Local time Processes & BDG inequality

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Definition 1. A process M is a **local martingale** with respect to (w.r.t.) the filtration $\mathcal{F}$ if:

1. M is adapted to the filtration $\mathcal{F}$, that is, $\forall t$, $M_t \in \mathcal{F}_t$.
2. There exists a sequence (τ_n) of stopping times such that $\tau_n \uparrow \infty$ almost surely (a.s.), and M^{τ_n} is a true martingale for each n.

i.e. M behaves like a martingale but only up to some stopping times."

Lévy's characterization of Brownian motion. Let $X = \{X_t\}_{t\geq 0}$ be a $\mathbb{R}^d$-valued, continuous, adapted process having zero mean and covariance matrix

$$\mathbb{E}\left[X_s^i X_t^j\right] = a_{ij}(s \wedge t), \quad 1 \leq i, j \leq d, \quad s, t \geq 0,$$

where $A = (a_{ij})$ is a positive definite symmetric $d \times d$-matrix. Then, the following statements are equivalent:

1. X is a Brownian motion with covariance A;
2. X is a martingale with $\langle X^i, X^j \rangle_t = a_{ij} t$, for each $1 \leq i, j \leq d, t \geq 0$;
3. For each $u \in \mathbb{R}^d$, the process $M(u) = \{M_t(u)\}_{t\geq 0}$ defined by
$$M_t(u) := \exp\left(i\langle u, X_t\rangle + \frac{t}{2}\langle u, Au\rangle\right) = \exp\left(i\sum_{k=1}^d \int_0^t u_k(s)\, dX_s^k + \frac{1}{2}\sum_{k=1}^d \int_0^t u_k^2(s)\, ds\right), \quad t \geq 0,$$
is a martingale.

Proof:

The implication from statement (1) to statement (2) is straightforward. To demonstrate that statement (2) implies statement (3), we consider the exponential martingale with a complex coefficient $\lambda = i$ applied to the local martingale

$$M_t = \sum_{k=1}^d \int_0^t u_k(s)\, dX_s^k.$$

This leads to $\mathcal{E}_t = \exp\left(iM_t - \frac{1}{2}\langle M, M\rangle_t\right)$ being a local martingale. Since $\mathcal{E}_t$ is bounded, it is a complex martingale.

Now, assume that statement (3) holds. By selecting $u_k = \xi \mathbf{1}_{[0,T]}$ for a specific $\xi \in \mathbb{R}^d$ and $T > 0$, we have

$$\mathcal{E}_t = \exp\left(i\langle \xi, X_{t\wedge T}\rangle - \frac{1}{2}|\xi|^2(t \wedge T)\right),$$

which is a martingale. For $s < t < T$, using the martingale property, we deduce that $X_t - X_s$ is independent of $\mathcal{F}_s$ and its Fourier transform is given by $\mathbb{E}(\exp(i\langle \xi, X_t - X_s\rangle)) = \exp\left(-\frac{|\xi|^2(t-s)}{2}\right)$. Therefore, X is indeed a Brownian motion.

Dambis, Dubins-Schwarz. Let $(M_t)_{t\geq 0}$ be a continuous martingale such that $M_0 = 0$ and $\langle M\rangle_\infty = +\infty$. There exists a Brownian motion $(B_t)_{t\geq 0}$ such that for every $t \geq 0$,

$$M_t = B_{\langle M\rangle_t}.$$

Proof: Take $r \in (0,\infty)$, define the pseudo-inverse of $\langle M\rangle_t$ as

$$\tau_r := \inf\{t \geq 0 \mid \langle M\rangle_t \geq r\}.$$

Then τ_r is an a.s. finite stopping time, $\forall r \in (0,\infty)$, τ_r is left-continuous, and increasing in r. Therefore, τ_{r-} and τ_{r+} exist, $\forall r \in (0,\infty)$. Define $\beta_r = M_{\tau_r}, r \in [0,\infty)$. Note M^{τ_r} is an L^2-bounded martingale, $\forall k \in (0,\infty)$. For $0 < q < r < k$, by the Optional Stopping Theorem, we have

$$\mathbb{E}(M^{\tau_k}_{\tau_r} \mid \mathcal{F}_{\tau_q}) = M^{\tau_k}_{\tau_q} \quad \text{a.s.}$$

Therefore,

$$\mathbb{E}(\beta_r \mid \mathcal{F}_{\tau_q}) = \beta_q \quad \text{a.s.}$$

Thus, $(\beta_r)_{r\geq 0}$ is a martingale w.r.t. $(\mathcal{G}_r = \mathcal{F}_{\tau_r})_{r\geq 0}$. Similarly, $(M_t^2 - \langle M\rangle_t)_{t\geq 0}$ is a continuous local martingale. Therefore,

$$\mathbb{E}(\beta_r^2 - \langle M\rangle_{\tau_r} \mid \mathcal{G}_q) = \beta_q^2 - \langle M\rangle_{\tau_q} \quad \text{a.s.}$$

We claim $\langle M\rangle_{\tau_r} = r, \forall r \in [0,\infty)$. Thus, $\mathbb{E}(\beta_r^2 - r \mid \mathcal{G}_q) = \beta_q^2 - q, \forall q > 0$.

