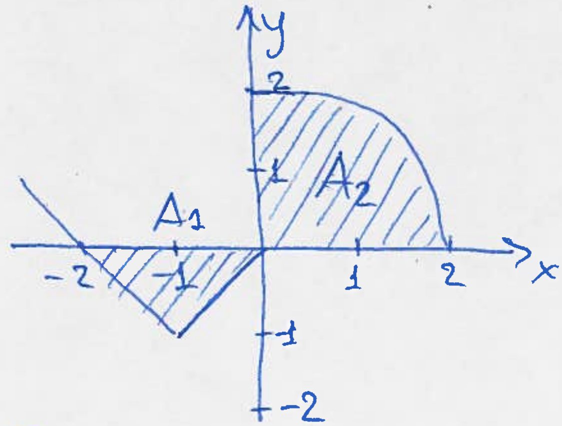
Math-205, Final Exam
June 2012
Solutions.

Q1. (a) Sketch the graph of the function

$$f(x) = \begin{cases} |x+1| - 1 & \text{if } x < 0, \\ \sqrt{4-x^2} & \text{if } 0 \le x \le 2 \end{cases}$$

Solution:



(b) Calculate $\int_{-2}^{2} f(x)dx$.

Solution: signed areas:

$$A_1 = \int_{-2}^{0} f(x)dx = -\frac{1}{2}\, 2\cdot 1 = -1.$$

$$A_2 = \frac{\pi}{4} 2^2 = \pi$$ (the area of $\frac{1}{4}$ of a circle with radius $r=2$.)

$$\int_{-2}^{2} f(x)dx = A_1 + A_2 = \underline{\pi - 1}.$$

(c) use fundamental theorem of calculus to calculate $F'(x)$, and find $F'(1)$.

Solution: $F(x) = \int_{\sqrt{x+3}}^{2} e^{4-t^2} dt \Rightarrow F'(x) = -e^{4-(x+3)} \cdot \left[(x+3)^{1/2}\right]' =$

$= -\frac{1}{2} e^{1-x} (x+3)^{-1/2}$; $\quad F'(1) = -\frac{1}{2} e^0 \cdot 4^{-1/2} = -\frac{1}{4}$.

Q2: Calculate indefinite integrals:

(a) $\int \frac{\sin t}{2-\cos^2 t}\,dt =$ $\left|\begin{array}{l} u=\cos t \\ du=-\sin t\,dt \end{array}\right.$

$= -\int \frac{du}{2-u^2} =$

$= +\int \frac{du}{(u-\sqrt{2})(u+\sqrt{2})} = \frac{1}{2\sqrt{2}}\int\left(\frac{1}{u-\sqrt{2}} - \frac{1}{u+\sqrt{2}}\right)du$

$= \frac{1}{2\sqrt{2}}\left(\ln|u-\sqrt{2}| - \ln|u+\sqrt{2}|\right) = \frac{1}{2\sqrt{2}}\ln\left|\frac{\cos(t)-\sqrt{2}}{\cos t+\sqrt{2}}\right| + C$

(b) $\int x\ln^2(x)\,dx = \frac{1}{2}\int \ln^2(x)\,d\,x^2 = \frac{x^2}{2}\ln^2 x - \int \frac{x^2}{2}\,d\ln^2 x$

$= \frac{x^2}{2}\ln^2(x) - \int x\ln x\,dx = \frac{x^2}{2}\ln^2 x - \int \ln x\,d\left(\frac{x^2}{2}\right) =$

$= \frac{x^2}{2}\ln^2 x - \frac{x^2}{2}\ln x + \int \frac{x^2}{2}\frac{1}{x}\,dx =$

$= \frac{x^2}{2}\ln^2 x - \frac{x^2}{2}\ln x + \frac{x^2}{4} + C = \frac{x^2}{2}\left(\ln^2 x - \ln x + \frac{1}{2}\right) + C$

(c) $\int 4\cos^4(x)\,dx = \int (1+\cos(2x))^2\,dx$ $\left\{2\cos^2 x = 1+\cos(2x)\right\}$

$= \int\left[1 + 2\cos(2x) + \cos^2(2x)\right]dx =$

$= x + 2\int \cos(2x)\,dx + \int \frac{1+\cos(4x)}{2}\,dx =$

$= x + \sin(2x) + \frac{x}{2} + \frac{1}{2}\int \cos(4x)\,dx =$

$= \frac{3}{2}x + \sin(2x) + \frac{1}{8}\sin(4x) + C$

Q3. Find the antiderivative of $f(x)$ that satisfies the given condition:

