COMP 232, Winter 2019 **CONCORDIA UNIVERSITY**

MATHEMATICS FOR COMPUTER SCIENCE.
Assignment 2.
Solution.

1. Consider the implication
$$p \wedge (\neg q \rightarrow \neg p) \;\Rightarrow q$$

   (a) Prove the implication by a direct proof, *i.e.* assume that the left hand side of the implication is True, and then show that the right hand side must be True also.

   SOLUTION: Assume the LHS is True. Then p is True and $\neg q \rightarrow \neg p$ is True. Next $\neg q \rightarrow \neg p$ is equivalent to $p \rightarrow q$ by contrapositive. Now since p is True and $p \rightarrow q$ is True it follows by the modus ponens law that q is True, *i.e.*, the RHS is True.

   (b) Prove the implication by contradiction.

   SOLUTION: Suppose the LHS is True, but the RHS is False. Thus p and $\neg q \rightarrow \neg p$ are True, but q is False. Equivalently, by contrapositive, p and $p \rightarrow q$ are True. Thus modus ponens law implies that q is true, but this contradicts an earlier assumption that q is False.

2. For each of the statements below state whether it is True or False. If True then give a proof. If False then give a counterexample.

   (a) If $a, b \in \mathbb{Z}^+$ are odd then the product ab is odd.

   SOLUTION: True. We can write $a = 2k_a + 1$ and $b = 2k_b + 1$, for nonnegative integers k_a and k_b. Then $ab = (2k_a + 1)(2k_b + 1) = 4k_ak_b + 2k_a + 2k_b + 1 = 2(2k_ak_b + k_a + k_b) + 1, 2k_ak_b + k_a + k_b \in \mathbb{Z}$, which shows that ab is odd.

   (b) If $a, b \in \mathbb{Z}^+$, where a is even and b is odd then $a + b$ is odd.

   SOLUTION: True. We can write $a = 2k_a$ and $b = 2k_b + 1$, for positive integer k_a and nonnegative integer k_b. Then $a + b = 2k_a + (2k_b + 1) = 2(k_a + k_b) + 1, k_a + k_b \in \mathbb{Z}$, which shows that $a + b$ is odd.

   (c) Let $a, b \in \mathbb{Z}^+$. If $a + b$ is even, then a^2 or b^2 is even.

   SOLUTION: False. Let $a = b = 3$. Then $a + b = 6$ is even, but both a^2 and b^2 are are odd, namely both equal 9.

   (d) For all $a \in \mathbb{Z}^+$, if $a > 3$ then $a^2 - 4$ is composite.

   SOLUTION: True: $a^2 - 4 = (a - 2)(a + 2)$, where both factors are greater than 1.

3. Let $n \in \mathbb{Z}^+$. Prove that the following statements are equivalent:

$$n^3 \text{ is odd}, \qquad n^2 \text{ is odd}, \qquad 1-n \text{ is even}, \qquad n^2+1 \text{ is even}.$$

PROOF: Re-order the above as

(1) $1-n$ is even , (2) n^2 is odd , (3) n^2+1 is even , (4) n^3 is odd .

To establish equivalence it suffices to prove that

$$(1) \Rightarrow (2)\ , \qquad (2) \Rightarrow (3)\ , \qquad (3) \Rightarrow (4)\ , \qquad \text{and} \qquad (4) \Rightarrow (1)\ .$$

(a) (1) $\Rightarrow$ (2) : If $1-n$ is even then $1-n = 2k_1$ for some integer k_1. Hence $n = -2k_1+1$ so that n is odd. Then $n^2 = (-2k_1+1)^2 = 2(2k_1^2 - 2k_1) + 1$, so n^2 is odd.

(b) (2) $\Rightarrow$ (3) : Since n^2 is odd we have $n^2 = 2k_2+1$. Thus $n^2+1 = 2k_2+2 = 2(k_2+1)$, so that n^2+1 is even.

