Université Paris Saclay

Analysis report

A SHORT AND VARSATILE FINITE ELEMENT MULTI SCALE CODE FOR HOMOGENIZATION PROBLEMS

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Contents

List of Figures

Introduction

This is a critical analysis of Assyr Abdulle and Achin Nonnenmacher's research paper "A Short and Versatile Finite Element Multiscale for Homogenization Problems".

The authors describe a multiscale finite element solver for parabolic or elliptic challenges characterized by highly oscillating coefficients. The researchers base their analysis of finite element on coupled macrosolvers and microsolvers due to the development of heterogeneous multiscale method. In brief, the authors discuss the implementation of the flexible and short finite element (FE) of the multiscale algorithm, which can withstand quadrilateral or simplicial FE and multiple coupling conditions for the constrained micro simulations. The authors also present several basic numerical examples including time dependent and three-dimensional problems which demonstrate the efficiency and versatility of the employed computational strategy.

Chapter 1

Paper Information

1.1 Title

Paper Title: A Shorty and Versatile Finite Element Multiscale for Homogenization Problems.

The chosen title of the paper directly speaks of the aim of the paper. The paper defines the aim of the paper, and that is to present a description of a multiscale finite element solver for parabolic or elliptic challenges. The title is therefore relevant, and readers looking for similar information are not likely to be disappointed because the content presents exactly what is described in the title.

1.2 Author

The article has been written and published by two authors; Assyr Abdulle and Achin Nonnenmacher.

1.3 Publication information

The article was published on Elsevier Journal, Mathematics Section. It was first received on 26th August 2008, received again in the revised form in 12th January 2009, accepted two months later and then published online on May 2009.

1.4 The Aim of the Study

The main aim of Assyr Abdulle and Achin Nonnenmacher?s study is to present a description of a multiscale finite element solver for parabolic or elliptic challenges characterized by highly oscillating coefficients.
The two scholars had great interest in the development of multiscale PDEs (partial differential equations) featuring the highly oscillating coefficients and they opted to venture in the sector. It is true that the classical numerical methods do lead to huge challenges and complexity for such problems.
Despite of the challenges experienced in the computational homogenization, only a limited number of open-box computer codes are available in the field. There is therefore a rare numerical comparison between the diverse strategies used.
The objective of the research is to present a short FE implementation for parabolic or multiscale elliptic problems. The paper proposes implementation through MATLAB, which permits for concise coding.

1.5 Paper Structure

This is a journal research paper. The paper was published on Elsevier journal database in the Mathematics section. The article was received on the Journal in 2008 and later published online on May 2009. Abstract, introduction, model problem, implementation, numerical examples, and the conclusion.

1.5.1 The Abstract

The scholars have used the abstract to present the aim of the study. The scholars use the abstract to show what is expected in the paper. Specifically, the paper abstract points out that the scholars will be describing multiple finite element. The abstract is also used to indicate some of the most important areas to be covered in the paper; discussion of flexible and short FE implementation.

1.5.2 Introduction

the introduction prepares the reader for the paper context. The authors use the first paragraph to identify the need to conduct the study in the first place. They first indicate that lately there have been an increase in studies in the finite element are. They also point out that despite the fact that there are lots of similar studies, there are very few open-box computer codes available for the subject area. Through the introduction, the authors introduce the FE-HMM algorithm and code which will be the proposed framework for the study. The authors use the introduction to introduce the challenges experienced in the study.

1.5.3 Model Problem

In this chapter, the scholars introduce the class of problems they are interested in finding solutions to. Specifically, they are interested in solving the elliptical problems featuring coefficient retrieved from some fine scale structure. It is in this section that they introduce the problems formulation. The model problem section also presents the homogenized problem; it also introduces the FE-HMM (FE homogeneous multiscale method). More details about these are presented later in the paper.

1.5.4 Implementation

The implementation section of the paper presents a simplified description of the implementation of the two-dimensional problems. First, the section gives data representation of the macro triangulation. It is this section that presents the core structure which is the assembly of the macro problem. Other subtopics discussed in this section include the micro solutions and assembly on the sampling domains, parabolic problems, and the three-dimensional problems,

1.5.5 Numerical Examples

Simply put, the authors present a number of numerical example to demonstrate the versatility, the use, and performance of the code they have adopted for the study. The first group go he numerical examples presents an elliptical problem that features non-uniformly coefficients, time dependent multiscale problem and elliptical problem featuring random tensors. The second group of the numerical examples examines three dimensional simulations. Most importantly, this section also presents the research results. The findings are presented in details.

1.5.6 Conclusion

The conclusion is the last part of the paper. The conclusion talks about the research findings, presents some interpretations in a simplified manner, and makes some recommendations.

Overall, the author has successfully presented a complete, valid and reliable research paper. However, this structure does not follow in the normal research studies formats. The paper structure looks a little complex, no paper chapters. A normal research paper, journal paper, would feature the research table of contents, introduction detailing on the research background, earlier research studies, detailed methodology with the hypotheses, results section, and discussion before providing a conclusion. It must however be appreciated that the scholars used some recommendable sources from different areas including journals, books, published thesis, etc. some of these sources seems to be outdated though, dating back to as far as 1978. This research was published in 2008, and it would be recommendable that the artist uses the most recent sources in respect to the year, say 5 years earlier.

Chapter 2

Analysis

2.1 Excessively narrow subject focus

While Assyr and Achin have done a great deal to focus on their area of study, I have a great feeling that this was overtly very narrow field of study. The information provided by the authors regarding the subject area is very narrow. There is a lot to discuss regarding Finite element. The readers deserve a better buildup of the research paper before actually narrowing down to the specific area of study. From my observation, the duo left out some very important aspects by failing to broadly discuss the research background. As a matter of fact, this research does not touch on similar research studies, to help the reader understand the earlier studies findings on the same field. I won't be wrong to argue that specific concentration on homogenization problems may lead to uncalled for, narrow, incomplete and may be biased outcomes and conclusions. This could be an indicator of lack of robust understanding of the subject areas to be covered.

2.2 Omission of potentially relevant information

The subject area of this research paper is rather a complex read and a sophisticated area of study. There are recognized culture of research papers to include abbreviations, definitions of difficult terms and explain complex terms for the reader. The fact that the paper was published in Elsevier implies that this article will be read by both expert researchers and mostly by unexperienced students interested in the subject area. The authors have plainly ignored definition of complex terms, no abbreviations, no basic tables of content, no tables of figures, and no disclamations. These are basic research paper requirements and any journal paper needs to be cautiously edited and have to be standard. The authors may have ignored these intentionally, but they are very important part of research practice.

