MATH 205, Section AA, Summer 2015, Midterm Test

[10 marks] 1. (a) Evaluate the definite integral

$$\int_{-1}^{3} (|x| - 1)\, dx$$

by interpreting it in terms of signed area.

(b) Use Fundamental Theorem of Calculus, Part 1 to evaluate the derivative $F'(x)$ of the function

$$F(x) = \int_{x^2}^{1} \sqrt{1+t}\, \sin\left(\frac{\pi}{2} t\right)\, dt.$$

and use it to determine whether $F(x)$ is increasing or decreasing at $x = 1$.

[5 marks] 2. Find the antiderivative $F(x)$ of the function

$$f(x) = \frac{2^x}{1+2^x}$$

such that $F(0) = 1$.

[15 marks] 3. Find the following indefinite integrals

(a) $\displaystyle\int \left(\sqrt{2x} + \frac{1}{\sqrt{x}}\right)^2 dx$ (b) $\displaystyle\int \frac{\sec^2(x)}{1+\tan(x)} dx$ (c) $\displaystyle\int x e^{2x} dx$.

[15 marks] 4. Evaluate the following definite integrals (*do not approximate, give the exact value*)

(a) $\displaystyle\int_0^1 x^2\sqrt{x^3+1}\,dx$ (b) $\displaystyle\int_0^{\pi/2} \sin(x)\cos^3(x)\,dx$ (c) $\displaystyle\int_1^2 x^2 \ln(x)\,dx$.

[15 marks] 5. (a) Sketch the curves $y = |x|$ and $y = 2 - x^2$, find their points of intersection and then, find the area of the region enclosed by the curves.

(b) Find the average value of the function $f(x) = \tan^2(x)$ on the interval $[0, \pi/4]$.

(c) Sketch the region bounded by the curves $y = x^3$ and $y = x^4$. Then, find the volume of the solid obtained by rotating this region about the x-axis.

[5 marks] Bonus question 1. Evaluate the limit by interpreting it as a definite integral

$$\lim_{n\to\infty} \sum_{i=1}^{n} \frac{i}{n^2+i^2}.$$

[5 marks] Bonus question 2. Find a function f **and a number** a **such that**

$$2+\int_a^x \frac{f(t)}{t^2}dt = 2\sqrt{x} \quad \textbf{for all} \quad x>0.$$

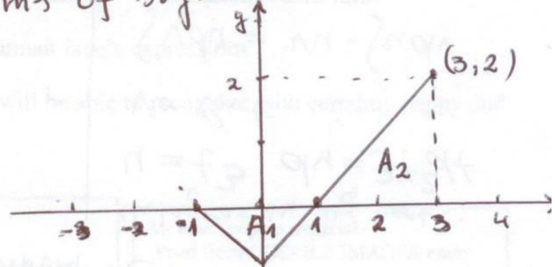
Hint. Differentiate the given equality.

MATH 205, Midterm, Solutions

① (a) Evaluate the definite integral

$$\int_{-1}^{3} (|x|-1)\,dx$$

by interpreting it in terms of signed area.



Solution. $f(x) = |x| - 1$,

$$f(x) = \begin{cases} x-1, & 0 \le x \le 3 \\ -x-1, & -1 \le x < 0 \end{cases}$$

$$\int_{-1}^{3} (|x|-1)\,dx = -A_1 + A_2 = -\frac{2\cdot 1}{2} + \frac{2\cdot 2}{2} = -1+2 = 1.$$

(b) Use FTC, Part 1 to evaluate the derivative $F'(x)$ of the function $F(x) = \int_{x^2}^{1} \sqrt{1+t}\, \sin\left(\frac{\pi}{2}t\right)dt$ and use it to determine whether $F(x)$ is increasing or decreasing at $x=1$.

