ENGR 391 - NUMERICAL METHODS IN ENGINEERING

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PROBLEMS FOR CHAPTER 3

2. Consider the following set of simultaneous equation

$A\bar{x} = b$ where

$$A = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 4 & -2 \\ 0 & -2 & 2 \end{bmatrix} \text{ and } b = [-1,1,2]^T$$

a) Check whether the matrix A is positive definite.

b) Obtain the l_2 norm of vector b

c) Solve equation $A\bar{x} = b$ by Gaussian Elimination method.

Solution:

(a) For matrix to be positive definite; the determinants of all the co-factor matrices must positive

$$A = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 4 & -2 \\ 0 & -2 & 2 \end{bmatrix}$$

The determinants of all the co-factor matrices are

$$\text{Det } |1| > 0$$

$$\text{Det } \begin{vmatrix} 1 & -1 \\ -1 & 4 \end{vmatrix} = 3 > 0$$

$$\text{Det } \begin{vmatrix} 1 & -1 & 0 \\ -1 & 4 & -2 \\ 0 & -2 & 2 \end{vmatrix} = 2 > 0$$

Since the determinants of all the cofactor matrices are greater than 0, matrix A is positive definite

(b) l_2 of $x_{n+1} = \|x_{n+1}\|_2 = \{(-1)^2 + (1)^2 + (2)^2\}^{1/2} = 2.4495....$

(c) The augmented matrix is given by

$$\tilde{A}^1 = \begin{bmatrix} 1 & -1 & 0 & M & -1 \\ -1 & 4 & -2 & M & 1 \\ 0 & -2 & 2 & M & 2 \end{bmatrix}$$

$$\text{Row } j - \frac{A_{ji}}{A_{ij}} \text{Row } i \rightarrow \text{New Row } j$$

In this case

for $\begin{matrix} i = 1 \\ j = 2 \end{matrix}$ $\quad \text{Row } 2 - \frac{-1}{1} \text{Row } 1 \rightarrow \text{New Row } 2$

$A_{21} - \frac{A_{21}}{A_{11}} A_{11} \rightarrow A_{21}(\text{New})$ $\quad$ -1 - $\frac{-1}{1}(1) \rightarrow 0$ $\quad$ A_{21} (New)

$A_{22} - \frac{A_{21}}{A_{11}} A_{12} \rightarrow A_{22}(\text{New})$ $\quad$ 4 - $\frac{-1}{1}(-1) \rightarrow 3$ $\quad$ A_{22} (New)

$A_{23} - \frac{A_{21}}{A_{11}} A_{13} \rightarrow A_{23}(\text{New})$ $\quad$ -2 - $\frac{-1}{1}(0) \rightarrow -2$ $\quad$ A_{23} (New)

$A_{24} - \frac{A_{21}}{A_{11}} A_{14} \rightarrow A_{24}(\text{New})$ $\quad$ 1 - $\frac{-1}{1}(-1) \rightarrow 0$ $\quad$ A_{24} (New)

The new matrix formed is referred to as the next augmented matrix.

$$\tilde{A}^2 = \begin{bmatrix} 1 & -1 & 0 & M & -1 \\ 0 & 3 & -2 & M & 0 \\ 0 & -2 & 2 & M & 2 \end{bmatrix}$$

for $\begin{matrix} i = 1 \\ j = 3 \end{matrix}$ $\quad \text{Row3} - \frac{A_{32}}{A_{22}} \text{ Row2} \rightarrow \text{ New Row3}$

$$-2 - \frac{-2}{3}\, 3 \rightarrow 0 \text{ (New)}$$

$$2 - \frac{-2}{3}\, (-2) \rightarrow 0.667 \text{ (New)}$$

$$2 - \frac{-2}{3}\, (0) \rightarrow 2 \text{ (New)}$$

Since all values below the pivot element become zero, the next augmented matrix becomes:

$$\tilde{A}^2 = \begin{bmatrix} 1 & -1 & 0 & \mathrm{M} & -1 \\ 0 & 3 & -2 & \mathrm{M} & 0 \\ 0 & 0 & 0.6777 & \mathrm{M} & 2 \end{bmatrix}$$

Back substitution:

$x_3 = 3$

$x_2 = 2$

$x_1 = 1$

10. Find the Crout decomposition of the matrix

$$A = \begin{bmatrix} 1 & 1 & 0 \\ -1 & 4 & -2 \\ 0 & 2 & 2 \end{bmatrix}$$

Solution

Factorize A into L * U by letting $U_{ii} = 1$.

$$\begin{bmatrix} 1 & 1 & 0 \\ -1 & 4 & -2 \\ 0 & 2 & 2 \end{bmatrix} = \begin{bmatrix} L_{11} & 0 & 0 \\ L_{21} & L_{22} & 0 \\ L_{31} & L_{32} & L_{33} \end{bmatrix} * \begin{bmatrix} 1 & U_{12} & U_{13} \\ 0 & 1 & U_{23} \\ 0 & 0 & 1 \end{bmatrix}$$

