# Midterm Exam II ENGR 213
# Applied Ordinary Differential Equations

**Winter 2016 Version A**
**Solutions**

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[10 points] **Problem 1.**

Determine which of the system of functions is linearly independent, which is linearly dependent.

$$(a)\ \{2, \frac{1}{x^2}, \frac{\ln x}{x^2}\} \qquad\qquad (b)\ \{\sin x, \cos(2x), 1 - \sin x - 2\sin^2(x)\}$$

Explain your answer.

**Solution.**
**(a)** Since

$$W = \begin{vmatrix} 2 & x^{-2} & x^{-2}\ln x \\ 0 & -2x^{-3} & x^{-3}(1-2\ln x) \\ 0 & 6x^{-4} & -x^{-4}(5-6\ln x) \end{vmatrix} = 2\begin{vmatrix} -2x^{-3} & x^{-3}(1-2\ln x) \\ 6x^{-4} & -x^{-4}(5-6\ln x) \end{vmatrix} =$$

$$= 2x^{-7}(5-6\ln x) - 6x^{-7}(1-2\ln x) = 4x^{-7} > 0$$

for all $x > 0$, then this system of functions is linearly-independent.
**(b)** Since

$$1 - \sin x - 2\sin^2(x) = 1 - \sin x - 2\frac{1-\cos(2x)}{2} =$$
$$= 1 - \sin x - 1 + \cos(2x) = -\sin x + \cos(2x),$$

the last function in the set is the linear combination of the other two functions. Therefore, this system of functions is linearly dependent.

[10 points] **Problem 2.**

Find the general solution of the given differential equation

$$16y^{(4)} - 72y'' + 81y = 0$$

**Solution.** Since the auxiliary equation

$$16m^4 - 72m^2 + 81 = (4m^2 - 9)^2 = (2m - 3)^2(2m + 3)^2 = 0$$

has two repeated real roots

$$m_1 = m_2 = \frac{3}{2}, \quad m_3 = m_4 = -\frac{3}{2},$$

the fundamental set of solutions consists of the functions

$$y_1(x) = e^{\frac{3}{2}x}, y_2(x) = xe^{\frac{3}{2}x}, y_3(x) = e^{-\frac{3}{2}x}, y_4(x) = xe^{-\frac{3}{2}x}.$$

Hence the general solution is

$$y(x) = e^{\frac{3}{2}x}(C_1 + C_2x) + e^{-\frac{3}{2}x}(C_3 + C_4x).$$

[10 points] **Problem 3.**

Use the method of undetermined coefficients to find the general solution of the following initial value problem

$$y'' - 4y' = 2xe^{4x}, \qquad y(0) = -\frac{1}{4}, \qquad y'(0) = \frac{1}{8}$$

**Solution.** It is a nonhomogeneous equation. Thus, its solution is

$$y(x) = y_c(x) + y_p(x),$$

where $y_c(x)$ is a general solution of the corresponding homogeneous equation

$$y_c'' - 4y_c' = 0$$

and $y_p(x)$ is a particular solution of the nonhomogeneous equation

$$y_p'' - 4y_p' = 2xe^{4x}.$$

First let us find a general solution of the homogeneous equation. The auxiliary equation for the corresponding homogeneous differential equation is the following

$$m^2 - 4m = m(m-4) = 0.$$

Thus, the roots of the auxiliary equation are $m_1 = 0$, $m_2 = 4$ and the general solution of the corresponding homogeneous equation is

$$y_c(x) = C_1 + C_2 e^{4x},$$

where $C_1$ and $C_2$ are arbitrary constants.
Since the right-hand side part $2xe^{4x}$ is a product of liner function and exponential function, and 4 is the root of the auxiliary equation, we find particular solution in the form

$$y_p(x) = x(Ax + B)e^{4x},$$

where $A$ and $B$ are unknown coefficients. To find them, we substitute $y_p(x)$ into the given nonhomogeneous equation. Since

