Solution Final Exam Fall 2016 MATH 203

Salik Bahar

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#1

(a) (i)

$$(f \circ g)(x) = f(g(x)) = \sqrt[3]{g(x) - 1} = \sqrt[3]{\left(1 + \left(\frac{x}{1+x^3}\right)^3\right) - 1} = \sqrt[3]{\left(\frac{x}{1+x^3}\right)^3} = \frac{x}{1+x^3}$$

(ii)

$$(g \circ f)(x) = g(f(x)) = 1 + \left(\frac{f(x)}{1+(f(x))^3}\right)^3 = 1 + \left(\frac{\sqrt[3]{x-1}}{1+(\sqrt[3]{x-1})^3}\right)^3$$

$$= 1 + \frac{x-1}{(1+(x-1))^3} = 1 + \frac{x-1}{x^3} = \frac{x^3+x-1}{x^3}$$

$$\begin{aligned}\text{Domain}(f \circ g) &= \text{Domain}(g) \cap \text{Domain}(\frac{x}{1+x^3}) \\ &= \{x \in \mathbb{R} \mid x \neq -1\} \cap \{x \in \mathbb{R} \mid x \neq -1\} \\ &= \{x \in \mathbb{R} \mid x \neq -1\} = (-\infty, -1) \cup (-1, \infty)\end{aligned}$$

$$\begin{aligned}\text{Domain}(g \circ f) &= \text{Domain}(f) \cap \text{Domain}(1 + \frac{x-1}{x^3}) \\ &= \mathbb{R} \cap \{x \in \mathbb{R} \mid x \neq 0\} \\ &= \{x \in \mathbb{R} \mid x \neq 0\} = (-\infty, 0) \cup (0, \infty)\end{aligned}$$

(b) Interchange y for x and solve for y;

$$
\begin{aligned}
y &= \sqrt{2^x - 2} \\
x &= \sqrt{2^y - 2} \\
x^2 &= (\sqrt{2^y - 2})^2 \\
x^2 &= 2^y - 2 \\
x^2 + 2 &= 2^y \\
\log_2(x^2 + 2) &= \log_2(2^y) \\
\log_2(x^2 + 2) &= y
\end{aligned}
$$

Therefore,

$$f^{-1}(x) = \log_2(x^2 + 2)$$

or

$$f^{-1}(x) = \frac{\ln(x^2 + 2)}{\ln(2)}$$

$$
\begin{aligned}
\text{Domain}(f) &= \{x \in \mathbb{R} \mid 2^x - 2 \geq 0\} \\
&= \{x \in \mathbb{R} \mid 2^x \geq 2\} \\
&= \{x \in \mathbb{R} \mid x \geq 1\} = \text{Range}(f^{-1})
\end{aligned}
$$

and

$$\text{Domain}(f^{-1}) = \{x \in \mathbb{R} \mid x^2 + 2 > 0\} = \mathbb{R} = \text{Range}(f)$$

#2

(a)

$$\lim_{x \to 5} \frac{\sqrt{2x-1} - 3}{x^3 - 125} = \lim_{x \to 5} \frac{\sqrt{2x-1} - 3}{x^3 - 125} \cdot \frac{\sqrt{2x-1} + 3}{\sqrt{2x-1} + 3}$$

$$= \lim_{x \to 5} \frac{(2x-1) - 9}{(x^3 - 125)(\sqrt{2x-1} + 3)}$$

$$= \lim_{x \to 5} \frac{2x - 10}{(x-5)(x^2 + 5x + 25)(\sqrt{2x-1} + 3)}$$

$$= \lim_{x\to 5} \frac{2(x-5)}{(x-5)(x^2+5x+25)(\sqrt{2x-1}+3)}$$

$$= \lim_{x\to 5} \frac{2}{(x^2+5x+25)(\sqrt{2x-1}+3)} = \frac{2}{(75)\cdot(6)} = \frac{1}{225}$$