Lemma. For every $0 \leq a < b < \infty$,

$$\{\langle M\rangle_b - \langle M\rangle_a = 0\} \,\Delta \left\{\sup_{t\in[a,b]} |M_t - M_a| = 0\right\}$$

is a null set.

From the above lemma, we have $\beta_{r+} = M_{\tau_{r+}} = M_{\tau_r} = \beta_r$. Further, note τ_r is left-continuous, we have $r \mapsto \beta_r$ is continuous a.s. Thus, $(\beta_r)_{r\geq 0}$ is a continuous process which is a $(\mathcal{G}_r)_{r\geq 0}$ martingale with quadratic variation $\langle\beta\rangle_r = r, \forall r > 0$. By Lévy's Characterization theorem, we have $(\beta_r)_{r\geq 0}$ is a $(\mathcal{G}_r)_{r\geq 0}$ Standard Brownian Motion. Finally, since $\beta_r = M_{\tau_r}$, we have $\beta_{\langle M\rangle_t} = M_t$ a.s.

Proof of Lemma When $\sup_{t\in[a,b]} |M_t - M_a| = 0$, from the approximation of $\langle M\rangle$, $\langle M\rangle_b - \langle M\rangle_a = 0$ is satisfied a.s. Now we prove the converse. Consider the continuous local martingale $Y_t := M_{t\wedge b} - M_{t\wedge a}$. Then $\langle Y\rangle_t = \langle M\rangle_{t\wedge b} - \langle M\rangle_{t\wedge a} = 0$. Therefore, $\sup_{t\in[a,b]} |M_t - M_a| = \sup_{s\in[0,\infty)} |Y_s| = 0$.

Burkholder-Davis-Gundy Inequalities

Theorem. For every $p \in [0,\infty]$, there exist two constants c_p and C_p such that, for all continuous local martingales M vanishing at zero,

$$c_p\mathbb{E}\left[\left(\langle M,M\rangle_\infty^{1/2}\right)^p\right] \leq \mathbb{E}\left[(M^*_\infty)^p\right] \leq C_p\mathbb{E}\left[\left(\langle M,M\rangle_\infty^{1/2}\right)^p\right].$$

Corollary. For any stopping time T

$$c_p\mathbb{E}\left[\left(\langle M,M\rangle_T^{1/2}\right)^p\right] \leq \mathbb{E}\left[(M^*_T)^p\right] \leq C_p\mathbb{E}\left[\left(\langle M,M\rangle_T^{1/2}\right)^p\right].$$

More generally, for any bounded predictable process H

$$c_p\mathbb{E}\left[\left(\int_0^T H_s^2 d\langle M,M\rangle_s\right)^{p/2}\right] \leq \mathbb{E}\left[\sup_{0\leq t\leq T}\left|\int_0^t H_s dM_s\right|^p\right] \leq C_p\mathbb{E}\left[\left(\int_0^T H_s^2 d\langle M,M\rangle_s\right)^{p/2}\right].$$

The proof of the theorem is broken up into several steps.

Proposition. For $p \geq 2$, there exists a constant C_p such that for any continuous local martingale M such that $M_0 = 0$,

$$\mathbb{E}\left[(M_\infty^*)^p\right] \leq C_p \mathbb{E}\left[\left(\langle M, M\rangle_\infty^{1/2}\right)^p\right].$$

Proof: By stopping, it is enough to prove the result for bounded M. The function $x \to |x|^p$ being twice differentiable, we may apply Itô's formula to the effect that

$$|M_\infty|^p = \int_0^\infty p|M_s|^{p-1}(\operatorname{sgn} M_s)dM_s + \frac{1}{2}\int_0^\infty p(p-1)|M_s|^{p-2}d\langle M, M\rangle_s.$$

Consequently,

$$\mathbb{E}[|M_\infty|^p] = \frac{p(p-1)}{2}\mathbb{E}\left[\int_0^\infty |M_s|^{p-2}d\langle M, M\rangle_s\right].$$

On the other hand, by Doob's inequality, we have $\|M_\infty^*\|_p \leq \frac{p}{p-1}\|M_\infty\|_p$, and the result follows from straightforward calculations.

