PROBLEMS FOR CHAPTER 8

1. Obtain the solution of the nonlinear equation given below using Newton's method. Use 3 iterations.

$$-x_1 + 2x_1^2 - 2x_1x_2 + x_2^2 = 1$$

$$x_1^2 - 2x_1x_2 - x_2 + x_2^2 = 0$$

Solution:

$$F(x) = \begin{bmatrix} -x_1 + 2x_1^2 - 2x_1x_2 + x_2^2 - 1 \\ x_1^2 - 2x_1x_2 - x_2 + x_2^2 \end{bmatrix}$$

Now the Jacobian matrix is formed.

$$J_{ij} = \frac{\partial f_i(x)}{\partial x_j}$$

$$J_{11} = \frac{\partial f_1(x)}{\partial x_1} = \frac{\partial(-x_1 + 2x_1^2 - 2x_1x_2 + x_2^2 - 1)}{\partial x_1} = 4x_1 - 2x_2 - 1$$

$$J_{12} = \frac{\partial f_1(x)}{\partial x_2} = \frac{\partial(-x_1 + 2x_1^2 - 2x_1x_2 + x_2^2 - 1)}{\partial x_2} = -2x_1 + 2x_2$$

$$J_{21} = \frac{\partial f_2(x)}{\partial x_1} = \frac{\partial(x_1^2 + 10x_2 - x_3 - 11.54)}{\partial x_1} = 2x_1 - 2x_2$$

$$J_{22} = \frac{\partial f_2(x)}{\partial x_2} = \frac{\partial(x_1^2 - 2x_1x_2 - x_2 + x_2^2)}{\partial x_2} = -2x_1 - 1 + 2x_2$$

Hence, the Jacobian matrix is

$$J(x) = \begin{bmatrix} 4x_1 - 2x_2 - 1 & -2x_1 + 2x_2 \\ 2x_1 - 2x_2 & -2x_1 - 1 + 2x_2 \end{bmatrix}$$

Now to evaluate $F(x_0)$

$$x_0 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \qquad F(x_0) = \begin{bmatrix} -1 + 2\times1^2 \ -2\times1\times1 + 1^2 - 1 \\ 1^2 \ -2\times1\times1 \ -1 + 1^2 \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \end{bmatrix}$$

To evaluate $J(x_0)$

$$x_0 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \qquad J(x_0) = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

The inverse is found using the cofactor method.

$$[J(x_0)]^{-1} = \frac{1}{\det \ J(x_0)} \begin{bmatrix} \text{confators} \\ \text{of} \\ J(x_0) \end{bmatrix}^T$$

$$\det \left|J(x_0)\right| = \begin{vmatrix} 1 & 0 \\ 0 & -1 \end{vmatrix} = -1$$

$$\text{cofactor matrix} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$

Then the inverse of $J(x_0)$ is

$$[J(x_0)]^{-1} = -1\times \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

The first iteration can now be performed

k = 0

$$[x_{k+1}] = [x_k] - [J(x_k)]^{-1} F(x_k)$$

$$x_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} -1 \\ -1 \end{bmatrix}$$

$$x_1 = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$$

After applying three iterations, we obtain the following table.

k	0	1	2	3
x_k	1	2	1.5263	1.3535
	1	0	0.4211	0.5514

2. Solve the following set of nonlinear equations using Newton's method. Use 3 iterations.

$$12x_1 - 3x_2^2 - 4x_3 = 7.17$$

$$x_1^2 + 10x_2 - x_3 = 11.54$$

$$x_2^3 + 7x_3 = 7.631$$

Solution:

$$F(x) = \begin{bmatrix} 12x_1 - 3x_2^2 - 4x_3 - 7.17 \\ x_1^2 + 10x_2 - x_3 - 11.54 \\ x_2^3 + 7x_3 - 7.631 \end{bmatrix}$$

Now the Jacobian matrix is formed.

$$J_{ij} = \frac{\partial f_i(x)}{\partial x_j}$$

$$J_{11} = \frac{\partial f_1(x)}{\partial x_1} = \frac{\partial(12x_1 - 3x_2^2 - 4x_3 - 7.17)}{\partial x_1} = 12$$

$$J_{12} = \frac{\partial f_1(x)}{\partial x_2} = \frac{\partial(12x_1 - 3x_2^2 - 4x_3 - 7.17)}{\partial x_2} = -6x_2$$

$$J_{13} = \frac{\partial f_1(x)}{\partial x_3} = \frac{\partial(12x_1 - 3x_2^2 - 4x_3 - 7.17)}{\partial x_3} = -4$$

$$J_{21} = \frac{\partial f_2(x)}{\partial x_1} = \frac{\partial(x_1^2 + 10x_2 - x_3 - 11.54)}{\partial x_1} = 2x_2$$

