Brownian Motion via Hilbert Spaces

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Introduction

Brownian motion, a cornerstone of modern physics and probability theory, refers to the random movement of small particles suspended in a fluid due to collisions with its molecules. This phenomenon was first observed in 1827 by the Scottish botanist Robert Brown, who noticed pollen grains in water exhibiting continuous, irregular motion under a microscope [3].

The theoretical understanding of Brownian motion took shape through the work of several key figures in the late 19th and early 20th centuries.In the early 20th century, Louis Bachelier developped a mathematical framework for understanding the probabilistic behavior of stock prices, which included the idea of a Brownian motion process [2]. In 1905, Albert Einstein provided a quantitative explanation by linking the random motion to molecular agitation, offering crucial evidence for the existence of atoms and molecules [6]. Simultaneously, Marian Smoluchowski further developed the statistical description of the phenomenon [12]. Later, in 1908, Jean Perrin's experiments validated Einstein's predictions, securing broader acceptance of atomic theory [11].

A significant extension of Brownian motion theory came through the contributions of French mathematician Paul Lévy in the early 20th century. Lévy's work focused on stochastic processes and probability theory, where he rigorously formalized Brownian motion as a continuous-time stochastic process with independent increments — a cornerstone in the field of modern probability theory [9]. Lévy's insights into the properties of Brownian motion, such as the distribution of particle displacements and the notion of paths with continuous yet nowhere-differentiable trajectories, greatly enriched the mathematical framework behind stochastic processes. His work laid the groundwork for what is now known as the Wiener process, named after Norbert Wiener, who independently developed a similar formulation. Additionally, Lévy's investigations extended into the study of stable distributions and the development of Lévy processes, broadening the scope of stochastic analysis and inspiring applications in diverse fields, from finance to quantum and statistical physics [1].

Brownian motion has since become a pivotal concept, bridging the realms of physics, chemistry, and mathematics. It not only deepened the understanding of molecular kinetics but also laid the groundwork for stochastic processes and diffusion theory. The historical construction of Brownian motion is thus a testament to the interplay between observation, theory, and experimental validation, marking a profound advancement in scientific thought. In this project, we will introduce brownian motion, its probabilistic properties and how such stochastic process was constructed via the usage functional analysis techniques.

Brownian Motion Properties

Definition 1 (Brownian Motion)**.** Let $B = \{B_t\}_{t\in[0,\infty)}$ is called a brownian motion or a Wiener process if ,

1. $B_0 = 0$ almost surely;
2. B_t is almost surely continuous;
3. For all $0 \leq s < t < \infty$, $B_t - B_s$ is $N(0, t-s)$;
4. For all $0 < t_1 < \ldots < t_n$, the increments $B_{t_1}, B_{t_2} - B_{t_1}, \ldots, B_{t_n} - B_{t_{n-1}}$ are independent.

Symmetries of the Brownian motion. Let $\{B_t\}_{t\in[0,\infty)}$ be a Brownian motion. Then the following processes are also Brownian motions :

1. $\{-B_t\}_{t\in[0,\infty)}$, (reflection).
2. $\{\frac{1}{\sqrt{\alpha}}B_{\alpha t}\}_{t\in[0,\infty)}$, for $\alpha > 0$, (scaling).
3. $\{X_t\}_{t\in[0,\infty)}$, where $X_0 = 0$ and $X_t = tB_{1/t}$, for $t > 0$, (inversion).
4. $\{B_1 - B_{1-t}\}_{t\in[0,1]}$, (time reversal).

Proof. **1. Reflection:** $\{-B_t\}_{t\in[0,\infty)}$

- Starts at 0: $-B_0 = -0 = 0$
- Independent increments: Let $0 \leq s < t$. Then $(-B_t) - (-B_s) = -(B_t - B_s)$. Since $B_t - B_s$ is independent of $B_u - B_v$ for any disjoint intervals $[s,t]$ and $[u,v]$, so is $-(B_t - B_s)$.
- Normally distributed increments: $B_t - B_s \sim N(0, t-s)$. Therefore, $-(B_t - B_s) \sim N(0, t-s)$ as well.

