Solution Final Exam Winter 2016 MATH 203

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#1

a)

$$\begin{aligned}
\log_2(x^2-4) - 2\log_2(x+2) &= -1 \\
\log_2(x^2-4) - \log_2(x+2)^2 &= -1 && \text{(since } n\log a = \log a^n\text{)} \\
\log_2\left(\frac{x^2-4}{(x+2)^2}\right) &= -1 \\
\frac{x^2-4}{(x+2)^2} &= 2^{-1} && \text{(Exponentiate by 2 on both sides since } 2^{\log_2 a} = a\text{)} \\
\frac{(x+2)(x-2)}{(x+2)^2} &= 2^{-1} \\
\frac{x-2}{x+2} &= \frac{1}{2} \\
2(x-2) &= x+2 \\
2x-4 &= x+2 \\
2x-x &= 4+2 \\
x &= 6
\end{aligned}$$

b) To compute the inverse of $f(x) = \frac{2\cdot 3^x}{4+3^x}$, change y for x and x for y then solve for

y

$$\begin{aligned}
x &= \frac{2 \cdot 3^y}{4 + 3^y} \\
x(4 + 3^y) &= 2 \cdot 3^y \\
4x + x \cdot 3^y &= 2 \cdot 3^y \\
4x &= 2 \cdot 3^y - x \cdot 3^y \\
4x &= 3^y(2 - x) \\
\frac{4x}{2 - x} &= 3^y \\
\ln\left(\frac{4x}{2 - x}\right) &= \ln\left(3^y\right) && \text{(Take the logarithm on both sides)} \\
\ln\left(\frac{4x}{2 - x}\right) &= y \ln\left(3\right) \\
\frac{\ln\left(\frac{4x}{2-x}\right)}{\ln 3} &= y
\end{aligned}$$

Therefore,

$$f^{-1}(x) = \frac{\ln\left(\frac{4x}{2-x}\right)}{\ln 3}$$

Observe that the domain of f, x can take all real numbers since the denominator of f will never be equal to zero. Then

$$\textbf{Domain}_f = (-\infty, \infty) = \textbf{Range}_{f^{-1}}$$

Now for the domain of f^{-1}, see that there a restriction on x for the numerator of f^{-1}.

$$\ln\left(\frac{4x}{2 - x}\right) = \ln 4x - \ln\left(2 - x\right)$$

We know that logarithmic functions can only take positive values, so the first term above $x > 0$, and for the second term $2 > x > 0$. Therefore,

$$\textbf{Domain}_{f^{-1}} = (0, 2)$$

#2

a) Note that

$$|x+2| = \begin{cases} x+2 & x \geq -2 \\ -(x+2) & x < -2 \end{cases}$$

If we first take the left hand limit, we get

$$\begin{aligned} \lim_{x \to -2^-} \frac{|x+2|}{x^2 - x - 6} &= \lim_{x \to -2^-} \frac{-(x+2)}{x^2 - x - 6} \\ &= \lim_{x \to -2^-} -\frac{(x+2)}{(x+2)(x-3)} \\ &= \lim_{x \to -2^-} -\frac{1}{(x-3)} = 1/5 \end{aligned}$$

On the other hand, the limit from the right is

$$\begin{aligned} \lim_{x \to -2^+} \frac{|x+2|}{x^2 - x - 6} &= \lim_{x \to -2^+} \frac{(x+2)}{x^2 - x - 6} \\ &= \lim_{x \to -2^+} \frac{(x+2)}{(x+2)(x-3)} \\ &= \lim_{x \to -2^+} \frac{1}{(x-3)} = -1/5 \end{aligned}$$

Since the limits from the right and from the left don't coincide

$$\lim_{x \to -2^-} \frac{|x+2|}{x^2 - x - 6} \neq \lim_{x \to -2^+} \frac{|x+2|}{x^2 - x - 6}$$

therefore the limit does not exist.

