ENGR-233 Final exam 2014 answers

1. $7x+12y+z=66$

2. (a) $l=\sqrt{\pi^2+1}$; (b) The mass $m=0$

3. $\int_C (3x^2-4xy)\,dx+(y^3-x^2)\,dy=-25$

4. $V=\frac{14}{3}\pi\left(1-\frac{1}{\sqrt{2}}\right)$

5. $\iint_S y^3 z\,dS = \frac{9}{2}(5\sqrt{5}-1)$

6. $\iint_S \boldsymbol{F}\cdot\boldsymbol{n}\,dS = -16\pi$

7. $\oint_C \boldsymbol{F}\cdot d\,\boldsymbol{r} = \frac{13}{2}$

8. $(\bar{x},\bar{y},\bar{z}) = \left(\frac{32}{15\pi},0,0\right)$

9. $g(x,z)=8x^2\sin z\cos z$

10. $\iint_R (x^2+y^2)(x^2-y^2)^5\,dA=0$

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#1) (1,5,-1) ⊥ line intersects: $-x+y-5z=4$

$2x-y-2z=0$

~~$x-7z=4$~~

~~$z=\frac{-4}{7} \wedge x=4 \wedge y=0$~~

$3(x-1)+5(y+12)-(z+1)=0$

$3x-3+5y+60-z-1=0$

$3x+5y+z+56=0$

$\langle -1, 1, -5\rangle$ and $\langle 2, -1, -2\rangle$

$$\begin{vmatrix} i & j & k \\ -1 & 1 & -5 \\ 2 & -1 & -2 \end{vmatrix}$$

$\langle -2+5, -(2+10), 1-2\rangle$

$\langle 3, -12, -1\rangle$ is ⊥ intersects

#2) $r(t) = \langle \cos(\pi t), t, \sin(\pi t)+5)$ $0 \le t \le 1$

a) length curve $\int \|r'\| dt$ $\int_0^1 \sqrt{\pi^2\sin^2(\pi t)+1+\pi^2\cos^2(\pi t}$

$r'(t) = \langle -\pi\sin(\pi t), 1, \pi\cos(\pi t)\rangle$

$\int_0^1 \sqrt{\pi^2+1}\, dt = \boxed{\sqrt{\pi^2+1}}$

b) mass w/ density $\rho = xz$ $\int_c \rho ds.$

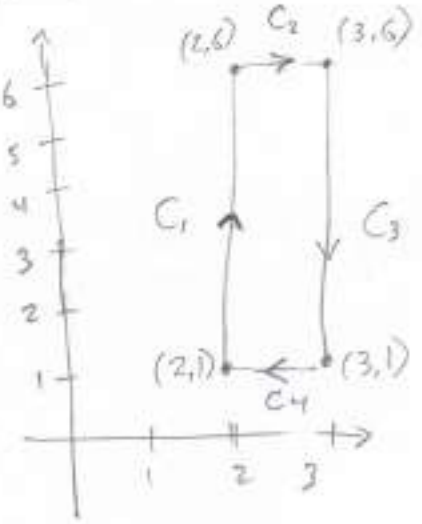
$\int_0^1 xz\, dt = \int_0^1 \cos(\pi t)\cdot(\sin(\pi t)+5)\, dt = \cos^2\sin^2 + 5\cos\pi t\, dt$

$\int_0^1 1+5\cos(\pi t)\, dt = t - 5\pi\sin(\pi t)\Big|_0^1 = 1-5\pi(0) = \boxed{1}$

