Online Portfolio Optimization under CV@R constraint using Stochastic Mirror Descent

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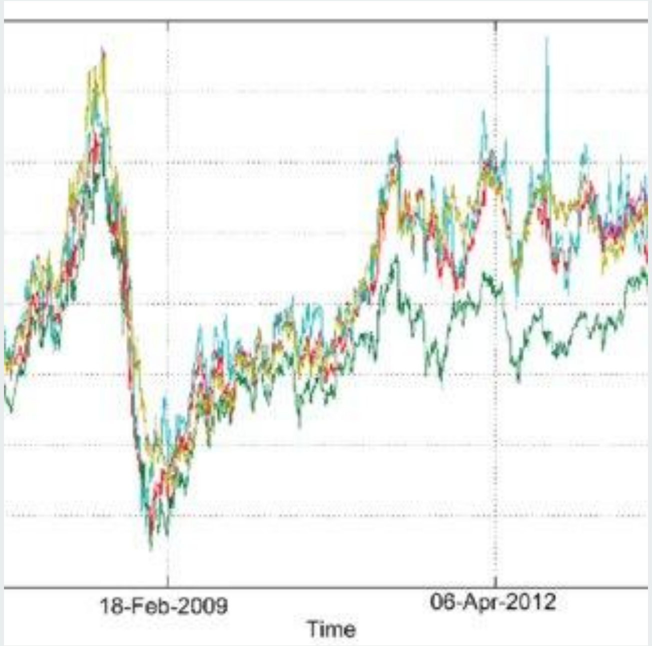
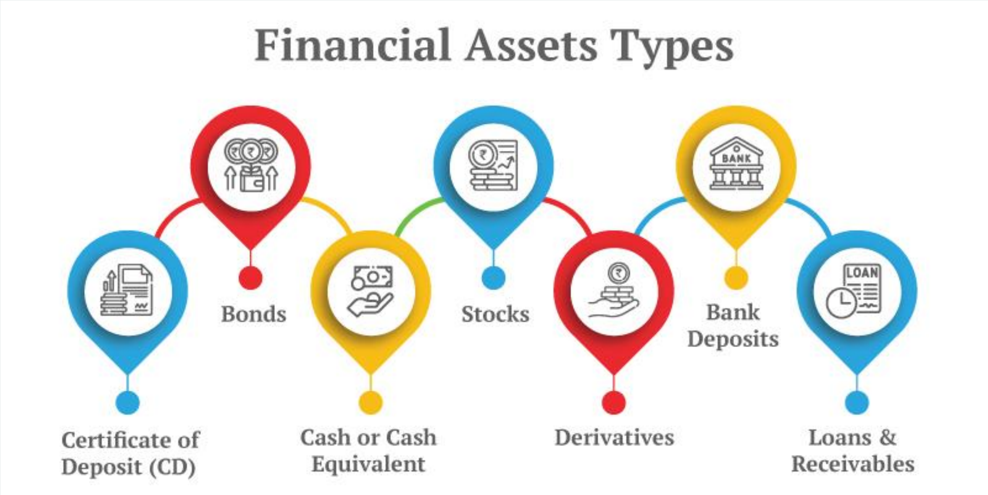
Financial Market: What is it?





Best investment strategy?

Best Investment Strategy?

Long run: Fully invest in asset with largest expected return

Short run: Issues because of high variability

Best Investment Strategy?

We define Z as the return vector of the assets:

$$Z_i := \frac{A_i(T)}{A_i(0)} - 1$$

An investment strategy is the vector of positive weights assigned to assets:

