \in C$. We denote by $\Delta_I \to \Delta$ the right fibration associated with the functor $\Delta^{op} \to \mathrm{Ani}$, $[n] \mapsto I^{n+1}$ by unstraightening. An object in Δ_I is a pair $([n] \in \Delta, i_\bullet \in I^{n+1})$ and a morphism $([n], i_\bullet) \to ([m], j_\bullet)$ in Δ_I is a morphism $\alpha : [n] \to [m]$ in Δ such that $i_k = j_{\alpha(k)}$ for all $k \in [n]$. We denote by $f_\bullet : X_\bullet \to X$ the Cech-nerve given by morphisms $f_{[n], i_\bullet} : X_{[n], i_\bullet} := X_{i_0} \times_X \dots \times_X X_{i_n} \to X$.

We will now show one of our main results, which is that the property of a morphism being Künneth can by checked !-locally on the source.

Proposition 3.21. Let $D : Corr(C, E) \to \mathrm{Pr}^L$ be a 6-functor formalism and $g : X \to Z$ be any morphism in E. Let $(f_i : X_i \to X)_{i \in I}$ be a universal !-cover. If $g \circ f_{[n], i_\bullet} : X_{i_0} \times_X \dots \times_X X_{i_n} \to Z$ is Künneth for all $([n], i_\bullet) \in \Delta_I$, then g is Künneth.

Proof. By !-codescend, we have an equivalence $D(X) \cong \mathrm{colim}_{!,([n],i_\bullet)\in\Delta_I} D(X_{[n],i_\bullet})$ in Pr^L. Using that the forgetful functor $\mathrm{Mod}_{D(Z)}(\mathrm{Pr}^L) \to \mathrm{Pr}^L$ is conservative and [HM24, Lemma 3.2.5] (by passing to the induced geometric setup $(C_{/Z}, E)$ of the slice category $C_{/Z}$) we see that this is also an equivalence in $\mathrm{Mod}_{D(Z)}(\mathrm{Pr}^L)$. Let $W \to Z$ be an arbitrary map. We claim that the following equivalences hold

$$\begin{aligned} D(X) \otimes_{D(Z)} D(W) &\cong \mathrm{colim}_{!,([n],i_\bullet)\in\Delta_I} D(X_{[n],i_\bullet}) \otimes_{D(Z)} D(W) \\ &\cong \mathrm{colim}_{!,([n],i_\bullet)\in\Delta_I} D(X_{[n],i_\bullet} \times_Z W) \\ &\cong D(X \times_Z W). \end{aligned}$$

In the first line we used that the functor $D(W) \otimes_{D(Z)} (-)$ commutes with colimits (see Lemma 2.9). The second line follows since the morphisms $X_i \to Z$ are Künneth. The last line follows by !-descent by noting that $(g_i : X_i \times_Z W \to X \times_Z W)_{i\in I}$ is a !-cover, as it is the base change of the universal !-cover $(X_i \to X)_{i\in I}$ and by observing that the morphisms $g_{[n],i_\bullet} : X_{[n],i_\bullet} \times_Z W \to X \times_Z W$ are given by the base change of $f_{[n],i_\bullet} : X_{[n],i_\bullet} \to X$ by $X \times_Z W \to X$. We now show that the equivalences

$$m_{[n],i_\bullet} : D(X_{[n],i_\bullet}) \otimes_{D(Z)} D(W) \cong D(X_{[n],i_\bullet} \times_Z W)$$

are compatible with the colimit systems: By considering Equation (3.15.1) for the morphims $h : X_{[n],i_\bullet} \to X_{[m],j_\bullet}$ in the colimit system we observe that the following diagram commutes.

$$\begin{array}{ccc} D(X_{[n],i_\bullet}) \otimes_{D(Z)} D(W) & \xrightarrow{h_! \otimes id_{D(W)}} & D(X_{[m],j_\bullet}) \otimes_{D(Z)} D(W) \\ {\scriptstyle m_{[n],i_\bullet}}\downarrow & & \downarrow{\scriptstyle m_{[m],j_\bullet}} \\ D(X_{[n],i_\bullet} \times_Z W) & \xrightarrow{(h \times id_W)_!} & D(X_{[m],j_\bullet} \times_Z W). \end{array}$$

This shows the compatibility of the equivalences $m_{[n],i_\bullet}$ with the the colimit systems. $\square$

Similarly, the notion of a morphism being Künneth can be checked !-locally on the target.

Proposition 3.22. Let $D : Corr(C,E) \to \mathrm{Pr}^L$ be a 6-functor formalism and $f : X \to Z$ be any morphism in E and $(g_i : Z_i \to Z)_{i\in I}$ be a universal !-cover. Denote by $f_{([n],i_\bullet)} : X_{([n],i_\bullet)} := X \times_Z Z_{[n],i_\bullet} \to Z_{[n],i_\bullet}$ the base change of f by $g_{[n],i_\bullet} : Z_{[n],i_\bullet} \to Z$. If $f_{[n],i_\bullet}$ is Künneth for all $([n],i_\bullet) \in \Delta_I$, then f is Künneth.

Proof. Let $W \to Z$ be an arbitrary map and $W_{([n],i_\bullet)} := W \times_Z Z_{[n],i_\bullet} \to Z_{[n],i_\bullet}$ the base change. Using !-descent for the !-covers $X_{([n],i_\bullet)} \to X, Z_{([n],i_\bullet)} \to Z, W_{([n],i_\bullet)} \to W$ we obtain equivalences

$$\begin{aligned} D(X) \otimes_{D(Z)} D(W) &\cong \mathrm{colim}_{!,([n],i_\bullet)\in\Delta_I^{op}} D(X_{([n],i_\bullet)}) \otimes_{\mathrm{colim}_{!,([l],j_\bullet)\in\Delta_I^{op}} D(Z_{([l],j_\bullet)})} \mathrm{colim}_{!,([m],k_\bullet)\in\Delta_I^{op}} D(W_{([m],k_\bullet)}) \\ &\cong \mathrm{colim}_{!,([n],i_\bullet)\in\Delta_I^{op}} D(X_{([n],i_\bullet)} \times_{Z_{([n],i_\bullet)}} W_{([n],i_\bullet)}) \\ &\cong \mathrm{colim}_{!,([n],i_\bullet)\in\Delta_I^{op}} D(X \times_Z W \times_Z Z_{([n],i_\bullet)}) \\ &\cong D(X \times_Z W) \end{aligned}$$

Here we used !-descent for the !-covers $X_{([n],i_\bullet)} \to X, Z_{([n],i_\bullet)} \to Z, W_{([n],i_\bullet)} \to W$ in the first line, the fact that the category Δ_I^{op} is sifted in the second line, the fact that $f_{[n],i_\bullet}$ is Küneth for all $([n],i_\bullet) \in \Delta_I$ in the third line and !-descent for $(X \times_Z W \times_Z Z_{([n],i_\bullet)} \to X \times_Z W$ in the last line. $\square$

In view of our applications to analytic stacks we investigate the following situation of an extension of a 6-functor formalism.