(b)

$$\lim_{x\to\infty} \frac{(x^3+1)(2x-3)^2}{(x+1)^2(3x+2)^3} = \lim_{x\to\infty} \frac{4x^5-12x^4+9x^3+4x^2-12x+9}{27x^5+108x^4+171x^3+134x^2+52x+8}$$

$$= \lim_{x\to\infty} \frac{x^5}{x^5} \cdot \frac{4-(12/x)+(9/x^2)+(4/x^3)-(12/x^4)+(9/x^5)}{27+(108/x)+(171/x^2)+(134/x^3)+(52/x^4)+(8/x^5)}$$

$$= \frac{4-0+0+0-0+0}{27+0+0+0+0+0} = \frac{4}{27}$$

We can also evaluate the limit without expanding the function

$$\text{Highest Term in Numerator} : x^3\cdot(2x)^2 = x^3\cdot 4x^2 = 4x^5$$

$$\text{Highest Term in Denominator} : x^2\cdot(3x)^3 = x^2\cdot 27x^3 = 27x^5$$

Thus, since the degree of the numerator and denominator are equal, we get

$$\lim_{x\to\infty} \frac{(x^3+1)(2x-3)^2}{(x+1)^2(3x+2)^3} = \frac{4}{27}$$

#3

(a) The function $f(x) = \frac{|x^2+4x-5|}{x^2-25}$ is undefined when $x = \pm 5$. Thus, we want evaluate

$$\lim_{x\to -5^-} f(x), \qquad \lim_{x\to -5^+} f(x)$$

$$\lim_{x\to 5^-} f(x), \qquad \lim_{x\to 5^+} f(x)$$

Note that

$$|x^2+4x-5| = \begin{cases} x^2+4x-5 & \text{if} \quad (x^2+4x-5)\geq 0 \quad \Rightarrow \quad x\in(-\infty,-5]\cup[1,\infty) \\ -(x^2+4x-5) & \text{if} \quad (x^2+4x-5)<0 \quad \Rightarrow \quad x\in(-5,1) \end{cases}$$

$$\lim_{x\to-5^-}\frac{|x^2+4x-5|}{x^2-25}=\lim_{x\to-5^-}\frac{x^2+4x-5}{x^2-25}=\lim_{x\to-5^-}\frac{(x+5)(x-1)}{(x-5)(x+5)}$$

$$=\lim_{x\to-5^-}\frac{x-1}{x-5}=\frac{-6}{-10}=\frac{3}{5}$$

$$\lim_{x\to-5^+}\frac{|x^2+4x-5|}{x^2-25}=\lim_{x\to-5^+}\frac{-(x^2+4x-5)}{x^2-25}=\lim_{x\to-5^+}\frac{-(x+5)(x-1)}{(x-5)(x+5)}$$

$$=\lim_{x\to-5^+}\frac{-(x-1)}{x-5}=\frac{-(-6)}{-10}=-\frac{3}{5}$$

$$\lim_{x\to5^-}\frac{|x^2+4x-5|}{x^2-25}=\lim_{x\to5^-}\frac{|x^2+4x-5|}{x^2-25}=\lim_{x\to5^-}\frac{x^2+4x-5}{x^2-25}=\lim_{x\to5^-}\frac{(x+5)(x-1)}{(x-5)(x+5)}$$

$$=\lim_{x\to5^-}\frac{x-1}{x-5}=-\infty$$

$$\lim_{x\to5^+}\frac{|x^2+4x-5|}{x^2-25}=\lim_{x\to5^+}\frac{|x^2+4x-5|}{x^2-25}=\lim_{x\to5^+}\frac{x^2+4x-5}{x^2-25}=\lim_{x\to5^+}\frac{(x+5)(x-1)}{(x-5)(x+5)}$$