Proposition. For $p \geq 4$, there exists a constant c_p such that

$$c_p \mathbb{E}\left[\left(\langle M, M\rangle_\infty^{1/2}\right)^p\right] \leq \mathbb{E}\left[(M_\infty^*)^p\right].$$

Proof: By stopping, it is enough to prove the result in the case where $\langle M, M\rangle$ is bounded. In what follows, a_p will always designate a universal constant, but this constant may vary from line to line. For instance, for two reals x and y

$$|x+y|^p \leq a_p(|x|^p + |y|^p).$$

From the equality $M_t^2 = 2\int_0^t M_s dM_s + \langle M, M\rangle_t$, it follows that

$$\mathbb{E}\left[\langle M, M\rangle_\infty^{p/2}\right] \leq a_p\left(\mathbb{E}\left[(M_\infty^*)^p\right] + \mathbb{E}\left[\left|\int_0^\infty M_s dM_s\right|^{p/2}\right]\right),$$

and applying the inequality of Proposition (4.3) to the local martingale $\int_0^\cdot M_s dM_s$, we get

$$\begin{aligned}\mathbb{E}\left[\langle M, M\rangle_\infty^{p/2}\right] &\leq a_p\left(\mathbb{E}\left[(M_\infty^*)^p\right] + \mathbb{E}\left[\left(\int_0^\infty M_s^2 d\langle M, M\rangle_s\right)^{p/4}\right]\right) \\ &\leq a_p\left(\mathbb{E}\left[(M_\infty^*)^p\right] + \left(\mathbb{E}\left[(M_\infty^*)^p\right]\right)^{1/2}\left(\mathbb{E}\left[\langle M, M\rangle_\infty^{p/2}\right]\right)^{1/2}\right).\end{aligned}$$

If we set $x = \mathbb{E}\left[\langle M, M\rangle_\infty^{p/2}\right]^{1/2}$ and $y = \mathbb{E}\left[(M_\infty^*)^p\right]^{1/2}$, the above inequality reads $x^2 - a_p xy - a_p y^2 \leq 0$, which entails that x is less than or equal to the positive root of the equation $x^2 - a_p xy - a_p y^2 = 0$, which is of the form $a_p y$. This establishes the proposition.

Definition 2 (Domination relation)**.** A positive, adapted right-continuous process X is dominated by an increasing process A, if

$$\mathbb{E}[X_T|\mathcal{F}_0] \leq \mathbb{E}[A_T|\mathcal{F}_0]$$

for any bounded stopping time T.

Lemma. If X is dominated by A and A is continuous, for x and $y > 0$,

$$P[X_T^* > x; A_\infty \le y] \le \frac{1}{x}\mathbb{E}[A_\infty \wedge y]$$

where $X_T^* = \sup_{s \le T} X_s$.

Proof: It suffices to prove the inequality in the case where $P(A_\infty \le y) > 0$ and, in fact, where $P(A_\infty \le y) = 1$, which may be achieved by replacing P by $P' = P(\cdot | A_\infty \le y)$ under which the domination relation is still satisfied.

Moreover, by Fatou's lemma, it is enough to prove that

$$P[X_T^* > x; A_\infty \le y] \le \frac{1}{x}\mathbb{E}[A_\infty \wedge y]$$

but reasoning on $[0, n]$ amounts to reasoning on $[0, \infty)$ and assuming that the r.v. X_∞ exists and the domination relation is true for all stopping times whether bounded or not. We define $R = \inf\{t : A_t > y\}$, $S = \inf\{t : X_t > x\}$, where in both cases the infimum of the empty set is taken equal to $+\infty$. Because A is continuous, we have $\{A_\infty \le y\} = \{R = +\infty\}$ consequently

$$P[X_T^* > x; A_\infty \le y] = P[X_T^* > x; R = +\infty] \le P[X_S \ge x; (S < \infty) \cap (R = +\infty)] \le P[X_S \ge x; S \le R] \le \frac{1}{x}\mathbb{E}[X_{S \wedge R}]$$

the last inequality being satisfied since, thanks to the continuity of A, and $A_0 \le y$ a.s., we have $A_{S \wedge R} \le A_\infty \wedge y$.