Proposition 3.23. [HM24, Theorem 3.4.11] Let $D : Corr(C, E) \to \mathrm{Pr}^{\mathrm{L}}$ be a sheafy 6-functor formalism, where C is a subcanoncial site. Then there is a collection of edges $\tilde{E}$ in $\mathcal{X} := \mathrm{Sh}(C)$ with the following properties:

(i) The inclusion $C \hookrightarrow \mathcal{X}$ defines a morphism of geometric setups $(C, E) \to (\mathcal{X}, \tilde{E})$ and D extends uniquely to a 6-functor formalism $\tilde{D}$ on $(\mathcal{X}, \tilde{E})$.

(ii) $\tilde{E}$ is $*$-local on the target: Let $f : X \to Y$ in C be map for which the pullback to any object in C lies in E, then f lies in $\tilde{E}$.

(iii) $\tilde{E}$ is !-local: Let $f : X \to Y$ be a map which is !-local on the source or target in $\tilde{E}$, then f lies in $\tilde{E}$.

(iv) $\tilde{E}$ is tame: Every map $f : X \to Y$ in $\tilde{E}$ with $Y \in C$ is !-locally on the source in E.

Moreover, there is a minimal choice of $\tilde{E}$.

Theorem 3.24. Let $D : Corr(C, E) \to \mathrm{Pr}^{\mathrm{L}}$ be a sheafy 6-functor formalism on a subcanonical site C. Let $S \in C$ and assume that $D_S : ((C_E)_{/S})^{op} \to \mathrm{Mod}_{D(S)}(\mathrm{Pr}^{\mathrm{L}})$ is symmetric monoidal. Then the extension $\tilde{D}_S : ((\mathrm{Shv}(C)_{\tilde{E}})_{/S})^{op} \to \mathrm{Mod}_{D(S)}(\mathrm{Pr}^{\mathrm{L}})$ is symmetric monoidal.

Proof. We review the inductive construction of $\tilde{E}$ in the proof of [HM24, Theorem 3.4.11.] and show that the property of morphisms being Künneth is preserved under each extension step. Let E_0 be the class of morphisms in $\mathcal{X}$ which are representable in E and by A the collections of edges E' such that E' satisfies (i) and (iv) in Proposition 3.23. For $E' \in A$, let $(E')' := E''$ be the class of morphisms in $\mathcal{X}$ which are $D^{!'}$-locally in E'. For $E' \in A$, we denote by E'_S the class of morphisms $X \to S$ in E' and by $Ku(E'_S) \subset E'_S$ the class of morphisms $f : X \to S$ in E'_S which are Künneth. Note that $E_0 \in A$. We first observe that $Ku(E_{0,S}) = E_{0,S}$. Indeed, by representability we have that any $f : X \to S$ in E_0 lies in E since $S \in C$, thus f is Künneth since $D_S : ((C_E)_{/S})^{op} \to \mathrm{Mod}_{D(S)}(\mathrm{Pr}^{\mathrm{L}})$ is symmetric monoidal. We show the following claim.

$$(*) \text{ If } E' \in A \text{ and } Ku(E'_S) = E'_S, \text{ then } Ku(E''_S) = E''_S$$

Indeed, let $f : X \to S$ be an element in E''_S. By construction, there exists a small $!-$cover $(X_i \to X)_{i \in I}$ with $g_i : X_i \to X$ in E' and the composition $f \circ g_i \in E'$. By stability under base change and composition of E' we have that all the morphisms $f \circ g_{[n],i_\bullet} : X_{[n],i_\bullet} \to S$ are in E'. It thus follows that f is Künneth by Proposition 3.21 and thus $Ku(E''_S) = E''_S$.
For $E' \in A$, we define $E'_! := \bigcup_{n \geq 2} E^n$ where $E^2 := E''$ and $E^{n+1} := (E^n)'$ for $n \geq 2$. By [HM24, Theorem 3.4.11] we have that $E'_! \in A$ and satisfies (iii). We have the following claim:

$$(*') \text{ If } E' \in A \text{ and } Ku(E'_S) = E'_S, \text{ then } Ku(E'_{!S}) = E'_{!S}$$

Indeed, let $f : X \to S$ be in $E'_{!S}$. By definition, we know that $f \in E^n_S$ for some $n \geq 2$, thus f is Künneth by applying $(*)$ repeatedly.

For $E' \in A$ satisfing (iii), we follow Heyer-Mann and define E'_* to be the class of morphisms $X \to Y$ in $\mathcal{X}$ which after pullback under any morphism $U \to Y$ for $U \in C$ lie in E'. By [HM24,

Theorem 3.4.11] this defines again an element $E'_* \in A$. Since $E'_{*S} = E'_S$, we obtain the following claim:

$$(**) \text{ If } E' \in A \text{ and } Ku(E'_S) = E'_S, \text{ then } Ku(E'_{*S}) = E'_{*S}$$

By construction we have $\tilde{E} \coloneqq \bigcup_{n \geq 1} \tilde{E}^n$, where $\tilde{E}^1 \coloneqq E_{0!*}$ and $\tilde{E}^{n+1} \coloneqq (\tilde{E}^n)_{!*}$. It follows from $(*), (*')$ and $(**)$ that $Ku(\tilde{E}_S) = \tilde{E}_S$. $\qquad\square$

Remark 3.25. Proposition 3.21 together with Proposition 3.19 can be used to construct dualisable comodule categories: For example, consider a 6-functor formalism D on $(\mathrm{Sh}(C), \tilde{E})$ as above and assume that the restriction of D to (C, E) satisfies Künneth. Let $G \in \mathrm{Sh}(C)$ be a commutative group object such that $h : G \to *$ is D-cohomologically proper and the natural map $f : * \to */G$ is a descendable D-cover. We note that in this case $h_* \cong h_!$ satisfies the projection formula. Then using Proposition 3.19 and the proper base change along the cartesian square

$$
\begin{array}{ccc}
G & \xrightarrow{\;h\;} & * \\
\downarrow & & \downarrow{\scriptstyle f} \\
* & \xrightarrow[\;f\;]{} & */G
\end{array}
$$

we obtain an equivalence $D(*/G) \cong \mathrm{coMod}_{h_*(1)} D(*)$. In fact, this equivalence is stable under arbitry base change, that is for any $\pi : Y \to *$ we obtain an equivalence

$$D(*/G \times_* Y) \cong \mathrm{coMod}_{\pi^* h_*(1)} D(Y).$$

Using Proposition 3.21 we note that we obtain a Künneth formula for comodules:

$$\mathrm{coMod}_{h_*(1)} D(*) \otimes_{D(*)} D(Y) \cong \mathrm{coMod}_{\pi^* h_*(1)} D(Y).$$

Although the corresponding statement for smooth D-covers reduces to a well-known statement about categories of modules (cf. [BFN10, Proposition 4.1]) a corresponding abstract statement for comodules is unknown to the author.

Theorem 3.26. Let $D : Corr(C, E) \to \mathrm{Pr}^{\mathrm{L}}$ be a 6-functor formalism which is Künneth and consider the extension $\tilde{D} : Corr(\mathrm{Shv}(C), \tilde{E}) \to \mathrm{Pr}^{\mathrm{L}}$. Assume that $S \in \mathrm{Shv}(C)$ admits a !-cover $(S_i \to S)_{i \in I}$ with $S_i \to S$ in E_0 and $S_i \in C$ for all $i \in I$. Then any morphism $f : X \to S$ in $\tilde{E}$ is Künneth.