$$=\lim_{x\to5^+}\frac{x-1}{x-5}=\infty$$

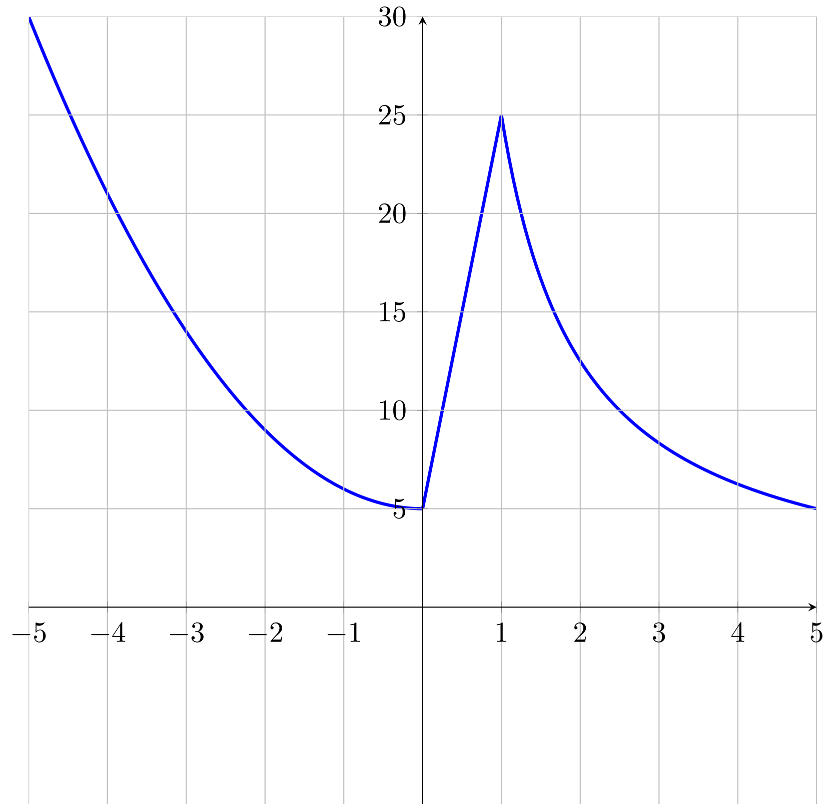
(b)

$$\lim_{x\to0^-}5+x^2=\lim_{x\to0^+}ax+b$$
$$5=b$$

and

$$\lim_{x\to1^-}ax+5=\lim_{x\to1^+}\frac{25}{x}$$
$$a+5=25$$
$$a=20$$

Therefore f is continuous everywhere when $a = 20$ and $b = 5$.



#4

(a) There are may ways to differentiate this function. I will only show one way using the Quotient Rule;

$$f'(x) = \frac{\frac{d}{dx}(\sqrt{x} + 3\sqrt[3]{x^2} + x^5) \cdot (2x\sqrt[3]{x}) - (\sqrt{x} + 3\sqrt[3]{x^2} + x^5) \cdot \frac{d}{dx}(2x\sqrt[3]{x})}{(2x\sqrt[3]{x})^2}$$

$$= \frac{(\frac{1}{2}x^{-1/2} + \frac{1}{3} \cdot 3(x^2)^{-1/2} \cdot 2x + 5x^4) \cdot (2x\sqrt[3]{x} - (\sqrt{x} + 3\sqrt[3]{x^2} + x^5) \cdot ((2)\sqrt[3]{x}) + 2x \cdot \frac{1}{3}x^{-2/3})}{4x^2 \cdot x^{2/3}}$$

$$= \frac{(\frac{1}{2}x^{-1/2} + (x^{-1} \cdot 2x) + 5x^4) \cdot (2x\sqrt[3]{x} - (\sqrt{x} + 3\sqrt[3]{x^2} + x^5) \cdot ((2)\sqrt[3]{x}) + 2x \cdot \frac{1}{3}x^{-2/3})}{4x^{8/3}}$$

(b)

$$
\begin{aligned}
f'(x) &= \frac{d}{dx}(x^3 + ex - \sin\pi) \cdot (\cos(2x)) + (x^3 + ex - \sin\pi) \cdot \frac{d}{dx}(\cos(2x)) \\
&= (3x^2 + e) \cdot (\cos(2x)) + (x^3 + ex - \sin\pi) \cdot (-2\sin(2x))
\end{aligned}
$$

(c) Note that

$$\ln^k(x) = (\ln(x))^k$$

for all $k \in \mathbb{R}$.

$$f'(x) = \frac{d}{dx}\Big(\ln^3(x^2 + \tan(3x))\Big) = 3(\ln(x^2 + \tan(3x)))^2 \cdot \frac{d}{dx}(\ln(x^2 + \tan(3x))$$

$$= 3(\ln(x^2 + \tan(3x)))^2 \cdot \frac{1}{x^2 + \tan(3x)} \cdot \frac{d}{dx}(x^2 + \tan(3x))$$

$$= 3(\ln(x^2 + \tan(3x)))^2 \cdot \frac{1}{x^2 + \tan(3x)} \cdot (2x + \sec^2(3x)) \cdot \frac{d}{dx}(3x)$$

$$= 3(\ln(x^2 + \tan(3x)))^2 \cdot \frac{1}{x^2 + \tan(3x)} \cdot (2x + \sec^2(3x) \cdot 3)$$

