Concordia University
Department of Electrical and Computer Engineering
ENGR 233: Applied Advanced Calculus

Fall 2007 **Mid-Term Exam Solution** **Time: 75 Minutes**

Question 1: Express the vector $\mathbf{x}$ in terms of the vectors $\mathbf{a}$ and $\mathbf{b}$ shown in the following figures.

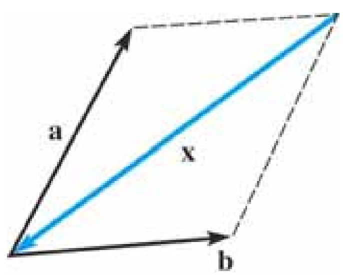

Figure 1(a)

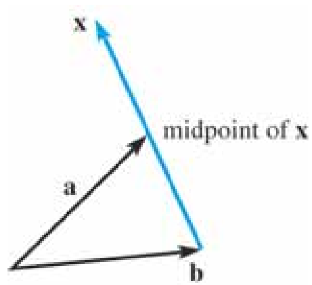

Figure 1(b)

Solution:

In Figure 1(a): $\mathbf{x} = -(\mathbf{a}+\mathbf{b}) = -\mathbf{a}-\mathbf{b}$

In Figure 1(b): $\mathbf{x} = 2(\mathbf{a}-\mathbf{b}) = 2\mathbf{a}-2\mathbf{b}$

Question 2: Find a vector $\mathbf{v} = \langle x,\ y,\ 1\rangle$ that is orthogonal to both $\mathbf{a} = \langle 3,\ 1,\ -1\rangle$ and $\mathbf{b} = \langle -3,\ 2,\ 2\rangle$.

Solution:

The equations of system are:

$3x + y - 1 = 0 \quad \cdots\cdots\cdots\cdots\cdots(1)$

$-3x + 2y + 2 = 0 \quad \cdots\cdots\cdots\cdots(2)$

Solving equations (1) and (2), we have $y = -\frac{1}{3}$ and $x = \frac{4}{9}$

Hence, the vector is $\mathbf{v} = \left\langle \frac{4}{9},\ -\frac{1}{3},\ 1\right\rangle$

Question 3: Find the volume of a parallelepiped for which the vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ given below are three edges.

$\mathbf{a} = 3\mathbf{i}+\mathbf{j}+\mathbf{k} \quad \mathbf{b} = \mathbf{i}+4\mathbf{j}+\mathbf{k}$ and $\mathbf{c} = \mathbf{i}+\mathbf{j}+5\mathbf{k}$

Solution:

We have, $\mathbf{b}\times\mathbf{c}=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ 1 & 4 & 1\\ 1 & 1 & 5\end{vmatrix}=\begin{vmatrix}4 & 1\\ 1 & 5\end{vmatrix}\cdot\mathbf{i}-\begin{vmatrix}1 & 1\\ 1 & 5\end{vmatrix}\cdot\mathbf{j}+\begin{vmatrix}1 & 4\\ 1 & 1\end{vmatrix}\cdot\mathbf{k}=19\mathbf{i}-4\mathbf{j}-3\mathbf{k}$

Hence, the volume of the parallelepiped is

$|\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})|=|(3\mathbf{i}+\mathbf{j}+\mathbf{k})\cdot(19\mathbf{i}-4\mathbf{j}-3\mathbf{k})|=|57-4-3|=50$ cu. units.

Question 4: Find the parametric and symmetric equations for the line through the point $P(4,\ 6,\ -7)$ and parallel to the vector $\mathbf{a}=\langle 3,\ \frac{1}{2},\ -\frac{3}{2}\rangle$

Solution:

The parametric equation of the line is
$x=4+3t$, $y=6+\frac{1}{2}t$, $z=-7-\frac{3}{2}t$

The symmetric equation of the line is
$$\frac{x-4}{3}=\frac{y-6}{\frac{1}{2}}=\frac{z+7}{-\frac{3}{2}}$$

Question 5: Express the vector equation of a circle $\mathbf{r}(t)=a\cos t\,\mathbf{i}+a\sin t\,\mathbf{j}$ as a function of arch length s. Verify that $\mathbf{r}'(s)$ is a unit vector.

Solution:

We have $\mathbf{r}'(t)=-a\sin t\,\mathbf{i}+a\cos t\,\mathbf{j}$ $\quad\therefore \|\mathbf{r}'(t)\|=\sqrt{a^2\sin^2 t+a^2\cos^2 t}=a,\quad a>0$

The length of the arch is $s=\int_0^t\|\mathbf{r}'(t)\|du=\int_0^t a\,du=at$

Thus, the vector function in terms of the arch length s is $\mathbf{r}(s)=a\cos\left(\frac{s}{a}\right)\mathbf{i}+a\sin\left(\frac{s}{a}\right)\mathbf{j}$

$$\therefore\quad \mathbf{r}'(s)=-\sin\left(\frac{s}{a}\right)\mathbf{i}+\cos\left(\frac{s}{a}\right)\mathbf{j}$$

Since $\|\mathbf{r}'(s)\|=\sqrt{\sin^2\left(\frac{s}{a}\right)+\cos^2\left(\frac{s}{a}\right)}=1$, $\mathbf{r}'(s)$ is a unit vector.

