Bachelor Thesis

Online Portfolio Optimization under CV@R Constraints with Stochastic Mirror Descent

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Abstract

We find a solution to the problem of portfolio optimization under risk-management constraints using an online method that estimates the portfolio parameters in parallel to approximating the optimal investment strategy. At each iteration, assuming the estimators of portfolio parameters are valid, we approximate the investment strategy with the highest return using a recently developed algorithm based on stochastic mirror descent. We keep updating the estimators making them converge to the true portfolio parameters in the long run, thus causing the strategy to converge to the real optimal investment. We establish results regarding the asymptotic properties of convergence of the estimators of portfolio parameters as well as the rate of convergence of the online method. We illustrate the behavior of our algorithm when ran numerically on simulated data and study its experimental convergence.

Contents

1 Introduction

The question of optimizing a financial portfolio under a risk management constraint has been addressed several times in the last few years with many different approaches by Ermoliev and Wets [EW88], Kall and Wallace [KWK94], Prekopa [PP95], Kan and Kibzun [KK97], Birge and Louveaux [BL11], Rockafellar and Uryasev [RU$^+$00], Costa, Gadat and Huang [CGH22], and others. A simple statement of the problem is the following: given a portfolio of financial assets, what is the best way to invest while satisfying a given risk constraint? The necessity of this constraint comes from the fact that even though an investment in the unique asset of highest expected return works best in the long run, it generally causes issues in the short term because of high variability. Hence, many questions arise.

First, a valid representation of the portfolio assets is needed to simulate a real-life financial market. Historically, financial assets have been modeled using stochastic differential equations (SDE). These equations have no explicit solution in the general case, which is why we need a discretization scheme to simulate a solution as discussed by Talay and Tubaro [TT90]. However, the procedure of discretization creates a bias in the stochastic approximation. In this work, we exclusively consider risky assets, which correspond to a geometric Brownian motion (GBM) [MR05] of drift μ and volatility σ, a stochastic process for which the natural logarithm follows a Brownian motion. In addition, we choose to neglect the bias caused by the discretization in order to focus on the long term optimization of the investment strategy.

Second, we need to treat the question of the risk management constraint. Many works have used different numerical algorithms to set this constraint ([EW88], [KWK94], [PP95], [KK97], [BL11], [CGH22]), but the most common way it to use the Conditional Value at Risk (CV@R). Intuitively, this can be thought of as the average loss in scenarios where the return is very low at a certain risk level. The CV@R has been discussed in [ADEH99] and proven to be a coherent risk measure thanks to its mathematical properties [Pfl00]. In this work, we compute the risk measure of a given investment strategy using the Monte-Carlo method developed by Rockafellar and Uryasev [RU$^+$00] to approximate the CV@R of a portfolio using a batch empirical mean, and the results from [BTT86] and [Pfl00] linking it to the minimum of a convex function. For more on the risk constraint, please refer to the subsection 2.2 below.

Third, in order to be able to approximate the optimal investment strategy in the long run, we need to impose a reinvestment condition, meaning that all the funds should be reallocated at each step. To be more specific, consider a portfolio of m assets and assume one owns an initial capital K.

Then at each step t, we need an investment strategy $(u_1)_t, \dots, (u_m)_t$ that allocates the same initial capital K, with $\sum_{i=1}^m (u_i)_t = 1$. This means that the amount $(u_i)_t K$ is invested in the i^{th} asset, for all i in $\{1, \dots, m\}$, thus allocating all the capital K. From a computational point of view, this condition means that the problem of optimizing the return of the portfolio should be solved under the constraint of the investment strategy belonging to the simplex $\Delta_m = \left\{u \in \mathbb{R}^m : \sum_{i=1}^m u_i = 1\right\}$. For that, one possible approach is to use an iterative algorithm to approximate the solution of the optimization problem in the whole space $\mathbb{R}^m$ and keep projecting the solution on the simplex Δ_m after each iteration [WCP13]. However, the projection on the simplex of probability distributions can be quite tricky, which is why in this work, we follow the footsteps of Costa, Gadat and Huang [CGH22] who use the stochastic mirror descent (SMD) algorithm developed by Nemirovsky and Yudin in 1983 [NY83] to overcome this issue. This is a generalization of gradient descent (GD) based on replacing the Euclidean norm in the second-order proximity term by the Bregman divergence of a certain convex function.

In their work, Costa et al. employ the SMD framework in order to approximate the optimal investment strategy in a financial portfolio managed by CV@R constraints [CGH22]. However, their investigation assumes that the parameters of the assets' motion (i.e. drift μ and volatility σ of the GBMs) are known, which is generally not true in real life. To overcome this issue, we develop an online algorithm that simultaneously estimates the parameters of the assets and approximates the optimal investment strategy corresponding to those estimations.

This article is organized as follows. In Section 2, we detail settings of the problem, the constraints and the asset modelling. We also give some mathematical background concerning stochastic mirror descent. In Section 3, we discuss an algorithm on which our work is based, that is the algorithm for portfolio optimization under CV@R constraints developed by Costa et al. in [CGH22]. Then, in Section 4, we present our results for the online method for estimating the portfolio parameters and approximating the optimal strategy and the rate of convergence bounds. Finally, in Section 5, we illustrate our results by using the online method we developed on simulated financial data.

2 Preliminaries

2.1 Problem

Consider m assets numbered from 1 to m. We define $A_i(t)$ as the price of asset i at time t. We then define Z as the return vector of the assets at a fixed time in the horizon T given by:

$$Z_i := \frac{A_i(T)}{A_i(0)} - 1.$$

An **investment strategy** is a vector of positive weights $u = (u_1, \dots, u_m)$; $u_i > 0$ s.t. $\sum_{i=1}^m u_i = 1$, i.e.

$$u \in \Delta_m := \Big\{ u \in \mathbb{R}_+^m : \sum_{i=1}^m u_i = 1 \Big\}.$$

Our goal here is to optimize the return of our strategy, given by

$$\langle Z, u \rangle = \sum_{i=1}^m u_i Z_i.$$

2.2 Risk constraint

Let $\alpha \in (0,1)$ be a risk level. We define the value at risk at level α of the strategy u as:

$$V@R_\alpha(u) = \sup \Big\{ q \in \mathbb{R} : \mathbb{P}\Big(\langle Z, u \rangle \leq q \Big) \leq \alpha \Big\}.$$

Intuitively, this can be understood as the worst value we can accept at this risk level. We then define the conditional value at this risk level as the mean of the loss for return values under the $V@R_\alpha(u)$:

$$CV@R_\alpha(u) = \mathbb{E}\Big[-\langle Z, u \rangle | \langle Z, u \rangle \leq V@R_\alpha(u) \Big].$$

At a risk level of α, we are interested in optimizing the return of our strategy while keeping conditional value at risk lower than a certain threshold M, that is:

$$\begin{aligned} P_M &= \mathrm{argmax}_{u \in \Delta_m} \Big\{ \sum_{i=1}^m u_i Z_i : CV@R_\alpha(u) \leq M \Big\} \\ &= \mathrm{argmax}_{\begin{cases} u \in \Delta_m \\ CV@R_\alpha(u) \leq M \end{cases}} \quad \langle Z, u \rangle \\ &= \mathrm{argmin}_{\begin{cases} u \in \Delta_m \\ CV@R_\alpha(u) \leq M \end{cases}} \quad -\langle Z, u \rangle. \end{aligned}$$

Using Lagrange multipliers, the problem above can be re-formulated as:

$$Q_\lambda = \mathrm{argmin}_{u \in \Delta_m} \Big\{ -\sum_{i=1}^m u_i Z_i + \lambda CV@R_\alpha(u) \Big\},$$

for any $\lambda > 0$. In fact, the collection of convex problems $(P_M)_{M>0}$ and $(Q_\lambda)_{\lambda>0}$ are equivalent, that is, for any $M > 0$ for which the P_M exists (feasible constraint),

$$\exists \lambda_M^\star > 0, \quad u_M^\star = \text{argmin}_{u\in\Delta_m}\Big\{ -\sum_{i=1}^{m} u_i Z_i + \lambda_M^\star CV@R_\alpha(u)\Big\}.$$

The expression of the $CV@R_\alpha(u)$ can as also be formulated as:

$$CV@R_\alpha(u) = \min_{\theta\in\mathbb{R}} \psi_\alpha(u,\theta),$$

where ψ is the convex **coercive** Lipschitz continuous and differentiable function defined by:

$$\psi_\alpha(u,\theta) = \theta + \frac{1}{1-\alpha}\mathbb{E}[\lfloor\langle Z,u\rangle - \theta\rfloor_+],$$

where $\lfloor x\rfloor_+ = \max(0,x)$ ([BTT86], [Pfl00]). Finally, the initial problem can be expressed as the convex unconstrained problem:

$$Q_\lambda = \text{argmin}_{(u,\theta)\in\Delta_m\times\mathbb{R}}\{p_\lambda(u,\theta)\},$$

where the objective function p_λ is defined by:

$$p_\lambda(u,\theta) = -\sum_{i=1}^{m} u_i\mathbb{E}[Z_i] + \lambda\psi_\alpha(\theta,u). \tag{1}$$

2.3 Portfolio structure

We consider a portfolio of m financial assets $S = (S^1,\dots,S^m)$. We assume that the portfolio is formed exclusively of risky assets encoded by geometric Brownian motions, that is:

$$\forall i \in \{1,\dots,m\}, \quad dS_t^i = \mu_i S_t^i dt + \sigma_i S_t^i dW_t,$$

which has the solution:

$$S_t^i = S_0^i e^{\left(\mu_i - \frac{\sigma_i^2}{2}\right)t + \sigma_i W_t},$$

or equivalently:

$$\log\frac{S_t^i}{S_0^i} = \left(\mu_i - \frac{\sigma_i^2}{2}\right)t + \sigma_i W_t,$$

where W_t is a Brownian motion, μ_i and σ_i are constants. We will often use the expression above, which will allow us to estimate the parameters of the geometric Brownian motion using an online method.

2.4 Stochastic Mirror Descent (SMD)

2.4.1 Gradient Descent (GD)

When looking to solve the unconstrained optimization problem:

$$\text{argmin}_{x \in \mathbb{R}^m} f(x),$$

we consider an iterative algorithm where at each iteration, we update the algorithm to the minimizer of an approximation of f in the neighborhood of the current position for which the minimizer can be easily determined.

Assuming $f \in C^1_L$, that is, f is differentiable with an L-Lipschitz gradient, we get the following result:

$$\forall (x,y) \in \left(\mathbb{R}^m\right)^2 \quad \left| f(y) - f(x) - \langle \nabla f(x), y - x \rangle \right| \leq \frac{L}{2} \|y - x\|^2,$$

then for any $x_0 \in \mathbb{R}^m$ we can define:

$$\phi_{x_0,+}(x) = f(x_0) + \langle \nabla f(x), x - x_0 \rangle + \frac{L}{2} \|y - x\|^2,$$

called the *surogate* function. It provides a 2^{nd} order approximation of f in the neighbourhood of x_0 and satisfies:

$$f(x) \leq \phi_{x_0,+}(x) \quad \forall x \in \mathbb{R}^n.$$

Our idea is to approximate, iteratively, our objective function f locally using $\phi_{x_k,+}$, and then use an explicit formula of the minimizer of $\phi_{x_k,+}$:

$$\begin{aligned} x_{k+1} &= \text{argmin}_{x \in \mathbb{R}^m} \phi_{x_k,+}(x) \\ &= x_k - \frac{1}{L} \nabla f(x_k), \end{aligned}$$

to estimate the minimizer of f. This corresponds to a gradient descent with a constant step-size equal to L^{-1}. It is called the Maximization-Minimization algorithm and is the optimal way of performing a gradient descent with a constant step-size [ZKY07].

In figure 1, we show an example of a gradient descent performed on a function defined on $\mathbb{R}^2$. We notice the convergence of the sequence of points $(x_k)_{k \geq 0}$ found by the algorithm to the minimizer $x^\star$ of the function.