(d)

$$f'(x) = \frac{\frac{d}{dx}(\arcsin^2(x)) \cdot \sqrt{1 - x^2} - (\arcsin^2(x)) \cdot \frac{d}{dx}(\sqrt{1 - x^2})}{(\sqrt{1 - x^2})^2}$$

$$= \frac{2\arcsin(x) \cdot \frac{1}{\sqrt{1-x^2}} \cdot \sqrt{1 - x^2} - (\arcsin^2(x)) \cdot \frac{1}{2}(1 - x^2)^{-1/2} \cdot (-2x)}{1 - x^2}$$

$$= \frac{2\arcsin x + \frac{x(\arcsin^2(x))}{\sqrt{1-x^2}}}{1 - x^2} = \frac{2\arcsin x}{1 - x^2} + \frac{x(\arcsin^2(x))}{(1 - x^2)^{3/2}}$$

(e) Set $y = f(x)$

$$y = (3x^2+5)^{\arctan(x)}$$

$$\ln(y) = \ln\left((3x^2+5)^{\arctan(x)}\right) \qquad \text{(Take the natural logarithm on both sides)}$$

$$\ln(y) = (\arctan(x))\ln(3x^2+5) \qquad (\text{since } \ln(x^r) = r\cdot\ln(x))$$

$$\frac{y'}{y} = \frac{d}{dx}\left(\arctan(x)\cdot\ln(3x^2+5)\right) \qquad (\text{Differentiate Implicitly with respect to } x)$$

$$\frac{y'}{y} = \frac{1}{1+x^2}\cdot\ln(3x^2+5) + \arctan x\cdot\frac{6x}{3x^2+5} \qquad \text{(Product Rule on the right side)}$$

$$y' = y\cdot\left(\frac{\ln(3x^2+5)}{1+x^2} + \frac{6x\arctan x}{3x^2+5}\right)$$

$$f'(x) = (3x^2+5)^{\arctan(x)}\cdot\left(\frac{\ln(3x^2+5)}{1+x^2} + \frac{6x\arctan x}{3x^2+5}\right)$$

#5

(a)

$$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{h\to 0}\frac{\sqrt{(x+h)^2+8}-\sqrt{x^2+8}}{h}$$

$$= \lim_{h\to 0}\frac{\sqrt{(x+h)^2+8}-\sqrt{x^2+8}}{h}\cdot\frac{\sqrt{(x+h)^2+8}+\sqrt{x^2+8}}{\sqrt{(x+h)^2+8}+\sqrt{x^2+8}}$$

$$= \lim_{h\to 0}\frac{((x+h)^2+8)-(x^2+8)}{h\cdot(\sqrt{(x+h)^2+8}+\sqrt{x^2+8})}$$

$$= \lim_{h\to 0} \frac{(x^2+2xh+h^2+8)-x^2-8}{h\cdot(\sqrt{(x+h)^2+8}+\sqrt{x^2+8})} = \lim_{h\to 0} \frac{2xh+h^2}{h\cdot(\sqrt{(x+h)^2+8}+\sqrt{x^2+8})}$$

$$= \lim_{h\to 0} \frac{h\cdot(2x+h)}{h\cdot(\sqrt{(x+h)^2+8}+\sqrt{x^2+8})} = \lim_{h\to 0} \frac{2x+h}{\sqrt{(x+h)^2+8}+\sqrt{x^2+8}}$$

$$= \frac{2x}{2\sqrt{x^2+8}} = \frac{x}{\sqrt{x^2+8}}$$

(b) Rewrite

$$f(x) = \sqrt{x^2+8} = (x^2+8)^{1/2}$$

Then apply the power rule and chain rule to differentiate;

$$f'(x) = \frac{1}{2}(x^2+8)^{-/12}\cdot(2x) = \frac{x}{\sqrt{x^2+8}}$$

(c) Recall that dx and dy are called the differentials of f, and

$$dx = \Delta x \qquad \text{and} \qquad \Delta y \approx dy = f'(x)dx$$

Then

$$dx = \frac{dy}{\frac{x}{\sqrt{x^2+8}}} \qquad \text{and} \qquad dy = \frac{x}{\sqrt{x^2+8}}\cdot dx$$

