Laplace functions

	$f(t)$	$F(s)$
Impulse	$\delta(t)$	1
Step (=1)	$u(t)$	$\frac{1}{s}$
Ramp (= t)	$tu(t)$	$\frac{1}{s^2}$
	$t^n u(t)$	$\frac{n!}{s^{n+1}}$
	$e^{-at}u(t)$	$\frac{1}{s+a}$
	$\sin \omega t u(t)$	$\frac{\omega}{s^2+\omega^2}$
	$\cos \omega t u(t)$	$\frac{s}{s^2+\omega^2}$
	A	A/s
	$\frac{d\delta(t)}{dt}$	s
	$\epsilon^{-at}sin(\omega t)$	$\frac{\omega}{(s+a)^2+\omega^2}$
	$\epsilon^{-at}cos(\omega t)$	$\frac{s+a}{(s+a)^2+\omega^2}$

Transform Properties

$\mathscr{L}[kf(t)] = kF(s)$

$\mathscr{L}[f_1(t)+f_2(t)] = F_1(s)+F_2(s)$

$\mathscr{L}[e^{-at}f(t)] = F(s+a)$

$\mathscr{L}[f(t-T)] = e^{-sT}F(s)$

$\mathscr{L}[f(at)] = \frac{1}{a}F\left(\frac{s}{a}\right)$

$\mathscr{L}\left[\frac{df}{dt}\right] = sF(s)-f(0-)$

$\mathscr{L}\left[\frac{d^2f}{dt^2}\right] = s^2F(s)-sf(0-)-f'(0-)$

$f(\infty) = \lim_{s\to 0} sF(s)$

$f(0+) = \lim_{s\to\infty} sF(s)$

$f^{(3)}(t) = s^3F(s)-s^2f(0)-sf'(0)-f''(0)$

Transfer function

G(s) = Output/Input = C(s)/ R(s)

Poles/Zeros

$$F(s) = \frac{N(s)}{D(s)}$$

Linearization

$\Delta x = x - x_0$

$\dot{x} = \Delta\dot{x}$

$\ddot{x} = \Delta\ddot{x}$

...

$f(x) = f(x_0) + f'(x_0)\Delta x$

General Second-Order system

$$G(s) = \frac{\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2}$$

Steady state error for negative unity feedback

	Step	Ramp	Parab.
N=0	$A/(1+k_p)$	∞	∞
N=1	0	A/k_v	∞
N=2	0	0	A/k_a

$k_p = \lim_{s\to 0} G_l(s)$

$k_v = \lim_{s\to 0} sG_l(s)$

$k_a = \lim_{s\to 0} s^2G_l(s)$

$E(s) = R(s) - Y(s)$

$e_{ss}(t) = \lim_{s\to 0} sE(s)$

Conversion to Open loop negative unity feedback

$$G_l(s) = \frac{N}{D-N}$$

Underdamped Second-Order Systems

$T_s = \frac{4}{\zeta\omega_n}$ **Settling time**

Percent Overshoot

$$\%OS = 100e^{-\zeta\pi/\sqrt{1-\zeta^2}}$$

$$\zeta = \frac{-ln(\%OS/100)}{\sqrt{\pi^2+ln^2(\%OS/100)}}$$

$T_p = \frac{\pi}{\omega_n\sqrt{1-\zeta^2}}$ **Time to peak**

$T_r = \frac{\pi - cos^{-1}\zeta}{\omega_n\sqrt{1-\zeta^2}}$ **Time to rise**

$\zeta = \cos\theta$ **critical damping angle**

First order system

G(s) = a/(s+a) or 1/(s+a)

Time constant: $T_c = 1/a$

Time to rise: $T_r = 2.2/a$

Settling Time: $T_s = 4/a$

Step response: C(s) = R(s) G(s)

Steady state val.: lim sC(s), s -> 0

Steady state error

lim sE(s), s -> 0

E(s) = R(s) - C(s)

matrix product

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} e & f \\ g & h \end{bmatrix} = \begin{bmatrix} ae+bg & af+bh \\ ce+dg & cf+dh \end{bmatrix}$$

Matrix Recap

$$A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$$

$$A^T = \begin{bmatrix} a & d & g \\ b & c & h \\ c & f & i \end{bmatrix}$$

$$cofactor(A) = \begin{bmatrix} \begin{vmatrix} e & f \\ h & i \end{vmatrix} & -\begin{vmatrix} d & f \\ g & i \end{vmatrix} & \begin{vmatrix} d & e \\ g & h \end{vmatrix} \\ -\begin{vmatrix} b & c \\ h & i \end{vmatrix} & \begin{vmatrix} a & c \\ g & i \end{vmatrix} & -\begin{vmatrix} a & b \\ g & h \end{vmatrix} \\ \begin{vmatrix} b & c \\ e & f \end{vmatrix} & -\begin{vmatrix} a & c \\ d & f \end{vmatrix} & \begin{vmatrix} a & b \\ d & e \end{vmatrix} \end{bmatrix}$$

Eigenvalues: $det(A - \lambda I) = 0$

Matrix inverse

$$A^{-1} = \frac{1}{det(A)}(Cofactor(A))^T$$

Derivative recap

$$\frac{d}{dx}[f(x)g(x)] = f(x)g'(x) + g(x)f'(x)$$

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{g(x)f'(x) - f(x)g'(x)}{[g(x)]^2}$$