2. Scaling: $\{\frac{1}{\sqrt{\alpha}}B_{\alpha t}\}_{t\in[0,\infty)}$, for $\alpha > 0$

- Starts at 0: $\frac{1}{\sqrt{\alpha}}B_{\alpha\cdot 0} = \frac{1}{\sqrt{\alpha}}B_0 = 0$
- Independent increments: Let $0 \leq s < t$. Then $\frac{1}{\sqrt{\alpha}}B_{\alpha t} - \frac{1}{\sqrt{\alpha}}B_{\alpha s} = \frac{1}{\sqrt{\alpha}}(B_{\alpha t} - B_{\alpha s})$. Since $B_{\alpha t} - B_{\alpha s}$ is independent of $B_{\alpha u} - B_{\alpha v}$ for disjoint intervals, so is the scaled increment.
- Normally distributed increments: $B_{\alpha t} - B_{\alpha s} \sim N(0, \alpha t - \alpha s) = N(0, \alpha(t-s))$. Therefore, $\frac{1}{\sqrt{\alpha}}(B_{\alpha t} - B_{\alpha s}) \sim N(0, \frac{\alpha(t-s)}{\alpha}) = N(0, t-s)$.

3. Inversion: $\{X_t\}_{t\in[0,\infty)}$, where $X_0 = 0$ and $X_t = tB_{1/t}$ for $t > 0$

- Starts at 0: $X_0 = 0$ by definition.

- Let $0 \leq s < t$

$$\begin{aligned}\operatorname{Cov}(X_s, X_t) &= \operatorname{Cov}\left(sB_{1/s}, tB_{1/t}\right) \\ &= st\mathbb{E}\left(B_{1/s}B_{1/t}\right) - st\mathbb{E}\left(B_{1/s}\right)\mathbb{E}\left(B_{1/t}\right) \\ &= st\mathbb{E}\left(B_{1/t}^2\right) \\ &= s = \min\{s, t\}.\end{aligned}$$

4. **Time Reversal:** $\{B_1 - B_{1-t}\}_{t\in[0,1]}$

- Starts at 0: $B_1 - B_{1-0} = B_1 - B_1 = 0$
- Independent increments: Let $0 \leq s < t \leq 1$. Then $(B_1 - B_{1-t}) - (B_1 - B_{1-s}) = B_{1-s} - B_{1-t}$. Since $1 - t < 1 - s$, the increments are independent.
- Normally distributed increments: $B_{1-s} - B_{1-t} \sim N(0, (1 - s) - (1 - t)) = N(0, t - s)$.

□

Definition 2 (High Dimension Brownian Motion)**.** A standard d-dimensions Wiener process is an $\mathbb{R}^d$-valued collection of r.v.'s $\{B_t\}_{t\geq 0}$ with density

$$\varphi(x) = \frac{\exp\left(-\frac{1}{2}\left(\mathbf{x}\right)^{\mathrm{T}} \mathbf{\Sigma}^{-1} \left(\mathbf{x}\right)\right)}{\sqrt{(2\pi)^d |\mathbf{\Sigma}|}}$$

where Σ is the covariance matrix. Thus, the existence of higher dimensions Wiener process is a straightforward result that follows from the existence of 1-dimensional Wiener process such that :

$$B_t = \left(B_t^{(1)}, B_t^{(2)}, \ldots, B_t^{(d)}\right)$$

is a d-dimensions Brownian motion.

Proposition. Let $(B_t)_{t\geq 0}$ be a Wiener process on $(\Omega, \mathcal{F}, P)$. Then with probability 1 the function $t \mapsto B_t$ for $t \geq 0$ is nowhere differentiable.

Proof. We will prove that for any fixed time $t \geq 0$, Brownian motion B_t is not differentiable at t with probability 1. Consider the difference quotient for Brownian motion at time t:

$$\frac{B_{t+h} - B_t}{h}$$

We want to show that the limit of this quotient as $h \to 0$ does not exist with probability 1.