2

#3) $\oint_C \overbrace{(3x^2-4xy)}^{P}dx + \overbrace{(y^3-x^2)}^{Q}dy$

C is (2,1), (3,1) (3,6) and (2,6) Clockwise



$\oint_C [Pdx+Qdy] = \iint_R [Q_x - P_y]dxdy.$

$[Q_x=-2x - P_y=-4x] = \oint -2x+4x$

$\int_1^6\int_2^3 2x\,dxdy = \int_1^6 x^2\Big|_2^3 dy$

$\int_1^6 9-4dy = 5y\Big|_1^6 = 30-5 = \boxed{25}$ ∴

#4) $z=(x^2+y^2)^{\frac{1}{2}}$ bounded by $z^2=x^2+y^2$
inside↗

$z^2+x^2+y^2=1$

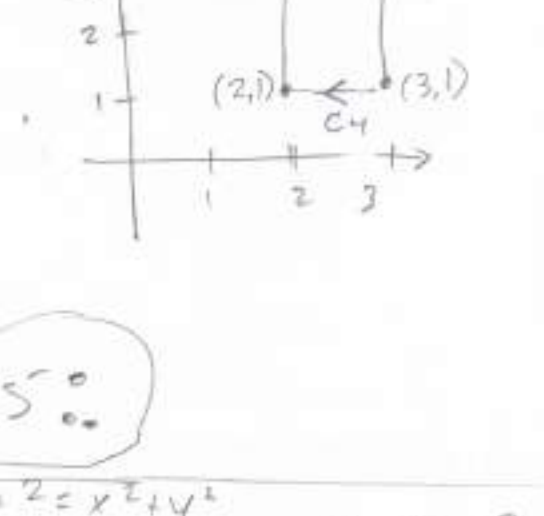
$z^2+x^2+y^2=4$

$\rho: 1 \to 2$

$\theta: 0 \to 2\pi$

$\phi: 0 \to \pi/4$

$\int_0^{2\pi}\int_0^{\pi/4}\int_1^2 \rho^2\sin\phi\, d\rho\, d\phi\, d\theta$



$\int_0^{2\pi}\int_0^{\pi/4} \frac{\rho^3}{3}\Big|_1^2 \sin\phi\, d\phi\, d\theta = \int_0^{2\pi}\int_0^{\pi/4} \frac{7}{3}\sin\phi\, d\phi\, d\theta$

$\frac{7}{3}\int_0^{2\pi} -\cos\phi\Big|_0^{\pi/4} d\theta = -\frac{7}{3}\int_0^{2\pi}\left(\frac{\sqrt{2}}{2}-1\right)d\theta$

$\frac{7}{3}\left(\frac{\sqrt{2}}{2}-1\right)\theta\Big|_0^{2\pi}$ $\boxed{-\frac{14\pi}{3}\left(\frac{\sqrt{2}}{2}-1\right)}$ ∴

#5) $\iint_S y^3 z\,dS$ $x=4+z^2$ bounded by $z=0$ $z=1$ $y=0$ $y=2$ $\underline{3}$

$z=\sqrt{4-x}$

$\int_0^1\int_0^2 y^3 z\sqrt{g_y+g_z}$

$g(x,z)=4-x+z^2$

$g_x=-1$

$g_z=2z$

$\int_0^2\int_0^1 y^3 z\sqrt{1+0+4z^2}\,dz\,dy$

$u=1+4z^2$

$du=8z\,dz$

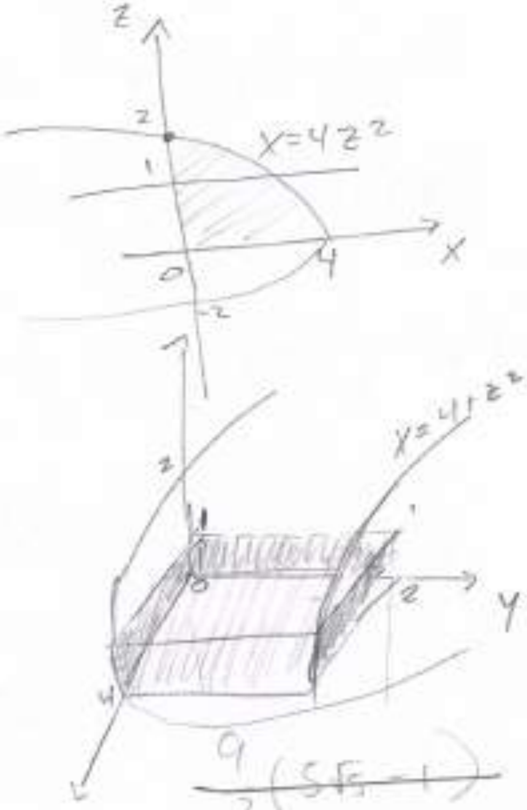
$\frac{du}{8}=z\,dz$

$\int_0^2\int_0^1 y^3\sqrt{u}\,\frac{du}{8}$

$\int_0^2 y^3\frac{2u^{\frac{3}{2}}}{3\cdot 8}\,du=\frac{1}{12}\int_0^2 y^3\left(1+4z^2\right)^{\frac{3}{2}}\Big|_0^1$