Solution. $F(x) = -\int_{1}^{x^2} \sqrt{1+t}\, \sin\left(\frac{\pi}{2}t\right)dt$

$$F'(x) = -\left[\left(\sqrt{1+t}\,\sin\left(\tfrac{\pi}{2}t\right)\right)\Big|_{t=x^2}\right](x^2)'$$

$$= -\sqrt{1+x^2}\,\sin\left(\tfrac{\pi}{2}x^2\right)(2x) = -2x\sqrt{1+x^2}\,\sin\left(\tfrac{\pi}{2}x^2\right).$$

$F'(1) = -2\sqrt{2}\,\sin\left(\frac{\pi}{2}\right) = -2\sqrt{2} < 0 \Rightarrow F(x)$ is decreasing at $x = 1$.

② Find antiderivative $F(x)$ of the function

$$f(x) = \frac{2^x}{1+2^x}$$

such that $F(0) = 1$.

Solution. The most general antiderivative

$$F(x) = \int \frac{2^x}{1+2^x}\,dx$$

$u = 1+2^x$

$du = 2^x \ln(2)\,dx$

$2^x dx = \frac{1}{\ln(2)}\,du$

$$= \int \frac{1}{u}\cdot\frac{1}{\ln(2)}\,du = \frac{1}{\ln(2)}\int\frac{du}{u}$$

$$= \frac{1}{\ln(2)}\ln|u| + C = \frac{\ln(1+2^x)}{\ln(2)} + C.$$

$$F(0) = \frac{\ln(1+2^0)}{\ln(2)} + C = \frac{\ln(2)}{\ln(2)} + C = 1 \Rightarrow 1 + C = 1$$

$$\Rightarrow C = 0. \quad \boxed{F(x) = \frac{\ln(1+2^x)}{\ln(2)}.}$$

Hence, the unique antiderivative $F(x)$ of the function $f(x) = \frac{2^x}{1+2^x}$ such that $F(0) = 1$ is

$$\boxed{F(x) = \frac{1}{\ln(2)}\ln(1+2^x).}$$

③ Find the following indefinite integrals

(a) $\int \left(\sqrt{2x} + \frac{1}{\sqrt{x}}\right)^2 dx.$

Solution.

$$\int \left(\sqrt{2x} + \frac{1}{\sqrt{x}}\right)^2 dx = \int \left[(\sqrt{2x})^2 + 2\sqrt{2x}\cdot\frac{1}{\sqrt{x}} + \left(\frac{1}{\sqrt{x}}\right)^2\right] dx$$

$$= \int \left(2x + 2\sqrt{2} + \frac{1}{x}\right) dx = \boxed{x^2 + 2\sqrt{2}\,x + \ln|x| + C}.$$

(b) $\int \frac{\sec^2(x)}{1+\tan(x)}\,dx$ $\qquad u = 1+\tan(x)$

$du = \sec^2(x)\,dx$

Solution.

$$\int \frac{\sec^2(x)}{1+\tan(x)}\,dx = \int \frac{1}{u}\,du = \ln|u| + C$$

$$= \boxed{\ln|1+\tan(x)| + C}.$$

(c) $\int x e^{2x}\,dx.$

Solution.

$$\int x e^{2x} dx = \int \underset{u}{x}\, d\underset{v}{\frac{e^{2x}}{2}} = x\cdot\frac{e^{2x}}{2} - \int \frac{e^{2x}}{2}\,dx$$

$$= \frac{xe^{2x}}{2} - \frac{1}{2}\int e^{2x} dx = \frac{xe^{2x}}{2} - \frac{1}{2}\cdot\frac{e^{2x}}{2} + C$$

$$= \boxed{\frac{x}{2}e^{2x} - \frac{1}{4}e^{2x} + C = \left(\frac{x}{2} - \frac{1}{4}\right)e^{2x} + C}$$

(4) Evaluate the definite integrals

(a) $\int_0^1 x^2 \sqrt{x^3+1}\, dx$.

Solution.