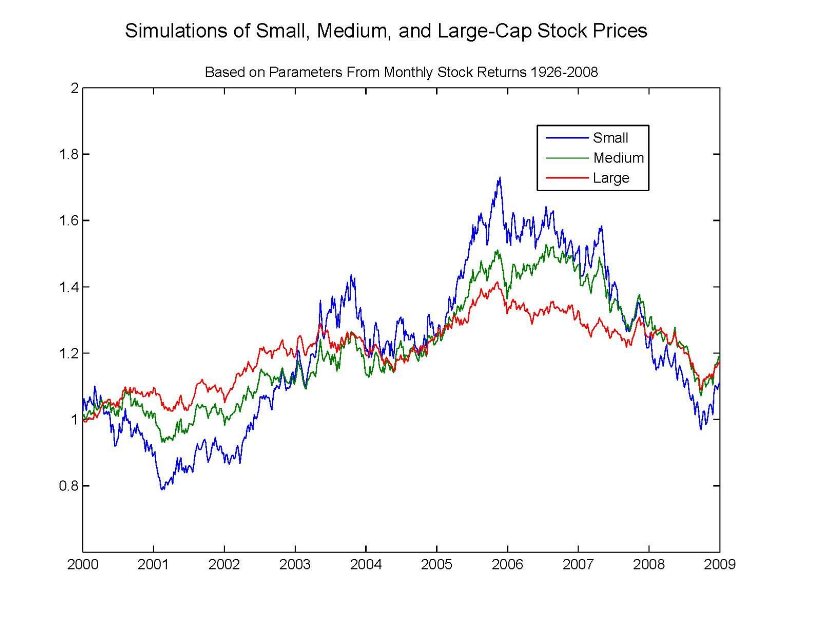
$$u \in \Delta_m := \left\{ u \in \mathbb{R}_+^m : \sum_{i=1}^{m} u_i = 1 \right\}$$

We want to optimize the total return of the strategy given by:

$$\langle Z, u \rangle = \sum_{i=1}^{m} u_i Z_i$$

Mathematical Model: Financial Assets

Stochastic Differential Equations (SDE)



Risky assets

↕

Geometric Brownian Motion (GBM)

$$dS_t = \mu S_t \, dt + \sigma S_t \, dW_t$$

μ : drift
σ : volatility
W_t: Brownian motion

$$S_t = S_0 \exp\left(\left(\mu - \frac{\sigma^2}{2}\right) t + \sigma W_t\right)$$

Mathematical Model: Risk Constraints

Conditional Value at Risk (CV@R)

At a risk level α:

V@R: worst value that can be accepted at this risk level,

$$V@R_\alpha(u) = \sup\Big\{q \in \mathbb{R} : \mathbb{P}\big(\langle Z, u\rangle \le q\big) \le \alpha\Big\}$$

CV@R: mean of the loss for return values under the **V@R:**

$$CV@R_\alpha(u) = \mathbb{E}\Big[-\langle Z, u\rangle \,|\, \langle Z, u\rangle \le V@R_\alpha(u)\Big]$$

Constrained Optimization Problem

$$P_M = \mathrm{argmin}_{\begin{cases} u \in \Delta_m \\ CV@R_\alpha(u) \leq M \end{cases}} - \langle Z, u \rangle$$

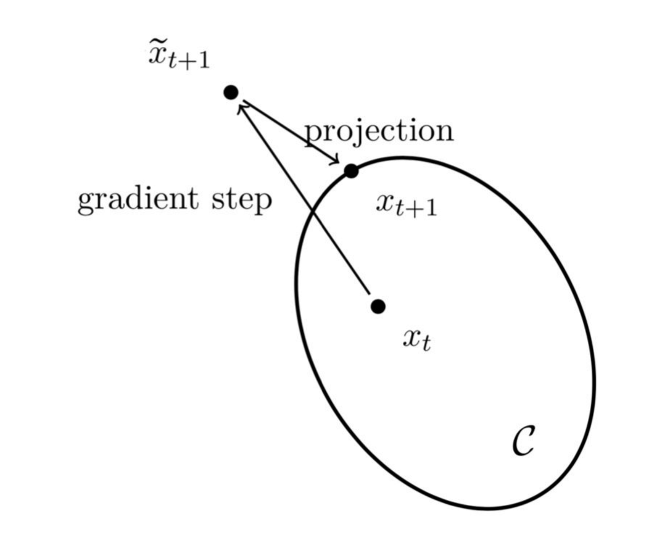
Lagrange multipliers

$$Q_\lambda = \mathrm{argmin}_{u \in \Delta_m} \left\{ - \sum_{i=1}^{m} u_i Z_i + \lambda CV@R_\alpha(u) \right\}$$

CV@R written as the minimum of a convex function [BTT86], [Pfl00]

$$Q_\lambda = \mathrm{argmin}_{(u,\theta) \in \Delta_m \times \mathbb{R}} \left\{ - \sum_{i=1}^{m} u_i \mathbb{E}[Z_i] + \lambda\theta + \frac{1}{1-\alpha} \mathbb{E}[\lfloor \langle Z, u \rangle - \theta \rfloor_+] \right\}$$

Projected Gradient Descent?