(d) The linearization of f at $a = 1$

$$L(x) = f(1) + f'(1)(x-1) = 3 + \frac{1}{3}(x-1) = \frac{1}{3}x + \frac{8}{3}$$

or by using differential notations

$$f(x+\Delta x) = f(1) + \Delta y \approx f(1) + dy = f(1) + f'(1)\Delta x = f(1) + f'(1)(x-1) = \frac{1}{3}x + \frac{8}{3}$$

Since

$$f(1) = \sqrt{9} = 3 \qquad \text{and} \qquad f(0.7) = \sqrt{(0.7)^2+8} = \sqrt{8.49}$$

then

$$\Delta x = dx = 0.7 - 1 = -0.3$$

Therefore,

$$f(0.7) \approx L(0.7) = \frac{1}{3}(0.7) + \frac{8}{3} = \frac{8.7}{3} = 2.9$$

Actual Value $\sqrt{8.49} \approx 2.9137$

#6

(a) Replace (x, y) with $(0, 1)$ into the equation

$$\begin{aligned} y^4 \tan(x) &= xy^3 + y - 1 \\ (1)^4 \tan(0) &= (0)(1)^3 + (1) - 1 \\ 0 &= 0 \end{aligned}$$

Therefore, $(0, 1)$ belongs to the curve of the equation. Differentiate implicitly with respect to x to find the slope of the tangent line (Isolate y')

$$\begin{aligned} \frac{d}{dx}(y^4 \tan(x)) &= \frac{d}{dx}(xy^3 + y - 1) \\ 4y^3 y' \cdot \tan(x) + y^4 \cdot \sec^2(x) &= y^3 + 3xy^2 y' + y' \\ 4y^3 y' \cdot \tan(x) - 3xy^2 y' - y' &= y^3 - y^4 \cdot \sec^2(x) \\ y'(4y^3 \tan(x) - 3xy^2 - 1) &= y^3 - y^4 \cdot \sec^2(x) \\ y' &= \frac{y^3 - y^4 \cdot \sec^2(x)}{4y^3 \tan(x) - 3xy^2 - 1} \\ y' &= \frac{(1)^3 - (1)^4 \sec^2(0)}{4(1)^3 \tan(0) - 3(0)(1)^2 - 1} = 0/-1 = 0 \end{aligned}$$

Therefore, the equation of the tangent line

$$y = 1$$

(b) $f(x) = 4x^{-5} - 3x^2$

$$\begin{aligned} f'(x) &= -20x^{-6} - 6x \\ f''(x) &= (f'(x))' = 120x^{-7} - 6 \\ f'''(x) &= (f''(x))' = -840x^{-8} \end{aligned}$$

(c) If we plug in $x = 0$ right away, then

$$\lim_{x\to 0} \frac{e^x - e^{-x} - 2x}{x - \sin(x)} = \frac{e^0 - e^{-0} - 2(0)}{0 - \sin(0)} = \frac{0}{0}$$

we get the indeterminate form of type $\frac{0}{0}$. Thus, we can apply the l'Hospital's Rule and get

$$\lim_{x\to 0} \frac{\frac{d}{dx}(e^x - e^{-x} - 2x)}{\frac{d}{dx}(x - \sin(x))} = \lim_{x\to 0} \frac{e^x + e^{-x} - 2}{1 - \cos(x)} = \frac{0}{0}$$

Again we have the indeterminate form $\frac{0}{0}$, so we apply l'Hospital's Rule a second time;

$$\lim_{x\to 0} \frac{e^x + e^{-x} - 2}{1 - \cos(x)} = \lim_{x\to 0} \frac{e^x - e^{-x}}{\sin(x)} = \frac{1-1}{0} = \frac{0}{0}$$

We have the indeterminate form $\frac{0}{0}$, so we apply the l'Hospital's Rule a third time:

$$\lim_{x\to 0} \frac{e^x - e^{-x}}{\sin(x)} = \lim_{x\to 0} \frac{e^x + e^{-x}}{\cos(x)} = \frac{2}{1} = 2$$