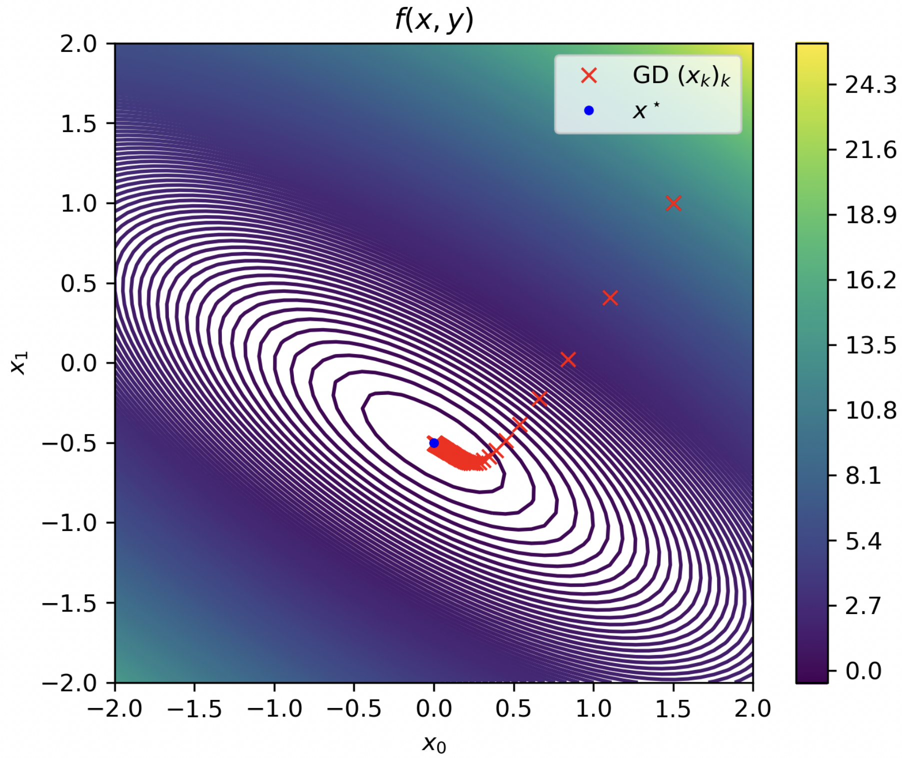

Figure 1: Example of gradient descent using a function f defined on $\mathbb{R}^2$. The colors represent the values taken by f according to the colorbar and the blue dot is its minimizer. The red crosses are the iterations of the GD.

2.4.2 Mirror Descent (MD)

Unlike the simple unconstrained context, solving the constrained optimization problem

$$\operatorname{argmin}_{x \in \mathcal{S}} f(x)$$

where $\mathcal{S} \subseteq \mathbb{R}^m$ might be tricky. One common solution to this problem is to keep projecting the estimation of the minimizer found by GD on the feasible region after each iteration. However, depending on the feasible region $\mathcal{S}$, this projection can be difficult or expensive. In the following, we present a generalization of GD that provides a solution to this problem.

For gradient descent, we used the iterative expression:

$$x_{t+1} = x_t - \eta \nabla f(x_t),$$

which can also be written as:

$$x_{t+1} = \operatorname{argmin}_{x \in \mathbb{R}^m} \left\{ f(x_t) + \langle \nabla f(x_t), x - x_t \rangle + \frac{1}{2\eta} \|x - x_t\|_2^2 \right\},$$

where term $f(x_t) + \langle \nabla f(x_t), x - x_t \rangle$ corresponds to a linear approximation of f around x_t and $\frac{1}{2\eta}\|x - x_t\|_2^2$ is a proximity term, corresponding to a penalization using the Euclidean norm.

The Mirror Descent (MD) algorithm, by Nemirovski and Yudin [NY83], considers changing the squared Euclidean norm in the penalization term seen above by another way of measuring the distance between x_t and x: given a strictly convex function $\phi : \mathbb{R}^m \to \mathbb{R}$ that we will call the *distance-generating function*, we consider replacing the squared Euclidean norm above with the Bregman divergence associated to ϕ, that is:

$$\begin{aligned} \mathcal{D}_\phi : \quad & \mathbb{R}^m \times \mathbb{R}^m \longrightarrow \mathbb{R}_+ \cup \{0\} \\ & (x, y) \longmapsto \phi(x) - \phi(y) - \langle \nabla \phi(y), x - y \rangle. \end{aligned}$$

When we do so, the iterative descent expression becomes:

$$x_{t+1} = \operatorname{argmin}_{x \in \mathbb{R}^m} \left\{ f(x_t) + \langle \nabla f(x_t), x - x_t \rangle + \frac{1}{\eta} \mathcal{D}_\phi(x, x_t) \right\}.$$

Please note that the $\frac{1}{2}$ in the penalization term is removed since it can be introduced in the Bregman divergence by getting it into the distance-generating function, which conserves its strict convexity.

We also note that mirror descent is a generalization of gradient descent as the squared Euclidean norm in the penalization term of GD is nothing but the Bregman divergence associated with the strictly convex function $\phi : x \mapsto \frac{1}{2}\|x\|_2^2$. For more details, please refer to Appendix A.1.

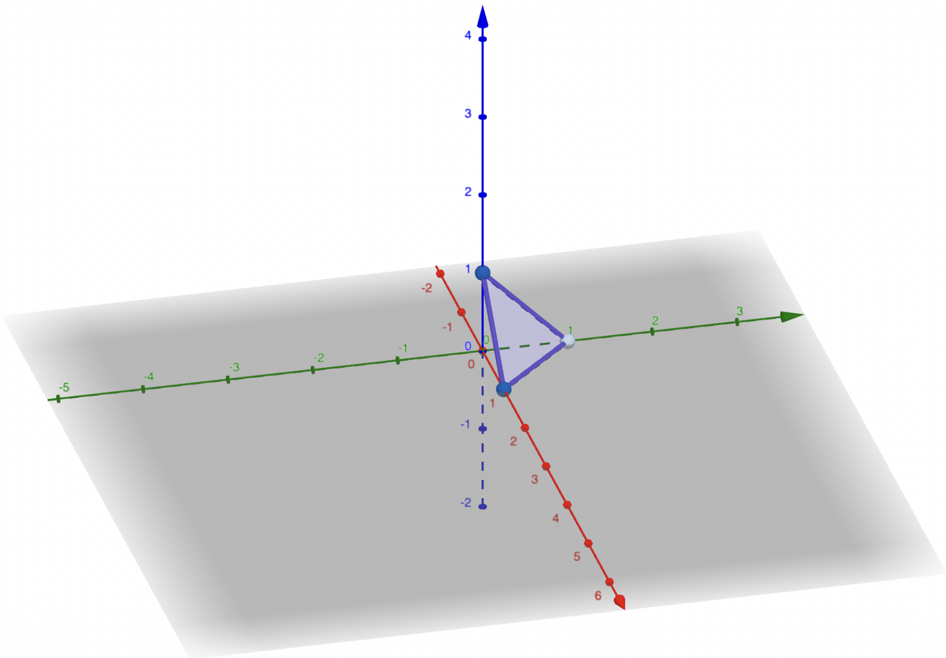

Figure 2: The 2-simplex.

We implement an example of a mirror descent to optimize a function f defined on $\mathbb{R}^3$ in the 2-simplex defined by:

$$\Delta_3 = \left\{ u \in \mathbb{R}^3 : \sum_{i=1}^{3} u_i = 1 \right\},$$

as shown in Figure 2. In Figure 3, we show the values taken by f on the plane of $\mathbb{R}^3$ containing the 2-simplex and plot the iterations of the mirror descent. We observe the convergence of the sequence of points $(x_k)_{k \geq 0}$ found by the algorithm to the minimizer of the f.

2.4.3 Stochastic Mirror Descent (SMD) for portfolio optimization

In our context of portfolio optimization, we wish to approximate the best investment strategy in a set of assets. For that, we take the feasible region $\mathcal{S}$ to be the $m-$dimensional simplex, that is:

$$\mathcal{S} = \Delta_m = \left\{ u \in \mathbb{R}^m : \sum_{i=1}^{m} u_i = 1 \right\}.$$

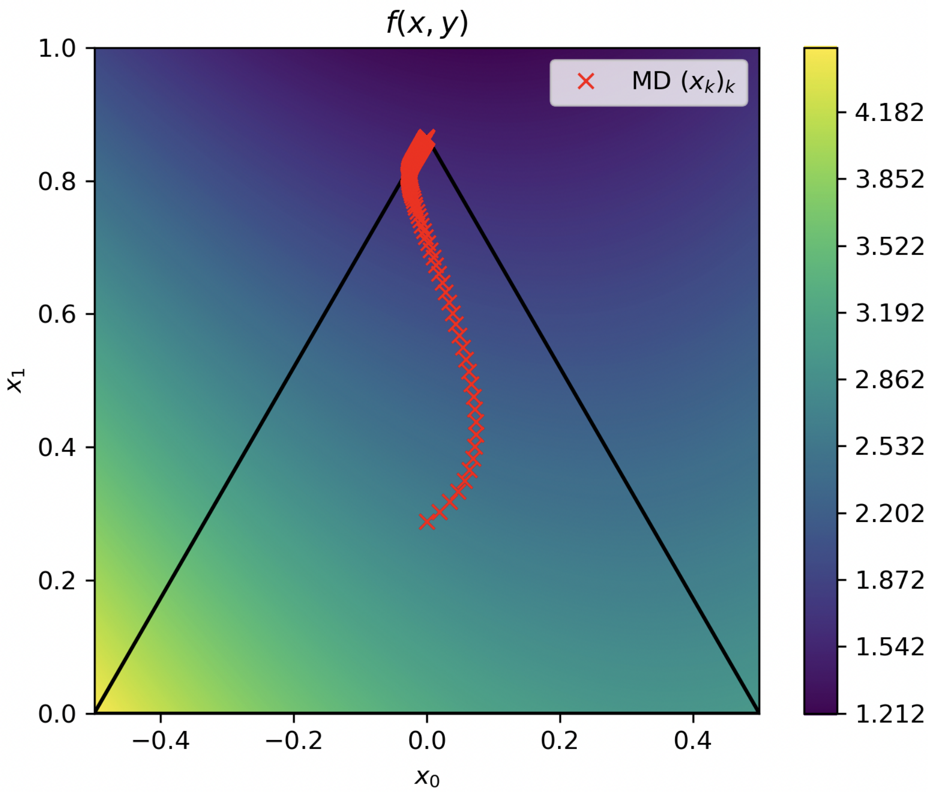

Figure 3: Example of MD using a function f defined on $\mathbb{R}^3$. The colors represent the values taken by f in plane containing the 2-simplex (shown in Figure 2). The black triangle represents the borders of the 2-simplex. The red crosses are the iterations of the MD.

Furthermore, we take the distance-generating function to be the entropy over the simplex of probability distributions, that is:

$$\forall u \in \Delta_m, \quad \phi(u) = \sum_{i=1}^{m} u_i \log u_i.$$

For simplicity, we also consider a constant step-size that we will denote η. In this setting, the iterative expression of mirror descent reads:

$$x_{t+1} = \mathrm{argmin}_{x \in \Delta_m} \Big\{ f(x_t) + \langle \nabla f(x_t), x - x_t \rangle + \frac{1}{\eta} \mathcal{D}_\phi(x, x_t) \Big\}.$$

This expression can be made explicit. Please refer to Appendix A.2 for the computations. We finally obtain:

$$x_{t+1} = \frac{x_t e^{-\eta \nabla f(x_t)}}{\|x_t e^{-\eta \nabla f(x_t)}\|_1}.$$

In our context of portfolio optimization, the objective function f is p_λ seen in (1), for which the gradient cannot be computed explicitly. Hence, we use a stochastic algorithm (Stochastic Mirror Descent) that approximates the value of the gradient by simulating the return vector Z. For this, we assume that the assets follow a GBM of parameters $\xi = (\mu, \sigma^2)$. In fact, we have

$$\begin{aligned} p_\lambda(u, \theta) &= -\sum_{i=1}^{m} u_i \mathbb{E}[Z_i] + \lambda \psi_\alpha(\theta, u) \\ &= -\sum_{i=1}^{m} u_i \mathbb{E}[Z_i] + \lambda \Big(\theta + \frac{1}{1-\alpha} \mathbb{E}[\lfloor \langle Z, u \rangle - \theta \rfloor_+] \Big) \\ &= \mathbb{E}\Big[-\sum_{i=1}^{m} u_i Z_i + \lambda\theta + \frac{\lambda}{1-\alpha} \lfloor \langle Z, u \rangle - \theta \rfloor_+ \Big], \end{aligned}$$

hence

$$\begin{aligned} \nabla p_\lambda(u, \theta) &= \nabla \mathbb{E}\Big[-\sum_{i=1}^{m} u_i Z_i + \lambda\theta + \frac{\lambda}{1-\alpha} \lfloor \langle Z, u \rangle - \theta \rfloor_+ \Big] \\ &= \mathbb{E}\Big[\nabla \Big(-\sum_{i=1}^{m} u_i Z_i + \lambda\theta + \frac{\lambda}{1-\alpha} \lfloor \langle Z, u \rangle - \theta \rfloor_+ \Big) \Big]. \end{aligned}$$

Therefore, the expression of the gradient of p_λ is given by $\nabla p_\lambda(u, \theta) = \big((\partial_u p_\lambda(u, \theta), \partial_\theta p_\lambda(u, \theta) \big)$ where:

$$\begin{cases} \partial_u p_\lambda(u, \theta) = \mathbb{E}\Big[\Big(-1 + \frac{\lambda}{1-\alpha} \mathbb{1}_{\{\langle Z, u \rangle \geq \theta\}} \Big) Z \Big] \\ \partial_\theta p_\lambda(u, \theta) = \mathbb{E}\Big[\lambda - \frac{\lambda}{1-\alpha} \mathbb{1}_{\{\langle Z, u \rangle \geq \theta\}} \Big] \end{cases},$$

and it can hence be estimated by $g_t = (g_{t,1}, g_{t,2})$ with:

$$\begin{cases} g_{t,1} := \Big(-1 + \frac{\lambda}{1-\alpha} \mathbb{1}_{\{\langle Z_k, u\rangle \geq \theta\}} \Big) Z_k \approx \partial_u p_\lambda(u,\theta) \\ g_{t,2} := \lambda - \frac{\lambda}{1-\alpha} \mathbb{1}_{\{\langle Z_k, u\rangle \geq \theta\}} \approx \partial_\theta p_\lambda(u,\theta) \end{cases}, \tag{2}$$

where

$$Z_k = e^{\mu_i - \frac{\sigma_i^2}{2} + \sigma_i N} - 1, \quad N \sim \mathcal{N}(0,1).$$

When we substitute this in the expression found above, we finally obtain the iterative step expression for SMD:

$$x_{t+1} = \frac{x_t e^{-\eta g_t}}{\|x_t e^{-\eta g_t}\|_1}. \tag{3}$$

The result above is used in the findings of Costa, Gadat and Huang [CGH22] that we discuss in the next section.

