DigComm - HW 1

Friday, January 26, 2024 2:05 PM

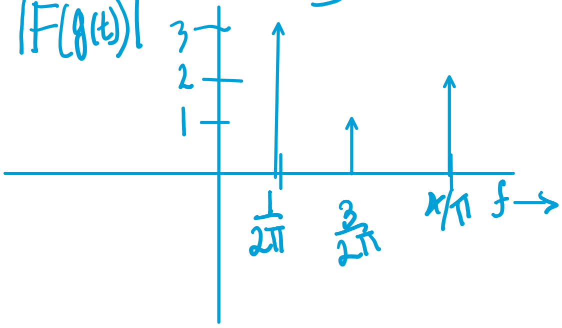
1. A periodic signal $g(t)$ is expressed by the following Fourier series:

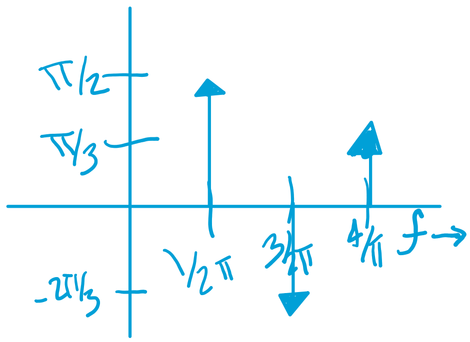
$$g(t) = 3\sin t + \cos\left(3t - \frac{2\pi}{3}\right) + 2\cos\left(8t + \frac{\pi}{3}\right)$$

(a) Sketch the amplitude and phase spectra for the trigonometric series.

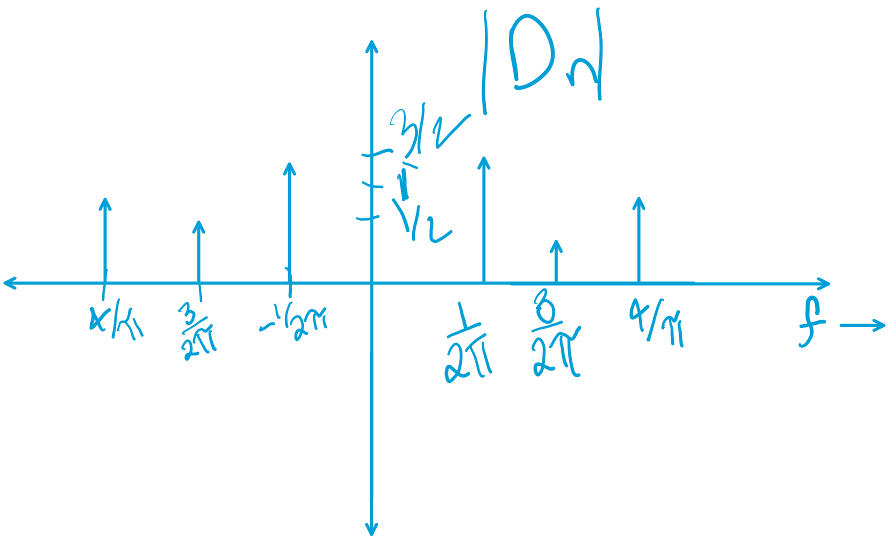
$f = \left[\frac{1}{2\pi}, \frac{3}{2\pi}, \frac{4}{\pi}\right]$