2.3 Lack of evidence to support conclusions

One of the major shortcomings I have observed in this journal is the heavy reliance on author's personal information and opinions instead of empirical information. For example, there are a number of formulas, calculations, and outright texts included in the journal, yet there is no critical backup of such information. A good example is in the introduction, first line and second line. The authors allege that there is an increase in the level of research in PDEs especially those with highly oscillating coefficients. This information is not backed up with credible resources, the authors does not provide references that are actually taking these researchers. The authors continue to claim that indeed, classical approaches often contribute towards to computational challenges, but there are no evidences for such numerical computational challenges. There are numerous of such allegations and personal opinions in the entire body of the text that lacks the proper evidence. Where there is an absence of supporting evidence, the reader does not have any way to judge the reliability and validity of the author's conclusions, and as a result, seriously undermines the value of the paper.

2.4 Exaggerated and unsubstantiated claims

When creating the model for the study, the author has suggested the use of the method FE-HMM. Well, reading through literature, am not sure there are other studies that have applied this model before. This brings in the question of building the research framework. What guides did the authors use to inform their frameworks? Definitely not clear. While the use of MATLAB is well documented, there is lacking evidence whether the model applied by the duo will yield the correct result or not. Again, this dents credibility of this research. It is very important that every study develop their framework from well documented and well researched frameworks that can be proved to work for specific cases. The authors define the basic FE-HMM algorithm as a macro finite element space and begins to construct the formula. One begins to ask, where does these formulas actually come from? How have the formulas been developed?

2.5 Insufficient detail to authorize for assessment of the conclusions validity and reliability

There are a lot of studies that have been conducted in respect to finite element subject area and to help with recommendations in the area of study. However, Assyr and Achin does not have strong depth in their explanations and discussions regarding both the data collection processed, methodology and sample selection criteria. This research paper has a lacking of adequate detail, and it has become very difficult to assess the reliability and validity of the research findings.

2.6 Numerical examples

The objective of the paper was to present some of the numerical examples. The authors did a recommendable job in presenting these. It would be important that we discuss and analyze the technicalities and to show how elaborate and thorough the authors were in presenting the numerical examples. Formulas and images used by the authors may be directly made available here. The authors basically present these examples to demonstrate how to use their code, to demonstrate the versatility, performance and capabilities of the code.

In the first group of examples, the authors present an elliptic problem which does feature a non-uniformly periodic coefficient, a time dependent parabolic problem and an elliptic problem. In their explanation for the problems, the authors categorically observe that the problem with the random tensors are widely applied in modelling of the porous media flows. In the second group of examples, the authors investigate 3 dimensional simulations which are basically based on the simplified engineering problems experienced in real life. Specifically, they have given us the steady heat distribution in microprocessor and in a heatsink in their examples. In the two examples, the authors make the assumptions that the heatsink and the processors are made from composite objects.

2.7 Non-uniformly periodic coefficients

In their first example, the authors conduct several numerical experiments to show the convergence behavior of the model, FE-HMM is diverse norms, with the intention of demonstrating the impact of the sampling size of the sampling domains on the numerical solution and to show how the micromeshes and macromeshes have to be packaged to reduce the computational complexity. They do not however perform serious numerical convergence for the other examples.

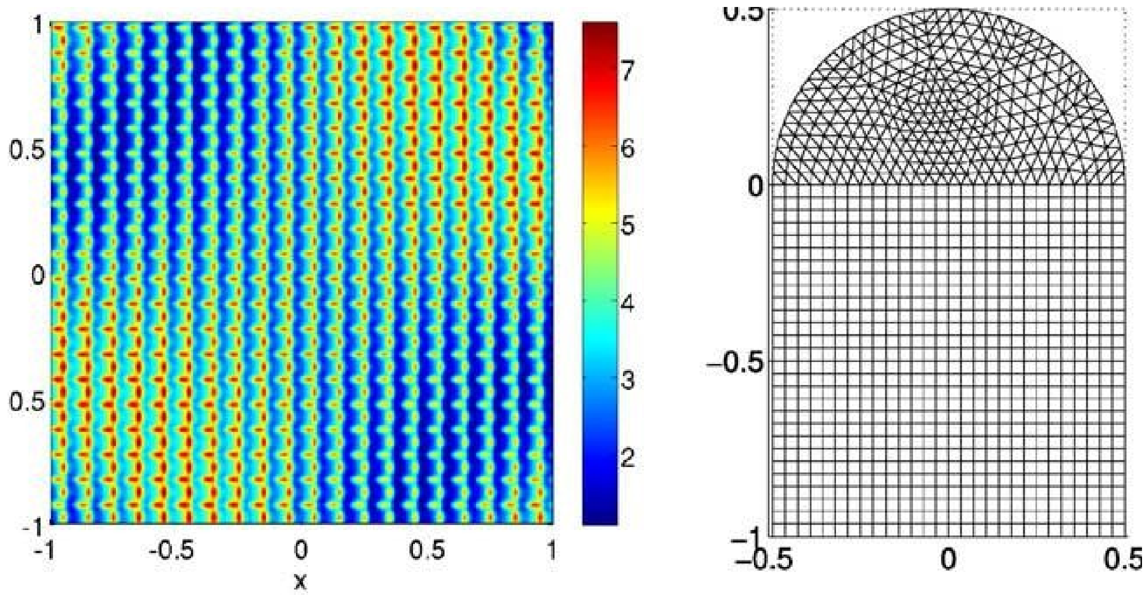


Figure 2.1: Snapshot of the conductivity tensor (left picture) and sketch of the computational mesh (right picture) for the elliptic problem 4.1

The authors bases their study on two scale problem formulated as follows;

$$-\nabla\cdot\left(a\left(x,\frac{x}{\varepsilon}\right)\nabla u^{\varepsilon}\right)=f(x)\quad\text{in }\Omega,$$

$$u^{\varepsilon}=0\quad\text{on }\partial\Omega_D,$$

$$-n\cdot\left(a\left(x,\frac{x}{\varepsilon}\right)\nabla u^{\varepsilon}\right)=0\quad\text{on }\partial\Omega_N,$$

where the conductivity tensor a^{ε} is given by

$$a\left(x,\frac{x}{\varepsilon}\right)=\frac{1.5\sin(2\pi x_1/\varepsilon)}{1.5\sin(2\pi x_2/\varepsilon)}+\frac{1.5\sin(2\pi x_2/\varepsilon)}{1.5\cos(2\pi x_1/\varepsilon)}+\sin(4x_1x_2)+1,$$

The authors further show the impact of micro discretization in the FE-HMM macro solution through varying N_{mic}. **Sampling Domain Size**: the researchers then report numerical results in the scenario when d, the domain sampling size, is not an integer multiple of the error term e. for instance, when d does not constitute an integer number of the error term e micro oscillations. The scenario also emerges for the periodic problems, where the researcher does not know the actual period size.

