COMP232 MATHEMATICS FOR COMPUTER SCIENCE

TERM TEST Wednesday October 28, 2015

NAME : ID : Total Points :

INSTRUCTIONS

Books, notes, calculators, and communication devices are not allowed.

Use a black or blue pen. Writing in pencil will be ignored !

Rough work can be done on the back of these pages, but it will not be reviewed.

Do not add or detach any pages !

All problems have equal value, namely 8 points.

Notation : $\mathbb{Z}$ denotes all integers, $\mathbb{Z}^+$ all integers greater than zero, and $\mathbb{R}$ all real numbers.

Table for Instructor's use

Problem	Max Points	Points	Problem	Max Points	Points
1	8		6	8	
2	8		7	8	
3	8		8	8	
4	8		9	8	
5	8		10	8	

Problem 1. For each of the following logical expressions, determine whether it is a tautology, a contradiction, or a contingency :

(a) $(p \rightarrow q) \leftrightarrow (q \rightarrow p)$

☐ Tautology ☐ Contradiction ☐ Contingency

(b) $(p \wedge q) \vee r \ \leftrightarrow \ p \wedge (q \vee r)$

☐ Tautology ☐ Contradiction ☐ Contingency

(c) $(p \rightarrow (q \rightarrow r)) \leftrightarrow ((p \rightarrow q) \rightarrow r)$

☐ Tautology ☐ Contradiction ☐ Contingency

(d) $\big((p \wedge \neg q) \vee (\neg p \wedge q)\big) \ \leftrightarrow \ \big((p \vee q) \wedge \neg(p \wedge q)\big)$

☐ Tautology ☐ Contradiction ☐ Contingency

Problem 2. Let $P(x, y, z)$ denote the statement

$$x + |y| < z^2, \qquad \text{where} \quad x, y, z \in \mathbb{Z}.$$

What is the truth value of each of the following two logical expressions? Checkmark only one box:

(a) $\forall x \forall y \exists z P(x, y, z)$ (b) $\forall x \forall z \exists y P(x, y, z)$

☐ both (a) and (b) are True ☐ both (a) and (b) are False

☐ (a) is True and (b) is False ☐ (a) is False and (b) is True

What is the truth value of each of the following two logical expressions? Checkmark only one box:

(c) $\forall y \exists x \exists z P(x, y, z)$ (d) $\forall z \exists x \exists y P(x, y, z)$

☐ both (c) and (d) are True ☐ both (c) and (d) are False

☐ (c) is True and (d) is False ☐ (c) is False and (d) is True

Problem 3. Determine the validity of the following two equivalence statements (*i.e.*, valid for *all* predicates P and Q, or invalid for some predicates P and Q).

Checkmark only one box:

(a) $\forall x[P(x) \land Q(x)] \iff \forall x P(x) \land \forall x Q(x)$ (b) $\forall x[P(x) \lor Q(x)] \iff \forall x P(x) \lor \forall x Q(x)$

☐ both (a) and (b) are Valid ☐ both (a) and (b) are Invalid

☐ (a) is Valid and (b) is Invalid ☐ (a) is Invalid and (b) is Valid

Determine the validity of the following two equivalence statements (*i.e.*, valid for *all* predicates P and Q, or invalid for some predicates P and Q).

Checkmark only one box:

(c) $\exists x[P(x) \land Q(x)] \iff \exists x P(x) \land \exists x Q(x)$ (d) $\exists x[P(x) \lor Q(x)] \iff \exists x P(x) \lor \exists x Q(x)$

☐ both (c) and (d) are Valid ☐ both (c) and (d) are Invalid

☐ (c) is Valid and (d) is Invalid ☐ (c) is Invalid and (d) is Valid

Problem 4.

For any predicates $A(x)$, $B(x)$, $C(x)$, and $D(x)$, consider these logical statements:

(a) $\forall x\Big(A(x) \to \neg B(x)\Big)$ (b) $\forall x\Big(C(x) \to B(x)\Big)$

(c) $\forall x\Big(D(x) \to A(x)\Big)$ (d) $\forall x\Big(D(x) \to \neg C(x)\Big)$

Of the following four choices, only one is valid. Which one is valid?