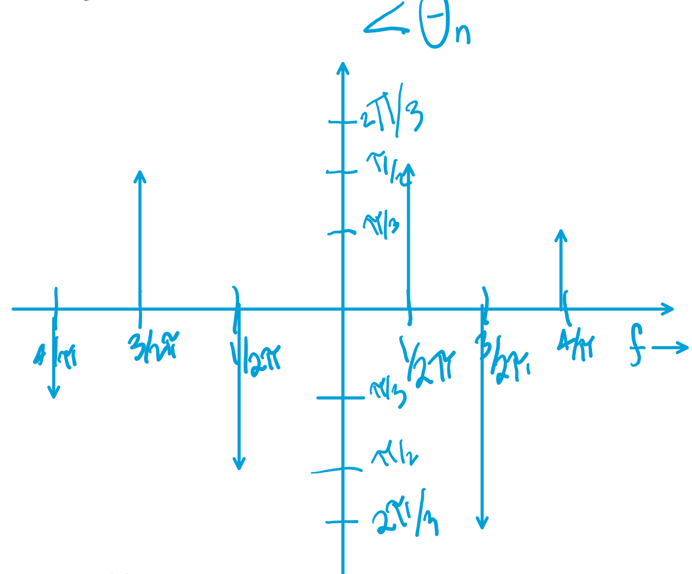
$\phi = \left[\pi/2, -\frac{2\pi}{3}, \pi/3\right]$





(b) By inspection of spectra in the first part, sketch the exponential Fourier series spectra.





(c) By inspection of spectra in the second part, write the exponential Fourier series for $g(t)$.

$$g(t) = D_0 + \sum_{n=1}^{\infty}\left(D_n e^{jn2\pi f_0 t} + D_{-n} e^{-jn2\pi f_0 t}\right)$$

$$D_n = \frac{1}{2}C_n e^{j\theta_n}$$

$$D_{-n} = \frac{1}{2}C_n e^{-j\theta_n}$$

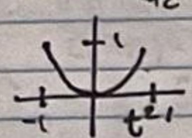
$g(t) = \frac{3}{2}e^{j\pi/2}e^{jt} + \frac{3}{2}e^{-j\pi/2}e^{jt}$

$+\frac{1}{2}e^{-j2\pi/3}e^{j3t} + \frac{1}{2}e^{j2\pi/3}e^{j3t} + e^{j\pi/3}e^{j8t} + e^{-j\pi/3}e^{j8t}$

2. (a) Sketch the signal $g(t) = t^2$ and find the exponential Fourier series to represent $g(t)$ over the interval $(-1, 1)$. Sketch the Fourier series for all values of t.

$\cos(\theta) = \frac{e^{j\theta} + e^{-j\theta}}{2}$ $\sin(\theta) = \frac{e^{j\theta} - e^{-j\theta}}{2j}$ $\frac{1}{j} = -j$

$D_n = \frac{1}{T_0} \int_{-T_0/2}^{T_0/2} f(t) e^{-jn\omega_0 t} \, dt \quad T_0 = 2 \quad f_0 = 1/2 \quad \omega_0 = \pi$



$D_n = \frac{1}{T_0} \int_{-1}^{1} t^2 e^{-j\pi n t} \, dt$

+	t^2	$e^{-j\pi n t}$
−	$2t$	$e^{-j\pi n t}/-j\pi n$
+	2	$e^{-j\pi n t}/(-j\pi n)^2$
−	0	$e^{-j\pi n t}/(-j\pi n)^3$

$T_0 D_n = \frac{t^2 e^{-jn\pi n t}}{-j\pi n} - \frac{2t e^{-j\pi n t}}{(-j\pi n)^2} + \frac{2e^{-j\pi n t}}{(-j\pi n)^3} \Big|_{-1}^{1}$

$= \frac{j t^2 e^{-jn\pi t}}{\pi n} + \frac{j 2t e^{-j\pi n t}}{(\pi n)^2} - \frac{j 2 e^{-j\pi n t}}{(\pi n)^3} \Big|_{-1}^{1}$

$= \frac{j e^{-jn\pi}}{\pi n} + \frac{j2 e^{-j\pi n}}{(\pi n)^2} - \frac{j2 e^{-j\pi n}}{(\pi n)^3} - \frac{j e^{jn\pi}}{\pi n}$

$+ \frac{j2 e^{jn\pi}}{(\pi n)^2} + \frac{j2 e^{j\pi n}}{(\pi n)^3}$

$= \frac{j}{\pi n}\left(e^{-jn\pi} - e^{jn\pi}\right) + \frac{2j}{(\pi n)^2}\left(e^{-j\pi n} + e^{jn\pi}\right)$