$\frac{1}{12}\int_0^2 y^3\left(5\sqrt{5}-1\right)=\frac{5\sqrt{5}-1}{12}\,\frac{y^4}{4}\Big|_0^2$

$\frac{16}{4}\,\frac{5\sqrt{5}-1}{12}=\boxed{\frac{1}{3}\left(5\sqrt{5}-1\right)}$ ∴



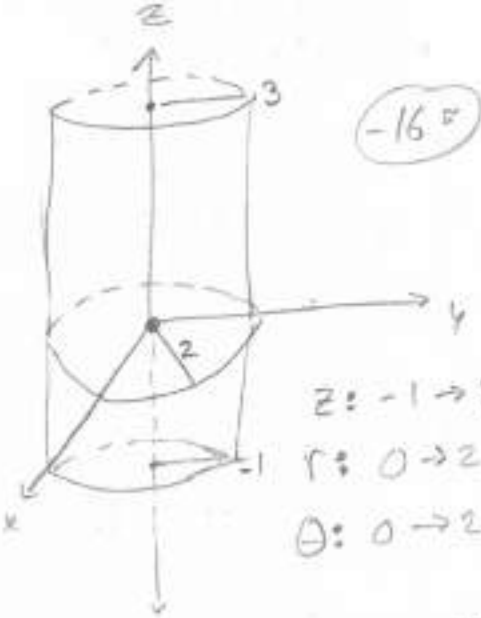
~~$\frac{9}{2}\left(5\sqrt{5}-1\right)$~~

≈ 45.8

#6) Divergence $\oiint_S F \cdot \hat{n}\, dS$ $\mathbb{F} = \langle 3x^2, 2x+y, x^2-z^2 \rangle$ \4

through cylinder $x^2+y^2=4$ $-1 \le z \le 3$

$\oiint_S F \cdot \hat{n}\, dS = \iiint_D \operatorname{div} F\, dV$



$\boxed{-16\pi}$

z: $-1 \to 3$
r: $0 \to 2$
θ: $0 \to 2\pi$

$\operatorname{div} F = (6x + 1 - 2z)$

$\iiint 6x + 1 - 2z\, dV$

$\int_{-1}^{3}\int_{0}^{2\pi}\int_{0}^{2} [6r\cos\theta + 1 - 2z] r\, dr\, d\theta\, dz$

$\int_{-1}^{3}\int_{0}^{2\pi}\int_{0}^{2} [6r^2\cos\theta + r - 2zr]\, dr\, d\theta\, dz = \int_{-1}^{3}\int_{0}^{2\pi} 2r^3\cos\theta + \frac{r^2}{2} - zr^2 \Big|_0^2 d\theta d$

$\int_{-1}^{3}\int_{0}^{2\pi} 16\cos\theta + 2 - 4z\, d\theta\, dz = \int_{-1}^{3} 16\sin\theta + 2\theta - 4z\theta \Big|_0^{2\pi} dz$

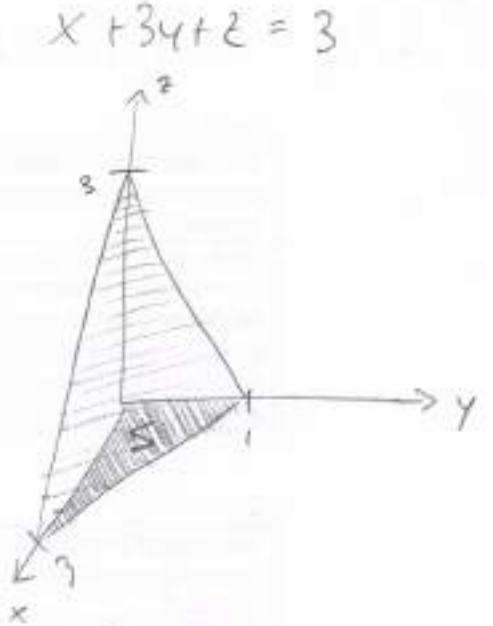
$\int_{-1}^{3} 0 + 4\pi - 8\pi z\, dz = [4z\pi - 4\pi z^2]_{-1}^{3} = 4z\pi - 4\pi z^2 \Big|_{-1}^{3}$

