COMP 361 Elementary Numerical Methods Section NN
TAKE-HOME TERM TEST 2 April 14, 2020 Duration: 8:00-24:00

FIRST NAME	LAST NAME	ID:

INSTRUCTIONS

- Books, notes, internet are allowed.
- The test has to be done individually. No copying from anybody and no soliciting of outside help. Violators of these rules will get zero marks for the exam and they may face additional penalties.
- This exam has two parts:
  - Part I has 6 multiple choice questions, worth 2 point each. You must mark your answer on the examination sheet.
  - Part II has 6 problems, worth 8 points each.
- The entire exam is worth 60 marks.
- The total number of pages is 9.
- When you are done scan your exam and submit the scan online on EAS and on Moodle no later than midnight April 14, 2020.
- Use a *dark black* or *dark blue* pen to enter your final answers.

Table for Instructor's use

PART I	Problem 7	Problem 8	Problem 9	Problem 10	Problem 11	Problem 12	TOTAL

PART I

In each Problem *circle the correct answers* (only one answer is correct)

Problem 1. For all vectors $\mathbf{x}$ in $\mathbb{R}^n$ we have

(a) $\| \mathbf{x} \|_2 \leq n \| \mathbf{x} \|_\infty$ (b) $||\mathbf{x} - \mathbf{y}|| \geq |||\mathbf{x}|| - ||\mathbf{y}|||$

(A) Both (a) and (b) are valid.

(B) (a) is valid and (b) is invalid.

(C) (a) is invalid and (b) is valid.

(D) Both (a) and (b) are invalid.

Problem 2. For all n by n non-singular matrices A we have

(a) $||A|| \geq \frac{1}{||A^{-1}||}$ (b) $||A||_1 = ||A^T||_\infty$

(A) Both (a) and (b) are valid.

(B) (a) is valid and (b) is invalid.

(C) (a) is invalid and (b) is valid.

(D) Both (a) and (b) are invalid.

Problem 3. For all square matrices A and B we have

(a) $\| A - B \|_\infty \leq \| A \|_\infty - \| B \|_\infty$ (b) $\| AB \|_\infty \leq \| A \|_\infty \| B \|_\infty$

(A) Both (a) and (b) are valid.

(B) (a) is valid and (b) is invalid.

(C) (a) is invalid and (b) is valid.

(D) Both (a) and (b) are invalid.

Problem 4. For all square, invertible matrices A and B we have

(a) $\mathrm{cond}(AB) \geq \mathrm{cond}(A)\ \mathrm{cond}(B)$ (b) $||A|| = \max_{||x||=1} ||Ax||$

(A) Both (a) and (b) are valid.

(B) (a) is valid and (b) is invalid.

(C) (a) is invalid and (b) is valid.

(D) Both (a) and (b) are invalid.

Problem 5. For all square, invertible matrices A :

(a) A can be **LU**-decomposed without pivoting.
(b) If A is diagonally dominant then it can be **LU**-decomposed without pivoting.

(A) Both (a) and (b) are valid.

(B) (a) is valid and (b) is invalid.

(C) (a) is invalid and (b) is valid.

(D) Both (a) and (b) are invalid.

Problem 6. What is the sum of the number of operations including multiplications, divisions, additions and subtractions needed to **LU**-decompose a general $n \times n$ matrix?

(A) $n^3/3 + n^2 - n/3$

(B) $\frac{n(n-1)(2n+5)}{6}$

(C) $\frac{2}{3}n + n^2/2 - \frac{7}{6}n$

(D) none of the above

PART II

Write your final answer in *black or blue pen* in the space provided below.

For a perfect score you must use correct methods, present all details of your work using meaningful notation, and obtain correct answers.

Problem 7.

Consider the fixed point iteration $x^{(k+1)} = f(x^{(k)})$, where

$$f(x) = \frac{x^2 + x}{2}.$$

(a) (6 points) Find all fixed points of this fixed point iteration and list them in the box below. For each fixed point determine whether it is attracting or repelling, and if attracting, determine if the convergence is linear or quadratic.

(b) (2 points) In the space below, draw the standard graphical interpretation of the fixed point iterations for all fixed points found in part (a). Make sure that your graphs are qualitatively accurate.

Problem 8.

(a) (2 points) Using precise mathematical notation, write down the specific form that Newton's method takes when applied to the numerical determination of the real root of the equation $x^3+x-1=0$. Enter your answer in the box below, making sure that your answer is written in a form that is useful in part (b) of this Problem.

$$\boxed{x^{(k+1)} = f(x^{(k)})\ , \quad \text{where} \quad f(x) = \qquad\qquad\qquad\qquad}$$

(b) (2 points) Compute the first 3 iterations starting at $x^{(0)} = -0.7$.

(c) (4 points) In the space below, draw a careful graphical interpretation of this fixed point iteration, showing the line $y = x$, the curve $y = f(x)$, and indicating the first three iterations, starting with $x^{(0)} = -0.7$. Determine whether the iterations converge. If they do determine whether convergence is linear or quadratic.

Problem 9.

(a) (4 points) Showing all details derive the *local Trapezoidal Rule*

$$\int_{-h/2}^{h/2} f(x)dx \;\cong\; \frac{h}{2}\left[f(-h/2) + f(h/2)\right],$$

for the reference interval $[-h/2, h/2]$. You must make use of the Lagrange interpolation polynomial that interpolates $f(x)$ at the two points, $x_0 = -h/2$ and $x_1 = h/2$, and the corresponding Lagrange basis functions, $\ell_0(x)$ and $\ell_1(x)$.

(b) (4 points) Showing all details use Taylor expansions to determine the error formula for the quadrature rule from part (a) of this Problem. Enter your answer in the box.

Problem 10.

Consider the Gauss-Legendre quadrature rule with three Gaussian points and three weights for the integral

$$\int_{-1}^{1} f(x)dx \approx A_0 f(x_0) + A_1 f(x_1) + A_2 f(x_2).$$

(a) (2 points) Determine the Gaussian points using the Legendre polynomial $e_3(x) = x^3 - \frac{3}{5}x$.

(b) (5 points) Determine the weights A_0, A_1, A_2 by means of the Lagrange interpolation.

(c) (1 point) Determine the order of accuracy of a general 3-point composite Gauss quadrature rule. Enter your answer in the box.

Problem 11. (8 points) Find the best function in the least-squares sense of the form $f(x) = a\sin(\pi x) + b\cos(\pi x)$ that fits the data $\{(-1,-1),(-1,0),(0,1),(0.5,2),(1,1)\}$. Provide all details of the computation.

Problem 12. (a) (4 points) Do there exist constants a, b, c, d such that the function

$$S(x) \quad = \quad \begin{cases} ax^3 + x^2 + cx & \text{when} \quad -1 \leq x \leq 0 \\ bx^3 + x^2 + dx & \text{when} \quad 0 \leq x \leq 1 \end{cases}$$

is a natural cubic spline that agrees with the function $|x|$ at the knots $-1, 0, 1$? State your answer and justify it.

(b (4 points) Determine the natural cubic spline that interpolates the function $f(x) = x^6$ over the interval $[0, 2]$ using knots $0, 1, 2$.