$+ \frac{2j}{(\pi n)^3}\left(-e^{-j\pi n} + e^{j\pi n}\right)$

$\cancel{D_n = \frac{1}{T_0}\left(\frac{j}{\pi n}(-1)\sin(\pi n)(2j)\right)}$

$D_n = \frac{1}{T_0}\left(\frac{-j}{\pi n}(2j)\sin(\pi n)\right) + \frac{1}{T_0}\left(\frac{2j}{(\pi n)^2}(2)\cos(\pi n)\right)$

$+ \frac{1}{T_0}\left(\frac{2j}{(\pi n)^3}(2j)(\sin(\pi n))\right)$

$= \frac{1}{T_0}\left[\frac{2}{\pi n}\cancel{\sin(\pi n)} + \frac{4j}{(\pi n)^2}\cos(\pi n) + \frac{-4}{(\pi n)^3}\cancel{\sin(\pi n)}\right]$

(b) Verify Parseval's theorem for this case, given that

$$\sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90}.$$

(a) $C_n = \frac{1}{2}\frac{-4j^2(-1)^2}{(\pi n)^2} = \frac{2(-1)^n}{(n\pi)^2}$

$C_0 = \frac{1}{2}\int_{-1}^{1} t^2\,dt = \frac{1}{3}$

(b) Parsavels theroem: $P = |C_0|^2 + \sum_{n=-\infty}^{\infty} |C_n|^2$

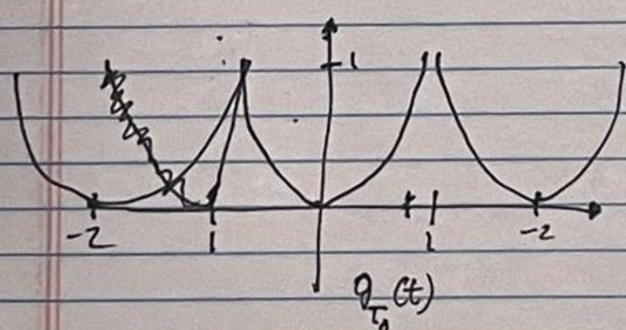
$P_{signal} = \frac{1}{T_0}\int_{-T_0/2}^{T_0/2} g(t)g^*(t)\,dt = \frac{1}{2}\int_{-1}^{1} t^4\,dt = \frac{1}{5}$

$P = \frac{1}{9} + 2\sum_{n=1}^{\infty}\frac{4}{(n\pi)^4} = \frac{1}{5}$

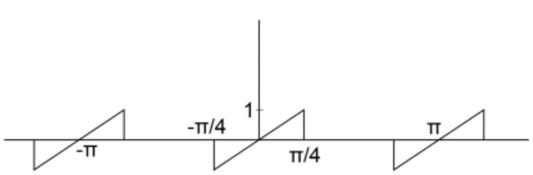
$\frac{1}{5} - \frac{1}{9} = \frac{(2)(4)}{\pi^4}\sum_{n=1}^{\infty}\frac{1}{n^4} = \frac{\pi^4}{8}\left(\frac{1}{5}-\frac{1}{9}\right)$

$\frac{9}{45} - \frac{5}{45} = \frac{4}{45}$. $\frac{4\pi^4}{360} = \frac{\pi^4}{90}$

$\sum_{n=1}^{\infty}\frac{1}{n^4} = \frac{\pi^4}{90}$



3. For the periodic signals shown in the figure, find the compact trigonometric Fourier series and sketch the amplitude and phase spectra. If either the sine or cosine terms are absent in the Fourier series explain why.