Now applying the procedure:

i = 1 Row1 * Column1 = A_{11}

j = 1 (of L) (of U)

$(L_{11} * 1 + 0 * 0 + 0 * 0) = 1$

$L_1 = 1$

i = 1 Row1 * Column2 = A_{12}

j = 2 (of L) (of U)

i = 1

j = 1, 2, 3

$(L_{11} * U_{12} + 0 * 1 + 0 * 0) = 1$ gives

$U_{12} = 1$ L_{11}, U_{12}, U_{13}

i = 1 Row1 * Column3 = A_{13}

j = 3 (of L) (of U)

$(L_{11} * U_{13} + 0 * U_{23} + 0 * 1) = 0$

$U_{13} = 0$

i = 2 Row2 * Column1 = A_{21}

j = 1 (of L) (of U)

$(L_{21} * 1 + L_{22} * 0 + 0 * 0) = -1$

$L_{21} = -1$

i = 2 Row2 * Column2 = A_{22} i = 2

j = 2 (of L) (of U) j = 1, 2, 3

$(L_{21} * U_{12} + L_{22} * 1 + 0 * 0) = 4$ gives

$-1+L_{22} = 4; L_{22} = 5$ L_{21}, L_{22}, U_{23}

i = 2 Row2 * Column3 = A_{23}

j = 3 (of L) (of U)

$(L_{21} * U_{13} + L_{22} * U_{23} + 0 * 1) = -2$

$1(0)+5(U_{23})=-2;\ U_{23}= -0.4$

i = 3 Row3 * Column1 = A_{31}

j = 1 (of L) (of U)

(L_{31} * 1 + L_{32} * 0 + L_{33} * 0) = 0

L_{31} = 0

i = 3 Row3 * Column2 = A_{32} i = 3

j = 2 (of L) (of U) j = 1, 2, 3

(L_{31} * U_{12} + L_{32}* 1 + L_{33} * 0) = 2 gives

L_{32} = 2 L_{31}, L_{32}, L_{33}

i = 3 Row3 * Column3 = A_{23}

j = 3 (of L) (of U)

(L_{31} * U_{13} + L_{32} * U_{23} + L_{33} * 1) = 2

2(-0.4)+ L_{33} = 5; L_{33} = 2.8

The system is now factorized

$$\begin{bmatrix} 1 & 1 & 0 \\ -1 & 4 & -2 \\ 0 & 2 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 5 & 0 \\ 0 & 2 & 2.8 \end{bmatrix} * \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & -0.4 \\ 0 & 0 & 1 \end{bmatrix}$$

16. If [A] = $[2,-4,1,3]^T$ find

$$\|A\|_1, \|A\|_E, \|A\|_\infty$$

Solution

$$A = \begin{bmatrix} 2 \\ -4 \\ 1 \\ 3 \end{bmatrix}$$

$l_1 = \|A\|_1$ = largest absolute column sum

$$\begin{matrix} |2| \\ |-4| \\ |1| \\ |3| \end{matrix}$$

= 10

$$l_1 = \|A\|_1 = 10$$

$$l_\infty = \|A\|_\infty = 4$$

$$l_E = \|A\|_E = \{2^2 + (-4)^2 + 1^2 + 3^2\}^{1/2}$$

$$l_E = \|A\|_E = \{30\}^{1/2} = 5.477$$

21. In problem 20, find the relative error of the solution vector after 4th iteration, using the 1_∞ norm, in both Jacobi method and Gauss-Seidel method.

Solution

<u>Jacobi</u>

$x^{(3)} = \begin{bmatrix} 4.406 \\ 5.875 \\ -5.469 \end{bmatrix}$ and $x^{(4)} = \begin{bmatrix} 1.594 \\ 2.828 \\ -4.531 \end{bmatrix}$

$x^{(4)} - x^{(3)} = \begin{bmatrix} -2.813 \\ -3.047 \\ 0.938 \end{bmatrix}$

l_∞ of $x^{(4)} - x^{(3)} = \|x^{(4)} - x^{(3)}\|_\infty = 3.047$

l_∞ of $x^{(3)} = \|x^{(3)}\|_\infty = 5.875$

Relative error = $\dfrac{\|x^{(4)} - x^{(3)}\|_\infty}{\|x^{(3)}\|_\infty} = 0.519$

<u>Gauss-Seidel</u>

$x^{(3)} = \begin{bmatrix} 3.088 \\ 3.927 \\ -5.018 \end{bmatrix}$ and $x^{(4)} = \begin{bmatrix} 3.055 \\ 3.954 \\ -5.011 \end{bmatrix}$

$x^{(4)} - x^{(3)} = \begin{bmatrix} -0.033 \\ 0.027 \\ 0.007 \end{bmatrix}$

l_∞ of $x^{(4)} - x^{(3)} = \|x^{(4)} - x^{(3)}\|_\infty = 0.033$

l_∞ of $x^{(3)} = \|x^{(3)}\|_\infty = 3.088$

$$\text{Relative error} = \frac{\|x^{(4)} - x^{(3)}\|_\infty}{\|x^{(3)}\|_\infty} = 0.011$$