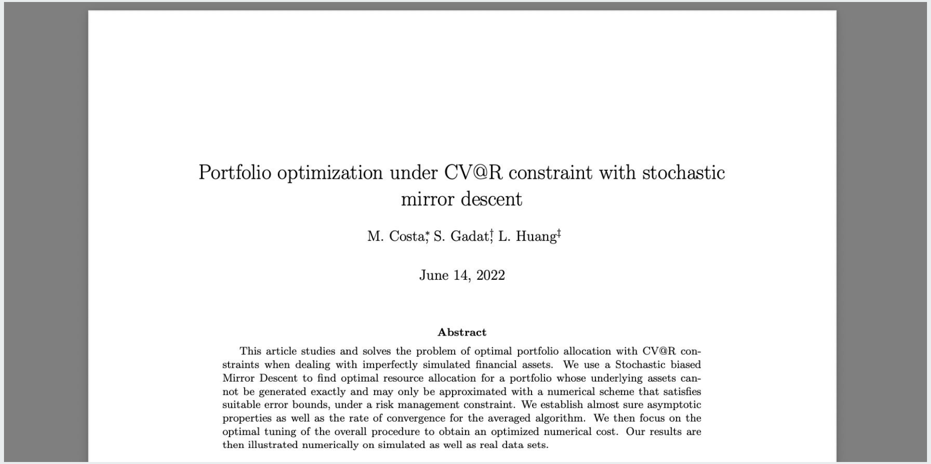
from [S+15]

Projection on the simplex is too complicated!

Previous Work

Portfolio optimization under CV@R constraint with stochastic mirror descent

by Costa, Gadat, Huang, 2022
[CGH22]



Portfolio optimization under CV@R constraint with stochastic mirror descent

M. Costa[*], S. Gadat[†], L. Huang[‡]

June 14, 2022

Abstract

This article studies and solves the problem of optimal portfolio allocation with CV@R constraints when dealing with imperfectly simulated financial assets. We use a Stochastic biased Mirror Descent to find optimal resource allocation for a portfolio whose underlying assets cannot be generated exactly and may only be approximated with a numerical scheme that satisfies suitable error bounds, under a risk management constraint. We establish almost sure asymptotic properties as well as the rate of convergence for the averaged algorithm. We then focus on the optimal tuning of the overall procedure to obtain an optimized numerical cost. Our results are then illustrated numerically on simulated as well as real data sets.

Portfolio parameters:
$(\boldsymbol{\mu}, \boldsymbol{\sigma})$

Mirror Descent

Optimal investment strategy

Mirror Descent (MD)

by Nemirovsky and Yudin [NY83]

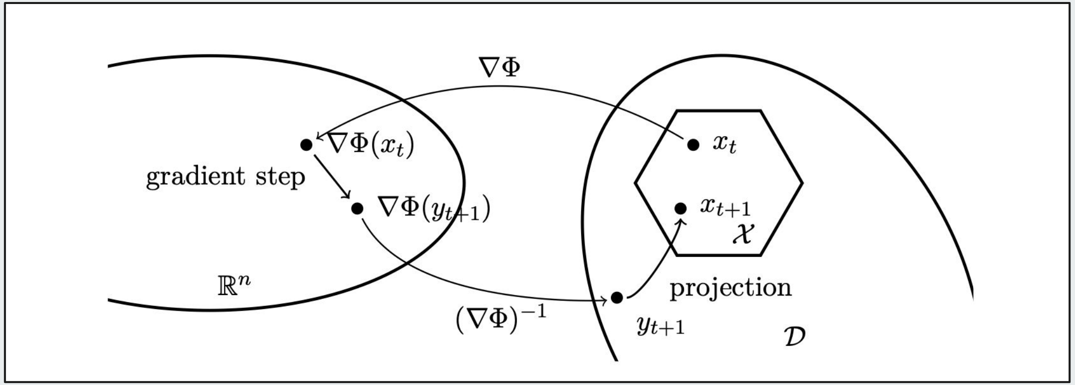
Gradient Descent

$$x_{t+1} = \text{argmin}_{x\in\mathbb{R}^m}\left\{f(x_t) + \langle\nabla f(x_t), x - x_t\rangle + \frac{1}{2\eta}\|x - x_t\|_2^2\right\}$$