#7 Note that

$$x = x(t) \qquad \text{and} y = y(t)$$

are both function of time t, where $t = \#seconds$. Also,

$$\frac{dx}{dt} = 5 \qquad \text{when} \qquad x = -1$$

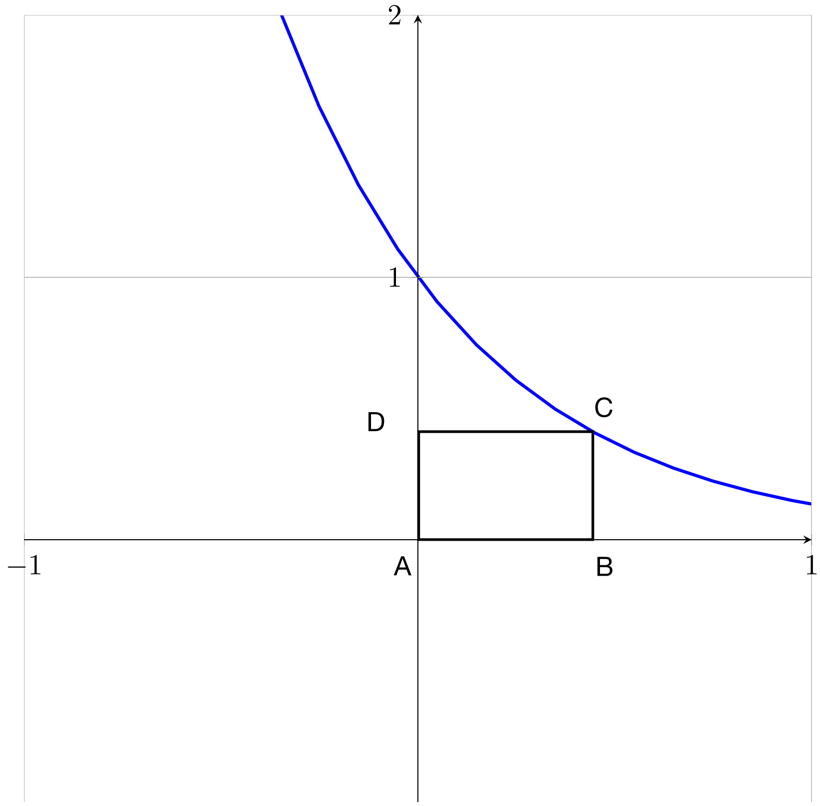
Now Differentiate both side of the equation with respect to t.

$$\begin{aligned} 2x^2 + 5y^2 &= 22 \\ 4x \cdot \frac{dx}{dt} + 10y \cdot \frac{dy}{dt} &= 0 \\ 4(-1) \cdot (5) + 10y \cdot \frac{dy}{dt} &= 0 \\ 10y \cdot \frac{dy}{dt} &= 20 \\ yy' &= 2 \end{aligned}$$

$$y' = \frac{dy}{dt} = \frac{2}{y}$$

If $y > 0$ then $\frac{dy}{dt} = \frac{2}{y} > 0$. Therefore, the y-coordinate is increasing when $y > 0$ at rate a $\frac{2}{y}$ *cm/sec*.

(b) Sketch a graph



Since the points are $A(0,0)$, $B(x,0)$, $C(x,e^{-2x})$ and $D(0,e^{-2x})$, then the area of the rectangle $ABCD$

$$|AB| \times |AD| = x \cdot e^{-2x}$$

$$\text{Maximize} : A(x) = x \cdot e^{-2x}$$
$$\text{Constraint} : x > 0 \quad \text{and} \quad y = e^{-2x} > 0$$

To maximize the Area, first compute $\frac{dA}{dx}$ and find its critical number

$$A'(x) = e^{-2x} - 2xe^{-2x} = 0$$
$$e^{-2x} = 2xe^{-2x}$$
$$x = \frac{1}{2}$$

Since $x \in (0, \infty)$, we apply the 'First Derivative Test for Absolute Extreme Values'. From the chart below,

interval	$(0, 1/2)$	$(1/2, \infty)$
x	$1/4$	1
$A'(x)$	$+$	$-$
$A(x)$	increasing	decreasing

we see that $A'(x) > 0$ for all $x \in (0, 1/2)$ and $A'(x) < 0$ for all $x \in (1/2, \infty)$. Therefore,

$$A(1/2) = \frac{1}{2} \cdot e^{-1} \approx 0.184 \text{ square units}$$

is the largest area possible when $x > 0$ and the coordinate point C is $(1/2, e^{-1}) = (1/2, 0.367)$

#8 $f(x) = \frac{2x}{x^2-9}$

(a) The domain of f:

$$\{x \mid x^2 - 9 \neq 0\} = \{x \mid x \neq \pm 3\} = (-\infty, -3) \cup (-3, 3) \cup (3, \infty)$$

The $x-$ and $y-$ intercepts are both 0.

