PROBLEMS FOR CHAPTER 1

1. For a function f (x) it is known that f (1.8) = -1.1664 and f'(1.8) = 3.8880. Find the approximate value of x when f (x) = 0, using Taylor series expansion.

Solution:

f(1.8)= -1.1664   f'(1.8)= 3.8880

Taylor Series:

$$f(x) = f(a) + (x-a)f'(a) + \frac{(x-a)^2\, f''(a)}{2!} ..... + \frac{(x-a)^n\, f^n(a)}{n!}$$

Therefore f(x) = f(1.8) + (x-1.8) f'(1.8) = 0

-1.1664 + 3.8880(x-1.8) = 0

x = 2.1

4. Given the Taylor series expansion of f(x) = sin(x) about x=0. From the obtain sin(π/4) to an accuracy of 4 digits.

Solution:

f(x) = sin(x)   Taylor series expansion about 0

$$f(x) = f(a) + (x-a)f'(a) + \frac{(x-a)^2 f''(a)}{2!} ...... + \frac{f^n(a)(x-a)^n}{n!}$$

f(a) = sin0 = 0

f'(a) = cos0 = 1

f''(a) = -sin0 = 0

f'''(a) = -cos0 = 1

therefore,

$$f(x) = 0 + x + 0 - \frac{x^3}{3!} + 0 + \frac{x^5}{5!} .....$$

$$= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} ....$$

$$\text{Sin}(\pi/4) = f(\pi/4) = \frac{\pi}{4} - \frac{(\pi/4)^3}{3!} + \frac{(\pi/4)^5}{5!} ....$$

$$= 0.7071$$

5. In problem 4, obtain the values of sin(π /4) considering 1,2,3,4,5,6,7,8,9 and 10 terms in the Taylor series expansion. The result must converge to the exact value of 1/√2. Check whether the convergence can be improved using Aitkin's Δ^2 process. Comment on the outcome.

Solution: Considering the first 10 terms of the as shown below

$$f(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \frac{x^{11}}{11!} + \frac{x^{13}}{13!} - \frac{x^{15}}{15!} + \frac{x^{17}}{17!} - \frac{x^{19}}{19!}$$

We have formed a table where

$$p_n^* = p_n - \frac{(p_{n+1} - p_n)^2}{p_{n+2} - 2p_{n+1} + p_n}$$

n	sin(π /4)	Δ^2
1	0.785398163	0.707068533
2	0.704652651	0.707106990
3	0.707143045	0.707106779
4	0.707106469	0.707106781
5	0.707106782	
6	0.707106781	
7		
8		
9		
10		

6. Obtain $(1/3)^{10}$ using

(i) 4 digit arithmetic (result after each multiplication rounded off to 4 digits).

(ii) 6 digit arithmetic

(iii) 8 digit arithmetic

Solution: the Table below is obtained by the value rounded up by (1/3)

E.g. (1/3)*(1/3)= 0.1111

0.1111*(1/3)= 0.03702 or $0.3702*10^{-1}$

And the table below is obtained.

$(1/3)^n$	Actual	4- digit	6 - digit	8 - digit
1	1/3	0.3333	0.333333	0.33333333
2	1/9	0.1111	0.111111	0.11111111
3	1/27	0.3702×10^{-1}	0.370369×10^{-1}	$0.37037036 \times 10^{-1}$
4	1/81	0.1234×10^{-1}	0.123456×10^{-1}	$0.12345679 \times 10^{-1}$
5	1/243	0.4113×10^{-2}	0.411521×10^{-2}	$0.41152261 \times 10^{-2}$
6	1/729	0.1371×10^{-2}	0.137173×10^{-2}	$0.13717420 \times 10^{-2}$
7	1/2187	0.4569×10^{-3}	0.457244×10^{-3}	$0.45724734 \times 10^{-3}$
8	1/6561	0.1523×10^{-3}	0.152415×10^{-3}	$0.15241578 \times 10^{-3}$
9	1/19683	0.5076×10^{-4}	0.508048×10^{-4}	$0.50805259 \times 10^{-4}$
10	1/59049	0.1692×10^{-4}	0.169349×10^{-4}	$0.16935086 \times 10^{-4}$

7. Approximate sin(x) using a polynomial
$P2(x) = a0 + a1\ x + a_2x^2$
In the range $0 \le x \le \pi/2$, by minimizing the squared error in the range, as the given by

$$E = \int_{0}^{\pi/2} [P^2(x) - \sin(x)^2]dx$$

The coefficients a_0, a_1 and a_2 are obtained by solving the equation
$\frac{\partial E}{\partial a1}$= 0, I=-, 1,2.
Compare $P_2(x)$ with the Taylor series expansion of sin(x) about x = 0, considering only three terms.

Solution: the value of $P_2(x) = a0 + a1\ x + a_2x^2$ is obtained by using the least squares method later taught in chapter 4

$$P(x) = a_0 + a_1x^1 + a_2x^2$$

Values for sin(x) in the range $0 \le x \le \pi/2$ are

0	π/6	π/3	π/2
0	0.5	0.866025	1

The matrix has the form:

$$\begin{bmatrix} n+1 & \sum_{i=0}^{n} x_i & \sum_{i=0}^{n} x_i^2 \ldots & \sum_{i=0}^{n} x_i^m \\ \sum_{i=0}^{n} x_i & \sum_{i=0}^{n} x_i^2 & \sum_{i=0}^{n} x_i^3 \ldots & \sum_{i=0}^{n} x_i^{m+1} \\ \sum_{i=0}^{n} x_i^2 & \sum_{i=0}^{n} x_i^3 & \sum_{i=0}^{n} x_i^4 \ldots & \sum_{i=0}^{n} x_i^{m+2} \\ \vdots & & & \\ \sum_{i=0}^{n} x_i^m & \sum_{i=0}^{n} x_i^{m+1} & \sum_{i=0}^{n} x_i^{m+2} & \sum_{i=0}^{n} x_i^{2m} \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ \vdots \\ a_m \end{bmatrix} = \begin{bmatrix} \sum_{i=0}^{n} f(x_i) \\ \sum_{i=0}^{n} x_i f(x_i) \\ \sum_{i=0}^{n} x_i^2 f(x_i) \\ \vdots \\ \sum_{i=0}^{n} x_i^m f(x_i) \end{bmatrix}$$

substituting the data gives

$$\begin{bmatrix} 3.1416 & 3.838 & 5.168 \\ 3.838 & 5.168 & 7.366 \\ 5.168 & 7.366 & 10.862 \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 2.739 \\ 3.554 \\ 4.942 \end{bmatrix}$$

Solving the matrix gives us

$$a_2 = -0.408$$

$$a_1 = 1.330$$

$$a_0 = -0.081$$

Hence $\quad P(x) = -0.408x^2 + 1.33x - 0.081$

Whereas the taylor series expansion of sin(x) about x=0 from problem 4 is

$$f(x) = 0 + x + 0 - \frac{x^3}{3!} + 0 + \frac{x^5}{5!}$$

Or

$$f(x) = 0 + x + 0 - 0.16667\, x^3 + 0 + 0.00833\, x^5$$