(c) (3) $\Rightarrow$ (4) : If n^2+1 is even then (easily) n^2 is odd, $n^2 = 2k_3+1$. Now if if n^2 is odd then n is odd (which follows easily from the contrapositive: if n is even then n^2 is even). Thus $n = 2k_4+1$. Hence $n^3 = n^2\ n = (2k_3+1)(2k_4+1) = 2(2k_3k_4+k_3+k_4)+1$, so that n^3 is odd.

(d) (4) $\Rightarrow$ (1) : We prove the contrapositive: If $1-n$ is odd then n^3 is even. Now if $1-n$ is odd then (easily) n is even, say, $n = 2k_5$. Hence $n^3 = 2(4k_5^3)$ is even.

4. (a) Use a proof by cases to show that 10 is not the square of a positive integer.

PROOF: We consider two cases. Case (i): Let $n \in \mathbb{Z}$ and $1 \le n \le 3$. Then $n^2 \le 9$, so $n^2 \ne 10$. Case (ii): Let $n \in \mathbb{Z}$ and $n \ge 4$. Then $n^2 \ge 16$, so $n^2 \ne 10$. As these two cases represent all possible values of n, and in neither case $n^2 = 10$, so 10 is not the square of an integer.

(b) Prove by contraposition that if n is an integer and $3n+2$ is even, then n is even.

PROOF: The contrapositive is: if n is odd then $3n+2$ is odd. So let n be odd. Then $n = 2k+1$ for some $k \in \mathbb{Z}$. Then $3n+2 = 3(2k+1)+2 = 6k+5 = 2(3k+2)+1$, and thus $3n+2$ is odd.

5. For each of the statements below state whether it is True or False. If True then give a proof. If False then explain why, *e.g.*, by giving a counterexample.

(a) The sum of the squares of any two rational numbers is a rational number.

SOLUTION: True: Suppose x and y are rational numbers :

$$x = \frac{p_1}{q_1} \qquad \text{and} \qquad y = \frac{p_2}{q_2}\ .$$

where $p_1,\ q_1$ and $p_2,\ q_2$ are integers, with q_1 and q_2 nonzero. Then

$$x^2 + y^2 = \frac{p_1^2}{q_1^2} + \frac{p_2^2}{q_2^2} = \frac{p_1^2 q_2^2 + p_2^2 q_1^2}{q_1^2 q_2^2}\ , \text{ which is rational.}$$

(b) For all positive $x \in \mathbb{R}$, if x is irrational then $\sqrt{x}$ is irrational.

SOLUTION: True. By contrapositive: suppose $\sqrt{x}$ is not irrational, *i.e.*, it is rational. Then we can write $\sqrt{x} = \frac{p}{q}$, for certain $p, q \in \mathbb{Z}^+$. It follows that $x = (\frac{p}{q})^2 = \frac{p^2}{q^2}$, which is rational, *i.e.*, not irrational.

(c) For all $x, y \in \mathbb{R}$, if x and y are irrational then $x^2 + y$ is irrational.

SOLUTION: False. Let $x = 2^{\frac{1}{4}}$ and $y = -2^{\frac{1}{2}}$. Then both x and y are irrational, but $x^2 + y = (2^{\frac{1}{4}})^2 - 2^{\frac{1}{2}} = 2^{\frac{1}{2}} - 2^{\frac{1}{2}} = 0$, where 0 is considered to be a rational number, as it can be written as $\frac{0}{1}$.

In the above proof we used the well-known fact that $2^{\frac{1}{2}}$ is irrational. For completeness we now show that $2^{\frac{1}{4}}$ is also irrational. Suppose on the contrary that $2^{\frac{1}{4}}$ is rational, *i.e.*, $2^{\frac{1}{4}} = \frac{p}{q}$, for certain $p, q \in \mathbb{Z}^+$. Then $2^{\frac{1}{2}} = (2^{\frac{1}{4}})^2 = (\frac{p}{q})^2 = \frac{p^2}{q^2}$, which contradicts the fact that $2^{\frac{1}{2}}$ is irrational.

(d) $\log_{10}(2)$ is irrational.

