ENGR 233 - Tut UA - Solutions to 1st Midterm Exam, February 2009

Problem 1(a). *Determine the work done by the constant force* $\vec{F} = 2\hat{\imath} + 3\hat{\jmath}$ *in moving a particle from the point* $(-1, 1)$ *to the point* $(4, 6)$ *along a straight line.*

Solution: Work $W = \vec{F} \cdot \vec{d}$. The displacement $\vec{d}$ is the vector from $(-1, 1)$ to $(4, 6)$ so $\vec{d} = (4 - (-1))\hat{\imath} + (6 - 1)\hat{\jmath} = 5\hat{\imath} + 5\hat{\jmath}$. Therefore

$$W = (2\hat{\imath} + 3\hat{\jmath}) \cdot (5\hat{\imath} + 5\hat{\jmath}) = 10 + 15 = 25$$

with appropriate units, which are not given in the problem (for example Nm).

(b) *Find the area of the triangle with vertices at the points* $(0, 1, 2)$, $(1, 2, 3)$ *and* $(3, 2, 1)$.

Solution: We determine two of the edges of the triangle, expressed as vectors, for example between the first two points $(0, 1, 2)$, $(1, 2, 3)$:

$$\vec{a} = (1 - 0)\hat{\imath} + (2 - 1)\hat{\jmath} + (3 - 2)\hat{k} = \hat{\imath} + \hat{\jmath} + \hat{k}$$

and between the first point $(0, 1, 2)$ and the last point $(3, 2, 1)$:

$$\vec{b} = (3 - 0)\hat{\imath} + (2 - 1)\hat{\jmath} + (1 - 2)\hat{k} = 3\hat{\imath} + 1\hat{\jmath} - 1\hat{k}$$

and take their cross-product

$$\vec{a} \times \vec{b} = (-1 - (-1))\hat{\imath} - (-1 - 3)\hat{\jmath} + (1 - 3)\hat{k} = -2\hat{\imath} + 4\hat{\jmath} - 2\hat{k}.$$

The magnitude $\|\vec{a} \times \vec{b}\|$ gives the area of the parallelogram spanned by the two vectors so the area of the triangle is half of that:

$$A = \frac{1}{2}\|\vec{a} \times \vec{b}\| = \frac{1}{2}\sqrt{(-2)^2 + 4^2 + (-2)^2} = \frac{\sqrt{24}}{2} = \sqrt{6}.$$

Another solution to the problem was discovered by a few students: if we take the magnitudes of the two edges we computed: $\|\vec{a}\| = \sqrt{3}$ and $\|\vec{b}\| = \sqrt{11}$ and then compute the magnitude of the third edge, the distance between $(1, 2, 3)$ and $(3, 2, 1)$, which is $\sqrt{2^2 + 0 + (-2)^2} = \sqrt{8}$, we see that they satisfy Pythagoras' theorem:

$$(\sqrt{3})^2 + (\sqrt{8})^2 = (\sqrt{11})^2$$

which means that this is a right triangle. In this case (BUT NOT IN GENERAL) we can find the area by

$$A = \frac{\sqrt{3}\sqrt{8}}{2} = \sqrt{6}.$$

Problem 2. *Let $\vec{r}(t) = 2\,\hat{\imath} + \cos t\,\hat{\jmath} + \sin t\,\hat{k}$ be the position vector of a moving particle.*
(a) *Describe the shape of the curve traced by the trajectory of the particle (hint: find an equation relating y and z).*
Solution: The x, y, z coordinates along the curve satisfy $x = 2$ (constant) and

$$y^2 + z^2 = \cos^2 t + \sin^2 t = 1.$$

That means the curve is a circle of radius 1, lying in the plane $x = 2$ (parallel to the yz plane), and centered at $(2, 0, 0)$,
(b) *Find the velocity vector $\vec{v}(t)$, the acceleration vector $\vec{a}(t)$, and the speed at any t.*
Solution: The velocity

$$\vec{v}(t) = \vec{r}\,'(t) = -\sin t\,\hat{\jmath} + \cos t\,\hat{k},$$

the speed

$$v(t) = \|\vec{r}\,'(t)\| = \sqrt{(-\sin t)^2 + (\cos t)^2} = 1,$$

and the acceleration

$$\vec{a}(t) = \vec{r}\,''(t) = -\cos t\,\hat{\jmath} - \sin t\,\hat{k}.$$

(c) *Find the tangential component of the acceleration at any t. What does this tell you about the motion?*
Solution: The tangential component of the acceleration

$$a_T = \frac{\vec{a} \cdot \vec{v}}{v} = \frac{(-\sin t\,\hat{\jmath} + \cos t\,\hat{k}) \cdot (-\cos t\,\hat{\jmath} - \sin t\,\hat{k})}{1} = (-\sin t)(-\cos t) + \cos t(-\sin t) = 0.$$

