Instructions for the Examination

- Answer all 4 Questions.
- Questions have equal weight.
- Show all the intermediate steps of your solution.
- Make reasonable assumptions if necessary.
- You can only write on the provided space.
- You can not separate the sheets.
- Standard type calculator can be used.
- The exam is closed book.
- Enter your name and ID number on both sheets.
- Available time is 70 minutes

Question 1:

A bin of 40 parts contains 5 that are defective. A sample of three is selected at random without replacement.

a) What is the number of the elements of the sample space of this experiment?
b) Let's consider an event such that all three selected parts are defective. Determine the number of elements of this event.
c) What is the probability that only two of these three selected parts are defective?

Solution 1:

Part a: (1.5 marks)

Elements of this sample space have 3 parts. These parts can be defective "*d*" or not defective "*n*". Therefore the number of the elements of this sample space is $2^3 = 8$ elements. Elements are {*ddd, ddn, dnd, ndd, nnd, ndn, dnn, nnn*}.The number of possible ways that 3 parts can be selected at random without replacement are: $\binom{40}{3} = \frac{40!}{3!\ \ 37!} = 9880$.

Part b: (1.5 marks)

This event includes all the elements which have all three parts defective and therefore we have only $1^3 = 1$ element in this event which is {*ddd*}. The number of possible ways that 3 defective parts can be selected at random without replacement are: $\binom{5}{3} = \frac{5!}{3!\ \ 2!} = 10$

Part c: (2.0 marks)

When only two elements are defective, we are concerned with 3 elements of the sample space in part a which are {*ddn, dnd, ndd*}. We have:

$$P(ddn \cup dnd \cup ndd) = P(ddn) + P(dnd) + P(ndd)$$

$$= \frac{5}{40}\times\frac{4}{39}\times\frac{35}{38} + \frac{5}{40}\times\frac{35}{39}\times\frac{4}{38} + \frac{35}{40}\times\frac{5}{39}\times\frac{4}{38} = \frac{35}{988} = 0.0354$$

Another way to solve is: $P(2 \text{ defectives}) = \frac{\binom{5}{2}\binom{35}{1}}{\binom{40}{3}} = \frac{\frac{5!}{2!\ \ 3!}\times\frac{35!}{1!\ \ 34!}}{\frac{40!}{3!\ \ 37!}} = \frac{10\times 35}{9880} = \frac{35}{9880} = 0.0354$

Question 2

A computer system uses passwords with 6 characters constructed from 26 letters (a-z) and 10 integers (0-9). Let's define following categories of passwords:

A: Passwords having right most character equal to an odd number.
B: Passwords having only letters as their 6 characters.
C: Passwords having first letter as a vowel (*a*, *e*, *i*, *o*, and *u*).

Suppose a hacker selects a password at random. Determine the following:

a) P(A), P(B), P(C).
b) $P(A\cap B)$, $P(A\cap C)$, $P(B\cap C)$.
c) Find out if any two of these events are independent from each other.
d) Find out if any two of these events are disjoint.

Solution 2:

Part a: (1.5 marks)

Number of 6-character passwords is 36^6.

Number of passwords having the right most character equal to an odd number is $36^5\times 5$, therefore

$$P(A)=\frac{36^5\times 5}{36^6}=\frac{5}{36}=0.1388.$$

Number of passwords having only letters for 6 characters is 26^6, therefore $P(B)=\dfrac{26^6}{36^6}=0.1419$.

Number of passwords having the first character to be a vowel is 5×36^5, therefore

$$P(C)=\frac{5\times 36^5}{36^6}=\frac{5}{36}=0.1388.$$

Part b: (1.5 marks)

There is no 6-character password having 6 letters with the right most character to be an odd number, therefore $P(A\,|\,B)=0$ and $P(A\cap B)=P(A\,|\,B)\times P(B)=0\times P(B)=0$.

Number of passwords having the first character as vowel and the last character an odd number is

$5\times 36^4\times 5$, therefore $P(A\cap C)=\dfrac{5\times 36^4\times 5}{36^6}=\dfrac{25}{1296}=0.0192$.

Number of passwords having all letters and the first letter being vowel is 5×26^5, therefore

$$P(B\cap C)=\frac{5\times 26^5}{36^6}=0.0272.$$

Part c: (1 mark)

$P(A\cap C)=0.0192$ and $P(A)\times P(C)=0.1388\times 0.1388=0.0192$, therefore $P(A\cap C)=P(A)\times P(C)$ and two events A & C are independent. A & B are not independent since $P(A\cap B)\neq P(A)\times P(B)$. B & C are not independent since $P(B\cap C)\neq P(B)\times P(C)$.

Part d: (1 mark)

Since $P(A\cap B)=0$, therefore A and B are disjoint (mutually exclusive).

$P(A\cap C)\neq 0$, therefore A & C are not disjoint.

$P(B\cap C)\neq 0$, therefore B & C are not disjoint.

Question 3:

A product is made by two different manufacturers. Manufacturers M1 and M2 produce 55% and 45% of the product, respectively. It is known from the past experience that 1% and 2% of the products manufactured by each manufacturer, respectively, are defective.

a) If a finished product is randomly selected, what is the probability that it is defective?
b) If a finished product is randomly selected and is found to be defective, what is the probability that the manufacturer M1 has manufactured this product? In this case, which manufacturer was most likely used?

Solution 3:

Part a: (2 marks)

Let "D" be the event that the selected product is defective.
Let "M1" be the event that the selected product is manufactured by manufacturer 1.
Let "M2" be the event that the selected product is manufactured by manufacturer 2.

