EMAT 233 Final Exam W2012

Solutions

Q1 (a) The plane contains the 2 vectors

$$\vec{v} = \overrightarrow{(2,1,0)(3,4,0)} = \langle 1,3,0 \rangle$$

$$\vec{w} = \overrightarrow{(2,1,0)(1,1,1)} = \langle -1,0,1 \rangle$$

and then is $\perp$ to $\vec{v} \times \vec{w} = \begin{vmatrix} i & j & k \\ 1 & 3 & 0 \\ -1 & 0 & 1 \end{vmatrix}$

$$= \langle 3,-1,3 \rangle$$

Then, using $\vec{N} = \vec{v} \times \vec{w}$ & the point $(2,1,0)$, we have

$$3(x-2) - (y-1) + 3(z-0) = 0$$

$$\iff 3x - y + 3z = 6 - 1 = 5$$

(b) $\vec{r}(t) = \overrightarrow{OP_0} + t\vec{v}$

$$= \langle 3,4,0 \rangle + t\langle 3,-1,3 \rangle$$

$$= \langle 3+3t, 4-t, 3t \rangle$$

Q2 (a) By the chain Rule

$$\frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt} + \frac{\partial w}{\partial z}\frac{dz}{dt}$$

$$= \left(3x^2 \cdot \frac{1}{2}(x^3+y)^{-1/2} + ze^{zx}\right)(2)$$

$$+ \frac{1}{2}(x^3+y)^{-1/2}(2t) + xe^{zx}\left(\frac{-1}{t^2}\right)$$

Using $t=1$, $x(1)=2$, $y(1)=1$ & $z(1)=1$,
we get

$$\frac{dw}{dt}(1) = \left(6(9)^{-1/2} + e^2\right)2 + \frac{1}{2}(9)^{-1/2}(2) + 2e^2(-1)$$

$$= (2+e^2)2 + \frac{1}{3} - 2e^2 = \frac{7}{3}$$

(b) $\vec{\nabla} T = \langle 10x + ye^{xy}, xe^{xy} \rangle$

$\vec{\nabla} T(2,3) = \langle 20 + 3e^6, 2e^6 \rangle = \vec{v}$

<u>unit direction</u> $\vec{u} = \dfrac{\vec{v}}{|\vec{v}|} = \dfrac{\langle 20+3e^6, 2e^6 \rangle}{\sqrt{(20+3e^6)^2 + 4e^{12}}}$

and the value of the directional derivative in the direction on $\vec{u}$ is

$$|\vec{\nabla} T(2,3)| = \sqrt{(20+3e^6)^2 + 4e^{12}}$$

$\boxed{Q3}$ For every point (x,y,z) on the surface, the tangent plane to $F(x,y,z) = z - 10 + x^2 + y^2 = 0$ has normal $\vec{\nabla} F(x,y,z) = \langle 2x, 2y, 1 \rangle$

We want $\langle 2x, 2y, 1 \rangle = k \langle 1, \frac{3}{2}, \frac{1}{2} \rangle$

$\iff x = \frac{k}{2}, \quad y = \frac{3k}{4}, \quad 1 = \frac{k}{2}$

Then, $k=2$ and $x=1$, $y=\frac{3}{2}$.

Plugging in the equation of the surface to find z, we get

$$z = 10 - 1^2 - \left(\frac{3}{2}\right)^2 \Leftrightarrow z = 9 - \frac{9}{4} = \frac{27}{4}$$

$\boxed{Q5}$ $f(x,y,z) = e^{x^2}\cos z + z^4 \sin y$

(a) $\text{grad} f = \vec{\nabla} f = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right\rangle$

$$= \left\langle 2x e^{x^2}\cos z,\ z^4\cos(y),\ -e^{x^2}\sin z + 4z^3 \sin y \right\rangle$$

(b) $\text{div}(\vec{\nabla} f) = \frac{\partial}{\partial x}\left(2x e^{x^2}\cos z\right) + \frac{\partial}{\partial y}\left(z^4 \cos y\right)$

$$+ \frac{\partial}{\partial z}\left(-e^{x^2}\sin z + 4z^3 \sin y\right)$$

$$= 2e^{x^2}\cos z + 4x^2 e^{x^2}\cos z - z^4 \sin y - e^{x^2}\cos z + 12z^2 \sin y$$

(c) div f does not make sense since f is a scalar function

(d) $\text{curl}(\vec{\nabla} f) = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2x e^{x^2}\cos z & z^4 \cos y & -e^{x^2}\sin z + 4z^3 \sin y \end{vmatrix}$

$$= \left\langle 4z^3\cos y - 4z^3 \cos y,\ -2xe^{x^2}\sin z + 2xe^{x^2}\sin z,\ 0 - 0 \right\rangle = \langle 0, 0, 0 \rangle$$

