Concordia University

Department of Computer Science & Software Engineering

COMP 232/4 Introduction to Discrete Mathematics Winter 2019

Solutions to Assignment 5

1. For each of the following relations on the set $\mathbb{Z}$ of integers, determine if the relation is reflexive, symmetric, anti-symmetric, or transitive. On the basis of these properties, state whether or not it is an equivalence relation or a partial order.

   (a) $R = \{(a,b) | a^2 = b^2\}$

   Answer: R is reflexive: $(a,a) \in R$ since $a^2 = a^2$ for all $a \in \mathbb{Z}$

   R is symmetric: $(a,b) \in R \Longrightarrow (b,a) \in R$, since $a^2 = b^2 \Longrightarrow b^2 = a^2$ for all $a, b \in \mathbb{Z}$

   R is not antisymmetric, because $(-1,1) \in R \wedge (1,-1) \in R$

   R is transitive: $((a,b) \in R \wedge (b,c) \in R) \Longrightarrow (a,c) \in R$ because if $a^2 = b^2$ and $b^2 = c^2$, then $a^2 = c^2$ for all $a, b, c \in \mathbb{Z}$

   Because R is reflexive, symmetric, and transitive, R is an equivalence relation.
   Because R is not antisymmetric, R is not a partial order.

   (b) $S = \{(a,b) | \ |a-b| \leq 1\}$

   Note that $|a-b| \leq 1 \Longleftrightarrow (a+1=b) \vee (a=b+1) \vee a=b \qquad \star$

   Answer: S is reflexive: $(a,a) \in S$ since $|a-a| \leq 1$ for all $a \in \mathbb{Z}$

   S is symmetric: $(a,b) \in S \Longrightarrow (b,a) \in S$ for all $a, b \in \mathbb{Z}$.
   If $(a,b) \in S$, then one of three cases holds, according to $\star$:

   Case1: $0 < a-b \leq 1 \Longleftrightarrow a = b+1 \Longleftrightarrow b-a = b?(b+1) = 1$ and $(b,a) \in S$

   Case2: $-1 \leq a-b < 0 \Longleftrightarrow a = b-1 \Longleftrightarrow b-a = b = (b-1) = -1$ and $(b,a) \in S$

   Case3: $a-b = 0 \Longleftrightarrow a = b$ and $(b,a) \in S$.

   S is not antisymmetric, because S is symmetric and not the identity relation.
   S is not transitive, because $(1,2) \in S$ and $(2,3) \in S$, but $(1,3) \notin S$.
   Because S is not transitive, it is neither an equivalence relation, nor a partial order.

2. Prove that $\{(x, y)|\ x - y \in \mathbb{Q}\}$ is an equivalence relation on the set of real numbers, where $\mathbb{Q}$ denotes the set of rational numbers.

   Solution:

   Let $S = \{(x, y)|\ x - y \in \mathbb{Q}\}$

   Reflexivity: $x - x = 0 \in \mathbb{Q}$, thus $(x, x) \in S$ for all $x \in \mathbb{R}$
   Symmetry: Let $x - y \in \mathbb{Q}$. Then, $y - x = -(x - y)$ is again a rational number.
   Transitivity: If $x - y \in \mathbb{Q}$ and $y - z \in \mathbb{Q}$, then their sum, namely $(x - y) + (y - z) = x - z$, is also a rational number (as the rational numbers are closed under addition).

   Because S is reflexive, symmetric, and transitive, S is an equivalence relation.

3. Prove or disprove the following statements:

   (a) Let R be a relation on the set $\mathbb{Z}$ of integers such that xRy if and only if $xy \geq 1$. Then, R is irreflexive.

   Answer: R is not irreflexive, as the pair $(1, 1)$ is in the relation.

   (b) Let R be a relation on the set $\mathbb{Z}$ of integers such that xRy if and only if $x = y + 1$ or $x = y - 1$. Then, R is irreflexive.

   Answer: R is irreflexive, as $n \neq n + 1$ and $n \neq n - 1$, for every integer n. Thus, for every integer n, the pair (n, n) is not in the relation.

   (c) Let R and S be reflexive relations on a set A. Then, $R - S$ is irreflexive.

   Answer: $R - S$ is indeed irreflexive. Given that R and S are reflexive, for any element $a \in A$, $(a, a) \in R$ and $(a, a) \in S$. This, in turn, implies that $(a, a) \notin \overline{S}$ and so $(a, a) \notin R \cap \overline{S}$. Now, $R \cap \overline{S} = R - S$. Therefore, $R - S$ is irreflexive.

