CONCORDIA UNIVERSITY

Department of Mathematics & Statistics

Course	Number	Section(s)
Mathematics	204	All
Examination	**Date**	**Pages**
Final	April 2009	3
Instructors		**Course Examiner**
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Special Instructions

- ▷ Ruled booklets to be used.
- ▷ **Only approved calculators are allowed.**

Answer ten questions. All questions have equal value.

1. Using the Gauss-Jordan method (i.e. reduced row echelon form method), find all the solutions of the following system of equations

$$\begin{aligned} 2x - 3y - 8z \quad &= 7 \\ -2x \quad + 2z - w &= -7 \\ x - y - 3z + w &= 6. \end{aligned}$$

2. Let $M = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 1 \\ 0 & 2 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 3 \\ 1 & 1 \\ 2 & 0 \end{bmatrix}$.

a) Calculate M^{-1}.

b) Find the matrix C such that $MC = B$.

3. a) Use Cramer's rule to solve the following system of equations

$$\begin{aligned} x + y \quad &= 1 \\ y + z &= 2 \\ x \quad + 2z &= -1. \end{aligned}$$

(No marks given if you don't use Cramer's rule.)

b) Calculate the determinant of the matrix $\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ -2 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}$.

4. Write the vector $(-7, 11)$ as a sum of two vectors $\vec{v}$ and $\vec{w}$ such that $\vec{v}$ is parallel to the vector $(10, -6)$ and $\vec{w}$ is orthogonal to $\vec{v}$.

5. Let $O = (0,0,0)$, $P = (1,2,-2)$, $Q = (0,2,-1)$ and $R = (1,0,1)$.

   a) Find the area of the triangle with vertices P, Q and R.

   b) Find the volume of the parallelepiped determined by the vectors $\overrightarrow{OP}$, $\overrightarrow{OQ}$, $\overrightarrow{OR}$.

6. Find the equation of the plane that contains the lines $(x, y, z) = (1, 2, -2) + t(1, 1, 0)$ and $(x, y, z) = (1, 2, -2) + t(0, 1, 2)$.

7. Let $\vec{v}_1 = (1, 0, 1)$, $\vec{v}_2 = (0, 1, 1)$ and $\vec{v}_3 = (-2, 3, 1)$.

   a) Show that the vectors $\vec{v}_1$, $\vec{v}_2$ and $\vec{v}_3$ are linearly dependent.

   b) Describe, geometrically, the subspace of $\mathbb{R}^3$ spanned by $\vec{v}_1$, $\vec{v}_2$ and $\vec{v}_3$.

   c) Find a vector $\vec{w}$ such that $\vec{v}_1$, $\vec{v}_2$ and $\vec{w}$ are linearly independent.

8. a) Give an example of a 3×3 homogeneous system of linear equations $AX = 0$ which has infinitely many solutions.

   b) Let

$$A = \begin{bmatrix} 1 & 0 & -3 & 0 & 1 & 3 \\ 0 & 1 & -2 & 0 & 2 & 4 \\ 0 & 0 & 0 & 1 & 5 & 2 \end{bmatrix} \text{ and } X = \begin{bmatrix} x \\ y \\ z \\ u \\ v \\ w \end{bmatrix}.$$

   Find a basis for the solution space of the homogeneous system of linear equations $AX = 0$.

9. Find the standard matrix for the composition of the following two linear operators on $\mathbb{R}^2$: A reflection about the line $y = x$, followed by a rotation counterclockwise of 45^o.

10. Let $A = \begin{bmatrix} 2 & 2 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$. Find an invertible matrix P and a diagonal matrix D such that $P^{-1}AP = D$.

11. For $n \geq 0$, let $X_n = \begin{bmatrix} a_n \\ b_n \\ c_n \end{bmatrix}$ where a_n, b_n and c_n are real numbers. Let

$M = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ 0 & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & 1 \end{bmatrix}$. Suppose that $X_n = MX_{n-1}$ for $n > 0$.

a) Calculate M^n for $n \geq 1$.

b) Write down the entries a_n, b_n, c_n of X_n in terms of a_0, b_0, c_0 and n.

c) Suppose that $X_0 = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$. What happens to a_n, b_n and c_n as n gets large?

(Hint: we have $P^{-1}MP = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{2} \end{bmatrix}$, with $P = \begin{bmatrix} 1 & 1 & 0 \\ 0 & -1 & -1 \\ 1 & 1 & 1 \end{bmatrix}$ and $P^{-1} = \begin{bmatrix} 0 & 1 & 1 \\ 1 & -1 & -1 \\ -1 & 0 & 1 \end{bmatrix}$.)