SOLUTION: True: This is most easily proved by contradiction: Suppose $\log_{10}(2)$ is rational. (We know that $\log_{10}(2)$ is positive.) Then $\log_{10}(2) = \frac{p}{q}$ for positive integers p and q. By definition of the logarithm function this means that $2 = 10^{p/q}$, or equivalently, $2^q = 10^p$. Now 2 is the only prime divisor of 2^q, while $10^p = 2^p 5^p$ has 5 as a prime divisor, so that 2^q and 10^p cannot be equal. Hence we have a contradiction.

6. Give a proof of each of the following:

(a) If the integers $1, 2, 3, \cdots, 7,$ are placed around a circle, in any order, then there exist two adjacent integers that have a sum greater than or equal to 9.

SOLUTION: The number 7 must be somewhere along the circle. The smallest neighbors that 7 can have are the integers 1 and 2. But the sum of 7 and 2 is 9, so the sum of 7 and its larger neighbor is greater than or equal to 9.

(b) If the integers $1, 2, 3, \cdots, 16$ are placed around a circle, in any order, then there exist three integers in consecutive locations around the circle that have a sum greater than or equal to 27.

SOLUTION: Suppose the sum of any three adjacent numbers is less than 27, *i.e.*, less than or equal to 26. The number 1 must be somewhere along the circle. The remaining 15 positions can be grouped in five non-intersecting groups of three adjacent positions. Then the total sum will be less than or equal to $5 \cdot 26 + 1 = 131$. However, we know that $1 + 2 + ... + 16 = 16 \cdot 17/2 = 136$, so that we have a contradiction.

7. For each of the following, determine whether it is valid or invalid. If valid then give a proof. If invalid then give a counter example.

   (a) $(A \cap B) \cup (C \cap D) \;=\; (A \cap D) \cup (C \cap B)$

   SOLUTION: False. Take A and D be non-empty, with $A = D$, and let both B and C be empty. Then $(A \cap B) \cup (C \cap D)$ is empty, but $(A \cap D) \cup (C \cap B) = A$ is not empty, so that they cannot be equal.

   (b) $A - (B \cup C) = (A - B) \cap (A - C)$

   SOLUTION: True:

$$\begin{aligned}
\{x \mid x \in A - (B \cup C)\} &= \{x \mid x \in A \;\wedge\; x \notin (B \cup C)\} \\
&= \{x \mid x \in A \;\wedge\; x \notin B \;\wedge\; x \notin C\} \\
&= \{x \mid x \in A \;\wedge\; x \notin B \;\wedge\; x \in A \;\wedge\; x \notin C\} \\
&= \{x \mid x \in (A - B) \;\wedge\; x \in (A - C)\} \\
&= \{x \mid x \in (A - B) \cap (A - C)\}
\end{aligned}$$

   (c) $B \cap C \subseteq A \quad \Rightarrow \quad (C - A) \cap (B - A)$ is empty

   SOLUTION: This statement is valid. We prove the contrapositive:

   If $(C - A) \cap (B - A)$ is not empty, then $B \cap C$ is not a subset of A.

   PROOF: Since $(C - A) \cap (B - A)$ is not empty, there must be an x such that $x \in (C - A) \cap (B - A)$, *i.e.*, $x \in C$ and $x \in B$ and $x \notin A$. Thus $x \in B \cap C$ and $x \notin A$. Thus $B \cap C$ is not a subset of A.

   (d) $(A \cup B) - (A \cap B) = A \quad \Rightarrow \quad B$ is empty

   SOLUTION:

   This statement is also valid. We prove it by contradiction. Assume that $(A \cup B) - (A \cap B) = A$, but B is not empty. Then there exists an element $b \in B$. There are two cases to consider:

   Case 1: $b \in A$: Then $b \in A \cup B$ and $b \in A \cap B$. Hence $b \notin (A \cup B) - (A \cap B)$. Thus, using our assumption that $(A \cup B) - (A \cap B) = A$ it follows that $b \notin A$, which is a contradiction.

   Case 2: $b \notin A$: In this case $b \in A \cup B$, but $b \notin A \cap B$. Hence $b \in (A \cup B) - (A \cap B)$. Using our assumption that $(A \cup B) - (A \cap B) = A$, it follows that $b \in A$, which is a contradiction.