Department of Computer Science & Software Engineering

Comp232 Mathematics for Computer Science

Fall 2020

Assignment 4. Due date: Friday December 4
SOLUTION

1. Use mathematical induction to solve the following:

   (a) Find a formula for $\frac{1}{1.2}+\frac{1}{2.3}+\cdots+\frac{1}{n(n+1)}$ by examining the values of this expression for small values of n.

   (b) Show that $7^n - 1$ is a multiple of 6 for all $n \in \mathbb{N}$

SOLUTION

(a) By computing the first few sums and getting the answers 1/2, 2/3, and 3/4, we can conjuncture that the sum is $n/(n+1)$.

Proof

Let $P(n)$ denotes that the sum is $n/(n+1)$

Basis step, $P(1) = 1/2$, the reader can verify that the conjecture is valid.

Inductive hypothesis, suppose $P(k)$ represented by $\frac{1}{1.2}+\frac{1}{2.3}+\cdots+\frac{1}{k(k+1)} = \frac{k}{k+1}$ is true

show that $P(k+1)$ represented by $\left[\frac{1}{1.2}+\frac{1}{2.3}+\cdots+\frac{1}{k(k+1)}\right]+\frac{1}{(k+1)(k+2)} = \frac{k+1}{k+2}$ is true

Starting from the left, we replace the quantity in brackets by $k/(k+1)$ (by the inductive hypothesis), and then do the algebra:

$$\frac{k}{(k+1)}+\frac{1}{(k+1)(k+2)} = \frac{k^2+2k+1}{(k+1)(k+2)} = \frac{k+1}{k+2}$$

yielding the desired expression.

(b) Let $P(n)$ denotes that $7^n - 1$ is a multiple of 6 for all $n \in \mathbb{N}$

Basis step, $P(1) = 7^1 - 1 = 6$, true since 6 is multiple of 6, $(6 = 1 \times 6)$.

Inductive hypothesis, suppose $P(k)$ represented by : $7^k - 1$ is a multiple of 6 for all $k \in \mathbb{N}$ is true.

show that $P(k+1)$ represented by : $7^{k+1} - 1$ is a multiple of 6 for all $k \in \mathbb{N}$ is true.

From the inductive hypothesis $7^k - 1$ is a multiple of 6 for all $k \in \mathbb{N}$.
Then $7^k - 1 = 6 \times C$, for some $C \in \mathbb{N}$
and $7^k = 6 \times C + 1$

$P(k+1)$ can be written as: $7^{k+1} - 1 = 7 \times 7^k - 1$, substitute $(6 \times C + 1)$ for 7^k yields:
$7^{k+1} - 1 = 7(6 \times C + 1) - 1 = 7 \times 6 \times C + 6 = 6\ (7 \times C + 1)$. This shows that $7^{k+1} - 1$ is multiple of 6 and $P(k+1)$ is true.

Thus, $P(n)$ is true , for all $n \in \mathbb{N}$

2. Suppose that a bank machine can dispense money in either 3\$ or 10\$ bills. Show that any amount over 17\$ could be dispensed with combinations of only the 3\$ or the 10\$ bills

SOLUTION

P(18): Eighteen dollars can be made using six 3-dollar bills.
$P(k) \to P(k+1)$: Suppose that k dollars can be formed, for some $k \geq 18$.

- If at least two 10-dollar bills are used, replace them by seven 3-dollar bills to form $k+1$ dollars.
- Otherwise (that is, at most one 10-dollar bill is used), at least three 3-dollar bills are being used, and three of them can be replaced by one 10-dollar bill to form $k+1$ dollars.

3. Use mathematical induction to show that n lines in the plane passing through the same point divide the plane to $2n$ parts.

SOLUTION

The basis step follows since one line divides the plane into 2 regions. Now assume that k lines passing through the same point divide the plane into $2k$ regions. Adding the $(k+1)st$ line splits exactly two of these regions into two parts each. Hence, the $k+1$ lines split the plane into $2k + 2 = 2(k+1)$ regions.

4. Let $a_1 = 2$, $a_2 = 9$, and $a_n = 2a_{n-1} + 3a_{n-2}$ for $n \geq 3$. Use strong induction to show that $a_n \leq 3^n$ for all positive integer n.

SOLUTION

Let $P(n)$ be the proposition that $a_n \leq 3^n$.
Basis step: $a_1 = 2 \leq 3 = 3^1$ and $a_2 = 9 \leq 9 = 3^2$.
Inductiove step: assume $P(j)$ is true for $1 \leq j \leq k$.
Then $a_j \leq 3^j$ for $1 \leq j \leq k$.
Hence $a_{k+1} = 2a_k + 3a_{k-1} \leq 2.3^k + 3.3^{k-1} = 2.3^k + 3^k = 3.3^k = 3^{k+1}$.

