Concordia University

Department of Computer Science & Software Engineering

COMP 232/4 Introduction to Discrete Mathematics Winter 2019

Assignment 4

Due date: Monday, April 1st, 2019

1. Find a formula for
$$\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\ldots+\frac{1}{n(n+1)}$$
by examining the values of this expression for small values of n. Use mathematical induction to prove your result.

Answer:

n	$f(n)$
1	$\frac{1}{2}$
2	$\frac{1}{2}+\frac{1}{6}=\frac{4}{6}=\frac{2}{3}$
3	$f(2)+\frac{1}{12}=\frac{9}{12}=\frac{3}{4}$
4	$f(3)+\frac{1}{20}=\frac{16}{20}=\frac{4}{5}$

Proof by induction:

Base case:
$$n=1: \frac{1}{2}=\frac{1}{2}$$

Inductive Hypothesis:
$$P(n): \frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\ldots+\frac{1}{n(n+1)}=\frac{n}{n+1}$$

Inductive Step:
$$P(n+1): \frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\ldots+\frac{1}{n(n+1)}+\frac{1}{(n+1)(n+2)}=\frac{n+1}{n+2}$$

LHS $=\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\ldots+\frac{1}{n(n+1)}+\frac{1}{(n+1)(n+2)}$

$=\frac{n}{n+1}+\frac{1}{(n+1)(n+2)}$ by inductive hypothesis

$=\frac{n(n+2)+1}{(n+1)(n+2)}=\frac{n^2+2n+1}{(n+1)(n+2)}=\frac{(n+1)^2}{(n+1)(n+2)}=\frac{n+1}{n+2}=$ RHS

2. Show that

$$1^3 + 2^3 + \ldots + n^3 = \left(\frac{n(n+1)}{2}\right)^2$$

whenever n is a positive integer.

Answer: Proof by induction

Base case: $n = 1$: $1^3 = 1^2$

Inductive Hypothesis:

$$1^3 + 2^3 + \ldots + n^3 = \left(\frac{n(n+1)}{2}\right)^2$$

Inductive Step:

$$1^3 + 2^3 + \ldots + n^3 + (n+1)^3 = \left(\frac{(n+1)(n+2)}{2}\right)^2$$

LHS $= 1^3 + 2^3 + \ldots + n^3 + (n+1)^3 = \left(\frac{n(n+1)}{2}\right)^2 + (n+1)^3$ by inductive hypothesis

$= \frac{(n(n+1))^2 + 4(n+1)^3}{4} = \frac{n^2(n+1)^2 + 4(n+1)(n+1)^2}{4} = \frac{(n^2+4n+4)(n+1)^2}{4} = \frac{(n+2)^2(n+1)^2}{4} = \left(\frac{(n+1)(n+2)}{2}\right)^2$

3. Use mathematical induction to show that 3 divides $n^3 + 2n$ whenever n is a nonnegative integer.

Answer:

Base Case: $n = 1$: $3|1 + 2$

Inductive Hypothesis: $P(n) : n^3 + 2n = 3k_1$ for some integer k_1.

Inductive Step: $P(n+1)$: $(n+1)^3 + 2(n+1) = 3k_2$ for some integer k_2.

LHS $= (n+1)^3 + 2(n+1) = n^3 + 3n^2 + 3n + 1 + 2n + 2 = (n^3 + 2n) + (3n^2 + 3n + 3)$

$= 3k_1 + 3(n^2 + n + 1)$ by inductive hypothesis

thus $3|n^2 + n + k_1 + 1$

Since $n^2 + n + k_1 + 1 \in \mathbb{Z}$, let $k_2 = n^2 + n + k_1 + 1$.

Thus $3|(n+1)^3 + 2(n+1)$ if $3|n^3 + 2n$.

4. Show that n lines separate the plane into $(n^2 + n + 2)/2$ regions if no two of these lines are parallel and no three pass through a common point.

Answer: Proof by induction

Base Case: $n = 1$: one line divides the plane into two regions. $(1^2 + 1 + 2)/2 = 4/2 = 2$

Inductive hypothesis: $P(n)$: n lines separate the plane into $(n^2+n+2)/2$ regions

Inductive step: $P(n+1)$: $n+1$ lines separate the plane into $((n+1)^2+(n+1)+2)/2$ regions

$((n+1)^2+(n+1)+2)/2 = (n^2+2n+1+n+1+2)/2 = ((n^2+n+2)+(2n+2))/2$

$= \frac{n^2+n+2}{2} + \frac{2n+2}{2} = \frac{n^2+n+2}{2} + (n+1)$

By inductive hypothesis, n lines divide the plane into $(n^2+n+2)/2$ regions.
Because line $n+1$ is neither parallel to any of the other n lines, nor are more than two lines allowed to intersect in this infinite plane, line $n+1$ intersects all other n lines. Call one direction of line $n+1$ north, the other south. Sweeping down line $n+1$ from north to south, the line segment north of the first intersection divides that subplane into two, and from then on, the line segment after each intersection divides each encountered subplane into two, ending with the infinite subplane south of the nth intersection. The line $n+1$ thus adds $n+1$ regions, as shown above.

