ENGR 233 - Tut UB - Solutions to 1st Midterm Exam, February 2009

Problem 1(a). *Find a normal vector $\vec{N}$ to the plane $x+y+z=0$, and give parametric equations for the normal line through the point $(0,0,0)$.*

Solution: From Chapter 7, we know that a normal vector to the plane given by an equation of the form $ax+by+cz=d$ is given by (a,b,c), so in this case we would get

$$\vec{N} = \hat{\imath} + \hat{\jmath} + \hat{k}.$$

Alternatively, we can view this plane as a level surface of the function $f(x,y,z) = x+y+z$ and then the gradient gives us a normal vector, which in this case is the same one:

$$\vec{\nabla} f = \frac{\partial f}{\partial x}\hat{\imath} + \frac{\partial f}{\partial y}\hat{\jmath} + \frac{\partial f}{\partial z}\hat{k} = \hat{\imath} + \hat{\jmath} + \hat{k}.$$

Now we use this vector as the direction vector $\vec{a}$ for a line through the initial point $(0,0,0)$, which gives the vector equation of the line as

$$\vec{r} = \vec{r}_0 + \vec{a}t = (0,0,0) + t(1,1,1) = t\hat{\imath} + t\hat{\jmath} + t\hat{k}.$$

The parametric equations are then $x=t$, $y=t$ and $z=t$.

(b) *Find the component $comp_{\vec{N}}\vec{F}$ of the vector $\vec{F} = 3\hat{\imath} + 2\hat{\jmath} + \hat{k}$ in the direction given by the vector $\vec{N}$ from part (a). The sign of your answer will depend on the direction of the vector $\vec{N}$ you have chosen.*

Solution: In order to compute the component in the direction of $\vec{N}$ we need to first have a unit vector in that direction,

$$\hat{N} = \frac{\vec{N}}{\|\vec{N}\|} = \frac{\hat{\imath} + \hat{\jmath} + \hat{k}}{\sqrt{3}}.$$

Note that this will be the same, up to sign, no matter which vector we chose for $\vec{N}$ in part (a). Now we use the formula

$$\mathrm{comp}_{\vec{N}}\vec{F} = \vec{F} \cdot \hat{N} = (3\hat{\imath} + 2\hat{\jmath} + \hat{k}) \cdot \frac{\hat{\imath} + \hat{\jmath} + \hat{k}}{\sqrt{3}} = \frac{6}{\sqrt{3}} = 2\sqrt{3}.$$

(c) *Find the volume of the parallelepiped formed by the vectors $\langle 0,1,-1\rangle$, $\langle 1,-1,0\rangle$ and $\langle 3,2,1\rangle$.*

Solution: The volume of the parallelepiped is given by the absolute value of the triple product of the three vectors, $\vec{a} \cdot (\vec{b} \times \vec{c})$, computed as the following determinant

$$\vec{a} \cdot (\vec{b} \times \vec{c}) = \begin{vmatrix} 0 & 1 & -1 \\ 1 & -1 & 0 \\ 3 & 2 & 1 \end{vmatrix} = 0 - 1(1) - 1(2+3) = -6.$$

Here we have expanded along the first row. Therefore $V = |\vec{a} \cdot (\vec{b} \times \vec{c})| = 6$.

Problem 2. *The following questions refer to the curve in the xy-plane described by the vector function* $\vec{r}(t) = t^2\hat{\imath} + t\hat{\jmath}$.

(a) *Sketch the curve (hint: express x in terms of y).*

Solution: Since $x = t^2$ and $y = t$, we have $x = y^2$ so we draw a parabola along the positive x axis going through the points $(0,0), (1,1)$ and $(1,-1)$.

(b) *Find a tangent vector to the curve at the point* $(1,1)$.

Solution: We know $\vec{r}\,'(t) = 2t\ \hat{\imath} + \hat{\jmath}$ is tangent at each point, so at $(1,1)$ we have $t = y = 1$ and this becomes

$$\vec{r}\,'(1) = 2\ \hat{\imath} + \hat{\jmath}.$$

(c) *Find the curvature κ of the curve at the points $(0,0)$ and at $(1,1)$. Which is larger? What does this tell you about the difference in the shape of the curve at these points?*

Solution: To find the curvature, in addition to the velocity

$$\vec{r}\,'(t) = 2t\ \hat{\imath} + \hat{\jmath}$$

we need to compute the acceleration

$$\vec{r}\,''(t) = 2\ \hat{\imath}$$

and the speed

$$v(t) = \|\vec{r}\,'(t)\| = \sqrt{(2t)^2 + 1} = \sqrt{4t^2 + 1}.$$

Using the formula,

$$\kappa = \frac{\|\vec{r}\,'(t) \times \vec{r}\,''(t)\|}{\|\vec{r}\,'(t)\|} = \frac{\|(0-2)\hat{k}\|}{\sqrt{4t^2+1}} = \frac{2}{\sqrt{4t^2+1}}.$$

At the point $(0,0)$, $t = y = 0$ so the curvature is

$$\kappa = \frac{2}{\sqrt{4 \cdot 0^2 + 1}} = 2.$$

At the point $(1,1)$, $t = y = 1$ so the curvature is

$$\kappa = \frac{2}{\sqrt{4 \cdot 1^2 + 1}} = \frac{2}{\sqrt{5}} < 2.$$

The fact that the curvature at $(1,1)$ is smaller than at $(0,0)$ means the curve is more "flat" there, more like a line and less "sharp" or turning.