Check only one box:

☐ (a) follows from (b), (c), and (d) ☐ (b) follows from (a), (c), and (d)

☐ (c) follows from (a), (b), and (d) ☐ (d) follows from (a), (b), and (c)

Problem 5. (a) In the open space immediately below below, write down the *contrapositive* of

$$[\ (\neg p \vee q)\ \wedge\ (\neg q \vee r)\ ]\ \ \Rightarrow\ \ [\ \neg p \vee r\ ]\ ,$$

appropriately written in a form to be useful in part (b). In particular, neither of the two logical expressions in your contrapositive should start with the negation symbol "$\neg$".

(b) Next prove this contrapositive by a clear, concise, and correct *direct proof*, using proper logical notation.

NOTE : The contrapositive will be of the form $L \Rightarrow R$. In a direct proof one assumes that L is True, and using this, one shows that R is True. In particular, *do not use logical equivalences* and *do not use truth tables*.

Problem 6.

(a) In the space provided below, write the statement

"Some students do the assignments, but do not attend classes",

in logical notation, using quantifiers and the following predicates :

$S(x) \equiv x \textit{ is a student}$, $C(x) \equiv x \textit{ attends classes}$, $A(x) \equiv x \textit{ does the assignments}$.

(b) If P is a logical statement then its *negation* is the logical statement $\neg P$. In the space provided below, write the negation of the statement in (a) in logical notation, using quantifiers and the predicates $S(x)$, $C(x)$, and $A(x)$. Write your negation is such way that it will be useful in Problem 7; in particular, your negation should not start with the symbol "$\neg$".

Problem 7. Which one of the following is a negation of the statement in Problem 6a :

Checkmark only one box:

(a) Students neither do the assignments nor attend classes.

☐ is a negation ☐ not a negation

(b) There is a student who neither does the assignments nor attends classes.

☐ is a negation ☐ not a negation

(c) Students attend classes or do not do the assignments.

☐ is a negation ☐ not a negation

(d) Students who attend classes do the assignments.

☐ is a negation ☐ not a negation

Problem 8. Using proper set-theoretic notation, prove the following *by contradiction* :

If $A \cap B \cap C \subseteq D$ then $(A - D) \cap (B - D) \cap (C - D)$ is empty.

NOTE : The above statement is of the form $L \Rightarrow R$. In a proof by contradiction one assumes that L is True and R is False, and then one shows that this leads to a contradiction.

Problem 9. Only one of the statements below is True. Indicate which one is True.

Checkmark only one box:

(a) if x and y are both nonzero and their product is rational then x and y are rational.

☐ True ☐ False

(b) if x is nonzero and rational and y is irrational then the product xy is irrational.

☐ True ☐ False

(d) If $\sqrt{x}$ is irrational then $x \in \mathbb{Z}^+$.

☐ True ☐ False

(d) If p_1 and p_2 are two consecutive prime numbers greater than 2 then $p_1 + p_2 - 1$ is also prime.

☐ True ☐ False

Problem 10.

For each of the definitions of f below, indicate whether f is a properly defined function, and if so then indicate whether f is one-to-one, whether f is onto, and whether f is has an inverse. If f is not a properly defined function then *only* check "*f is not a function*".

(a) $f \ : \ \mathbb{R} \longrightarrow \mathbb{R}$, given by $f(x) = 1 + \sqrt{|\ x\ |}$.

☐ f is not a function ☐ f is a function ☐ one-to-one ☐ onto ☐ invertible

(b) $f \ : \ \mathbb{Z}^+ \longrightarrow \mathbb{Z}$, given by $f(n) = (-1)^n \ n$.

☐ f is not a function ☐ f is a function ☐ one-to-one ☐ onto ☐ invertible

(c) $f \ : \ \mathbb{Z} \longrightarrow \mathbb{Z}^+$, given by $f(n) = |\ n\ | - n + 1$.

☐ f is not a function ☐ f is a function ☐ one-to-one ☐ onto ☐ invertible

(d) $f \ : \ \mathbb{Z}^+ \longrightarrow \mathbb{Z}$, given by $f(x) \ = \ \begin{cases} \frac{n-1}{2} & \text{if } \ n \text{ is odd ,} \\ -\frac{n}{2} & \text{if } \ n \text{ is even ,} \end{cases}$

☐ f is not a function ☐ f is a function ☐ one-to-one ☐ onto ☐ invertible