MID TERM

CMPS 130 - Winter 14

Warmuth

NAME:____________________

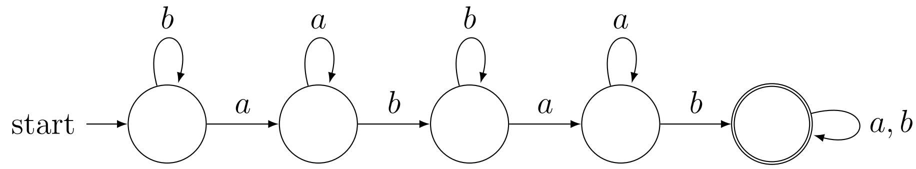
Student ID:____________

This exam is closed book and notes. Show partial solutions to get partial credit.
If your answers are not written legibly, you won't get full credit.
Clarity and succinctness will be rewarded.

Question 1:____________(out of 10)
Question 2:____________(out of 20)
Question 3:____________(out of 10)
Question 4:____________(out of 10)
Question 5:____________(out of 10)
Question 6:____________(out of 10)
Question 7:____________(out of 10)
Question 8:____________(out of 10)
Question 9:____________(out of 10)
Extra Credit:____________(out of 10)

Total:____________(out of 100+10)

1. Give a *deterministic finite state automaton* (FA) that accepts all strings over $\{a, b\}$ that contain the sub-word ab at least twice. Thus $aabaaaabab, abbbbabb$ are in the language and $baabb, abbba$ are not.



2. **a)** Is the following true or false? "For every regular language L, every subset of L is regular as well." If false, then give a simple counterexample.

False. Counterexample: $L = \{a, b\}^*$. L is regular, but has non-regular subset: $\{a^n b^n \mid n \in \mathbb{N}\}$.

b) Is the following true or false? "Every non-regular language is infinite!" Justify with a sentence.

True. The proof is by contraposition: must show that every finite language is regular. But this is obvious (because if $L = \{w_1, \dots, w_n\}$, then its regular expression is $w_1 + \ldots + w_n$).

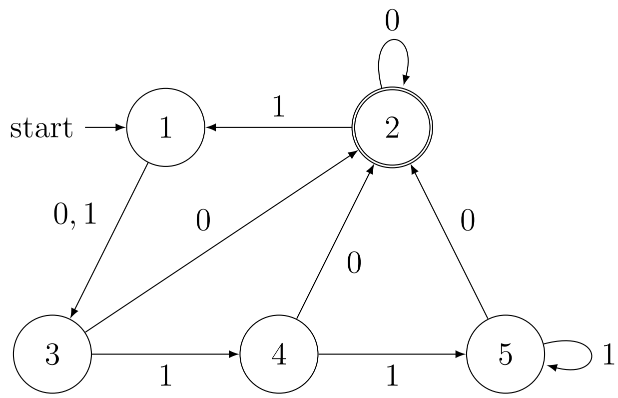
c) Is the following true or false? "The intersection of any two non-regular languages is non-regular." If false, then give a counterexample.

False. Counterexample: $L_1 = \{a^n b^n \mid n \in \mathbb{N}\}, L_2 = \{b^n a^n \mid n \in \mathbb{N}\}$. Then $L_1 \cap L_2 = \{\Lambda\}$.

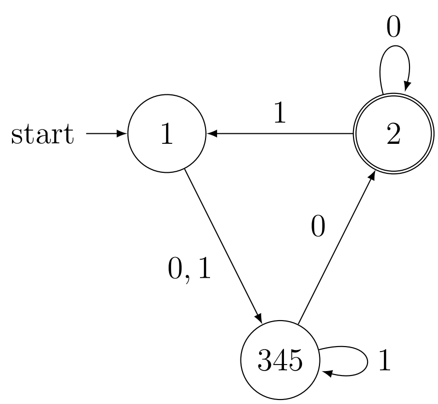
d) Is the following always true, or sometimes false? "If each of the languages $L_1, L_2, \ldots$ is regular, then $\bigcup_{i=1}^{\infty} L_i$ is regular as well." Give reasons why this is true, or give a counter example.

False. Counterexample: $L_i = \{a^i b^i\}$. Then $\bigcup_{i=1}^{\infty} L_i = \{a^n b^n \mid n \in \mathbb{N}^+\}$.