3 Previous Work

Recent results developed an algorithm to find optimal resource allocation for a financial portfolio under a risk management constraint using stochastic mirror descent. It considers financial assets following a Geometric Brownian Motion (GBM) of drift vector μ and volatility vector σ. This method is described in Algorithm 1.

Therefore we can, given the parameters (μ, σ^2) of the portfolio, use SMD to estimate the optimal strategy U of investment. Let $\xi := (\mu, \sigma^2)$. Then we will denote by

$$\Gamma_{\eta, X_0, \alpha, \lambda, k_{\max}}(\xi)$$

or, depending on the context, simply

$$\Gamma(\xi)$$

the strategy returned by SMD when using the step-size sequence $\eta = \{\eta_n, n \in \mathbb{N}\}$, risk-level α, Lagrange multiplier λ and number of iterations $k_{\max}$ and $X_0 = \big(U_0, \theta_0\big)$.

However, this method assumes that the parameters (μ, σ) of the assets' motion are known. This is generally not true in real life, which is why we address the problem of estimating the GBM parameters parallel to optimizing the investment strategy. The next section presents our results.

Algorithm 1 Portfolio hedging under CV@R$_\alpha$ constraints using Stochastic Mirror Descent [CGH22, Algorithm 1]

Data: step-size sequence $\{\eta_n, n \in \mathbb{N}\}$ and $U_0 \in \mathbb{R}^m$, $\theta_0 \in \mathbb{R}$; $\alpha \in (0,1)$; number of iterations $k_{\max}$

Result: Two sequences: $X_k = (U_k, \theta_k)_{0 \le k \le k_{\max}}$

for $k = 0, \ldots, k_{\max} - 1$ **do**

Simulate the return random vector Z^{k+1}

Compute a stochastic approximation $\hat{g}_{k+1}$ of $\nabla p_\lambda(U_k, \theta_k)$ with:

$$\begin{cases} \hat{g}_{k+1,1} \leftarrow -Z^{k+1} + \frac{\lambda}{1-\alpha} Z^{k+1} \mathbf{1}_{\langle Z^{k+1}, U_k \rangle \ge \theta_k} \\ \hat{g}_{k+1,2} \leftarrow \lambda \Big[1 - \frac{1}{1-\alpha} \mathbf{1}_{\langle Z^{k+1}, U_k \rangle \ge \theta_k} \Big] \end{cases} .$$

Update the algorithm $X_{k+1} \leftarrow \operatorname{argmin}_{x \in \Delta_m \times \mathbb{R}} \Big\{ \langle \hat{g}_{k+1}, x - X_k \rangle + \frac{1}{\eta_{k+1}} \mathcal{D}_\Phi(x, X_k) \Big\}$ using:

$$X_{k+1} \leftarrow (U_{k+1}, \theta_{k+1}), \qquad \begin{cases} U^{k+1} \leftarrow \frac{U^k e^{-\eta_{k+1} \hat{g}_{k+1,1}}}{\| U^k e^{-\eta_{k+1} \hat{g}_{k+1,1}} \|_1} \\ \theta^{k+1} \leftarrow \theta^k - \eta_{k+1} \hat{g}_{k+1,2} \end{cases} .$$

4 Our Results

4.1 Estimating the portfolio parameters

In order to approximate the optimal investment strategy, our idea relies on estimating the parameters (μ, σ) of the assets' motion and then using Algorithm 1. In the following, we present our result for estimating these parameters. In Theorem 1, give the expression of the estimators.

Theorem 1. *Let $\mu \in \mathbb{R}^m$ and $\sigma \in \mathbb{R}_+^m$. Consider the portfolio formed of risky assets following geometric Brownian motions with parameters μ and σ, that is:*

$$\forall i \in \{1, \dots, m\}, \quad dS_t^i = \mu_i S_t^i dt + \sigma_i S_t^i dW_t.$$

Further, assume we are given an $n-$sample $(X_1, \dots, X_n)$ of random vectors such that

$$\forall t \in \{1, \dots, n\}, \quad X_t^i = \log \frac{S_{t+1}^i}{S_t^i},$$

then the values of $(\hat{\mu}_n)_{n\geq 0}$ and $(\hat{\sigma}_n^2)_{n\geq 0}$ computed iteratively as follows:

$$\begin{cases} \hat{\mu}_{n+1} = \frac{n}{n+1}\hat{\mu}_n - \frac{1}{2n(n+1)}\hat{\sigma}_n^2 + \frac{1}{n+1}X_{n+1} + \frac{1}{2(n+1)}\Big(\hat{\mu}_n - \frac{1}{2}\hat{\sigma}_n^2 - X_{n+1}\Big)^2 \\ \hat{\sigma}_{n+1}^2 = \frac{n-1}{n}\hat{\sigma}_n^2 + \frac{1}{n+1}\Big(\hat{\mu}_n - \frac{1}{2}\hat{\sigma}_n^2 - X_{n+1}\Big)^2, \end{cases}$$

form two sequences of unbiased estimators converging respectively to μ and σ^2).

Proof. From the expression of a geometric Brownian motion seen above:

$$\log \frac{S_t^i}{S_0^i} = \Big(\mu_i - \frac{\sigma_i^2}{2}\Big)t + \sigma_i W_t,$$

where W_t is a Browninan motion and μ_i and σ_i are constants, we get:

$$\log \frac{S_{t+1}^i}{S_t^i} = \Big(\mu_i - \frac{\sigma_i^2}{2}\Big) + \sigma_i\Big(W_{t+1} - W_t\Big) \sim \mathcal{N}\Big(\mu_i - \frac{\sigma_i^2}{2}, \sigma_i^2\Big),$$

hence

$$\begin{cases} \mathbb{E}\Big[\log \frac{S_{t+1}^i}{S_t^i}\Big] = \mu_i - \frac{\sigma_i^2}{2} \\ \mathbb{V}\mathrm{ar}\Big[\log \frac{S_{t+1}^i}{S_t^i}\Big] = \sigma_i^2. \end{cases}$$

Using the Law of Large Numbers, these two can be approximated given an $n-$sample $(X_1, \dots, X_n)$ of random vectors such that

$$\forall t \in \{1, \dots, n\}, \quad X_t^i = \log \frac{S_{t+1}^i}{S_t^i},$$

with the empirical mean and the empirical variance using the formulae:

$$\begin{cases} \bar{X}_n = \frac{1}{n}\sum_{t=1}^{n} X_t \longrightarrow_{n\to+\infty} \mathbb{E}\Big[\log\frac{S_{t+1}}{S_t}\Big] \\ \frac{1}{n-1}\sum_{t=1}^{n}\Big(X_t - \bar{X}_n\Big)^2 \longrightarrow_{n\to+\infty} \mathbb{V}\mathrm{ar}\Big[\log\frac{S_{t+1}}{S_t}\Big]. \end{cases}$$

Therefore, we will approximate given an $n-$sample $(X_1, \ldots, X_n)$ where $X_t = \log\frac{S_{t+1}}{S_t}$ the parameters μ and σ using the estimators $\hat{\mu}_n$ and $\hat{\sigma}_n$ such that:

$$\begin{cases} \bar{X}_n = \hat{\mu}_n - \frac{1}{2}\hat{\sigma}_n^2 \\ \frac{1}{n-1}\sum_{j=1}^{n}\Big(X_j - \bar{X}_n\Big)^2 = \hat{\sigma}_n^2. \end{cases}$$

We obtain the following iterative expressions for the sequence of estimators $(\hat{\mu}_n, \hat{\sigma}_n^2)_{n\geq 0}$:

$$\begin{cases} \hat{\mu}_{n+1} = \frac{n}{n+1}\hat{\mu}_n - \frac{1}{2n(n+1)}\hat{\sigma}_n^2 + \frac{1}{n+1}X_{n+1} + \frac{1}{2(n+1)}\Big(\hat{\mu}_n - \frac{1}{2}\hat{\sigma}_n^2 - X_{n+1}\Big)^2 \\ \hat{\sigma}_{n+1}^2 = \frac{n-1}{n}\hat{\sigma}_n^2 + \frac{1}{n+1}\Big(\hat{\mu}_n - \frac{1}{2}\hat{\sigma}_n^2 - X_{n+1}\Big)^2. \end{cases}$$

The result follows. For a more thorough proof, please refer to Appendix A.3. □

We present in Theorem 2 our result concerning the rates of convergence of $(\hat{\mu}_n)_{n\geq 0}$ and $(\hat{\sigma}_n^2)_{n\geq 0}$.

Theorem 2. *Let $\mu \in \mathbb{R}^m$ and $\sigma \in \mathbb{R}_+^m$. Consider the sequence $(\hat{\mu}_n, \hat{\sigma}_n^2)_{n\geq 0}$ of estimators described above. Then the variances of the estimators are given by the expressions:*

$$\begin{cases} \mathbb{E}\Big[|\hat{\mu}_n - \mu|^2\Big] = \frac{\sigma^2}{n} + \frac{\sigma^4}{2(n-1)} \\ \mathbb{E}\Big[|\hat{\sigma}_n^2 - \sigma^2|^2\Big] = \frac{2\sigma^4}{n-1}, \end{cases}$$

meaning that they converge to 0 in polynomial time.

Proof. At time $t \in \{1, \ldots, n\}$, we have:

$$\begin{aligned} X_t^i &= \log\frac{S_{t+1}^i}{S_t^i} \\ &= \Big(\mu - \frac{\sigma^2}{2}\Big) + \sigma\Big(W_{i+1} - W_i\Big) \sim \mathcal{N}\Big(\mu - \frac{\sigma^2}{2}, \sigma^2\Big). \end{aligned}$$

In order to estimate the parameters μ and σ, we use the sample mean and the sample variance of $(X_1, \ldots, X_n)$ which are known to converge to the mean and variance of the random variables, that is:

$$\begin{cases} \bar{X}_n = \frac{1}{n}\sum_{i=1}^{n} X_i \longrightarrow \mathbb{E}[X_i] = \mu - \frac{\sigma^2}{2} \\ S_n^2 = \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \bar{X}_n)^2 \longrightarrow \mathbb{V}\mathrm{ar}[X_i] = \sigma^2 \end{cases}.$$

and of which we can compute the distribution. In fact, since X_i follows a normal distribution, then it is simple to compute the distribution of the sample mean:

$$\bar{X}_n \sim \mathcal{N}\Big(\mu - \frac{\sigma^2}{2}, \frac{\sigma^2}{n}\Big).$$

For the sample variance, we obtain the following distribution:

$$S_n^2 = \frac{\sigma^2}{n-1}\mathcal{C}, \text{ where } \mathcal{C} \sim \chi^2(n-1)$$

Finally, this gives the second-moment errors of the estimators in terms of n:

$$\begin{cases} \mathbb{E}\Big[\Big|\hat{\mu}_n - \mu\Big|^2\Big] = \frac{\sigma^2}{n} + \frac{\sigma^4}{2(n-1)} \\ \mathbb{E}\Big[\Big|\hat{\sigma}_n^2 - \sigma^2\Big|^2\Big] = \frac{2\sigma^4}{n-1} \end{cases},$$

and the result follows. Please refer to Appendix A.4 for a more thorough proof. □

4.2 Online Portfolio Optimization

4.2.1 Algorithm

We wish to use the methods seen above to create an online method that will improve our estimators of the portfolio parameters over time, hence improving our investment strategy. To be more specific, we want an online method to estimate the minimizer $x^\star$ of $p_{\lambda,\xi}$ where $\xi = (\mu, \sigma^2)$, that is:

$$x^\star = \mathrm{argmin}_{(u,\theta)\in\Delta_m\times\mathbb{R}}\{p_{\lambda,\xi}(u,\theta)\},$$

where:

$$p_{\lambda,\xi}(u,\theta) = -\sum_{i=1}^{n} u_i\mathbb{E}\big[Z_i\big] + \lambda\theta + \frac{\lambda}{1-\alpha}\mathbb{E}\big[\lfloor\langle Z, u\rangle - \theta\rfloor_+\big], \tag{4}$$

and Z is the simulated return vector satisfying:

$$\forall i \in \{1, \ldots, m\} \quad Z_i = e^{(\mu - \frac{\sigma^2}{2}) + \sigma N_i}, \quad N_i \sim \mathcal{N}(0, 1).$$

For that, our idea is to keep updating our estimator $\hat{\xi}_n = \left(\hat{\mu}_n, \hat{\sigma}_n^2\right)$ of ξ at any time n, and estimating the minimizer of $p_{\lambda, \hat{\xi}_n}$ using SMD, that is:

$$\begin{aligned} \hat{x}_n^\star &= \Gamma(\hat{\xi}_n) \\ &\approx \operatorname{argmin}_{(u,\theta) \in \Delta_m \times \mathbb{R}} \{p_{\lambda, \hat{\xi}_n}(u, \theta)\}, \end{aligned}$$

where:

$$p_{\lambda, \hat{\xi}_n}(u, \theta) = -\sum_{i=1}^{n} u_i \mathbb{E}\left[Z_i'\right] + \lambda\theta + \frac{\lambda}{1-\alpha} \mathbb{E}\left[\lfloor \langle Z', u \rangle - \theta \rfloor_+\right], \tag{5}$$

where Z' is the simulated return vector satisfying:

$$\forall i \in \{1, \ldots, m\} \quad Z_i' = e^{\left(\hat{\mu}_n - \frac{\hat{\sigma}_n^2}{2}\right) + \hat{\sigma}_n N_i}, \quad N_i \sim \mathcal{N}(0, 1).$$

Our procedure is described in detail in Algorithm 2.

