ENGR233 QA TERM TEST 1. OCTOBER 21, 2019, 10:45am–11:45am.

Closed book exam. Faculty approved calculators (SHARP EL-531 or CASIO FX-300MS).

1. Find $\vec{r}(t)$ such that $\vec{r}\,'(t) = \langle 1, 2t\rangle$ and $\vec{r}(0) = \langle 0, 0\rangle$. **[2 marks]**

This is a question from Quiz 1. You're welcome.

Solution. We have

$$\vec{r}(t) = \int_0^t \vec{r}'(u)\,\mathrm{d}u + \vec{r}(0) = \int_0^t \langle 1, 2u\rangle\,\mathrm{d}u = \langle u, u^2\rangle\Big|_0^t = \langle t, t^2\rangle.$$

2. Compute the curvature of the circular helix given by $\vec{r}(t) = \langle 3\cos t, 3\sin t, 4t\rangle$.

[5 marks]

This is similar to WeBWork Assignment 2, question 10.

Solution. *First solution.* We have $\vec{r}'(t) = \langle -3\sin t, 3\cos t, 4\rangle$, and so

$$\|\vec{r}'(t)\| = \sqrt{(-3\sin t)^2 + (3\cos t)^2 + 4^2} = \sqrt{3^2 + 4^2} = 5.$$

(We have used the Pythagorean identity $\sin^2 t + \cos^2 t = 1$.) Thus,

$$\vec{T}(t) = \frac{1}{5}\langle -3\sin t, 3\cos t, 4\rangle, \quad \vec{T}'(t) = \frac{1}{5}\langle -3\cos t, -3\sin t, 0\rangle, \quad \|\vec{T}'(t)\| = \frac{3}{5}$$

(again by the Pythagorean identity). Finally, the curvature at the point corresponding to t is

$$\kappa(t) = \frac{\|\vec{T}'(t)\|}{\|\vec{r}'(t)\|} = \frac{3/5}{5} = \frac{3}{25}.$$

That is, the circular helix has constant curvature equal to $3/25$.

The decimal representation of this number is 0.12, and this is acceptable as a final answer.

Second solution. We have $\vec{r}'(t) = \langle -3\sin t, 3\cos t, 4\rangle$, and so

$$\|\vec{r}'(t)\| = \sqrt{(-3\sin t)^2 + (3\cos t)^2 + 4^2} = \sqrt{3^2+4^2} = 5, \quad \|\vec{r}'(t)\|^3 = 5^3 = 125.$$

Also, $\vec{r}''(t) = \langle -3\cos t, -3\sin t, 0\rangle$, and so

$$\begin{aligned}
\vec{r}'(t)\times\vec{r}''(t) &= \begin{vmatrix} \vec{\imath} & \vec{\jmath} & \vec{k} \\ -3\sin t & 3\cos t & 4 \\ -3\cos t & -3\sin t & 0 \end{vmatrix} \\
&= \langle 12\sin t, 12\cos t, 9\sin^2 t + 9\cos^2 t\rangle \\
&= \langle 12\sin t, 12\cos t, 9\rangle \\
&= 3\langle 4\sin t, 4\cos t, 3\rangle,
\end{aligned}$$

by the Pythagorean identity $\sin^2 t + \cos^2 t = 1$. Thus (using the Pythagorean identity again),

$$\|\vec{r}'(t)\times\vec{r}''(t)\| = 3\sqrt{4^2+3^2} = 15.$$

Finally,

$$\kappa(t) = \frac{\|\vec{r}'(t)\times\vec{r}''(t)\|}{\|\vec{r}'(t)\|^3} = \frac{15}{125} = \frac{3}{25}.$$

That is, the circular helix has constant curvature equal to $3/25$.

Both $3/25$ and $15/125$ are acceptable as a final answer, as are equivalent expressions, within reason. The decimal representation of this number is 0.12. This is also acceptable.

3. Let $f(x,y) = x^2 - y^2$. Find a **unit** vector $\vec{u}$ for which $D_{\vec{u}}f(3,4) = 0$, where $D_{\vec{u}}f(3,4)$ is the directional derivative of f, in the direction of $\vec{u}$, at the point $(3,4)$.

[5 marks]

This is similar to textbook Section 9.5, problem 33(a), which is a recommended problem in the course outline.

Solution. We have $f_x(x,y) = 2x$, $f_y(x,y) = -2y$, and so

$$\nabla f(3,4) = \langle f_x(3,4), f_y(3,4)\rangle = \langle 6, -8\rangle.$$

Let $\vec{u} = \langle u_1, u_2 \rangle$ be a unit vector. Then

$$D_{\vec{u}} f(3,4) = \nabla f(3,4) \cdot \vec{u} = \langle 6, -8 \rangle \cdot \langle u_1, u_2 \rangle = 6u_1 - 8u_2.$$

We want $D_{\vec{u}} f(3,4) = 0$, i.e. $u_2 = (6/8)u_1 = (3/4)u_1$. Since $\vec{u}$ is a unit vector, we also have

$$1 = \|\vec{u}\|^2 = u_1^2 + u_2^2 = u_1^2 + \left(\frac{3}{4}\right)^2 u_1^2 = u_1^2\left(1 + \frac{9}{16}\right) = u_1^2\left(\frac{25}{16}\right),$$

giving $u_1^2 = 16/25$, i.e. $u_1 = \pm 4/5$. We choose $u_1 = 4/5$, so that $u_2 = (3/4)u_1 = 3/5$. Thus,

$$\vec{u} = \left\langle \frac{4}{5}, \frac{3}{5} \right\rangle.$$

We could also have chosen $u_1 = -4/5$, giving $u_2 = -3/5$ and

$$\vec{u} = \left\langle -\frac{4}{5}, -\frac{3}{5} \right\rangle.$$

This is also perfectly correct and acceptable as an answer. However, these are the only two possible correct answers. There are no other unit vectors for which $D_{\vec{u}} f(3,4) = 0$.