Proposition. Under the hypothesis of Lemma , for any $k \ge 0, 1$,

$$\mathbb{E}[(X_T^*)^k] \le 2^{k+1}\mathbb{E}[A_\infty^k].$$

Proof: Let F be a continuous increasing function from $\mathbb{R}_+$ into $\mathbb{R}_+$ with $F(0) = 0$. By Fubini's theorem and the above lemma,

$$\mathbb{E}[F(X_T^*)] = \mathbb{E}\left[\int_0^\infty 1_{\{X_T^* > x\}} dF(x)\right] \le \int_0^\infty \mathbb{E}[1_{\{X_T^* > x\}}] dF(x) \le \int_0^\infty \left(\frac{1}{x}\mathbb{E}[A_\infty \wedge x] + P[A_\infty > x]\right) dF(x)$$

$$\le \int_0^\infty \left(\frac{2}{x}\mathbb{E}[A_\infty \wedge x] + P[A_\infty > x]\right) dF(x) = 2\mathbb{E}[F(A_\infty)] + \mathbb{E}\left[A_\infty \int_x^\infty \frac{dF(x)}{x}\right] = 2\mathbb{E}[F(A_\infty)]$$

if we set $\tilde{F}(x) = 2F(x) + x\int_x^\infty \frac{dF(u)}{u}$. Taking $F(x) = x^k$, we obtain the desired result.

Remark: For $k \ge 1$ and $f(x) = x^k$, $\tilde{F}$ is identically $+\infty$ and the above reasoning under the hypothesis of the proposition to find a universal constant c such that $\mathbb{E}[(X_T^*)^k] \le c\mathbb{E}[A_\infty^k]$. This actually follows also from the case where X is a positive martingale which is not in $\mathcal{H}^1$ as $X_t = X_s$ for every s.

To finish the proof of Theorem , it is now enough to use the above result with $X = (M_t^2)^k$ and $A = c_1(\langle M, M\rangle_t)$ for the left-hand side inequality. The necessary domination relations follow from Propositions at time T.

Theorem. For any continuous semimartingale X, there exists a modification of the local time process $(L_t^a)_{t \ge 0}$ which is continuous in t and càdlàg in a and moreover we have

$$L_t^a - L_s^a = 2\int_s^t \mathbf{1}_{X_u = a}\, dX_u = 2\int_s^t \mathbf{1}_{X_u = a}\, dV_u,$$

where V is the finite variation part of the semimartingale X.

In particular, if X is a local martingale, then the local time has a bicontinuous version.

Proof: Define:

$$\hat{M}_t^a := \int_0^t \mathbf{1}_{X_s > a}\, dM_s.$$

We want to apply Kolmogorov's continuity theorem to $a \in \mathbb{R} \mapsto (\hat{M}_t^a)_{t\in[0,T]} \in C([0,T];\mathbb{R})$ seen as a random variable with values on $C_T = C([0,T];\mathbb{R})$ with norm $\|f\|_{C_T} = \sup_{t\in[0,T]} |f(t)|$. Recall that Kolmogorov's continuity theorem states that a stochastic process $Y : \mathbb{R} \to \mathcal{B}$ has a continuous version if

$$\mathbb{E}[\|Y(a) - Y(b)\|_{\mathcal{B}}^p] \le C_L |a-b|^{1+\epsilon}$$

for some $p, \epsilon > 0$ and $a, b \in [0, L]$ for all L with some finite C_L. Moreover, a consequence of the theorem is also that the process Y can be chosen to be locally Hölder continuous with index $\gamma \in (0, \epsilon/p)$, namely for any $L > 0$

$$\|Y(a)(\omega) - Y(b)(\omega)\|_{\mathcal{B}} \le K_L(\omega)|a-b|^{\gamma}, \quad a, b \in [0, L]$$

almost surely.

In our case we take $\mathcal{B} = C_T$ and then we need to estimate for some $p \ge 2$

$$\mathbb{E}\left[\sup_{t\in[0,T]} |\hat{M}_t^a - \hat{M}_t^b|^p\right].$$

By Burkholder-Davis-Gundy (BDG) inequality, (see next exercise sheet) take $b > a$,

$$\mathbb{E}\left[\sup_{t\in[0,T]} |\hat{M}_t^a - \hat{M}_t^b|^p\right] \le C_p \mathbb{E}\left[\left(\int_0^T \mathbf{1}_{a < X_s \le b}\, d\langle M, M\rangle_s\right)^{p/2}\right].$$

By the occupation time formula,

$$\le C_p \mathbb{E}\left[\left(\int_a^b L_T^x\, dx\right)^{p/2}\right].$$

By Jensen's inequality,

$$\le C_p (b-a)^{p/2} \mathbb{E}\left[\int_a^b (L_T^x)^{p/2} \frac{dx}{b-a}\right] \lesssim_p (b-a)^{p/2} \sup_{x\in\mathbb{R}} \mathbb{E}[(L_T^x)^{p/2}].$$