4.2.2 Rate of Convergence

In the following, we present our rate of convergence upper bound for the online method presented in Algorithm 2. Our main result, Theorem 3, gives an upper bound for the difference between the minimums of $p_{\lambda,\xi}$ and $p_{\lambda,\hat{\xi}_n}$ defined in (4) and (5).

Theorem 3. *Given $\hat{\xi}_n = (\hat{\mu}_n, \hat{\sigma}_n^2)$ the estimator of the real GBM parameters $\xi = (\mu, \sigma^2)$ of the online method, let Z and Z' denote respectively the true return vector and its estimator, that is $Z := e^{\mu - \frac{\sigma^2}{2} + \sigma N}$ and $Z' := e^{\hat{\mu}_n - \frac{\hat{\sigma}_n^2}{2} + \sigma N}$ where $N \sim \mathcal{N}(0, 1)$. Let $\hat{\alpha}_n := \hat{\mu}_n - \frac{\hat{\sigma}_n^2}{2}$ and $\alpha := \mu - \frac{\sigma^2}{2}$. We define $p_{\lambda,\xi}$ and $p_{\lambda,\hat{\xi}_n}$ as described above. Then*

$$\left| \min_{(u,\theta) \in \Delta_m \times \mathbb{R}} p_{\lambda,\xi} - \min_{(u,\theta) \in \Delta_m \times \mathbb{R}} p_{\lambda,\hat{\xi}_n} \right| \leq \left\| \mathcal{A}_n \right\|_\infty + \left\| \mathcal{B}_n \right\|_\infty,$$

where

$$\mathcal{A}_n := e^\mu \left(\left| \hat{\mu}_n - \mu \right| + \frac{e}{2} \left(\hat{\mu}_n - \mu \right)^2 \right),$$

Algorithm 2 Online method for Portfolio Optimization under CV@R constraints.

Data: Initial estimator $\hat{\xi}_0 = \left(\hat{\mu}_0, \hat{\sigma}_0^2\right) \in \mathbb{R} \times \mathbb{R}_+$, number of days $n_{\max}$.

Result: Optimal investment strategy for GBM parameters $\xi = \left(\mu_0, \sigma_0^2\right)$

Algorithm:

At time $n = 0$:

- Observe the initial values of the portfolio assets: $S_0 \leftarrow \left(S_0^1, \ldots, S_0^m\right)$.
- Using SMD, set the strategy to $\Gamma(\hat{\xi}_0)$.

for $n = 0, \ldots, n_{\max} - 1$ **do**

Observe the values of the portfolio assets: $S_{n+1} \leftarrow \left(S_{n+1}^1, \ldots, S_{n+1}^m\right)$.

Compute $X_{n+1} = \left(X_{n+1}^1, \ldots, X_{n+1}^m\right)$ where:

$$X_{n+1}^i = \log \frac{S_{n+1}^i}{S_n^i}, \quad \forall i \in \{1, \ldots, m\}.$$

Update the algorithm $\hat{\xi}_{n+1} = \left(\hat{\mu}_{n+1}, \hat{\sigma}_{n+1}^2\right)$ using the iterative expression:

$$\begin{cases} \hat{\mu}_{n+1} = \frac{n}{n+1}\hat{\mu}_n - \frac{1}{2n(n+1)}\hat{\sigma}_n^2 + \frac{1}{n+1}X_{n+1} + \frac{1}{2(n+1)t}\left(\hat{\mu}_n - \frac{1}{2}\hat{\sigma}_n^2 - X_{n+1}\right)^2 \\ \hat{\sigma}_{n+1}^2 = \frac{n-1}{n}\hat{\sigma}_n^2 + \frac{1}{(n+1)t}\left(\hat{\mu}_n - \frac{1}{2}\hat{\sigma}_n^2 - X_{n+1}\right)^2 \end{cases}.$$

Using SMD, set the strategy to $\Gamma(\hat{\xi}_{n+1})$.

Return $\Gamma(\hat{\xi}_{n_{\max}})$.

and

$$\mathcal{B}_n := \Big[|\hat{\alpha}_n - \alpha| + \frac{e}{2}(\hat{\alpha}_n - \alpha)^2 + |\hat{\sigma}_n - \sigma|\Big(\frac{\sqrt{2}}{\sqrt{\pi}} + \sigma\Big) + \frac{e}{2}(\hat{\sigma}_n - \sigma)^2(1+\sigma^2)$$
$$+ e(\hat{\alpha}_n - \alpha)(\hat{\sigma}_n - \sigma)\sigma\Big]e^{\alpha + \frac{\sigma^2}{2}},$$

with probability at least:

$$1 - \epsilon_n = \prod_{i=1}^{m} \int_{-1}^{1} f_{\mathcal{N}((\hat{\alpha}_n - \alpha)_i, (\hat{\sigma}_n - \sigma)_i^2)}(t)dt + \prod_{i=1}^{m} \Big\{1 - \Big(\frac{\sigma_i^2}{n} + \frac{\sigma_i^4}{2(n-1)}\Big)\Big\} - 1.$$

Furthermore, $1 - \epsilon_n \xrightarrow[n \to +\infty]{} 1$*.*

The proof of this result relies on the observation that the distance between $p_{\lambda,\xi}$ and $p_{\lambda,\hat{\xi}_n}$, that is $\left|p_{\lambda,\xi}(x) - p_{\lambda,\hat{\xi}_n}(x)\right|$, is bounded by a constant for all x with high probability converging to 1. For a complete proof of Theorem 3, please refer to Appendix A.5.

5 Experiments

In this section, we run a simulation of a financial portfolio formed of 4 assets (i.e. $m = 4$) following a GBM of drift vector $\mu = (\mu_1, \mu_2, \mu_3, \mu_4)$ and of volatility vector $\sigma = (\sigma_1, \sigma_2, \sigma_3, \sigma_4)$.

In particular, we run the online algorithm introduced in the section above (Algorithm 2) and study the convergence of the estimators of the portfolio parameters, $\hat{\mu}_n$ and $\hat{\sigma}_n^2$. In the following, we explain the experimental settings and give the results of our simulation of the online method.

5.1 Experimental Settings

In order to simulate the behavior of the portfolio, we start by setting the parameters of the assets' motion, that is the drift μ and the volatility σ of the GBM. We know that the expectation of the return of the portfolio is given by:

$$\begin{aligned} \mathbb{E}[Z] &= \mathbb{E}\Big[e^{(\mu - \frac{\sigma}{2}) + \sigma N} - 1\Big] \\ &= e^{\mu} - 1. \end{aligned}$$

Based on this, we want to simulate a portfolio with assets having positive and negative returns evenly distributed around 0. That is, we want a portfolio

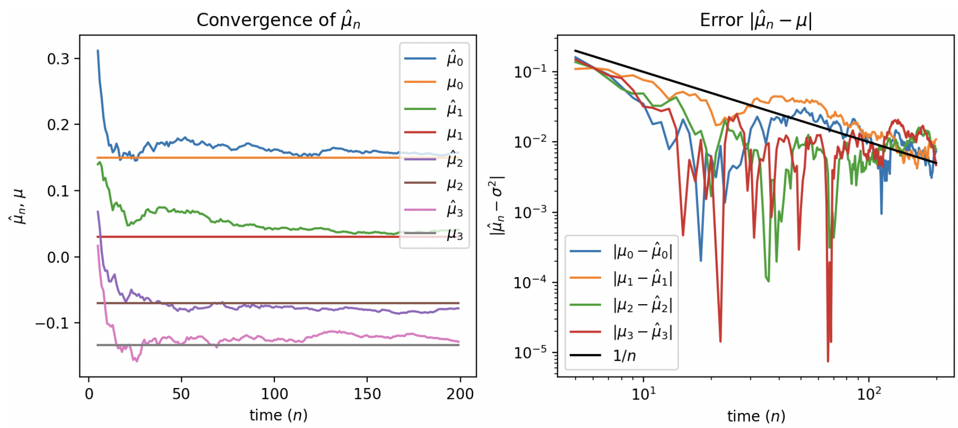

Figure 4: Convergence of $\hat{\mu}_n$ towards μ.

such that the evenly distributed strategy (i.e. the strategy where we invest equally in all assets) is expected to return 0. We choose the following drift vector:

$$\mu = \begin{pmatrix} \mu_1 \\ \mu_2 \\ \mu_3 \\ \mu_4 \end{pmatrix} = \begin{pmatrix} 0.15 \\ 0.03 \\ -0.07 \\ -0.133 \end{pmatrix},$$

for which the expected return of the evenly distributed strategy is:

$$\sum_{i=1}^{m} \frac{1}{m}(e^{\mu_i} - 1) = \sum_{i=1}^{4} \frac{1}{4}(e^{\mu_i} - 1) \approx 0.$$

We expect the online method in this setting to return a strategy that would almost fully invest in the two first assets (with proportions depending on their volatility), as they are the only ones with positive expected return). For more on the experimental settings, please refer to Appendix B.

5.2 Results

We simulate the financial portfolio described above and apply the online method. We keep track of the values of the estimators $\hat{\mu}_n$ and $\hat{\sigma}_n^2$ and the errors $|\hat{\mu}_n - \mu|$ and $|\hat{\sigma}_n^2 - \sigma^2|$.

We plot the values of the estimators $\hat{\mu}_n$ and $\hat{\sigma}_n^2$ respectively in Figure 4 and Figure 5, as well as the errors on a log-log scale. On the error plots of $|\hat{\mu}_n - \mu|$ and $|\hat{\sigma}_n^2 - \sigma^2|$, we observe that the errors behave like $n \mapsto \frac{1}{n}$. We deduce from this that the estimators $\hat{\mu}_n$ and $\hat{\sigma}_n^2$ converge towards their respective limits, μ and σ^2, in $\Omega\left(\frac{1}{n}\right)$ time.

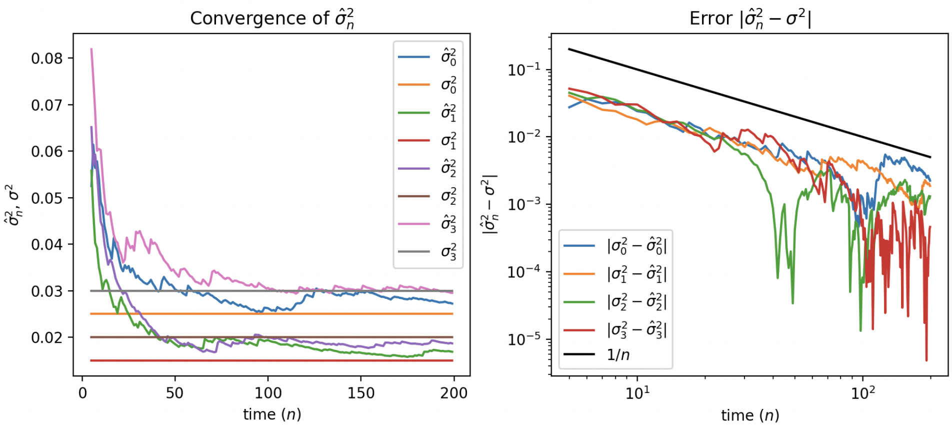

Figure 5: Convergence of $\hat{\sigma}_n^2$ towards σ^2.

6 Conclusion & Future Work

In conclusion, we propose the online method (Algorithm 2) as a solution to the problem of portfolio optimization under risk management constraints. This method approximates the optimal investment strategy in parallel to estimating the portfolio parameters. The solution converges to the optimal strategy according to the speed of convergence upper bound seen in Theorem 3.