5. Give an example of the following relations:

(a) A relation on $\{a, b, c\}$ that is reflexive and transitive, but not antisymmetric

(b) A relation on $\{1, 2\}$ that is symmetric and transitive, but not reflexive.

(c) A relation on $\{1, 2, 3\}$ that is reflexive and transitive, but not symmetric.

SOLUTION

(a) $\{(a, a), (b, b)(c, c), (a, b), (b, a)\}$

(b) $\{(1, 1)\}$

(c) $\{(1, 1), (2, 2), (3, 3), (1, 2)\}$

6. Give a recursive definition of the sequence $\{a_n\}$. where $n = 1, 2, 3, \ldots$ if

   a) $a_n = 4n - 2$

   b) $a_n = 1 + (-1)^n$

   c) $a_n = n(n + 1)$

   In all parts of this question you must proof and verify your answer.

   SOLUTION

   a) Each term is 4 more than the term before it. We can therefore define the sequence by $a_1 = 2$ and $a_{n+1} = a_n + 4$ for all $n \geq 1$.

   b) We note that the terms alternate: 0, 2, 0, 2, and so on. Thus we could define the sequence by $a_1 = 0$, $a_2 = 2$, and $a_n = a_{n-2}$ for all $n \geq 3$.

   c) The sequence starts out 2, 6, 12, 20, 30, and so on. The differences between successive terms are 4, 6, 8, 10, and so on. Thus the n^{th} term is $2n$ greater than the term preceding it; in symbols: $a_n = a_{n-1} + 2n$. Together with the initial condition $a_1 = 2$, this defines the sequence recursively.

7. Consider the following relations on the set of positive integers.

   $R_1 = \{(x, y) \mid x + y > 10\}$

   $R_2 = \{(x, y) \mid y \text{ divides } x\}$

   $R_3 = \{(x, y) \mid gcd(x, y) = 1)\}$

   $R_4 = \{(x, y) \mid x \text{ and } y \text{ have the same prime divisors }\}$

   Which of these relations are: reflexive, symmetric, antisymmetric or transitive? Justify your answer.

   SOLUTION

   Reflexive:
   R_1 is not reflexive since $1 + 1 < 10$, so $(1, 1)$ is not in R_1
   R_2 is reflexive since $x|x$ for every positive integer x.

R_3 is not reflexive since $gcd(2,2) = 2$, so $(2,2)$ is not in R_3.
R_4 is reflexive since x and x have the same prime divisors for every integer x, so $(x,x) \in R_4$ for all x.

Symmetric:
R_1 is symmetric, since $x + y > 10$ implies $y + x > 10$.
R_2 is not symmetric since $1 \mid 2$, but $2 \nmid 1$.
R_3 is symmetric since $gcd(x,y) = 1$ implies $gcd(y,x) = 1$.
R_4 is symmetric since x any y have the same prime divisors if and only if y and x have the same prime divisors.

Antisymmetric:
R_1 is not antisymmetric, since $(2,9)$ and $(9,2)$ both belong to R_1.
R_2 antisymmetric since $x \mid y$, and $y \mid x$. imply that $x = y$ if x and y are positive integers.
R_3 is not antisymmetric since $gcd(2,1) = gcd(1,2) = 1$.
R_4 is not antisymmetric since 12 any 18 have the same prime divisors, namely 2 and 3, and 18 and 12 have the same prime divisors.

Transitive:
R_1 is not transitive since $(2,9) \in R_1$ and $(9,3) \in R_1$ but $(2,3) \notin R_1$.
R_2 is transitive since $x \mid y$ and $y \mid z$ imply that $x \mid z$.
R_3 is not transitive since $gcd(4,5) = 1$ and $gcd(5,6) = 1$ but $gcd(4,6) = 2$.
R_4 is transitive, for if x and y have the same prime divisors and y and z have the same prime divisors, then x and z have the same prime divisors.

8. Suppose A is the set composed of all ordered pairs of positive integers. Let R be the the relation defined on A where $(a,b)R(c,d)$ means that $ad = bc$. Show that R is an equivalence relation.

SOLUTION

A relation on a set A is called an equivalence relation if it is reflexive, symmetric, and transitive.

R is reflexive: $((a,b),(a,b)) \in R$ since $ab = ba$.

R is symmetric: if $((a,b),(c,d)) \in R$ then $ad = bc$, which also means that $cb = da$, so $((c,d),(a,b)) \in R$.

R is transitive: if $((a,b),(c,d)) \in R$ and $((c,d),(e,f)) \in R$ then $ad = bc$ and $cf = de$.
Multiplying these equations gives $acdf = bcde$, and since all these numbers are nonzero, we have $af = be$, so $((a,b),(e,f)) \in R$.

Thus, R is an equivalence relation.