4. Let R be the relation on $\mathbb{Z}^+$ defined by xRy if and only if $x < y$. Then, in Set Builder Notation, $R = \{(x, y)|\ y - x > 0\}$.

   (a) Use Set Builder Notation to express the transitive closure of R.

   Note that $R = \{(x, y)|\ y - x \geq 1\}$ and $R^2 = \{(x, y)|\ y - x \geq 2\}$ and $R^n = \{(x, y)|\ y - x \geq n\}$.

   Answer: Thus $R^* = \bigcup_{i=1}^{\infty} R^i = R = \{(x, y)|\ y - x > 0\}$.

   (b) Use Set Builder Notation to express the composite relation R^n , where n is a positive integer.
   Answer: See note above, $R^n = \{(x, y)|\ y - x \geq n\}$.

5. Give the transitive closure of the relation $R = \{(a, c), (b, d), (c, a), (d, b), (e, d)\}$ on domain $A = \{a, b, c, d, e\}$.

   Answer:

$$M_R = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} \quad M_{R^2} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{bmatrix} \quad M_{R^3} = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} = M_R$$

$M_{R^4} = M_{R^2}$, $M_{R^5} = M_R$

Thus $M_{R^*} = M_R \cup M_{R^2} = \begin{bmatrix} 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 \end{bmatrix}$ and $R^* = \{(a,a),(a,c),(b,b),(b,d),(c,a),(c,c),(d,b),(d,d),(e,b),(e,d)\}$.

6. Give an example to show that when the symmetric closure of the reflexive closure of the transitive closure of a relation is formed, the result is not necessarily an equivalence relation.

   Answer: Let Q be a relation on $\{a,b,c\}$ with $Q = \{(b,a),(b,c)\}$.
   The transitive closure is $t(Q) = \{(b,a),(b,c)\}$
   The reflexive closure of the transitive closure is $r(t(Q)) = \{(a,a),(b,a),(b,b),(b,c),(c,c)\}$
   The symmetric closure of the reflexive closure of the transitive closure is
   $s(r(t(Q)) = \{(a,a),(a,b),(b,a),(b,b),(b,c),(c,b),(c,c)\}$.

   $(c,b) \in s(r(t(Q))$ and $(b,a) \in s(r(t(Q))$, but $(c,a) \notin s(r(t(Q))$.

7. Show that the symmetric closure of the union of two relations is the union of their symmetric closures.

   Answer: The symmetric closure of a relation R is $s(R) = R \cup R^{-1} = \{(a,b) | (a,b) \in R \lor (b,a) \in R\}$

   The union of the symmetric closures of two relations R, S is:

   $$s(R) \cup s(S) = \{(a,b) | (a,b) \in R \lor (b,a) \in R \lor (a,b) \in S \lor (b,a) \in S\}$$

   The symmetric clusure of the union of S and R is

   $$s(R \cup S) = s(\{(a,b) | (a,b) \in R \lor (a,b) \in S\} = \{(a,b) | (a,b) \in R \lor (a,b) \in S \lor (b,a) \in R \lor (b,a) \in S\}$$

   We see that $s(R \cup S) = s(R) \cup s(S)$.

8. Let $S = \{1,2,3,4\}$. With respect to the lexicographic order based on the usual less than relation,

   (a) find all pairs in $S \times S$ less than $(2,3)$

   Answer: $\{(1,1),(1,2),(1,3),(1,4),(2,1),(2,2)\}$

   (b) find all pairs in $S \times S$ greater than $(3,1)$

   Answer: $\{(3,2),(3,3),(3,4),(4,1),(4,2),(4,3),(4,4)\}$

   (c) draw the Hasse diagram of the poset $(S \times S, \preceq)$.

Answer:

$$
\begin{tikzcd}[row sep=small]
(4,4) \arrow[d, dash] \\
(4,3) \arrow[d, dash] \\
(4,2) \arrow[d, dash] \\
(4,1) \arrow[d, dash] \\
(3,4) \arrow[d, dash] \\
(3,3) \arrow[d, dash] \\
(3,2) \arrow[d, dash] \\
(3,1) \arrow[d, dash] \\
(2,4) \arrow[d, dash] \\
(2,3) \arrow[d, dash] \\
(2,2) \arrow[d, dash] \\
(2,1) \arrow[d, dash] \\
(1,4) \arrow[d, dash] \\
(1,3) \arrow[d, dash] \\
(1,2) \arrow[d, dash] \\
(1,1)
\end{tikzcd}
$$