The authors further study the impact of similar modeling error for the HMM. They select $\delta = \frac{5}{3}\epsilon$. And then make a comparison of the yielded HMM solution to the Dirichlet or periodic boundary conditions for the microproblem.

The authors report that they obtained much better answers with the periodic boundary conditions. While the good behavior observed through the periodic boundary, they are not yet fully researched on and well understood. Figure 4.6 illustrates the N_{mic} cross-sectional plots.

It is important to note that the authors put more emphasis on the fact that while it is more comfortable to compare the HMM solution with a resolved fine scale answer, the former $\epsilon - 0$ does not automatically converge to the latter in the energy norm or the H^1.

2.8 Method Performance

The authors also examine the dependence of H1 and L2 error terms on the selection of the micro and macromeshes and provides indicators of the CPU times for the model ? FE-HMM. For the author to compare L2 and H1 with ease for the different sizes of the meshes, they put into c0onsideration the following quasi ID problem on a square domain where the homogenized tensor and particular solution can be calculated analytically:

$$
\begin{aligned}
&-\nabla \cdot \left(a\left(\frac{x}{\varepsilon}\right) \nabla u^{\varepsilon} \right) = f(x) \quad \text{in } \Omega = (0,1)^2, \\
&u^{\varepsilon}|_{\Gamma_D} = 0 \quad \text{on } \Gamma_D := \{x_1 = 0\} \cup \{x_1 = 1\}, \\
&n \cdot \left(a\left(\frac{x}{\varepsilon}\right) \nabla u^{\varepsilon} \right) \Big| \Gamma_D = 0 \quad \text{on } \Gamma_N := \partial\Omega \setminus \Gamma_N.
\end{aligned}
$$

Assyr Abdulle and Achin Nonnenmacher?s chose micro-discretization and macro-discretization.

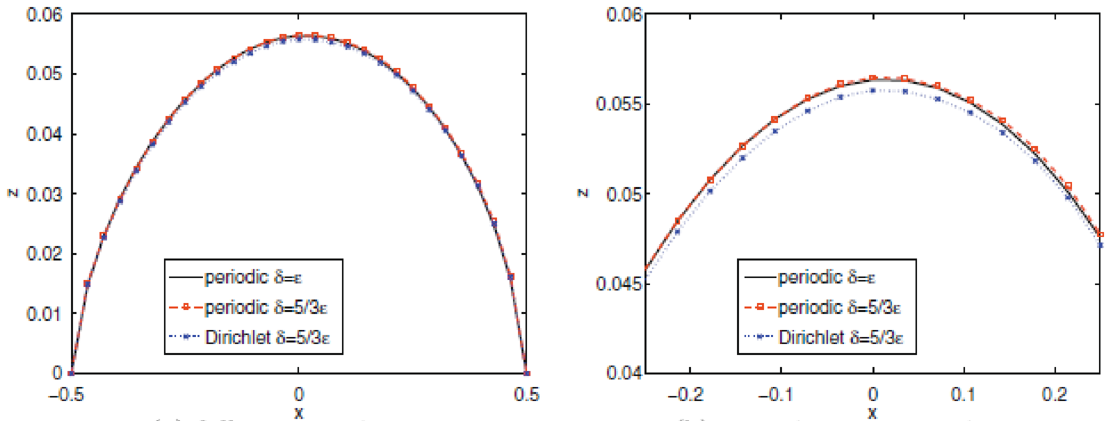
The macro- and micromeshes optimal refinement strategy: In what follows,

N_{mac} is used to represent the number of discretization points in every direction of the domain $\Omega[0,1]^2$ while we recall that N_{mac} refers to the number of discretization points in each direction of the d-dimensional sampling domains (here $d = 2$). Thus, the macromesh H is given by $H = \delta/N_{mac}$ and the micromesh by $H = \delta/N_{mic}$ (in the following computations we choose $\delta = e$). As the sampling domain is of size ?, we will also consider the scaled micromesh given by ?(Mmac). The macro and micro DOF are then given by $\alpha(M_{mac})$ and $\alpha = M_{mic}$, respectively, The authors assumes that the floating point operation costs of the method is directly proportional to the total DOF. For the problem above, they chose a sequence of macromeshes and the used the micromesh hs depending on the criteria (the sequence is $H = 1/32, 164, 1/128$). This provides the following micro and macromeshes written in regards to M_{mic} and N_{mac}($H = 1/Nmac, hs = 1/Nmac$).

Nmic (L2-micro refinement strategy) 16 32 64 128

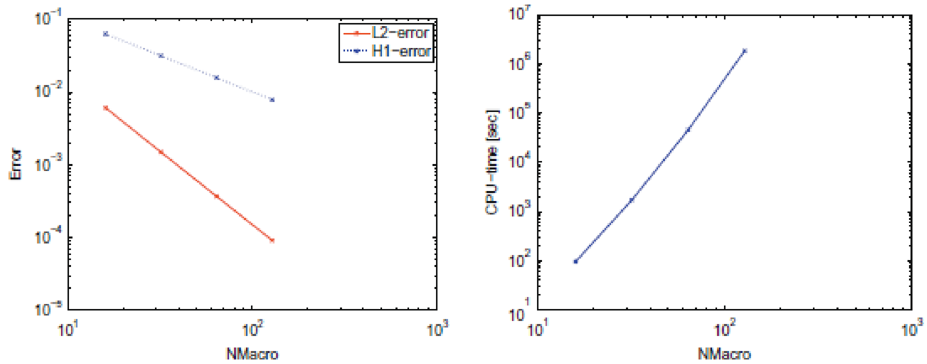
Nmic (H1-micro refinement strategy) 4 6 8 11

		$N_{mic} = 4$	$N_{mic} = 8$	$N_{mic} = 16$	$N_{mic} = 32$	Finescale
Periodic	$\|u\|_A$	0.2085	0.2119	0.2130	0.2133	0.2146
	$\|u\|_\infty$	0.0702	0.0720	0.0726	0.0728	0.0737
Dirichlet	$\|u\|_A$	0.2080	0.2107	0.2117	0.2119	0.2146
	$\|u\|_\infty$	0.0699	0.0714	0.0719	0.0720	0.0737



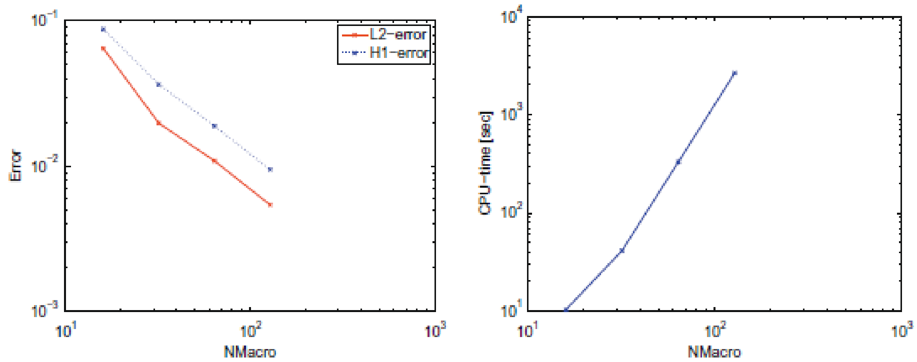
(a) full cross-section (b) zoom into cross-section

Remark. Because of the constant homogenized tensor, we could solve the microproblem just once. However, to assess realistic computing times for two-scale problems, we provide the CPU-times that would be needed to solve the full problem.



