PART I

Problem 1.

(a) For any two vectors $\mathbf{x}_1$ and $\mathbf{x}_2$ in $\mathbb{R}^n$ we have $\| \mathbf{x}_1 + \mathbf{x}_2 \|_1 \leq \| \mathbf{x}_1 \|_1 + \| \mathbf{x}_2 \|_1$.

☐ True ☐ False ☐ Don't know

(b) For any two square matrices A and B we have $\| A + B \|_\infty \leq \| A \|_\infty + \| B \|_\infty$.

☐ True ☐ False ☐ Don't know

(c) The *spectral radius* of a square matrix A (*i.e.*, the magnitude of the largest eigenvalue of A) is a matrix norm.

☐ True ☐ False ☐ Don't know

(d) A square matrix that is diagonally dominant is invertible.

☐ True ☐ False ☐ Don't know

(e) A *general n by n tridiagonal system* can be solved in $5n - 4$ operations; counting multiplications and divisions only.

☐ True ☐ False ☐ Don't know

(f) Newton's method for computing the square root of 2 converges quadratically if the initial guess is close enough.

☐ True ☐ False ☐ Don't know

(g) If one interpolates a smooth function $f(x)$ by a polynomial of degree n or less at $n+1$ equally spaced interpolation points in a given interval $[a, b]$, then $\max_{[a,b]} \mid f(x) - p_n(x) \mid$ can be made arbitrarily small by taking n large enough.

☐ True ☐ False ☐ Don't know

(h) The specific function $f(x) = e^x$ can be approximated to arbitary accuracy (assuming exact arithmetic) in the interval $[0, 1]$ by a Taylor polynomial of sufficiently high degree about the point $x = 0$.

☐ True ☐ False ☐ Don't know

(i) Simpson's Rule for integrating a sufficiently smooth function $f(x)$ over a given interval has a higher order of accuracy than the Trapezoidal Rule.

☐ True ☐ False ☐ Don't know

PART II

Problem 2.

Suppose $\mathbf{T}$ is a *general* n *by* n *tridiagonal* matrix. Thus you cannot make any assumptions about the entries in $\mathbf{T}$, except that $\mathbf{T}$ is tridiagonal, nonsingular, and that it can be **LU**-decomposed. For each of the following, determine how many operations are needed, when only multiplications and divisions are counted:

(a) computing the **LU**-decomposition of $\mathbf{T}$,

(b) solving $\mathbf{Lg} = \mathbf{f}$ for $\mathbf{g} \in \mathbb{R}^n$, where $\mathbf{f} \in \mathbb{R}^n$ is a given vector,

(c) solving $\mathbf{Ux} = \mathbf{g}$ for $\mathbf{x} \in \mathbb{R}^n$, where $\mathbf{g} \in \mathbb{R}^n$ is the vector obtained in step (b),

(d) The total of steps (a), (b), and (c).

In each case the number of operations is a function of n. Enter your answers in the boxes below.

(a)	(b)	(c)	(d)

Problem 3.

(a) For the fixed point iteration $x^{(k+1)} = f(x^{(k)})$, where $f(x) = 2(x - x^2)$, determine all fixed points. For each fixed point determine whether it is attracting or repelling, and if attracting, determine if the convergence is quadratic. Enter your answers in the Table below:

fixed point	attracting?	quadratic convergence?
$x^* =$	Yes / No	Yes / No / NA
$x^* =$	Yes / No	Yes / No / NA
$x^* =$	Yes / No	Yes / No / NA

(b) In the space below, draw the standard graphical interpretation of this fixed point iteration, for x in the interval $[0, 1]$. Take $x^{(0)} = \frac{1}{4}$, and show enough iterations to illustrate the behavior of this fixed point iteration. Make sure that $f(x)$ is drawn accurately, with correct slope at the fixed points.

Problem 4.

(a) Using precise mathematical notation, write down the specific form that Newton's method takes when applied to the numerical determination of the *square root* of 3. Enter your answer in the box below, making sure that your answer is written in a form that is useful in Part (b) of this Problem.

$x^{(k+1)} = f(x^{(k)})$, where $f(x) =$

(b) In the space below, draw a careful graphical interpretation of this fixed point iteration, showing the line $y = x$, the curve $y = f(x)$, and indicating the first few iterations, starting with $x^{(0)} = \frac{1}{4}$

Problem 5.

Suppose $p_n \in \mathbb{P}^n$ interpolates the function $f(x) = e^x$ at $n+1$ Chebyshev points in $[-1, 1]$. Determine the smallest value of n for which $| f(x) - p_n(x) | < 10^{-3}$ everywhere in $[-1, 1]$. Enter your answer in the box.

Problem 6.

(a) Derive the formula for the Taylor polynomial $p_n(x)$ of degree n for $f(x) = e^x$ about the point $x_0 = 0$. Enter you answer in the box below.

$p_n(x) =$

(b) Again for $f(x) = e^x$, what is the smallest value of n so that $| f(x) - p_n(x) | < 10^{-3}$ everywhere in the interval $[-1, 1]$? Enter your answer in the box.

Problem 7. Suppose we use *local interpolation* to approximate $\sin(x)$ in the interval $[0, \pi]$, using N intervals of equal size $h = \pi/N$, and interpolating $\sin(x)$ in each interval by a *local polynomial* $p \in \mathbb{P}^2$ at three *local Chebyshev points* . What is the smallest value of N (the number of intervals) so that $|\sin(x) - p(x)|$ is less than 10^{-3} in each of the N intervals? Enter your answer in the box.

Problem 8. For the numerical differentiation formula

$$f''(0) \cong \frac{f(h) - 2f(0) + f(-h)}{h^2},$$

use Taylor expansions to determine the leading error term. Enter your answer in the box.

Problem 9. Showing all details, derive the numerical differentiation formula in Problem 8. You must make use of the Lagrange interpolation polynomial that interpolates $f(x)$ at the three points, $x_0 = -h$, $x_1 = 0$, and $x_2 = h$, and the Lagrange basis functions, $\ell_0(x)$, $\ell_1(x)$, and $\ell_2(x)$.

Problem 10. The formula

$$\int_{-\frac{h}{2}}^{\frac{h}{2}} f(x)\,dx \approx \frac{h}{6}\,f(-\frac{h}{2}) + \frac{4h}{6}f(0) + \frac{h}{6}\,f(\frac{h}{2}),$$

defines the *local Simpson's Rule* for the reference interval $[-\frac{h}{2}, \frac{h}{2}]$.

Showing all details, derive this local integration formula using the Lagrange interpolation polynomial that interpolates $f(x)$ at the three points, $x_0 = -\frac{h}{2}$, $x_1 = 0$, and $x_2 = \frac{h}{2}$, and the Lagrange basis functions, $\ell_0(x)$, $\ell_1(x)$, and $\ell_2(x)$.