Let's consider a sequence $h_n = 2^{-n}$ for $n = 1, 2, 3, \ldots$.

$$D_n = \frac{B_{t+h_n} - B_t}{h_n} = \frac{B_{t+2^{-n}} - B_t}{2^{-n}}$$

Using the stationary increments property of Brownian motion, $B_{t+2^{-n}} - B_t$ has the same distribution as $B_{2^{-n}}$, and $B_{2^{-n}} \sim N(0, 2^{-n})$. Therefore, we can write $B_{2^{-n}} = \sqrt{2^{-n}} Z_n$, where $Z_n \sim N(0,1)$. Thus,

$$D_n = \frac{\sqrt{2^{-n}} Z_n}{2^{-n}} = 2^{n/2} Z_n$$

So, $D_n = 2^{n/2} Z_n$, where Z_n has the standard normal distribution.

Now, let's fix $M > 0$ and consider the event $E_n = \{|D_n| \leq M\}$, which means that the difference quotient is bounded by M for $h = 2^{-n}$.

$$E_n = \left\{ \left| \frac{B_{t+2^{-n}} - B_t}{2^{-n}} \right| \leq M \right\} = \{|B_{t+2^{-n}} - B_t| \leq M2^{-n}\}$$

We calculate the probability of E_n:

$$P(E_n) = P(|B_{t+2^{-n}} - B_t| \leq M2^{-n}) = P(|B_{2^{-n}}| \leq M2^{-n})$$

Let $Z \sim N(0,1)$. Then $\frac{B_{2-n}}{\sqrt{2^{-n}}} \sim N(0,1)$, so $B_{2^{-n}} \sim \sqrt{2^{-n}} Z$.

$$P(E_n) = P(|\sqrt{2^{-n}} Z| \leq M2^{-n}) = P(|Z| \leq \frac{M2^{-n}}{\sqrt{2^{-n}}}) = P\left(|Z| \leq \frac{M}{\sqrt{2^n}}\right)$$

Let $\phi(x)$ be the probability density function of the standard normal distribution. Then for small x, $P(|Z| \leq x) = \int_{-x}^{x} \phi(z)dz \approx 2x\phi(0) = 2x\frac{1}{\sqrt{2\pi}}$. For large n, $\frac{M}{\sqrt{2^n}}$ is small, so we can approximate:

$$P(E_n) \approx 2\frac{M}{\sqrt{2^n}}\frac{1}{\sqrt{2\pi}} = \frac{2M}{\sqrt{2\pi}}2^{-n/2} = C2^{-n/2}$$

where $C = \frac{2M}{\sqrt{2\pi}}$ is a constant.

Consider the sum $\sum_{n=1}^{\infty} P(E_n) = \sum_{n=1}^{\infty} P\left(|Z| \leq \frac{M}{\sqrt{2^n}}\right)$. Since $P(E_n) \approx C2^{-n/2}$ and $\sum_{n=1}^{\infty} 2^{-n/2}$ is a convergent geometric series (with ratio $2^{-1/2} < 1$), the series $\sum_{n=1}^{\infty} P(E_n)$ converges:

$$\sum_{n=1}^{\infty} P(E_n) < \infty$$

By the first Borel-Cantelli Lemma, $P(E_n \text{ i.o.}) = 0$. This means that with probability 1, only finitely many events E_n occur. In other words, for almost every ω, there exists some $N(\omega)$ such that for all $n > N(\omega)$, the event E_n does not occur. This means for all $n > N(\omega)$, $|D_n| = \left|\frac{B_{t+2-n} - B_t}{2^{-n}}\right| > M$.

Since this holds for any $M > 0$, it implies that for almost every ω, the difference quotient $\frac{B_{t+h} - B_t}{h}$ does not converge to a finite limit as $h \to 0$ along the sequence $h_n = 2^{-n}$. Therefore, B_t is not differentiable at t with probability 1. Since this holds for any fixed $t \geq 0$, we conclude that with probability 1, the function $t \mapsto B_t$ is nowhere differentiable for $t \geq 0$.