(a) Errors (b) CPU-time

Fig. 4.7. Dependence of errors and CPU-time on the macro DOF for the test-case problem described in Section 4.1.1 (L^2-micro refinement strategy).



(a) Errors (b) CPU-time

Fig. 4.8. Dependence of errors and CPU-time on the macro DOF for the test-case problem described in Section 4.1.1 (H^1-micro refinement strategy).

As predicted by the a-priori-estimates, the chosen refinement strategies achieve quadratic and linear convergence rates for the error in the L2 and H1 norms, respectively. We list in **Table 4.3** the detailed errors.

We see in **Table 4.3** that an appropriate refinement strategy is crucial to achieve the optimal quadratic convergence rate in the L2 norm. We also see (in accordance with the theoretical estimates) that we can use a much larger micromesh for H1 approximations than for L2 approximations. We emphasize that a correct refinement strategy is crucial for efficient computations. Finally, we observe as expected that the convergence rates are independent of e.

Remark. A comparison with a fine-scale solution is not useful in the present context of performance-evaluation. Indeed, as the performance of the FE-HMM is independent of the fine scale e, this parameter can be tuned arbitrarily for the fine scale problem, leading to arbitrary (high) CPU-time for the corresponding fine scale solution.

Parallel implementation of the FE-HMM. We remark that an implementation on a parallel computer is easily possible and leads ? due to the independence of the different microproblems ? to a near-optimal speedup. A parallel version of the FE-HMM (on eight nodes) has been used for the 3d-problems in Sections 4.4 and 4.5.

2.9 Problems with random tensor

The authors also consider a problem with random tensor, ?? (x). the ?, which is the domain is similar to the one in the earlier example, only that the zero Dirichlet boundary conditions are selected on the entire boundary. Triangles are also used in this problem specifically in the circular section of the domain while the quadrilaterals are adopted for the other parts of the computational domain. The scholars report they noticed the random variables for the fine scale conditions were most of the time used for problems constituting pressure equation in the porous media flow.

While the stochastic field for the problem in question is generated from the computer, the code adopted for this paper could be used alongside a real-life tensor retrieved from imaging methods, for instance using the microscoy or scanning technique.

The long-term normal stochastic field realization with zero and variance $r = 1$, is generated through moving ellipse average technique. The authors set the lengths of the correlation of the stochastic field to be $\epsilon_y = 0.02$ and $\epsilon_x = 0.01$.

Assyr and Achin calculate a reference solution on the fine grid of about 106df and make a comparison of the solution to the study model FE-HMM on the macro coarse grid within a range of 1100df. In this case of our model, FE-HMM, the authors present results for different sizes of the sampling domain. First, they select $\delta = 0.06$ and $\delta = 0.02$. it can be then observed that the δ closer to the solution profile to the reference solution as the sampling domain features more correlation length of the random field. Similar observation has been detailed in Table 4.4 when making a comparison of the energy norm of the different solution which have been achieved with the experiment model FE-HMM to the energy norm for the reference answer.

2.10 Parabolic problems

As earlier mentioned, the code by the authors can easily handle the parabolic challenges without much difficulty. They considered the following, as an example;

$$\frac{\partial u}{\partial t} = \nabla \cdot (a^{\varepsilon}(x)u^{\varepsilon}) + f \quad \text{in } \Omega \times [0,T].$$

They considered an implicit Euler for simplicity purposes for the time integration. they chose a time step of change in $t = 0.1$ and $T = 1$.
Then they kept similar boundary conditions and domain as in earlier problem, section 4.1. set of $f = 0$ and taking into consideration the original condition

$$\mu_0 = -10(x?0.5).(x + 0.5).(y + 1)$$

. The tensor adapted for the problem is as follows;

$$a\left(x, \frac{x}{\varepsilon}\right) = 0.1 \cdot \left(\frac{1.5\sin(2\pi x_1/\varepsilon)}{1.5\sin(2\pi x_2/\varepsilon)} + \frac{1.5\sin(2\pi x_2/\varepsilon)}{1.5\cos(2\pi x_1/\varepsilon)} + \sin(4x_1x_2) + 1\right).$$

A time-independent tensor was considered by the authors in this case, and they only had to solve the microproblems only one time, and can use similar stiffness matrix whenever they solve the linear equation that emerges from the implicit Euler scheme.

The corresponding code to this challenge is provided in $fe\_hmm2d\_para.m$.

Additional problems with the tensors taking the form $\alpha(xx/e, t)$ can be calculated in a similar way, but in the current situation the stiffness matrix must be updated every step of the way.

The parabolic problem results that are solved through the FE-HMM model code have been duly sketched and shown in figure 4.10. the results at $t = \frac{1}{4}$ point are very close to one another and are not possibly distinguishable to the eye. Table 4.5 shows the comparison of the infinity norms and the energy for different numerical solutions. Again, the authors confirm observation of the fact that the FE-HMM model is independent of the error term $\mathbf{e}$.

Chapter 3

3-D Problem: Steady State distribution of Heat in a Heatsink

In this section, the authors present a three dimensional multiscale problem, which is the conduction of steady state of heatsink manufactured from layered material. Important for the efficiency of the current microprocessors is a complex cooling process which is normally achieved with the help of a heatsink, and it is here that the advanced composite materials are applied to get the most efficient cooling process. The heatsink are generally mounted on top of the microprocessors with the intention of dissipating away the heat from it, usually through an inbuilt fan at the top section. However, since the authors were mainly interested in avoiding discussion of the model challenges, they did away with the effects that comes from the fans and better models.

