Concordia University

ENGR 233: Applied Advanced Calculus

Fall 2021 Final Examination

Wednesday, 08 December 2021 ***Duration: 3 hrs*** ***Maximum Marks: 100***

Note: All questions are compulsory and carry marks indicated thereof in brackets.

Q1. Find the equation of the plane containing the point $(1,7,-1)$ that is perpendicular to the line of intersection of the two planes $-x+y-8z=4$ and $3x-y+2z=0$. **[5]**

Q2. If the position vector of a moving particle is given by $\vec{r}(t)=\frac{1}{2}t^2\hat{\imath}+\frac{1}{3}t^3\hat{\jmath}-\frac{1}{2}t^2\,\hat{k}$, find the tangential and normal components of the acceleration vector at any time $t\geq 0$. Then find its curvature. **[5]**

Q3. A thin lamina is bounded by the graphs of $y=x^2$ and $y=x^3$ on the xy −plane. If the density at any point P in the lamina is directly proportional to the square of its distance from the origin, how far away is its center of mass from the x-axis? **[10]**

Q4. Use the Green's Theorem to evaluate the line integral $\oint_C\,(2xy-x^2)dx+(x+y^2)dy$, where C is the boundary of the region determined by the graphs of $x=y^2$ and $y=x^2$. **[10]**

Q5. Test if the vector field $\vec{F}(x,y,z)=(z\cos xz)\hat{\imath}+e^y\hat{\jmath}+(x\cos xz)\hat{k}$ is conservative. If so, find its potential function and then the work done in moving a particle from $(0,0,0)$ to $\left(\frac{\pi}{2},2,\frac{1}{2}\right)$ along any path. **[10]**

Q6. Reverse the order of integration of $\int_0^1\left(\int_{\sqrt{y}}^{\sqrt{2-y^2}}f(x,y)\,dx\right)dy$. **[10]**

Q7. Compute $\iint_\Omega\,\sqrt{6-x^2-y^2-2x+4y}\,dxdy$, where Ω is the disk of radius 1 centered at $(-1,2)$. **[10]**

Q8. Compute the integral $\iiint_V\,zdxdydz$, where V is the part of the intersection $C_1\cap C_2$ of the two infinite round cylinders $C_1: x^2+z^2\leq 1$ and $C_2: y^2+z^2\leq 1$, lying over the plane $z=0$. **[10]**

Q9. Compute the surface integral $\iint_\Sigma\,xydS$, where Σ is the part of the unit sphere $x^2+y^2+z^2=1$ lying in the octant $x\geq 0, y\geq 0, z\geq 0$. **[15]**

Q10. Find the outward flux of the vector field $\vec{F}(x,y,z)=(x+yz)\hat{\imath}+(y+zx^2)\hat{\jmath}+(z+x^3y^2)\hat{k}$ through the surface of the ellipsoid $\frac{x^2}{4}+\frac{y^2}{9}+\frac{z^2}{16}=1$. **[15]**

$\cos\theta = \vec{a} \bullet \vec{b} / (\|\mathbf{a}\| \|\mathbf{b}\|)$

$\mathrm{comp}_{\mathbf{b}}\mathbf{a} = \|\mathbf{a}\| \cos\theta = \mathbf{a} \cdot \hat{\mathbf{b}}$

$\mathrm{proj}_{\mathbf{b}}\mathbf{a} = (\vec{\mathbf{a}} \cdot \hat{\mathbf{b}})\hat{\mathbf{b}}$

Area of a parallelogram = $\|\boldsymbol{a} \times \boldsymbol{b}\|$

Volume of a parallelepiped = $|\boldsymbol{a} \bullet (\boldsymbol{b} \times \boldsymbol{c})|$

Equation of a line : $\vec{r} = \vec{r}_2 + t(\vec{r}_2 - \vec{r}_1) = \vec{r}_2 + t\vec{a}$