$\boxed{u = x^3+1}$

$du = 3x^2\, dx$

$x^2\, dx = \frac{1}{3}\, du$

$x = 0 \Rightarrow u = 1$

$x = 1 \Rightarrow u = 2$

$$\int_0^1 x^2 \sqrt{x^3+1}\, dx = \int_1^2 \sqrt{u}\left(\frac{1}{3}\, du\right)$$

$$= \frac{1}{3}\int_1^2 u^{1/2}\, du = \frac{1}{3}\left[\frac{2}{3} u^{3/2}\Big|_1^2\right]$$

$$= \frac{1}{3}\left[\frac{2}{3} 2^{3/2} - \frac{2}{3} 1^{3/2}\right] = \boxed{\frac{2}{9}\left(2\sqrt{2}-1\right).}$$

(b) $\int_0^{\pi/2} \sin(x)\cos^3(x)\, dx$. Solution 1.

$\boxed{u = \sin(x)}$

$du = \cos(x)\, dx$

$\cos^2(x) = 1 - \sin^2(x)$

$x = 0 \Rightarrow u = 0$

$x = \frac{\pi}{2} \Rightarrow u = 1$

$$= \int_0^{\pi/2} \sin(x)\cos^2(x)\cos(x)\, dx$$

$$= \int_0^{\pi/2} \sin(x)\left(1-\sin^2(x)\right)\cos(x)\, dx$$

$$= \int_0^1 u(1-u^2)\, du = \int_0^1 (u-u^3)\, du = \left(\frac{u^2}{2}-\frac{u^4}{4}\right)\Big|_0^1$$

$$= \frac{1}{2}-\frac{1}{4} = \frac{1}{4}.$$

Solution 2. $u = \cos(x)$; $du = -\sin(x)\, dx$; $x = 0 \Rightarrow u =$; $x = \frac{\pi}{2} \Rightarrow u =$

$$\int_0^{\pi/2} \sin(x)\cos^3(x)\, dx = \int_1^0 u^3(-du) = \int_0^1 u^3\, du$$

$$= \frac{u^4}{4}\Big|_0^1 = \frac{1}{4}.$$

(c) $\int_1^2 x^2 \ln(x)\, dx.$

Solution.

$$\int_1^2 x^2 \ln(x)\, dx = \int_1^2 \underbrace{\ln(x)}_{u}\, d\underbrace{\frac{x^3}{3}}_{v}$$

$$= \left[\frac{x^3}{3}\ln(x)\Big|_1^2\right] - \int_1^2 \frac{x^3}{3}\, d\ln(x)$$

$$= \frac{2^3}{3}\ln(2) - \frac{1}{3}\ln(1) - \int_1^2 \frac{x^3}{3}\cdot\frac{1}{x}\, dx$$

$$= \frac{8}{3}\ln(2) - \frac{1}{3}\int_1^2 x^2 dx = \frac{8}{3}\ln(2) - \frac{1}{3}\left[\frac{x^3}{3}\Big|_1^2\right]$$

$$= \frac{8}{3}\ln(2) - \frac{1}{3}\left[\frac{2^3}{3} - \frac{1^3}{3}\right] = \frac{8}{3}\ln(2) - \frac{1}{3}\left[\frac{8}{3} - \frac{1}{3}\right]$$

$$= \boxed{\frac{8}{3}\ln(2) - \frac{7}{9}.}$$

Hence,

$$\boxed{\int_1^2 x^2 \ln(x)\, dx = \frac{8}{3}\ln(2) - \frac{7}{9}.}$$

(5) (a) Sketch the curves $y = |x|$, $y = 2 - x^2$, find their points of intersection and then, find the area of the region enclosed by the curves.

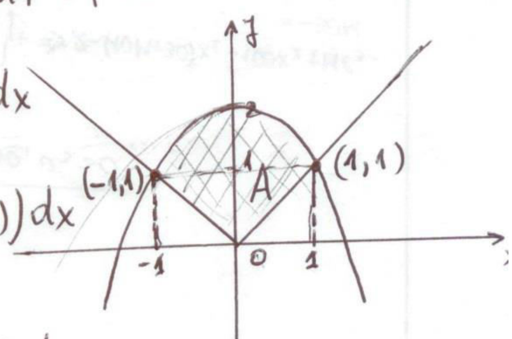
<u>Solution.</u> Let $x > 0$. Then $|x| = x$ and

$$2 - x^2 = x \Rightarrow x^2 + x - 2 = 0 \Rightarrow (x+2)(x-1) = 0$$

$\Rightarrow x = 1,\ y = 1$ is a point of intersection.