$$y_p'(x) = e^{4x}(4Ax^2 + 4Bx + 2Ax + B)$$

and

$$y_p''(x) = e^{4x}(16Ax^2 + 8(A + 2B)x + 4B + 8Ax + 2A + 4B) =$$
$$= e^{4x}(16Ax^2 + 16(A + B)x + 2A + 8B)$$

we receive the following identity

$$8Ax + 2A + 4B \equiv 2x$$

from the given differential equation. As a result we get the following system of algebraic equations $8A = 2$ and $2A + 4B = 0$. Thus $A = 1/4$ and $B = -1/8$. Therefore

$$y_p(x) = e^{4x}\left(\frac{1}{4}x^2 - \frac{1}{8}x\right).$$

Hence, the general solution of the nonhomogeneous equation is

$$y(x) = C_1 + C_2 e^{4x} + e^{4x}\left(\frac{1}{4}x^2 - \frac{1}{8}x\right) = C_1 + e^{4x}\left(\frac{1}{4}x^2 - \frac{1}{8}x + C_2\right).$$

Next we find unknown constants $C_1$ and $C_2$ from the initial conditions

$$y(0) = C_1 + C_2 = -\frac{1}{4}.$$

Since

$$y'(x) = e^{4x}\left(x^2 - \frac{1}{2}x + 4C_2 + \frac{1}{2}x - \frac{1}{8}\right) = e^{4x}\left(x^2 + 4C_2 - \frac{1}{8}\right)$$

then

$$y'(0) = 4C_2 - \frac{1}{8} = \frac{1}{8} \text{ and } C_2 = \frac{1}{16}.$$

Then $C_1 = -\frac{1}{4} - C_2 = -\frac{5}{16}$. Therefore, we get the unique solution of the given initial-value problem

$$y(x) = e^{4x}\left(\frac{1}{4}x^2 - \frac{1}{8}x + \frac{1}{16}\right) - \frac{5}{16}.$$

[10 points] **Problem 4.**

Using the method of variation of parameters, solve the boundary value problem

$$\frac{d^2y}{dx^2} + y = 2\sec^3 x, \qquad y(0) = -2, \qquad y(\frac{\pi}{4}) = 0$$

**Solution.** First let us find a general solution of the homogeneous equation. The auxiliary equation for the corresponding homogeneous differential equation is the following

$$m^2 + 1 = 0.$$

Thus, the roots of the auxiliary equation are $m_1 = i$, $m_2 = -i$ and the fundamental set of solutions consist of two functions

$$y_1(x) = \cos x \text{ and } y_2(x) = \sin x.$$

Thus, we are looking for the general solution of the nonhomogeneous equation in the form

$$y(x) = C_1(x)\cos x + C_2(x)\sin x$$

By the method of variation of parameters,

$$C_1'(x) = \frac{W_1(x)}{W(x)} \qquad \text{and} \qquad C_2'(x) = \frac{W_2(x)}{W(x)},$$

where

$$W = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix} = \cos^2 x + \sin^2 x = 1,$$

$$W_1 = \begin{vmatrix} 0 & \sin x \\ 2\sec^3 x & \cos x \end{vmatrix} = -2\sin x \sec^3 x$$

and

$$W_2 = \begin{vmatrix} \cos x & 0 \\ -\sin x & 2\sec^3 x \end{vmatrix} = 2\sec^2 x.$$

Then

$$C_1'(x) = \frac{W_1}{W} = -2\frac{\sin x}{\cos^3 x} \qquad \text{and} \qquad C_2'(x) = \frac{W_2}{W} = \sec^2 x.$$

