Department of Computer Science & Software Engineering

Comp232 Mathematics for Computer Science

Fall 2020

Assignment 2. Solutions.

1. Let $P(x, y, z)$ denote the statement $x + y \le z$, where $x, y, z \in \mathbb{Z}^+$. What is the truth value of each of the following? Explain your answers.

   (a) $\forall x \exists y \exists z P(x, y, z)$.

   Solution: True. Let x be any integer. By choosing $y = 1$ and $z = x + 1$, we see that the statement is true.

   (b) $\forall y \exists x \forall z P(x, y, z)$.

   Solution: False. If $y = 10$ and $z = 1$ then there is no x such that $10 + x \le 1$

   (c) $\exists z \exists y \forall x P(x, y, z)$.

   Solution: False. x could be arbitrarily large.

2. For each of the premise-conclusion pairs below, give a valid step-by-step argument (proof) along with the name of the inference rule used in each step. For examples, see pages 73 and 74 in textbook.

   (a) Premise: $\{\neg p \vee q \rightarrow r,\ s \vee \neg q,\ \neg t,\ p \rightarrow t,\ \neg p \wedge r \rightarrow \neg s\}$, conclusion: $\neg q$.

   Solution:

Step	Conclusion	Reason
1.	$p \rightarrow t$	Premise
2.	$\neg t$	Premise
3.	$\neg p$	Modus Tollens using (1) and (2)
4.	$\neg p \vee q$	Addition using (3)
5.	$\neg p \vee q \rightarrow r$	Premise
6.	r	Modus Ponens using (4) and (5)
7.	$\neg p \wedge r$	Conjunction using (3) and (6)
8.	$\neg p \wedge r \rightarrow \neg s$	Premise
9.	$\neg s$	Modus Ponens using (7) and (8)
10.	$s \vee \neg q$	Premise
11.	$\neg q$	Disjunctive Syllogism using (9) and (10)

(b) Premise: $\{\neg p \to r \wedge \neg s,\ t \to s,\ u \to \neg p,\ \neg w,\ u \vee w\}$, conclusion: $\neg t \vee w$.

Solution:

Step	Conclusion	Reason
1.	$u \vee w$	Premise
2.	$\neg w$	Premise
3.	u	Disjunctive Syllogism using (1) and (2)
4.	$u \to \neg p$	Premise
5.	$\neg p$	Modus Ponens using (3) and (4)
6.	$\neg p \to r \wedge \neg s$	Premise
7.	$r \wedge \neg s$	Modus Ponens using (5) and (6)
8.	$\neg s$	Simplification using (7)
9.	$t \to s$	Premise
10.	$\neg t$	Modus Tollens using (8) and (9)
11.	$\neg t \vee w$	Addition using (10)

(c) Premise: $\{p \vee q,\ q \to r,\ p \wedge s \to t,\ \neg r,\ \neg q \to u \wedge s\}$, conclusion: t.

Solution:

Step	Conclusion	Reason
1.	$\neg r$	Premise
2.	$q \to r$	Premise
3.	$\neg q$	Modus Tollens using (1) and (2)
4.	$\neg q \to u \wedge s$	Premise
5.	$u \wedge s$	Modus Ponens using (3) and (4)
6.	s	Simplification using (5)
7.	$p \vee q$	Premise
8.	p	Disjunctive Syllogism using (3) and (7)
9.	$p \wedge s$	Conjunction using (6) and (8)
10.	$p \wedge s \to t$	Premise
11.	t	Modus Ponens using (9) and (10)

3. For each of the following, determine whether argument is valid. You may use a counterexample, equivalence transformations or truth-tables to justify your answer.

   (a) $p \to q$
   $\neg p$
   $\therefore\ \neg q$

   Solution: Invalid. Counterexample: p is *False* and q is *True*.

   (b) $\neg p \to \neg q$
   $\therefore\ (\neg p \to q) \to p$

   Solution: Valid. We show using a truth-table that

   $$\underbrace{\big(\neg p \to \neg q\big)}_{\mathbf{a}} \to \underbrace{\big(\underbrace{(\neg p \to q)}_{\mathbf{b}} \to p\big)}_{\mathbf{c}} \text{ is a tautology.}$$

p	q	$\neg p$	$\neg q$	**a** $\neg p \to \neg q$	**b** $\neg p \to q$	**c** $(\mathbf{b} \to p)$	$\mathbf{a} \to \mathbf{c}$
T	T	F	F	T	T	T	T
T	F	F	T	T	T	T	T
F	T	T	F	F	T	F	T
F	F	T	T	T	F	T	T

   (c) $p \to r$
   $q \to r$
   $\neg(p \vee q)$
   $\therefore\ \neg r$

   Solution: Invalid. Counterexample: p and q are *False* and r is *True*.