$\square$

Brownian Motion as a Gaussian Process

Gaussian characterization of Brownian motion. Let $B_{t\geq 0}$ be an $\mathbb{R}^d$-valued process defined on $(\Omega, \mathcal{F}, P)$ with a.s. continuous sample paths:

$B_{t\geq 0}$, is a centered Gaussian process with $\Sigma(s,t) = (s \wedge t) I_{d\times d} \iff B_{t\geq 0}$ is a a std. Wiener process

Proof. (see [5] chap 8) since (B_t) is a brownian motion then for $0 = t_0 < t_1 < \cdots < t_n$, the increments $B_{t_i} - B_{t_{i-1}}$, $1 \leq i \leq n$, are i.i.d rv's normally distributed $N(0, t_i - t_{i-1})$. Hence,

$$(B_{t_1} - B_{t_0}, \ldots, B_{t_n} - B_{t_{n-1}}) \sim \bigotimes_{i=1}^{n} N(0, t_i - t_{i-1}),$$

thus B_t is normally distributed, thus (B_t) is a Gaussian process. Moreover, since $B_t = B_t - B_0$, we have $E[B_t] = 0$ and

$$\mathrm{Cov}[B_s, B_t] = \mathrm{Cov}[B_s, B_t - B_0] = \mathrm{Cov}[B_s, B_t - B_s] = \mathrm{Var}[B_s] = s.$$

In the other direction, if (B_t) is a centered Gaussian , then for any $0 = t_0 < t_1 < \cdots < t_n$, the vector $(B_{t_1} - B_{t_0}, \ldots, B_{t_n} - B_{t_{n-1}})$ has a multivariate gaussian distribution with

$$E[B_{t_i} - B_{t_{i-1}}] = E[B_{t_i}] - E[B_{t_{i-1}}] = 0,$$

and

$$\mathrm{Cov}[B_{t_i} - B_{t_{i-1}}, B_{t_j} - B_{t_{j-1}}] = \min(t_i, t_j) - \min(t_i, t_{j-1}) - \min(t_{i-1}, t_j) + \min(t_{i-1}, t_{j-1}) = (t_i - t_{i-1}) \cdot \delta_{i,j}$$

for any $i, j = 1, \ldots, n$.
the increments are independent with distribution $N(0, t_i - t_{i-1})$, thus (B_t) is a Wiener process.

□

Brownian Motion as a Markov Process

Markov Property. A Wiener process $(B_t)_{t\geq 0}$ is Markov process in $\mathbb{R}^d$, thus for $X \subseteq \mathbb{R}^d$ and $0 \leq s < t$,

$$P[B_t \in X \mid \mathcal{F}_s^B] = \int_X p_{t-s}(B_s, y)\, dy \quad \text{a.s.}$$

Proof. For $0 \leq s < t$ we have $B_t = B_s + (B_t - B_s)$ where B_s is $\mathcal{F}_s^B$-measurable, and $B_t - B_s$ is independent of $\mathcal{F}_s^B$ by the independents increments. Hence

$$\begin{aligned} P[B_t \in X \mid \mathcal{F}_s^B] &= P[B_s + B_t - B_s \in X] \\ &= N(B_s, (t-s) \cdot I_d)[X] \\ &= \int_X (2\pi(t-s))^{-d/2} \cdot \exp\left(-\frac{|y - B_s|^2}{2(t-s)}\right) dy \\ &= \int_X p_{t-s}(B_s, y)\, dy \end{aligned}$$

This constitutes a weak version of the markov property. see [10] chap. 2 for more details for the strong markov property which generalize this property for not only determenestic times but also random stopping times. $\square$

Brownian Motion as a Martingale

Definition 3. A real-valued stochastic process $\{X(t) : t \geq 0\}$ is a martingale with respect to the filtration $\{\mathcal{F}(t) : t \geq 0\}$ if

- X_t it is adapted to $\{\mathcal{F}(t) : t \geq 0\}$,
- $\mathbb{E}[|X(t)|] < \infty$ for all $t \geq 0$, and
- for all $s \leq t$, $\mathbb{E}[X(t)|\mathcal{F}(s)] = X(s)$, almost surely.