This algorithm opens many new doors for investigation. For instance, this article only considers risky assets which we model using GBMs for simplicity. We suggest the addition of risk-less assets, which would allow modelling debt obligations. This can be done by simulating an interest rate using a Cox-Ingersoll-Ross [CIJR05] and using it as the growth ratio of an asset, as done by Costa et al. in [CGH22]. In addition, as stated above, our work does not take into account the bias caused by the discretization of SDEs. In the future, this should be considered in order to improve the behavior of our method. Finally, this work assumes that the assets are independent from one another, which is generally not true in real life. Future research should introduce a more generalized framework that would allow for a correlation matrix between the different assets.

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A Omitted proofs

A.1 Gradient Descent and Mirror Descent

In the setting of mirror descent introduced above, we have seen that

$$\forall (x,y) \in \mathbb{R}^m \times \mathbb{R}^m, \quad \mathcal{D}_\phi(x,y) = \phi(x) - \phi(y) - \langle \nabla\phi(y), x-y\rangle.$$

In particular, if we consider the strictly convex function

$$\begin{aligned} \phi : \quad \mathbb{R}^m &\longrightarrow \mathbb{R}_+ \cup \{0\} \\ x &\longmapsto \frac{1}{2}\|x\|_2^2, \end{aligned}$$

then we have $\forall (x,y) \in \mathbb{R}^m \times \mathbb{R}^m$:

$$\begin{aligned} \mathcal{D}_\phi(x,y) &= \frac{1}{2}\|x\|_2^2 - \frac{1}{2}\|y\|_2^2 - \langle \nabla \frac{1}{2}\|y\|_2^2, x-y\rangle \\ &= \frac{1}{2}\langle x,x\rangle - \frac{1}{2}\langle y,y\rangle - \langle \nabla \frac{1}{2}\langle y,y\rangle, x-y\rangle \\ &= \frac{1}{2}\langle x,x\rangle - \frac{1}{2}\langle y,y\rangle - \frac{1}{2}\langle 2y, x-y\rangle \\ &= \frac{1}{2}\langle x,x\rangle + \frac{1}{2}\langle y,y\rangle - \langle x,y\rangle \\ &= \frac{1}{2}\langle x-y, x-y\rangle \\ &= \frac{1}{2}\|x-y\|_2^2. \end{aligned}$$

In this case, the expression of the iterative step for mirror descent is:

$$\begin{aligned} x_{t+1} &= \mathrm{argmin}_{x\in\mathbb{R}^m}\Big\{ f(x_t) + \langle \nabla f(x_t), x - x_t\rangle + \frac{1}{\eta}\mathcal{D}_\phi(x, x_t)\Big\} \\ &= \mathrm{argmin}_{x\in\mathbb{R}^m}\Big\{ f(x_t) + \langle \nabla f(x_t), x - x_t\rangle + \frac{1}{2\eta}\|x - x_t\|_2^2\Big\}, \end{aligned}$$

which corresponds to gradient descent. Therefore, gradient descent is the particular case of mirror descent of distance-generating function $\phi : x \mapsto \frac{1}{2}\|x\|_2^2$.

A.2 Explicit expression of MD iterative step

We have seen above that the iterative step of mirror descent for portfolio optimization is given by

$$x_{t+1} = \mathrm{argmin}_{x\in\Delta_m}\Big\{\frac{1}{\eta}\phi(x) + \langle \nabla f(x_t) - \frac{1}{\eta}\nabla\phi(x_t), x\rangle\Big\},$$

which can be simplified as follows:

$$
\begin{aligned}
x_{t+1} &= \operatorname{argmin}_{x\in\Delta_m}\Big\{\langle \nabla f(x_t), x - x_t\rangle + \frac{1}{\eta}\mathcal{D}_\phi(x, x_t)\Big\} \\
&= \operatorname{argmin}_{x\in\Delta_m}\Big\{\langle \nabla f(x_t), x\rangle - \langle \nabla f(x_t), x_t\rangle + \frac{1}{\eta}\mathcal{D}_\phi(x, x_t)\Big\} \\
&= \operatorname{argmin}_{x\in\Delta_m}\Big\{\langle \nabla f(x_t), x\rangle + \frac{1}{\eta}\Big(\phi(x) - \phi(x_t) - \langle \nabla\phi(x_t), x - x_t\rangle\Big)\Big\} \\
&= \operatorname{argmin}_{x\in\Delta_m}\Big\{\langle \nabla f(x_t), x\rangle + \frac{1}{\eta}\Big(\phi(x) - \langle \nabla\phi(x_t), x\rangle - \langle \nabla\phi(x_t), x_t\rangle\Big)\Big\} \\
&= \operatorname{argmin}_{x\in\Delta_m}\Big\{\langle \nabla f(x_t), x\rangle + \frac{1}{\eta}\Big(\phi(x) - \langle \nabla\phi(x_t), x\rangle\Big)\Big\} \\
&= \operatorname{argmin}_{\sum_{i=1}^m x_i = 1}\Big\{\frac{1}{\eta}\phi(x) + \Big\langle \nabla f(x_t) - \frac{1}{\eta}\nabla\phi(x_t), x\Big\rangle\Big\}.
\end{aligned}
$$

We introduce the Lagrangian function:

$$
\begin{aligned}
\mathcal{L}(x,\lambda) &= \frac{1}{\eta}\phi(x) + \Big\langle \nabla f(x_t) - \frac{1}{\eta}\nabla\phi(x_t), x\Big\rangle + \lambda\Big(\sum_{i=1}^m x_i - 1\Big) \\
&= \frac{1}{\eta}\sum_{i=1}^m x_i \log x_i + \Big\langle \nabla f(x_t) - \frac{1}{\eta}\nabla\phi(x_t), x\Big\rangle + \lambda\Big(\sum_{i=1}^m x_i - 1\Big)
\end{aligned}
$$

Hence, the stationary point (x_{t+1}, λ^*) of $\mathcal{L}$ is such that:

$$
\begin{cases}
\frac{\partial \mathcal{L}(x,\lambda)}{\partial x_i}\Big|_{(x_{t+1},\lambda^*)} = 0 \\
\frac{\partial \mathcal{L}(x,\lambda)}{\partial \lambda}\Big|_{(x_{t+1},\lambda^*)} = 0.
\end{cases}
$$

Therefore

$$
\begin{cases}
\frac{1}{\eta}\Big(1 + \log{(x_{t+1})_i}\Big) + \Big(\nabla f(x_t) - \frac{1}{\eta}\nabla\phi(x_t)\Big)_i + \lambda^* = 0 \\
\sum_{i=1}^m (x_{t+1})_i - 1 = 0,
\end{cases}
$$

hence

$$
\begin{cases}
\log{(x_{t+1})_i} = \nabla\phi(x_t)_i - \eta\nabla f(x_t)_i - 1 - \eta\lambda^* \\
\sum_{i=1}^m (x_{t+1})_i = 1,
\end{cases}
$$

thus

$$
\begin{cases}
(x_{t+1})_i = e^{\nabla\phi(x_t)_i - 1} e^{-\eta\nabla f(x_t)_i - \eta\lambda^*} = (x_t)_i e^{-\eta\nabla f(x_t)_i - \eta\lambda^*} \\
\sum_{i=1}^m (x_{t+1})_i = 1.
\end{cases}
$$

Now substituting the expression of $(x_{t+1})_i$ into $\sum_{i=1}^m (x_{t+1})_i$, we get

$$\begin{aligned}\sum_{i=1}^m (x_{t+1})_i &= \sum_{i=1}^m (x_t)_i e^{-\eta\nabla f(x_t)_i - \eta\lambda^*} \\ &= e^{-\eta\lambda^*} \sum_{i=1}^m (x_t)_i e^{-\eta\nabla f(x_t)_i}.\end{aligned}$$

Substituting this in the second equation of the system above, we get:

$$e^{\eta\lambda^*} = \sum_{i=1}^m (x_t)_i e^{-\eta\nabla f(x_t)_i}.$$

Hence, we get $\forall i \in \{1, \ldots, m\}$:

$$(x_{t+1})_i = \frac{(x_t)_i e^{-\eta\nabla f(x_t)_i}}{\sum_{i=1}^m (x_t)_i e^{-\eta\nabla f(x_t)_i}},$$

which gives

$$x_{t+1} = \frac{x_t e^{-\eta\nabla f(x_t)}}{\left\| x_t e^{-\eta\nabla f(x_t)} \right\|_1}.$$

A.3 Iterative expression of $(\hat{\mu}_n, \hat{\sigma}_n^2)_{n\geq 0}$

We have seen above that the estimators $\hat{\mu}_n$ and $\hat{\sigma}_n$ of the parameters μ and σ of the GBM describing the motion of the assets are such that:

$$\begin{cases} \bar{X}_n = \hat{\mu}_n - \frac{1}{2}\hat{\sigma}_n^2 \\ \frac{1}{n-1}\sum_{j=1}^n \left(X_j - \bar{X}_n\right)^2 = \hat{\sigma}_n^2; \end{cases}$$

where $(X_1, \ldots, X_n)$ is an $n-$sample such that $X_t = \log \frac{S_{t+1}}{S_t}$ for all $t \in \{1, \ldots, n\}$. Then the following holds:

Lemma 4. *Denote by α_n and β_n respectively the empirical mean and empirical variance of our sample, that is:*

$$\begin{cases} \alpha_n := \bar{X}_n = \hat{\mu}_n - \frac{1}{2}\hat{\sigma}_n^2 \\ \beta_n := \frac{1}{n-1}\sum_{j=1}^n \left(X_j - \bar{X}_n\right)^2 = \hat{\sigma}_n^2 \end{cases}$$

Then

$$\begin{cases} \alpha_{n+1} = \frac{n}{n+1}\alpha_n + \frac{1}{n+1}X_{n+1} \\ \beta_{n+1} = \frac{n-1}{n}\beta_n + \frac{1}{n+1}\left(\alpha_n - X_{n+1}\right)^2. \end{cases}$$

Proof. We have:

$$\begin{cases} \alpha_n = \bar{X}_n \\ \beta_n = \frac{1}{n-1}\sum_{j=1}^{n}\left(X_j - \bar{X}_n\right)^2 = \frac{1}{n-1}\sum_{j=1}^{n} X_j^2 - \frac{n}{n-1}\bar{X}_n^2 \end{cases}.$$

From the first equality, we get:

$$(n+1)\alpha_{n+1} = n\alpha_n + X_{n+1},$$

which gives:

$$\alpha_{n+1} = \frac{n}{n+1}\alpha_n + \frac{1}{n+1}X_{n+1}.$$

From the second equality, we get:

$$n\left(\beta_{n+1} + \frac{n+1}{n}\bar{X}_{n+1}^2\right) = (n-1)\left(\beta_n + \frac{n}{n-1}\bar{X}_n^2\right) + X_{n+1}^2,$$

hence

$$\begin{aligned}
\beta_{n+1} &= \frac{n-1}{n}\beta_n + \bar{X}_n^2 + \frac{1}{n}X_{n+1}^2 - \frac{n+1}{n}\bar{X}_{n+1}^2 \\
&= \frac{n-1}{n}\beta_n + \alpha_n^2 - \frac{n+1}{n}\alpha_{n+1}^2 + \frac{1}{n}X_{n+1}^2 \\
&= \frac{n-1}{n}\beta_n + \alpha_n^2 - \frac{n+1}{n}\left(\frac{n}{n+1}\alpha_n + \frac{1}{n+1}X_{n+1}\right)^2 + \frac{1}{n}X_{n+1}^2 \\
&= \frac{n-1}{n}\beta_n + \alpha_n^2 - \frac{n}{n+1}\alpha_n^2 - \frac{2}{n+1}\alpha_n X_{n+1} - \frac{1}{n(n+1)}X_{n+1}^2 + \frac{1}{n}X_{n+1}^2 \\
&= \frac{n-1}{n}\beta_n + \frac{1}{n+1}\alpha_n^2 - \frac{2}{n+1}\alpha_n X_{n+1} + \frac{1}{n+1}X_{n+1}^2 \\
&= \frac{n-1}{n}\beta_n + \frac{1}{n+1}\left(\alpha_n - X_{n+1}\right)^2.
\end{aligned}$$

The result follows. $\square$

Now we find the iterative expressions of $\hat{\mu}_n$ and $\hat{\sigma}_n^2$. Using Lemma 4, we have:

$$\begin{cases} \alpha_n = \hat{\mu}_n - \frac{1}{2}\hat{\sigma}_n^2 \\ \beta_n = \hat{\sigma}_n^2 \end{cases},$$

then

$$\begin{cases} \hat{\mu}_n = \alpha_n + \frac{1}{2}\beta_n \\ \hat{\sigma}_n^2 = \beta_n \end{cases}.$$