(e) $\text{grad} f = \vec{\nabla} f$ is a vector function, so grad (grad f) does not make sense

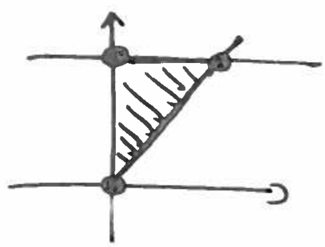
$\boxed{Q4}$ Find the moment of inertia about the y-axis of the lamina that has the given shape & density

<u>bounded</u> by $x=0, y=x, y=1$

$$\rho(x,y) = \sqrt{1+y^4}$$

$$I_y = \iint_R x^2 \rho(x,y)\, dA$$

<u>sol</u> R



$$\left.\begin{array}{l} 0 \le y \le 1 \\ 0 \le x \le y \end{array}\right] \quad \underline{\text{type 2}}$$

$$\left.\begin{array}{l} 0 \le x \le 1 \\ x \le y \le 1 \end{array}\right] \quad \underline{\text{type 1}}$$

$$I_y = \iint_R x^2 \sqrt{1+y^4}\, dA$$

$$= \int_0^1 \int_x^1 x^2 \sqrt{1+y^4}\, dy\, dx \quad \text{Hum}$$

But $\displaystyle\int_0^1 \int_0^y x^2 \sqrt{1+y^4}\, dx\, dy$

$$= \int_0^1 \left.\frac{x^3}{3}\right|_0^y \sqrt{1+y^4}\, dy$$

$$= \frac{1}{3} \int_0^1 y^3 \sqrt{1+y^4}\, dy$$

$u = y^4 + 1$

$du = 4y^3\, dy$

$\frac{du}{4} = y^3\, dy$

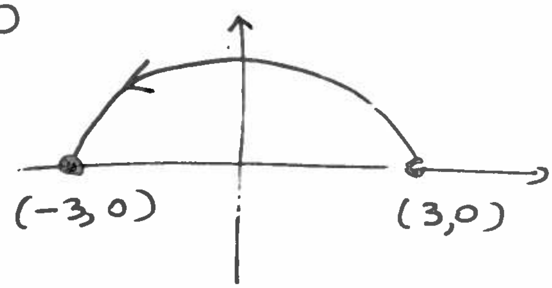
$$\frac{1}{4}\int \sqrt{u}\, du = \frac{1}{4}\,\frac{u^{3/2}}{3/2} = \frac{1}{6}\,u^{3/2}$$

Then $\frac{1}{18}\,(1+y^4)^{3/2}\Big|_0^1 = \frac{1}{18}\left(2^{3/2} - 1^{3/2}\right)$

$\boxed{Q6}$ Compute the line integral $\int -y\,dx + x\,dy$

for a) helix $\frac{x^2}{9} + \frac{y^2}{4} = 1$ from $(3,0)$ to $(-3,0)$

for $y \geq 0$



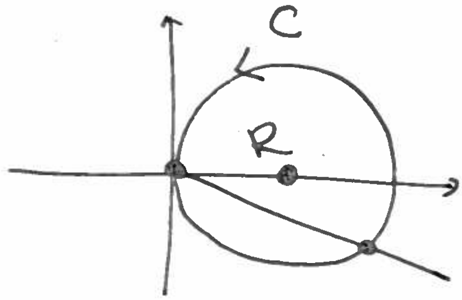
sol $\vec{x}(t) = \langle 3\cos t,\ 2\sin t \rangle \qquad 0 \leq t \leq \pi$

check $\frac{x^2}{9} + \frac{y^2}{4} = \frac{9\cos^2 t}{9} + \frac{4\sin^2 t}{4} = 1$ ok

$$\int_C -y\,dx + x\,dy = \int_0^\pi -2\sin t\,(-3\sin t) + (3\cos t)(2\cos t)\,dt$$

$$= 6\int_0^\pi \sin^2 t + \cos^2 t\,dt = 6\int_0^\pi dt = 6\pi$$

b) $r = 2\cos\theta \quad -\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$



$$r = 2\cos\theta$$
$$\Leftrightarrow r^2 = 2r\cos\theta$$
$$\Leftrightarrow x^2 + y^2 = 2x$$
$$\Leftrightarrow (x-1)^2 - 1 + y^2 = 0$$
$$\Leftrightarrow (x-1)^2 + y^2 = 1$$

$$\left[\begin{array}{l} r = a\cos\theta \\ \quad \Longleftrightarrow \left(x - \frac{a}{2}\right)^2 + y^2 = \left(\frac{a}{2}\right)^2 \end{array} \right]$$

Not so easy to parametrise....