Martingale property. Brownian motion is a martingale.

Proof. Since $B_t \sim \mathcal{N}(0, t)$, we have

$$\mathbb{E}[|B_t|] < \infty,$$

so $B_t \in L^1$ for all $t \geq 0$
For $0 \leq s \leq t$,

$$\mathbb{E}[B_t \mid \mathcal{F}_s] = \mathbb{E}[B_s + (B_t - B_s) \mid \mathcal{F}_s] = B_s + \mathbb{E}[B_t - B_s \mid \mathcal{F}_s] = B_s + 0 = B_s.$$

$\square$

Theorem. (Lévy)
If B is any continuous $(\mathcal{F}_t)$-martingale with $B_0 = 0$ and $B_t^2 - t$ a martingale, then B is an $(\mathcal{F}_t)$-Brownian motion.

Proof. Define the process $Z_t = e^{\lambda B_t - \frac{\lambda^2}{2} t}$ for $\lambda \in \mathbb{R}$. We aim to show Z_t is a martingale. For $s < t$, compute $\mathbb{E}[Z_t \mid \mathcal{F}_s]$:

$$\mathbb{E}[e^{\lambda B_t - \frac{\lambda^2}{2} t} \mid \mathcal{F}_s] = e^{\lambda B_s - \frac{\lambda^2}{2} s} \cdot \mathbb{E}[e^{\lambda (B_t - B_s) - \frac{\lambda^2}{2}(t-s)} \mid \mathcal{F}_s].$$

Since B is a martingale, $\mathbb{E}[B_t - B_s \mid \mathcal{F}_s] = 0$, and $\mathbb{E}[(B_t - B_s)^2 \mid \mathcal{F}_s] = t - s$ (from $B_t^2 - t$ being a martingale). Expanding the exponential and matching terms shows:

$$\mathbb{E}[e^{\lambda (B_t - B_s)} \mid \mathcal{F}_s] = e^{\frac{\lambda^2}{2}(t-s)}.$$

Thus,

$$\mathbb{E}[Z_t \mid \mathcal{F}_s] = Z_s \implies Z_t \text{ is a martingale.}$$

Since Z_t is a martingale, $\mathbb{E}[Z_t] = Z_0 = 1$. This implies:

$$\mathbb{E}[e^{\lambda B_t}] = e^{\frac{\lambda^2}{2} t}.$$

This is the characteristic function of a normal random variable with mean 0 and variance t. Hence, $B_t \sim \mathcal{N}(0,t)$.

For $s < t$, the increment $B_t - B_s$ satisfies:

$$\mathbb{E}[e^{\lambda(B_t - B_s)} \mid \mathcal{F}_s] = e^{\frac{\lambda^2}{2}(t-s)}.$$

This shows $B_t - B_s$ is independent of $\mathcal{F}_s$ and $B_t - B_s \sim \mathcal{N}(0, t-s)$. By induction, all increments are independent. $\square$

Brownian Motion Construction

This construction is due to Ciesielsky in [4] using Wiener isometry idea which was observed by Wiener in [13].
The foundation of Wiener's approach lies in a key insight: constructing a Brownian motion process on a probability space $(\Omega, \mathcal{F}, P)$ inherently establishes a linear isometry between two Hilbert spaces. Specifically, the Hilbert space $L^2((0,1), \lambda)$ (with Lebesgue measure λ) can be isometrically embedded into $L^2(P)$, the space of square-integrable random variables. This connection arises from the covariance structure of the Brownian maotion. For indicator functions $\mathbf{1}_{[0,t]}$ and $\mathbf{1}_{[0,s]}$ in $L^2((0,1))$, their inner product $\langle \mathbf{1}_{[0,t]}, \mathbf{1}_{[0,s]} \rangle = \min(s,t)$ mirrors the covariance $\mathbb{E}[W_t W_s] = \min(s,t)$ of a Wiener process $\{W_t\}$. Wiener recognized that mapping $\mathbf{1}_{[0,t]} \mapsto W_t$ preserves this structure and can be extended linearly to a full isometry, now termed the *Wiener integral.* This isometry, I_W, bridges deterministic functions and stochastic integrals, providing a framework to represent the Wiener process explicitly through orthonormal bases in $L^2((0,1))$. Such bases enable decompositions of the Brownian motion, leveraging the interplay between its covariance properties and the geometry of L^2 spaces.

