4. Let l be the line with parametric equations

$x = -2+t \quad y = -1+3t \quad z = 1+5t$

and let $v = (1,1,1)$. Find w_1 & w_2 s.t. $v = w_1 + w_2$ and w_1 is parallel to l and w_2 is perpendicular to l.

we can rewrite l as

$$l = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -2 \\ -1 \\ 1 \end{bmatrix} + \underbrace{\begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix}}_{\text{vector component.}} t$$

Since w_1 is parallel it will be some scalar multiple of the vector component of l hence

$(1,1,1) = \alpha(1,3,5) + w_2$ now $w_2 \cdot l = 0$ hence dotting everything we arrive at.

$$l \cdot (1,1,1) = \alpha(1,3,5) \cdot l + w_2 \cdot l$$
$$(1,3,5) \cdot (1,1,1) = \alpha(1,3,5) \cdot (1,3,5) + 0$$
$$1+3+5 \quad \alpha(1+9+25)$$
$$9 = \alpha 35$$
$$\frac{9}{35} = \alpha$$

$$\boxed{\begin{aligned} &\text{So } w_1 = 9/35(1,3,5) \text{ \& } w_2 = v - w_1 = (1,1,1) - 9/35(1,3,5) \\ &= 1/35(26, 8, 10) \end{aligned}}$$

5. let $P_1(1,1,1)$. $P_2(1,3,4)$ $P_3(2,1,5)$

(a) find the area of the triangle with vertices $P_1\ P_2\ P_3$

we start by finding two vectors emminating from a specific point

$$\vec{P_1P_2} = (1,3,4) - (1,1,1) = (0,2,3)$$
$$\vec{P_1P_3} = (2,1,5) - (1,1,1) = (1,0,4)$$

taking the cross product we get.

$$\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 0 & 2 & 3 \\ 1 & 0 & 4 \end{vmatrix} = \begin{matrix} \vec{i}(8) - \vec{j}(3) + \vec{k}(-2) \\ 8\vec{i} - 3\vec{j} - 2\vec{k} \end{matrix}$$

finding the norm $\sqrt{8^2 + (-3)^2 + (-2)^2} = \sqrt{64+9+4} = \sqrt{77}$

the area of the triangle is $\boxed{\dfrac{\sqrt{77}}{2}}$

(b) Find the equation of the plane trough $P_1\ P_2\ P_3$

we already found the normal from (a) hence the equation is then.

$$\boxed{8(x-1) - 3(y-1) - 2(z-1)}$$

6. Let $P(1,2,3)$ be a point let $n = (1,3,4)$

(a) Find the point-normal equation of the plane through P with normal n

$$\boxed{1(x-1) + 3(y-2) + 4(z-3) = 0}$$

b) Express the equation of the plane in the form $ax+by+cz+d=0$.

$$x - 1 + 3y - 6 + 4z - 12 = 0$$

$$\boxed{x + 3y + 4z - 19 = 0}$$

7. Let $v_1 = (1,-2,3)$ and $v_2 = (2,0,4)$

(a) Find numbers x & y so that $xv_1 + yv_2 = (0,-4,2)$

$$x(1,-2,3) + y(2,0,4) = (0,-4,2)$$

$$\left.\begin{array}{l} x + 2y = 0 \\ -2x = -4 \end{array}\right\} \left(\begin{array}{cc|c} 1 & 2 & 0 \\ -2 & 0 & -4 \end{array}\right) \begin{array}{l} R_1 + \tfrac{1}{2}R_2 \\ -R_2/2 \end{array}$$

$$\left(\begin{array}{cc|c} 0 & 2 & -2 \\ 1 & 0 & 2 \end{array}\right) \left.\begin{array}{l} R_1/2 \\ R_2 \leftrightarrow R_1 \end{array}\right\} = \left(\begin{array}{cc|c} 1 & 0 & 2 \\ 0 & 1 & -1 \end{array}\right)$$

$$\boxed{x = 2 \quad y = -1}$$

(b) Find v_3 so that $\{v_1, v_2, v_3\}$ is a basis of $\mathbb{R}^3$

All we need is a third vector that is linearly independent relative to v_1 & v_2 and is in $\mathbb{R}^3$. One way of finding this vector is by performing a cross product. The other is just by observing that $(0,1,0)$ is linearly indepedent relative to v_1 & v_2