(a) $f(x) = (1+e^x)^2$; $F(0) = 2$:

$$F(x) = \int (1 + 2e^x + e^{2x})dx = x + 2e^x + \int e^{2x}dx =$$

$$= x + 2e^x + \frac{1}{2}e^{2x} + C;$$

$$F(0) = 2 \Rightarrow 2e^0 + \frac{1}{2}e^0 + C = 2 \Rightarrow C = -\frac{1}{2}$$

$$F(x) = x + 2e^x + \frac{1}{2}e^{2x} - \frac{1}{2}.$$

(b). $f(x) = \dfrac{x}{x^2-2x-3}$; $F(1) = 0$.

Solution:

$$f(x) = \frac{x}{(x-3)(x+1)} = \frac{A}{x-3} + \frac{B}{x+1} \Rightarrow \underline{x = A(x+1) + B(x-3)}$$

at $x = 3 \Rightarrow 3 = 4A$, $A = \frac{3}{4}$;

at $x = -1 \Rightarrow -1 = -4B$, $B = \frac{1}{4}$;

$$\Rightarrow F(x) = \int \left(\frac{3}{4}\frac{1}{x-3} + \frac{1}{4}\frac{1}{x+1}\right)dx =$$

$$= \frac{3}{4}\ln|x-3| + \frac{1}{4}\ln|x+1| + C.$$

$$F(1) = 0 \Rightarrow \frac{3}{4}\ln 2 + \frac{1}{4}\ln 2 + C = 0.$$

$$\boxed{C = -\ln 2}$$

Q4: Evaluate the following definite integrals:

(a) $$\int_0^{\pi/4} \frac{\sec^2(x)}{4+\tan^2(x)}\,dx \;;$$

$$\boxed{\begin{aligned} u &= \tan(x) \\ du &= \sec^2(x)\,dx \\ u(0) &= 0 \\ u\left(\tfrac{\pi}{4}\right) &= 1 \end{aligned}}$$

$$= \int_0^1 \frac{du}{u^2+4} =$$

$$= \frac{1}{2}\arctan\left(\frac{u}{2}\right)\Big|_0^1 = \frac{1}{2}\arctan\left(\frac{1}{2}\right)$$

(b) $$\int_0^3 t\sqrt{1+t}\,dt = ;$$

$$\left| \begin{aligned} x &= \sqrt{1+t}\;; \quad t = x-1 \\ dx &= dt \\ x(0) &= 1\,; \\ x(3) &= 4 \end{aligned}\right.$$

$$= \int_1^4 (x-1)\,x^{1/2}\,dx =$$

$$= \int_1^4 \left(x^{3/2} - x^{1/2}\right)dx =$$

$$= \left(\frac{2}{5}x^{5/2} - \frac{2}{3}x^{3/2}\right)\Big|_1^4 = \left(\frac{2}{5}\cdot 2^5 - \frac{2}{3}2^3 - \frac{2}{5} + \frac{2}{3}\right) =$$

$$= \frac{62}{5} - \frac{14}{3} = \frac{116}{15}$$

Q5: Improper integrals:

(a) $$\int_2^\infty \frac{dx}{x\ln^2 x} = \lim_{t\to\infty}\int_2^t \frac{dx}{x\ln^2 x} = \lim_{t\to\infty}\int_2^t \frac{d\ln x}{\ln^2 x} =$$

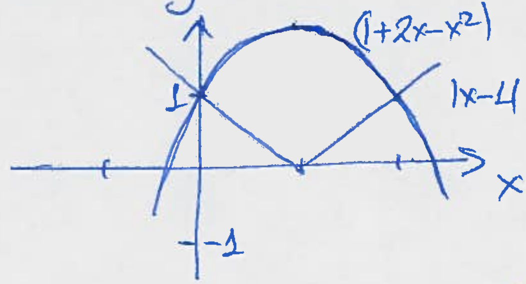
$$= \lim_{t\to\infty}\left(\frac{-1}{\ln x}\right)\Big|_2^t = \frac{1}{\ln 2} - \lim_{t\to\infty}\frac{1}{\ln t} = \frac{1}{\ln 2} - 0$$

(b) $$\int_1^2 \frac{dx}{(x-1)^{3/2}} = -2\lim_{t\to 1^+}\frac{1}{\sqrt{x-1}}\Big|^2 = -2\lim\left(1 - \frac{1}{\sqrt{t-1}}\right) \to \infty$$

DNE.

