1. Solve the system by Gauss-Jordan elimination

Dec 2017 EXAM.

$$\begin{aligned} x - 2y + z &= 2 \\ 2x - 4y + 2z &= 4 \\ 5x - y + 2z &= 13. \end{aligned}$$

Linear system

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Solution; Start with the augmented matrix and transform it to RREF [Reduced Row Echelon Form] by applying a set of EROS

$$\begin{bmatrix} 1 & -2 & 1 & 2 \\ 2 & -4 & 2 & 4 \\ 5 & -1 & 2 & 13 \end{bmatrix} \xrightarrow{\frac{1}{2}R_2} \begin{bmatrix} 1 & -2 & 1 & 2 \\ 1 & -2 & 1 & 2 \\ 5 & -1 & 2 & 13 \end{bmatrix} \xrightarrow[R_3-5R_1]{R_2-R_1} \begin{bmatrix} 1 & -2 & 1 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 9 & -3 & 3 \end{bmatrix}$$

$$\xrightarrow[R_2 \leftrightarrow R_3]{\frac{1}{9}R_3 \text{ then}} \begin{bmatrix} 1 & -2 & 1 & 2 \\ 0 & 1 & -1/3 & 1/3 \\ 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{R_1+2R_2} \begin{bmatrix} 1 & 0 & \frac{1}{3} & \frac{8}{3} \\ 0 & 1 & -1/3 & 1/3 \\ 0 & 0 & 0 & 0 \end{bmatrix} \Longrightarrow$$

$$\left.\begin{aligned} x &= -\frac{1}{3}z + \frac{8}{3} \\ y &= \frac{1}{3}z + 1/3 \end{aligned}\right\} \text{ or } \begin{aligned} x &= -\frac{1}{3}t + 8/3 \\ y &= \frac{1}{3}t + 1/3 \\ z &= t \end{aligned} \left\{ \begin{array}{l} \text{where } t \text{ is} \\ \text{a parameter} \\ \text{with } -\infty < t < \infty \end{array} \right.$$

Rewrite as vector:

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -\frac{1}{3}t + 8/3 \\ 1/3 t + 1/3 \\ t + 0 \end{bmatrix} \Longrightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{t}{3}\begin{bmatrix} -1 \\ 1 \\ 3 \end{bmatrix} + \frac{1}{3}\begin{bmatrix} 8 \\ 1 \\ 0 \end{bmatrix}$$

Where parameter t has values between $(-\infty, \infty)$. Particular values (solutions) can be obtained by assigning specific or particular values for t. Examples

For $t = 0$ $x = 8/3$, $y = 1/3$, $z = 0$

$t = 3$, $x = -1 + \frac{8}{3} = \frac{5}{3}$, $y = \frac{2}{3}$, $z = 3$.

etc.

(1) Use the Gauss-Jordan method to find all solutions of the following system:

OCTOBER 2018 MIDTERM.

$$\begin{aligned} 2x_1 + 2x_2 - 2x_3 - x_4 - 2x_5 &= 0 \quad ① \\ 4x_1 \qquad\qquad\qquad\qquad - x_5 &= 0 \quad ② \\ 8x_1 \qquad - x_3 \qquad\qquad &= 0 \quad ③ \\ 10x_1 \qquad\qquad - 2x_4 \qquad &= 0 \quad ④ \end{aligned}$$

Linear system ②

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Solution: From equations ②, ③ and ④ we can obtain: $x_3 = 8x_1$, $x_4 = 5x_1$, $x_5 = 4x_1$

Now replace x_3, x_4 and x_5 in equation ①

Thus $2x_1 + 2x_2 - 2(8x_1) - 5x_1 - 2(4x_1) = 0$

or $2x_2 = 27x_1 \Rightarrow x_2 = \frac{27}{2}x_1$

We oblain parametric solution for the above system

$$\begin{cases} x_2 = \frac{27}{2}x_1 \\ x_3 = 8x_1 \\ x_4 = 5x_1 \\ x_5 = 4x_1 \end{cases} \Rightarrow$$

Now let $x_1 = t$, $-\infty < t < \infty$

Thus the general solution of the system is:

$x_1 = t$, $x_2 = \frac{27}{2}t$, $x_3 = 8t$

$x_4 = 5t$ and $x_5 = 4t$

where $-\infty < t < \infty$.