While not necessary, we can check out solution. First of all,

$$\|\vec{u}\|^2 = \left(\frac{4}{5}\right)^2 + \left(\frac{3}{5}\right)^2 = \frac{4^2 + 3^2}{5^2} = 1,$$

so $\vec{u}$ is indeed a unit vector. Secondly,

$$D_{\vec{u}} f(3,4) = \nabla f(3,4) \cdot \vec{u} = \langle 6, -8 \rangle \cdot \left\langle \frac{4}{5}, \frac{3}{5} \right\rangle = 6\left(\frac{4}{5}\right) - 8\left(\frac{3}{5}\right) = \frac{24}{5} - \frac{24}{5} = 0,$$

as required. Similarly with $\langle -4/5, -3/5 \rangle$.

4. Consider the circular helix given by $\vec{r}(t) = \langle 3\cos t, 3\sin t, 4t \rangle$. Give an equation for the osculating plane at $t = \pi/2$. Write your equation in the form $ax + by + cz + d = 0$.

(Recall that the osculating plane at t is the plane having $\vec{B}(t)$ as a normal vector, where $\vec{B}(t)$ is the unit binormal vector at t.)

[6 marks]

This is similar to WeBWork Assignment 2, problem 10, and examples from class, question 13.

Solution. We have $\vec{r}'(t) = \langle -3\sin t, 3\cos t, 4\rangle$, and so

$$\|\vec{r}'(t)\| = \sqrt{(-3\sin t)^2 + (3\cos t)^2 + 4^2} = \sqrt{3^2 + 4^2} = 5.$$

(We have used the Pythagorean identity $\sin^2 t + \cos^2 t = 1$.) Thus,

$$\vec{T}(t) = \frac{1}{5}\langle -3\sin t, 3\cos t, 4\rangle,$$

and so

$$\vec{T}'(t) = \frac{1}{5}\langle -3\cos t, -3\sin t, 0\rangle = \frac{3}{5}\langle -\cos t, -\sin t, 0\rangle.$$

Notice that $\|\langle -\cos t, -\sin t, 0\rangle\| = 1$. Therefore, $\vec{N}(t) = \langle -\cos t, -\sin t, 0\rangle$, and

$$\vec{B}(t) = \vec{T}(t) \times \vec{N}(t) = \frac{1}{5}\begin{vmatrix} \vec{\imath} & \vec{\jmath} & \vec{k} \\ -3\sin t & 3\cos t & 4 \\ -\cos t & -\sin t & 0 \end{vmatrix} = \frac{1}{5}\langle 4\sin t, -4\cos t, 3\rangle.$$

(We have again used the Pythagorean identity in the third component.)

Now, when $t = \pi/2$ we have

$$\vec{r}(t) = \vec{r}(\pi/2) = \langle 0, 3, 2\pi\rangle \quad \text{and} \quad \vec{B}(t) = \vec{B}(\pi/2) = \frac{1}{5}\langle 4, 0, 3\rangle.$$

We seek the plane containing the point $(0, 3, 2\pi)$ with normal vector $\langle 4, 0, 3\rangle$. (Any vector parallel to $\vec{B}(\pi/2)$ will serve as a normal vector.) The point-normal equation of our plane is

$$\langle 4, 0, 3\rangle \cdot \langle x - 0, y - 3, z - 2\pi\rangle,$$

i.e.

$$4x + 3(z - 2\pi) = 0,$$

i.e.

$$4x + 3z - 6\pi = 0.$$

We could take any vector parallel to $\vec{B}(\pi/2)$ as a normal vector, i.e. any vector of the form $\langle 4\alpha, 0, 3\alpha\rangle$, where α is a nonzero scalar. Any equation of the form

$$4\alpha x + 3\alpha z - 6\alpha\pi = 0,$$

where α is a nonzero scalar, is equivalent to the above, and is acceptable as a final answer.

Note that the question asks for an answer in the form $ax + by + cz + d = 0$, which is not the same form as, e.g., $4x + 3(z - 2\pi) = 0$,

5. Is the vector field $\vec{F}(x, y, z) = \langle x(z^2 - y^2), y(x^2 - z^2), z(y^2 - x^2)\rangle$ incompressible? Justify your answer.

[2 marks]

This is similar to WeBWork Assignment 4, questions 6,7,8.

Solution. Yes, $\vec{F}$ is incompressible because its divergence is equal to 0 throughout its domain:

$$\begin{aligned}
\nabla \cdot \vec{F} &= \langle \partial/\partial x, \partial/\partial y, \partial/\partial z\rangle \cdot \langle x(z^2 - y^2), y(x^2 - z^2), z(y^2 - x^2)\rangle \\
&= \frac{\partial}{\partial x} x(z^2 - y^2) + \frac{\partial}{\partial y} y(x^2 - z^2) + \frac{\partial}{\partial x} z(y^2 - x^2) \\
&= (z^2 - y^2) + (x^2 - z^2) + y^2 - x^2) \\
&= 0.
\end{aligned}$$

The question asks for justification, so a correct response without justification will earn at most one mark.

END OF EXAM