Proof. Since $S_i \to S$ in E_0 we have that $S_{[n],i_\bullet} \in C$ for all $([n], i_\bullet) \in \Delta_I$. The result follows now by Theorem 3.24 and Proposition 3.22. $\qquad\square$

Remark 3.27. The previous corollary applies for example if $S \in \mathrm{Shv}(C)_{\tilde{E}/*}$ has representable diagonal, $\Delta_S : S \to S \times S \in E_0$. In the context of derived algebraic geometry and with $D(-)$ given by $\mathrm{QCoh}(-)$ this corresponds to the affineness condition on the diagonal of the base in the definition of perfect stacks (c.f [BFN10, Definition. 3.2]). One could slightly generalise this to the affineness condition to allow "quasi-affine" diagonals (to reproduce the results of [Ste23, Theorem 1.0, 3]) using the fact that open immersion are Künneth Proposition 3.12.

We include the following lemma

Lemma 3.28. Let $\mathrm{Sh}(C)$ be the category of sheaves of anima on a small site C, Z an object in $\mathrm{Sh}(C)$ and $X, Y \in \mathrm{Sh}(C)_{/Z}$. Then the functor given by taking fibre products

$$X \times_Z (-) : \mathrm{Sh}(C)_{/Z} \to \mathrm{Sh}(C)_{/Z}, \ Y \mapsto X \times_Z Y$$

commutes with all colimits. In particular, it has a left adjoint which is given by the Hom-stack

$$\mathrm{Hom}_Z(X, Y) : C_{/Z} \to \mathrm{Ani}, \ U \mapsto \mathrm{Hom}_Z(X \times_Z U, Y).$$

Here, the category $C_{/Z} \hookrightarrow \mathrm{Sh}(C)_{/Z}$ is given by pullback of the forgetful functor $\mathrm{Sh}(C)_{/Z} \to \mathrm{Sh}(C)$ along the fully faithful inclusion $C \hookrightarrow \mathrm{Sh}(C)$ via Yoneda.

Proof. The fact that the functor $X \times_Z (-)$ commutes with all colimits is immediate as $\mathrm{Sh}(C)$ and hence $\mathrm{Sh}(C)_{/Z}$ are $\infty-$topoi. Since $\mathrm{Sh}(C)_{/Z}$ is presentable, we have a right adjoint L_X which is given on $U \in C_{/Z}$ by

$$\begin{aligned}
L_X(U) &\cong \mathrm{Hom}_Z(U, L_X) \\
&\cong \mathrm{Hom}_Z(X \times_Z U, Y) \\
&\cong \mathrm{Hom}_Z(X, Y)(U).
\end{aligned}$$

$\square$

Lemma 3.29. Let $D_0 : Corr(C_0, E_0) \to \mathrm{Pr}^L$ be a 6-functor formalism on a subcanonical site C_0 and consider an extension $D : Corr(\mathrm{Sh}(C_0), \tilde{E}_0) \to \mathrm{Pr}^L$ as in Proposition 3.23. Then $f : X \to Y$ in E_0 is Künneth with respect to D_0 if and only if f is Künneth with respect to D.

Proof. If f is Künneth with respect to D then it is also Künneth with respect to D_0 since $C_0 \subset \mathrm{Sh}(C_0)$. Conversely, since $f \in E_0$ we obtain $f \in \tilde{E}_0$ by the definition of the extension $\tilde{E}_0$. Let $W \to Z$ be a morphism in $\mathrm{Sh}(C_0)$ and write $D(W) \cong \lim_{n \in J} D_0(W_n)$ with $W_n \in C_0$. By Corollary 3.15.1 we obtain that $D_0(X) \in \mathrm{Mod}_{D_0(Z)}(\mathrm{Pr}^L)$ is dualisable. We claim that there is a chain of equivalences

$$\begin{aligned}
D_0(X) \otimes_{D_0(Y)} D(W) &\cong \lim_{n \in J} D_0(X) \otimes_{D_0(Y)} D_0(W_n) \\
&\cong \lim_{n \in J} D_0(X \times_Y W_n) \\
&\cong D(X \times_Y W).
\end{aligned}$$

Indeed, the first line follows by Lemma 2.10, the second equivalence follows from the fact that f is Künneth with respect to D_0 and the last one follows from Lemma 3.28. $\square$

Remark 3.30. Let $D : Corr(C, E) \to \mathrm{Pr}^L$ be a 6-functor formalism. By Lemma 3.7, Lemma 3.5 and Proposition 3.21 we see that the class K of Künneth morphisms in C has good properties such as closure under composition, arbitrary base change and !-locality on the source. Note however, that (C, K) does not define a geometric setup in the sense of [HM24, Definition 2.1.1]. The problem is that the class K is not right cancellative in general, or in view of [HM24, Remark 2.1.2] that for $f : X \to Y$ Künneth the diagonal $\Delta_f : X \to X \times_Y X$ is not necessarily Künneth.

3.5 Consequences: Tannakian reconstruction

Let $D : Corr(C, E) \to \mathrm{CAlg}(\mathrm{Pr}^\mathrm{L})$ be a 6 functor formalism and let $S \in C$. It is an interesting question whether one can recover maps $\mathrm{Hom}_{C_{/S}}(X, Y)$ in $C_{/S}$ from the category of morphisms $\mathrm{Fun}_{D(S)}^{L,\otimes}(D(Y), D(X))$ in $\mathrm{CAlg}(\mathrm{Mod}_{D(S)}(\mathrm{Pr}^\mathrm{L}))$. Our arguments basically follows [Ste23].

Definition 3.31. Let $D : Corr(C, E) \to \mathrm{CAlg}(\mathrm{Pr}^\mathrm{L})$ be a 6-functor formalism and $f : X \to S$ in C. We call f Tannakian if the functor

$$\mathrm{Hom}_{C_{/S}}(Y, X) \cong \mathrm{Fun}_{D(S)}^{L,\otimes}(D(X), D(Y))$$

is an equivalence for all $Y \in C_{/S}$. We call D Tannakian if all morphisms in C are Tannakian.

Given a 6-functor formalism $D : Corr(C, E) \to \mathrm{CAlg}(\mathrm{Pr}^\mathrm{L})$ on a subcanonical site C which is Tannakian and Künneth, we will investigate which morphisms $f : X \to S$ in the extension $\tilde{D} : Corr(\mathrm{Shv}(C), \tilde{E}) \to \mathrm{CAlg}(\mathrm{Pr}^\mathrm{L})$ given by Proposition 3.23 are Tannakian. Our arguments basically follows [Ste23]. However, instead of proving a 1-affiness result for the class of !-covers in Theorem 3.26, we show directly that they satisfy 2-descent. More generally we show the following lemma.

Lemma 3.32. Let $D : Corr(C, E) \to \mathrm{CAlg}(\mathrm{Pr}^\mathrm{L})$ be a 6 functor formalism and $f : X' \to X$ in E satisfying Künneth. Then 1) implies 2) implies 3):

1. f is a universal !-cover

2. For all $M \in \mathrm{Mod}_{D(X)} \mathrm{Pr}^\mathrm{L}$ we have that the natural map $M \cong \lim^* M \otimes_{D(X)} D(X'^{n/X})$ is an isomorphism. In particular f satisfies *-descent.

3. f satisfies 2-descent, i.e. we have an isomorphism $\mathrm{Mod}_{D(X)} \mathrm{Pr}^\mathrm{L} \cong \lim_n \mathrm{Mod}_{D(X'^{n/X})} \mathrm{Pr}^\mathrm{L}$.