In order to show that $\sup_{a\in\mathbb{R}} \mathbb{E}[(L_T^a)^{p/2}]$ is finite we observe that since

$$|X_{T\wedge a} - (X_0 - a)_+| \le |X_T - X_0|,$$

$$\mathbb{E}[(L_T^a)^{p/2}] = \mathbb{E}\left[\left(2\int_0^T \mathbf{1}_{X_s > a}\, dM_s - \int_0^T \mathbf{1}_{X_s > a}\, dV_s\right)^{p/2}\right]$$

$$\lesssim_p \mathbb{E}[|X_T - X_0|^p] + \mathbb{E}\left[\left(\int_0^T \mathbf{1}_{X_s > a}\, dV_s\right)^{p/2}\right].$$

This shows that $\sup_{a\in\mathbb{R}} \mathbb{E}[(L_T^a)^{p/2}]$ is finite provided L_T is finite. In this case, Kolmogorov's continuity criterion tells us that $a \mapsto M_t^a$ is continuous in a uniformly in t. If the quantity L_T is not finite, then we introduce a suitable sequence of stopping times $(T_n)_{n\geq 0}$ and look at that stopped martingale $(\hat{M}_t^a)^{T_n}$. For example, take

$$T_n = \inf\left\{ t \geq 0 : \sup_{s\in[0,t]} |X_s - X_0| + [M]_t + \int_0^t |dV_s| \geq n \right\}.$$

so that we now know that $(t, a) \mapsto (\hat{M}_t^a)^{T_n}$ is continuous in both variables and then taking the limit as $n \to \infty$ we deduce that $(t, a) \mapsto \hat{M}_t^a$ is also continuous in both variables since $T_n \to \infty$ almost surely.

Actually from this proof one could also deduce that the process $a \mapsto \hat{M}_t^a$ for fixed t is locally Hölder continuous for any $\gamma < 1/2$, i.e.

$$\sup_{t\in[0,T]} |\hat{M}_t^a - \hat{M}_t^b| \leq C_L(\omega)|b - a|^\gamma, \quad a, b \in [0, L],$$

holds almost surely for some random constant C_L which can be taken to be

$$C_L(\omega) = C_L^{N_t}(\omega),$$

where $N_t := \inf\{n \geq 0 : T_n > t\}$ where $C_L^{N_t}(\omega)$ is the constant appearing in the bound

$$\sup_{t\in[0,T]} |\hat{M}_t^a - \hat{M}_t^b| \leq C_L^{N_t}(\omega)|b - a|^\gamma, \quad a, b \in [0, L],$$

which holds for any $n \geq 0$ by considering the stopped process.

As far as $\int_0^t \mathbf{1}_{X_s>a}\, dV_s$ is concerned we have letting

$$\hat{V}_t^a := \int_0^t \mathbf{1}_{X_s>a}\, dV_s,$$

and using dominated convergence

$$\hat{V}_t^{a+} := \lim_{b\to a^+} \hat{V}_t^b = \int_0^t \mathbf{1}_{X_s\geq a}\, dV_s = \hat{V}_t^a,$$

since $\lim_{b\to a} \mathbf{1}_{X_s>b} = \mathbf{1}_{X_s>a}$. However we have $\lim_{b\to a^-} \mathbf{1}_{X_s>b} = \mathbf{1}_{X_s\geq a}$ so

$$\hat{V}_t^{a-} := \lim_{b\to a^-} \hat{V}_t^b = \int_0^t \mathbf{1}_{X_s\geq a}\, dV_s \neq \hat{V}_t^a.$$

So the process $a \mapsto \hat{V}_t^a$ is almost surely càdlàg. Additionally

$$\hat{V}_t^{a-} - \hat{V}_t^a = \int_0^t \mathbf{1}_{X_s=a}\, dV_s = \int_0^t \mathbf{1}_{X_s=a}\, dX_s,$$

since by the occupation time formula and Itô isometry, we have $\int_0^t \mathbf{1}_{X_s=a}\, dM_s = 0$, since

$$\mathbb{E}\left[\int_0^T \mathbf{1}_{X_s=a}\, dM_s\right]^2 = \mathbb{E}\left[\int_0^T \mathbf{1}_{X_s=a}\, d\langle M, M\rangle_s\right] = \mathbb{E}\left[\int_0^T \mathbf{1}_{X_s=a}\, d[X]_s\right] = \int_{\mathbb{R}} L_T^x\, dx = 0,$$

almost surely. Putting all together we have proven the following theorem.

References

[1] Revuz, D., Yor, M. (2013). Continuous martingales and Brownian motion (Vol. 293). Springer Science Business Media.