(3)

40

Department of Mathematics & Statistics

Course	Number	Section(s)
Mathematics	204/2	All, except EC
Examination	**Date**	**Pages**
Final	Fall 2009	3
Instructors		**Course Examiner**
A. Kokotov, R. Mearns, M. Singh		F. Thaine

Special Instructions: ▷ Only approved calculators are allowed.

Answer 10 questions. All questions have equal value.

1. Using the Gauss-Jordan method (i.e. reduced row echelon form method), find all the solutions of the following system of equations

$$\begin{aligned} 3x \quad\quad + z + 5u &= 7 \\ 3x - 2y \quad\quad - 5u &= 0 \\ x + y - z + 3u &= 4. \end{aligned}$$

2. Let $M = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 2 & 3 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 4 \\ 2 & 3 \\ 1 & 1 \end{bmatrix}$.

   a) Calculate M^{-1}.

   b) Find the matrix C such that $MC = B$.

3. a) Use Cramer's rule to solve the following system of equations

$$\begin{aligned} x \quad\quad + 2z &= 2 \\ x + y + z &= 0 \\ 2x + y \quad\quad &= 1. \end{aligned}$$

   (No marks given if you don't use Cramer's rule.)

   b) Calculate the determinant of the matrix $\begin{bmatrix} 1 & 1 & 2 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 5 & 1 \\ 2 & 0 & 3 & 1 \end{bmatrix}$.

Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX

4. Write the vector $(-14, 23)$ as a sum of two vectors $\vec{\mathbf{v}}$ and $\vec{\mathbf{w}}$ such that $\vec{\mathbf{v}}$ is parallel to the vector $(3, 4)$ and $\vec{\mathbf{w}}$ is orthogonal to $\vec{\mathbf{v}}$.

5. Let $P = (3, 0, 2)$, $Q = (1, 2, -1)$ and $R = (2, -1, 1)$.

   a) Find the area of the triangle with vertices P, Q and R.

   b) Find a nonzero vector perpendicular to the triangle with vertices P, Q and R.

   c) Show if $\vec{\mathbf{u}}$ and $\vec{\mathbf{v}}$ are orthogonal vectors in $\mathbb{R}^3$ then $||\vec{\mathbf{u}} + \vec{\mathbf{v}}||^2 = ||\vec{\mathbf{u}}||^2 + ||\vec{\mathbf{v}}||^2$.

6. a) Find parametric equations for the line in $\mathbb{R}^3$ that contains the point $(4, -2, 1)$ and is perpendicular to the plane $3x + y - 5z - 2 = 0$.

   b) Find the equation of the plane that contains the points $(1,0,0)$, $(0, 2, 0)$ and $(0, 0, 3)$.

7. Let $\vec{\mathbf{v}}_1 = (3, 5, 1)$, $\vec{\mathbf{v}}_2 = (4, 3, 2)$ and $\vec{\mathbf{v}}_3 = (7, -3, 5)$.

   a) Show that the vectors $\vec{\mathbf{v}}_1$, $\vec{\mathbf{v}}_2$ and $\vec{\mathbf{v}}_3$ are linearly dependent.

   b) Describe, geometrically, the subspace of $\mathbb{R}^3$ spanned by $\vec{\mathbf{v}}_1$, $\vec{\mathbf{v}}_2$ and $\vec{\mathbf{v}}_3$.

8. Let

$$A = \begin{bmatrix} 1 & 0 & 1 & 0 & 4 & 2 \\ 0 & 1 & 3 & 0 & 5 & 2 \\ 0 & 0 & 0 & 1 & 3 & 4 \end{bmatrix} \quad \text{and } X = \begin{bmatrix} x \\ y \\ z \\ u \\ v \\ w \end{bmatrix}.$$

   Find a basis for the solution space of the homogeneous system of linear equations $AX = 0$.

9. Find the standard matrix for the composition of the following two linear operators on $\mathbb{R}^2$: A rotation counterclockwise of 30^o, followed by a reflection about the x axis.

10. Let $A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 1 & 1 & 3 \end{bmatrix}$. Find an invertible matrix P and a diagonal matrix D such that $P^{-1}AP = D$.

11. For $n \geq 0$, let $X_n = \begin{bmatrix} a_n \\ b_n \\ c_n \end{bmatrix}$ where a_n, b_n and c_n are real numbers. Let

$M = \begin{bmatrix} \frac{1}{3} & -\frac{2}{3} & \frac{2}{3} \\ 0 & \frac{1}{3} & 0 \\ 0 & -\frac{2}{3} & 1 \end{bmatrix}$. Suppose that $X_n = MX_{n-1}$ for $n > 0$.

a) Calculate M^n for $n \geq 1$.

b) Write down the entries a_n, b_n, c_n of X_n in terms of a_0, b_0, c_0 and n.

c) Suppose that $X_0 = \begin{bmatrix} 3 \\ 4 \\ 2 \end{bmatrix}$. What happens to a_n, b_n and c_n as n gets large?

(Hint: we have $P^{-1}MP = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{3} & 0 \\ 0 & 0 & \frac{1}{3} \end{bmatrix}$, with $P = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$ and $P^{-1} = \begin{bmatrix} 0 & -1 & 1 \\ -1 & 0 & 1 \\ 1 & 1 & -1 \end{bmatrix}$.)