5. Show by strong induction that any integer $n \geq 6$ can be written as $3a+4b$ for some non-negative integers a and b.

   Answer: Proof by strong induction

   Base case: $P(6)$: $6 = 3\cdot 2 + 4\cdot 0$
   $P(7)$: $7 = 3\cdot 1 + 4\cdot 1$
   $P(8)$: $8 = 3\cdot 0 + 4\cdot 2$

   Inductive hypothesis: $P(n-2) \wedge P(n-1) \wedge P(n)$: $n-2 = 3a_1 + 4b_1 \wedge n-1 = 3a_2 + 4b_2 \wedge n = 3a+4b$

   Inductive step: $P(n+1)$: $n+1 = 3i+4j$ for integers i, j

   Since $n+1 = (n-2)+3$ and by inductive hypothesis, we get $n+1 = 3a_1 + 4b_1 + 3 = 3(a_1+1) + 4b_1$. Since $a_1 + 1 \in \mathbb{Z}$, $a_1+1 = i$ and $b_1 = j$.

6. Use strong induction to show that every positive integer can be written as a sum of distinct powers of two.

   Hint: In the inductive step, consider separately the cases when $n+1$ is even and $n+1$ is odd.

   Answer: Proof by strong induction

   Intuition: This clearly has to be true when you consider binary notation for the positive integers. Remember, that the binary encoding of powers of two has $\lfloor \log_2 k \rfloor$ digits, all zeroes except the first. Adding one to an odd integer requires to carry: $111 + 1 =_2 1000$

   Base case: $P(1)$: $1 = \sum_{i=0}^{0} a_{1i} \cdot 2^i = 1 \cdot 2^0$. $P(2)$: $2 = \sum_{i=0}^{1} a_{2i} \cdot 2^i = 0 \cdot 2^0 + 1 \cdot 2^1 = 2^1$ for $a_2 i \in \{0,1\}$

   Inductive hypothesis: Assume $P(n)$: $k = \sum_{i=0}^{\lfloor \log_2 k \rfloor} a_{ki} \cdot 2^i$ is true for all $k \leq n, a_{ki} \in \{0,1\}$

Inductive step: $P(n+1)$: $n+1 = \sum_{i=0}^{\lfloor \log_2 (n+1) \rfloor} a_{(n+1)i} \cdot 2^i$

Proof by cases:

Case n is even : then $\lfloor \log_2 (n+1) \rfloor = \lfloor \log_2 n \rfloor$.

By inductive hypothesis, $n = \sum_{i=0}^{\lfloor \log_2 n \rfloor} a_{ni} 2^i = \sum_{i=0}^{\lfloor \log_2 (n+1) \rfloor} a_{ni} \cdot 2^i$, and $a_{n0} = 0$.

Thus by inductive hypothesis, $n+1 = 1 + \sum_{i=0}^{\lfloor \log_2 n \rfloor} a_{ni} 2^i = \sum_{i=0}^{\lfloor \log_2 (n+1) \rfloor} a_{(n+1)i} \cdot 2^i$ with $a_{(n+1)0} = 1$ and $a_{(n+1)t} = a_{nt}$ for $t = 1, \ldots, \lfloor \log_2 (n+1) \rfloor$.

Case n is odd and $(n+1)$ is a power of two: then adding 1 means turning all $\lfloor \log_2 n \rfloor$ successive ones to zero and adding a leading one in position $\lfloor \log_2 n + 1 \rfloor$.

$n = \sum_{i=0}^{\lfloor \log_2 n \rfloor} a_{ni} 2^i$ and $a_{ni} = 1$ for all $i = 0, \ldots, \lfloor \log_2 n \rfloor$.

Thus again, by hypothesis, $n+1 = 1 + \sum_{i=0}^{\lfloor \log_2 n \rfloor} a_{ni} 2^i = \sum_{i=0}^{\lfloor \log_2 (n+1) \rfloor} a_{(n+1)i} \cdot 2^i$, where

$a_{(n+1)\lfloor \log_2 (n+1) \rfloor} = 1$ and $a_{(n+1)s} = 0$, for $s = 0, \ldots, \lfloor \log_2 n \rfloor$.

Case n is odd and $(n+1)$ is not a power of two: then adding 1 means turning all p successive trailing ones to zero and turning to one the rightmost zero in position $p+1$.

$n = \sum_{i=0}^{\lfloor \log_2 n \rfloor} a_{ni} 2^i$ and $a_{ni} = 1$ for $i = 0, \ldots, p$ for integer p with $0 \le p \le \lfloor \log_2 n + 1 \rfloor$.