b)

$$\lim_{x\to 1}\frac{x-1}{3-\sqrt{x^2+8}} = \lim_{x\to 1}\frac{x-1}{3-\sqrt{x^2+8}}\cdot\frac{3+\sqrt{x^2+8}}{3+\sqrt{x^2+8}}$$

$$= \lim_{x\to 1}\frac{(x-1)(3+\sqrt{x^2+8})}{9-x^2-8}$$
$$= \lim_{x\to 1}\frac{(x-1)(3+\sqrt{x^2+8})}{1-x^2}$$

$$= \lim_{x\to 1}\frac{(x-1)(3+\sqrt{x^2+8})}{-(x^2-1)}$$

$$= \lim_{x\to 1}-\frac{(x-1)(3+\sqrt{x^2+8})}{(x-1)(x+1)}$$

$$= \lim_{x\to 1}-\frac{(3+\sqrt{x^2+8})}{(x+1)}$$

$$= -\frac{(3+\sqrt{1+8})}{((1+1))} = -3 \qquad \text{(plug in } x=1\text{)}$$

c)

$$= \lim_{x\to\infty}\frac{x\sqrt{1+9x^4}}{(3+2x)(4x+x^2)} = \lim_{x\to\infty}\frac{x\sqrt{1+9x^4}}{7x+9x^2+2x^3}$$

$$= \lim_{x\to\infty}\frac{x\sqrt{x^4((1/x^4)+9)}}{7x+9x^2+2x^3} = \lim_{x\to\infty}\frac{x\cdot x^2\sqrt{(1/x^4)+9}}{7x+9x^2+2x^3}$$

$$= \lim_{x\to\infty}\frac{x^3}{x^3}\frac{\sqrt{(1/x^4)+9}}{(7/x^2)+(9/x)+2} = \lim_{x\to\infty}\frac{\sqrt{(1/x^4)+9}}{(7/x^2)+(9/x)+2} = \frac{\sqrt{9}}{2} = 3/2$$

#3

We see that the function is undefined when $x = 2$ since the denominator will be equal to zero. Also you can verify that, as x approaches 2 from the left, $f(x)$ goes to negative infinity, and if x approaches 2 from the right, then $f(x)$ goes to infinity. Therefore, $x = 2$ is a **vertical asymptote** of f.

For the horizontal asymptote,

$$\begin{aligned}
\lim_{x\to\infty} \frac{3^{x+1}}{3^x - 9} &= \lim_{x\to\infty} \frac{3^{x+1}}{3^x - 3^2} \\
&= \lim_{x\to\infty} \frac{3^x}{3^x} \cdot \frac{3}{1 - 3^{2-x}} && \text{(Factor out } 3^x \text{ from the numerator and denominator)} \\
&= 3 && \text{(since the } \lim_{x\to\infty} 3^{2-x}\text{)}
\end{aligned}$$

and

$$\lim_{x\to-\infty} \frac{3^{x+1}}{3^x - 9} = \lim_{x\to-\infty} \frac{3^{x+1}}{3^x - 3^2} = 0 \qquad \text{(since the } \lim_{x\to-\infty} 3^{x+1} = 0 \text{ and } \lim_{x\to-\infty} 3^x = 0\text{)}$$

Therefore, the **horizontal asymptote** of f are $y = 3$ and $y = 0$.

#4

a) Simplify $f(x)$ first then derive

$$\begin{aligned}
f(x) &= x^{1/2}(\sqrt{x} - x^{-3/2})e^{2x} \\
&= x^{1/2}(x^{1/2} - x^{-3/2})e^{2x} \\
&= (x - x^{-1})e^{2x}
\end{aligned}$$

Now, we compute the derivative of f using product rule,

$$\begin{aligned}
f'(x) &= [\frac{d}{dx}(x - x^{-1})]cdote^{2x} + (x - x^{-1})cdot[\frac{d}{dx}e^{2x}] \\
&= \left(1 - (-\frac{1}{x^2})\right) \cdot e^{2x} + (x - x^{-1}) \cdot (2e^{2x}) \\
&= \left(1 + \frac{1}{x^2}\right) \cdot e^{2x} + (x - x^{-1}) \cdot (2e^{2x})
\end{aligned}$$

b)

$$f'(x) = \frac{\frac{d}{dx}\left(\frac{x^4}{x+3}\right)}{\frac{x^4}{x+3}} \qquad \text{(since } \frac{d}{dx}\ln u = \frac{u'}{u}\text{)}$$

$$= \frac{\frac{3x^4+12x^3}{(x+3)^2}}{\frac{x^4}{x+3}}$$

$$= \frac{3x^4+12x^3}{x^4(x+3)}$$

c) Use Quotient Rule,

$$f'(x) = \frac{(\tan x - x)\cdot \frac{d}{dx}\arctan x - \frac{d}{dx}(\tan x - x)\cdot \arctan x}{(\tan x - x)^2}$$

