CONCORDIA UNIVERSITY

Department of Mathematics & Statistics

Course	**Number**	**Section(s)**
Mathematics	204	All
Examination	**Date**	**Pages**
Final	December 2017	2
Instructors		**Course Examiner**
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Special Instructions

- ▷ Only approved calculators are allowed.
- ▷ Justify all your answers.
- ▷ Answer all questions. All questions have equal value.

1. Solve the system by Gauss-Jordan elimination

$$\begin{array}{rcrcrcr} x & - & 2y & + & z & = & 2 \\ 2x & - & 4y & + & 2z & = & 4 \\ 5x & - & y & + & 2z & = & 13. \end{array}$$

2. Find the inverse of the matrix $A = \begin{pmatrix} 0 & -1 & -2 \\ 1 & 3 & 0 \\ 4 & 0 & -6 \end{pmatrix}$

   and solve for X if $AX = \begin{pmatrix} 1 & 0 & 1 & 2 \\ 2 & 1 & 1 & 3 \\ 1 & 2 & 4 & 5 \end{pmatrix}$.

3. Find the matrix of cofactors of $A = \begin{pmatrix} 1 & 0 & 2 \\ 3 & 7 & 8 \\ 4 & -1 & 4 \end{pmatrix}$

   and find the value of $a_{11}C_{21} + a_{12}C_{22} + a_{13}C_{13}$.

4. (a) Find the determinant of $A = \begin{pmatrix} 6 & -1 & 0 & 4 \\ 3 & 3 & -2 & 0 \\ 0 & 1 & 8 & 6 \\ 2 & 3 & 0 & 4 \end{pmatrix}$

   (b) Solve for x if $\begin{vmatrix} x & 1 & -2 \\ 1 & -1 & 1 \\ -1 & 0 & 2 \end{vmatrix} = 7$.

5. (a) Find an equation of the plane containing the point $P(2,1,-1)$ and the line $x = 3t-2, y = -t+4, 7z = 2t+1$.

(b) Find the point of intersection of the line $x = 2t-1, y = 3t-2, z = t+4$ and the plane $x+2y+3z = -4$.

6. Show that the points $P(1,2,4)$, $Q(-1,0,3)$, $R(-2,-4,1)$, $S(-3,-2,2)$ lie in the same plane.

7. Find the standard matrix for the linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ given that $T(1,2) = (1,6)$, $T(2,1) = (1,4)$.

8. Given the vectors $x_1 = (1,2,3,4), x_2 = (1,0,1,4), x_3 = (2,1,0,3), x_4 = (1,1,2,6)$, $x_5 = (-1,2,-3,4)$, find a basis of the subspace of $\mathbb{R}^4$ spanned by x_1, x_2, x_3, x_4, x_5.

9. In question 8, find coefficients a, b, c, d, e (not all zero) such that $ax_1+bx_2+cx_3+dx_4+ex_5 = 0$.

10. Let $A = \begin{pmatrix} 1 & 3 & 3 \\ -3 & -5 & -3 \\ 3 & 3 & 1 \end{pmatrix}$. Find an invertible matrix P and a diagonal matrix D such that $P^{-1}AP = D$.

MATH. 204.

Final Exam Solutions. Dec. 2017.

1.
$$\left[\begin{array}{ccc|c} 1 & -2 & 1 & 2 \\ 2 & -4 & 2 & 4 \\ 5 & -1 & 2 & 13 \end{array}\right] \xrightarrow[-5R_1+R_3\to R_3]{-2R_1+R_2\to R_2} \left[\begin{array}{ccc|c} 1 & -2 & 1 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 9 & -3 & 3 \end{array}\right]$$

$$\xrightarrow{\frac{1}{9}R_3 \leftrightarrow R_2} \left[\begin{array}{ccc|c} 1 & -2 & 1 & 2 \\ 0 & 1 & -1/3 & 1/3 \\ 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow{2R_2+R_1\to R_1}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 1/3 & 8/3 \\ 0 & 1 & -1/3 & 1/3 \\ 0 & 0 & 0 & 0 \end{array}\right] \Longrightarrow \begin{aligned} x &= \frac{8}{3} - \frac{1}{3}z \\ y &= \frac{1}{3} + \frac{1}{3}z \end{aligned}$$