Equations: heat gets transferred in three different ways including convection, conduction and radiation. Radiation is however ignored here because it is very minimal means of energy transfer. Fourier?s law can be used to describe heatsink heat conduction:

$$-\Delta. (\alpha \ \Delta\mu) = f$$

Where,
u is the body temperature, a is the tensor thermal conductivity and f is the source of the heat. The surrounding air transfer of the conductive heat can be expressed through the robin boundary condition as follows:

$$n \cdot (a\nabla u) = q_0 + \alpha(u_{\mathrm{amb}} - u),$$

Where,

U_{amb} is the ambient temperature, a is the heat transfer coefficient, qo is the heat flux gaining entry to the domain.,

Adopting the described heat transfer model above, the authors come up with the following problem which is a description of the steady state temperature distribution by the heatsink:

$$\begin{aligned}
&-\nabla \cdot (a^\varepsilon \nabla u^\varepsilon) = f \quad \text{in } \Omega, \\
&n \cdot (a^\varepsilon \nabla u^\varepsilon) + \alpha u^\varepsilon = g_R \quad \text{on } \partial\Omega_R, \\
&n \cdot (a^\varepsilon \nabla u^\varepsilon) = g_N \quad \text{on } \partial\Omega_N,
\end{aligned}$$

Where,

Ω represents the entire domain, $\partial\Omega n$ and $\partial\Omega r$ are the heatsink surfaces with the Neumann and Robin boundary conditions, respectively. The Robin hand boundary conditions right-hand is expressed as

$$gR = q0 + \alpha\mu amb$$

.

The Robin boundary conditions constitutes additional terms $-\int \partial\Omega r$ and $\int \partial\Omega r \; gRvds$ to be integrated in the weak form.

And for the case of the FE-HMM model, this leads to an additional contribution R to the matrix A stiffness, expressed in the following manner;

$$R_{ij} = \int_{\partial\Omega_R} \alpha \varphi_i^H \varphi_j^H \, ds,$$

Again, another contribution to the load vector I is expressed in the following manner;

Table 4.5 Energy and maximum errors for the FE-HMM and the fine-scale solutions of the parabolic problem described in Section 4.3.

$$r_j = \int_{\partial\Omega_R} g_R \varphi_j^H \, ds.$$

		$N_{mic} = 4$	$N_{mic} = 8$	$N_{mic} = 16$	$N_{mic} = 32$	Finescale
$\varepsilon = 0.005$	$\|u\|_A$	0.8354	0.8491	0.8525	0.8533	0.8697
	$\|u\|_\infty$	1.0839	1.1060	1.1117	1.1130	1.1379
$\varepsilon = 10^{-5}$	$\|u\|_A$	0.8313	0.8490	0.8525	0.8533	–
	$\|u\|_\infty$	1.0773	1.1058	1.1117	1.1130	–

While the authors considered the two-scale tensors in their experiment with the periodicity in the microscale, other tensors for realistic materials got through imaging methods and given ass data points could be easily considered.

3.1 Setup

The heatsink side mounted on top of the processor have some level of heat originating from the heat flux that is modelled through Neumann boundary conditions;

$$n \cdot (a\nabla u) = g_N \quad \text{on } \partial\Omega_N,$$

The entering heat flux $\partial\Omega N$ is expressed as;

$$g_N(x) = 3493\, e^{-1000(x^2+y^2)} \ \frac{\mathrm{W}}{\mathrm{m}^2},$$

Where, The numerical Rvalues are selected in a manner that we have a total incoming power of

$$P = \int_{\partial\Omega_N} g_N \, dx = 10\,\mathrm{W}$$

In their calculation, the authors assume that the heatsink is manufactured from a layered material and components. The benefit of selecting such type of materials is that an analytical formula can be used for the homogenized material. This gave them the chance to make a comparison of the Fe-HMM solution with an exact solution. It is important to say that for majority of the of the materials adopted for cooling procedures, for instance, the analytical formulas are not available and I is unavoidable to use the computational procedures as described in this experiment. From my research, and from the recommendations by Archin and colleague, earlier research studies mainly adopt carbon nanotubes as them heatsinks.

Remark: the authors emphasizes on the fact that the selected tensor, there is no benefit in using the FE-HMM model since the homogenized tensor is constant and can be computed analytically. It is an open knowledge that for such problems, the authors need only to call for the microsolver once when using the research model FE-HMM, and no knowledge regarding the correct averaging process is required. The reason for using such a simple tensor is that the research has the ability to make a precise comparison with different averaging processes. A more general tensor is used in the section 4.5 of the paper, where the scholars studies the heat distribution in a microprocessor.

The microscale tensor is set to be as follows;

$$a^{\varepsilon}(x) = \begin{pmatrix} a_{11}^{\varepsilon}(x) & 35 & 0 \\ 35 & a_{22}^{\varepsilon}(x) & 0 \\ 0 & 0 & 200 \end{pmatrix},$$

Where,

$$a_{11}^{\varepsilon}(x) = [500/(5+3.5\cdot\sin(2\pi x_1/\varepsilon))]\cdot e^{10\cdot x_3},$$
$$a_{22}^{\varepsilon}(x) = [500/(5+3.5\cdot\cos(2\pi x_1/\varepsilon))]\cdot e^{10\cdot x_3}.$$

It is well known that the analytical formulas for such tensors are available for the homogenized tensor. The authors got the following for the 46 tensor;

$$a^{0}(x) \approx \begin{pmatrix} 100.0\cdot e^{10\cdot x_3} & 35 & 0 \\ 35 & 140.0\cdot e^{10\cdot x_3} & 0 \\ 0 & 0 & 200 \end{pmatrix}$$

For the tensors that are homogenized, in the numerical tests, the heatsink has the dimensions 87:5 91:9 50 mm3. CUBIT model was used to generate the macromesh and constitutes 70000 tetrahedra featuring 17000 grid points and the average tetrhedra volume is estimated to be 2:30 mm^3.

Harmonic and arithmetic mean: the authors also calculate the FE-HMM results with the numerical results got from tensor average using the more naïve averaging processes as the harmonic or arithmetic means. To calculate these average tensors, the authors keep the macro variables fixed and the average the tensor they have, the element-wise over the micro-scale only.

$$a_{ij}^{\text{arithmetic}}(x_k) = \int_Y a_{ij}(x_k, y)\,dy,$$
$$a_{ij}^{\text{geometric}}(x_k) = \left(\int_Y (a_{ij}(x_k, y))^{-1}\,dy\right)^{-1},$$

Where Y is the three dimensional unit cube. This yields the following tensors;

$$a^{\text{arithmetic}}(x) = \begin{pmatrix} 140.0 \cdot e^{10x_3} & 35 & 0 \\ 35 & 140.0 \cdot e^{10x_3} & 0 \\ 0 & 0 & 200 \end{pmatrix},$$
$$a^{\text{harmonic}}(x) = \begin{pmatrix} 100.0 \cdot e^{10x_3} & 35 & 0 \\ 35 & 100.0 \cdot e^{10x_3} & 0.0 \\ 0 & 0 & 200 \end{pmatrix}.$$

When as comparison is made to the homogenized tensor, the authors clearly observe that the obtained tensor through the arithmetic average over-estimates the material conductivity while the tensor that is obtained through the arithmetic average under-estimates the material conductivity.

