Midterm Exam II ENGR 213
Applied Ordinary Differential Equations

Winter 2016 Version B
Solutions

[10 points] **Problem 1.**

Determine which of the system of functions is linearly independent, which is linearly dependent.

$$(a)\ \{-3, \frac{1}{x}, \frac{e^x}{x}\}\quad x > 0 \qquad\qquad (b)\ \{1, \sin(2x), (\sin x - \cos x)^2\}$$

Explain your answer.

Solution.
(a) Since

$$W = \begin{vmatrix} -3 & x^{-1} & x^{-1}e^x \\ 0 & -x^{-2} & e^x(x^{-1} - x^{-2}) \\ 0 & 2x^{-3} & e^x(x^{-1} - 2x^{-2} + 2x{-}3) \end{vmatrix} = -3 \begin{vmatrix} -x^{-2} & e^x x^{-1}(1 - x^{-1}) \\ 2x^{-3} & e^x x^{-1}(1 - 2x^{-1} + 2x^{-2}) \end{vmatrix} =$$

$$= 3e^x x^{-3}(1 - 2x^{-1} + 2x^{-2} + 2x^{-1} - 2x^{-2}) = 3e^x x^{-3} > 0$$

for all $x > 0$, then this system of functions is linearly-independent.
(b) Since

$$(\sin x - \cos x)^2 = \sin^2 x - 2\sin x \cos x + \cos^2 x = 1 - 2\sin x \cos x = 1 - \sin(2x),$$

the last function in the set is the linear combination of the other two functions. Therefore, this system of functions is linearly dependent.

[10 points] **Problem 2.**

Find the general solution of the given differential equation

$$y^{(4)} - 16y = 0$$

Solution. Since the auxiliary equation

$$m^4 - 16 = (m^2 - 4)(m^2 + 4) = (m - 2)(m + 2)(m - 2i)(m + 2i) = 0$$

has four roots

$$m_1 = 2,\ m_2 = -2,\ m_3 = 2i,\ m_4 = -2i,$$

the fundamental set of solutions consists of the functions

$$y_1(x) = e^{2x}, \quad y_2(x) = e^{-2x}, \quad y_3(x) = \cos(2x), \quad y_4(x) = \sin(2x).$$

Hence the general solution is

$$y(x) = C_1 e^{2x} + C_2 e^{-2x} + C_3 \cos(2x) + C_4 \sin(2x).$$

[10 points] **Problem 3.**

Use the method of undetermined coefficients to find the general solution of the following initial value problem

$$y'' + 10y' + 25y = 4e^{-5x}, \qquad y(0) = 2, \qquad y'(0) = -2$$

Solution. It is a non-homogeneous equation. Thus, its solution is

$$y(x) = y_c(x) + y_p(x),$$

where $y_c(x)$ is a general solution of the corresponding homogeneous equation

$$y_c'' + 10y_c' + 25y_c = 0$$

and $y_p(x)$ is a particular solution of the nonhomogeneous equation

$$y_p'' + 10y_p' + 25y_p = 4e^{-5x}$$

First let us find a general solution of the homogeneous equation. The auxiliary equation for the corresponding homogeneous differential equation is

$$m^2 + 10m + 25 = (m + 5)^2 = 0$$

Thus, the roots of the auxiliary equation are $m_1 = -5$, $m_2 = -5$ and the general solution of the corresponding homogeneous equation is

$$y_c(x) = C_1 e^{-5x} + C_2 x e^{-5x},$$

where C_1 and C_2 are arbitrary constants.

Since the right-hand side of the equation $4e^{-5x}$ is a multiple of the exponential function, and -5 is a repeated root of the auxiliary equation, we seek for a particular solution of the form

$$y_p(x) = Ax^2 e^{-5x},$$

where A is unknown coefficient. In order to find it, we substitute $y_p(x)$ into the given nonhomogeneous equation.

Since

$$y_p'(x) = e^{-5x}(2Ax - 5Ax^2)$$

and

$$y_p''(x) = e^{-5x}(2A - 20Ax + 25Ax^2)$$

Plugging into the equation and multiplying by e^{5x} we get

$$25Ax^2 - 20Ax + 2A + 20Ax - 50Ax^2 + 25Ax^2 = 2A = 4.$$

So $A = 2$ and $y_p(x) = 2x^2 e^{-5x}$.

Hence, the general solution of the nonhomogeneous equation is

$$y(x) = C_1 e^{-5x} + C_2 x e^{-5x} + 2x^2 e^{-5x}$$

Next we find unknown constants C_1 and C_2 from the initial conditions. On the one hand,

$$y(0) = C_1 = 2.$$

On the other hand,

$$y'(x) = e^{-5x}(C_2 + 4x - 5C_1 - 5C_2 x - 10x^2),$$

and consequently

$$y'(0) = C_2 - 5C_1 = -2.$$

Therefore, $C_1 = 2, \quad C_2 = 8$. So, we get the unique solution of the given initial-value problem

$$y(x) = e^{-5x}(2x^2 + 8x + 2).$$

[10 points] **Problem 4.**

Using the method of variation of parameters, solve the boundary value problem

$$\frac{d^2y}{dx^2} - 4y = \frac{32x^2}{e^{2x}}.$$

Solution. First let us find a general solution of the homogeneous equation. The auxiliary equation for the corresponding homogeneous differential equation is the following