Generalization

Mirror Descent

$$x_{t+1} = \text{argmin}_{x\in\mathbb{R}^m}\left\{f(x_t) + \langle\nabla f(x_t), x - x_t\rangle + \frac{1}{\eta}\mathcal{D}_\phi(x, x_t)\right\}$$



from [S+15]

Bregman Divergence:

$$\begin{aligned}\mathcal{D}_\phi :\quad & \mathbb{R}^m \times \mathbb{R}^m \longrightarrow \mathbb{R}_+ \cup \{0\}\\ & (x, y) \longmapsto \phi(x) - \phi(y) - \langle\nabla\phi(y), x - y\rangle\end{aligned}$$

ϕ strictly convex

Entropy on simplex

$$\phi(u) = \sum_{i=1}^{m} u_i \log u_i$$

Stochastic Mirror Descent (SMD) for Portfolio Optimization

MD applied to the our constrained optimization problem:

$$X_{k+1} = \arg\min_{x \in \mathbf{\Delta}_m \times \mathbb{R}} \left\{ \langle \nabla p_\lambda(X_k), x - X_k \rangle + \frac{1}{\eta_{k+1}} \mathcal{D}_\Phi(x, X_k) \right\}$$

> A remarkable feature is that this latter minimization can be made explicit :
>
> $$X_{k+1} = \begin{pmatrix} U^{k+1} \\ \theta^{k+1} \end{pmatrix} \quad \text{with} \quad \begin{cases} U^{k+1} = \dfrac{U^k e^{-\eta_{k+1} \partial_u p_\lambda(U^k, \theta^k)}}{\| U^k e^{-\eta_{k+1} \partial_u p_\lambda(U^k, \theta^k)} \|_1}, \\ \theta^{k+1} = \theta^k - \eta_{k+1} \partial_\theta p_\lambda(U^k, \theta^k) \end{cases}$$

— Costa, Gadat, Huang [CGH22]

"Stochastic": stochastic approximation of the gradient

What's Missing?

Expression of GBM:

$$S_t = S_0 \exp\left(\left(\mu - \frac{\sigma^2}{2}\right)t + \sigma W_t\right)$$

Portfolio parameters

Portfolio parameters:
$(\boldsymbol{\mu}, \boldsymbol{\sigma})$

Costa, Gadat, Huang [CGH22]

Optimal investment strategy

These parameters are never given in real life!

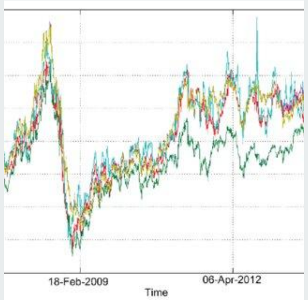
Our solution: Online Method

SMD

Estimate portfolio parameters (μ, σ)

Online method

Find optimal investment strategy



Next day:
Read asset price

Our results

Online estimation of portfolio parameters

- Each day, we are given the value X of the assets prices.
- Using this value as well as the previous ones, we estimate the portfolio parameters.
- These estimators are unbiased and converge to the true parameter values.

Theorem 1. *Let $\mu \in \mathbb{R}^m$ and $\sigma \in \mathbb{R}_+^m$. Consider the portfolio formed of risky assets following geometric Brownian motions with parameters μ and σ, that is:*

$$\forall i \in \{1, \ldots, m\}, \quad dS_t^i = \mu_i S_t^i dt + \sigma_i S_t^i dW_t.$$

Further, assume we are given an $n-$sample $(X_1, \ldots, X_n)$ of random vectors such that

$$\forall t \in \{1, \ldots, n\}, \quad X_t^i = \log \frac{S_{t+1}^i}{S_t^i},$$

then the values of $(\hat{\mu}_n)_{n \geq 0}$ and $(\hat{\sigma}_n^2)_{n \geq 0}$ computed iteratively as follows:

$$\begin{cases} \hat{\mu}_{n+1} = \frac{n}{n+1}\hat{\mu}_n - \frac{1}{2n(n+1)}\hat{\sigma}_n^2 + \frac{1}{n+1}X_{n+1} + \frac{1}{2(n+1)}\left(\hat{\mu}_n - \frac{1}{2}\hat{\sigma}_n^2 - X_{n+1}\right)^2 \\ \hat{\sigma}_{n+1}^2 = \frac{n-1}{n}\hat{\sigma}_n^2 + \frac{1}{n+1}\left(\hat{\mu}_n - \frac{1}{2}\hat{\sigma}_n^2 - X_{n+1}\right)^2, \end{cases}$$

form two sequences of unbiased estimators converging respectively to μ and σ^2).