Let $z = t$.

$$X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \frac{8}{3} \\ \frac{1}{3} \\ 0 \end{bmatrix} + \begin{bmatrix} -\frac{1}{3} \\ \frac{1}{3} \\ 1 \end{bmatrix} t.$$

2.

$$A^{-1} = \begin{bmatrix} -1 & -\frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{4}{9} & -\frac{1}{9} \\ -\frac{2}{3} & -\frac{2}{9} & \frac{1}{18} \end{bmatrix}.$$

$$X = \begin{bmatrix} -\frac{4}{3} & \frac{1}{3} & 0 & -\frac{4}{3} \\ \frac{10}{9} & \frac{2}{9} & \frac{1}{3} & \frac{13}{9} \\ -\frac{19}{18} & -\frac{1}{9} & -\frac{2}{3} & -\frac{31}{18} \end{bmatrix}.$$

3. Cofactor matrix:

$$\begin{bmatrix} 36 & 20 & -31 \\ -2 & -4 & 1 \\ -14 & -2 & 7 \end{bmatrix}.$$

$\therefore$) $(1)(-2)+(0)(-4)+(2)(-31) = -64.$

4. A.

$\det(A) = 560.$

B. $\begin{bmatrix} x & 1 & -2 \\ 1 & -1 & 1 \\ -1 & 0 & 2 \end{bmatrix} = 7$

$[-2x-1+0]-[-2+0+2] = 7$

$-2x-1=7$

$x=-4.$

5.A. $P(2,1,-1)$.

$$\bar{d} = \left(3, -1, \frac{2}{7}\right).$$

$$Q = \left(-2, 4, \frac{1}{7}\right).$$

$$\vec{PQ} = \left(-4, 3, \frac{8}{7}\right).$$

$$\bar{n} = \bar{d} \times \vec{PQ} = \begin{vmatrix} 3 & -1 & \frac{2}{7} \\ -4 & 3 & \frac{8}{7} \end{vmatrix}$$

$$\bar{n} = \left(-2, \frac{-32}{7}, 5\right).$$

$\therefore$,

$$-2(x-2) - \frac{32}{7}(y-1) + 5(z+1) = 0$$

$$-2x - \frac{32}{7}y + 5z = -\frac{95}{7}.$$

B.

$$x + 2y + 3z = -4$$

$$(2t-1) + 2(3t-2) + 3(t+4) = -4$$

$$11t = -11$$

$$t = -1 \Rightarrow x = -3$$

$$y = -5$$

$$z = 3.$$

6. $\overrightarrow{PQ} = (-2,-2,-1)$.

$\overrightarrow{PR} = (-3,-6,-3)$.

$\overrightarrow{PS} = (-4,-4,-2)$.

$\overrightarrow{PQ} \cdot (\overrightarrow{PR} \times \overrightarrow{PS})$

$\overrightarrow{PQ} \cdot (0, 6, -12)$

$(-2,-2,-1) \cdot (0,6,-12) = 0$.

7. Let $T = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$.

$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 6 \end{bmatrix} \Rightarrow \begin{array}{l} a+2b=1 \\ c+2d=6. \end{array}$

$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \end{bmatrix} \Rightarrow \begin{array}{l} 2a+b=1 \\ 2c+d=4 \end{array}$

$\therefore$, $a = \frac{1}{3}$ $c = \frac{2}{3}$ $\therefore$, $T = \frac{1}{3}\begin{bmatrix} 1 & 1 \\ 2 & 8 \end{bmatrix}$.

$b = \frac{1}{3}$ $d = \frac{8}{3}$.