$$J_{22} = \frac{\partial f_2(x)}{\partial x_2} = \frac{\partial(x_1^2 + 10x_2 - x_3 - 11.54)}{\partial x_2} = 10$$

$$J_{23} = \frac{\partial f_2(x)}{\partial x_3} = \frac{\partial(x_1^2 + 10x_2 - x_3 - 11.54)}{\partial x_3} = -1$$

$$J_{31} = \frac{\partial f_3(x)}{\partial x_1} = \frac{\partial(x_2^3 + 7x_3 - 7.631)}{\partial x_1} = 0$$

$$J_{32} = \frac{\partial f_3(x)}{\partial x_2} = \frac{\partial(x_2^3 + 7x_3 - 7.631)}{\partial x_2} = 3x_2^2$$

$$J_{33} = \frac{\partial f_3(x)}{\partial x_3} = \frac{\partial(x_2^3 + 7x_3 - 7.631)}{\partial x_3} = 7$$

Hence, the Jacobian matrix is

$$J(x) = \begin{bmatrix} 12 & -6x_2 & -4 \\ 2x_1 & 10 & -1 \\ 0 & 3x_2^2 & 7 \end{bmatrix}$$

Now to evaluate $F(x_0)$

$$x_0 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \qquad F(x_0) = \begin{bmatrix} 12\times 1 - 3\times 1^2 - 4\times 1 - 7.17 \\ 1^2 + 10\times 1 - 1 - 11.54 \\ 1^3 + 7\times 1 - 7.631 \end{bmatrix} = \begin{bmatrix} -2.17 \\ -1.54 \\ 0.369 \end{bmatrix}$$

To evaluate $J(x_0)$

$$x_0 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \qquad J(x_0) = \begin{bmatrix} 12 & -6 & -4 \\ 2 & 10 & -1 \\ 0 & 3 & 7 \end{bmatrix}$$

Then the inverse of $J(x_0)$ is found

$$[J(x_0)]^{-1} = \begin{bmatrix} 0.078 & 0.0321 & 0.0491 \\ -0.015 & 0.0897 & 0.0043 \\ 0.0064 & -0.0385 & 0.1410 \end{bmatrix}$$

The first iteration can now be performed

k = 0

$$[x_{k+1}] = [x_k] - [J(x_k)]^{-1} F(x_k)$$

$$x_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} - \begin{bmatrix} 0.078 & 0.0321 & 0.0491 \\ -0.015 & 0.0897 & 0.0043 \\ 0.0064 & -0.0385 & 0.1410 \end{bmatrix} \begin{bmatrix} -2.17 \\ -1.54 \\ 0.369 \end{bmatrix}$$

$$x_1 = \begin{bmatrix} 1.2005 \\ 1.1042 \\ 0.9026 \end{bmatrix}$$

After applying three iterations, we obtain the following table.

k	0	1	2	3
	1	1.2005	1.2	1.2
	1	1.1042	1.1	1.1
x_k	1	0.9026	0.9	0.9

3. Solve the following nonlinear system using Newton's method. Use 3 iterations.

$$x_1 + \cos(x_1x_2x_3) - 1 = 0$$

$$(1-x_1)^{1/4} + x_2 + 0.05\,x_3^2 - 0.15x_3 - 1 = 0$$

$$-x_1^2 - 0.1\,x_2^2 + 0.01x_2 + x_3 - 1 = 0$$

Solution:

$$F(x) = \begin{bmatrix} x_1 + \cos(x_1x_2x_3) - 1 \\ (1-x_1)^{1/4} + x_2 + 0.05x_3^2 - 0.15x_3 - 1 \\ -x_1^2 - 0.1x_2^2 + 0.01x_2 + x_3 - 1 \end{bmatrix}$$

Now the Jacobian matrix is formed.

$$J_{ij} = \frac{\partial f_i(x)}{\partial x_j}$$

$$J_{11} = \frac{\partial f_1(x)}{\partial x_1} = \frac{\partial(x_1 + \cos(x_1x_2x_3) - 1)}{\partial x_1} = 1 - x_2x_3 sin(x_1x_2x_3)$$

$$J_{12} = \frac{\partial f_1(x)}{\partial x_2} = \frac{\partial(x_1 + \cos(x_1x_2x_3) - 1)}{\partial x_2} = -x_1x_3 sin(x_1x_2x_3)$$

$$J_{13} = \frac{\partial f_1(x)}{\partial x_3} = \frac{\partial(x_1 + \cos(x_1 x_2 x_3) - 1)}{\partial x_3} = -x_1 x_2 sin(x_1 x_2 x_3)$$