The functions $y = |x|$ and $y = 2 - x^2$ are even hence,

$x = -1,\ y = 1$ is also a point of intersection.



<u>Solution 1.</u> $A = \int_{-1}^{1} (2 - x^2 - |x|)\,dx$

$$= \int_0^1 (2 - x^2 - x)\,dx + \int_{-1}^{0} (2 - x^2 - (-x))\,dx$$

$$= \int_{-1}^{0} (2 - x^2 + x)\,dx + \int_0^1 (2 - x^2 - x)\,dx$$

$$= \left(2x - \frac{x^3}{3} + \frac{x^2}{2}\right)\Big|_{-1}^{0} + \left(2x - \frac{x^3}{3} - \frac{x^2}{2}\right)\Big|_0^1$$

$$= -\left(2\cdot(-1) - \frac{(-1)^3}{3} + \frac{(-1)^2}{2}\right) + \left(2\cdot 1 - \frac{1^3}{3} - \frac{1^2}{2}\right)$$

$$= -\left(-2 + \frac{1}{3} + \frac{1}{2}\right) + \left(2 - \frac{1}{3} - \frac{1}{2}\right) = -\left(-\frac{7}{6}\right) + \frac{7}{6} = \frac{14}{6} = \frac{7}{3}$$

<u>Solution 2.</u>

$$A = 2\int_0^1 (2 - x^2 - |x|)\,dx = 2\int_0^1 (2 - x^2 - x)\,dx$$

$$= 2\left(2x - \frac{x^3}{3} - \frac{x^2}{2}\right)\Big|_0^1 = 2\left(2 - \frac{1}{3} - \frac{1}{2}\right) = 2\cdot\frac{7}{6} = \frac{7}{3}$$

(b) Find the average value of $f(x) = \tan^2(x)$ on $[0, \frac{\pi}{4}]$.

Solution.

$$f_{ave} = \frac{1}{\frac{\pi}{4} - 0} \int_0^{\pi/4} \tan^2(x)\,dx \qquad \tan^2(x) = \sec^2(x) - 1$$

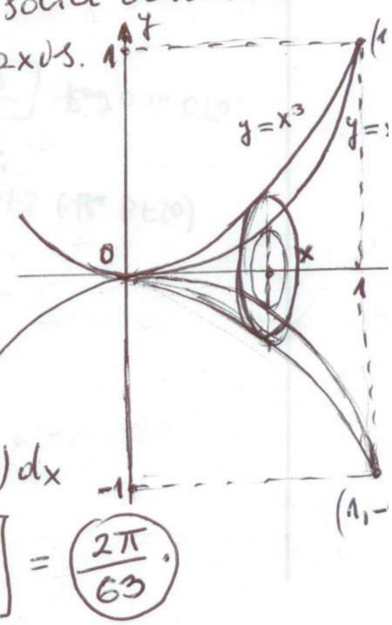
$$= \frac{4}{\pi} \int_0^{\pi/4} \left(\sec^2(x) - 1\right) dx = \frac{\pi}{4} \left(\tan(x) - x\right)\Big|_0^{\pi/4}$$

$$= \frac{4}{\pi} \left[\left(\tan\left(\frac{\pi}{4}\right) - \frac{\pi}{4}\right) - \left(\tan(0) - 0\right) \right]$$

$$= \frac{4}{\pi} \left(1 - \frac{\pi}{4}\right) = \frac{\cancel{4}}{\pi} \cdot \frac{4 - \pi}{\cancel{4}} = \frac{4 - \pi}{\pi}.$$

$$\boxed{f_{ave} = \frac{4 - \pi}{\pi}.}$$

(c) Sketch the region bounded by the curves $y = x^3$ and $y = x^4$. Then, find the volume of the solid obtained by rotating this region about the x-axis.