But closed curve $\longrightarrow$ Green's thm

$$\int \overset{-y}{P}\,dx + \overset{x}{Q}\,dy = \iint_R \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\, dA$$

Here $\dfrac{\partial Q}{\partial x} - \dfrac{\partial P}{\partial y} = 1 - (-1) = 2$

Then $2\iint_R dA = 2\,\text{area}(R) = 2\pi$

as a shifted circle has the same area than a circle centered at the origin

<u>or</u> compute the double integral

<u>polar coordinates</u> $\quad -\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$

$0 \le r \le 2\cos\theta$

$$2\int_{-\pi/2}^{\pi/2}\int_0^{2\cos\theta} r\,dr\,d\theta$$

$$= 2\int_{-\pi/2}^{\pi/2} \frac{r^2}{2}\Big|_0^{2\cos\theta} d\theta$$

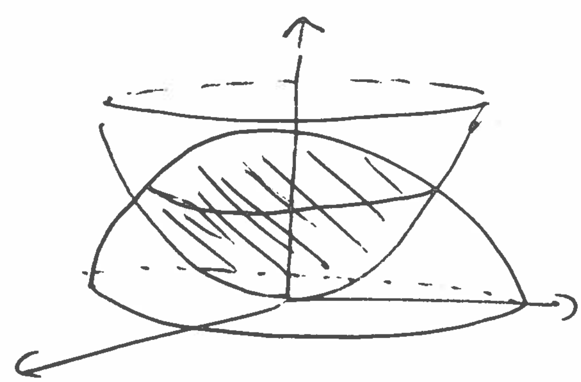
$$= 4\int_{-\pi/2}^{\pi/2} \cos^2\theta\, d\theta = 4\int_{-\pi/2}^{\pi/2} \frac{1}{2}(1+\cos 2\theta)\,d\theta$$

$$= 2\left[\theta + \frac{\sin(2\theta)}{2}\Big|_{-\pi/2}^{\pi/2}\right] = 2\pi$$

Q8. Find the flux outward of the radial vector field $\vec{F}(x,y,z) = xi + yj + zk$ through the boundary of the region in $\mathbb{R}^3$ given by

$x^2+y^2+z^2 \le 2 \quad \& \quad z \ge x^2+y^2$

<u>sol</u>



<u>boundary</u>

$\rightsquigarrow$ use divergence thm

$$\text{Flux} = \iint_S (\vec{F} \cdot \vec{n})\, dS$$

Parametrise S & compute $\vec{n}$. Here S_1 & S_2.

<u>or</u>

$$\iint_S (\vec{F} \cdot \vec{n})\, dS = \iiint_D \operatorname{div} \vec{F}\, dV$$

where D is the region inside S.

Then <u>S needs to be closed</u>.

Here $\vec{F} = \langle x, y, z \rangle$

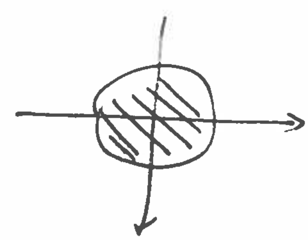
$$\operatorname{div} \vec{F} = \vec{\nabla} \cdot \vec{F} = \frac{\partial}{\partial x}(x) + \frac{\partial}{\partial y}(y) + \frac{\partial}{\partial z}(z) = 3$$

<u>and get</u>

$$3 \iiint_D dV$$

$$x^2 + y^2 \leq z \leq \sqrt{2 - x^2 - y^2}$$

$$(x, y) \in R$$



circle of intersection of

$$x^2 + y^2 + z^2 = 2 \quad \& \quad x^2 + y^2 = z$$

$$\Leftrightarrow \quad z^2 + z - 2 = 0$$

$$\Leftrightarrow \quad (z - 1)(z + 2) = 0 \quad \boxed{z = 1}, \; z = -2$$

$\boxed{x^2+y^2=1}$ use polar coordinates

$$r^2 \leq z \leq \sqrt{2-r^2}$$

$$0 \leq \theta \leq 2\pi$$

$$0 \leq r \leq 1$$

Then $3\int_0^{2\pi}\int_0^1\int_{r^2}^{\sqrt{2-r^2}} r\,dz\,dr\,d\theta$

$$= 3(2\pi)\int_0^1 \sqrt{2-r^2}\,r - r^3\,dr \qquad u = 2-r^2$$

$$du = -2r\,dr$$

$$\frac{du}{-2} = r\,dr$$

$$= \frac{\cancel{3}(2\pi)}{-2}\,\frac{(2-r^2)^{3/2}}{\cancel{3}/2}\Big|_0^1 - 6\pi\,\frac{r^4}{4}\Big|_0^1$$

