Stein's Method

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Introduction

Stein's method, a powerful and general approach in probability theory, was introduced by Stanford statistician Charles Stein in the early 1970s. The. His new approach, fundamentally different from classical methods, was presented in a seminal paper at the sixth Berkeley Symposium in 1970 and subsequently published in 1972.

At its core, the method aims to obtain bounds on the distance between two probability distributions with respect to a probability metric, a goal that goes beyond simply proving asymptotic convergence. The method initially focused on normal approximation, but its components were rapidly extended to other distributions, most notably the Poisson distribution, by Stein's Ph.D. student Louis Chen[1].

Distances Notations

The total variation distance $d_{TV}(\mu, \nu)$ between the measures μ and ν on $\mathbb{R}$ is defined by

$$d_{TV}(\mu, \nu) = \sup_A |\mu(A) - \nu(A)|,$$

where the supremum is over measurable sets A. This is equivalent to

$$d_{TV}(\mu, \nu) = \frac{1}{2} \sup_f \left| \int f(t) d\mu(t) - \int f(t) d\nu(t) \right|,$$

where the supremum is taken over continuous functions which are bounded by 1 and vanish at infinity; this is the definition most commonly used in what follows. The total variation distance between two random variables X and Y is defined to be the total variation distance between their distributions:

$$d_{TV}(X, Y) = \sup_A |\mathbb{P}(X \in A) - \mathbb{P}(Y \in A)| = \frac{1}{2} \sup_f |\mathbb{E}f(X) - \mathbb{E}f(Y)|.$$

If the Banach space of signed measures on $\mathbb{R}$ is viewed as dual to the space of continuous functions on $\mathbb{R}$ vanishing at infinity, then the total variation distance is (up to the factor of $\frac{1}{2}$ the norm distance on that Banach space.

The Wasserstein distance $d_W(X, Y)$ between the random variables X and Y is defined by : $d_W(X, Y) = \sup_{||g||_L \leq 1} |\mathbb{E}g(X) - \mathbb{E}g(Y)|$,

Stein's Method

Lemma F. or fixed $z \in \mathbb{R}$ and $F(z) = P(Y \leq z)$, the cumulative distribution function of Z, the unique bounded solution $g(w)$ of the equation

$$g'(w) - wg(w) = \mathbf{1}_{(-\infty,z]}(w) - F(z)$$

is given by

$$g(w) = \begin{cases} \sqrt{2\pi} e^{\frac{w^2}{2}} F(z)(1 - F(z)), & w \leq z \\ \sqrt{2\pi} e^{\frac{w^2}{2}} F(z)(1 - F(w)), & w > z \end{cases}$$

Proof: Multiply both sides of the equation by $e^{-\frac{w^2}{2}}$:

$$e^{-\frac{w^2}{2}}[g'(w) - wg(w)] = e^{-\frac{w^2}{2}}[\mathbf{1}_{(-\infty,z]}(w) - F(z)].$$

The above equation yields:

$$\left(g(w)e^{-\frac{w^2}{2}}\right)' = e^{-\frac{w^2}{2}}[\mathbf{1}_{(-\infty,z]}(w) - F(z)].$$

Integration yields:

$$g(w)e^{-\frac{w^2}{2}} = \int_{-\infty}^{w} e^{-\frac{x^2}{2}}[\mathbf{1}_{(-\infty,z]}(x) - F(z)]\,dx,$$

and since g is unique, there is no constant term in the above equation. Then multiply both sides by $e^{\frac{w^2}{2}}$, we have:

$$g(w) = e^{\frac{w^2}{2}} \int_{-\infty}^{w} e^{-\frac{x^2}{2}}[\mathbf{1}_{(-\infty,z]}(x) - F(z)]\,dx.$$

