Answer 10 questions. All questions have equal value.
Only approved calculators are allowed.

1. Find all the solutions of the following system of equations

$$\begin{aligned} 2x + 4y - 3z - 2w &= -7 \\ x + 2y - 2z - w &= -5 \\ -2x - 4y + z + w &= 3. \end{aligned}$$

2. Let $v_1 = (1,2,0)$, $v_2 = (0,1,1)$ and $v_3 = (1,1,1)$.

   a) Show that the vectors v_1, v_2 and v_3 are linearly independent.

   b) Write the vector $(1,-4,2)$ as a linear combination of v_1, v_2 and v_3.

3. Let $M = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 3 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & -1 \\ 1 & 1 & 1 \end{bmatrix}$.

   a) Calculate M^{-1}.

   b) Find the matrix C such that $MC = B$.

4. Let $M = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 3 \end{bmatrix}$. Write M and M^{-1} as products of elementary matrices.

5. a) Consider the following system of equations

$$\begin{aligned} x + \quad 2z &= 6 \\ 2y + z &= -3 \\ x + 2y \quad &= 0. \end{aligned}$$

   Use Cramer's rule to solve for z. No marks if you don't use Cramer's rule.

   b) Calculate the determinant of the matrix $\begin{bmatrix} 1 & 1 & 2 & 0 \\ 0 & 0 & 3 & 1 \\ 1 & 0 & 2 & 1 \\ 2 & 2 & 0 & 1 \end{bmatrix}$.

6. a) Find the orthogonal projection of the vector $(1,2,4)$ on the vector $(1,2,2)$.

   b) Find the distance from the point $(2,5)$ to the line $3x - 4y - 6 = 0$.

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7. Let $O = (0,0,0)$, $P = (1,0,2)$, $Q = (0,1,2)$ and $R = (1,-1,6)$.

   a) Find the volume V of the parallelepiped determined by the vectors $\overrightarrow{OP}$, $\overrightarrow{OQ}$, $\overrightarrow{OR}$.

   b) Find the area A of the parallelogram determined by the vectors $\overrightarrow{OP}$, $\overrightarrow{OQ}$.

   c) Find the distance h from R to the plane spanned by the vectors $\overrightarrow{OP}$, $\overrightarrow{OQ}$. (Hint: use the results of parts (a) and (b).)

8. Find a basis for the solution space of the following system of equations

$$\begin{bmatrix} 1 & 5 & 0 & 0 & 3 \\ 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 1 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ u \\ v \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

9. Let $W = \{(x,y,z) \in \mathbb{R}^3 : x + y + z = 0\}$.

   a) Show that W is a subspace of $\mathbb{R}^3$.

   b) Find a basis of W.

10. Let $A = \begin{bmatrix} 1 & 2 & 2 \\ 0 & 3 & 0 \\ 2 & -2 & 1 \end{bmatrix}$. The characteristic polynomial of A is $(\lambda - 3)^2(\lambda + 1)$.
    Find an invertible matrix P and a diagonal matrix D such that $P^{-1}AP = D$.

11. For $n \geq 0$, let $X_n = \begin{bmatrix} a_n \\ b_n \\ c_n \end{bmatrix}$ where a_n, b_n and c_n are real numbers. Let

    $M = \begin{bmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & 1 & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} & 1 \end{bmatrix}$. Suppose that $X_n = MX_{n-1}$ for $n > 0$.

    a) Write down the entries a_n, b_n, c_n of X_n in terms of a_0, b_0, c_0 (and n).

    b) Suppose that $X_0 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$. What happens to a_n, b_n and c_n as n gets large?

    (Hint: we have $P^{-1}MP = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{2} \end{bmatrix}$, with $P = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$ and $P^{-1} = \begin{bmatrix} -1 & 1 & 1 \\ 1 & 0 & -1 \\ 1 & -1 & 0 \end{bmatrix}$.)