Proof. We start by showing that 1) implies 2). Let $M \in \mathrm{Mod}_{D(X)} \mathrm{Pr}^\mathrm{L}$, then we have equivalences

$$\begin{aligned}
M &\cong \mathrm{Hom}_{D(X)}(D(X), M) \\
&\cong \mathrm{Hom}_{D(X)}(\mathrm{colim}_{n \in \Delta^{op}} D(X'^{n/X}), M) \\
&\cong \lim_{n \in \Delta} \mathrm{Hom}_{D(X)}(D(X'^{n/X}), M) \\
&\cong \lim_{n \in \Delta} \mathrm{Hom}_{D(X)}(D(X), D(X'^{n/X}) \otimes_{D(X)} M) \cong \lim_{n \in \Delta} M \otimes_{D(X)} D(X'^{n/X})
\end{aligned}$$

Here we used that $f : X' \to X$ and hence also the basechanges $f : X'^{n/X} \to X$ are Künneth, which implies that $D(X'^{n/X}) \in \mathrm{Mod}_{D(X)} \mathrm{Pr}^\mathrm{L}$ are self-dual by Corollary 3.15.1 in the fourth line. Taking $M = D(X)$ it follows that f satisfies $*$-descent.
To see that 2) implies 3), note that the natural functor

$$B : \mathrm{Mod}_{D(X)} \mathrm{Pr}^\mathrm{L} \to \lim_n \mathrm{Mod}_{D(X'^{n/X})} \mathrm{Pr}^\mathrm{L} \; M \mapsto M_n := M \otimes_{D(X)} D(X'^{n/X})$$

admits a right adjoint given by $\lim_n \mathrm{Mod}_{D(X'^{n/X})} \mathrm{Pr}^\mathrm{L} \to \mathrm{Mod}_{D(X)} \mathrm{Pr}^\mathrm{L}$, $M_n \mapsto \lim_n f_{n,*} M_n$ where $f_{n,*} : \mathrm{Mod}_{D(X'^{n/X})} \mathrm{Pr}^\mathrm{L} \to \mathrm{Mod}_{D(X)} \mathrm{Pr}^\mathrm{L}$ is the forgetful functor. By 2), the unit of this adjunction

is an equivalence. Thus, it suffices to verify that the counit is an equivalence. Since $D(X'^{n/X}) \in \mathrm{Mod}_{D(X)}\mathrm{Pr}^{\mathrm{L}}$ are dualisable , we calculate

$$\lim_n \lim_n f_{n,*} M_n \otimes_{D(X)} D(X'^{m/X}) \cong \lim_n (f_{n,*} M_n \otimes_{D(X)} D(X'^{m/X})) \cong M_m.$$

<div align="right">□</div>

We can now formulate an extension lemma for Tannkian morphisms. The idea of the proof is adapted from [Ste23, Theorem 4.2.1].

Theorem 3.33. Let C be a subcanonical site and $D : Corr(C, E) \to \mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L}})$ be a Tannakian sheafy 6-functor formalism satisfying Künneth. Consider its extension $D : Corr(\mathrm{Shv}(C), \tilde{E}) \to \mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L}})$. Let $f : X \to S$ be a morphism in $\tilde{E}$ and assume that there is a !-cover $g : S' \to S$ with $S' \in C$ and $g \in E_0$. Then f is Tannakian.

Proof. By iterating the following argument we can assume $S = * \in C$ the final object. Consider the following commutative diagram

$$
\begin{array}{ccc}
(C_{/X})^{op} & \xrightarrow{\ D\ } & \mathrm{CAlg}(\mathrm{Mod}_{D(*)}(\mathrm{Pr}^{\mathrm{L}}))_{D(X)/} \\
\downarrow & & \downarrow \\
C^{op} & \xhookrightarrow{\ D\ } & \mathrm{CAlg}(\mathrm{Mod}_{D(*)}(\mathrm{Pr}^{\mathrm{L}}))
\end{array}
$$

Note that the lower horizontal functor is fully faithful since $D : Corr(C, E) \to \mathrm{CAlg}(\mathrm{Mod}_{D(*)}(\mathrm{Pr}^{\mathrm{L}}))$ is Tannakian by assumption. It suffices to show that this is a pullback diagram, the morphism

$$\mathrm{Hom}_{C/*}(Y, X) \to \mathrm{Fun}_{D(*)}(D(X), D(Y))$$

arises as the fibre over $D(X) \in \mathrm{CAlg}(\mathrm{Mod}_{D(*)}(\mathrm{Pr}^{\mathrm{L}}))$ of the left fibration $\mathrm{CAlg}(\mathrm{Mod}_{D(*)}(\mathrm{Pr}^{\mathrm{L}}))_{D(X)/} \to \mathrm{CAlg}(\mathrm{Mod}_{D(*)}(\mathrm{Pr}^{\mathrm{L}}))$. Let P be the pullback of the diagram, which we can identify as as full sub-category $\mathrm{Mod}_{D(X)}^{aff}\mathrm{Pr}^{\mathrm{L}}$ of $\mathrm{Mod}_{D(X)}\mathrm{Pr}^{\mathrm{L}}$ consisting of those $M \in \mathrm{Mod}_{D(X)}\mathrm{Pr}^{\mathrm{L}}$ for which $M \cong D(Z)$ for some $Z \in C$. Let

$$\Theta : (C_{/X})^{op} \to \mathrm{Mod}_{D(X)}^{aff}\mathrm{Pr}^{\mathrm{L}}$$

be the induced functor. We want to show it is an equivalence. Note that since $S' \to S'$ satisfies !-descent and is Künneth by Theorem 3.26, this is also true for $X' := X \times_S S' \to X$ by Lemma 3.5. By Lemma 3.32 we have 2-descent for $X' \to X$ and since $g \in E_0$ we see that the equivalence from 3) in Lemma 3.32 descents to an equivalence

$$\mathrm{Mod}_{D(X)}^{aff}\mathrm{Pr}^{\mathrm{L}} \cong \lim_n \mathrm{Mod}_{D(X'^{n/X})}^{aff}\mathrm{Pr}^{\mathrm{L}}.$$

. We thus obtain a diagram

$$
\begin{array}{ccc}
(C_{/X})^{op} & \xrightarrow{\quad \Theta \quad} & \mathrm{CAlg}(\mathrm{Mod}_{D(X)}^{aff}\mathrm{Pr}^{\mathrm{L}}) \\
\downarrow & & \downarrow \\
\lim_n (C_{/X'^{n/X}})^{op} & \xrightarrow[\lim_n \Theta_n]{} & \lim_n \mathrm{CAlg}(\mathrm{Mod}_{D(X'^{n/X})}^{aff}\mathrm{Pr}^{\mathrm{L}})
\end{array}
$$

Note that for any n, the functor Θ_n is an equivalence, hence the limit $\lim_n \Theta_n$ is an equivalence. It follows that Θ is fully faithful. We now show essential surjectivity of Θ. Let $D(X) \to D(Z) \in \mathrm{CAlg}(\mathrm{Mod}_{D(X)}^{aff} \mathrm{Pr}^{\mathrm{L}})$, then by fully faithfullness of the right vertical arrow and the equivalence $\lim_n \Theta_n$ we obtain compatible arrows $Z \times_X X'^{n/X} \to X'^{n/X}$. But this gives an arrow $Z \to X$ by !-descent. so Θ is an equivalence.

