Midterm Math 205/2 - 2014

1. (a) Graph $f(x) = 2 - |2x - 4|$ and use it to evaluate $\int_{-1}^{3} f(x)\,dx$.

   (b) Let $\int_{-1}^{1} f(x)\,dx = -2$, $\int_{-1}^{6} f(x)\,dx = 5$, $\int_{-1}^{6} g(x)\,dx = 4$. Evaluate $\int_{1}^{6} [3f(x) - 2g(x)]dx$, or explain why it is impossible.

2. Use the Fundamental Theorem of Calculus to evaluate the derivative $F'(x)$ if $F(x) = \int_{x^2-1}^{x^2} t^2\sqrt{1+t^2}dt$.

3. Evaluate the following indefinite integrals:

   (a) $\int \dfrac{e^x dx}{e^{2x}+9}$;

   (b) $\int x^3 \ln^2 x dx$.

4. Evaluate the antiderivative $F(t)$ of $f(t) = 4\sec^2 t\left(1 + \tan^3 t\right)$ such that $F(0) = 3$

5. Evaluate the following definite integrals (*do not approximate*):

   (a) $\int_{0}^{3} \dfrac{x^3}{\sqrt{x^2+16}}dx$;

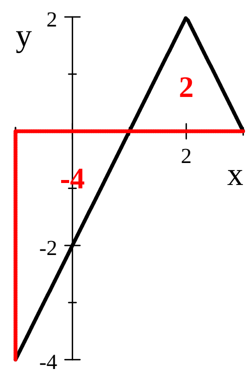
   (b) $\int_{0}^{\pi/2} \cos^3 x \sin^2 x dx$.

6. Calculate the average value of $f(x) = x\sin x$ on $[0, \pi]$ (*do not approximate*).

Bonus : Evaluate $\int_{0}^{4} \sqrt{4x - x^2}dx$ without integration.

Solutions:

1. (a) Graph $f(x) = 2 - |2x-4|$ and use it to evaluate $\int\limits_{-1}^{3} f(x)\,dx$.



$\rightarrow \int\limits_{-1}^{3} f(x)\,dx = -4+2=-2$

(b) Let $\int\limits_{-1}^{1} f(x)\,dx = -2$, $\int\limits_{-1}^{6} f(x)\,dx = 5$, $\int\limits_{-1}^{6} g(x)\,dx = 4$. Evaluate $\int\limits_{-1}^{6} [3f(x) - 2g(x)]dx$, or explain why it is impossible: $\int\limits_{1}^{6} [3f(x) - 2g(x)]dx = 3\int\limits_{1}^{6} f(x)\,dx - 2\int\limits_{1}^{6} g(x)\,dx = 3\left(\int\limits_{1}^{-1} f(x)\,dx + \int\limits_{-1}^{6} f(x)\,dx\right) - 2\int\limits_{1}^{6} g(x)\,dx = 3(2+5)-4 = 17$

2. Use the Fundamental Theorem of Calculus to evaluate the derivative $F'(x)$ if $F(x) = \int\limits_{x^2-1}^{x^2} t^2\sqrt{1+t^2}dt$:

As $\frac{d}{dt}F(t) = t^2\sqrt{1+t^2}$ and $F'(x) = \frac{d}{dx}\left(F(x^2) - F\left(x^2-1\right)\right) = \frac{dF\left(x^2\right)}{dt}\frac{dt}{dx} - \frac{dF\left(x^2-1\right)}{dt}\frac{dt}{dx} = \left(x^2\right)^2\sqrt{1+\left(x^2\right)^2}2x - \left(x^2-1\right)^2\sqrt{1+\left(x^2-1\right)^2}2x = 2x^5\sqrt{x^4+1} - 2x\sqrt{\left(x^2-1\right)^2+1}\left(x^2-1\right)^2$.

3. Evaluate the following indefinite integrals:

(a) $\int \frac{e^x dx}{e^{2x}+9} = \left| \; t = e^x \quad dt = e^x dx \; \right| = \int \frac{dt}{t^2+9} = \frac{1}{3}\arctan\frac{t}{3} + C =$ $\frac{1}{3}\arctan\frac{e^x}{3} + C.$

(b) $\int x^3 \ln^2 x dx = \left| \begin{array}{ll} u = \ln^2 x & u' = \frac{2\ln x}{x} \\ v' = x^3 & v = \frac{x^4}{4} \end{array} \right| = \frac{x^4 \ln^2 x}{4} - \frac{1}{2}\int x^3 \ln x dx =$ $\left| \begin{array}{ll} u = \ln x & u' = \frac{1}{x} \\ v' = x^3 & v = \frac{x^4}{4} \end{array} \right| = \frac{x^4\left(8\ln^2 x - 4\ln x + 1\right)}{32} + C.$

4. Evaluate the antiderivative $F(t)$ of $f(t) = 4\sec^2 t\left(1+\tan^3 t\right)$ such that $F(0) = 3$:

$F(t) = \int 4\sec^2 t\left(1+\tan^3 t\right) dt = 4\int\left(\sec^2 t + \sec^2 t\tan^3 t\right) dt = 4\left(\tan t + \int \sec^2 t \tan^3 t dt = \right| \begin{array}{r} x = \tan \\ dx = \sec^2 \end{array}$
$4\tan t + \int x^3 dx = 4\tan t + x^4 + C = 4\tan t + \tan^4 t + C.$ Since $F(0) =$ $C = 3 \rightarrow F(t) = 4\tan t + \tan^4 t + 3.$

5. Evaluate the following definite integrals (*do not approximate*):

(a) $\int\limits_0^3 \frac{x^3}{\sqrt{x^2+16}} dx = \left| \begin{array}{cc} t^2 = x^2+16 & x = 0 \rightarrow t = 4 \\ tdt = xdx & x = 3 \rightarrow t = 5 \end{array} \right| = \int\limits_4^5 \frac{\left(t^2-16\right)\cancel{t}dt}{\cancel{t}} =$ $\frac{13}{3}$

(b) $\int\limits_0^{\pi/2} \cos^3 x \sin^2 x dx = \int\limits_0^{\pi/2} \cos^3 x\left(1-\cos^2 x\right) dx = \int\limits_0^{\pi/2} \cos^3 x dx - \int\limits_0^{\pi/2} \cos^5 x dx \overset{Wallis\ formulae}{=}$ $\frac{2}{3} - \frac{4\times 2}{5\times 3} = \frac{2}{15}$

6. Calculate the average value of $f(x) = x\sin x$ on $[0,\pi]$ (*do not approximate*):

$f_{ave} = \frac{1}{\pi}\int\limits_0^{\pi} x\sin x dx = \left| \begin{array}{cc} u = x & u' = 1 \\ v' = \sin x & v = -\cos x \end{array} \right| = \frac{1}{\pi}\left(-x\cos x]_0^{\pi} + \int\limits_0^{\pi} \cos x\right) =$ $\frac{1}{\pi}(\pi + 0) = 1.$

Bonus : Evaluate $\int\limits_0^4 \sqrt{4x-x^2} dx$ without integration.

$y = \sqrt{4x-x^2} \rightarrow y^2 + (x-2)^2 = 4$ is a circle with center $\mathbf{C}(2,0)$ and radius $r = 2$. Since $y = \sqrt{4x-x^2} \geq 0$ it is only the upper circle with area

$$\int_0^4 \sqrt{4x - x^2}\,dx = 2\pi.$$