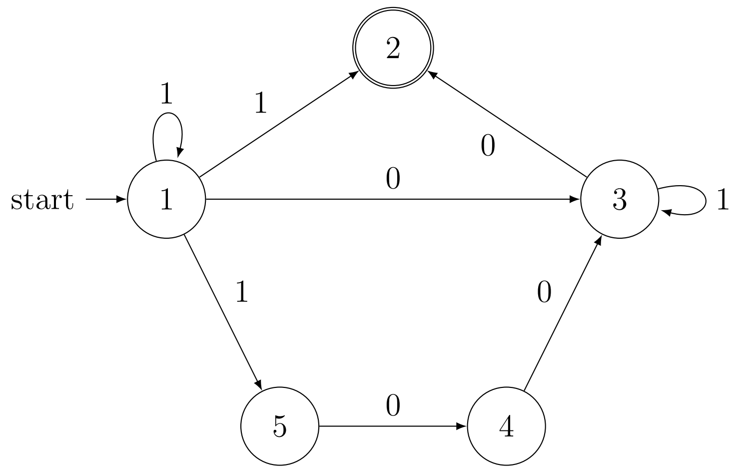
3. Minimize the following FA. Show your work (give the table).
   Show the resulting FA (if the number of states was reduced).



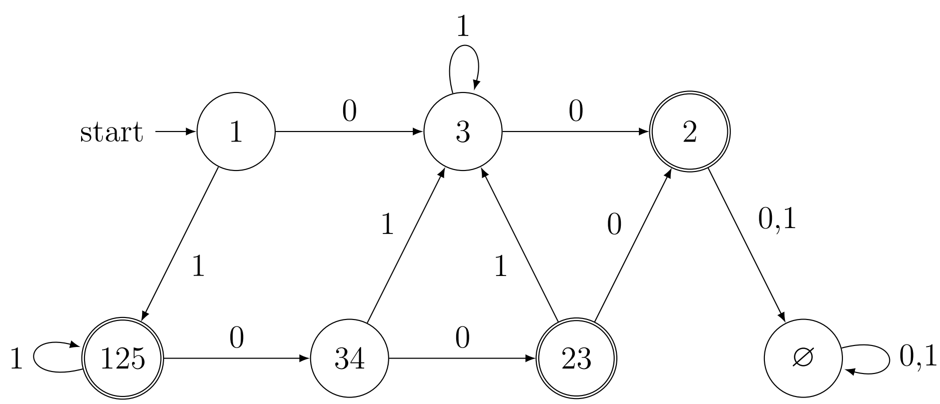
2	1			
3	2	1		
4	2	1		
5	2	1		
	1	2	3	4



4. Use the "subset construction method" to convert the following NFA to an FA. Label the FA you produce with the corresponding subsets of the states of the original NFA.



State	0	1
$\rightarrow$ 1	3	125
3	2	3
$*$ 125	34	125
$*$ 2	$\varnothing$	$\varnothing$
34	23	3
$\varnothing$	$\varnothing$	$\varnothing$
$*$ 23	2	3



5. Show that the language $L = \{0^i 1^j : i \geq j\}$ is non-regular by either

   (a) by exhibiting an infinite set of pairwise distinguishable words
   (you need to show that any pair in the infinite set is distinguishable)

   (b) or by using the Pumping Lemma.

   As a reminder, the Pumping Lemma says:

   - For every regular language L there is a constant N such that each word $x \in L$ of length at least N can be written as uvw such that the following holds:

     $|uv| \leq N$,
     v is not the empty word and
     for all $i \geq 0$, $uv^iw \in L$.

   Hint: Make sure that you are pumping in the right direction.

   Solutions:

   (a) Pick $S = \{\Lambda, 0, 00, 000, \ldots\}$. I will show that the members of S are pairwise L-distinguishable.
   Let $x = 0^n$ and $y = 0^m$, where $n \neq m$. Without loss of generality, assume $n > m$.
   To distinguish x and y, let $z = 1^n$. Then $xz = 0^n1^n \in L$ because $n \geq n$, but $yz = 0^m1^n \notin L$ because $m < n$.

   (b) Suppose to the contrary that L is regular, so there exists a machine with n states accepting L. Then pick $x = 0^n1^n$; by the Pumping Lemma, we know that $x = uvw$ for some u, v, w satisfying

      i. $|uv| \leq n$,
      ii. $v \neq \Lambda$, and
      iii. $uv^iw \in L$ for all $i \in \mathbb{N}$.