$\square$

4 Künneth morphisms for analytic stacks

4.1 The 6-functor formalism for analytic stacks

In this subsection we recall the construction of the 6-functor formalism of analytic stacks, following Clausen-Scholze [AS, Lecture 16-20]. Since we are mainly interested in formal questions of 6-functor formalisms, we will not recall all the relevant definitions and refer to [AS], [CS22]. We denote by $D(\mathbb{Z}) := D(\mathrm{CondAb})$ be the derived category of light condensed abelian groups.

Definition 4.1. [AS, Lecture 8] An analytic ring $A = (A^\triangleright, D(A))$ is the datum of a condensed animated ring $A^\triangleright$ and a fully faithful subcategory $D(A) \overset{j_A}{\subset} \mathrm{Mod}_{A^\triangleright} D(\mathbb{Z})$ such that

1. The category $D(A)$ is stable under all limits and colimits in $\mathrm{Mod}_{A^\triangleright} D(\mathbb{Z})$ and there is a left adjoint of the inclusion $j_A : D(A) \subset \mathrm{Mod}_{A^\triangleright} D(\mathbb{Z})$ which we denote by $(-)_A^\wedge$.

2. For all $M \in D(\mathbb{Z})$ and $N \in D(A)$, we have that $R\underline{\mathrm{Hom}}_{\mathbb{Z}}(M, N) \in D(A)$.

3. The left adjoint $(-)_A^\wedge$ sends connective to connective objects.

4. The underlying condensed animated ring $A^\triangleright$ lies in $D(A)$.

We denote by AnRing be the category of analytic rings, by $\mathrm{Aff} := \mathrm{AnRing}^{op}$ its opposite category and by $\mathrm{AnSpec}(-) := (-)^{op} : \mathrm{AnRing} \to \mathrm{Aff}$ the opposite functor. Condition (1) in Definition 4.1 implies that $D(A)$ is an accessible localisation of $\mathrm{Mod}_{A^\triangleright} D(\mathbb{Z})$ and thus a presentable category. For any morphism $f : \mathrm{AnSpec}(B) \to \mathrm{AnSpec}(A)$ in Aff, there is an associated functor $f^* : D(A) \to D(B)$ in Pr^{L} which is defined by the diagram

$$
\begin{array}{ccc}
\mathrm{Mod}_{A^\triangleright} D(\mathbb{Z}) & \overset{f^{\triangleright *}}{\longrightarrow} & \mathrm{Mod}_{B^\triangleright} D(\mathbb{Z}) \\
j_A \uparrow & & \downarrow (-)_B^\wedge \\
D(A) & \underset{f^*}{\longrightarrow} & D(B)
\end{array}
$$

Here we denote by $f^{\triangleright,*} := B^\triangleright \otimes_{A^\triangleright} (-)$ the base change functor. Denoting the right adjoint of f^* by f_* we thus obtain a functor

$$
D : \mathrm{Aff}^{op} \to \mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L}}), \ \mathrm{AnSpec}(A) \mapsto D(A)
$$
$$
(f : \mathrm{AnSpec}(B) \to \mathrm{AnSpec}(A)) \mapsto f^* : D(A) \to D(B)
$$

Remark 4.2. In order to apply Theorem 3.33, it will be convenient to work with (condensed) $\mathbb{E}_\infty$-rings as opposed to (condensed) animated rings and to consider analytic $\mathbb{E}_\infty$-rings. We denote by AnCR the category of analytic complete $\mathbb{E}_\infty$-rings as in [Cam24, Definition 2.1.1]. Indeed, by [Cam24, Lemma 2.1.3] the resulting 6-functor formalism on AnCR is Tannakian. Everything we say in the following should be correct as well for AnCR replacing AnRing.

Our aim is now to apply the construction result [HM24, Proposition 3.3.3]. As a first step we thus need an appropriate class of morphisms I, P and E in Aff.

Definition 4.3. [AS, lecture 17] Let $\pi : X \to Y$ be a map in Aff.

1. We say π is proper if π_* commutes with small colimits and satisfies the projection formula.

2. We say π is an open immersion if it is a D-open immersion in the sense of Definition 3.11.

3. We say π is !-able if it can by written as a composition $\pi = p \circ j$ of an open immersion j and a proper map p.

We denote by E the class of !-able maps and by $I, P \subset E$ the class of open immersions and of proper maps, respectively.

We will need the following alternative description of proper maps.

Lemma 4.4. Let $\pi : \mathrm{AnSpec}(A) \to \mathrm{AnSpec}(B)$ be a map in Aff.

1. π is proper if and only if A carries the induced analytic ring structure of B, i.e. $A = A^{\triangleright}_{/B} := (A^{\triangleright}, \mathrm{Mod}_{A^{\triangleright}} D(B))$.

2. If $\pi : X \to Y$ is proper then it satisfies proper base change, i.e. for any Cartesian square

$$
\begin{array}{ccc}
X' & \xrightarrow{\pi'} & Z \\
\downarrow{f'} & & \downarrow{f} \\
X & \xrightarrow{\pi} & Y
\end{array}
$$

the natural morphism of functors $f^* \pi_* \to f'^* \pi'_*$ is an equivalence. The base change of a proper map is again proper. Any composition of proper maps is proper.

Proof. 1) Assume A carries the induced analytic ring structure. Then $\pi_* : \mathrm{Mod}_{A^{\triangleright}} D(B) \to D(B)$ is the forgetful functor, and hence commutes with colimits and is $D(B)$-linear and thus satisfies the projection formula. Now assume that π is proper. By the projection formula we have for any $N \in D(B)$ equivalences

$$
\pi_*(A^{\triangleright} \otimes_A \pi^*(N)) \cong \pi_*(A^{\triangleright}) \otimes_B N
$$
$$
\cong (\pi_*(A^{\triangleright}) \otimes_{B^{\triangleright}} N)^{\wedge}_B.
$$

However, the left hand side identifies to $\pi_*((A^{\triangleright} \otimes_B N)^{\wedge}_A)$. We thus obtain that the element $A^{\triangleright} \otimes_B N \in \mathrm{Mod}_{A^{\triangleright}} D(B)$ lies automatically in $D(A)$, since the former is generated by colimits of the elements $A^{\triangleright} \otimes_B N$ for $N \in D(B)$ we obtain the desired equivalence $D(A) \cong \mathrm{Mod}_{A^{\triangleright}} D(B)$.
2) Assume that π is proper. Let $f : \mathrm{AnSpec}(C^{\triangleright}, D(C)) \to \mathrm{AnSpec}(B^{\triangleright}, D(B))$ any morphism of affine analytic stacks, then we obtain by the description of pushouts in AnRing !!!REF!!! that the fibre product is given by $\mathrm{AnSpec}(A) \times_{\mathrm{AnSpec}(C)} \mathrm{AnSpec}(B) \cong \mathrm{AnSpec}((A^{\triangleright} \otimes_{B^{\triangleright}} C^{\triangleright}, \mathrm{Mod}_{A^{\triangleright} \otimes_{B^{\triangleright}} C^{\triangleright}} D(C))$. Let $\tilde{\pi}$ be the base change of π along f. The base change diagram reduces to the diagram

$$
\begin{array}{ccc}
\mathrm{Mod}_{A^{\triangleright} \otimes_{B^{\triangleright}} C^{\triangleright}} D(C) & \xrightarrow{\tilde{\pi}_*} & D(C) \\
\uparrow{\tilde{f}^*} & & \uparrow{f^*} \\
\mathrm{Mod}_{A^{\triangleright}} D(B) & \xrightarrow{\pi_*} & D(B)
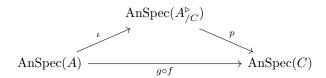\end{array}
$$

which indeed commutes. In particular we see that the fibre product carries the induced analytic ring structure from $(C^{\triangleright}, D(C))$, which shows that $\tilde{\pi}$ is proper by part 1). The claim that a composition of proper maps is again proper can be checked easily from the definition of proper maps. $\square$

Lemma 4.5. Let $\pi : \mathrm{AnSpec}(A) \to \mathrm{AnSpec}(B)$ be a map in Aff. If π is an open immersion then it satisfies proper base change. The base change of an open immersion is again an open immersion. Any composition of open immersions is an open immersion.

