COMP361 NUMERICAL METHODS FINAL EXAM

The total duration of this Final Exam is 180 minutes, which includes an estimated 15 minutes for downloading and preparation, and 25 minutes for scanning and for uploading the completed Final Exam to the **Quiz 2** category of the EAS. All 10 Problems have the same value, namely 10 points.

You may consult the Lecture Notes during the Final Exam, including the version with solutions.

You will have to work quite efficiently in order to complete all 10 Problems within the allotted time.

It is strongly recommended that you first work on the solution of a Problem on scratch paper. Thereafter write your solution in a very clear fashion on the sheets of white paper that you will scan (or take a picture of) and upload to the **Quiz 2** category of the EAS as a single PDF file. Do not submit the scratch paper, but keep it in case issues regarding authenticity arise.

The solutions you submit must contain sufficient details of the complete solution procedure, and not just the final result. In most Problems you will be asked to put key results in a hand-drawn box.

Communicating with others is strictly prohibited during the Final Exam. Not adhering to this rule can have severe consequences. Clear instances of potential copying were observed on the Term Test, for which disciplinary action has not yet been initiated.

This course is about computation with real arithmetic. However, many of the real numbers that appear in this Final Exam are in fact integers or rational numbers. This allows most computations to be done with exact arithmetic, thereby avoiding extensive computations with a calculator.

Problem 1. For each of the following state whether or not it is valid for all n by n matrices $\mathbf{A}$:

(a) $\| \mathbf{A} \|_1 \leq \| \mathbf{A} \|_\infty$, (b) $\| \mathbf{A} \|_\infty \leq \| \mathbf{A} \|_1$, (c) $\| \mathbf{A} \|_1 = \| \mathbf{A} \|_\infty$.

Put your answer in a hand-drawn box.

Problem 2. Make use of the Banach Lemma to show that the following matrix is invertible:

$$\mathbf{A} = \begin{pmatrix} 4 & 1 & -1 & 1 \\ 1 & 5 & 1 & -2 \\ -2 & 1 & 5 & 1 \\ 1 & -1 & 1 & 4 \end{pmatrix} .$$

Also use the Banach Lemma to find a bound on $\| \mathbf{A}^{-1} \|_\infty$. Show the details of your computations, using *proper mathematical notation* . Put your bound on $\| \mathbf{A}^{-1} \|_\infty$ in a hand-drawn box.

Also put a corresponding bound on the *condition number* , $\mathrm{cond}(\mathbf{A})$, in a separate hand-drawn box.

Problem 3. Suppose that solving a *tridiagonal system* of linear equations of dimension $n = 10^3$ takes 0.001 second on a certain computer. Based only on the necessary number of multiplications and divisions, estimate how much time it would take to solve a *general system* of linear equations of the same dimension $n = 10^3$ on that computer.

Show all details of how you arrive at your answer, and put your final answer in a hand-drawn box.

Problem 4. Consider the fixed point iteration

$$x^{(k+1)} = f(x^{(k)}) \ , \quad k = 0, 1, 2, 3, \cdots \ , \quad \text{where } f(x) = \frac{4x^5 + 5}{5x^4} \ .$$

Analytically determine all fixed points of this iteration and determine if they are attracting or repelling. If attracting then also determine if the convergence is linear or quadratic. Show the details of how you arrive at your answers, and put your final answers in a hand-drawn box.

Do you recognize where this iteration arises? Put your answer in a separate hand-drawn box.

Also give the standard graphical interpretation of the iteration, showing the curves $y = f(x)$ and $y = x$, as well as several iterations, starting with $x^{(0)} \approx 1.0$. Make sure that your graph is qualitatively correct.

What happens if $x^{(0)}$ is negative ? Are there values for $x^{(0)}$ that do not result in convergence ?

Problem 5. Carry out the first iteration of Newton's method for solving the system

$$x_1^2 + x_2^2 - 4 = 0 \ ,$$

$$e^{x_1} - x_2 = 0 \ .$$

using $x_1^{(0)} = 0$, $x_2^{(0)} = 1$ as initial guesses. Show all details, and put your values of $x_1^{(1)}$ and $x_2^{(1)}$ in a hand-drawn box.

Also give a hand-drawn sketch of the situation by drawing the curves $x_1^2 + x_2^2 - 4 = 0$ and $e^{x_1} - x_2 = 0$ in the same x_1, x_2 - plane, indicating their intersection points, as well as the initial guess $(x_1^{(0)}\ ,\ x_2^{(0)})$, and the point $(x_1^{(1)}\ ,\ x_2^{(1)})$. Make sure to put labels "x_1" and "x_2" along the respective axes.