In their calculation, the authors assume that the heatsink is manufactured from a layered material and components. The benefit of selecting such type of materials is that an analytical formula can be used for the homogenized material. This gave them the chance to make a comparison of the Fe-HMM solution with an exact solution. It is important to say that for majority of the of the materials adopted for cooling procedures, for instance, the analytical formulas are not available and I is unavoidable to use the computational procedures as described in this experiment. From my research, and from the recommendations by Archin and colleague, earlier research studies mainly adopt carbon nanotubes as them heatsinks.

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Chapter 4

TP4 - Mathematical modeling of our problem

4.1 Recap of the Fe-HMM method

4.1.1 Introduction

We have introduced in this article a new quasi-periodic environment which tries to extend the found results in our previous works (TP). Such places present some kind of heterogeneity like the size is too small compared to the size of the domain in question. From the mathematical side, our problem can be defined as:

We need to find $u_\epsilon \in H_0^1(\Omega)$

$$\int_\Omega A_\epsilon(x)\nabla u_\epsilon.\nabla v\,dx = \int_\Omega f(x)v(x)\,dx \qquad \forall v \in \mathbb{H}_0^1(\Omega)$$

where,

$$A_\epsilon(x) = A(x, x/\epsilon)$$

The difference compared to what we have been previously doing is the fact that the construction of elementary matrices will be a little different: by solving a problem called "micro-problem" whose resolution is done by finite elements method and by adding penalization (as in the previous TP).

4.1.2 Theoretical modeling of the method

We introduce the expression of A^{eff}:

$\forall$ x fixed,

$$A^{\text{eff}}_{jk}(\bar{x}) = \int_Y A(\bar{x}, y)(e_k + \nabla_y w_k(\bar{x}, y)) \cdot (e_j + \nabla_y w_j(\bar{x}, y))\, dy.$$

In order to do a P1 Finite element discretisation of our current problem, we introduce:
- V_h^0 space as the internal approximation space of $H_0^1(\Omega)$
- V_h space as the discretisation space of $H^1(\Omega)$
- $N = dim(V_h)$
- $N_0 = dim(V_h^0)$

The discretized problem will be written as following:

Find $u_h \in V_h^0$ that satisfies:

$$\int_\Omega A^{\text{eff}} \nabla u^{\text{eff}} \nabla v_h dx = \int_\Omega f(x) v(x) dx \quad \forall \quad v_h \in \mathbb{V}_h^0$$

$$\begin{aligned}\int_\Omega A^{\text{eff}} \nabla u^{\text{eff}} \nabla v_h dx &= \sum_l \int_{T_l} A^{\text{eff}} \nabla u^{\text{eff}} \nabla v_h dx \\ &= \sum_k^K w_{k,l} A^{\text{eff}}(x_{k,l}) \nabla u^{\text{eff}}(x_{k,l}) \nabla v_h(x_{k,l})\end{aligned}$$

where,
- $x_{k,l}$ integration points in the triangle number l.
- $w_{k,l}$ the associated weights.

4.2 The local problem

TP4 - Questions 1 and 2.

We define $Y_\epsilon = (-\epsilon/2, \epsilon/2)^2$
Then A^{eff} will become:

$$A^{\text{eff}}(x)_{i,j} = \int_\Omega A(x,y)(e_j + \nabla_y w_j(x,y)) \,.\, ((e_i + \nabla_y w_i(x,y))dy$$

We define also $y = x/\epsilon = \varphi(x)$ and $J(\varphi)(x,y) = \frac{1}{\epsilon^2} = \frac{1}{|Y_\epsilon|}$
Then A^{eff} will become:

$$A^{\text{eff}}(\bar{x}) = \frac{1}{|Y_\epsilon|} \int_{Y_\epsilon} A(\bar{x}, x/\epsilon)(e_j + \nabla_x \eta_j(\bar{x}, x/\epsilon)) \,.\, ((e_i + \nabla_x \eta_i(\bar{x}, x/\epsilon))dx$$

with $Y_\epsilon(\overline{x}) = \overline{x} + Y_\epsilon$
We change then the variable x to $x - \overline{x}$

$$A^{\text{eff}}(\bar{x}) = \frac{1}{|Y_\epsilon|} \int_{Y_\epsilon} A(\bar{x}, x/\epsilon)(e_j + \nabla_x \tilde{\eta}_j(\bar{x}, x/\epsilon)) \,.\, ((e_i + \nabla_x \tilde{\eta}_i(\bar{x}, x/\epsilon))dx$$

with $\tilde{\eta}(\overline{x}, \frac{x}{\epsilon}) = \eta(\overline{x}, \frac{x+\overline{x}}{\epsilon})$
We are interested now in finding the equation satisfied by $\tilde{\eta}$. We start then by the equation

$$\int_Y A(x,y)\nabla_y w_i(x,y)\nabla_y v \, dy = -\int_Y A(x,y)e_i.\nabla_y v \, dy \quad \forall v \in \mathbb{V}$$

We arrive finally to the final equation satisfied by $\tilde{\eta}_i$ where we have used the same changing of variables like below

$$\int_{Y_\epsilon(\bar{x})} A(\bar{x}, x/\epsilon)\nabla_x\tilde{\eta}_i(\bar{x}, x/\epsilon)\nabla_x v = -\int_{Y_\epsilon(\bar{x})} A(\bar{x}, x/\epsilon)e_i.\nabla_x v \qquad \forall v \in V_\epsilon(\bar{x})$$

with $V_\epsilon(\overline{x})$ is defined as

$$V_\epsilon(\bar{x}) = \left\{ v \in \mathbb{H}^1_\#(Y_\epsilon(\bar{x})), \int_{Y_\epsilon(\bar{x})} v\, dx = 0 \right\}$$

TP4 - Questions 3.