   (d) $p \to q$
   $p \to (q \to \neg p)$
   $\therefore\ \neg p$

   Solution: Valid. We show using a truth-table that

   $$\Big(\underbrace{(p \to q)}_{\mathbf{a}} \wedge \underbrace{\big(p \to \underbrace{(q \to \neg p)}_{\mathbf{b}}\big)}_{\mathbf{c}}\Big) \to \neg p \text{ is a tautology.}$$

p	q	$\neg p$	**a** $p \to q$	**b** $q \to \neg p$	**c** $p \to \mathbf{b}$	$\mathbf{a} \wedge \mathbf{c}$	$\mathbf{a} \wedge \mathbf{c} \to \neg p$
T	T	F	T	F	F	F	T
T	F	F	F	T	T	F	T
F	T	T	T	T	T	T	T
F	F	T	T	T	T	T	T

4. For each of the arguments below, indicate whether it is valid or invalid.

   (a) A convertible car is fun to drive.
   Isaac's car is not a convertible.
   ∴ Isaac's car is not fun to drive.

   Solution: Invalid. The argument is of the form
   $CC(\text{Isaac}) \to FD(\text{Isaac})$
   $\neg CC(\text{Isaac})$
   $\therefore \neg FD(\text{Isaac})$
   After applying universal instantiation, the argument contains the **fallacy of denying the hypothesis**.

   (b) All healthy people eat an apple a day.
   Herbert is not a healthy person.
   ∴ Herbert does not eat an apple a day.

   Solution: Invalid. The argument is of the form
   $HP(\text{Herbert}) \to EA(\text{Herbert})$
   $\neg HP(\text{Herbert})$
   $\therefore \neg EA(\text{Herbert})$
   It is an **inverse** error.

   (c) If a product of two real numbers is 0, then at least one of the numbers is 0.
   For a particular real number x, neither $(x-1)$ nor $(x+1)$ equals 0.
   ∴ The product $(x-1)(x+1)$ is not 0.

   Solution: Valid. Let's use a instead of $x-1$ and b instead of $x+1$. The argument is of the form
   $\forall x \forall y \big(x \cdot y = 0 \to (x = 0 \vee y = 0)\big)$
   $a \neq 0 \wedge b \neq 0$
   $\therefore\ a \cdot b \neq 0.$
   The argument is an instance of *Universal Modus Tollens*.

5. Use rules of inference to show that if $\forall x\big(P(x) \to Q(x)\big)$, $\forall x\big(Q(x) \to R(x)\big)$, and $\exists x\big(\neg R(x)\big)$ are true, then $\exists x\big(\neg P(x)\big)$ is true.

Solution:

(1)	$\forall x\big(P(x) \to Q(x)\big)$	Premise
(2)	$P(c) \to Q(c)$	Universal instantiation from (1)
(3)	$\forall x\big(Q(x) \to R(x)\big)$	Premise
(4)	$Q(c) \to R(c)$	Universal instantiation from (3)
(5)	$P(c) \to R(c)$	Hypothetical Syllogism from (2) and (4)
(6)	$\exists x\big(\neg R(x)\big)$	Premise
(7)	$\neg R(c)$	Existential instantiation from (6)
(8)	$\neg P(c)$	Modus Tollens from (7) and (8)
(6)	$\exists x\big(\neg P(x)\big)$	Existential Generalization from (7)

6. (a) Give a direct proof of: "If x is an odd integer and y is an even integer, then $x+y$ is odd."

Solution: If x is odd, then $x = 2k+1$ for some $k \in \mathbb{Z}$, and if y is even, then $y = 2k'$ for some $k' \in \mathbb{Z}$. Consequently $x+y = 2k+1+2k' = 2(k+k')+1$ which means that $x+y$ is odd.

(b) Give a proof by contradiction of: "If n is an odd integer, then n^2 is odd."