By integrating, we get

$$C_1(x) = -2\int \frac{\sin x}{\cos^3 x} dx = 2\int \frac{d(\cos x)}{\cos^3 x} = -\frac{1}{\cos^2 x} + C_1 = -\sec^2 x + C_1$$

and

$$C_2(x) = 2\int \sec^2 x dx = 2\tan x + C_2.$$

Hence, the general solution of the nonhomogeneous equation is

$$y(x) = \cos x(-\sec^2 x + C_1) + \sin x(2\tan x + C_2) =$$

$$= C_1 \cos x + C_2 \sin x + 2\sin x \tan x - \sec x.$$

From the initial conditions

$$y(0) = C_1 - 1 = -2 \;\text{ and }\; y(\frac{\pi}{4}) = \frac{\sqrt{2}}{2}C_1 + \frac{\sqrt{2}}{2}C_2 + \sqrt{2} - \sqrt{2} = \frac{\sqrt{2}}{2}(C_1 + C_2) = 0$$

Therefore $C_1 = -1$ and $C_2 = 1$ and

$$y(x) = \sin x - \cos x + 2\sin x \tan x - \sec x$$

[10 points] **Problem 5.**

Solve the Cauchy-Euler equation

$$x^2\frac{d^2y}{dx^2} - 3x\frac{dy}{x} + 13y = 0$$

**Solution.** By looking for a solution in the form $y(x) = x^m$, we reduce the differential equation to quadratic equation with respect to $m$. From

$$y'(x) = mx^{m-1} \quad \text{and} \quad y''(x) = m(m-1)x^{m-2}$$

we get the auxiliary equation

$$m(m-1) - 3m + 13 = m^2 - 4m + 13 = (m-2)^2 + 9 = 0.$$

Since the roots of the auxiliary equation are $m_1 = 2 + 3i$ and $m_2 = 2 - 3i$, the general solution of the given equation

$$y(x) = x^2(C_1 \cos(3 \ln x) + C_2 \sin(3 \ln x).$$

[5 points] **Bonus question.**

For which real value(s) of parameters $a$ and $b$ all solutions of the differential equation

$$y'' + ay' + by = 0, \quad y(0) = 0$$

approaches 0 as $x \to \infty$.

**Solution.** General solution of the differential equation depends on the roots of the auxiliary equation

$$m^2 + am + b = (m + \frac{a}{2})^2 + b - \frac{a^2}{4} = 0.$$

In general, we can write the roots in the form

$$m_1 = -\frac{a}{2} + \sqrt{\frac{a^2}{4} - b} \quad \text{and} \quad m_2 = -\frac{a}{2} - \sqrt{\frac{a^2}{4} - b}.$$

Therefore,

(a) If both roots are real numbers, the general solution is a linear combination of two exponential functions. It approaches 0 as $x$ approaches $\infty$ only if both exponents are negative, that is

$$a > 0 \quad \text{and} \quad 0 < \frac{a^2}{4} - b < \frac{a^2}{4}$$

Hence, $0 < b < \frac{a^2}{4}$.

(b) If there is one real repeated root $m = -\frac{a}{2}$, that is $b = \frac{a^2}{4}$, then

$$y(x) = e^{-\frac{a}{2}x}(C_1 x + C_2)$$

and

$$\lim_{x\to\infty} e^{-\frac{a}{2}x}(C_1 x + C_2) = \lim_{x\to\infty} \frac{C_1 x + C_2}{e^{\frac{a}{2}x}} = \lim_{x\to\infty} \frac{2C_1}{ae^{\frac{a}{2}x}} = 0$$

if $a > 0$.

(c) If roots are complex numbers, that is $b - \frac{a^2}{4} > 0$, then

$$y(x) = e^{-\frac{a}{2}x}(C_1 \cos(\sqrt{b - \frac{a^2}{4}}x) + C_2 \sin(\sqrt{b - \frac{a^2}{4}}x))$$

and

$$\lim_{x\to\infty} e^{-\frac{a}{2}x}(C_1 \cos(\sqrt{b - \frac{a^2}{4}}x) + C_2 \sin(\sqrt{b - \frac{a^2}{4}}x)) = 0$$

if $a > 0$.

Therefore, all solutions of the differential equation approaches 0 as $x \to \infty$ for $a > 0$ and $b > 0$.