$$= \frac{(\tan x - x)\cdot \frac{1}{1+x^2} - (\sec^2 x - 1)\cdot \arctan x}{(\tan x - x)^2}$$

d)

$$\begin{aligned}
f'(x) &= \cos(x^2 + \cos(2x)x)\cdot \frac{d}{dx}(x^2 + \cos(2x)x) \\
&= \cos(x^2 + \cos(2x)x)\cdot (2x + \frac{d}{dx}(\cos(2x)x)) \\
&= \cos(x^2 + \cos(2x)x)\cdot (2x + (-2\sin(2x))x + \cos(2x)\cdot(1)) \\
&= \cos(x^2 + \cos(2x)x)\cdot (2x - 2\sin(2x))x + \cos(2x))
\end{aligned}$$

e)

$$
\begin{aligned}
f(x) &= (1+2x)^{x^2} \\
\ln(f(x)) &= \ln\left((1+2x)^{x^2}\right) && \text{(Take the logarithm on both sides)} \\
\ln(f(x)) &= x^2 \ln((1+2x)) \\
\frac{f'(x)}{f(x)} &= 2x \cdot \ln((1+2x)) + x^2 \cdot \frac{2}{1+2x} && \text{(By implicit differentiation on the left and Product Rule on the right)} \\
\frac{f'(x)}{f(x)} &= 2x \cdot \ln((1+2x)) + \frac{2x^2}{1+2x} \\
f'(x) &= (2x \cdot \ln((1+2x)) + \frac{2x^2}{1+2x}) \cdot f(x) \\
f'(x) &= (2x \cdot \ln((1+2x)) + \frac{2x^2}{1+2x}) \cdot (1+2x)^{x^2}
\end{aligned}
$$

#5

a) Replace (x, y) with $(2, 1)$, then

$$
\begin{aligned}
xy + 2\sqrt{3+y^2} &= x^3 - 2 \\
(2)(1) + 2\sqrt{3+(1)^2} &= (2)^3 - 2 \\
2 + 2(2) &= 8 - 2 \\
6 &= 6
\end{aligned}
$$

Therefore, the point $(2, 1)$ belongs to the given curve. Next we find y' when $x = 2$ and $y = 1$ and the equation of the tagent line $l(x)$ at this point. By implicit differentiation,

$$
\begin{aligned}
xy + 2\sqrt{3+y^2} &= x^3 - 2 \\
(y + xy') + 2\left(\frac{1}{2}(3+y^2)^{-1/2}\right) \cdot 2yy' &= 3x^2 \\
1 + 2y' + \frac{1}{\sqrt{3+1^2}} \cdot 2(1)y' &= 3(2)^2 && \text{(Replace } (x, y) \text{ by } (2, 1)) \\
1 + \frac{2y'}{2} &= 12 \\
y' &= 11
\end{aligned}
$$

Hence, the slope of the tangent line $m = 11$ and

$$l(x) = 11x + b$$

To solve for b (the y-intercept) just replace $(x, l(x)) = (2, 1)$, since the tangent line passes through that point.

$$1 = 11(2) + b \Rightarrow b = -21$$

Therefore the equation of the tangent line to the curve $y + x\sqrt{1+y^2} + 2 = x^2$ at the point $(2, 0)$.

$$l(x) = 11x - 21$$

b) Let

$$x : \text{'distance between car } A \text{ and it's starting point' at some time } t \text{ in seconds}$$
$$y : \text{'distance between ship } B \text{ and it's starting point' at some time } t \text{ in seconds}$$
$$D : \text{'distance between ship } A \text{ and } B\text{' in } t \text{ seconds}$$

Then,

$$D^2 = x^2 + y^2$$

$$2D \cdot \frac{dD}{dt} = 2x\frac{dx}{dt} + 2y\frac{dy}{dt} \qquad (\frac{dx}{dt} = 12 \text{ and } \frac{dy}{dt} = 16)$$

For $t = 5$, $x = 60$ and $y = 80$, then

$$D = \sqrt{60^2 + 80^2} = 100$$

Therefore

$$\frac{dD}{dt} = \frac{2 \cdot (60) \cdot 12 + 2 \cdot (80) \cdot 16}{2(100)} = 20km/hr$$

c) Evaluating the limit directly, we see that

$$\lim_{x \to 0} \frac{e^{x^2} - 1}{1 - cos(2x)} = \frac{0}{0}$$

We have an indeterminate form of type "$\frac{0}{0}$", so l'Hopital's Rule applies, giving

$$\lim_{x \to 0} \frac{e^{x^2} - 1}{1 - cos(2x)} = \lim_{x \to 0} \frac{2xe^{x^2}}{2\sin(2x)} = \frac{0}{0}$$