This means there is no acceleration in the direction of motion, i.e. all the acceleration is in the direction normal to the curve. This corresponds to the fact that the speed is constant so the acceleration only measures the change in the direction of the velocity. We know this is the case in circular motion with constant speed, where we have centripetal acceleration.
(d) *Give parametric equations for the tangent line to the curve at time $t = \pi/2$.*
Solution: We first determine the position at $t = \pi/2$, which is

$$\vec{r}(\pi/2) = 2\,\hat{\imath} + \cos(\pi/2)\,\hat{\jmath} + \sin(\pi/2)\,\hat{k} = (2, 0, 1).$$

This will be the initial position $\vec{r}_0$ of the line. Then we find a tangent vector to the curve at this point by

$$\vec{r}\,'(\pi/2) = -\sin(\pi/2)\,\hat{\jmath} + \cos(\pi/2)\,\hat{k} = -\hat{\jmath}.$$

This will be the direction vector $\vec{a}$ of the line (not to be confused with the acceleration).

Now we introduce a new parameter t for the line and write the vector equation of the line as

$$\vec{r} = \vec{r}_0 + \vec{a}t = (2, 0, 1) + t(0, -1, 0) = 2\hat{\imath} - t\hat{\jmath} + \hat{k}.$$

This gives the parametric equations $x = 2$, $y = -t$ and $z = 1$.

Problem 3. *The following questions refer to the function*

$$z = f(x, y) = e^{1-x^2-y^2}.$$

(a) *In the xy-plane, sketch the level curves corresponding to $f(x, y) = C$ for different values of the constant C $(0 < C \leq e)$.*

Solution: The level curves satisfy the equation $f(x, y) = e^{1-x^2-y^2} = C$, a constant. Taking the logarithm on both sides (we can do that since $C > 0$) gives

$$1 - x^2 - y^2 = \ln C$$

which is the same as

$$x^2 + y^2 = 1 - \ln C.$$

Since $C \leq e$ we know $\ln C \leq 1$ so this is the equation of a circle of radius $\sqrt{1 - \ln C}$ centered at the origin in the xy plane. For example, for the value $C = e$ we draw a circle of radius 0, which is a single point at the origin. For $C = 1$ we draw a circle of radius 1. For $C = 1/e$ we draw a circle of radius $\sqrt{2}$. As C tends to 0 we get larger and larger circles.

(b) *If $x = \cos t$ and $y = \sin t$, find $\frac{dz}{dt}$ at $t = \pi$.*

Solution: By the chain rule

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt} = e^{1-x^2-y^2}(-2x)(-\sin t) + e^{1-x^2-y^2}(-2y)(\cos t).$$

At $t = \pi$ we have $x = \cos \pi = -1$ and $y = \sin \pi = 0$ so this gives

$$\left.\frac{dz}{dt}\right|_{t=\pi} = e^{1-(-1)^2}(-2)(-1)(-\sin \pi) + e^{1-(-1)^2-y^2}(-2)(0)(\cos \pi) = 0.$$

This makes sense since $(\cos t, \sin t)$ parameterizes a circle of radius one centered at the origin and we know that that's a level curve of f, so on this curve f (i.e. z) is constant with respect to t, which means the derivative is zero.

(c) *Give the direction along which the function f increases most rapidly at the point $(0, 1)$, and find the maximum rate of increase.*

Solution: We compute the gradient of f:

$$\vec{\nabla} f(x,y) = \frac{\partial f}{\partial x}\hat{\imath} + \frac{\partial f}{\partial y}\hat{\jmath} = e^{1-x^2-y^2}(-2x)\hat{\imath} + e^{1-x^2-y^2}(-2y)\hat{\jmath}.$$

At the point $(0,1)$ this gives $\vec{\nabla} f(0,1) = -2\hat{\jmath}$. The direction of the gradient is the direction along which the function increases most rapidly so in this case it corresponds to the unit vector $-\hat{\jmath}$. The maximum rate of increase is the magnitude of the gradient, i.e.

$$\|\vec{\nabla} f(0,1)\| = \|-2\hat{\jmath}\| = 2.$$

Problem 4. *Give the equation of the tangent plane to the graph of*

$$z = 25 - x^2 - y^2$$

at the point $(3,-4,0)$.

Solution: This is a level surface of the function $f(x,y,z) = x^2 + y^2 + z$ so the gradient $\vec{\nabla} f = 2x\hat{\imath} + 2y\hat{\jmath} + \hat{k}$ is a normal vector to the surface. At the point $(3,-4,0)$ we get

$$\vec{N} = \vec{\nabla} f(3,-4,0) = 6\hat{\imath} - 8\hat{\jmath} + \hat{k}.$$

The equation of the plane is therefore $6x - 8y + z = d$ where we obtain d by plugging in the point $(3,-4,0)$ so $d = 18 + 32 + 0 = 50$. The equation of the plane becomes

$$6x - 8y + z = 50.$$