$[12\pi - 36\pi] - [-4\pi - 4\pi] = -24\pi - -8\pi = 16\pi$ ∴

$\underline{Q_7}$ Stokes $\oint F \cdot dr = \iint_S \text{curl} F \cdot n\, dS$

$F = \langle 4z + x,\ -2xy,\ x^2 - y \rangle$ plane $x + 3y + z = 3$

$$\text{CURL} F = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ f_x & f_y & f_z \\ 4z+x & -2xy & x^2-y \end{vmatrix}$$



$\nabla \times F = \langle -1, -2x + 4, -2y \rangle$

Stokes shortcut

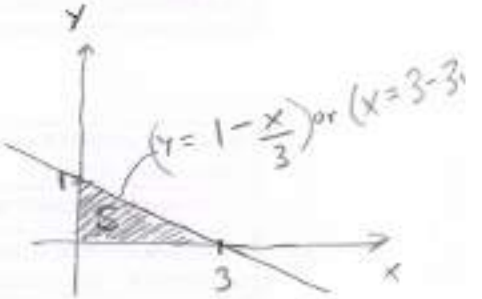
$\iint_S \text{curl} F \cdot \nabla g$

$g(x,y,z) = x + 3y + z - 3$

$\nabla g = \langle 1, 3, 1 \rangle$

$f(x,y) = z = 3 - x - 3y$

$dS = \sqrt{1 + f_x^2 + f_y^2} = \sqrt{1 + 1 + 9}$



$$\iint_S \langle -1, -2x+4, -2y \rangle \cdot \langle 1, 3, 1 \rangle = \iint -1 - 6x + 12 - 2y\ dA$$

$$\int_0^3 \int_0^{1-\frac{x}{3}} 11 - 6x - 2y\ dy\,dx = \int_0^3 11y - 6yx - y^2 \Big|_0^{1-\frac{x}{3}} dx = \int_0^3 11\left(1 - \frac{x}{3}\right) - 6\left(1 - \frac{x}{3}\right) - \left(1 - \frac{x}{3}\right)$$

$$\int_0^3 \cancel{11} - \frac{11x}{3} - \cancel{6} + \frac{6x}{3} - \left(\frac{x^2}{9} - \frac{2x}{3} + 1\right) dx = \int_0^3 5 - \cancel{\frac{5}{3}}x - \frac{x^2}{9} + \cancel{\frac{2x}{3}} - 1\ dx$$

$$\int_0^3 4 - x - \frac{x^2}{9}\, dx = 4x - \frac{x^2}{2} - \frac{x^3}{27} \Big|_0^3 \quad 12 - \frac{9}{2} - 1 = \frac{24 - 9 - 2}{2} = \boxed{\frac{13}{2}}$$

#8) Center mass $\overset{\text{half}}{\text{cylinder}}$ $x^2+y^2 \le 4$ $\rho - x^2$ \6

$x \ge 0$

$-1 \le z \le 1$

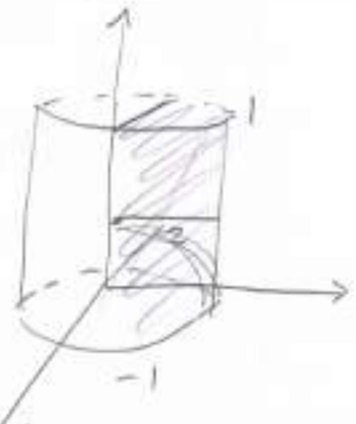
$$\iiint \rho dV = x^2 dV = r^2\cos^2\theta \, r \, dz dr d\theta$$

$$\frac{1}{2}\int_0^{2\pi}\int_0^{2}\int_{-1}^{1} r^3\cos^2\theta \, dz dr d\theta$$