Proof. Let $I \in D(B)$ be the idempotent algebra corresponding to the open immersion π, so $D(A) \subset D(B)$ identifies with the full subcategory of elements $M \in D(B)$ for which $I \otimes M \cong 0$. Let $f : \mathrm{AnSpec}(C) \to \mathrm{AnSpec}(B)$ be any map in Aff. By the definition of pushouts in AnRing, we know that $C \otimes_B A$ is given by the completion of the analytic ring structure on $C^\triangleright$ such that module $M \in \mathrm{Mod}_{C^\triangleright \otimes_{B^\triangleright} A^\triangleright} D(\mathbb{Z})$ lies in $D(C \otimes_B A)$ if and only its restriction to $\mathrm{Mod}_{C^\triangleright} D(\mathbb{Z})$ respectively $\mathrm{Mod}_{A^\triangleright} D(\mathbb{Z})$ lies in $D(C)$ respectively $D(A)$. Thus $M \in D(C)$ lies in $D(C \otimes_B A)$ if and only $M \otimes I \cong 0$. This shows that the base change is again an open immersion. The claim that proper base change holds follows by the construction the equivalence $D(C \otimes_B A) \cong \{M \in D(C) \mid f^*(I) \otimes M \cong 0\}$. The claim that a composition of open immersions is again an open immersion can be checked easily from the definition of open immersions. $\qquad\square$

Lemma 4.6. The pair (Aff, E) is a geometric setup

Proof. Since open immersions and proper maps are stable under arbitrary base change in Aff (cf. Lemma 4.4, Lemma 4.5), the same is true for any !-able map. We now check stability under composition. Let $f : \mathrm{AnSpec}(A) \to \mathrm{AnSpec}(B)$ and $g : \mathrm{AnSpec}(B) \to \mathrm{AnSpec}(C)$ be !-able maps and write $f = p_1 \circ j_1$ and $g = p_2 \circ j_2$ for $j_i \in I$ and $p_i \in P$. We consider the factorisation

$$
\begin{array}{ccc}
 & \mathrm{AnSpec}(A^\triangleright_{/C}) & \\
{}^{\iota}\nearrow & & \searrow^{p} \\
\mathrm{AnSpec}(A) \xrightarrow{\qquad g \circ f \qquad} & & \mathrm{AnSpec}(C)
\end{array}
$$

Here the map p is clearly proper, since it is given by the induced analytic ring structure from C. It thus suffices to check that ι is an open immersion. Note that we have a cartesian diagram:

$$
\begin{array}{ccc}
\mathrm{AnSpec}(A^\triangleright_{/B}) & \xhookrightarrow{\tilde{j_2}} & \mathrm{AnSpec}(A^\triangleright_{/C}) \\
\downarrow{\scriptstyle \tilde{p_1}} & & \downarrow{\scriptstyle p_1} \\
\mathrm{AnSpec}(B) & \xhookrightarrow{j_2} & \mathrm{AnSpec}(B^\triangleright_{/C})
\end{array}
$$

It thus follows that $\iota \cong \tilde{j_2} \circ j_1$ is an open immersion since it is a composition of open immersions (cf. Lemma 4.5). $\qquad\square$

Lemma 4.7. The classes $I, P \subset E$ satisfy the following properties:

1. I and P contain all identity morphisms and are stable under compositions and arbitrary base change in Aff.

2. every $f \in E$ can be written as a composition $f = p \circ j$ with $p \in P$, $j \in I$.

3. I and P are right cancellative. i.e. if $p, p \circ q \in I$ (respectively in P) then $q \in I$ (respectively in P)

4. Every morphism $f \in I \cap P$ is n-truncated for some $n \geq -2$ (possibly depending on f).

Proof. 1) An isomorphism $X \cong Y$ in Aff is clearly an open immersion and proper. The claim that I and P are stable under base change and composition is Lemma 4.4 and Lemma 4.5. The condition 2) is satisfied by definition of the class E.

3) Consider the diagram

$$\begin{array}{ccc} \text{AnSpec}(A) & \xrightarrow{q} & \text{AnSpec}(B) \\ & {\scriptstyle p\circ q}\searrow & \downarrow{\scriptstyle p} \\ & & \text{AnSpec}(C) \end{array}$$

and consider the proper case. Note that for any proper map $p : \text{AnSpec}(A^{\triangleright}, \text{Mod}_{A^{\triangleright}} D(B)) \to \text{AnSpec}((B^{\triangleright}, D(B)))$ the functor $p_* : \text{Mod}_{A^{\triangleright}} D(B) \to D(B)$ is the forgetful functor and hence conservative (cf. Lemma 4.4). Assume $p, p \circ q$ are proper. Then q_* commutes with all colimits since p_* commutes with colimits and $p_* \circ q_*$ is conservative. The fact that q_* satisfies the projection formula follows from Example 2.1.

We now consider the case of open immersions. Let $j := p \circ q$ and consider $D := \text{cofib}(j_!(1) \to 1) \in D(C)$ the idempotent algebra corresponding to the open immersion j. Then $p^*(D) \in D(B)$ is again idempotent and we claim that it defines an open immersion via Lemma 4.4. We thus need to show

$$\ker(q^*) \cong \text{Mod}_{p^*(D)} D(B)$$

as full subcategories of $D(B)$. If $M \in \text{Mod}_{p^*(D)} D(B)$, then $p^*(D) \otimes_B M \cong M$ so applying $q^*(-)$ we obtain $q^* p^*(D) \otimes_A q^*(M) \cong q^*(M) \cong 0$ since $q^* p^*(D) \cong j^*(D) \cong 0$, so $M \in \ker(q^*)$. Conversely, let $M \in \ker(q^*)$, and let $X := \text{AnSpec}(A), Y := \text{AnSpec}(B), Z := \text{AnSpec}(C)$. Consider the following cartesian diagram:

$$\begin{array}{ccccc} & & \overset{\tilde{j}}{\overbrace{}} & & \\ X \times_Z Y & \xrightarrow{\tilde{q}} & Y \times_Z Y & \xrightarrow{\tilde{p}} & Y \\ \downarrow{\scriptstyle h} & \lrcorner & \downarrow{\scriptstyle \tilde{p}} & \lrcorner & \downarrow{\scriptstyle p} \\ X & \xrightarrow{q} & Y & \xrightarrow{p} & Z \\ & & \underset{j}{\underbrace{}} & & \end{array}$$

