CONCORDIA UNIVERSITY

DEPARTMENT OF COMPUTER SCIENCE AND SOFTWARE ENGINEERING

COMP232 MATHEMATICS FOR COMPUTER SCIENCE

ASSIGNMENT 3 Winter 2019

Due Date: March 22, 2019

1. Let the function $f : \mathbb{R} \longrightarrow \mathbb{R}$ be given by

$$f(x) = \frac{x+1}{x-1} \quad \text{if } x \neq 1 \ , \qquad f(x) = 1 \quad \text{if } x = 1 \ .$$

   Draw the graph of f versus the values of x. Is f a bijection (*i.e.*, one-to-one and onto)? If yes then give a proof and derive a formula for f^{-1}. If no then explain why not.

2. Let $f : \mathbb{Z}^2 \longrightarrow \mathbb{Z}^2$ be defined as $f(m,n) = (m-n,n)$. Is f indeed a properly defined function from $\mathbb{Z}^2$ to $\mathbb{Z}^2$? Is f a bijection, *i.e.*, one-to-one and onto? If yes then give a proof and derive a formula for f^{-1}. If no then explain why not.

   Also derive a formula for the composite function f_k, for $k \in \mathbb{Z}^+$. Here f_2 denotes the composite function $f \circ f$, f_3 denotes the composite function $f \circ f \circ f$, *etc.* (You are asked to derive the formula for f_k for general $k \in \mathbb{Z}^+$.) Is f_k a bijection? If yes then give a proof and derive a formula for its inverse f_k^{-1}. If no then explain why not.

3. If A and B are sets and $f : A \longrightarrow B$, then for any subset S of A we define

$$f(S) \ = \ \{b \in B : b = f(a) \text{ for some } a \in S\} \ .$$

   Similarly, for any subset T of B we define the *pre-image* of T as

$$f^{-1}(T) \ = \ \{a \in A : f(a) \in T\} \ .$$

   Note that $f^{-1}(T)$ is well defined even if f does not have an inverse !

   Now let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be defined as $f(x) = x^2$. Let S_1 denote the closed interval $[-2,1]$, that is all $x \in \mathbb{R}$ that satisfy $-2 \leq x \leq 1$, and let S_2 be the open interval $(-1,2)$, that is all $x \in \mathbb{R}$ that satisfy $-1 < x < 2$. Also let $T_1 = S_1$ and $T_2 = S_2$.

   Determine

$$f(S_1 \cup S_2) \ , \ f(S_1) \cup f(S_2) \ , \ f(S_1 \cap S_2) \ , \ f(S_1) \cap f(S_2) \ ,$$

   and

$$f^{-1}(T_1 \cup T_2) \ , \ f^{-1}(T_1) \cup f^{-1}(T_2) \ , \ f^{-1}(T_1 \cap T_2) \ , \text{ and } f^{-1}(T_1) \cap f^{-1}(T_2) \ .$$

4. (a) Prove that $\lfloor -x \rfloor = -\lceil x \rceil$ and $\lceil -x \rceil = -\lfloor x \rfloor$.
   (b) Give a proof by cases that $\lfloor 4x \rfloor = \lfloor x \rfloor + \lfloor x + \frac{1}{4} \rfloor + \lfloor x + \frac{1}{2} \rfloor + \lfloor x + \frac{3}{4} \rfloor$.

5. (a) Use the Euclidean algorithm to determine whether or not the years 1812 and 2013 are relatively prime.
   (b) Let $k, m, n \in \mathbb{Z}^+$, where k and m are relatively prime. Prove that if $k|mn$ then $k|n$.

6. (a) Prove that if $n \in \mathbb{Z}^+$ is odd then $n^2 \equiv 1 \pmod 8$.
   (b) Prove that for any $m, n \in \mathbb{Z}^+$ the number $\gcd(m+n, mn) - \gcd(m, n)$ is even.
   Hint: Consider the cases that arise depending on whether n and m are both even, both odd, or one is even and the other odd.

7. (a) Without computing the value of 100! , determine how many zeros are at the end of this number when it is written in decimal form. Justify your answer.
   (b) Find all solutions to $m^2 - n^2 = 105$, for which both m and n are integers.

   Hint: Both proofs rely on the Factorization Theorem.

8. (a) Suppose that Hilbert's Grand Hotel is fully occupied, but the hotel closes all the even numbered rooms for maintenance. Show that all guests can remain in the hotel.
   (b) Show that a countably infinite number of guests arriving at Hilbert's fully occupied Grand Hotel can be given rooms without evicting any current guest.

9. Determine whether each of these sets is countable or uncountable. For those that are countably infinite, exhibit a one-to-one correspondence between the set of positive integers and that set.
   (a) integers not divisible by 3
   (b) integers divisible by 5 but not by 7
   (c) the real numbers with decimal representations consisting of all 1s
   (d) the real numbers with decimal representations of all 1s or 9s