8. Let $A = \begin{pmatrix} 1 & -3 & 0 & 0 & 2 & 5 \\ 0 & 0 & 1 & 0 & 1 & 6 \\ 0 & 0 & 0 & 1 & 1 & 1 \end{pmatrix}$ $X = \begin{pmatrix} x \\ y \\ z \\ u \\ v \\ w \end{pmatrix}$ Find a basis of the solution space of the homogeneous system of linear equation $AX = 0$.

$$\begin{aligned} x - 3y \quad + 2v + 5w &= 0 \\ z \quad + v + 6w &= 0 \\ u + v + w &= 0 \end{aligned}$$

$$\left.\begin{aligned} x &= 3y - 2v - 5w \\ y &= y \\ z &= -v - 6w \\ u &= v - w \\ v &= v \\ w &= w \end{aligned}\right\} \begin{pmatrix} 3 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} y + \begin{pmatrix} 2 \\ 0 \\ -1 \\ 1 \\ 1 \\ 0 \end{pmatrix} v + \begin{pmatrix} -5 \\ 0 \\ -6 \\ -1 \\ 0 \\ 1 \end{pmatrix} w$$

$$\boxed{\text{The basis is then } \left\{ \begin{pmatrix} 3 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 0 \\ -1 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -5 \\ 0 \\ -6 \\ -1 \\ 0 \\ 1 \end{pmatrix} \right\}}$$

9. Let $A=\begin{pmatrix}1&1&1\\-2&-2&-1\\0&0&-1\end{pmatrix}$. Find an invertible matrix P and a diagonal matrix D s.t. $P^{-1}AP=D$

To solve this problem we start by finding the eigenvalues, followed by the eigenvectors.

$$\det\left(\begin{pmatrix}1&1&1\\-2&-2&-1\\0&0&-1\end{pmatrix}-t\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}\right)=\det\begin{pmatrix}1-t&1&1\\-2&-2-t&-1\\0&0&-1-t\end{pmatrix}$$

$$(-1-t)\big((1-t)(-2-t)+2\big)\Rightarrow(-1-t)(\cancel{-2}-t+2t+t^2\cancel{+2})$$
$$(-1-t)(t^2+t)$$
$$=-t(t+1)(t+1)$$

The eigenvalues are $\lambda_1=0$ & $\lambda_2=-1$ repeated twice.
Let us now find the eigenvectors.

$$\begin{pmatrix}1&1&1\\-2&-2&-1\\0&0&-1\end{pmatrix}-0\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}=\begin{pmatrix}1&1&1\\-2&-2&-1\\0&0&-1\end{pmatrix}$$

$$\begin{pmatrix}1&1&1\\-2&-2&-1\\0&0&-1\end{pmatrix}-(-1)\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}=\begin{pmatrix}2&1&1\\-2&-1&-1\\0&0&0\end{pmatrix}$$

$$\begin{pmatrix}1&1&1\\-2&-2&-1\\0&0&-1\end{pmatrix}\begin{matrix}\\R_2+2R_1\\-R_3\end{matrix}=\begin{pmatrix}1&1&1\\0&0&1\\0&0&1\end{pmatrix}\begin{matrix}R_1-R_2\\R_2\\R_3-R_2\end{matrix}$$

$$\begin{pmatrix}2&1&1\\-2&-1&-1\\0&0&0\end{pmatrix}\begin{matrix}R_1/2\\R_2+R_1\\ \end{matrix}=\begin{pmatrix}1&1/2&1/2\\0&0&0\\0&0&0\end{pmatrix}$$

$$\begin{pmatrix}1&1&0\\0&0&1\\0&0&0\end{pmatrix}\Rightarrow\begin{cases}x_1+x_2=0\\x_3=0\end{cases}\quad\begin{aligned}x_1&=-x_2\\x_2&=x_2\\x_3&=0\end{aligned}$$

The eigenvector is $\begin{pmatrix}-1\\1\\0\end{pmatrix}x_3$

$$\left\{x_1+\tfrac{1}{2}x_2+\tfrac{1}{2}x_3=0\right.\Rightarrow\begin{aligned}x_1&=-\tfrac{1}{2}x_2-\tfrac{1}{2}x_3\\x_2&=x_2\\x_3&=x_3\end{aligned}$$

