ENGR 213 Final Exam (Fall 2013)

Problem 1:

Given that $y = -\frac{2}{x} + x,$ is a solution of the differential equation $xy' + y = 2x$. Find x_0 and the largest interval I for which y(x) is a solution of the initial-value problem:

$$\mathrm{xy}' + \mathrm{y} = 2\mathrm{x}, \quad \mathrm{y}(\mathrm{x}_0) = 1$$

Problem 2:

Show that the given differential equation is exact and obtain the solution:

$$(x^3 + y^3)dx + 3xy^2 dy = 0$$

Problem 3:

Find all cube roots of z = 27i.

Problem 4:

Solve the following initial value problem

$$y'' + 6y' + 8y = \sin(2x), \qquad y(0) = 0, \quad y'(0) = 0$$

by using undetermined coefficients.

Problem 5:

Find general solution by variation of parameters:

$$x^2 y'' - xy' + y = x$$

Problem 6:

Find the first eight coefficients ($i.e.\ a_0, a_1, a_2, ...a_7$) of the power series expansion

$$y = \sum_{n=0}^{\infty} a_n x^n$$

of the solution to the differential equation

$$y'' + xy' + y = 0$$

subject to the initial value conditions $y(0) = 0, \; y'(0) = 1$

Problem 7:

Write in matrix form and find the general solution of the system:

$$\frac{dx}{dt} = 6x - y$$

$$\frac{dy}{dt} = 5x + 2y$$

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Final Examination

1. $y = -\frac{2}{x} + x$ is solution of $xy' + y = 2x$

$y(x_0) = 1$.

a) $1 = -\frac{2}{x_0} + x_0$ i.e. $x_0^2 - x_0 - 2 = 0$

$x_0 = 2$ or -1.

b) Largest interval is

$$-1 < x_0 < \infty$$

2. $(x^3 + y^3)\,dx + 3xy^2\,dy = 0$.

$M\,dx + N\,dy = 0$.

$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \longrightarrow 3y^2 = 3y^2$ $\therefore$ Exact DE.

$$f = \int M\,dx + g(y)$$

$$= \frac{x^4}{4} + xy^3 + g(y)$$

$$\frac{\partial f}{\partial y} = 3xy^2 + g'(y) = N = 3xy^2$$

$\therefore g'(y) = 0$, $g(y) = $ constant.

$$f = \frac{x^4}{4} + xy^3 + C$$

3. Cube root of $27i$

$$27i = 27\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right)$$

$$\therefore \sqrt[3]{27i} = 3\left[\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right],$$

$$3\left[\cos\frac{5\pi}{6} + i\sin\frac{5\pi}{6}\right],$$

$$3\left[\cos\frac{9\pi}{6} + i\sin\frac{9\pi}{6}\right].$$

4. $y'' + 6y' + 8y = \sin 2x \qquad y(0)=0,\ y'(0)=0.$

$y = e^{mx}$

$m^2 + 6m + 8 = 0$ Homogeneous part.

$(m+2)(m+4) = 0 \qquad m_1 = -2,\ m_2 = -4$

$y_c = c_1 e^{-2x} + c_2 e^{-4x}$

$y_p = A\cos 2x + B\sin 2x$

$y_p' = -2A\sin 2x + 2B\cos 2x$

$y_p'' = -4A\cos 2x - 4B\sin 2x$

Substituting

$-4A\cos 2x - 4B\sin 2x - 12A\sin 2x$

$+12B\cos 2x + 8A\cos 2x + 8B\sin 2x = \sin 2x$

$\therefore \quad -4B - 12A + 8B = 1$

$-4A + 12B + 8A = 0$

Solving $B = \frac{1}{40}$, $A = -\frac{3}{40}$.

$$\therefore y = c_1 e^{-2x} + c_2 e^{-4x} - \frac{3}{40}\cos 2x + \frac{1}{40}\sin 2x.$$