$$J_{21} = \frac{\partial f_2(x)}{\partial x_1} = \frac{\partial((1 - x_1)^{1/4} + x_2 + 0.05x_3^2 - 0.15x_3 - 1)}{\partial x_1} = -\frac{1}{4(1 - x_1)^{3/4}}$$

$$J_{22} = \frac{\partial f_2(x)}{\partial x_2} = \frac{\partial((1 - x_1)^{1/4} + x_2 + 0.05x_3^2 - 0.15x_3 - 1)}{\partial x_2} = 1$$

$$J_{23} = \frac{\partial f_2(x)}{\partial x_3} = \frac{\partial((1 - x_1)^{1/4} + x_2 + 0.05x_3^2 - 0.15x_3 - 1)}{\partial x_3} = 0.1x_3 - 0.15$$

$$J_{31} = \frac{\partial f_3(x)}{\partial x_1} = \frac{\partial(- x_1^2 - 0.1 x_2^2 + 0.01x_2 + x_3 - 1)}{\partial x_1} = - 2 x_1$$

$$J_{32} = \frac{\partial f_3(x)}{\partial x_2} = \frac{\partial(- x_1^2 - 0.1 x_2^2 + 0.01x_2 + x_3 - 1)}{\partial x_2} = 0.01 - 0.2x_2$$

$$J_{33} = \frac{\partial f_3(x)}{\partial x_3} = \frac{\partial(- x_1^2 - 0.1 x_2^2 + 0.01x_2 + x_3 - 1)}{\partial x_3} = 1$$

Hence, the Jacobian matrix is

$$J(x) = \begin{bmatrix} 1 - x_2 x_3 sin(x_1 x_2 x_3) & -x_1 x_3 sin(x_1 x_2 x_3) & -x_1 x_2 sin(x_1 x_2 x_3) \\ -4(1 - x_1)^{-3/4} & 1 & 0.1x_3 - 0.15 \\ - 2 x_1 & 0.01 - 0.2x_2 & 1 \end{bmatrix}$$

Now to evaluate $F(x_0)$

$$x_0 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \qquad F(x_0) = \begin{bmatrix} 0 + \cos(0 \times 1 \times 1) - 1 \\ (1 - 0)^{1/4} + 1 + 0.05 \times 1^2 - 0.15 \times 1 - 1 \\ - 0^2 - 0.1 \times 1^2 + 0.01 \times 1 + 1 - 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0.9 \\ -0.09 \end{bmatrix}$$

To evaluate $J(x_0)$

$$x_0 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \qquad J(x_0) = \begin{bmatrix} 1 & 0 & 0 \\ -4 & 1 & -0.05 \\ 0 & -0.19 & 1 \end{bmatrix}$$

Then the inverse of $J(x_0)$ is found

$$[J(x_0)]^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 4.0384 & 1.0096 & 0.0505 \\ 0.7673 & 0.1918 & 1.0096 \end{bmatrix}$$

The first iteration can now be performed

k = 0

$$[x_{k+1}] = [x_k] - [J(x_k)]^{-1} F(x_k)$$

$$x_1 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \\ 4.0384 & 1.0096 & 0.0505 \\ 0.7673 & 0.1918 & 1.0096 \end{bmatrix} \begin{bmatrix} 0 \\ 0.9 \\ -0.09 \end{bmatrix}$$

$$x_1 = \begin{bmatrix} 0 \\ 0.0959 \\ 0.9182 \end{bmatrix}$$

let k = 1 and perform the next iteration.

$$x_1 = \begin{bmatrix} 0 \\ 0.0959 \\ 0.9182 \end{bmatrix} \quad \text{Hence} \quad F(x_1) = \begin{bmatrix} 0 \\ 0.0003 \\ -0.0817 \end{bmatrix}$$

and

$$J(x_1) = \begin{bmatrix} 1 & 0 & 0 \\ -4 & 1 & -0.0582 \\ 0 & -0.0092 & 1 \end{bmatrix},$$

$$[J(x_0)]^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 4.0021 & 1.0005 & 0.0582 \\ 0.0367 & 0.0092 & 1.0005 \end{bmatrix}$$

Now the second iteration can be done.

$$x_2 = \begin{bmatrix} 0 \\ 0.0959 \\ 0.9182 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \\ 4.0021 & 1.0005 & 0.0582 \\ 0.0367 & 0.0092 & 1.0005 \end{bmatrix} \begin{bmatrix} 0 \\ 0.0003 \\ -0.0817 \end{bmatrix}$$

$$x_2 = \begin{bmatrix} 0 \\ 0.1003 \\ 1 \end{bmatrix}$$

After applying three iterations, we obtain the following table.

k	0	1	2	3
	0	0	0	0
	1	0.0959	0.1003	0.1
x_k	1	0.9182	1	1