Using total probability:

$$P(D) = P(D \mid M1)P(M1) + P(D \mid M2)P(M2) = 0.01\times 0.55 + 0.02\times 0.45 = 0.0145$$

Part b: (3 marks)

Using Bayes' theorem:

$$P(M1 \mid D) = \frac{PD \mid M1)\times P(M1)}{P(D)} = \frac{0.01\times 0.55}{0.0145} = \frac{11}{29} = 0.379$$

$$P(M2 \mid D) = 1 - P(M1 \mid D) = 1 - \frac{11}{29} = \frac{18}{29} = 0.621$$

Since $P(M2 \mid D) > P(M1 \mid D)$, therefore most likely the manufacturer 2 has been used.

Question 4:

A dice is biased so that the probability mass function of the face value is as follows:

face value	1	2	3	4	5	6
probability mass function	0.20	0.25	0.25	0.10	0.15	0.05

a) Evaluate the expected value and standard deviation of the face value of the dice, if we roll it once. Discuss the result.
b) If the dice is rolled once what is the probability that the face value is less than 3? What is the probability that the face value is larger than or equal to 2?
c) If a dice is rolled twice, what is the probability mass function of the average value of the faces?

Solution 4:

Part a: (2 marks)

Let the random variable X be the face value in the above table, then $f(x)$ will have the values of the probability mass function as shown in the second row of the table.

$$\mu = E(X) = \sum_{i=1}^{6} x_i f_X(x_i) = 1\times f_X(1) + 2\times f_X(2) + 3\times f_X(3) + 4\times f_X(4) + 5\times f_X(5) + 6\times f_X(6)$$

$$= 1\times 0.20 + 2\times 0.25 + 3\times 0.25 + 4\times 0.10 + 5\times 0.15 + 6\times 0.05 = \mathbf{2.90}$$

$$\sigma^2 = V(X) = E(X^2) - \mu^2 = \left[\sum_{i=1}^{6} x_i^2 f_X(x_i)\right] - \mu^2$$

$$= 1^2\times f_X(1) + 2^2\times f_X(2) + 3^2\times f_X(3) + 4^2\times f_X(4) + 5^2\times f_X(5) + 6^2\times f_X(6) - 2.90^2$$

$$= 1^2\times 0.20 + 2^2\times 0.25 + 3^2\times 0.25 + 4^2\times 0.10 + 5^2\times 0.15 + 6^2\times 0.05 - 2.90^2 = \mathbf{2.19}$$

$$\sigma = \sqrt{2.19} = \mathbf{1.48}$$

If we throw the dice many times we expect the average of the faces to be equal to $\mu = 2.90$ with deviation of $\sigma = 1.48$.

Part b: (1.5 marks)

$$P(X < 3) = P(X = 1) + P(X = 2) = 0.20 + 0.25 = \mathbf{0.45}$$

$$P(X \geq 2) = 1 - P(X = 1) = 1 - 0.20 = \mathbf{0.80}$$

Part c: (1.5 marks)

Let random variable Y be the average value of faces if the dice is rolled twice. Then, $Y = \{1.0,\ 1.5,\ 2.0,\ 2.5,\ 3.0,\ 3.5,\ 4.0,\ 4.5,\ 5.0,\ 5.5,\ 6.0\}$. Now we should find $f_Y(y)$ for different values of y.

$$f_Y(1.0) = f_X(1) f_X(1) = 0.20\times 0.20 = \mathbf{0.04}$$

$$f_Y(1.5) = 2 f_X(1) f_X(2) = 2\times 0.20\times 0.25 = \mathbf{0.10}$$

$$f_Y(2.0) = f_X(2) f_X(2) + 2 f_X(1) f_X(3) = 0.20^2 + 2\times 0.20\times 0.25 = \mathbf{0.1625}$$

$$f_Y(2.5) = 2 f_X(2) f_X(3) + 2 f_X(1) f_X(4) = 2(0.25\times 0.25 + 0.20\times 0.10) = \mathbf{0.165}$$

$$f_Y(3.0) = f_X(3) f_X(3) + 2 f_X(2) f_X(4) + 2 f_X(1) f_X(5) = 0.25^2 + 2(0.25\times 0.10 + 0.20\times 0.15) = \mathbf{0.1725}$$

$$f_Y(3.5) = 2 f_X(3) f_X(4) + 2 f_X(2) f_X(5) + 2 f_X(1) f_X(6) = 2(0.25\times 0.10 + 0.25\times 0.15 + 0.20\times 0.05) = \mathbf{0.145}$$

$$f_Y(4.0) = f_X(4) f_X(4) + 2 f_X(3) f_X(5) + 2 f_X(2) f_X(6) = 0.10^2 + 2(0.25\times 0.15 + 0.25\times 0.05) = \mathbf{0.11}$$

$$f_Y(4.5) = 2 f_X(4) f_X(5) + 2 f_X(3) f_X(6) = 2(0.10\times 0.15 + 0.25\times 0.05) = \mathbf{0.055}$$

$$f_Y(5.0) = f_X(5) f_X(5) + 2 f_X(4) f_X(6) = 0.15^2 + 2\times 0.10\times 0.05 = \mathbf{0.0325}$$

$$f_Y(5.5) = 2 f_X(5) f_X(6) = 2\times 0.15\times 0.05 = \mathbf{0.015}$$

$$f_Y(6.0) = f_X(6) f_X(6) = 0.05^2 = \mathbf{0.0025}$$

Note that $f_Y(y)$ values add up to 1.