Again we have the same indeterminate form, so we apply l'Hopital's Rule a second time:

$$\lim_{x\to 0} \frac{2e^{x^2} + 4x^2 e^{x^2}}{4\cos(2x)} = \frac{2}{4} = 1/2$$

#6

a) Since $f(x) = 3 + x + 3x^2 - x^3$ is continuous on $[0, 3]$ and differentiable on $(0, 3)$ then by MVT, there exist some c in $[0, 3]$ such that

$$f'(c) = \frac{f(3) - f(0)}{3 - 0} = \frac{6 - 3}{3} = 1$$

where $f'(c) = 1$ is the slope of the secant line joining $(0, f(0))$ and $(3, f(3))$.

b) The derivative of $f(x)$

$$f'(x) = 1 + 6x - 3x^2$$

By MVT,

$$\begin{aligned}
f'(c) &= 1 + 6c - 3c^2 = 1 \\
6c - 3c^2 &= 0 \\
2c - c^2 &= 0 \\
c^2 - 2c &= 0 \\
c^2 - 2c + 1 &= 1 && \text{(By completing the square)} \\
(c-1)^2 &= 1 \\
c - 1 &= \pm 1 && \text{(Square root both sides)} \\
c &= 1 \pm 1
\end{aligned}$$

Therefore, $c = 2$ and $c = 0$. Note that both points lie on the interval $[0, 3]$.

7

a) Remember that the definition of derivative

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

Then, for $f(x)=\sqrt{2x+1}$

$$f'(x)=\lim_{h\to 0}\frac{\sqrt{2(x+h)+1}-\sqrt{2x+1}}{h}$$

$$=\lim_{h\to 0}\frac{\sqrt{2(x+h)+1}-\sqrt{2x+1}}{h}\cdot\frac{\sqrt{2(x+h)+1}+\sqrt{2x+1}}{\sqrt{2(x+h)+1}+\sqrt{2x+1}} \quad \text{(By multiplying and dividing the conjugate of the numerator)}$$

$$=\lim_{h\to 0}\frac{2(x+h)+1-2x-1}{h(\sqrt{2(x+h)+1}+\sqrt{2x+1})}=\lim_{h\to 0}\frac{2h}{h(\sqrt{2(x+h)+1}+\sqrt{2x+1})}$$

$$=\lim_{h\to 0}\frac{2}{\sqrt{2(x+h)+1}+\sqrt{2x+1}}=\frac{2}{2\sqrt{2x+1})}=\frac{1}{\sqrt{2x+1}}$$

b) Recall that the linearization of f at a is

$$L(x)=f(a)+f'(a)(x-a)$$

The linearization of $f(x)=\sqrt{2x+1}$ at $a=4$.

$$\begin{aligned}L(x)&=\sqrt{9}+\frac{1}{\sqrt{9}}(x-4)\\&=\frac{x}{3}+3-\frac{4}{3}\\&=\frac{x}{3}+\frac{5}{3}\end{aligned}$$

c) Linearization claims that $f(x)\approx L(x)$ when x is near a. Then,

$$\begin{aligned}f(x)&\approx f(a)+f'(a)(x-a)\\ \sqrt{2x+1}&\approx\frac{x}{3}+\frac{5}{3}\\ \sqrt{2(3)+1}&\approx\frac{3}{3}+\frac{5}{3} && (\text{since } \sqrt{2(3)+1}=\sqrt{7})\\ \sqrt{7}&\approx\frac{8}{3}\approx 2.67\end{aligned}$$

Note that the actual value of $\sqrt{30}\approx 2.645\ldots$, so the linear approximation is pretty strong.

8

a) To find the absolute extrema of $f(x)$ on a closed interval $[0, 3]$, we use **The Closed Interval Method**. First, we find the critical numbers of f in (a, b). Using Quotient Rule, the derivative of f is

$$f'(x) = \frac{2 - 2x^2}{(x^2 + x + 1)^2}$$

Now set $f'(x) = 0$ and solve for x

$$\begin{aligned} \frac{2 - 2x^2}{(x^2 + x + 1)^2} &= 0 \\ 2 - 2x^2 &= 0 \\ 2 &= 2x^2 \\ x &= \pm 1 \end{aligned}$$

.