NOTE : This Problem is meant to illustrate Newton's Method for solving a system of equations.

Thus do not reduce the two equations to a single equation.

Problem 6. Write down the Lagrange interpolating polynomial $p(x)$ that interpolates the function $f(x) = \sin(x)$ at the points $x_0 = 0$, $x_1 = \frac{\pi}{4}$, and $x_2 = \frac{\pi}{2}$. You must write $p(x)$ using the correct expressions for the Lagrange basis functions (also called "*Lagrange interpolating coefficients*") for the given values of x_0, x_1, x_2 . Enter your formula for $p(x)$ in a hand-drawn box.

Use the interpolating polynomial $p(x)$ determined above, with the correct values of $f(x) = \sin(x)$ at the three interpolation points, to approximate the value of $f(x) = \sin(x)$ at the point $x = 1.0$. Enter this value in a separate hand-drawn box.

Problem 7. Suppose we want to use polynomial interpolation at $n+1$ *Chebyshev points* to approximate the function $f(x) = \sin(x)$ by *a single polynomial* $p_n(x)$ in the entire interval $[-\frac{\pi}{2}\ ,\ \frac{\pi}{2}]$. Thus $p_n(x)$ has degree n or less, *i.e.*, $p_n(x) \in \mathbb{P}_n$. The Chebyshev points, which are usually given for the standard interval $[-1, 1]$, are now scaled to the current interval $[-\frac{\pi}{2}\ ,\ \frac{\pi}{2}]$.

Determine the formula for the bound on the maximum error $\mid p_n(x) - f(x) \mid$ in the interval $[-\frac{\pi}{2}, \frac{\pi}{2}]$, and enter it in a hand-drawn box. Make sure to use the fact that the $n+1$ Chebyshev points are now with respect to the interval $[-\frac{\pi}{2}\ ,\ \frac{\pi}{2}]$.

What is the smallest value of n for which the bound is less than 10^{-3} ? Enter your answer in an additional hand-drawn box.

Problem 8. Suppose that we use *local polynomial interpolation* to approximate the function $f(x) = \sin(x)$ in the interval $[-\frac{\pi}{2}\ ,\ \frac{\pi}{2}]$.

More precisely, we subdivide the interval $[-\frac{\pi}{2}\ ,\ \frac{\pi}{2}]$ into N subintervals of equal size $h = \frac{\pi}{N}$. In each subinterval the function $f(x)$ is interpolated by a local polynomial of degree 2 or less at three equally spaced points, namely the midpoint and the two endpoints of the subinterval.

Determine the smallest number of subintervals N that is needed so that the maximum interpolation error is less than 10^{-6} . Make use of the Table on the Page numbered 183 of the Lecture Notes, which is for a standard interval $[-1\ ,\ 1]$. Be sure to take into account that the equally spaced local interpolation points are now for subintervals of size h , where $h = \frac{\pi}{N}$.

Show the details of your computations, and enter your value of N in a hand-drawn box.

Problem 9. Given the points $x_0 = -h,\ x_1 = 0$, and $x_2 = h$, make use of *Lagrange basis functions* to determine the *numerical differentiation formula* that approximates the derivative $f'(0)$ of a function $f(x)$ in terms of $f(-h)$, $f(0)$, and $f(h)$. Show all details of your computations, and put the formula that you obtain in a hand-drawn box.

Next use *Taylor expansions* to determine the *leading error term* of your approximation formula. Show all details of your computations, and put your leading error term in a separate hand-drawn box.

For the special case where $f(x) = \sin(x)$, still with $x_0 = -h,\ x_1 = 0$, and $x_2 = h$, determine how small h must be so that the leading error term is less than 10^{-6} in absolute value. Put this value of h in a hand-drawn box.

Problem 10. Suppose we want to use *Simpson's Rule* to evaluate the definite integral

$$\int_{-1}^{1} f(x)\, dx\ , \quad \text{where} \quad f(x) \ = \ e^{(x^2)}\ .$$

Based on the *bound on the leading error term of the composite Simpson's Rule* on Page numbered 284 of the Lecture Notes, determine the minimal number of equally spaced intervals N needed so that the absolute value of this bound is less than 10^{-6}.

Show all details of your computations, and put your value of N in a hand-drawn box. Make sure to use a *correct upper bound* on $\| f'''' \|_\infty$, and show this bound in a separate hand-drawn box.