The eigenvector are $\begin{pmatrix}-1/2\\1\\0\end{pmatrix}x_2$ and $\begin{pmatrix}-1/2\\0\\1\end{pmatrix}x_3$

so $P=\begin{pmatrix}-1&-1/2&-1/2\\1&1&0\\0&0&1\end{pmatrix}$ $P^{-1}=\begin{pmatrix}-2&-1&-1\\2&2&1\\0&0&1\end{pmatrix}$

$$\boxed{\begin{pmatrix}0&0&0\\0&-1&0\\0&0&-1\end{pmatrix}=\begin{pmatrix}-2&-1&-1\\2&2&1\\0&0&1\end{pmatrix}\begin{pmatrix}1&1&1\\-2&-2&-1\\0&0&-1\end{pmatrix}\begin{pmatrix}-1&-1/2&-1/2\\1&1&0\\0&0&1\end{pmatrix}}$$

10. Let $A=\begin{pmatrix}-3 & 4\\ -2 & 3\end{pmatrix}$ compute A^{100}

we begin by finding the eigenvalues, then the eigenvectors and diagonalizing. This will simplify our calculations, since we will have $A=PDP^{-1}$

$$\det\left(\begin{pmatrix}-3 & 4\\ -2 & 3\end{pmatrix}-t\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\right)\Rightarrow \det\left(\begin{pmatrix}-3-t & 4\\ -2 & 3-t\end{pmatrix}\right)=(-3-t)(3-t)+8.$$

$$\begin{aligned}(-3-t)(3-t)+8 &= -9+\cancel{3t}-\cancel{3t}+t^2+8\\ &= -1+t^2\\ &= t^2-1\\ &= (t+1)(t-1)\end{aligned}$$

the eigenvalues are then $\lambda_1=-1,\ \lambda_2=1$

to find the eigenvectors we have.

$$\begin{pmatrix}-3 & 4\\ -2 & 3\end{pmatrix}-(-1)\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}=\begin{pmatrix}-2 & 4\\ -2 & 4\end{pmatrix}$$

$$\begin{pmatrix}-2 & 4\\ -2 & 4\end{pmatrix}\begin{matrix}R_1/2\\ R_2+R_1\end{matrix}=\begin{pmatrix}1 & -2\\ 0 & 0\end{pmatrix}$$

$$x_1-2x_2=0 \Rightarrow \left.\begin{matrix}x_1=2x_2\\ x_2=x_2\end{matrix}\right\}\begin{pmatrix}2\\ 1\end{pmatrix}x_2$$

the eigenvector is $\begin{pmatrix}2\\ 1\end{pmatrix}$

$$\begin{pmatrix}-3 & 4\\ -2 & 3\end{pmatrix}-1\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}=\begin{pmatrix}-4 & 4\\ -2 & 2\end{pmatrix}$$

$$\begin{pmatrix}-4 & 4\\ -2 & 2\end{pmatrix}\begin{matrix}R_1/4\\ R_2-1/2R_1\end{matrix}=\begin{pmatrix}1 & -1\\ 0 & 0\end{pmatrix}$$

$$x_1-x_2=0\ \left.\begin{matrix}x_1=x_2\\ x_2=x_2\end{matrix}\right\}\begin{pmatrix}1\\ 1\end{pmatrix}x_1$$

the eigenvector is $\begin{pmatrix}1\\ 1\end{pmatrix}x_1$

our matrix P is now $\begin{pmatrix}2 & 1\\ 1 & 1\end{pmatrix}$ and $P^{-1}=\frac{1}{1}\begin{pmatrix}1 & -1\\ -1 & 2\end{pmatrix}=\begin{pmatrix}1 & -1\\ -1 & 2\end{pmatrix}$

hence

$$A=PDP^{-1}\Rightarrow\begin{pmatrix}2 & 1\\ 1 & 1\end{pmatrix}\begin{pmatrix}-1 & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}1 & -1\\ -1 & 2\end{pmatrix}$$

$$A^{1000}=\begin{pmatrix}2 & 1\\ 1 & 1\end{pmatrix}\begin{pmatrix}-1^{1000} & 0\\ 0 & 1^{1000}\end{pmatrix}\begin{pmatrix}1 & -1\\ -1 & 2\end{pmatrix}=\boxed{\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}}$$

$$=P\,\mathbb{1}\,P^{-1}=PP^{-1}=\mathbb{1}$$