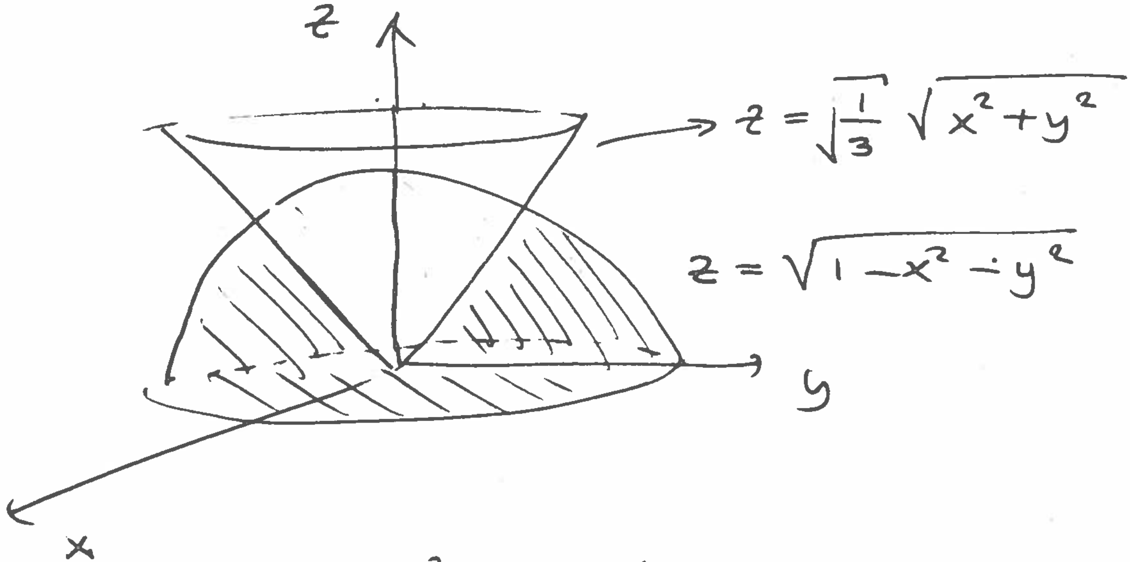
$$= -2\pi\left(1^{3/2} - 2^{3/2}\right) - \frac{6\pi}{4}$$

$$= \pi\left(2\cdot 2^{3/2} - 2 - \frac{3}{2}\right)$$

$\boxed{Q9}$ Find the volume of the region bounded by $x^2+y^2+z^2 \leq 1$, $4z^2 \leq x^2+y^2+z^2$ and $z \geq 0$

Sol $\quad 4z^2 \leq x^2+y^2+z^2 \iff 3z^2 \leq x^2+y^2$

outside the cone



$0 \leq \theta \leq 2\pi$

$0 \leq \rho \leq 1$

cone $\leq \phi \leq \pi/2$

spherical

$$z^2 = a(x^2+y^2) \iff \rho^2\cos^2\phi = a\,\rho^2\sin^2\phi$$

$$\iff \frac{\sin^2\phi}{\cos^2\phi} = \tan^2\phi = \frac{1}{a}$$

$$\iff \phi = \arctan\left(\sqrt{\frac{1}{a}}\right) \text{ or } \arctan\left(-\sqrt{\frac{1}{a}}\right)$$

top bottom

Here $a = \frac{1}{3} \Rightarrow \frac{1}{a} = 3$

$$\arctan(\sqrt{3}) = \frac{\pi}{3}$$

$$\frac{\sin\phi}{\cos\phi} = \sqrt{3}$$

ie $\sin\phi = \frac{\sqrt{3}}{2}$

$\cos\phi = 1/2$

and $\phi = \pi/3$

Then get $\iiint_D 1 \, dV$

$$= \int_0^{2\pi} \int_0^1 \int_{\pi/3}^{\pi/2} \rho^2 \sin\phi \, d\phi \, d\rho \, d\theta$$