Question 6: Let $\mathbf{r}(t)=t^2\,\mathbf{i}+(t^3-2t)\mathbf{j}+(t^2-5t)\mathbf{k}$ be the position vector of a moving particle. At what points does the particle pass through the xy-plane? What are the velocities and accelerations at these points?

Solution:

The particle passes through xy-plane, i.e., when $z = 0$

Thus, at these points $t^2 - 5t = 0 \quad \therefore t = 0\,,5$; i.e., the particle pass through the xy-plane when the points are $(0\,,0\,,0)$ and $(25\,,115\,,0)$

The velocity of the particle is $\mathbf{v}(t) = \mathbf{r}'(t) = 2t\,\mathbf{i} + (3t^2 - 2)\,\mathbf{j} + (2t - 5)\,\mathbf{k}$

Thus, the velocities of the two pints are $\mathbf{v}(0) = -2\,\mathbf{j} - 5\,\mathbf{k}$ and $\mathbf{v}(5) = 10\mathbf{i} + 73\mathbf{j} + 5\,\mathbf{k}$

The acceleration of the particle is $\mathbf{a}(t) = \mathbf{v}'(t) = 2\,\mathbf{i} + 6t\,\mathbf{j} + 2\,\mathbf{k}$

Thus, the accelerations of the two pints are $\mathbf{a}(0) = 2\mathbf{i} + 2\,\mathbf{k}$ and $\mathbf{a}(5) = 2\mathbf{i} + 30\mathbf{j} + 2\,\mathbf{k}$

Question 7: Find the curvature of an elliptical orbit that is described by $\mathbf{r}(t) = a\cos t\,\mathbf{i} + b\sin t\,\mathbf{j} + c\,\mathbf{k}$; $a > 0,\quad b > 0,\quad c > 0$.

Solution:

The velocity is $\mathbf{v}(t) = \mathbf{r}'(t) = -a\sin t\mathbf{i} + b\cos t\mathbf{j} \quad \therefore \|\mathbf{v}\| = \sqrt{a^2\sin^2 t + b^2\cos^2 t}$

The acceleration is $\mathbf{a}(t) = \mathbf{r}''(t) = -a\cos t\mathbf{i} - b\sin t\mathbf{j}$

Since $\mathbf{v}\times\mathbf{a} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -a\sin t & b\cos t & 0 \\ -a\cos t & -b\sin t & 0 \end{vmatrix} = ab\mathbf{k} \quad \therefore \|\mathbf{v}\times\mathbf{a}\| = ab$

The curvature is $\kappa = \dfrac{\|\mathbf{v}\times\mathbf{a}\|}{\|\mathbf{v}\|^3} = \dfrac{ab}{(a^2\sin^2 t + b^2\cos^2 t)^{3/2}}$

Question 8: Given that $z = u^2\cos 4v$, $u = x^2y^3$, $v = x^3 + y^3$, find $\dfrac{\partial z}{\partial x}$ and $\dfrac{\partial z}{\partial y}$ using the Chain Rule.

Solution:

We have,

$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial x} = (2u\cos 4v)(2xy^3) - (4u^2\sin 4v)(3x^2) = 4xy^3u\cos 4v - 12x^2u^2\sin 4v$$

$$\frac{\partial z}{\partial y} = \frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial y} + \frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial y} = (2u\cos 4v)(3x^2y^2) - (4u^2\sin 4v)(3y^2) = 6x^2y^2u\cos 4v - 12y^2u^2\sin 4v$$

Question 9: Find the directional derivatives of the function

$$F(x,\ y,\ z) = 2x - y^2 + z^2$$

at the point $P(4,\ -4,\ 2)$ in the direction of $\overrightarrow{PO}$, where O is the origin of the 3-D space.

Solution:

We have $\nabla F(x, y, z) = 2\mathbf{i} - 2y\mathbf{j} + 2z\mathbf{k}$ $\quad \therefore \nabla F(4, -4, 2) = 2\mathbf{i} + 8\mathbf{j} + 4\mathbf{k}$

The given vector $\overrightarrow{PO} = -4\mathbf{i} + 4\mathbf{j} - 2\mathbf{k}$ $\quad \therefore \left\|\overrightarrow{PO}\right\| = \sqrt{4^2 + 4^2 + 2^2} = \sqrt{36} = 6$

The unit vector in the given direction is, $\mathbf{u} = -\frac{2}{3}\mathbf{i} + \frac{2}{3}\mathbf{j} - \frac{1}{3}\mathbf{k}$

$\therefore$ The directional derivative is $D_{\mathbf{u}}F(4, -4, 2) = -\frac{4}{3} + \frac{16}{3} - \frac{4}{3} = \frac{8}{3}$

Question 10: Find the equation of the tangent plane to the graph of the equation $xy + yz + zx = 7$ at the point $P(1,\ -3,\ -5)$.

Solution:

Let the equation of the graph is $F(x, y, z) = xy + yz + zx - 7$
Hence, the derivative $\nabla F(x, y, z) = (y + z)\mathbf{i} + (z + x)\mathbf{j} + (x + y)\mathbf{k}$
At the given point $P(1,\ -3,\ -5)$ the derivative is $\nabla F(1, -3, -5) = -8\mathbf{i} - 4\mathbf{j} - 2\mathbf{k}$

Hence, at the given point the equation of the tangent plane is
$-8(x-1) - 4(y+3) - 2(z+5) = 0$

$\Rightarrow -8x - 4y - 2z = 14$

$\Rightarrow 4x + 2y + z = -7$