Solution: Suppose n is odd and n^2 is even. Then $n = 2k+1$, for some $k \in \mathbb{Z}$, and $n^2 = (2k+1)(2k+1) = 4k^2+4k+1 = 2(2k^2+2k)+1$, which contradicts the assumption that n^2 is even.

(c) Give an indirect proof of: "If x is an odd integer, then $x+2$ is odd."

Solution: We show that if $x+2$ is even, then x is even. If $x+2$ is even, then $x+2 = 2k$, for some $k \in \mathbb{Z}$. Therefore $x = 2k-2 = 2(k-1)$, which means that x is even.

(d) Use a proof by cases to show that there are no solutions in positive integers to the equation $x^4+y^4 = 100$.

Solution: The powers are $1^4 = 1,\ 2^4 = 16,\ 3^4 = 81,\ 4^4 = 256$.

Therefore the possible cases for (x,y) where $x^4+y^4 \leq 100.$ are

- $(1,1)$ which gives 2.
- $(1,2)$ or $(2,1)$ which gives 17.
- $(1,3)$ or $(3,1)$ which gives 82.
- $(2,2)$ which gives 32.
- $(2,3)$ or $(3,2)$ which gives 97.

Consequently the equation $x^4+y^4 = 100$ has no solutions im positive integers.

(e) Prove that given a nonnegative integer n, there is a unique nonnegative integer m, such that $m^2 \leq n < (m+1)^2$.

Solution:

Establish existence: Choose $m = \lfloor \sqrt{n} \rfloor$. Then $\sqrt{n} = m+\epsilon$, where $0 \leq \epsilon < 1$.

Thus $m^2 \leq (m+\epsilon)^2 = \sqrt{n}^2 = n = \sqrt{n}^2 = (m+\epsilon)^2 < (m+1)^2$.

Show uniqueness: Suppose $p^2 \leq n < (p+1)^2$ for some integer p. Then $p \leq \sqrt{n} < (p+1)$, and since $\sqrt{n} = m+\epsilon$ we get $p \leq m+\epsilon < p+1$, which, since $0 \leq \epsilon < 1$, is true only if $p = m$.

7. For each of the statements below state whether it is True or False. If True then give a proof. If False then explain why, e.g., by giving a counterexample.

   (a) The difference of any two odd integers is odd.

   Solution: The claim is false. Counterexample: $n = 7, m = 3, m - n = 4$.

   (b) Let a and b be integers. If $a + b$ is even, then either a or b is even.

   Solution: The claim is false. Counterexample: $1 + 1 = 2$.

   (c) For all positive integers n, it holds that n is even if and only if $3n^2 + 8$ is even.

   Solution: Only if direction: Suppose n is even. Then $n = 2k$ for some $k \in \mathbb{Z}$. Consequently $3n^2 + 8 = 3(2k)^2 + 8 = 12k^2 + 8 = 2(6k + 4)$ which means that $3n^2 + 8$ is even.

   If direction: Proof by contraposition: Suppose n is odd. Then $n = 2k + 1$ for some $k \in \mathbb{Z}$. Consequently $3n^2 + 8 = 3(2k + 1)^2 + 8 = 3(4k^2 + 4k + 1) + 8 = 12k^2 + 12k + 3 + 8 = 2(6k^2 + 6k) + 10 + 1 = 2(6k^2 + 6k + 5) + 1$, which means that $3n^2 + 8$ is odd.

   (d) For all positive $x, y \in \mathbb{R}$, if x is irrational and y is irrational then $x + y$ is irrational.

   Solution: The statement is false. We first show that $2+\sqrt{2}$ is irrational. Suppose it is not. Then $2 + \sqrt{2} = \frac{a}{b}$, for some integers a and b, with $b \neq 0$. But then $\sqrt{2} = \frac{a}{b} - 2 = \frac{a-2b}{b}$ which would mean that $\sqrt{2}$ is rational; a contradiction since we already have shown that $\sqrt{2}$ is irrational. Similarly we see that $2 - \sqrt{2}$ is irrational. We now have a counter-example, since $(2 + \sqrt{2}) + (2 - \sqrt{2}) = 4$, which is rational. Note that $2 - \sqrt{2}$ is positive.

   (e) $\forall x, y \in \mathbb{R}$, if x is irrational and y is rational then xy is irrational.

   Solution: The statement is false. Counterexample: Let $x = \sqrt{2}$, which is irrational, and $y = 0$, which is rational. Then $xy = 0$, which is rational.