$x = -1$ is not in the interval $(0, 3)$. Hence, our only critical number is $x = 1$, and $f(1) = 2/3$.
Now we find the values of f at the endpoint of the interval $[0, 3]$. So $f(0) = 0$ and $f(3) = 6/13$. Therefore, the **absolute max** is $f(1) = 2/3$ and **absolute min** is $f(0) = 0$.

b) Let x be half the length of the rectangle (i.e, the distance between the origin and the bottom tip of the rectangle, $0 \leq x \leq \infty$) and y the width of the rectangle. Then the area of the rectangle

$$\begin{aligned} A &= 2xy \\ &= 2x(12 - x^2) \\ &= 24x - 2x^3 \end{aligned}$$

Next, we'll compute the derivative of A w.r.t x and apply the First Derivative Test for abs. max/min values.

$$\begin{aligned} A' &= 24 - 6x^2 \\ 24 - 6x^2 &= 0 \\ x &= \pm 2 \end{aligned}$$

Since $0 \leq x \leq \infty$ then $x = 2$ is the only critical number for A. Also, $A' > 0$ on the interval $(0, 2)$. Similarly, $A' < 0$ on the interval $(2, \infty)$. Therefore by the FD test, $x = 2$ is the absolute max value for A and the dimension for the largest area of rectangle inscribed in the parabola are $2x = 4$ and $y = (12 - 2^2) = 8$.

#9

a) Since there are no restrictions on x for f, then the domain of f are all real numbers.
Also, $f(x) = f(-x)$ for all $x \in \mathbb{R}$. Therefore f is an even function and is symmetric about the y-axis.

b) The derivative of $f(x) = 2x^2 - x^4$

$$f'(x) = 4x - 4x^3$$

Apply the increasing/decreasing test to check on which interval f is increasing and decreasing.

$$\begin{aligned} 4x - 4x^3 &> 0 \\ 4x &> 4x^3 && \text{(since is always positive)} \\ x &> x^3 \\ 1 &> x^2 \\ x < 1 & \qquad\qquad x > -1 \end{aligned}$$

Since $f'(x) > 0$ for all $x \in (-1, 1)$ then f is increasing on $(-1, 1)$. Similarly, $f'(x) < 0$ for all x in the intervals $(-\infty, -1)$ and $(1, \infty)$ then f is decreasing on $(-\infty, -1)$ and $(1, \infty)$

Note that, f' changes sign from negative to positive at $x = -1$. Therefore by FD test, $f(-1) = 1$ is the local minimum of f. Also, f' changes sign from positive to negative at $x = 1$. Therefore by FD test, $f(1) = 1$ is the local maximum of f.

c) The second derivative of f

$$f''(x) = 4 - 12x^2$$

Now apply the Concavity test,

$$\begin{aligned}
4 - 12x^2 &> 0 \\
4 &> 12x^2 \\
\frac{4}{12} &> x^2 \\
\frac{1}{3} &> x^2 \\
\frac{1}{\sqrt{3}} &> x \qquad\qquad -\frac{1}{\sqrt{3}} < x \qquad\qquad \text{(by square rooting both sides)}
\end{aligned}$$