   By (i) and (ii) (and because the first n symbols in x are all 0), we can conclude that $v = 0^k$ for some k where $1 \leq k \leq n$.
   Now pick $i = 0$, so $uv^iw = uw = 0^{n-k}1^n$. Then $uw \notin L$ because $n - k < n$. But this contradicts (iii).

6. **a)** Give six closure properties of regular languages.

If L_1, L_2 are regular languages over the alphabet Σ, then the following languages are regular:

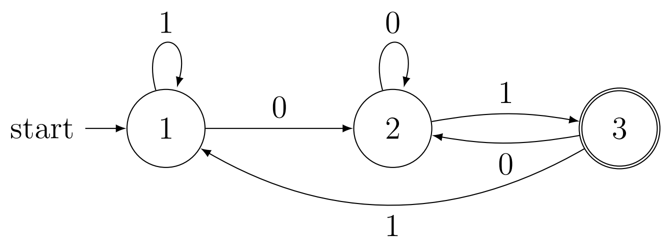
(a) $L_1 \cup L_2$
(b) $L_1 \cap L_2$
(c) $\overline{L_1}$
(d) $L_1 - L_2$
(e) $L_1 L_2$
(f) L_1^*
(g) $Prefix(L_1) = \{w : \exists u : wu \in L_1\}$
(h) $Suffix(L_1) = \{w : \exists u : uw \in L_1\}$
(i) $Subword(L_1) = \{w : \exists u, v : uwv \in L_1\}$

b) Use the closure properties of regular languages to show that the following language is not regular:

The set of non-palindromes,
i.e. the language $\{a, b\}^* - PAL$,
where $PAL := \{w \in \{a, b\}^* : w = w^R\}$.
(You can use the fact that PAL is known to be non-regular.)

Suppose to the contrary that $\{a, b\}^* - PAL$ is regular. Then $\overline{\{a, b\}^* - PAL} = PAL$ is regular, due to closure under complement. But this contradicts the non-regularity of PAL.

7. For the following FA:



(a) Show that there are three pairwise distinguishable words w.r.t. the accepted language. Give three words and then show that each pair is distinguishable by giving a witness with the right properties.

Pick $x_1 = \Lambda$, $x_2 = 0$, $x_3 = 01$. Then pick $z_{12} = 1$, $z_{13} = \Lambda$, $z_{23} = \Lambda$. Then:

$$x_1 z_{12} = 1 \notin L$$
$$x_2 z_{12} = 01 \in L$$

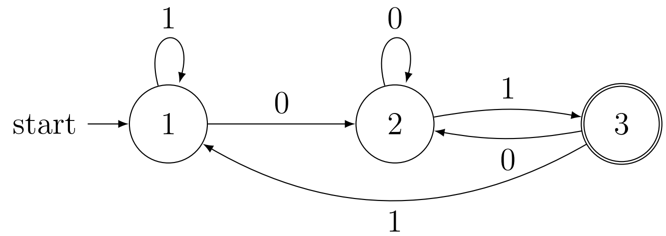
$$x_1 z_{13} = \Lambda \notin L$$
$$x_3 z_{13} = 01 \in L$$

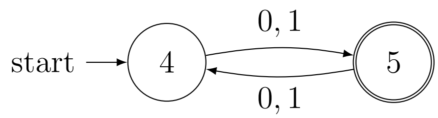
$$x_2 z_{23} = 1 \notin L$$
$$x_3 z_{23} = 01 \in L.$$