This system has infinitely many solutions including the zero solution ~~for~~ $t = 0$. That is when $t = 0$

$x_1 = x_2 = x_3 = x_4 = x_5 = 0$. Also,

when $t = 2$, $x_1 = 2$, $x_2 = 27$, $x_3 = 16$

$x_4 = 10$ and $x_5 = 8$

MARKS

1. Use the Gauss-Jordan method to find all the solutions of the system:

$$x_1 + x_2 + 2x_3 = 8$$
$$-x_1 - 2x_2 + 3x_3 = 1$$
$$3x_1 - 7x_2 + 4x_3 = 10.$$

linear System

2. Determine the values of a for which the system has no solution, exactly one solution or infinitely many solutions:

$$x + 2y - 3z = 4$$
$$3x - y + 5z = 2$$
$$4x + y + (a^2 - 14)z = a + 2.$$

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1 Soln: start with the augmented matrix and apply a set of EROS to oblain RREF.

$$\begin{bmatrix} 1 & 1 & 2 & 8 \\ -1 & -2 & 3 & 1 \\ 3 & -7 & 4 & 10 \end{bmatrix} \xrightarrow[R_3-3R_1]{R_2+R_1} \begin{bmatrix} 1 & 1 & 2 & 8 \\ 0 & -1 & 5 & 9 \\ 0 & -10 & -2 & -14 \end{bmatrix} \xrightarrow[R_3-10R_1]{R_1+R_2} \begin{bmatrix} 1 & 0 & 7 & 17 \\ 0 & -1 & 5 & 9 \\ 0 & 0 & -52 & -104 \end{bmatrix}$$

$$\xrightarrow[-\frac{1}{52}R_3]{} \begin{bmatrix} 1 & 0 & 7 & 17 \\ 0 & -1 & 5 & 9 \\ 0 & 0 & 1 & 2 \end{bmatrix} \xrightarrow[R_2-5R_3]{R_1-7R_3} \begin{bmatrix} 1 & 0 & 0 & 3 \\ 0 & -1 & 0 & -1 \\ 0 & 0 & 1 & 2 \end{bmatrix} \Rightarrow \begin{matrix} x_1 = 3 \\ x_2 = 1 \\ x_3 = 2. \end{matrix}$$

$=R_2$

2. Aug. matrix: $\begin{bmatrix} 1 & 2 & -3 & 4 \\ 3 & -1 & 5 & 2 \\ 4 & 1 & a^2-14 & a+2 \end{bmatrix} \xrightarrow[R_2-3R_1]{R_3-4R_1} \begin{bmatrix} 1 & 2 & -3 & 4 \\ 0 & -7 & 14 & -10 \\ 0 & -7 & a^2-2 & a-14 \end{bmatrix}$

$$\xrightarrow{R_3-R_2} \begin{bmatrix} 1 & 2 & -3 & 4 \\ 0 & -7 & 14 & -10 \\ 0 & 0 & a^2-16 & a-4 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 2 & -3 & 4 \\ 0 & -7 & 14 & -10 \\ 0 & 0 & (a+4)\cdot(a-4) & a-4 \end{bmatrix}$$

Note that R_3 is a zero row when $a-4=0$
Also when $a+4=0$, Row 3: $0 \; 0 \; 0 \; -8 \Rightarrow 0=-8$.
This implies the system has no solution when $a=-4$. Finally when $a \neq \pm 4$ The system will have unique soln.

Linear System: P4

APRIL 2012 Exam.

MARKS

1. Use the Gauss-Jordan method to find all the solutions of the system:

$$x_1 + x_2 + 2x_3 = 8$$
$$-x_1 - 2x_2 + 3x_3 = 1$$
$$3x_1 - 7x_2 + 4x_3 = 10.$$

2. Determine the values of a for which the system has no solution, exactly one solution or infinitely many solutions:

$$x + 2y - 3z = 4$$
$$3x - y + 5z = 2$$
$$4x + y + (a^2 - 14)z = a + 2.$$

Solution: Apply a set of EROS to the augmented matrix of the system:

$$\begin{bmatrix} 1 & 2 & -3 & 4 \\ 3 & -1 & 5 & 2 \\ 4 & 1 & a^2-14 & a+2 \end{bmatrix} \xrightarrow{R_3-(R_1+R_2)} \begin{bmatrix} 1 & 2 & -3 & 4 \\ 3 & -1 & 5 & 2 \\ 0 & 0 & a^2-16 & a-4 \end{bmatrix} \longrightarrow$$

$$\begin{bmatrix} 1 & 2 & -3 & 4 \\ 3 & -1 & 5 & 2 \\ 0 & 0 & (a-4)(a+4) & a-4 \end{bmatrix}$$