Theorem Wiener's Isometry. Let $\{B_t\}_{t\geq 0}$ be a standard Brownian motion defined on $(\Omega, \mathcal{F}, P)$. Let $I = (0,T]$ be a nonempty interval in $\mathbb{R}_+$. then the mapping $\mathbf{1}_{(s,t]} \mapsto B_t - B_s$ extends to a linear isometry $I_B : L^2(I) \to L^2(\Omega, \mathcal{F}, P)$, and for $\psi \in L^2(I), I_B(\psi)$ is centered gaussian.

Proof. Let $I \subseteq \mathbb{R}_+$ be a nonempty interval. A *simple function* in $L^2(I)$ can be expressed as:

$$\varphi = \sum_{i=1}^{n} a_i \mathbf{1}_{(s_i,t_i]},$$

where $(s_i, t_i] \subseteq J$ are disjoint intervals. Define the Wiener integral for such φ as:

$$I_B(\psi) = \sum_{i=1}^{n} a_i (B_{t_i} - B_{s_i}).$$

For two simple functions $\varphi = \sum_{i=1}^{n} a_i \mathbf{1}_{(s_i,t_i]}$ and $\psi = \sum_{j=1}^{m} b_j \mathbf{1}_{(u_j,v_j]}$, compute the $L^2(J)$ inner product:

$$\langle \varphi, \psi \rangle_{L^2(J)} = \sum_{i,j} a_i b_j \lambda\left((s_i, t_i] \cap (u_j, v_j]\right).$$

The covariance in $L^2(\Omega)$ is:

$$\mathbb{E}\left[I_B(\varphi)I_B(\psi)\right] = \sum_{i,j} a_i b_j \mathbb{E}\left[(B_{t_i} - B_{s_i})(B_{v_j} - B_{u_j})\right].$$

For a Brownian motion process, $\mathbb{E}\left[(B_t - B_s)(B_v - B_u)\right] = \lambda\left((s,t] \cap (u,v]\right)$ (the overlap length). Thus,

$$\mathbb{E}\left[I_B(\varphi)I_B(\psi)\right] = \langle \varphi, \psi \rangle_{L^2(I)}.$$

since simple functions are dense in $L^2(I)$. For $\varphi \in L^2(I)$, there exist a sequence of simple functions $\{\varphi_n\}$ with $\varphi_n \to \varphi$ in $L^2(I)$. Define:

$$I_B(\varphi) = \lim_{n\to\infty} I_B(\varphi_n) \quad \text{in } L^2(\Omega).$$

The limit exists because $\{I_B(\varphi_n)\}$ is Cauchy in $L^2(\Omega)$:

$$\mathbb{E}\left[|I_B(\varphi_n) - I_B(\varphi_m)|^2\right] = \|\varphi_n - \varphi_m\|^2_{L^2(I)} \to 0.$$

By continuity of the inner product:

$$\mathbb{E}\left[I_B(\varphi)I_B(\psi)\right] = \lim_{n,m\to\infty} \mathbb{E}\left[I_B(\varphi_n)I_B(\psi_m)\right] = \lim_{n,m\to\infty} \langle \varphi_n, \psi_m \rangle_{L^2(I)} = \langle \varphi, \psi \rangle_{L^2(I)}.$$

- For simple φ, $I_B(\varphi)$ is a linear combination of independent Gaussian increments, hence Gaussian. Its mean is zero because $\mathbb{E}[B_t - B_s] = 0$.
- For general $\varphi \in L^2(I)$, $I_B(\varphi)$ is the L^2-limit of Gaussian random variables. Since Gaussianity is preserved under L^2-limits, $I_B(\varphi)$ is Gaussian and mean-zero.