This is equivalent to:

$$g(w) = -e^{\frac{w^2}{2}} \int_{w}^{\infty} e^{-\frac{x^2}{2}}[\mathbf{1}_{(-\infty,z]}(x) - F(z)]\,dx,$$

which is equivalent to statement of the lemma, since two forms of solution come from the fact that

$$\int_{-\infty}^{\infty} [\mathbf{1}_{(-\infty,z]}(x) - F(z)]e^{-\frac{x^2}{2}}\,dx = 0.$$

a. $w \leq z$

$$g(w) = e^{\frac{w^2}{2}} \int_{-\infty}^{w} [\mathbf{1}_{(-\infty,z]}(x) - F(z)]e^{-\frac{x^2}{2}}\,dx$$

$$= e^{\frac{w^2}{2}} \left[(1 - F(z))\sqrt{2\pi} \int_{-\infty}^{w} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}\,dx\right]$$

$$= \sqrt{2\pi} e^{\frac{w^2}{2}} F(z)(1 - F(z))$$

b. $w > z$

$$g(w) = -e^{\frac{w^2}{2}} \int_w^\infty [\mathbf{1}_{(-\infty,z]}(x) - F(z)] e^{-\frac{x^2}{2}}\,dx$$

$$= e^{\frac{w^2}{2}} \int_{-\infty}^z e^{-\frac{x^2}{2}} F(z)\,dx$$

$$= \sqrt{2\pi} e^{\frac{w^2}{2}} F(z) \int_{-\infty}^z \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}\,dx$$

$$= \sqrt{2\pi} e^{\frac{w^2}{2}} F(z)(1 - F(w)).$$

Stein's equation. If W has a standard normal distribution, then

$$E[f'(W)] = E[Wf(W)],$$

for all absolutely continuous functions $f : \mathbb{R} \to \mathbb{R}$ with $E|f'(Z)| < \infty$. Conversely, if it holds for all bounded, continuous and piecewise continuously differentiable functions f with $E|f'(Z)| < \infty$, then W has a standard normal distribution.

Proof. Suppose that $W \sim \mathcal{N}(0,1)$, and $\mathbb{E}|f'(W)| < \infty$, we can then write $\mathbb{E}f'(W)$ as an integral and exchange the order of integral to get the final result using Fubini theorem:

$$\mathbb{E}f'(W) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^\infty f'(z) e^{-\frac{z^2}{2}}\,dz$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^\infty f'(z) e^{-\frac{z^2}{2}}\,dz + \frac{1}{\sqrt{2\pi}} \int_{-\infty}^\infty f'(z) e^{-\frac{z^2}{2}}\,dz$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^\infty f'(z) \left(\int_{-\infty}^z (-x) e^{-\frac{x^2}{2}}\,dx \right) dz + \frac{1}{\sqrt{2\pi}} \int_{-\infty}^\infty f'(z) \left(\int_z^\infty x e^{-\frac{x^2}{2}}\,dx \right) dz$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^\infty f'(z) \left(\int_{-\infty}^z (-x) e^{-\frac{x^2}{2}}\,dx \right) dz + \frac{1}{\sqrt{2\pi}} \int_{-\infty}^\infty f'(z) \left(\int_z^\infty x e^{-\frac{x^2}{2}}\,dx \right) dz$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^\infty \left(f(z) - f(-\infty) \right) z e^{-\frac{z^2}{2}}\,dz$$

$$= \mathbb{E}Zf(W) - \frac{1}{\sqrt{2\pi}} f(0) \int_{-\infty}^\infty z e^{-\frac{z^2}{2}}\,dz$$

$$= \mathbb{E}Zf(W)$$

which implies that if $W \sim \mathcal{N}(0,1)$, then we have $\mathbb{E}f'(W) = \mathbb{E}Zf(W)$.
" $\Longleftarrow$ " Suppose that $\mathbb{E}f'(W) = \mathbb{E}Wf(W)$.
Recall Stein's equation:

$$f'(w) - wf(w) = g(w) - \mathbb{E}g(W),$$

where $W \sim \mathcal{N}(0,1)$.

Taking $g(w) = \mathbf{1}_{(-\infty,z]}$, the solution implied in Lemma 1 satisfies the conditions of Lemma 2 (Stein's Identity), thus we have

$$0 = \mathbb{E}[f'(W) - Wf(W)] = \mathbb{E}[\mathbf{1}_{(-\infty,z]}(W) - \Phi(z)] = P(W \le z) - P(Z \le z).$$

Therefore, $W \sim \mathcal{N}(0,1)$. This finish the proof.