Solution. $x^3 = x^4 \Rightarrow x^3(1 - x) = 0$

$\Rightarrow (x = 0, y = 0)$ and $(x = 1, y = 1)$ are the points of intersection of the given curves.

$$V = V_2 - V_1 = \int_0^1 \pi (x^3)^2 dx - \int_0^1 \pi (x^4)^2 dx$$

$$= \int_0^1 \pi x^6 dx - \int_0^1 \pi x^8 dx = \pi \int_0^1 (x^6 - x^8)\,dx$$

$$= \pi \left(\frac{x^7}{7} - \frac{x^9}{9}\right)\Big|_0^1 = \pi \left[\left(\frac{1}{7} - \frac{1}{9}\right) - 0\right] = \frac{2\pi}{63}.$$

B.q. 1. Evaluate the limit by interpreting it as a definite integral

$$\lim_{n\to\infty} \sum_{i=1}^{n} \frac{i}{n^2+i^2}.$$

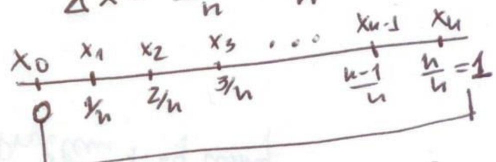
Solution:

$$\lim_{n\to\infty} \sum_{i=1}^{n} \frac{i}{n^2+i^2} = \lim_{n\to\infty} \sum_{i=1}^{n} \frac{1}{n^2} \frac{i}{1+\left(\frac{i}{n}\right)^2}$$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} \frac{1}{n} \frac{i/n}{1+\left(\frac{i}{n}\right)^2}$$

$$= \lim_{n\to\infty} \frac{1}{n}\left[\frac{1/n}{1+(1/n)^2} + \frac{2/n}{1+(2/n)^2} + \cdots + \frac{n/n}{1+(n/n)^2}\right]$$

$$\Delta x = \frac{1-0}{n} = \frac{1}{n}$$



$$= \int_0^1 \frac{x}{1+x^2}\,dx$$

$$u = 1+x^2$$
$$du = 2x\,dx$$
$$\frac{1}{2}du = x\,dx$$
$$x=0 \Rightarrow u=1$$
$$x=1 \Rightarrow u=2$$

$$= \int_1^2 \frac{1}{u}\left(\frac{1}{2}du\right)$$

$$= \frac{1}{2}\int_1^2 \frac{1}{u}\,du = \frac{1}{2}\left(\ln(u)\Big|_1^2\right)$$

$$= \frac{1}{2}\left(\ln(2)-\ln(1)\right) = \frac{\ln(2)}{2}.$$

B.q.2. Find a function f and a number a such that

$$2 + \int_a^x \frac{f(t)}{t^2}\,dt = 2\sqrt{x}\ , \quad x > 0.$$

Solution. Differentiate the given equality to obtain

$$\frac{f(x)}{x^2} = \frac{1}{\sqrt{x}} \Rightarrow f(x) = \frac{x^2}{\sqrt{x}} = x^{3/2}.$$

Hence, $\boxed{f(x) = x^{3/2} = x\sqrt{x}\ , \quad x > 0.}$

Take $x = a$ in the given equality

$$2 + \int_a^a \frac{f(t)}{t^2}\,dt = 2\sqrt{a} \Rightarrow 2\sqrt{a} = 2$$

$$\Rightarrow \sqrt{a} = 1 \Rightarrow \boxed{a = 1}$$

Check.

$$2 + \int_1^x \frac{t^{3/2}}{t^2}\,dt \overset{?}{=} 2\sqrt{x}$$

$$\Leftrightarrow 2 + \int_1^x t^{-1/2}\,dt \overset{?}{=} 2\sqrt{x}$$

$$\Leftrightarrow 2 + \left(2t^{1/2}\Big|_1^x\right) \overset{?}{=} 2\sqrt{x}$$

$$\Leftrightarrow 2 + \left(2x^{1/2} - 2\right) \overset{?}{=} 2\sqrt{x}$$

$$\Leftrightarrow \cancel{2} + 2\sqrt{x} - \cancel{2} \overset{?}{=} 2\sqrt{x} \quad \text{(holds!)}.$$