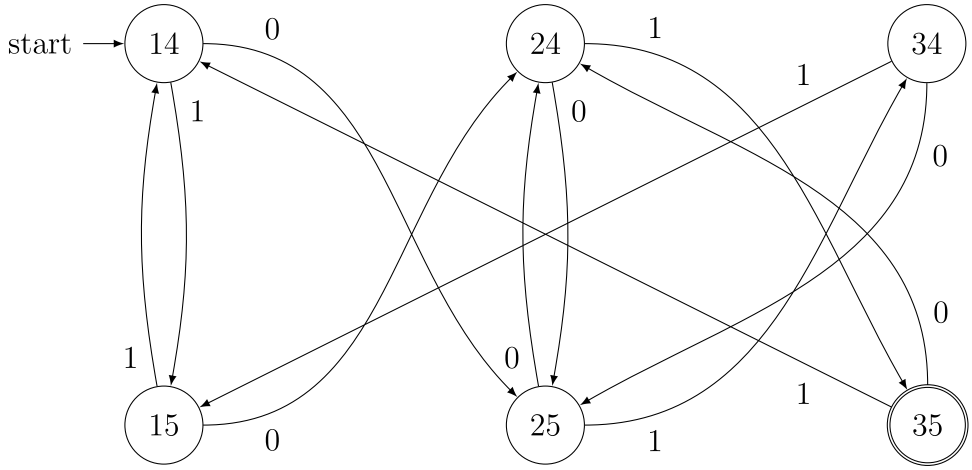
(b) What does this mean about the FA?

The FA has the minimum number of states.

8. Using the "cross-product construction" method, build an FA for the intersection of the regular languages accepted by the following two FA's.







9. Show that for any regular language $L \subseteq \{0,1\}^*$, the following language is regular:

$$\widetilde{L} = \{w \in L : w \text{ begins with } 0\}.$$

Solution 1. Notice that $\widetilde{L} = L \cap L_2$, where L_2 is the language of the regular expression $0(0+1)^*$. Therefore, by the closure property for $\cap$, we conclude that $\widetilde{L}$ is regular.

Solution 2. Let M be a FA for L. Then we can define a machine $\widetilde{M}$ for $\widetilde{L}$ by modifying M as follows:

- Add two new non-accepting states (call them $\widetilde{q_0}, \widetilde{q_1}$) to the machine.
- $\widetilde{q_0}$ is the new start state. Its transitions are:

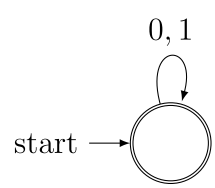
$$\widetilde{q_0} \xrightarrow{0} q_1$$
$$\widetilde{q_0} \xrightarrow{1} \widetilde{q_1},$$

  where q_1 is the state such that $q_0 \xrightarrow{0} q_1$ (where q_0 is the old start state of M).
- $\widetilde{q_1}$ is just a trap state.

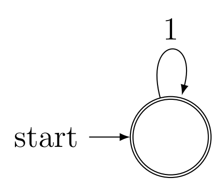
This machine $\widetilde{M}$ works because if the input string starts with 0, then it will end up in the same state as M. On the other hand, if the input string doesn't start with 0, then it will end up in either $\widetilde{q_0}$ or $\widetilde{q_1}$, neither of which are final states.

A false solution: Let M be an FA accepting L. Simply delete the 0-transition out of the start state.

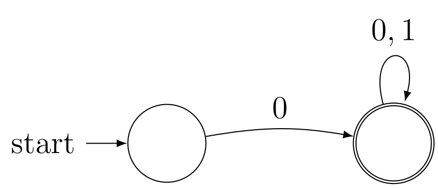
An example where this gives the in-correct answer:
Let $L = (0+1)^*$. The following FA accepts L.



If you delete the 0-transition then you get the NFA



which accepts 1^* and this is not $0L$. However the correct answer is the following NFA accepting $0(0+1)^*$.



Extra credit: Show that there are non-regular languages L such that L^* is non-regular (Don't forget to prove that L^* is non-regular).

Pick $L = \{x \in \{a,b\}^* \mid n_a(x) = n_b(x)\}$.
(L is non-regular because regular languages are closed under intersection and $L \cap a^*b^* = \{a^n b^n : n \geq 0\}$ which is non-regular.)

Claim: $L^* = L$.
Proof: $L^* \supseteq L^1 = L$ is trivial and $L^* \subseteq L$ follows from the fact that for all $i \geq 0$, L^i clearly only contains words with an equal number of a's and b's. $\square$

Finally observe that L^* is again non-regular because L was so.

Alternate proof: Let L be any non-regular language over some finite alphabet Σ. Let \$ be a new symbol not in Σ. Then clearly, the language $\$L$ is non-regular as well.

Claim: $(\$L)^*$ is non-regular.
Proof: Assume it is regular. Since regular languages are closed under intersection, $(\$L)^* \cap \$\Sigma^*$ is regular. However $(\$L)^* \cap \$\Sigma^*$ is the non-regular language $\$L$ and we have a contradiction. $\square$