Let $\tilde{x} = x_{T,k}$ and $\tilde{\eta} = \begin{pmatrix} \tilde{\eta}_1 \\ \tilde{\eta}_2 \end{pmatrix}$ and we try to compute

$$A^{\text{eff}}(x_{k,l})\nabla w_j(x_{k,l}).\nabla w_i(x_{k,l})$$

We have the assumption that

$$\nabla_x \tilde{w}_i(x_{T,k}, x) = (I_2 + \nabla_x\tilde{\eta}(x_{T,k}, x/\varepsilon))\nabla w_i(x_{T,k})$$

we know that the cells of the gradient of $\tilde{\eta}$ are the gradient of $\tilde{\eta}_i$; the equation becomes then

$$\frac{1}{|Y_\epsilon|}\int_{Y_\epsilon} [\sum_p A(x_{k,l}, x/\epsilon)(e_p + (\nabla\tilde{\eta})_p)\partial_p w_j].[\sum_h A(x_{k,l}, x/\epsilon)(e_h + (\nabla\tilde{\eta})_h)\partial_h w_i] dx$$

which is equal to

$$\frac{1}{|Y_\epsilon|}\int_{Y_\epsilon} A(x_{k,l}, x/\epsilon)(\mathbb{I}_2 + \nabla\tilde{\eta})\nabla w_j.(\mathbb{I}_2 + \nabla\tilde{\eta})\nabla w_i dx$$

TP4 - Questions 4.

Let $v \in V_\epsilon(\overline{x})$

$$\int_{Y_\epsilon(\bar{x})} A(\bar{x}, x/\epsilon)\nabla_x\tilde{\eta}_i(\bar{x}, x/\epsilon)\nabla_x v = -\int_{Y_\epsilon(\bar{x})} A(\bar{x}, x/\epsilon)e_i.\nabla_x v \qquad \forall v \in V_\epsilon(\bar{x})$$

We multiply it then by $\partial_i W_j$, we arrive to the conclusion that

$$\int_{Y_\epsilon(\bar{x})} A(\bar{x}, x/\epsilon)\nabla\tilde{w}_i(\bar{x}, x/\epsilon)\nabla_x v = 0 \qquad \forall v \in V_\epsilon(\bar{x})$$

We define the function $w_{lin} = w_i(x_{k,l}) + \nabla w_i x_{k,l}.x$
We conclude at the end that the problem which $\tilde{w}_i$ satisfies as a solution is defined as:

Find $\tilde{w}_i \in w_{lin} + V_\epsilon(\tilde{x})$

$$\int_{Y_\epsilon(\bar{x})} A(\bar{x}, x/\epsilon)\nabla\tilde{w}_i(\bar{x}, x/\epsilon)\nabla_x v = 0 \qquad \forall v \in V_\epsilon(\bar{x})$$

These kind of problems are called the micro-problems. Which in general, as mentioned in the TP4, they don't have an analytical solution so a finite element resolution method is needed then in this case to provide a numerical approximation of the desired solution.

4.3 Algorithm, validation and numerical implementation

4.3.1 General notes

In this section we try to solve the problem which is defined as

$$\mathbb{K}^{\text{eff}}\mathbb{U}^{\text{eff}} = \mathbb{F}$$

And K^{eff} is defined as

$$\mathbb{K}^{\text{eff}}_{i,j} = \sum_{T\in\mathcal{T}_H}\sum_{k=1}^{K}\frac{\omega_{T,k}}{|Y_\varepsilon(x_{T,k})|}\int_{Y_\varepsilon(x_{T,k})} A(x_{K,j}, x/\varepsilon)\nabla\tilde{w}_{j,h}(x)\cdot\nabla\tilde{w}_{i,h}(x)\,dx$$

And $w_{j,h}$ is solution of the equation below

$$\forall\phi_h \in V_{h,\#}(Y_\varepsilon(x_{T,k})),\quad \int_{Y_\varepsilon(x_{T,k})} A(x_{T,k}, x/\varepsilon)\,\nabla_x\tilde{w}_{i,h}(\bar{x}, x)\cdot\nabla_x\phi_h(x)\,dx = 0$$

Remark: during TP2, K, called as " matrice de rigidité ", was computed via a projection matrix in the V_h space. The code we currently have tries to build a local mesh for each integration point and then resolve the local equation of w_i. During TP2, we have used the penalisation option for resolution while in this case (the article) we have used the saddle point problem called as "problème de point selle".

4.3.2 Validation and results

TP4 - Questions 5.

Numerical method implementation.

TP4 - Questions 6.

To validate our calculations, we proceed to the obtained results from my previous TP.
Let $A(x, y)$ be defined as $A(x,y) = \begin{pmatrix} (2+sin(2\pi x))(4+sin(2\pi y)) & 0 \\ 0 & (2+sin(2\pi x))(4+sin(2\pi y)) \end{pmatrix}$