Our results

Rate of convergence

We prove that the second moment of each of the estimators exhibits the same behavior as $1/n$ when n approaches infinity.

Theorem 2. *Let $\mu \in \mathbb{R}^m$ and $\sigma \in \mathbb{R}_+^m$. Consider the sequence $(\hat{\mu}_n, \hat{\sigma}_n^2)_{n\geq 0}$ of estimators described above. Then the variances of the estimators are given by the expressions:*

$$\begin{cases} \mathbb{E}\left[|\hat{\mu}_n - \mu|^2\right] = \frac{\sigma^2}{n} + \frac{\sigma^4}{2(n-1)} \\ \mathbb{E}\left[|\hat{\sigma}_n^2 - \sigma^2|^2\right] = \frac{2\sigma^4}{n-1}, \end{cases}$$

meaning that they converge to 0 in polynomial time.

Our results

Online Method: Algorithm

1. Each day, we are given the value X of the asset prices.
2. Using this value as well as the previous ones, we estimate the portfolio parameters.
3. We compute the optimal investment strategy corresponding to the estimators, then invest accordingly.

Algorithm 2 Online method for Portfolio Optimization under CV@R constraints.

Data: Initial estimator $\hat{\xi}_0 = \left(\hat{\mu}_0, \hat{\sigma}_0^2\right) \in \mathbb{R} \times \mathbb{R}_+$, number of days $n_{\max}$.

Result: Optimal investment strategy for GBM parameters $\xi = \left(\mu_0, \sigma_0^2\right)$

Algorithm:

At time $n = 0$:

- Observe the initial values of the portfolio assets: $S_0 \leftarrow \left(S_0^1, \ldots, S_0^m\right)$.
- Using SMD, set the strategy to $\Gamma(\hat{\xi}_0)$.

for $n = 0, \ldots, n_{\max} - 1$ **do**

Observe the values of the portfolio assets: $S_{n+1} \leftarrow \left(S_{n+1}^1, \ldots, S_{n+1}^m\right)$.

Compute $X_{n+1} = \left(X_{n+1}^1, \ldots, X_{n+1}^m\right)$ where:

$$X_{n+1}^i = \log \frac{S_{n+1}^i}{S_n^i}, \quad \forall i \in \{1, \ldots, m\}.$$

Update the algorithm $\hat{\xi}_{n+1} = \left(\hat{\mu}_{n+1}, \hat{\sigma}_{n+1}^2\right)$ using the iterative expression:

$$\begin{cases} \hat{\mu}_{n+1} = \frac{n}{n+1}\hat{\mu}_n - \frac{1}{2n(n+1)}\hat{\sigma}_n^2 + \frac{1}{n+1}X_{n+1} + \frac{1}{2(n+1)t}\left(\hat{\mu}_n - \frac{1}{2}\hat{\sigma}_n^2 - X_{n+1}\right)^2 \\ \hat{\sigma}_{n+1}^2 = \frac{n-1}{n}\hat{\sigma}_n^2 + \frac{1}{(n+1)t}\left(\hat{\mu}_n - \frac{1}{2}\hat{\sigma}_n^2 - X_{n+1}\right)^2 \end{cases}.$$

Using SMD, set the strategy to $\Gamma(\hat{\xi}_{n+1})$.

Return $\Gamma(\hat{\xi}_{n_{\max}})$.

Our results

Online Method: Convergence

- The estimated optimal strategy at day *n* converges to the true optimal investment strategy as *n* approaches infinity.
- We give a convergence upper bound to the difference between the estimated strategy and the optimal one.