3. (a) $C_n = \sqrt{a_n^2 + b_n^2}$ $\quad \theta_n = \tan^{-1}\left(\frac{-b_n}{a_n}\right)$

$f(t) = \frac{4}{\pi} t \quad \{ -\pi/4 \le t \le \pi/4$

$T_0 = \pi \quad f_0 = \frac{1}{\pi} \quad \omega_0 = 2$

$a_n = \frac{2}{T_0}\int f(t)\cos(n\omega_0 t)\,dt = 0$ (odd function)

$b_n = \frac{2}{\pi}\int_{-\pi/4}^{\pi/4} (4/\pi) t \sin(2nt)\,dt$; $u = t \quad du = dt$

$dv = \sin(2nt)$

$= \frac{8}{\pi^2}\left[\frac{-t\cos(2nt)}{2n} + \frac{1}{2n}\int \cos(2nt)\,dt\right. \quad V = \frac{-\cos(2nt)}{2n}$

$= \frac{8}{\pi^2}\left[\frac{-t\cos(2nt)}{2n} + \frac{1}{(2n)^2}\sin(2nt)\right]_{-\pi/4}^{\pi/4}$

$= \frac{8}{\pi^2}\left[\frac{-(\pi/4)\cos(2n(\pi/4))}{2n} + \frac{\sin(2n(\pi/4))}{4n^2}\right]$

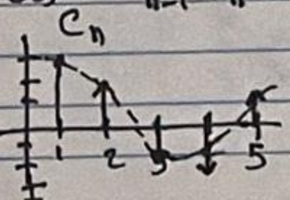
$- \frac{8}{\pi^2}\left[\frac{\pi/4\cos(2n(\pi/4))}{2n} - \frac{\sin(2n\pi/4)}{(2n)^2}\right]$

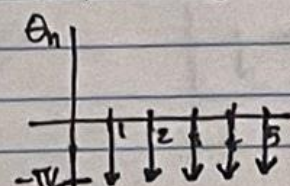
$= \frac{8}{\pi^2}\left[\frac{-\pi/2\cos(\frac{n\pi}{2})}{2n} + \frac{2\sin(\frac{n\pi}{2})}{(2n)^2}\right]$

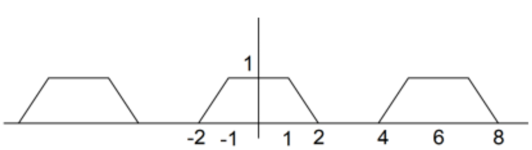
$= \frac{-2\cos(\frac{n\pi}{2})}{n\pi} + \frac{4\sin(\frac{n\pi}{2})}{(n\pi)^2}$

$C_n = b_n \quad \theta_n = -\pi/2 \quad a_0 = 0$ (avg area is 0)

$g(t) = \sum_{n=1}^{\infty} b_n \cos(2nt - \pi/2)$







3.b $T_0 = 6 \quad f_0 = 1/6 \quad \omega_0 = \pi/3$

$$a_n = \frac{2}{3}\int_0^1 \cos(\pi/3\, n t)\,dt + \frac{2}{3}\int_1^2 (-t+2)\cos(\pi/3\, n t)\,dt$$

$$a_n = \frac{2\sin(\pi/3\, n t)}{3(\pi/3)n}\Big|_0^1 + \frac{4}{3}\frac{\sin(\pi/3\, n t)}{\pi/3\, n}\Big|_1^2$$

$$-\frac{2(\pi n t\sin(\frac{\pi n t}{3}) + 3\cos(\frac{\pi n t}{3}))}{\pi^2 n^2}\Big|_1^2$$

$$a_n = \frac{-6\cos(\frac{2\pi n}{3}) + 2\pi n\sin(\frac{\pi n}{3}) - 6\cos(\frac{\pi n}{3})}{\pi^2 n^2} + \frac{2\sin(\pi n/3)}{\pi n}$$

$b_n = 0$ (even function)

$$C_0 = \frac{1}{6}(2)\int_0^1 1\,dt + \frac{2}{6}\int_1^2 (-t+2)\,dt = 1/2$$

$C_n = \sqrt{a_n^2 + b_n^2} \quad \theta_n = \tan^{-1}(\frac{-b_n}{a_n})$

$C_n = a_n \quad \theta_n = 0$

$$g(t) = 1/2 + \sum_{n=1}^{\infty} C_n \cos(\frac{n\pi t}{3})$$