Results of the current code - TP4

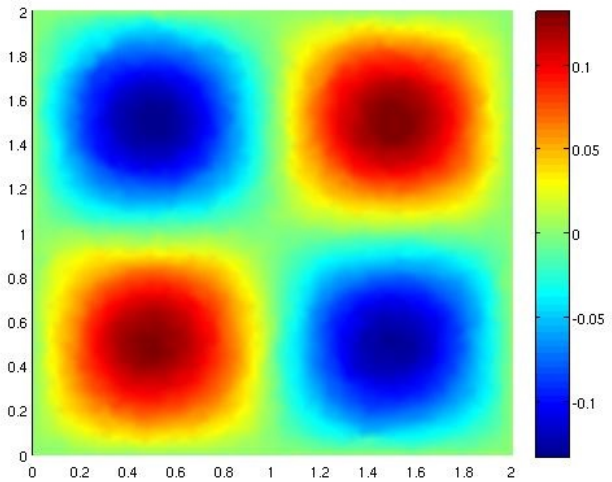

Figure 4.1: Solution A^{eff}

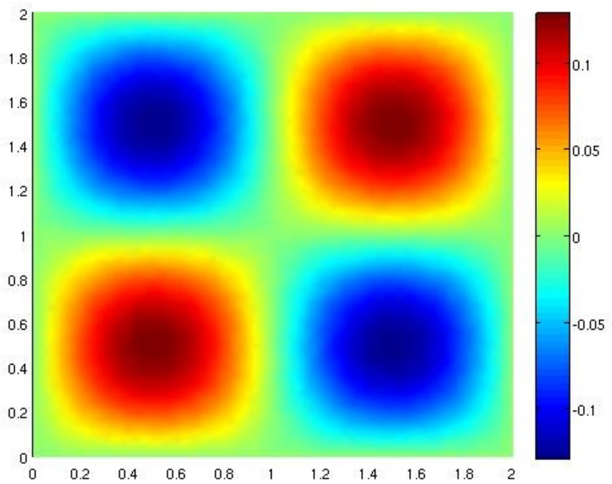

Figure 4.2: Solution found by the method Fe-HMM

Error analysis

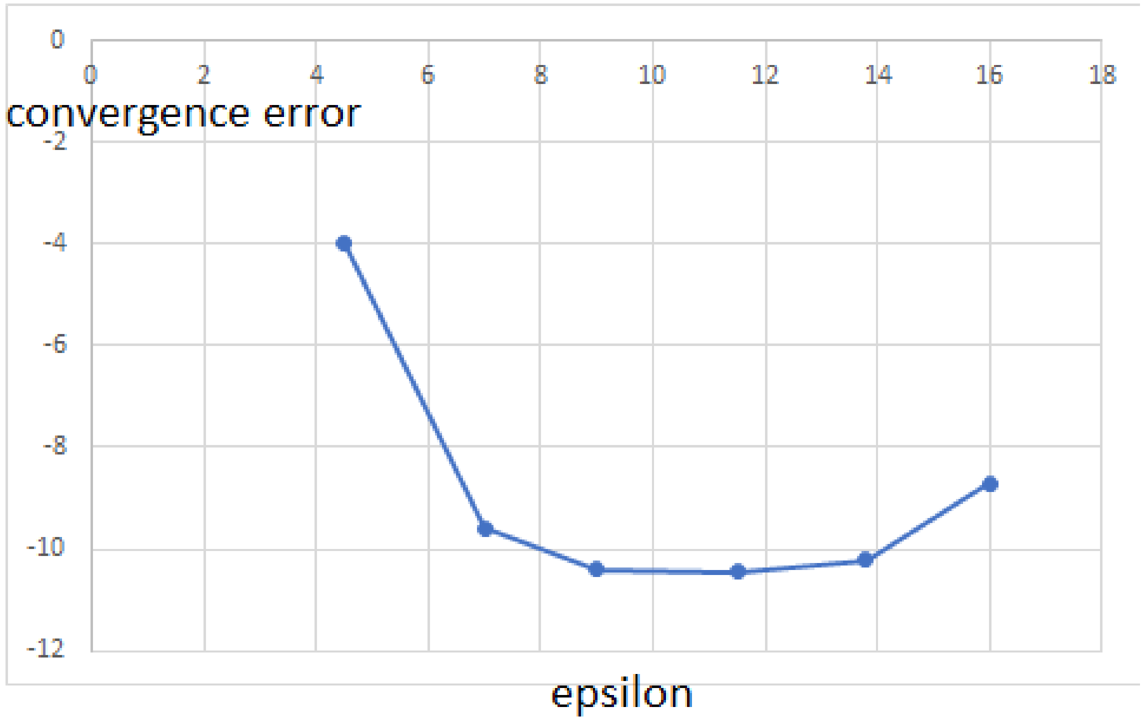

Figure 4.3: Evolution of the convergence error in L_2 norm - h=0.05

General notes

In order to test the convergence of the algorithm, we compute the error and make our observations.

According to both the visualized solutions, we see that the solution established with the FeHMM method converges to the exact solution.

On the other hand, we can see that there's a non-convergence in $H^1(\Omega)$, when we observe the evolution of the error in function of ϵ, we can easily say that the error goes down then up again which reflects a side of instability in the method itself.

We need to mention the case when ϵ have bigger values, which leads to an overlap in the existing meshes associated to the micro problem and this overlap leads to information loss.

In real life problems, the value of ϵ can't be determined with exactitude.

Conclusion

In this research paper, the authors have presented a versatile and short multiscale FE solver for the parabolic and elliptic PDEs featuring highly oscillating coefficients. I have the overall feeling that the researchers have done recommendable work except a few hitches. While there are several multiscale strategies and techniques that have earlier been developed specifically for the multiscale PDEs, only a few of these studies have gone into greater depths as in the case of this study, especially in the numerical examples implementations. This study based the algorithm on the heterogeneous multiscale technique, which provides the methodology with numerous efficiency micro and macrosolvers. The authors have successfully discussed in detail the multiscale FE solver implementation and have demonstrated that the multiscale strategy can be developed on the primary structure of a normal FE cord. The code also permits for a different triangulations and can be parallelized trivially (quadrilateral and simplical). The scholars have also presented a detailed discussion on the implementation of the time-dependent problems in the paper.

To give an illustration and a clear demonstration of the code they have used in the study, the scholars have presented a series of numerical examples for both the three and two dimensional problems with non-uniformly periodic, periodic and random tensors. A number of coupling conditions and boundary conditions have also been presented and discussed between the micro and macro FE solvers. The authors have also presented the time-dependent problems in their discussions. All these results are indicators that even though the code adopted for the research is simple, it can handle the complex and challenging problems. Therefore, a reader can have the belief that the research paper versatility and complexity of the proposed code could be important for further developments in other computational techniques for multiscale PDEs. Further, the code by the scholars could be easily integrated as a subroutine for a more general muslti-scale computational problems.

In regards to research formulation, thesis, and the structure, the scholars have done recommendable job.

This was my analysis of the research paper ?A Short and Versatile Finite Element Multiscale for Homogenization Problems? by Assyr Abdulle and Achin Nonnenmacher?s. The authors have successfully conducted their study and presented in a scholarly manner. They have described a multiscale finite element solver for parabolic or elliptic challenges characterized by highly oscillating coefficients. The researchers have based their analysis of finite element on coupled macrosolvers and microsolvers due to the development of heterogeneous multiscale method. In brief, the authors discuss the implementation of the flexible and short finite element (FE) of the multiscale algorithm, which can withstand quadrilateral or simplicial FE and multiple coupling conditions for the constrained micro simulations. The authors also present several basic numerical examples including time dependent and three-dimensional problems which demonstrate the efficiency and versatility of the employed computational strategy.

Overall, I have the feeling Assyr and Achin have done quite recommendable job except a few things. The study generally adheres to journal article standards, the research is referenced, even though some elements lacks appropriate sources. Some of the challenges I have identified with the paper include excessively narrow subject focus- while Assyr and Achin have done a great deal to focus on their area of study, I have a great feeling that this was overtly very narrow field of study. The information provided by the authors regarding the subject area is very narrow. Omission of potentially relevant information - the subject area of this research paper is rather a complex read and a sophisticated area of study. There are recognized culture of research papers to include abbreviations, definitions of difficult terms and explain complex terms for the reader. Lack of evidence to support conclusions - One of the major shortcomings I have observed in this journal is the heavy reliance on author?s personal information and opinions instead of empirical information. Lack of current data - the are a number of evidences that this research study failed in terms of using the most current data in respect to the time it was written and published, that is 2008. These are some of the major challenges with the paper. However, note that these mistakes are generally structural. I do not have major problem with the research techniques, the examples illustrated, even though I have difficult time finding sources used, to confirm the validity of the claims.

The authors have developed their thesis, gone ahead and proved it through their suggested model FE-HMM. This model seems to have worked perfectly well to give illustrations of the finite elements. More research needs to be done, using the same model, to prove whether there were computational errors. More research also needs to bone on the same line, using same method, same materials and similar conditions, but introduce non-heterogeneous materials.

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