Q6. (a) Sketch the curves and find the area enclosed: $y = 1+2x-x^2$, $y=|x-1|$.

Solution:



Both curves are symmetric with respect to $x=1$.
Points of intersection: at $x>1 \Rightarrow 1+2x-x^2 = x-1$
$\Rightarrow x^2 - x - 2 = 0$, $(x-2)(x+1)=0$; $\Rightarrow$ $\underline{x=2}$

(the other point is <1, so should not be considered).

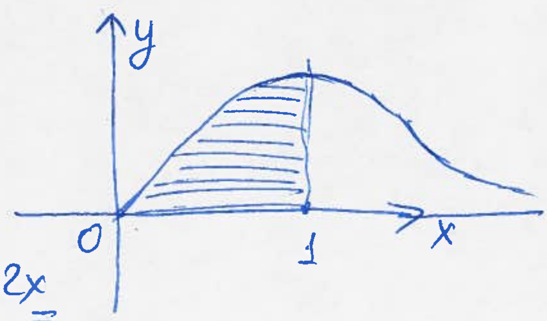
for $x<1$ the solution (point of intersection) is $x=0$

$$\Rightarrow A = \int_0^2 (1+2x-x^2-|x-1|)dx = 2\int_0^1 (1+2x-x^2-1+x)dx$$

$$= 2\int_0^1 (3x-x^2)dx = 2\left(\frac{3}{2}x^2 - \frac{1}{3}x^3\right)\Big|_0^1 = 2\left(\frac{3}{2}-\frac{1}{3}\right) = \frac{7}{3}$$

(b) Find the volume of a solid obtained by rotating the region bounded by $y=xe^{-x}$, $y=0$, $x=0$, $x=1$ about x-axis

Solution:



$$V = \pi\int_0^1 (xe^{-x})^2 dx =$$

$$= \pi\int_0^1 x^2 e^{-2x} dx = -\frac{\pi}{2}\int_0^1 x^2 de^{-2x} =$$

$$= -\frac{\pi}{2}x^2e^{-2x}\Big|_0^1 + \pi\int_0^1 e^{-2x}x\,dx = -\frac{\pi}{2}e^{-2} - \left(\frac{\pi}{2}xe^{-2x}\right)\Big|_0^1 +$$

$$+ \frac{\pi}{2}\int_0^1 e^{-2x}dx = -\frac{\pi}{2}e^{-2} - \frac{\pi}{2}e^{-2} - \left(\frac{\pi}{4}e^{-2x}\right)\Big|_0^1 = \frac{\pi}{4}(1-5e^{-2}).$$

Q6(c) Find the average value of $f(x) = \sin(x) \cdot \cos^3 x$ on the interval $[0, \pi/2]$.

$$\bar{f} = \frac{1}{\frac{\pi}{2}-0} \int_0^{\pi/2} \sin(x) \cos^3(x)\, dx = \qquad \left[\begin{array}{l} t = \cos(x) \\ dt = -\sin(x)dx \\ t(0) = 1 \\ t\left(\frac{\pi}{2}\right) = 0 \end{array}\right.$$

$$= \frac{2}{\pi} \int_1^0 -t^3 dt = \frac{1}{2\pi} t^4 \Big|_0^1 = \frac{1}{2\pi}$$

Q7. Find the limits:

(a) $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} \left(2 - \frac{1}{n}\right)\sqrt{\frac{3n+1}{n-1}} = \lim\limits_{n\to\infty}\left(2-\frac{1}{n}\right)\lim \sqrt{\frac{3+1/n}{1-1/n}}$

$= 2 \cdot \sqrt{3}$

(b) $\lim\limits_{n\to\infty} \frac{\ln(n^3-1)}{n+1} = \lim\limits_{x\to\infty} \frac{\ln(x^3-1)'}{(x+1)'} = \lim\limits_{x\to\infty} \frac{3x^2}{(x^3-1)} = 0$

Q8. Determine whether the series is divergent or convergent, and if convergent, absolutely or conditionally.