Substituting initial conditions and solving for c_1 & c_2

$$c_1 = \frac{1}{8}, \quad c_2 = -\frac{1}{20}.$$

$$y = \frac{1}{8}e^{-2x} - \frac{1}{20}e^{-4x} - \frac{3}{40}\cos 2x + \frac{1}{40}\sin 2x.$$

5.

$$x^2 y'' - xy' + y = x$$

$y = x^m$. Homogeneous case $\rightarrow$

$$\therefore m^2 - 2m + 1 = 0 = (m-1)^2$$

$$\therefore y_c = c_1 x + c_2 x \ln x \qquad \text{i.e. } c_1 y_1 + c_2 y_2.$$

Expressing in standard form

$$y'' - \frac{1}{x}y' + \frac{1}{x^2}y = \underbrace{\frac{1}{x}}_{f(x)}.$$

$$W = \begin{vmatrix} x & x\ln x \\ 1 & 1+\ln x \end{vmatrix}$$

$$u_1' = \frac{-y_2 f(x)}{W} = -\frac{\ln x}{x}$$

$$u_2' = \frac{y_1 f(x)}{W} = \frac{1}{x}$$

Integrating

$$u_1 = -\int \frac{\ln x}{x}\, dx = -\frac{\ln x^2}{2} + c$$

$$u_2 = \int \frac{1}{x}\, dx = \ln x + c$$

$$y_p = u_1 y_1 + u_2 y_2 = -\frac{\ln x^2}{2} \cdot x + \ln x \,(x \ln x)$$

$$\therefore y = c_1 x + c_2 x \ln x - x \cdot \frac{\ln x^2}{2} + x(\ln x)^2 .$$

6. $y'' + xy' + y = 0$

$$y = \sum_{n=0}^{\infty} a_n x^n$$

$$y' = \sum_{n=1}^{\infty} a_n n \cdot x^{n-1}$$

$$y'' = \sum_{n=2}^{\infty} a_n (n)(n-1) x^{n-2}$$

$$\therefore \sum_{n=2}^{\infty} a_n n \cdot (n-1) x^{n-2} + \sum_{n=1}^{\infty} a_n \cdot n x^n + \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{k=0}^{\infty} a_{k+2} \cdot (k+2)(k+1) x^k + \sum_{k=1}^{\infty} a_k k \cdot x^k + \sum_{k=0}^{\infty} a_k x^k = 0 .$$

$$\therefore 2a_2 + a_0 + \sum_{k=1}^{\infty} \left[a_{k+2} (k+2)(k+1) + (k+1) a_k \right] x^k = 0 .$$

$$2a_2 + a_0 = 0$$

$$a_{k+2} = \frac{-(k+1)\,a_k}{(k+2)(k+1)} = -\frac{a_k}{k+2}$$

$$\therefore\ a_2 = -\frac{a_0}{2} \quad — (1)$$

$$a_3 = \frac{-a_1}{3} \quad (2)$$

$$a_4 = \frac{-a_2}{4} \quad -(3)$$

$$\therefore\ y = a_0\left[1 - \frac{1}{2}\cdot x^2 + \cdots\right] + a_1\left[x - \frac{1}{3}x^3 + \cdots\right]$$

$$y(0) = a_0 = 1$$

$$y'(0) = a_1 = 1$$

$$\therefore\ y_0 = \left[1 - \frac{x^2}{2} + \cdots\right] + \left[x - \frac{x^3}{3} + \cdots\right]$$

7.

$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} 6 & -1 \\ 5 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} \quad \text{Matrix form.}$$

$$\begin{vmatrix} (6-\lambda) & -1 \\ 5 & (2-\lambda) \end{vmatrix} = 0. \qquad \lambda^2 - 8\lambda + 17 = 0$$

$$\lambda = \frac{8 \pm \sqrt{64-68}}{2} = 4 \pm i$$

$$(6-\lambda)x - y = 0$$

$$\therefore\ (6-\lambda)x = y.$$

$$\underline{\underline{\lambda = 4 - i}} \rightarrow (6-4+i)x = y$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 2+i \end{bmatrix}$$

$$\lambda = 4 + i \rightarrow \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 2-i \end{bmatrix}$$