COMP361 NUMERICAL METHODS TERM TEST

This Term Test consists of 10 Problems, worth 10 points each.

The total duration of this Term Test is 150 minutes, including an estimated 15 minutes for downloading and preparation, and 15 minutes for scanning and uploading the completed Term Test.

You may consult the Lecture Notes during the Term Test, including the version with solutions.

Problems may be solved in any order that is most convenient to you. You will have to work quite efficiently in order to complete all problems within the allotted time.

It is strongly recommended that you first work on the solution of a problem on scratch paper. Thereafter write your solution in a very clear fashion on the sheet(s) of paper that you will scan (or take a picture of) and upload to EAS. Submitting your solutions as a PDF file is also strongly recommended.

The solutions that you submit must contain sufficient details of the complete solution procedure, and not just the final result. Do not submit the scratch paper, but you must keep it in case issues regarding authenticity arise.

Communicating with others is strictly prohibited during the Term Test. Not adhering to this requirement will have severe consequences.

This course is about computation with real arithmetic. However, many of the real numbers that appear in this Term Test are in fact integers or rational numbers. This allows most computations to be done with exact arithmetic, thereby avoiding extensive computations with a calculator.

Problem 1. Which of of the following are valid for all $\mathbf{x} \in \mathbb{R}^n$?

(a) $\| \mathbf{x} \|_1 \le \| \mathbf{x} \|_2$, (b) $\| \mathbf{x} \|_1 \le \| \mathbf{x} \|_\infty$, (c) $\| \mathbf{x} \|_2 \le \| \mathbf{x} \|_\infty$, (d) $\| \mathbf{x} \|_\infty \le \| \mathbf{x} \|_1$.

Problem 2. For the matrix

$$\mathbf{A} = \begin{pmatrix} 4 & 6 & 0 & 1 \\ 1 & 7 & -5 & 3 \\ 2 & -1 & 7 & 8 \\ 1 & 3 & 2 & -1 \end{pmatrix} ,$$

find a nonzero vector $\mathbf{x}$ such that

$$\| \mathbf{Ax} \|_\infty = \| \mathbf{A} \|_\infty \ \| \mathbf{x} \|_\infty .$$

Also find a nonzero vector $\mathbf{x}$ such that

$$\| \mathbf{Ax} \|_1 = \| \mathbf{A} \|_1 \ \| \mathbf{x} \|_1 .$$

Problem 3. Make use of the Banach Lemma to show that the following matrix is invertible:

$$\mathbf{A} = \begin{pmatrix} 3 & 1 & 1 \\ -1 & -4 & 2 \\ 2 & 2 & 5 \end{pmatrix} .$$

Also use the Banach Lemma to find a bound on $\| \mathbf{A}^{-1} \|_\infty$.

Problem 4. Determine the LU-decomposition of the matrix

$$\mathbf{A} = \begin{pmatrix} 1 & 1 & 2 \\ 1 & 3 & 5 \\ -2 & 4 & 8 \end{pmatrix} .$$

Using only $\mathbf{L}$, $\mathbf{U}$, and $\mathbf{f}$, solve the system $\mathbf{Ax} = \mathbf{f}$ for $\mathbf{x}$, where $\mathbf{f} = (\ 3\ ,\ 6\ ,\ 6\ )^T$.

Make sure to show **L**, **U**, and the intermediate vector called $\mathbf{g}$ in the Lecture Notes.

Problem 5. Suppose we solve a system of linear equations of the form $\mathbf{Ax} = \mathbf{f}$, where

$$\mathbf{A} = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 2 \\ 1 & 3 & 2 \end{pmatrix} ,$$

by Gauss Elimination with *Row Pivoting* .

Show the result of each step in the Gauss Elimination with Row Pivoting.

What are the values of the three multipliers that arise?

(As a result of pivoting these multipliers should be at most 1 in magnitude.)

NOTE: In order to minimize the computations that you have to do, no right-hand side vector $\mathbf{f}$ is given, and hence no solution vector $\mathbf{x}$ need to be computed.

Problem 6.

What is the customary measure of *ill-conditioning* of a matrix ?

Explain what the main disadvantage is of computing with ill-conditioned matrices.

Also mention what possible remedies there are when the ill-conditioning is only moderate.

Problem 7. Show how to use Newton's Method to find the smallest positive solution of the equation

$$\tan(x) \ = \ e^{-x} \ .$$

Carry out the first iteration, with $x^{(0)} = 0$, and write down the value of $x^{(1)}$.

Also give a small hand-drawn sketch of the situation, that shows the curves $y \ = \ \tan(x)$ and $y \ = \ e^{-x}$ in the same x, y - plane, for x in the interval $[-\frac{\pi}{2} \ , \ \frac{\pi}{2}]$, and indicating their intersection point.

Problem 8. Carry out the first iteration of Newton's method for solving the system

$$x_1^2 \ + \ x_2^2 \ - \ 4 \ = \ 0 \ ,$$

$$x_2^2 \ - \ x_1 \ + \ 1 \ = \ 0 \ .$$

using $x_1^{(0)} = 1$, $x_2^{(0)} = 1$ as initial guesses. Show all details, including the values of $x_1^{(1)}$ and $x_2^{(1)}$.

Also give a small hand-drawn sketch of the situation by drawing the curves $x_1^2 \ + \ x_2^2 \ - \ 4 = 0$ and $x_2^2 \ - \ x_1 \ + 1 = 0$ in the same x, y - plane, indicating their intersection points, as well as the initial guess and the point $(\ x_1^{(1)} \ , \ x_2^{(1)} \ )$.

NOTE : This problem is meant to illustrate Newton's Method for solving a system of equations.

Thus do not reduce the two equations to a single equation.

Problem 9. Consider the *fixed point iteration* $x^{(k+1)} \ = \ f(x^{(k)})$, where

$$f(x) \ = \ c \, x(1 - x^2) \ , \quad \text{with} \quad c \ = \ \frac{3}{2} \ .$$

Analytically determine all fixed points, and for each of these determine analytically whether it is attracting for sufficiently close initial guess, or whether it is repelling.

Problem 10.

Again consider the *fixed point iteration* $x^{(k+1)} \ = \ f(x^{(k)})$ from the preceding problem, where

$$f(x) \ = \ c \, x(1 - x^2) \ , \quad \text{with} \quad c = \frac{3}{2} \ .$$

Give a hand-drawn graphical interpretation of this fixed point iteration, showing the line $y = x$, the curve $y = f(x)$, and indicating several iterations, starting with $x^{(0)} \approx 0.1$. Make sure that your sketch is qualitatively accurate.