CLT

Let $X_1, X_2, \dots, X_n$ be independent random variables with $\mathbb{E}[X_i] = 0$, $\mathbb{E}[X_i^2] = 1$, and $\mathbb{E}[|X_i|^3] < \infty$. Let

$$W = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} X_i,$$

and $W \sim \mathcal{N}(0,1)$.

Proof. Take any function $f \in C^1$ with f' absolutely continuous, and satisfying $|f| \le 1$, $|f'| \le \sqrt{\frac{\pi}{2}}$, $|f''| \le 2$.

Let $W_i = W - \frac{X_i}{\sqrt{n}}$, which implies that W_i is independent of X_i (denoted by "$\perp\!\!\!\perp$"). Note that

$$\mathbb{E}\left[X_i f(W)\right] = \mathbb{E}\left[X_i(f(W) - f(W_i))\right] + \mathbb{E}\left[X_i f(W_i)\right].$$

Note that $f''(x) = \frac{f(x+h)-2f(x)+f(x-h)}{h^2} + o(h^2)$, and thus we can obtain

$$\mathbb{E}[Wf(W)] = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \mathbb{E}[X_i f(W)],$$

and

$$|f(W) - f(W_i) - (W - W_i)f'(W_i)| \le \frac{1}{2}|f''(\xi_i)||X_i|^2 \le |X_i|^2,$$

where ξ_i is between W and W_i. Thus,

$$\mathbb{E}\left[X_i(f(W) - f(W_i) - (W - W_i)f'(W_i))\right] \le \frac{1}{2}\mathbb{E}[|f''(\xi_i)||X_i|^2] \le \frac{1}{2}\mathbb{E}[|X_i|^2|X_i|] \le \frac{1}{n^{3/2}}\mathbb{E}[|X_i|^3].$$

Again,

$$\mathbb{E}[X_i(W - W_i)f'(W_i)] = \frac{1}{n}\mathbb{E}[X_i^2 f'(W_i)] = \frac{1}{n}\mathbb{E}[X_i^2]\mathbb{E}[f'(W_i)] = \frac{1}{n}\mathbb{E}[f'(W_i)],$$

since W_i is independent of X_i. Based on the above calculations, we can get

$$\left|\mathbb{E}[Wf(W)] - \frac{1}{n}\sum_{i=1}^{n}\mathbb{E}[f'(W_i)]\right| \le \frac{1}{n^{3/2}}\sum_{i=1}^{n}\mathbb{E}[|X_i|^3].$$

Finally, note that

$$\left|\frac{1}{n}\sum_{i=1}^{n}\mathbb{E}[f'(W_i)] - \mathbb{E}[f'(W)]\right| \le \frac{1}{n}\sum_{i=1}^{n}|\mathbb{E}[f'(W_i) - f'(W)]| \le \frac{2}{n^2}\sum_{i=1}^{n}\mathbb{E}[|X_i|].$$

Combining all these together, we obtain that

$$\left|\mathbb{E}[f'(W) - Wf(W)]\right| \leq \frac{1}{n^{3/2}} \sum_{i=1}^{n} \mathbb{E}[|X_i|^3] + \frac{2}{n^2} \sum_{i=1}^{n} \mathbb{E}[|X_i|].$$

Since $\mathbb{E}[X_i^2] = 1$, we can say that $\mathbb{E}[|X_i|^3] \geq 1$, and $\mathbb{E}[|X_i|] \leq (\mathbb{E}[X_i^2])^{1/2} = 1$. Therefore,

$$\mathrm{Wass}(W, Z) \leq \frac{3}{n^{3/2}} \sum_{i=1}^{n} \mathbb{E}[|X_i|^3].$$

References

[1] Chen, L.H.Y., Goldstein, L., Shao, QM. (2011). Introduction. In: Normal Approximation by Stein's Method. Probability and Its Applications. Springer

[2] Chatterjee, S., Meckes, E. (2007). Multivariate normal approximation using exchangeable pairs. arXiv preprint math/0701464.