Thus, I_B is a linear isometry, and $I_B(\varphi)$ is mean-zero Gaussian for all $\varphi \in L^2(I)$.

$\square$

Definition 4. let $\psi : \mathbb{R} \to \{-1, 1\}$ be the function

$$\psi(t) = \begin{cases} 1 & \text{if } 0 \le t \le \frac{1}{2}; \\ -1 & \text{if } \frac{1}{2} < t \le 1; \\ 0 & \text{otherwise.} \end{cases} \tag{1}$$

furthermore for $n \ge 0$ and $0 \le k < 2^n$ we can generalize the equation 1 by defining the (n,k)th partionned Haar function by

$$\psi_{n,k}(t) = 2^{n/2}\psi(2^n t - k). \tag{2}$$

Proposition. The Haar functions $\psi_{n,k}$ form an orthonormal basis of $L^2([0,1]$, with respect to the Lebesgue measure .

Proof. These functions form an orthonormal set in $L^2(\mathbb{R})$ because if $j = j'$ and $k \neq k'$ then

$$\int_{\mathbb{R}} \psi_{nk}(x)\psi_{n'k'}(x)dx = 2^j \int_{\mathbb{R}} \psi(2^j x - k)\psi(2^j x - k')dx = 0 \tag{3}$$

because $\psi(y-k)\psi(y-k') = 0$ for any $y \in \mathbb{R}$ and $k \neq k'$.Now assume wlog, $n < n'$, then

$$\begin{aligned}
\int_{\mathbb{R}} \psi_{nk}(x)\psi_{n'k'}(x)dx &= 2^{n/2+n'/2} \int_{\mathbb{R}} \psi(2^n x - k)\psi(2^{n'} x - k')dx \\
&= 2^{n/2-n'/2} \int_{\mathbb{R}} \psi(y)\psi(2^{n'-n}y + 2^{n'-n}k - k')dy \\
&= 2^{n'/2-n/2} \int_0^{1/2} \psi(2^{n'-n}y + 2^{n'-n}k - k')dy - 2^{n'/2-n/2} \int_{1/2}^{1} \psi(2^{n'-n}y + 2^{n'-n}k - k')dy \\
&= 0
\end{aligned}$$

Furthermore we have :

$$\int_{\mathbb{R}} |\psi_{nk}(x)|^2 = 2^n \int_{\mathbb{R}} |\psi(2^n x - k)|^2 dx = \int_{\mathbb{R}} |\psi(x-k)|^2 dx = 1.$$

The Haar coefficients of a function $f \in L^2(\mathbb{R})$ are defined as the inner products

$$c_{nk} = \int_{\mathbb{R}} f(x)\psi_{nk}(x)dx,$$

and the Haar series of f is

$$\sum_{n,k\in\mathbb{N}} c_{nk}\psi_{nk}(x).$$

□

For the Haar basis, these inner products define the **Schauder functions** $G_{n,k}$, which are defined as the indefinite integrals of the Haar functions:

$$G_{n,k}(t) = \langle 1_{[0,t]}, \psi_{n,k} \rangle = \int_0^t \psi_{n,k}(s)\, ds$$

Theorem. Lévy (again)
Let $\xi_{m,k}$ i.i.d. rv's such that $\xi_{m,k} \sim \mathcal{N}(0,1)$, then with probability one, the infinite series

$$B(t) := \xi_{0,1}t + \sum_{m=1}^{\infty} \sum_{k=0}^{2^m-1} \xi_{m,k} G_{m,k}(t)$$

converges uniformly for $0 \leq t \leq 1$ to a standard brownian motion process.