For $a=4$ Row 3 becomes a row of zeros. This reduces the system to 2-equations in 3-unknows. The system has infinitely many solutions. For $a+4=0$ or $a=-4$ ~~row~~ 3 becomes $[0 \ 0 \ 0 \ -8]$ or $0=-8 \Rightarrow$ No solution.

For $a \neq 4, -4$ the system will have unique solution as the Row 3 will have the form $[0 \ 0 \ m \ n]$ where m and n are non zero numbers.

1. **Using the Gauss-Jordan method (i.e. reduced row echelon form method). find all the solutions of the following system of equations**

Dec 2012 EXAM.

$$\begin{aligned} 2x - 2y \qquad + 2u + 3v &= 1 \\ 3x - 3y - z + 5u + 2v &= 3 \\ 2x - 2y - 2z + 6u \qquad &= -2. \end{aligned}$$

linear system

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solution: This is a non-homogeneous linear system of 3-equations in 5-unknowns. The system has infinitely many solutions. I start with augmented matrix of the system and apply a set of EROS (Elelentary Row operations) to transform the system into RREF [Reduced Row Echelon Form]

$$\begin{bmatrix} 2 & -2 & 0 & 2 & 3 & 1 \\ 3 & -3 & -1 & 5 & 2 & 3 \\ 2 & -2 & -2 & 6 & 0 & -2 \end{bmatrix} \xrightarrow{\frac{1}{2}R_3} \begin{bmatrix} 2 & -2 & 0 & 2 & 3 & 1 \\ 3 & -3 & -1 & 5 & 2 & 3 \\ 1 & -1 & -1 & 3 & 0 & -1 \end{bmatrix} \xrightarrow[R_2-3R_3]{R_1-2R_3}$$

$$\begin{bmatrix} 0 & 0 & 2 & -4 & 3 & 3 \\ 0 & 0 & 2 & -4 & 2 & 6 \\ 1 & -1 & -1 & 3 & 0 & -1 \end{bmatrix} \xrightarrow{\frac{1}{2}R_2} \begin{bmatrix} 0 & 0 & 2 & -4 & 3 & 3 \\ 0 & 0 & 1 & -2 & 1 & 3 \\ 1 & -1 & -1 & 3 & 0 & -1 \end{bmatrix} \xrightarrow[R_3+R_2]{R_1-2R_2}$$

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 1 & -3 \\ 0 & 0 & 1 & -2 & 1 & 3 \\ 1 & -1 & 0 & 1 & 1 & 2 \end{bmatrix} \xrightarrow[R_2-R_1]{R_3-R_1} \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & -3 \\ 0 & 0 & 1 & -2 & 0 & 6 \\ 1 & -1 & 0 & 1 & 0 & 5 \end{bmatrix} \Rightarrow$$

$$\left.\begin{aligned} x - y + u &= 5 \\ z - 2u &= 6 \\ v &= -3 \end{aligned}\right\} \overset{or}{\Longrightarrow} \left.\begin{aligned} x &= y - u + 5 \\ z &= 2u + 6 \\ v &= -3 \end{aligned}\right\}$$

Replace $y = t_1$, $u = t_2$, $-\infty < t_1, t_2 < \infty$

Thus the ~~general~~ general solⁿ:

$$\left.\begin{aligned} x &= t_1 - t_2 + 5 \\ y &= t_1 \\ z &= 2t_2 + 6 \\ u &= t_2 \\ v &= -3 \end{aligned}\right\} \Rightarrow \begin{bmatrix} x \\ y \\ z \\ u \\ v \end{bmatrix} = t_1 \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + t_2 \begin{bmatrix} -1 \\ 0 \\ 2 \\ 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 5 \\ 0 \\ 6 \\ 0 \\ -3 \end{bmatrix}$$

General solution is L.Comb of three vectors.