$$= 2\pi \int_0^1 \rho^2 \left(-\cos\phi \Big|_{\pi/3}^{\pi/2} \right) d\rho$$

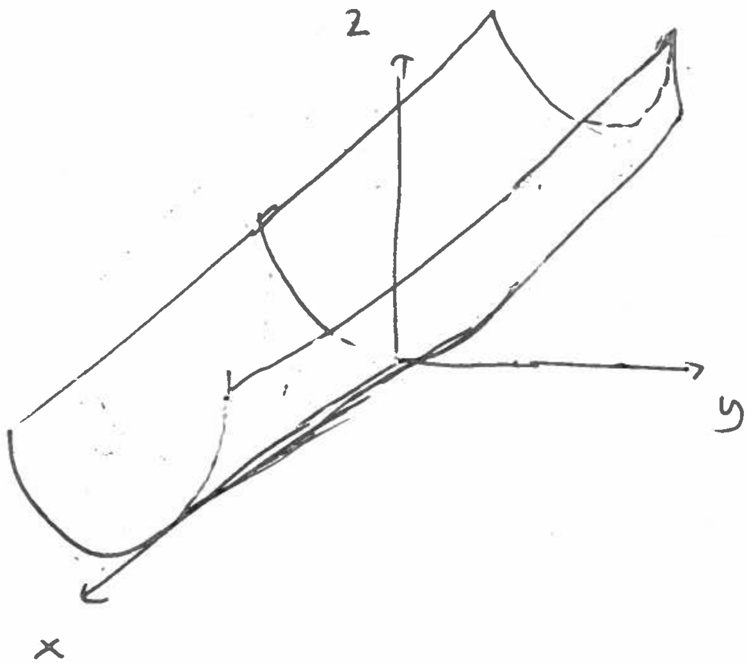
$$= 2\pi \int_0^1 \rho^2 \left(0 - \left(-\tfrac{1}{2}\right) \right) d\rho$$

$$= \pi \left. \frac{\rho^3}{3} \right|_0^1 = \frac{\pi}{3}$$

$\boxed{Q10}$ Let $\vec{F}(x,y,z)$ be the vector field

$\vec{F}(x,y,z) = \langle yz, xz, xy \rangle$.

a) Solve $\int_C \vec{F} \cdot d\vec{r}$ where C is the curve of intersection of $x^2+y^2=1$ & $z=y^2$ using Stokes' thm



$z = y^2$

and cut with

cylinder $x^2+y^2=1$

The curve C then encloses the portion of the surface $z=y^2$ contained in the cylinder

counterclockwise on C means $\vec{n}$ upwards for S.

<u>Surface</u> $z = f(x,y) = y^2$

<u>$\forall x, y \in R$</u> $x^2+y^2 \le 1$

<u>Stokes' thm</u>

$$\int_C \vec{F} \cdot d\vec{r} = \iint_S \text{curl}\,\vec{F} \cdot \vec{n}\, dS$$

where the normal on S & the orientation on C are given by <u>RH Rule</u>

$$\text{curl}\,\vec{F} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ yz & xz & xy \end{vmatrix}$$

$$= \langle x - x,\ y - y,\ z - z \rangle = \vec{0}$$

Then $\displaystyle\int_C \vec{F}\cdot d\vec{r} = \iint_S \vec{0}\, dS = 0$

In this case, this is a conservative vector field

for $\vec{F}(x,y,z) = \langle yz,\ xz,\ xy \rangle$,

$\vec{F} = \vec{\nabla f}$ where $f(x,y,z) = xyz$.

Then, the integral around any closed loop is 0!

$\boxed{\text{Q7}}$ Show

$$I = \int_C (1+e^{-y})\,dx - (xe^{-y} + 4y)\,dy$$

is independent of path & evaluate for any path between $(1,0)$ & $(2,1)$

<u>Sol</u> <u>Check</u> $\dfrac{\partial P}{\partial y} = -e^{-y}$

$$\frac{\partial Q}{\partial x} = -e^{-y} \quad \underline{\text{OK}}$$

Sol'n ① $\frac{\partial f}{\partial x} = 1 + e^{-y}$

② $\frac{\partial f}{\partial y} = -xe^{-y} - 4y$

$$f(x,y) = \int (1+e^{-y})\,dx = x + xe^{-y} + C(y)$$

$$\Rightarrow \frac{\partial f}{\partial y} = -xe^{-y} + C'(y) = ②$$

$$= -xe^{-y} - 4y$$

$$\Rightarrow C'(y) = \int -4y\,dy = -4\frac{y^2}{2} + C$$

$$= -2y^2 + C$$

Then $f(x,y) = x + xe^{-y} - 2y^2 + C$

& $\displaystyle\int_{(1,0)}^{(2,1)} (1+e^{-y})\,dx + (-xe^{-y} - 4y)\,dy$

$$= x + xe^{-y} - 2y^2 \Big|_{(1,0)}^{(2,1)}$$

$$= (2 + 2e^{-1} - 2) - (1+1)$$

$$= 2e^{-1} - 2.$$