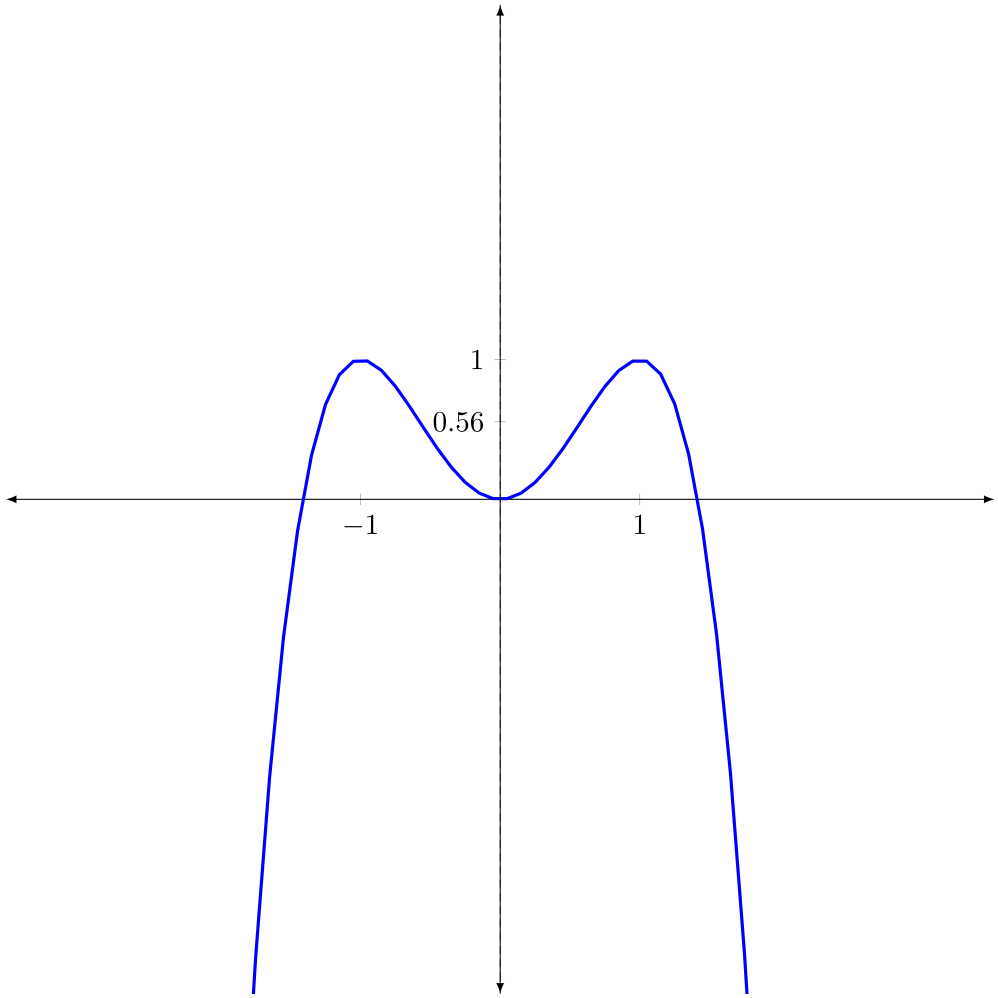
Therefore, f is concave upward on the interval $\left(-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$.

$$\begin{aligned}
4 - 12x^2 &< 0 \\
4 &< 12x^2 \\
\frac{4}{12} &< x^2 \\
\frac{1}{3} &< x^2 \\
\frac{1}{\sqrt{3}} &< x \qquad\qquad -\frac{1}{\sqrt{3}} > x \qquad\qquad \text{(by square rooting both sides)}
\end{aligned}$$

Therefore, f is concave downward on the intervals $\left(-\infty, -\frac{1}{\sqrt{3}}\right)$ and $\left(\frac{1}{\sqrt{3}}, \infty\right)$.

You can verify that the point of inflections occur at $f\left(\frac{1}{\sqrt{3}}\right) = 5/9$ and $f\left(-\frac{1}{\sqrt{3}}\right) = 5/9$ since the change of concavity occur at those points.

d) Before sketching the curve, verify that the x-intercept is $x = 0$ and y-intercept is $y = 0$. Also, there are no asymptotes.



(Make sure to sketch the asymptotes (if any) as dashed lines and mark all the intercepts, maximum and minimum points, and inflection points on your graph.)

Bonus Question Note that $\frac{d^2y}{dx^2} = y''$ and $y = f(u) = f(g(x))$. Apply the chain rule to compute the first derivative of y is

$$y' = [f(g(x))]' = f'(g(x)) \cdot g'(x) = f'(u) \cdot u' = \frac{df}{du} \cdot \frac{dg}{dx}$$

Now apply the chain rule and product rule to get the second derivative.

$$
\begin{aligned}
y'' &= [f'(g(x)) \cdot g'(x)]' \\
&= [\frac{d}{dx}(f'(g(x)))] \cdot g'(x) + f'(g(x)) \cdot \frac{d}{dx}g'(x) \qquad \text{(starting with the product rule)} \\
&= [f''(g(x)) \cdot g'(x)] \cdot g'(x) + f'(g(x)) \cdot g''(x) \\
&= f''(g(x)) \cdot [g'(x)]^2 + f'(g(x)) \cdot g''(x) \\
&= f''(u) \cdot [g'(x)]^2 + f'(u) \cdot g''(x) \\
&= \frac{d^2 f}{du^2} \cdot \left(\frac{dg}{dx}\right)^2 + \frac{df}{du} \cdot \frac{d^2 g}{dx^2}
\end{aligned}
$$