Hence, we have

$$\begin{cases}\alpha_{n+1} = \frac{n}{n+1}\hat{\mu}_n - \frac{n}{2(n+1)}\hat{\sigma}_n^2 + \frac{1}{n+1}X_{n+1} \\ \beta_{n+1} = \frac{n-1}{n}\hat{\sigma}_n^2 + \frac{1}{n+1}\Big(\hat{\mu}_n - \frac{1}{2}\hat{\sigma}_n^2 - X_{n+1}\Big)^2\end{cases}.$$

Therefore

$$\begin{cases}\hat{\mu}_{n+1} = \alpha_{n+1} + \frac{1}{2}\beta_{n+1} = \frac{n}{n+1}\hat{\mu}_n - \frac{n}{2(n+1)}\hat{\sigma}_n^2 + \frac{1}{n+1}X_{n+1} + \frac{1}{2}\Big(\frac{n-1}{n}\hat{\sigma}_n^2 + \frac{1}{n+1}\Big(\hat{\mu}_n - \frac{1}{2}\hat{\sigma}_n^2 - X_{n+1}\Big)^2\Big) \\ \hat{\sigma}_{n+1}^2 = \beta_{n+1} = \frac{n-1}{n}\hat{\sigma}_n^2 + \frac{1}{n+1}\Big(\hat{\mu}_n - \frac{1}{2}\hat{\sigma}_n^2 - X_{n+1}\Big)^2,\end{cases}$$

which simplifies to:

$$\begin{cases}\hat{\mu}_{n+1} = \frac{n}{n+1}\hat{\mu}_n - \frac{1}{2n(n+1)}\hat{\sigma}_n^2 + \frac{1}{n+1}X_{n+1} + \frac{1}{2(n+1)}\Big(\hat{\mu}_n - \frac{1}{2}\hat{\sigma}_n^2 - X_{n+1}\Big)^2 \\ \hat{\sigma}_{n+1}^2 = \frac{n-1}{n}\hat{\sigma}_n^2 + \frac{1}{n+1}\Big(\hat{\mu}_n - \frac{1}{2}\hat{\sigma}_n^2 - X_{n+1}\Big)^2\end{cases}.$$

A.4 Rate of convergence of $(\hat{\mu}_n)_{n\geq 0}$ and $(\hat{\sigma}_n^2)_{n\geq 0}$

We have seen above that

$$\bar{X}_n \sim \mathcal{N}\Big(\mu - \frac{\sigma^2}{2}, \frac{\sigma^2}{n}\Big),$$

and

$$S_n^2 = \frac{\sigma^2}{n-1}\mathcal{C}, \text{ where } \mathcal{C} \sim \chi^2(n-1).$$

Hence

$$\begin{aligned}\mathbb{E}\Big[\Big|\bar{X}_n - \Big(\mu - \frac{\sigma}{2}\Big)\Big|^2\Big] &= \mathbb{V}\mathrm{ar}\Big[\bar{X}_n\Big] \\ &= \frac{\sigma^2}{n}.\end{aligned}$$

For S_n^2, we have:

$$\begin{aligned}\mathbb{E}[S_n^2] &= \mathbb{E}\Big[\frac{\sigma^2}{n-1}\mathcal{C}\Big] \\ &= \frac{\sigma^2}{n-1}\mathbb{E}\Big[\sum_{i=1}^{n-1} N_i^2\Big] \\ &= \frac{\sigma^2}{n-1}\sum_{i=1}^{n-1}\mathbb{E}[N_i^2] \\ &= \frac{\sigma^2}{n-1}\sum_{i=1}^{n-1}\mathbb{V}\mathrm{ar}[N_i] \\ &= \sigma^2,\end{aligned}$$

for i.i.d. $N_i \sim \mathcal{N}(0,1)$ for all $i \in \{1, \dots, n-1\}$. Hence, we have

$$\begin{aligned}
\mathbb{E}\Big[\Big(S_n^2 - \sigma^2\Big)^2\Big] &= \mathbb{V}\mathrm{ar}\Big[S_n^2\Big] \\
&= \mathbb{V}\mathrm{ar}\Big[\frac{\sigma^2}{n-1}\mathcal{C}\Big] \\
&= \frac{\sigma^4}{(n-1)^2}\mathbb{V}\mathrm{ar}[\mathcal{C}] \\
&= \frac{\sigma^4}{(n-1)^2}\mathbb{E}\Big[\sum_{i=1}^{n-1} N_i^2\Big] \\
&= \frac{\sigma^4}{(n-1)^2}\sum_{i=1}^{n-1}\mathbb{V}\mathrm{ar}[N_i^2] \\
&= \frac{\sigma^4}{(n-1)^2}\sum_{i=1}^{n-1}\Big\{\mathbb{E}[N_i^4] - \mathbb{E}[N_i^2]^2\Big\} \\
&= \frac{\sigma^4}{(n-1)^2}\sum_{i=1}^{n-1}\Big\{\mathbb{E}[N_i^4] - \mathbb{E}[N_i^2]^2\Big\} \\
&= \frac{\sigma^4}{(n-1)^2}\sum_{i=1}^{n-1}\Big\{3 - 1^2\Big\} \\
&= \frac{2\sigma^4}{n-1}.
\end{aligned}$$

We want to estimate the parameters (μ, σ^2) of the assets' GBM using the estimators $(\hat{\mu}_n, \hat{\sigma}_n^2)$ such that:

$$\begin{cases}
\hat{\mu}_n = \bar{X}_n + \frac{1}{2}S_n^2 \xrightarrow[n \to +\infty]{} \mu \\
\hat{\sigma}_n^2 = S_n^2 \xrightarrow[n \to +\infty]{} \sigma^2
\end{cases}$$

The second-moment error of $\hat{\mu}_n$ in terms of n is:

$$\begin{aligned}
\mathbb{E}\Big[\Big|\hat{\mu}_n - \mu\Big|^2\Big] &= \mathbb{V}\mathrm{ar}[\hat{\mu}_n] \\
&= \mathbb{V}\mathrm{ar}\Big[\bar{X}_n + \frac{1}{2}S_n^2\Big],
\end{aligned}$$

which by independence of the sample mean $\bar{X}_n$ and the sample variance S_n^2 gives:

$$\begin{aligned}
\mathbb{E}\Big[\Big|\hat{\mu}_n - \mu\Big|^2\Big] &= \mathbb{V}\mathrm{ar}[\bar{X}_n] + \frac{1}{4}\mathbb{V}\mathrm{ar}\Big[S_n^2\Big] \\
&= \frac{\sigma^2}{n} + \frac{\sigma^4}{2(n-1)}.
\end{aligned}$$

For $\hat{\sigma}_n^2$, the second-moment error in terms of n is:

$$\begin{aligned}\mathbb{E}\Big[\Big|\hat{\sigma}_n^2 - \sigma^2\Big|^2\Big] &= \mathbb{V}\text{ar}\Big[\hat{\sigma}_n^2\Big] \\ &= \mathbb{V}\text{ar}\Big[S_n^2\Big] \\ &= \frac{2\sigma^4}{n-1}.\end{aligned}$$

A.5 Rate of convergence of the online method

In this section, we provide a complete proof of our main result on the rate of convergence of the online method, i.e. Theorem 3. We start by introducing some results that will allow us to make the proof. We denote by $\mathbb{E}_{\hat{\xi}_n}[.]$ the expectation conditioned on the value of $\hat{\xi}_n = (\hat{\mu}_n, \hat{\sigma}_n^2)$. All operations on vectors are to be understood component-wise.

Lemma 5. *Given μ and σ^2, let Z denote the return vector given by $Z := e^{\mu - \frac{\sigma^2}{2} + \sigma N}$, where $N \sim \mathcal{N}(0,1)$. Then:*

$$\mathbb{E}[Z] = e^{\mu}.$$

Proof. We note that

$$\begin{aligned}\mathbb{E}[Z] &= \mathbb{E}[e^{\mu - \frac{\sigma^2}{2} + \sigma N}] \\ &= e^{\mu - \frac{\sigma^2}{2}} \mathcal{M}_N(\sigma) \\ &= e^{\mu - \frac{\sigma^2}{2}} e^{\frac{\sigma^2}{2}} \\ &= e^{\mu},\end{aligned}$$

where $\mathcal{M}_N \colon t \mapsto \mathbb{E}[e^{tN}] = e^{\frac{1}{2}t^2}$ denotes the moment generating function of the standard normal distribution. □

Lemma 6. *Given $\hat{\xi}_n = (\hat{\mu}_n, \hat{\sigma}_n^2)$ the estimator of the real GBM parameters $\xi = (\mu, \sigma^2)$ of the online method, let Z and Z' denote respectively the true and the estimator return vectors, that is $Z := e^{\mu - \frac{\sigma^2}{2} + \sigma N}$ and $Z' := e^{\hat{\mu}_n - \frac{\hat{\sigma}_n^2}{2} + \sigma N}$ where $N \sim \mathcal{N}(0,1)$. Then for all $u \in \Delta_m$, we have:*

$$\begin{aligned}\Big|\mathbb{E}_{\hat{\xi}_n}\Big[\langle u, Z\rangle\Big] - \mathbb{E}_{\hat{\xi}_n}\Big[\langle u, Z'\rangle\Big]\Big| &\leq \Big\langle u, \mathcal{A}_n\Big\rangle \\ &\leq \Big\|\mathcal{A}_n\Big\|_\infty\end{aligned}$$

where

$$\mathcal{A}_n := e^{\mu}\Big(\Big|\hat{\mu}_n - \mu\Big| + \frac{e}{2}\Big(\hat{\mu}_n - \mu\Big)^2\Big),$$

with probability at least:

$$1 - \epsilon_n \geq \prod_{i=1}^{m}\Big\{1 - \Big(\frac{\sigma_i^2}{n} + \frac{\sigma_i^4}{2(n-1)}\Big)\Big\}.$$

Furthermore, $1 - \epsilon_n \xrightarrow[n\to\infty]{} 1$.

Proof. We have

$$\Big|\mathbb{E}_{\hat{\xi}_n}\Big[\langle u, Z\rangle\Big] - \mathbb{E}_{\hat{\xi}_n}\Big[\langle u, Z'\rangle\Big]\Big| = \Big|\Big\langle u, \mathbb{E}_{\hat{\xi}_n}[Z]\Big\rangle - \Big\langle u, \mathbb{E}_{\hat{\xi}_n}[Z']\Big\rangle\Big|.$$

Using Lemma 5, we get

$$\Big|\mathbb{E}_{\hat{\xi}_n}\Big[\langle u, Z\rangle\Big] - \mathbb{E}_{\hat{\xi}_n}\Big[\langle u, Z'\rangle\Big]\Big| = \Big|\Big\langle u, e^{\mu}\Big\rangle - \Big\langle u, e^{\hat{\mu}_n}\Big\rangle\Big|,$$

which gives

$$\begin{aligned}\Big|\mathbb{E}_{\hat{\xi}_n}\Big[\langle u, Z\rangle\Big] - \mathbb{E}_{\hat{\xi}_n}\Big[\langle u, Z'\rangle\Big]\Big| &= \Big|\langle u, e^{\mu} - e^{\hat{\mu}_n}\rangle\Big| \\ &\leq \Big\langle u, |e^{\mu} - e^{\hat{\mu}_n}|\Big\rangle \\ &= \Big\langle u, e^{\mu}|e^{\hat{\mu}_n - \mu} - 1|\Big\rangle.\end{aligned}$$

Hence

$$\Big|\mathbb{E}_{\hat{\xi}_n}\Big[\langle u, Z\rangle\Big] - \mathbb{E}_{\hat{\xi}_n}\Big[\langle u, Z'\rangle\Big]\Big| \leq \Big\langle u, e^{\mu}|e^{\hat{\mu}_n - \mu} - 1|\Big\rangle. \tag{6}$$

We have

$$(\hat{\mu}_n)_i - \mu_i \in (0, 1) \quad \forall i \in \{1, \ldots, m\} \tag{7}$$

with probability

$$\begin{aligned}1 - \epsilon_n &:= \mathbb{P}\Big[(\hat{\mu}_n)_i - \mu_i \in (0,1), \forall i \in \{1, \ldots, m\}\Big] \\ &= \prod_{i=1}^{m} \mathbb{P}\Big[(\hat{\mu}_n)_i - \mu_i \in (0,1)\Big] \\ &= \prod_{i=1}^{m}\Big\{1 - \mathbb{P}\Big[|(\hat{\mu}_n)_i - \mu_i| \geq 1\Big]\Big\},\end{aligned}$$

by independence of the components of μ (independence of the assets). Using the corollary of Markov's inequality that states:

$$\mathbb{P}\Big[|X| \geq 1\Big] \leq \frac{\mathbb{E}[|X|^n]}{a^n}; \quad a \geq 0,\, n \in \mathbb{N} - \{0\},$$

with $n = 2$ and $a = 1$, we obtain:

$$\begin{aligned} 1 - \epsilon_n &\geq \prod_{i=1}^{m} \left\{ 1 - \mathbb{E}\Big[|(\hat{\mu}_n)_i - \mu_i|^2 \Big] \right\} \\ &= \prod_{i=1}^{m} \left\{ 1 - \left(\frac{\sigma_i^2}{n} + \frac{\sigma_i^4}{2(n-1)} \right) \right\}. \end{aligned}$$