Proof. define:

$$a_m(\omega) = \sup_{t\in[0,1]} \left| \sum_{k<2^m} G_{m,k}(t)\xi_{m,k} \right| \leq 2^{-(\frac{m}{2}+1)} \sup_{k<2^m} |\xi_{m,k}|.$$

since ξ are $\mathcal{N}(0,1)$-distributed:

$$P[|\xi| > a] \leq \sqrt{\frac{2}{\pi}} \frac{1}{a} \exp\left\{ -\frac{a^2}{2} \right\}, \text{ for } a > 0,$$

thus,

$$\sum_{m\geq 1} P\Big[\sup_{0\leq k<n2^m} |\xi_{m,k}| > \sqrt{2m} \Big] \leq \sum_{m\geq 1} \sqrt{\frac{2}{\pi}} \frac{n2^m}{\sqrt{2m}} e^{-m} < \infty.$$

Thus, Borel Cantelli's lemma implies that for P-almost every ω, there exists $m_0(\omega)$ such that for $m \geq m_0(\omega)$, $\sup_{k<n_0 2^m} |\xi_{m,k}(\omega)| \leq \sqrt{2m}$. As a result:

$$P\text{-a.s.}, \qquad \sum_{m\geq m_0(\omega)} a_m(\omega) \leq \sum_{m\geq m_0(\omega)} 2^{-(\frac{m}{2}+1)}\sqrt{2m} < \infty.$$

It follows that P-a.s., $B(\cdot,\omega)$ converges uniformly, and therefore is continuous.
For $0 < s < t$:

$$\begin{aligned} E[B(t)B(s)] &= E\Bigg[(\xi_{0,1} \int_0^t \psi_0 dx + \sum_{n=1}^{\infty} \sum_{k<2^{-n}} \xi_{m,k} \int_0^t \psi_{n,k}(x)dx) \\ &\qquad \times (\xi_{0,1} \int_0^s \psi_0 dx + \sum_{n=1}^{\infty} \sum_{k<2^{-n}} \xi_{m,k} \int_0^s \psi_{n,k}(x)dx) \Bigg] \\ &= \int_0^t \psi_0 dx \int_0^s \psi_0 dx + \sum_{n=1}^{\infty} \int_0^t \psi_{n,k}(x)dx \int_0^s \psi_{n,k}(x)dx \\ &= \int_0^1 \chi_{[0,t]}(x)\chi_{[0,s]}(x)dx = \min\{s,t\}. \end{aligned}$$

for the same s, t as above :

$$\begin{aligned} \mathbb{E}(X_t - X_s)^2 &= \mathbb{E}\left(\sum_{n=1}^{\infty} \sum_{k<2^{-n}} \xi_{m,k} \int_s^t \psi_{n,k}(x)dx) \right)^2 \\ &= \sum_{k=0}^{\infty} \sum_{k<2^{-n}} \left(\int_s^t \int_s^t \psi_{n,k}(x)dx \right)^2 \\ &= \sum_{k=0}^{\infty} \sum_{k<2^{-n}} \langle \chi_{[s,t]}, \psi_{n,k} \rangle^2 \\ &= \|\chi_{[s,t]}\|_{L^2}^2 = t - s, \end{aligned}$$

hence the increments $X_t - X_s$ have the correct variance. Let us show that they are independent: for $0 \leq t_0 < t_1 \leq t_2 < t_3 \leq 1$:

$$\begin{aligned}\mathbb{E}\left((X_{t_3} - X_{t_2})(X_{t_1} - X_{t_0})\right) &= \sum_{n=1}^{\infty} \sum_{k<2^{-n}} \langle \chi_{[t_2,t_3]}, \psi_{n,k} \rangle \langle \chi_{[t_0,t_1]}, \psi_{n,k} \rangle \\ &= \langle \chi_{[t_2,t_3]}, \chi_{[t_0,t_1]} \rangle = 0.\end{aligned}$$

As the variables $X_t - X_s$ are jointly Gaussian, independence of the increments follows. Thus $B(t)_{0\leq t\leq 1}$ is a brownian motion $\square$

Conclusion

In this project, we introduced a rich class of stochastic processes with fascinating properties that constitute a cornerstone of the study of multiple stochastic processes such as Gaussian, Lévy and Poisson processes. We also managed to construct such random process using the material of the functional analysis course. Due to the time and length constraints, we could have established the relationship between such random processes and pdes through the heat equation or the Feynman-Kac equation.

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