Thus again, by hypothesis, $n+1 = 1 + \sum_{i=0}^{\lfloor \log_2 n \rfloor} a_{ni} 2^i = \sum_{i=0}^{\lfloor \log_2 (n+1) \rfloor} a_{(n+1)i} \cdot 2^i$, where

$a_{(n+1)i} = 0$ for $i = 0, \ldots, p$, $a_{(n+1)(p+1)} = 1$ and $a_{(n+1)t} = a_{nt}$ for $t = (p+2), \ldots, \lfloor \log_2 (n+1) \rfloor$.

Case n is odd and $(n+1)$ is a power of two: then adding 1 means turning all $\lfloor \log_2 n \rfloor$ successive ones to zero and adding a leading one in position $\lfloor \log_2 n + 1 \rfloor$.

$n = \sum_{i=0}^{\lfloor \log_2 n \rfloor} a_{ni} 2^i$ and $a_{ni} = 1$ for all $i = 0, \ldots, \lfloor \log_2 n \rfloor$.

Thus again, by hypothesis, $n+1 = 1 + \sum_{i=0}^{\lfloor \log_2 n \rfloor} a_{ni} 2^i = \sum_{i=0}^{\lfloor \log_2 (n+1) \rfloor} a_{(n+1)i} \cdot 2^i$, where

$a_{(n+1)\lfloor \log_2 (n+1) \rfloor} = 1$ and $a_{(n+1)s} = 0$, for $s = 0, \ldots, \lfloor \log_2 n \rfloor$.

7. The Fibonacci numbers are defined as follows: $f_1 = 1$, $f_2 = 1$, and $f_{n+2} = f_n + f_{n+1}$ whenever $n \ge 1$.

   (a) Characterize the set of integers n for which f_n is even and prove your answer using induction.

   Answer:

   We observe that $f_1 = 1, f_2 = 1, f_3 = 2, f_4 = 3, f_5 = 5, f_6 = 8, f_7 = 13, f_8 = 21, f_9 = 34, \ldots$

We hypothesize that $f_n = 2i$ for integers n, i, where $3|n$

Proof by strong induction:

Base case: $f_1 = 1, f_2 = 1, f_3 = 2$

Inductive hypothesis: P(n): $f_k = 2i$ for integers $k \leq n, i$, where $3|k$ and $f_k = 2i + 1$ for integers $k \leq n, i$, where $3 \not| k$

Inductive step: P(n+1): $f_{n+1} = 2i$ for integers n, i, where $3|(n+1)$ and $f_{n+1} = 2i+1$ for integers n, i, where $3 \not|(n+1)$.

$f_{n+1} = f_n + f_{n-1}$ by definition. We distinguish three cases:

Case 1 Both, f_n and f_{n-1} are odd and $3 \not| n$, $3 \not|(n-1)$ by inductive hypothesis.

Then their sum is even and also $3|\,(n+1)$.

Case 2 Only one of f_n and f_{n-1} are odd and $(3|n) \oplus (3|(n-1))$. Then their sum cannot be even and $3 \not|(n+1)$.

(b) Use induction to prove that $\sum_{i=1}^{n} i f_i = n f_{n+2} - f_{n+3} + 2$ for all $n \geq 1$.

Proof by strong induction:

Base case:

$P(1)$: LHS = 1, and RHS = $1 \cdot 2 - 3 + 2 = 1$

Inductive hypothesis: $P(n)$: $\sum_{i=1}^{k} i f_i = k f_{k+2} - f_{k+3} + 2$ for $1 \leq k \leq n$

Inductive step: $P(n+1)$: $P(n+1)$: $\sum_{i=1}^{(n+1)} i f_i = (n+1) f_{n+3} - f_{n+4} + 2$

LHS $= \sum_{i=1}^{(n+1)} i f_i = (n+1) f_{(n+1)} + \sum_{i=1}^{n} i f_i = (n+1) f_{(n+1)} + n f_{n+2} - f_{n+3} + 2$ by inductive hypothesis.

$(n+1) f_{(n+1)} + n f_{n+2} - f_{n+3} + 2 = f_{(n+1)} + n f_{(n+1)} + n f_{n+2} - f_{n+3} + 2$

$= f_{(n+1)} + n(f_{(n+1)} + f_{n+2}) - f_{n+3} + 2 = f_{(n+1)} + n f_{(n+3)} - f_{(n+3)} + 2$ by definition of Fibonnacci

$= f_{(n+1)} + n f_{(n+3)} - f_{(n+3)} + 2 = f_{(n+1)} + (n-1) f_{(n+3)} + 2$

RHS $= (n+1) f_{n+3} - f_{n+4} + 2 = (n+1) f_{n+3} - [f_{n+2} + f_{n+3}] + 2$ by def. Fibonnacci

$= (n+1) f_{n+3} - f_{n+2} - f_{n+3}] + 2 = (n-1) f_{n+3} + f_{(n+3)} - f_{n+2} + 2$

$= (n-1) f_{n+3} + [f_{(n+1)} + f_{(n+2)}] - f_{n+2} + 2 = (n-1) f_{n+3} + f_{(n+1)} + 2 = LHS$