Now in the case where (7) holds, we use the following bound of e^x when x is close to 0:

$$\forall x \in [-1, 1], \quad |e^x - 1| \leq |x| + \frac{e}{2} x^2. \tag{8}$$

By substituting the result above in (6), and using the fact that $u \in \Delta_m$, we obtain:

$$\begin{aligned} \Big| \mathbb{E}_{\hat{\xi}_n} \Big[\langle u, Z \rangle \Big] - \mathbb{E}_{\hat{\xi}_n} \Big[\langle u, Z' \rangle \Big] \Big| &\leq \Big\langle u, e^{\mu} \Big(|\hat{\mu}_n - \mu| + \frac{e}{2} (\hat{\mu}_n - \mu)^2 \Big) \Big\rangle \\ &\leq \Big\| e^{\mu} \Big(|\hat{\mu}_n - \mu| + \frac{e}{2} (\hat{\mu}_n - \mu)^2 \Big) \Big\|_{\infty}, \end{aligned}$$

with probability $1 - \epsilon_n$. The result follows. □

Lemma 7. *Let $X \sim \mathcal{N}(0, \sigma^2)$. The following results hold:*

(i)

$$\mathbb{E}\Big[e^X \Big] = e^{\frac{\sigma^2}{2}},$$

(ii)

$$\mathbb{E}\Big[X e^X \Big] = \sigma^2 e^{\frac{\sigma^2}{2}},$$

(iii)

$$\mathbb{E}\Big[|X| e^X \Big] = \frac{\sqrt{2}}{\sqrt{\pi}} \sigma + \sigma^2 e^{\frac{\sigma^2}{2}},$$

(iv)

$$\mathbb{E}\Big[X^2 e^X \Big] = \sigma^2 (1 + \sigma^2) e^{\frac{\sigma^2}{2}}.$$

Proof. (i) We have

$$\begin{aligned} \mathbb{E}[e^X] &= \mathcal{M}_X(1) \\ &= e^{\frac{1}{2}\sigma^2}, \end{aligned}$$

where $\mathcal{M}_X : t \mapsto \mathbb{E}[e^{tX}] = e^{\frac{1}{2}\sigma^2 t^2}$ denotes the moment generating function of X.

(ii)

$$\begin{aligned}
\mathbb{E}[Xe^X] &= \int_{-\infty}^{+\infty} xe^x \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{x^2}{2\sigma^2}}\,dx \\
&= -\frac{\sigma}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \Big(1 - \frac{1}{\sigma^2}x\Big) e^{x-\frac{x^2}{2\sigma^2}}\,dx + \sigma^2 \int_{-\infty}^{+\infty} \frac{e^x}{\sqrt{2\pi}\sigma} e^{-\frac{x^2}{2\sigma^2}}\,dx \\
&= -\frac{\sigma}{\sqrt{2\pi}} \Big[e^{x-\frac{x^2}{2\sigma^2}}\Big]_{-\infty}^{+\infty} + \sigma^2 \mathbb{E}[e^X] \\
&= \sigma^2 e^{\frac{1}{2}\sigma^2}.
\end{aligned}$$

(iii)

$$\begin{aligned}
\mathbb{E}\Big[|X|e^X\Big] &= \int_{-\infty}^{+\infty} |x|e^x \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{x^2}{2\sigma^2}}\,dx \\
&= \int_{0}^{+\infty} xe^x \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{x^2}{2\sigma^2}}\,dx - \int_{-\infty}^{0} xe^x \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{x^2}{2\sigma^2}}\,dx \\
&= \Big[-\frac{\sigma}{\sqrt{2\pi}} \Big[e^{x-\frac{x^2}{2\sigma^2}}\Big]_{0}^{+\infty} + \sigma^2 \int_{0}^{+\infty} \frac{e^x}{\sqrt{2\pi}\sigma} e^{-\frac{x^2}{2\sigma^2}}\,dx \Big] \\
&\quad - \Big[-\frac{\sigma}{\sqrt{2\pi}} \Big[e^{x-\frac{x^2}{2\sigma^2}}\Big]_{-\infty}^{0} + \sigma^2 \int_{-\infty}^{0} \frac{e^x}{\sqrt{2\pi}\sigma} e^{-\frac{x^2}{2\sigma^2}}\,dx \Big] \\
&= \frac{\sqrt{2}\sigma}{\sqrt{\pi}} + \sigma^2 \int_{0}^{+\infty} \frac{e^x}{\sqrt{2\pi}\sigma} e^{-\frac{x^2}{2\sigma^2}}\,dx - \sigma^2 \int_{-\infty}^{0} \frac{e^x}{\sqrt{2\pi}\sigma} e^{-\frac{x^2}{2\sigma^2}}\,dx \\
&= \frac{\sqrt{2}\sigma}{\sqrt{\pi}} + \sigma^2 \int_{0}^{+\infty} \frac{e^x}{\sqrt{2\pi}\sigma} e^{-\frac{x^2}{2\sigma^2}}\,dx + \sigma^2 \int_{0}^{+\infty} \frac{e^{-x}}{\sqrt{2\pi}\sigma} e^{-\frac{x^2}{2\sigma^2}}\,dx \\
&= \frac{\sqrt{2}\sigma}{\sqrt{\pi}} + \sigma^2 \int_{0}^{+\infty} \frac{e^x + e^{-x}}{\sqrt{2\pi}\sigma} e^{-\frac{x^2}{2\sigma^2}}\,dx \\
&= \frac{\sqrt{2}\sigma}{\sqrt{\pi}} + \frac{\sigma^2}{2} \mathbb{E}\Big[e^X + e^{-X}\Big] \\
&= \frac{\sqrt{2}\sigma}{\sqrt{\pi}} + \frac{\sigma^2}{2} \mathbb{E}[e^X] + \frac{\sigma^2}{2} \mathbb{E}[e^{-X}] \\
&= \frac{\sqrt{2}\sigma}{\sqrt{\pi}} + \sigma^2 e^{\frac{1}{2}\sigma^2}.
\end{aligned}$$

(iv)

$$\begin{aligned}
\mathbb{E}\Big[X^2e^X\Big] &= \int_{-\infty}^{+\infty} x^2e^x\frac{1}{\sqrt{2\pi}\sigma}e^{-\frac{x^2}{2\sigma^2}}dx \\
&= \frac{-\sigma}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} x\Big(1-\frac{1}{\sigma^2}x\Big)e^{x-\frac{x^2}{2\sigma^2}}dx + \sigma^2\int_{-\infty}^{+\infty} xe^x\frac{1}{\sqrt{2\pi}\sigma}e^{-\frac{x^2}{2\sigma^2}}dx \\
&= \frac{-\sigma}{\sqrt{2\pi}}\Bigg[\Big[xe^{x-\frac{x^2}{2\sigma^2}}\Big]_{-\infty}^{+\infty} - \int_{-\infty}^{+\infty} e^{x-\frac{x^2}{2\sigma^2}}dx\Bigg] + \sigma^2\mathbb{E}\Big[Xe^X\Big] \\
&= \sigma^2\int_{-\infty}^{+\infty} e^x\frac{e^{-\frac{x^2}{2\sigma^2}}}{\sqrt{2\pi}\sigma}dx + \sigma^2\mathbb{E}\Big[Xe^X\Big] \\
&= \sigma^2e^{\frac{\sigma^2}{2}} + \sigma^4e^{\frac{\sigma^2}{2}} \\
&= \sigma^2(1+\sigma^2)e^{\frac{\sigma^2}{2}}.
\end{aligned}$$

This concludes the proof. ☐

Lemma 8. *The function*

$$\begin{aligned}
\lfloor . \rfloor_+ : \mathbb{R} &\longrightarrow \mathbb{R}_+ \\
x &\longmapsto \max{(x,0)}
\end{aligned}$$

is $1-$*Lipshitz, that is:*

$$\Big|\lfloor x\rfloor_+ - \lfloor y\rfloor_+\Big| \leq \Big|x-y\Big|, \quad \forall x,y\in\mathbb{R}.$$

Proof. Without loss of generality, we only have to check that the expression above holds in the three following cases:

- First case, $x, y \geq 0$:

$$\Big|\lfloor x\rfloor_+ - \lfloor y\rfloor_+\Big| = \Big|x-y\Big|.$$

- Second case, $x \geq 0$, $y \leq 0$:

$$\begin{aligned}
\Big|\lfloor x\rfloor_+ - \lfloor y\rfloor_+\Big| &= \Big|x-0\Big| \\
&= x \\
&\leq x-y \\
&= \Big|x-y\Big|.
\end{aligned}$$

- Third case, $x, y \leq 0$:

$$\begin{aligned}
\Big|\lfloor x\rfloor_+ - \lfloor y\rfloor_+\Big| &= 0 \\
&\leq \Big|x-y\Big|.
\end{aligned}$$

This concludes the proof. $\square$

Lemma 9. *Given $\hat{\xi}_n = (\hat{\mu}_n, \hat{\sigma}_n^2)$ the estimator of the real GBM parameters $\xi = (\mu, \sigma^2)$ of the online method, let Z and Z' denote respectively the true return vector and its estimator, that is $Z \coloneqq e^{\mu - \frac{\sigma^2}{2} + \sigma N}$ and $Z' \coloneqq e^{\hat{\mu}_n - \frac{\hat{\sigma}_n^2}{2} + \sigma N}$ where $N \sim \mathcal{N}(0,1)$. Let $\hat{\alpha}_n \coloneqq \hat{\mu}_n - \frac{\hat{\sigma}_n^2}{2}$ and $\alpha \coloneqq \mu - \frac{\sigma^2}{2}$. Then for all $u \in \Delta_m$ and $\theta \in R$, we have:*

$$\left|\mathbb{E}_{\hat{\xi}_n}\left[\left\lfloor\langle u, Z\rangle - \theta\right\rfloor_+\right] - \mathbb{E}_{\hat{\xi}_n}\left[\left\lfloor\langle u, Z'\rangle - \theta\right\rfloor_+\right]\right| \leq \left\|\mathcal{B}_n\right\|_\infty$$

where

$$\mathcal{B}_n \coloneqq \Big[|\hat{\alpha}_n - \alpha| + \frac{e}{2}(\hat{\alpha}_n - \alpha)^2 + |\hat{\sigma}_n - \sigma|\Big(\frac{\sqrt{2}}{\sqrt{\pi}} + \sigma\Big) + \frac{e}{2}(\hat{\sigma}_n - \sigma)^2(1 + \sigma^2) \\ + e(\hat{\alpha}_n - \alpha)(\hat{\sigma}_n - \sigma)\sigma\Big] e^{\alpha + \frac{\sigma^2}{2}},$$

with probability at least:

$$1 - \epsilon_n = \prod_{i=1}^{m} \int_{-1}^{1} f_{\mathcal{N}((\hat{\alpha}_n - \alpha)_i, (\hat{\sigma}_n - \sigma)_i^2)}(t)dt,$$

where $f_{\mathcal{N}((\hat{\alpha}_n - \alpha)_i, (\hat{\sigma}_n - \sigma)_i^2)}$ denotes the p.d.f. of the i^{th} component of $\hat{\alpha}_n - \alpha + (\hat{\sigma}_n - \sigma)N$.
Furthermore, $1 - \epsilon_n \xrightarrow[n\to\infty]{} 1$.