Equation of a plane : $a\,x + b\,y + cz + d = 0$
also: $[(\vec{r}_2 - \vec{r}_1) \times (\vec{r}_3 - \vec{r}_1)] \bullet (\vec{r} - \vec{r}_1) = 0$

$$\frac{d\vec{r}(s)}{dt} = \frac{d\vec{r}}{ds}\frac{ds}{dt}$$

Length of a curve: $s = \int_{t_1}^{t_2} |\vec{r}\,'(t)|\,dt$

$$\kappa = \left\|\frac{d\vec{\mathbf{T}}}{ds}\right\| = \left\|\frac{d^2\vec{\mathbf{r}}}{ds^2}\right\| = \frac{|\vec{\mathbf{T}}'|}{|\vec{\mathbf{r}}'|} = \frac{\|\mathbf{r}'(t) \times \mathbf{r}''(t)\|}{\|\mathbf{r}'(t)\|^3}$$

$$\vec{\mathbf{a}}(t) = \kappa v^2\hat{\mathbf{N}} + \frac{dv}{dt}\hat{\mathbf{T}} = a_N\hat{\mathbf{N}} + a_T\hat{\mathbf{T}}$$

$$\hat{\mathbf{N}} = \frac{d\mathbf{T}/dt}{\|d\mathbf{T}/dt\|}$$

$$\hat{\mathbf{T}} = \frac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|}$$

The Binormal

$$\hat{\mathbf{B}} = \hat{\mathbf{T}} \times \hat{\mathbf{N}}$$

$$a_T = \frac{dv}{dt} = \frac{\mathbf{v} \bullet \mathbf{a}}{\|\mathbf{v}\|} \;\&\; a_N = kv^2 = \frac{\|\mathbf{v} \times \mathbf{a}\|}{\|\mathbf{v}\|}$$

$$\nabla = \mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}$$

$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial f}{\partial v}\frac{\partial v}{\partial x} \;\&\; \frac{\partial f}{\partial y} = \frac{\partial f}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial f}{\partial v}\frac{\partial v}{\partial y}$$

$D_u(F) = \nabla F \bullet \hat{u},\ \hat{u} = \text{unit vector}$

Equation of Tangent Plane: $\vec{n}_o \bullet (\vec{r} - \vec{r}_o) = 0,\ \vec{n}_o = \nabla F \text{ at P}$

$W = \int_C \vec{F} \bullet d\vec{r}$

equation of normal line to a surface: $\vec{n}_o \times (\vec{r} - \vec{r}_o) = 0,\ \vec{n}_o = \nabla F \text{ at P}$

Line integral : $\int_C F(x,y,z)ds = \int_a^b F(f(t), g(t), h(t))\sqrt{[f']^2 + [g']^2 + [h']^2}\ dt$

Surface integral : $\iint_S \vec{F} \bullet \hat{n}\ dS = \iint_S \vec{F} \bullet \hat{n}\ \sqrt{1 + (z_x)^2 + (z_y)^2}\,dxdy$

$$\oint_C \vec{F} \bullet d\vec{r} = \iint_S (curl\,\vec{F}) \bullet \hat{n}dS$$

$$\oiint_S (\vec{F} \bullet \hat{n})dS = \iiint_D (div\,\vec{F})dV$$

$$\oint_C [Pdx + Qdy] = \iint_R \left[\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right]dxdy$$

$$\tilde{x} = \frac{\iiint_D x\rho(x,y,z)dV}{m},$$

$$m = \iiint_D \rho(x,y,z)dV$$

$$I_x = \iiint_D (y^2 + z^2)\rho(x,y,z)dV;$$

$$x = r\cos\theta,\ y = r\sin\theta;\ z = z;\ r = \sqrt{x^2 + y^2},\ \theta = \tan^{-1}(y/x)$$

$$J(u,v) = \frac{\partial(x,y)}{\partial(u,v)}$$

$$x = \rho\sin\phi\cos\theta,\ y = \rho\sin\phi\sin\theta, z = \rho\cos\phi,$$
$$\rho = \sqrt{x^2 + y^2 + z^2},\ \theta = \tan^{-1}(y/x),\quad \phi = \tan^{-1}(\sqrt{x^2 + y^2}/z)$$

$$dV = r\ dr\,d\theta\,dz$$
$$dV = \rho^2\sin\phi\ d\rho d\phi\,d\theta$$