8 and 9.

$$\begin{bmatrix} 1 & 1 & 2 & 1 & -1 \\ 2 & 0 & 1 & 1 & 2 \\ 3 & 1 & 0 & 2 & -3 \\ 4 & 4 & 3 & 6 & 4 \end{bmatrix} \xrightarrow[-4R_1+R_4 \to R_4]{\substack{-2R_1+R_2 \to R_2 \\ -3R_1+R_3 \to R_3}}$$

$$\begin{bmatrix} 1 & 1 & 2 & 1 & -1 \\ 0 & -2 & -3 & -1 & 4 \\ 0 & -2 & -6 & -1 & 0 \\ 0 & 0 & -5 & 2 & 8 \end{bmatrix} \xrightarrow[-R_2+R_1 \to R_1]{\substack{-\frac{1}{2}R_2 \to R_2 \\ -R_2+R_3 \to R_3}}$$

$$\begin{bmatrix} 1 & 0 & 1/2 & 1/2 & 1 \\ 0 & 1 & 3/2 & 1/2 & -2 \\ 0 & 0 & -3 & 0 & -4 \\ 0 & 0 & -5 & 2 & 8 \end{bmatrix} \xrightarrow[\substack{-\frac{3}{2}R_3+R_2 \to R_2 \\ -\frac{1}{2}R_3+R_1 \to R_1}]{\substack{-\frac{1}{3}R_3 \to R_3 \\ 5R_3+R_4 \to R_4}}$$

$$\begin{bmatrix} 1 & 0 & 0 & 1/2 & 1/3 \\ 0 & 1 & 0 & 1/2 & -4 \\ 0 & 0 & 1 & 0 & 4/3 \\ 0 & 0 & 0 & 2 & \frac{44}{3} \end{bmatrix} \xrightarrow[-\frac{1}{2}R_4+R_2 \to R_2]{\substack{\frac{1}{2}R_4 \to R_4 \\ -\frac{1}{2}R_4+R_1 \to R_1}}$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 & -10/3 \\ 0 & 1 & 0 & 0 & -23/3 \\ 0 & 0 & 1 & 0 & 4/3 \\ 0 & 0 & 0 & 1 & \frac{22}{3} \end{bmatrix}.$$

Basis vectors: $\bar{x}_1, \bar{x}_2, \bar{x}_3, \bar{x}_4$.

$\therefore$

$$a = -\frac{10}{3}$$

$$b = -\frac{23}{3}$$

$$c = \frac{4}{3}$$

$$d = \frac{22}{3}$$

$$e = +1.$$

Here is -1

10.

$$A = \begin{bmatrix} 1 & 3 & 3 \\ -3 & -5 & -3 \\ 3 & 3 & 1 \end{bmatrix}.$$

$$\det(\lambda I - A) = 0$$

$$\det\left(\begin{bmatrix} (\lambda-1) & -3 & -3 \\ 3 & (\lambda+5) & 3 \\ -3 & -3 & (\lambda-1) \end{bmatrix}\right) = 0$$

$$[(\lambda-1)^2(\lambda+5) + 27 + 27] - [9(\lambda+5) - 9(\lambda-1) - 9(\lambda-1)] = 0$$

$(\lambda-1)(\lambda+2)^2 = 0$ C.E.

$\lambda = 1; -2.$

$\lambda = 1$:

$$\begin{bmatrix} 0 & -3 & -3 \\ 3 & 6 & 3 \\ -3 & -3 & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow$$

$$X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = t \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}.$$

E.V. for $\lambda = 1$: $\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$.

$\lambda = -2$: $\begin{bmatrix} -3 & -3 & -3 \\ 3 & 3 & 3 \\ -3 & -3 & -3 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow$

$$X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = s \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}.$$

E.V. for $\lambda = -2$: $\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$; $\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$.

$$P = \begin{bmatrix} 1 & -1 & -1 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}; \quad D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{bmatrix}$$

$$P^{-1} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 1 \\ -1 & -1 & 0 \end{bmatrix} \Rightarrow P^{-1}AP = D.$$