Since $f(-x) = \frac{2(-x)}{(-x)^2-9} = -\frac{2x}{x^2+9} = -f(x)$, the function f is odd. The curve is symmetric about the line $y = x$

Horizontal and Vertical Asymptotes:

$$\lim_{x \to \pm\infty} \frac{2x}{x^2 - 9} = 0$$

Therefore the line $y = 0$ is the horizontal asymptote of f.

Since the denominator is 0 when $x = \pm 3$, we compute the following limits:

$$\lim_{x \to 3^+} \frac{2x}{x^2 - 9} = \infty \qquad \lim_{x \to 3^-} \frac{2x}{x^2 - 9} = -\infty$$

$$\lim_{x \to -3^+} \frac{2x}{x^2 - 9} = \infty \qquad \lim_{x \to -3^-} \frac{2x}{x^2 - 9} = -\infty$$

Therefore the lines $x = 3$ and $x = -3$ are the vertical asymptotes of f.

(b) First compute the derivative of f;

$$f'(x) = \frac{(2)(x^2 - 9) - 2x \cdot (2x)}{(x^2 - 1)^2} = \frac{-18 - 2x^2}{(x^2 - 9)^2}$$

we can see that $f'(x) < 0$ for all $x \in \mathbb{R}$, thus f is decreasing for all $x \in \text{Domain}(f) = (-\infty, -3) \cup (-3, 3) \cup (3, \infty)$ and f has no local extrema's

(c) Compute and simplify $f''(x)$;

$$f''(x) = \frac{4x(x^2+27)}{(x^2-9)^3}$$

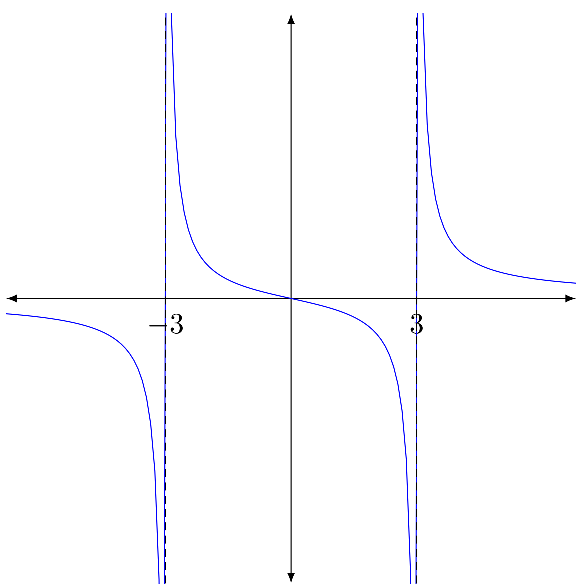
$f''(x) = 0$ when $x = 0$, and $f'(x)$ is undefined when $x = \pm 3$. Divide the domain of f into 4 intervals.

From the chart below,

interval	$(-\infty, -3)$	$(-3, 0)$	$(0, 3)$	$(3, \infty)$
x	-4	-2	2	4
$f''(x)$	$-$	$+$	$-$	$+$
concavity f	downward	upward	downward	upward

we see that the curve is concave upward on the $(-3, 0) \cup (3, \infty)$ and concave downward on $(-\infty, -3) \cup (0, 3)$. The function f has a point of inflection at $x = 0 \Rightarrow (0, 0)$

(d)



Bonus:

(a) Let $f(x) = x^5 + 5x - 5$. Since f is a polynomial then it is continuous everywhere, hence continuous on $[0, 1]$. Also, since $f(0) = -5$ and $f(1) = 1$ then $f(0) < 0 < f(1)$. Therefore by IVT, there exist $a \in [0, 1]$ such that $f(a) = 0$.

We proof by contradiction and assume otherwise. Suppose there exist at least two solutions, say, a and $b \in [0, 1]$ such that $f(a) = f(b) = 0$ where $a < b$. Since f is continuous $[a, b]$ and differentiable on (a, b), the Mean Value Theorem implies that there exist a $c \in [a, b]$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a} = 0$$

However, this is impossible since $f'(x) = 5x^4 + 5 > 0$ for all $x \in \mathbb{R}$. **Contradiction!**

Therefore, $x^5 + 5x = 5$ has only one solution between 0 and 1.