Solutions: (a) $\sum\limits_{n=1}^{\infty} (-1)^n \sqrt{\frac{4n-1}{1+n^2}}$;

check absolute convergence:

$\sum\limits_{n=1}^{\infty} \left(\frac{4n-1}{n^2+1}\right)^{1/2}$; compare with $\sum\limits_{n=1}^{\infty} \frac{1}{\sqrt{n}} \left(= \sum b_n\right)$

$\lim\limits_{n\to\infty} \frac{|a_n|}{b_n} = \lim\limits_{n\to\infty}\left(\sqrt{\frac{4n-1}{n^2+1}} \Big/ \frac{1}{\sqrt{n}}\right) = \lim\limits_{n\to\infty}\left(\frac{4n^2-n}{n^2+1}\right)^{1/2} = 2$

the series $\sum \frac{1}{n^{1/2}}$ is divergent p-series with $p = 1/2$.

$\Rightarrow$ the original series is absolutely divergent.

For conditional convergence, check for the alternating series test:

(a) $\lim\limits_{n\to\infty} |a_n| = \lim\limits_{n\to\infty} \left(\frac{4n-1}{n^2+1}\right)^{1/2} = \lim \frac{1}{n^{1/2}} \left(\frac{4-1/n}{1+1/n^2}\right)^{1/2} = 0$

(b) for monotonicity:

$$(a_x^2)' = \left(\frac{4x-1}{x^2+1}\right)' = \frac{4(x^2+1)-(4x-1)2x}{(x^2+1)^2} = \frac{4-4x^2+2x}{(x^2+1)^2} < 0$$

for $\underline{x > 10}$ (for example); $\Rightarrow$ a_n is monotonically decreasing for $\underline{\text{(at least)}}$ $n > 10$.

The series is convergent by alternating test;
$\Rightarrow$ it is $\underline{\text{conditionally convergent}}$

Q8 b. $a_n = n\, e^{-n^2}$;
applying the integral test:
$f(x) = x e^{-x^2}$; $f'(x) = e^{-x^2} - 2x^2 e^{-x^2} = (1-2x^2) e^{-x^2} < 0$
for $x \geq 1$ $\Rightarrow$ a_n is monotonically decresing.

$$\int_1^\infty x e^{-x^2} dx = -\frac{1}{2} e^{-x^2} \Big|_1^\infty = -0 + \frac{1}{2} e^{-1} < \infty$$

$\Rightarrow$ the series is convergent.
It is absolutely convergent because $\underline{a_n > 0}$

$\underline{\text{Q 9(a)}}$. Find the radius of convergence and the interval of convergence of the series $\sum\limits_{n=0}^{\infty} \frac{(x+2)^{3n}}{8^n}$

Solution: $\left|\frac{a_{n+1}}{a_n}\right| = \left| \frac{(x+2)^{3(n+1)}}{8^{n+1}} \frac{8^n}{x^{3n}} \right| = |x+2|^3 \frac{1}{8} < 1$

$\Rightarrow$ $|x+2| < 2$; the radius is $R=2$.

the interval: $\underline{-4 < x < 0}$;

Check at the end points:

at $x = 0 \Rightarrow \sum_{n=1}^{\infty} \frac{2^{3n}}{8^n} = \sum_{n=1}^{\infty} 1 \to \infty$

at $x = -4 \Rightarrow$ the same; $\sum_{n=1}^{\infty} (-1)^{3n} \to$ divergent.

$\Rightarrow$ the interval of convergence is $(-4, 0)$.

<u>9b.</u> Find the Maclaurin series for $f(x) = \frac{\ln(1+x^2)}{x}$.

Solution: $\ln(1+z) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{z^n}{n}$ because

$$\ln(1+z)' = \frac{1}{1+z} \to 1 \ (\text{at } z = 0)$$

$$\ln(1+z)'' = -\frac{1}{(1+z)^2} \to -1$$

$$\ln(1+z)''' = \frac{2}{(1+z)^3} \to 2!$$

$$\ln(1+z)^{(n)} = \frac{(n-1)!(-1)^{n+1}}{(1+z)^n} \to \frac{(-1)^{n+1}(n-1)!}{1}$$

$\Rightarrow$ for $z = x^2 \Rightarrow \ln(1+x^2) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} x^{2n}}{n}$

$$\frac{\ln(1+x^2)}{x} = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} x^{2n-1}}{n}$$

Bonus: Determin the domain $[a,b]$ of $f(x) = \sqrt{4x - x^2}$ and graph the function:

$[a,b] = [0,4]$ since $4x - x^2 = x(4-x) \underline{\geq 0}$

$f(x) = \sqrt{4x - x^2} = \sqrt{4 - (x-2)^2}$ — it is a semicircle.

$\Rightarrow \int_0^4 f(x)\,dx = \frac{\pi}{2} r^2 = \frac{\pi}{2} 2^2 = 2\pi$