Derivative	Integral (Antiderivative)
$\frac{d}{dx}n = 0$	$\int 0\, dx = C$
$\frac{d}{dx}x = 1$	$\int 1\, dx = x + C$
$\frac{d}{dx}x^n = nx^{n-1}$	$\int x^n\, dx = \frac{x^{n+1}}{n+1} + C$
$\frac{d}{dx}e^x = e^x$	$\int e^x\, dx = e^x + C$
$\frac{d}{dx}\ln x = \frac{1}{x}$	$\int \frac{1}{x}\, dx = \ln x + C$
$\frac{d}{dx}n^x = n^x \ln x$	$\int n^x\, dx = \frac{n^x}{\ln n} + C$
$\frac{d}{dx}\sin x = \cos x$	$\int \cos x\, dx = \sin x + C$
$\frac{d}{dx}\cos x = -\sin x$	$\int \sin x\, dx = -\cos x + C$
$\frac{d}{dx}\tan x = \sec^2 x$	$\int \sec^2 x\, dx = \tan x + C$
$\frac{d}{dx}\cot x = -\csc^2 x$	$\int \csc^2 x\, dx = -\cot x + C$
$\frac{d}{dx}\sec x = \sec x \tan x$	$\int \tan x \sec x\, dx = \sec x + C$
$\frac{d}{dx}\csc x = -\csc x \cot x$	$\int \cot x \csc x\, dx = -\csc x + C$

Closed loop transfer function neg feedback

$$TF = \frac{G(s)}{1+G(s)H(s)}$$

Transfer function -> State-Space

$\dot{x} = Ax + Bu$ and $y = Cx + Du$

$$\frac{d^ny}{dt^n} + a_{n-1}\frac{d^{n-1}y}{dt^{n-1}} + ... + a_1\frac{dy}{dt} + a_0t = b_0u$$

$s^2Y(s) = \ddot{y}$

$sY(s) = \dot{y}$

$Y(s) = y$

$x_1 = y,\ x_2 = \dot{y},\ x_3 = \ddot{y}$

$\dot{x}_1 = x_2,\ \dot{x}_2 = x_3,\ \dot{x}_3 = y$ ...

$$y = \underbrace{\begin{bmatrix} 1 & 0 & 0 \end{bmatrix}}_{C}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \underbrace{\begin{bmatrix} 0 & 0 & 0 \end{bmatrix}}_{D}\begin{bmatrix} u \\ \dot{u} \\ \ddot{u} \end{bmatrix}$$

$$\dot{x} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = A\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \underbrace{\begin{bmatrix} 0 \\ 0 \\ b_0 \end{bmatrix}}_{B} r$$

Routh-Hurwitz Criterion of characteristic eq.

S^n a_1 ... a_3 ...

$\vdots$ a_2 ...

S^0

$$u_1 = \frac{a_2a_3 - a_4a_1}{a_2}$$

$$u_3 = \frac{a_2a_5 - a_6a_1}{a_2}$$

Electricity, translation + rotation

Electrical Network Transfer Functions V(s) - (...) I(s) = 0

Impedance: Z(s) = V(s)/I(s) **Admittance:** Y(s) = 1/Z(s)

Impedance equations:

Resistor: R, Capacitor: 1/(Cs), Inductor: Ls

Translational Mechanical System Transfer Functions

Element Law equations:

Spring: F=kx, Viscous Damper: F=C$\dot{x}$, Mass: m$\ddot{x}$

Admittance equations:

Spring: K, Viscous Damper: Cs, Mass: Ms^2

Y(s)X(s) = F(s)

Rotational Systems Transfer Functions

M = J

Spring: K, Viscous Damper: Cs, Inertia: Js^2

$$\left(\sum Z_1(s)\right)X_1(s) - \left(\sum Z_{1,2}(s)\right)X_2(s) = F_1(s)$$

Root locus characteristic equation

$$1 + KG_l(s) = 0$$

State-space -> Transfer function

$$\frac{Y(s)}{U(s)} = C(sI-A)^{-1}B + D$$

Root locus Poles and Zeros

$$G_l(s) = \frac{N(s)}{D(s)}$$

Partial fraction expansion

$$F(S) = \frac{a}{(s+b)(s+c)^2} = \frac{K_1}{s+b} + \frac{K_2}{(s+c)^2} + \frac{K_3}{s+c}$$

$$F(s) = \frac{a}{s(s^2+bs+c)} = \frac{K_1}{s} + \frac{sK_2+K_3}{s^2+bs+c}$$

Root locus asymptote location

$$\sigma_a = \frac{\sum poles - \sum zeros}{n-m}$$

Root locus asymptote angle

$$\varphi_{ap} = \frac{2p+1}{n-m}180°, p = 0, 1, 2, ...(n-m-1)$$

Proportional plus Derivative (PD) Compensator

$$G_c(s) = K_c(s+a)$$

Phase Lag Compensator

$0 < p_0 < z_0$

Phase Lead Compensator

$$G_c(s) = \frac{K(s+z_0)}{s+p_0}$$

$0 < z_0 < p_0$

Root locus break-away/break-in points

K = ... dK/ds = 0

PID Compensator

$$G_c(s) = K_P + \frac{K_I}{s} + K_Ds$$

Root locus determine departure/arrival angle

$$\sum \theta_z - \sum \theta_p = -180°$$

Proportional plus Integral (PI) Compensator

$$G_c(s) = K_P + \frac{K_I}{s}$$