We want to verify that $p^*(D) \otimes_B M \cong M$. Note that

$$p^*(D) \otimes_B M \cong \text{cofib}(p^*(j_!(1)) \otimes_B M \to M)$$
$$\cong \text{cofib}(\tilde{j}_! \tilde{j}^*(M) \to M).$$

Here we used the symmetric monoidality of $p^*(-)$ in the first line and the fact that $\tilde{j}$ an open immersion, as it is a base change of the open immersion j and hence satisfies the projection formula for $\tilde{j}_!$. It thus suffices to show that $\tilde{j}_!(\tilde{j}^*(M)) \cong 0$, which follows by the following diagram chase:

$$\tilde{j}_! \tilde{j}^*(M) \cong q_! h_! \tilde{q}^* \tilde{p}^*(M)$$
$$\cong q_! q^* \tilde{p}_! \tilde{p}^*(M)$$
$$\cong q_! q^* p^* p_!(M)$$
$$\cong q_! q^*(M) \cong 0.$$

Here we used the definition of j and the commutativity of the left square in the first line, proper base change for the left square in the second line, proper base change for the right square in the third line and fully faithfulness of $p_!$ and that $M \in \ker(q^*)$ in the last line.

4) For any open immersion $j : U \to X$ we have (for example by the description in terms of idempotent algebras in Proposition 3.9) that the diagonal $\Delta_j : U \to U \times_X U$ is an isomorphism, thus j is a monomorphism in Aff and hence (-1)-truncated. $\qquad \square$

Definition 4.8. [HM24, Definition 3.3.2] Let (C, E) be a geometric setup such that C admits finite products. A suitable decomposition of E is a pair $I, P \subset E$ such that the conditions 1)-4) of the previous lemma are satisfied.

Thus the pair $I, P \subset E$ of open immersions and proper maps define a suitable decomposition of E in Aff.

Lemma 4.9. Any proper map $p : \mathrm{AnSpec}(B) \to \mathrm{AnSpec}(A)$ is Künneth.

Proof. Let $g : \mathrm{AnSpec}(C) \to \mathrm{AnSpec}(A)$ be any morphism in Aff. By Lemma 4.4, we have an equivalence $D(B) \cong \mathrm{Mod}_{B^\triangleright} D(A)$. Since any base change of a proper map is proper by Lemma 4.4, we obtain that $D(C \otimes_A B) \cong \mathrm{Mod}_{g^*(B^\triangleright)} D(C)$. On the other hand we have equivalences

$$D(C) \otimes_{D(A)} D(B) \cong D(C) \otimes_{D(A)} \mathrm{Mod}_{B^\triangleright} D(A) \cong \mathrm{Mod}_{g^*(B^\triangleright)} D(C).$$

$\qquad \square$

Lemma 4.10. Consider a cartesian diagram in Aff

$$
\begin{array}{ccc}
\mathrm{AnSpec}(D) & \xrightarrow{\ j'\ } & \mathrm{AnSpec}(C) \\
\downarrow{\scriptstyle p'} & \ulcorner & \downarrow{\scriptstyle p} \\
\mathrm{AnSpec}(A) & \xrightarrow{\ j\ } & \mathrm{AnSpec}(B)
\end{array}
$$

with $j \in I$ and $p \in P$. Then the natural map $j_! p'_* \to p_* j'_!$ is an isomorphism of functors.

Proof. Since j' is an open immersion the functor j'^* is essentially surjective. It thus suffices to show the isomorphism for $M := j'^*(N)$, $N \in D(X')$. Let $I \in D(B)$ be the idempotent algebra corresponding to j, then since p (and hence p') is proper we have that $C \cong C^\triangleright_{/B}$ the idempotent algebra corresponding to j' is given by by $p^*(I) \cong I \otimes_B C^\triangleright$. We claim that there are the following equivalences:

$$
\begin{aligned}
p_* j'_! j'^*(N) & \cong p_*(\mathrm{fib}(N \to N \otimes_C p^*(I))) \\
& \cong \mathrm{fib}(p_*(N) \to p_*(N) \otimes_B I)
\end{aligned}
$$

Here we used the definition of $j'_!$ in the first line and that p_* commutes with limits and satisfies proper base change in the second line. On the other hand, we obtain $j_! p'_* j'^*(N) \cong j_! j^* p_*(N)$ by proper base change. Using the definition of $j_! j^*$ we thus obtain

$$
\begin{aligned}
j_! p'_* j'^*(N) & \cong j_! j^* p_*(N) \\
& \cong \mathrm{fib}(p_*(N) \to p_*(N) \otimes_B I)
\end{aligned}
$$

which shows that the natural map $j_! p'^* \to p_* j'_!$ is an equivalence. $\qquad \square$

Proposition 4.11. [AS, lecture 17] The pair (Aff, E) defines a geometric setup and the functor $D : \mathrm{Aff}^{op} \to \mathrm{Pr}^{\mathrm{L}}$ extends to a 6-functor formalism $D : Corr(\mathrm{Aff}, E) \to \mathrm{Pr}^{\mathrm{L}}$ which is Künneth (and Tannakian if we work with AnCR).

Proof. To prove that the functor D extends to a 6-functor formalism we need to show that the assumptions a), b) and c) in [HM24, Proposition 3.3.3] are verified. The conditions a) and b) are satisfied by definition of the classes I and P and by the base change results for categorical open immersions and proper maps Lemma 4.4, Lemma 4.5. The last condition c) is satisfied by Lemma 4.10. By [HM24, Proposition 3.3.3] we thus obtain a 6-functor formalism

$$D : Corr(\mathrm{Aff}, E) \to \mathrm{Pr}^{\mathrm{L}}$$

such that for all $p \in P$ the functor $p_!$ is right adjoint to p^* and for all $j \in I$ the functor $j_!$ is left adjoint to j^*.

We will now prove that any !-able map $f : \mathrm{AnSpec}(A) \to \mathrm{AnSpec}(R)$ is Künneth. By definition $f \cong p \circ j$ for p a proper map and j an open immersion. By Lemma 4.9 and Proposition 3.12, open immersion and proper maps are Künneth, so we conclude by Lemma 3.4. The claim that D is Tannakian is [Cam24, Lemma 2.1.3]. $\qquad\square$

We work with the following definition of analytic stacks which is slightly simplified compared to the correct definition in [AS, Lecture 19].

Definition 4.12. [AS, Lecture 19] An analytic stack is a sheaf $X \in \mathrm{Shv}(\mathrm{Aff}, \mathrm{Ani})$ for the !-topology on Aff.

By Proposition 4.11 we have a sheafy 6-functor formalism $D : Corr(\mathrm{Aff}, E) \to \mathrm{Pr}^{\mathrm{L}}$ on the geometric setup (Aff, E), with E the class of !-able maps. Using the extension result [HM24, Theorem 3.4.11] we obtain a sheafy 6-functor formalism

$$D_{qc} : Corr(\mathrm{AnSt}, \tilde{E}) \to \mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L}})$$

on the category of analytic stacks with $\tilde{E}$ the class of morphisms as in Proposition 3.23.

Corollary 4.12.1. Let $f : X \to S$ be a morphism of analytic stacks in $\tilde{E}$ and assume that there is a !-cover $g : S' \to S$ with $S' \in C$ and $g \in E_0$. Then f is Künneth (and Tannakian if we work with analytic $\mathbb{E}_\infty$-rings).