$z: -1 \to 1$

$r: 0 \to 2$

$\theta: 0 \to 2\pi$



$$\cancel{\frac{1}{2}}\int_0^{2\pi}\int_0^{2} \cancel{z}\Big|_{-1}^{1} r^3\cos^2\theta \, dr d\theta = \int_0^{2\pi}\int_0^{2} r^3\cos^2\theta \, dr d\theta$$

$$\int_0^{2\pi} \frac{r^4}{4}\Big|_0^2 \cos^2\theta \, d\theta = \int_0^{2\pi} 4\cos^2\theta \, d\theta = 4\int_0^{2\pi} \frac{1+\cos 2\theta}{2} d\theta$$

$$2\left[\theta + \frac{\sin 2\theta}{2}\right]_0^{2\pi} = 2\theta + \sin 2\theta \Big|_0^{2\pi} = 4\pi + 0 - 0 + \theta = 4\pi$$

$$\tilde{x} = \iiint x\rho dv = x^3 dv = r^3\cos^3\theta = \int_0^{2\pi}\int_0^{2}\int_{-1}^{1} r^{\overset{4}{\cancel{3}}}\cos^3\theta \, r \, dz dr d\theta$$

$$2\int_0^{2\pi}\int_0^{2} r^4\cos^3\theta \, dr d\theta = 2\int_0^{2\pi} \frac{r^5}{5}\Big|_0^2 \cos^3\theta \, d\theta = \frac{32}{5} \cdot 2\int_0^{2\pi} \cos^3\theta d\theta$$

$$\frac{64}{5}\int_0^{2\pi} \cos\theta(1-\sin^2\theta)d\theta \quad \begin{matrix} u = \sin\theta \\ du = \cos\theta d\theta \end{matrix} \quad \frac{64}{5}\int_0^{2\pi} 1-u^2 du = \frac{64}{5}\int_{-}^{2\pi} u - \frac{u^3}{3}$$

$$\frac{64}{5}\left[\sin\theta - \frac{\sin^3\theta}{3}\right]_0^{2\pi} = \frac{64}{5}[0] = 0$$

#8) $\tilde{y} = \int\int\int y x^2 r \, dz dr d\theta$ $\qquad u = \cos\theta$

$\tilde{y} = \int\int\int r \sin\theta \, r^2 \cos^2\theta \, r \, d\theta dz dr$ $\qquad -du = \sin\theta d\theta$

$= r^4 \sin\theta \cos^2\theta \, d\theta dz dr = \int\int\int -u^2 du$

$$= \int_0^2\int_{-1}^1 \frac{-u^3}{3}\Big|_0^{2\pi} r^4 dz dr = \int_0^2\int_{-1}^1 -\frac{(\cos^3\theta)}{3}\Big|_0^{2\pi} = -\frac{1}{3}\int_0^2\int_0^1 \cos^3\theta\Big|_0^2$$

$$-\frac{1}{3} r^4 \int_0^2\int_0^1 0$$

$$\tilde{z} = \int_0^2\int_{-1}^1\int_0^{2\pi} z x^2 r \, d\theta dz dr = z r^2 \cos^2\theta \, r \, d\theta dz dr$$

$$z r^3 \int_0^2\int_{-1}^1\int_0^{2\pi} \cos^2\theta \, d\theta dz dr = z r^3 \int^2\int_{-1}^1\int_0^{2\pi} \frac{1+\cos 2\theta}{2} d\theta dz dr$$

$$\frac{z r^3}{2}\int_0^2\int_0^1 \theta + \frac{\sin 2\theta}{2}\Big|_0^{2\pi} = \int_0^2\int_{-1}^1 \frac{z r^3}{3}(2\pi) dz dr$$

$$\frac{r^3}{3}\int_0^2 \frac{z^2}{2}\Big|_{-1}^1 dr = \frac{1}{2} - \frac{1}{2}$$

center is at $\langle 0, 0, 0 \rangle$

#9) $F = \langle 8x\cos y \sin^2 z + 2ze^{2xz}, -4x^2 \underset{\sin^2(z)}{\sin(y)}, g(x,z)\cos y + 2xe^{2x}$

is conservative

$$\varphi_x = 8x\cos y \sin^2 z + 2ze^{2xz} \longmapsto 4x^2\cos y \sin^2 z + e^{2xz} + C(x$$

$$\varphi_y = -4x^2 \sin y \sin^2(z) \longmapsto -4x^2 \cos y \sin^2 z + C_y$$

$$\varphi_z = g(x,y)\cos(y) + 2xe^{2xz} \longmapsto (g(x,y))\cos(y) + e^{2xz} + C(z)$$

$$g(x,z) = (\sin^2 z)(4x^2) \qquad (8x^2 \sin z \cos z)$$

$$\varphi = 4x^2 \sin^2 z \cos(y) + e^{2xz}$$

#10)

Chang variables not on exam

Suck it!