$$m^2 - 4 = 0.$$

Thus, the roots of the auxiliary equation are $m_1 = 2$, $m_2 = -2$ and the fundamental set of solutions consist of two functions

$$y_1(x) = e^{2x} \quad \text{and} \quad y_2(x) = e^{-2x}.$$

Thus, we are looking for the general solution of the nonhomogeneous equation in the form

$$y(x) = C_1(x)e^{2x} + C_2(x)e^{-2x}$$

By the method of variation of parameters,

$$C_1'(x) = \frac{W_1(x)}{W(x)} \qquad \text{and} \qquad C_2'(x) = \frac{W_2(x)}{W(x)},$$

where

$$W = \begin{vmatrix} e^{2x} & e^{-2x} \\ 2e^{2x} & -2e^{-2x} \end{vmatrix} = -4,$$

$$W_1 = \begin{vmatrix} 0 & e^{-2x} \\ \frac{32x^2}{e^{2x}} & -2e^{-2x} \end{vmatrix} = -32x^2e^{-4x}$$

and

$$W_2 = \begin{vmatrix} e^{2x} & 0 \\ 2e^{2x} & \frac{32x^2}{e^{2x}} \end{vmatrix} = 32x^2.$$

Then

$$C_1'(x) = \frac{W_1}{W} = 8x^2e^{-4x} \qquad \text{and} \qquad C_2'(x) = \frac{W_2}{W} = -8x^2.$$

By integrating, we get

$$C_1(x) = 8\int x^2e^{-4x}dx = e^{-4x}(-2x^2 - x - 1/4) + D_1$$

and

$$C_2(x) = \int -8x^2dx = -\frac{8}{3}x^3 + D_2.$$

Hence, the general solution of the nonhomogeneous equation is

$$y(x) = e^{2x}(e^{-4x}(-2x^2 - x - 1/4) + D_1) + e^{-2x}(-\frac{8}{3}x^3 + D_2) =$$

$$= D_1e^{2x} + D_2e^{-2x} - e^{-2x}(\frac{8}{3}x^3 + 2x^2 + x).$$

[5 points] **Problem 5.**

Solve the Cauchy-Euler equation

$$x^2\frac{d^2y}{dx^2} - 3x\frac{dy}{x} + 4y = 0$$

$$y(1) = \ln(2), \quad y(4) = 0.$$

Solution. By looking for a solution in the form $y(x) = x^m$, we reduce the differential equation to quadratic equation with respect to m. From

$$y'(x) = mx^{m-1} \quad \text{and} \quad y''(x) = m(m-1)x^{m-2}$$

we get the auxiliary equation

$$m(m-1) - 3m + 4 = m^2 - 4m + 4 = (m-2)^2 = 0.$$

Since the roots of the auxiliary equation are $m_1 = 2$ and $m_2 = 2$, the general solution of the given equation

$$y(x) = C_1x^2 + C_2\ln(x)x^2.$$

From the initial values, $y(1) = C_1 = \ln 2$ and $y(4) = 16\ln 2 + 16C_2\ln 4 = 0$. So $C_2 = -\frac{1}{2}$. Final answer:

$$y(x) = \ln(2)x^2 - 1/2\ln(x)x^2 = x^2\ln\frac{2}{\sqrt{x}}.$$

[5 points] **Bonus question.**

For which real value(s) of parameters a and b all solutions of the differential equation

$$y'' + ay' + by = 0, \qquad y(0) = 0$$

approaches 0 as $x \to \infty$.

Solution. General solution of the differential equation depends on the roots of the auxiliary equation

$$m^2 + am + b = (m + \frac{a}{2})^2 + b - \frac{a^2}{4} = 0.$$

In general, we can write the roots in the form

$$m_1 = -\frac{a}{2} + \sqrt{\frac{a^2}{4} - b} \quad \text{and} \quad m_2 = -\frac{a}{2} - \sqrt{\frac{a^2}{4} - b}.$$

Therefore,

(a) If both roots are real numbers, the general solution is a linear combination of two exponential functions. It approaches 0 as x approaches ∞ only if both exponents are negative, that is

$$a > 0 \quad \text{and} \quad 0 < \frac{a^2}{4} - b < \frac{a^2}{4}$$

Hence, $0 < b < \frac{a^2}{4}$.

(b) If there is one real repeated root $m = -\frac{a}{2}$, that is $b = \frac{a^2}{4}$, then

$$y(x) = e^{-\frac{a}{2}x}(C_1 x + C_2)$$

and

$$\lim_{x\to\infty} e^{-\frac{a}{2}x}(C_1 x + C_2) = \lim_{x\to\infty} \frac{C_1 x + C_2}{e^{\frac{a}{2}x}} = \lim_{x\to\infty} \frac{2C_1}{ae^{\frac{a}{2}x}} = 0$$

if $a > 0$.

(c) If roots are complex numbers, that is $b - \frac{a^2}{4} > 0$, then

$$y(x) = e^{-\frac{a}{2}x}(C_1 \cos(\sqrt{b - \frac{a^2}{4}}x) + C_2 \sin(\sqrt{b - \frac{a^2}{4}}x))$$

and

$$\lim_{x\to\infty} e^{-\frac{a}{2}x}(C_1 \cos(\sqrt{b - \frac{a^2}{4}}x) + C_2 \sin(\sqrt{b - \frac{a^2}{4}}x)) = 0$$

if $a > 0$.

Therefore, all solutions of the differential equation approaches 0 as $x \to \infty$ for $a > 0$ and $b > 0$.