Proof. This follows from Proposition 4.11 together with Theorem 3.26. $\qquad\square$

4.2 Examples of analytic stacks and !-covers

We will give some examples of analytic stacks and !-covers.

Example 4.1. [AS, Lecture 19] For any $A \in \mathrm{AnRing}$, the functor

$$\mathrm{AnSpec}(A) : F : \mathrm{AnRing} \to \mathrm{Ani}, \ B \mapsto \mathrm{Hom}(A, B)$$

is an analytic stack.

Example 4.2. [AS, Lecture 19] Let $S \in \mathrm{Prof}^{\mathrm{light}}$ be a light profinite set, then we associate the analytic stack $\mathrm{AnSpec}(C(S, \mathbb{Z}))$, where $C(S, \mathbb{Z})$ is the ring of $\mathbb{Z}$-valued continuous functions on S, with its discrete analytic ring structure. This gives a functor

$$(-)_{\mathrm{Betti}} : \mathrm{Prof}^{\mathrm{light}} \to \mathrm{AnSt} .$$

One can check that this sends surjections of light profinite sets to !-covers and thus gives a functor from the category of compact Hausdorff spaces LCHaus to analytic stacks.

$$(-)_{\mathrm{Betti}} : \mathrm{CHaus} \to \mathrm{AnSt} .$$

Example 4.3. Let us sketch how to realise algebraic stacks in the category AnSt. Let Ring be the category of animated commutative rings. For $R \in \mathrm{Ring}$ we can associate to it the analytic ring $R^{\triangleright} := (\underline{R}, \mathrm{Mod}_{\underline{R}} D(\mathbb{Z})))$, giving a functor

$$(-)^{\triangleright} : \mathrm{Ring} \to \mathrm{AnRing}$$

This functor sends fpqc covers to !-covers, we thus obtain a functor

$$(-)^{\triangleright} : \mathrm{Shv}_{fpqc}(\mathrm{Ring}, \mathrm{Ani}) \to \mathrm{AnSt}$$

Example 4.4. Let (A, A^+) be a sheafy analytic Huber pair, then by [And21, Proposition 3.34] we can associate to it an analytic ring $(A, A^+)_{\square}$. Denoting by AnH the category of complete sheafy analytic Huber pairs we thus obtain a functor

$$(-)_{\square} : \mathrm{AnH} \to \mathrm{AnSt}, \ (A, A^+) \mapsto \mathrm{AnSpec}((A, A^+)_{\square}).$$

This functor sends open covers to !-covers, inducting a functor from the category AdicSp of analytic adic spaces to analytic stacks.

$$(-)_{\square} : \mathrm{AdicSp} \to \mathrm{AnSt}$$

4.3 A p-adic version of Drinfeld's lemma

In this subsection we want to give an application of our results which we hope to be of interest for a p-adic version of the work of Fargues and Scholze (cf. [FS21]). A central object of the latter is the following analytic adic space

$$Y_{(0,\infty),S} := \mathrm{Spa}(W(R^+)) \backslash V([\pi]p),$$

where $W(-)$ denotes the p-typical Witt vectors, $S = \mathrm{Spa}(R, R^+)$ an affinoid perfectoid space in characteristic p and $\pi \in R^+$ a pseudo-uniformiser. One main insight of the work of Anschütz, Le Bras and Mann ([ABM24]), is that the p-adic pro-étale cohomology of S can be understood by quasi-coherent cohomology of the quotient of the analytic adic space $Y_{(0,\infty),S}$ by the Frobenius φ induced by R^+

$$Y_{(0,\infty),S}/\varphi^{\mathbb{Z}}.$$

The Drinfeld lemma in the ℓ-adic setting of Fargues Scholze deals with the v-stack $\mathrm{Div}_k^1 := (\mathrm{Spa}(\mathbb{Q}_p^{\mathrm{cycl}})/\mathbb{Z}_p^{\times})/\varphi^{\mathbb{Z}}$ and the full subcategory category $D_{dlb}(Y, \Lambda) \subset D_{et}(Y, \Lambda)$ of dualisable

objects in the category of étale sheaves with coefficients in $\Lambda := \bar{\mathbb{Q}}_\ell$ on a small v-stack Y. It can be phrased as an equivalence of idempotent complete stable categories

$$D_{dlb}(\mathrm{Div}_k^1, \Lambda) \otimes_{D_{dlb}(\mathrm{Spd}(k), \Lambda)} D_{dlb}(W, \Lambda) \cong D_{dlb}(\mathrm{Div}_k^1 \times_{\mathrm{Spd}(k)} W, \Lambda)$$

for W any small v-stack (cf. [FS21, Chapter IV.7.3]). Guided by the work [ABM24] and motivated to obtain a p-adic version of Drinfeld's Lemma we consider the following analytic stacks

$$\mathrm{Div}_{\mathbb{Q}_p}^{1,\mathrm{ct}} := ((Y_{(0,\infty), \mathbb{Q}_p^{\mathrm{cycl},b}})_\square / (\mathbb{Z}_p^\times)_{\mathrm{Betti}}) / \varphi^{\mathbb{Z}}$$

$$\mathrm{Div}_{\mathbb{Q}_p}^{1,\mathrm{la}} := (\lim_{T \mapsto (T+1)^p - 1} \mathbb{D}_\square \backslash \{0\}) / \mathbb{Q}_p^{\times,\mathrm{la}}.$$

By Theorem 3.24, any morphism $X \to S$ of analytic stacks in $\tilde{E}$ with $S \in \mathrm{AnSt}$ such that Δ_S is representable and admitting a !-cover $U \to S$ with $U \in C$ give examples of Künneth morphisms for the 6-functor formalism $D_{qc} : Corr(\mathrm{AnSt}, \tilde{E}) \to \mathrm{Pr}^L$. In particular we obtain the following example.

Corollary 4.12.2. The structural morphism $\mathrm{Div}_{\mathbb{Q}_p}^{1,?} \to \mathrm{AnSpec}(\mathbb{Q}_{p,\square})$ for $? \in \{\mathrm{ct}, \mathrm{la}\}$ lies in $\tilde{E}$ and is thus Künneth.

We end with an example for a morphism which is not Künneth. One way to construct such a morphism is to consider stacks S with non-affine diagonal:

Example 4.5. Let k be an algebraically closed field and consider $E \to \mathrm{Spec}(k)$ an elliptic curve. We consider $E^\triangleright \to \mathrm{AnSpec}(k^\triangleright) := *$ with its discrete induced analytic structure. Then the morphism $f : * \to */E^\triangleright$ is not Künneth. Indeed, note that it lies in $\tilde{E}$ as it has a section. We argue that f is descendable and weakly cohomologically proper and that $f_*(1) \cong 1$. We thus obtain an isomorphism

$$D_{qc}(*/E^\triangleright) \cong \mathrm{coMod}_{f^* f_*(1)} D_{qc}(*) \cong D_{qc}(*).$$

In particular, we observe that the morphism

$$D_{qc}(*) \otimes_{D_{qc}(*/E^\triangleright)} D_{qc}(*) \cong D(*) \to D_{qc}(* \times_{*/E^\triangleright} *) \cong D_{qc}(E^\triangleright)$$

is not an isomorphism. Note that f has non-representable diagonal Δ_f as the pullback along itself gives $* \times_{*/E^\triangleright} * \cong E^\triangleright$ which does not lie in Aff.

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