Theorem 3. *Given $\hat{\xi}_n = (\hat{\mu}_n, \hat{\sigma}_n^2)$ the estimator of the real GBM parameters $\xi = (\mu, \sigma^2)$ of the online method, let Z and Z' denote respectively the true return vector and its estimator, that is $Z := e^{\mu - \frac{\sigma^2}{2} + \sigma N}$ and $Z' := e^{\hat{\mu}_n - \frac{\hat{\sigma}_n^2}{2} + \sigma N}$ where $N \sim \mathcal{N}(0,1)$. Let $\hat{\alpha}_n := \hat{\mu}_n - \frac{\hat{\sigma}_n^2}{2}$ and $\alpha := \mu - \frac{\sigma^2}{2}$. We define $p_{\lambda,\xi}$ and $p_{\lambda,\hat{\xi}_n}$ as described above. Then*

$$\left| \min_{(u,\theta)\in\Delta_m\times\mathbb{R}} p_{\lambda,\xi} - \min_{(u,\theta)\in\Delta_m\times\mathbb{R}} p_{\lambda,\hat{\xi}_n} \right| \leq \left\|\mathcal{A}_n\right\|_\infty + \left\|\mathcal{B}_n\right\|_\infty,$$

where

$$\mathcal{A}_n := e^{\mu}\left(\left|\hat{\mu}_n - \mu\right| + \frac{e}{2}\left(\hat{\mu}_n - \mu\right)^2\right),$$

and

$$\mathcal{B}_n := \Big[|\hat{\alpha}_n - \alpha| + \frac{e}{2}(\hat{\alpha}_n - \alpha)^2 + |\hat{\sigma}_n - \sigma|\Big(\frac{\sqrt{2}}{\sqrt{\pi}} + \sigma\Big) + \frac{e}{2}(\hat{\sigma}_n - \sigma)^2(1+\sigma^2) \\ + e(\hat{\alpha}_n - \alpha)(\hat{\sigma}_n - \sigma)\sigma\Big]e^{\alpha + \frac{\sigma^2}{2}},$$

with probability at least:

$$1 - \epsilon_n = \prod_{i=1}^{m} \int_{-1}^{1} f_{\mathcal{N}((\hat{\alpha}_n - \alpha)_i, (\hat{\sigma}_n - \sigma)_i^2)}(t)dt + \prod_{i=1}^{m} \left\{1 - \left(\frac{\sigma_i^2}{n} + \frac{\sigma_i^4}{2(n-1)}\right)\right\} - 1.$$

Furthermore, $1 - \epsilon_n \xrightarrow[n\to+\infty]{} 1.$

Experiments

- We simulate the online method on a financial portfolio of 4 assets.

- We observe that the errors of estimators of μ and σ^2 exhibit the same behavior as $1/n$ when n approaches infinity, which confirms our theoretical results.

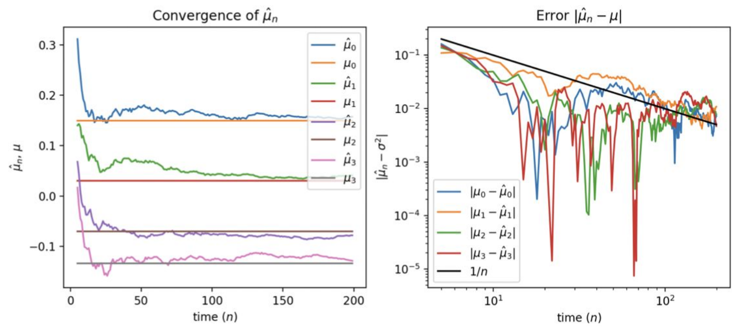


Figure 4: Convergence of $\hat{\mu}_n$ towards μ.

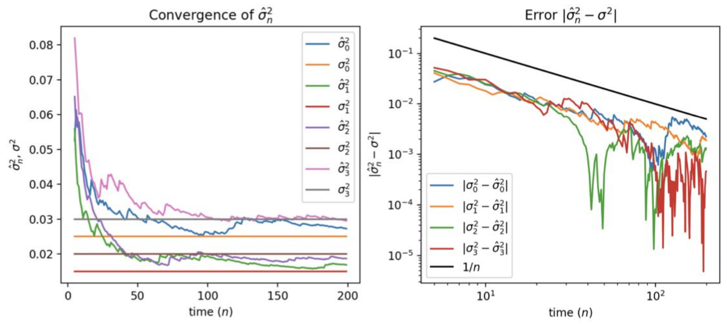


Figure 5: Convergence of $\hat{\sigma}_n^2$ towards σ^2.

Conclusion

- We suggest the algorithm of the online method as a solution to the problem of optimization under risk management constraints.

- We give a convergence upper bound, according to which the solution given by the online method converges to the optimal investment strategy.

- We run experiments by simulating the online method, which confirm our theoretical findings concerning the asymptotic behavior of the portfolio parameter estimators.

Thank you!

References

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