(d) *Compute the gradient $\vec{\nabla} f$ of the function $f(x,y) = x - y^2$ and show it is perpendicular to this curve at the point $(1,1)$. Explain why this will be true at all other points along the curve.*

Solution: The gradient of f is

$$\vec{\nabla} f(x,y) = \frac{\partial f}{\partial x}\hat{\imath} + \frac{\partial f}{\partial y}\hat{\jmath} = \hat{\imath} - 2y\hat{\jmath}$$

so at the point $(1,1)$ we have

$$\vec{\nabla} f(1,1) = \hat{\imath} - 2\hat{\jmath}.$$

Recall from part (b) that at the same point a tangent vector to the curve is given by

$$\vec{r}\,'(1) = 2\,\hat{\imath} + \hat{\jmath}.$$

Taking the dot product we see that

$$\vec{\nabla} f(1,1) \cdot \vec{r}\,'(1) = (1)(2) + (-2)(1) = 0$$

which means that the two vectors are perpendicular, i.e. $\vec{\nabla} f$ is perpendicular to the curve at this point. Similarly, we can show that for any value of t, at the point $\vec{r}(t) = (t^2, t)$,

$$\vec{\nabla} f(t^2, t) \cdot \vec{r}\,'(t) = (\hat{\imath} - 2t\hat{\jmath}) \cdot (2t\,\hat{\imath} + \hat{\jmath}) = 0$$

so the gradient is perpendicular to f at all points. This is true because the curve is a level curve of f, since as we saw in part (a) this curve is just the parabola $x = y^2$ and $f(x,y) = x - y^2 = 0$ there. We know that the gradient of a function is always perpendicular to the level curves.

Problem 3. *The following questions refer to the function*

$$w = f(x,y,z) = \cos(x + y + z).$$

(a) *Describe the level surfaces corresponding to $f(x,y,z) = C$ for different values of the constant C $(-1 \le C \le 1)$.*

Solution: The level curves satisfy the equation $f(x, y) = \cos(x+y+z) = C$, a constant. Taking the inverse cosine on both sides (we can do that since $-1 \leq C \leq 1$) gives

$$x + y + z = \cos^{-1} C.$$

Let d be a value of $\cos^{-1} C$ (in fact there are infinitely many possible values for each C). Then

$$x + y + z = d$$

is the equation of a plane perpendicular to the vector $(1, 1, 1)$ (as in problem 1). For example, for the value $C = 1$ we can take $d = \cos^{-1} 1 = 0$ so we get the plane $x+y+z = 0$ which goes through the origin. Alternatively we could take $\cos^{-1} 1 = 2\pi$ and get the plane $x+y+z = 2\pi$. Thus for $C = 1$ we get infinitely many parallel planes. Similarly, for $C = 0$ we get infinitely many planes corresponding to the different values $d = \cos^{-1}(0)$, such as $x + y + z = \pi/2$ or $x + y + z = 3\pi/2$.

(b) *Give the directional derivative of f at the point $(\pi/2, 0, 0)$ in the direction of the vector $3\hat{\imath} + 4\hat{\jmath}$.*

Solution: To find the directional derivative we first compute the gradient

$$\vec{\nabla} f(x, y, z) = \frac{\partial f}{\partial x}\hat{\imath} + \frac{\partial f}{\partial y}\hat{\jmath} + \frac{\partial f}{\partial z}\hat{k} = -\sin(x + y + z)\hat{\imath} - \sin(x + y + z)\hat{\jmath} - \sin(x + y + z)\hat{k}.$$

At the point $(\pi/2, 0, 0)$ we have $\sin(\pi/2 + 0 + 0) = 1$ so

$$\vec{\nabla} f(\pi/2, 0, 0) = -\hat{\imath} - \hat{\jmath} - \hat{k}.$$

Taking a unit vector $\hat{u}$ in the direction of $3\hat{\imath} + 4\hat{\jmath}$:

$$\hat{u} = \frac{3\hat{\imath} + 4\hat{\jmath}}{\sqrt{9 + 16}} = \frac{3}{5}\hat{\imath} + \frac{4}{5}\hat{\jmath}.$$

This gives

$$D_{\hat{u}} f(\pi/2, 0, 0) = \vec{\nabla} f(\pi/2, 0, 0) \cdot \hat{u} = -\frac{3}{5} - \frac{4}{5} = -\frac{7}{5}.$$

(c) *If $x = u^2$, $y = 2uv$, $z = v^2$, find $\frac{\partial w}{\partial u}$ when $u = v = \sqrt{\frac{\pi}{2}}$.*

Solution: By the chain rule

$$\frac{\partial w}{\partial u} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial u} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial u} = -\sin(x + y + z)(2u) - \sin(x + y + z)(2v).$$

When $u = v = \sqrt{\frac{\pi}{2}}$ we have $x = u^2 = \pi/2$, $y = 2uv = \pi$ and $z = v^2 = \pi/2$ so

$$\frac{\partial w}{\partial u} = -\sin(2\pi)(4\sqrt{\frac{\pi}{2}}) = 0.$$