Proof. We have

$$\left|\mathbb{E}_{\hat{\xi}_n}\left[\left\lfloor\langle u, Z\rangle - \theta\right\rfloor_+\right] - \mathbb{E}_{\hat{\xi}_n}\left[\left\lfloor\langle u, Z'\rangle - \theta\right\rfloor_+\right]\right| = \left|\mathbb{E}_{\hat{\xi}_n}\left[\left\lfloor\langle u, Z\rangle - \theta\right\rfloor_+ - \left\lfloor\langle u, Z'\rangle - \theta\right\rfloor_+\right]\right| \\ \leq \mathbb{E}_{\hat{\xi}_n}\left[\left|\left\lfloor\langle u, Z\rangle - \theta\right\rfloor_+ - \left\lfloor\langle u, Z'\rangle - \theta\right\rfloor_+\right|\right].$$

Using Lemma 8, we also know that:

$$\left|\left\lfloor\langle u, Z\rangle - \theta\right\rfloor_+ - \left\lfloor\langle u, Z'\rangle - \theta\right\rfloor_+\right| \leq \left|\langle u, Z\rangle - \langle u, Z'\rangle\right|,$$

and by substituting this into the inequality above then using the fact that $u \in \Delta_m$, we get:

$$\left|\mathbb{E}_{\hat{\xi}_n}\left[\left\lfloor\langle u, Z\rangle - \theta\right\rfloor_+\right] - \mathbb{E}_{\hat{\xi}_n}\left[\left\lfloor\langle u, Z'\rangle - \theta\right\rfloor_+\right]\right| \leq \mathbb{E}_{\hat{\xi}_n}\left[\left|\langle u, Z\rangle - \langle u, Z'\rangle\right|\right] \\ \leq \mathbb{E}_{\hat{\xi}_n}\left[\left\langle u, |Z - Z'|\right\rangle\right] \\ = \left\langle u, \mathbb{E}_{\hat{\xi}_n}\left[|Z - Z'|\right]\right\rangle.$$

In order to bound the term above, we proceed by bounding $|Z - Z'|$. We have:

$$|Z - Z'| = \left| e^{\mu - \frac{\sigma^2}{2} + \sigma N} - e^{\hat{\mu}_n - \frac{\hat{\sigma}_n^2}{2} + \hat{\sigma}_n N} \right|,$$

which gives

$$|Z - Z'| = e^{\alpha + \sigma N} \left| e^{\hat{\alpha}_n - \alpha + (\hat{\sigma}_n - \sigma) N} - 1 \right|. \tag{9}$$

Now we know that $\hat{\alpha}_n - \alpha + (\hat{\sigma}_n - \sigma)N \sim \mathcal{N}\Big(\hat{\alpha}_n - \alpha, (\hat{\sigma}_n - \sigma)^2\Big)$, hence we have:

$$\Big(\hat{\alpha}_n - \alpha + (\hat{\sigma}_n - \sigma)N\Big)_i \in (0,1) \quad \forall i \in \{1, \dots, m\} \tag{10}$$

with probability

$$\begin{aligned} 1 - \epsilon_n &:= \mathbb{P}\Big[\Big(\hat{\alpha}_n - \alpha + (\hat{\sigma}_n - \sigma)N\Big)_i \in (0,1) \quad \forall i \in \{1, \dots, m\}\Big] \\ &= \prod_{i=1}^{m} \mathbb{P}\Big[\Big(\hat{\alpha}_n - \alpha + (\hat{\sigma}_n - \sigma)N\Big)_i \in (0,1)\Big] \end{aligned}$$

by independence of the components of μ, σ and N (independence of the assets). This gives:

$$1 - \epsilon_n = \prod_{i=1}^{m} \int_{-1}^{1} f_{\mathcal{N}((\hat{\alpha}_n - \alpha)_i, (\hat{\sigma}_n - \sigma)_i^2)}(t) dt,$$

where $f_{\mathcal{N}((\hat{\alpha}_n - \alpha)_i, (\hat{\sigma}_n - \sigma)_i^2)}$ denotes the p.d.f. of the i^{th} component of $\hat{\alpha}_n - \alpha + (\hat{\sigma}_n - \sigma)N$. We note that

$$1 - \epsilon_n \xrightarrow[n \to +\infty]{} 1$$

since $(\hat{\alpha}_n - \alpha)_i \longrightarrow 0$ and $(\hat{\sigma}_n - \sigma)_i^2 \longrightarrow 0$ for all $i \in \{1, \dots, m\}$.
In the case where (10) holds, we use (8) in (9) to obtain:

$$\begin{aligned} |Z - Z'| &\le e^{\alpha + \sigma N} \left(\left| \hat{\alpha}_n - \alpha + (\hat{\sigma}_n - \sigma)N \right| + \frac{e}{2} \Big(\hat{\alpha}_n - \alpha + (\hat{\sigma}_n - \sigma)N \Big)^2 \right) \\ &\le e^{\alpha + \sigma N} \left(\left| \hat{\alpha}_n - \alpha \right| + \left| (\hat{\sigma}_n - \sigma)N \right| + \frac{e}{2} \Big(\hat{\alpha}_n - \alpha + (\hat{\sigma}_n - \sigma)N \Big)^2 \right) \\ &= \Big(|\hat{\alpha}_n - \alpha| + \frac{e}{2} (\hat{\alpha}_n - \alpha)^2 \Big) e^{\alpha} e^{X} + \left| \frac{\hat{\sigma}_n - \sigma}{\sigma} \right| e^{\alpha} |X| e^{X} \\ &\quad + \frac{e}{2} \frac{(\hat{\sigma}_n - \sigma)^2}{\sigma^2} e^{\alpha} X^2 e^{X} + e(\hat{\alpha}_n - \alpha) \frac{\hat{\sigma}_n - \sigma}{\sigma} e^{\alpha} X e^{X}. \end{aligned}$$

where $X = \sigma N \sim \mathcal{N}(0, \sigma^2)$. Hence, by taking the expected value we get:

$$\begin{aligned} \mathbb{E}_{\hat{\xi}_n}\Big[|Z - Z'|\Big] \le \Big(|\hat{\alpha}_n - \alpha| + \frac{e}{2} (\hat{\alpha}_n - \alpha)^2 \Big) e^{\alpha} \mathbb{E}_{\hat{\xi}_n}\Big[e^{X}\Big] + \left| \frac{\hat{\sigma}_n - \sigma}{\sigma} \right| e^{\alpha} \mathbb{E}_{\hat{\xi}_n}\Big[|X| e^{X}\Big] \\ + \frac{e}{2} \frac{(\hat{\sigma}_n - \sigma)^2}{\sigma^2} e^{\alpha} \mathbb{E}_{\hat{\xi}_n}\Big[X^2 e^{X}\Big] + e(\hat{\alpha}_n - \alpha) \frac{\hat{\sigma}_n - \sigma}{\sigma} e^{\alpha} \mathbb{E}_{\hat{\xi}_n}\Big[X e^{X}\Big]. \end{aligned}$$

When we plug the explicit expression of the expectations seen in Lemma 7 in the expression above, we finally obtain:

$$\mathbb{E}_{\hat{\xi}_n}\Big[|Z - Z'|\Big] \leq \mathcal{B}_n,$$

where

$$\mathcal{B}_n = \Big[|\hat{\alpha}_n - \alpha| + \frac{e}{2}(\hat{\alpha}_n - \alpha)^2 + |\hat{\sigma}_n - \sigma|\Big(\frac{\sqrt{2}}{\sqrt{\pi}} + \sigma\Big) + \frac{e}{2}(\hat{\sigma}_n - \sigma)^2(1 + \sigma^2) \\ + e(\hat{\alpha}_n - \alpha)(\hat{\sigma}_n - \sigma)\sigma\Big] e^{\alpha + \frac{\sigma^2}{2}}.$$

This implies that

$$\left|\mathbb{E}_{\hat{\xi}_n}\Big[\big\lfloor\langle u, Z\rangle - \theta\big\rfloor_+\Big] - \mathbb{E}_{\hat{\xi}_n}\Big[\big\lfloor\langle u, Z'\rangle - \theta\big\rfloor_+\Big]\right| \leq \Big\langle u, \mathcal{B}_n\Big\rangle \quad \forall u \in \Delta_m, \theta \in \mathbb{R}.$$

The result follows. □

Now that we have the results above, the proof of Theorem **??** becomes straightforward thanks to Lemma 6 and Lemma 9.

Proof of Theorem 3. Assume we are given the estimator $\hat{\xi}_n = (\hat{\mu}_n, \hat{\sigma}_n^2)$ of the real GBM parameters $\xi = (\mu, \sigma^2)$ of the online method. Using the expressions of $p_{\lambda,\xi}$ and $p_{\lambda,\hat{\xi}_n}$ seen respectively in (4) and (5), we have for all $u \in \Delta_m$ and $\theta \in \mathbb{R}$:

$$\begin{aligned}
\Big|p_{\lambda,\xi}(u,\theta) - p_{\lambda,\hat{\xi}_n}(u,\theta)\Big| &= \Big| - \Big\langle u, \mathbb{E}_{\hat{\xi}_n}[Z_i]\Big\rangle + \Big\langle u, \mathbb{E}_{\hat{\xi}_n}[Z_i']\Big\rangle + \frac{\lambda}{1-\alpha}\mathbb{E}_{\hat{\xi}_n}\Big[\lfloor\langle Z, u\rangle - \theta\rfloor_+\Big] \\
&\quad - \frac{\lambda}{1-\alpha}\mathbb{E}_{\hat{\xi}_n}\Big[\lfloor\langle Z', u\rangle - \theta\rfloor_+\Big]\Big| \\
&\leq \Big|\mathbb{E}_{\hat{\xi}_n}\Big[\langle u, Z\rangle\Big] - \mathbb{E}_{\hat{\xi}_n}\Big[\langle u, Z'\rangle\Big]\Big| \\
&\quad + \frac{\lambda}{1-\alpha}\Big|\mathbb{E}_{\hat{\xi}_n}\Big[\big\lfloor\langle u, Z\rangle - \theta\big\rfloor_+\Big] - \mathbb{E}_{\hat{\xi}_n}\Big[\big\lfloor\langle u, Z'\rangle - \theta\big\rfloor_+\Big]\Big|.
\end{aligned}$$

Substituting the inequalities from Lemma 6 and Lemma 9, we get:

$$\Big|p_{\lambda,\xi}(u,\theta) - p_{\lambda,\hat{\xi}_n}(u,\theta)\Big| \leq \Big\|\mathcal{A}_n\Big\|_\infty + \Big\|\mathcal{B}_n\Big\|_\infty.$$

We deduce from the inequality above that difference between the minimal values cannot be larger than $\Big\|\mathcal{A}_n\Big\|_\infty + \Big\|\mathcal{B}_n\Big\|_\infty$. In fact, using a union bound on the failing cases, we obtain:

$$\begin{aligned}
\Big|\min_{(u,\theta)} p_{\lambda,\xi} - \min_{(u,\theta)} p_{\lambda,\hat{\xi}_n}\Big| &\leq \sup_{(u,\theta)}\Big|p_{\lambda,\xi}(u,\theta) - p_{\lambda,\hat{\xi}_n}(u,\theta)\Big| \\
&\leq \Big\|\mathcal{A}_n\Big\|_\infty + \Big\|\mathcal{B}_n\Big\|_\infty,
\end{aligned}$$

with a probability at least:

$$\prod_{i=1}^{m}\int_{-1}^{1} f_{\mathcal{N}((\hat{\alpha}_n-\alpha)_i,(\hat{\sigma}_n-\sigma)_i^2)}(t)dt+\prod_{i=1}^{m}\left\{1-\left(\frac{\sigma_i^2}{n}+\frac{\sigma_i^4}{2(n-1)}\right)\right\}-1.$$

The result follows. $\square$

B Experimental Settings

We have seen above that we use the drift vector:

$$\mu=\begin{pmatrix}\mu_1\\ \mu_2\\ \mu_3\\ \mu_4\end{pmatrix}=\begin{pmatrix}0.15\\ 0.03\\ -0.07\\ -0.133\end{pmatrix}.$$

Moreover, we take the volatility vector:

$$\sigma^2=\begin{pmatrix}\sigma_1^2\\ \sigma_2^2\\ \sigma_3^2\\ \sigma_4^2\end{pmatrix}=\begin{pmatrix}0.025\\ 0.015\\ 0.02\\ 0.03\end{pmatrix}.$$

The parameters are set to the following:

- Problem parameters:
  - number of assets: $m = 4$,
  - lagrange multiplier: $\lambda = 0.7$,
  - risk level: $\alpha = 0.05$.
- Online method parameters:
  - initial estimator:
  $$\begin{cases}\hat{\mu}_0=(1,\ldots,1)^T\\ \hat{\sigma}_0^2=(1,\ldots,1)^T\end{cases},$$
  - maximal number of days: $n_{\max} = 200$.
- SMD parameters:
  - initial point:
  $$\begin{cases}U_0=\left(\frac{1}{m},\ldots,\frac{1}{m}\right)\\ \theta_0=1\end{cases},$$

- maximal number of iterations: $k_{\max} = 10^4$,
- step-size sequence: $\eta = \left\{ \eta_k = \frac{1}{\log(k+2)} \right\}_{k \geq 0}$.
- We also add a stopping criterion to the implementation of SMD (Algorithm 1) on the proximity of the points given by the following: if

$$\frac{|\hat{p}(X_{k+1}) - \hat{p}(X_k)|}{\|X_{k+1} - X_k\|} < \epsilon,$$

then we stop iterating. In the expression above, $\hat{p}(x)$ is a stochastic approximation of the value of $p(x)$ based on simulations similar to the ones in (2). In our simulation of the online method, we set $\epsilon := 10^{-3}$.

Remark
The stochastic mirror descents performed in the online method are not made independently. In fact, for each of the performed SMDs, the initial point is the last point reached by the previous SMD, and the step-size sequence is the sequence remaining when we remove the terms already used. For example, assume the first performed SMD has an initial point $\left(U_0, \theta_0\right)$, the step-size sequence $\eta = \{\eta_k\}_{k \geq 0}$ and takes $K \in \mathbb{N}$ iterations so that the last reached point is $\left(U_K, \theta_K\right)$. Then the second SMD is performed with $\left(U_K, \theta_K\right)$ as initial point and $\eta = \{\eta_k\}_{k \geq K}$ as step-size sequence. This way, the SMDs are done one after the other so that they form a unique descent where each time we stop and update the estimators of μ and σ^2.

For more on the experiments, all the simulations are available at https://github.com/youssef-chaabouni/Online-portfolio-optimization-under-CVAR-constraint-with-stochastic-mirror-descent.

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