[03] 1. Construct a Truth table for the following proposition. State whether it is a Tautology, Contradiction or a Contingency.

$[(\neg p \lor q) \land (p \lor \neg r)] \rightarrow (r \rightarrow q)$

TAUTOLOGY

[03] 2. Determine using logic equivalences without a truth table whether the following proposition is a Tautology, Contradiction or a Contingency

$(\neg p \lor \neg q) \land (\neg q \rightarrow \neg p) \lor p$

$\equiv (\neg p \lor \neg q) \land (\neg\neg q \lor \neg p) \lor p$	IMPLICATION with OR
$\equiv (\neg p \lor \neg q) \land (q \lor \neg p) \lor p$	double NEGATION
$\equiv \neg p \lor (\neg q \land q) \lor p$	DISTRIBUTIVE
$\equiv \neg p \lor F \lor p$	$\neg x \land x \equiv F$
$\equiv \neg p \lor p$	$\neg x \lor F \equiv \neg x$
$\equiv T$	$\neg x \lor x \equiv T$

$\Longrightarrow$ TAUTOLOGY

3. Is the follwing argument valid or invalid ? If valid prove your conclusion. If invalid list a set of values for P, Q, M, R that shows a counter example.

[06]

Hypothesis: $P \vee Q$

$M \to \neg Q$

$M \vee R$

Conclusion: $P \vee R$

TWO METHODS LISTED:

METHOD 1		
①	$M \to \neg Q$	GIVEN
② $\Rightarrow$	$\neg M \vee \neg Q$	$\to$ In terms of OR
③ Also	$M \vee R$	GIVEN
④ $\Rightarrow$	$R \vee \neg Q$	RESOLUTION Line 2-3
⑤	$P \vee Q$	GIVEN
⑥ $\Rightarrow$	$P \vee R$	RESOLUTION line 4-5

METHOD 2 PROOF (by CASES)

CASE 1 Assume $P = F$ (You CAN'T Assume $P=T$)

$P \vee Q$ given $T \Rightarrow Q = T$

$M \to \neg Q$ given $\Rightarrow \neg\neg Q \to \neg M \Rightarrow Q \to \neg M \Rightarrow \neg M = T$
$\Rightarrow M = F$

$M \vee R$ given AND $M = F \Rightarrow R$ is T

CASE 2 Assume. $R = F$

$M \vee R$ given $\Rightarrow M = T$

$M \to \neg Q$ given $\Rightarrow \neg Q = T \Rightarrow Q = F$

$P \vee Q$ given $\Rightarrow$ $P = T$

In two CASES we have At least one of P or $R = T$

$\Rightarrow P \vee R = T \Rightarrow$ Argument is Valid

4. For each Predicate below state whether it is true or false. If false show a counter example, if true explain why.

[08]

a) $\forall x \forall y \exists z P(x, y, z)$ where $P(x, y, z)$ represents $z < xy, x, y, z \epsilon Z^+$

If $x=1, y=1 \Rightarrow xy = 1 \Rightarrow \neg\exists z \in Z^+, z < 1$

$\Rightarrow \forall x \forall y \exists z\, P(x,y,z)$ is False

b) $\forall x \exists y \exists z Q(x, y, z)$ where $Q(x, y, z)$ represents $z = xy, x, y, z \epsilon Z^+$

Suppose $x = a,\ a \in Z^+$

Choose $y = b,\ b \in Z^+ \Rightarrow z = ab$ Note $a \in Z^+$, $b \in Z^+$ $\Rightarrow ab \in Z^+$

$\Rightarrow \forall x \exists y \exists z Q(x,y,z)$ is TRUE

c) $\forall z \exists x \exists y R(x, y, z)$ where $R(x, y, z)$ represents $x + y^2 < z^3, x, y, z \in Z^+$

Suppose $z = 1 \Rightarrow z^3 = 1$, then $\neg\exists x \exists y \in Z^+,\ x + y^2 < 1$

$\Rightarrow \forall x \exists x \exists y R(x,y,z)$ is False

d) $\forall x \forall z \exists y R(x, y, z)$ where $R(x, y, z)$ represents $x + y^2 < z^3, x, y, z \in Z^+$

Note: $y^2 < z^3 - x$

Choose $x = 10 \in Z^+$, $z = 2$

$\Rightarrow z^3 - x < 0$ AND $y^2 > 0$

$\neg\exists y \Rightarrow \forall x \forall z \exists y R(x,y,z)$ is False

[08] **5.** Suppose the variable x represents people, $H(x)$ represents x is a good hockey player, $S(x)$ represents x is storng, $Q(x)$ represents x is quick. Note: "neither A nor B" $\equiv \neg A \wedge \neg B$. Below the words of each statement state an equivalent quantified statement from the following list and state its number in list. OR WRITE None of these.

1. $\exists x\ \neg H(x) \wedge \neg[S(x) \vee Q(x)]$
2. $\forall x\ H(x) \wedge \neg[(S(x) \vee Q(x)]$
3. $\neg\exists x\ [H(x) \wedge S(x)] \vee [H(x) \wedge Q(x)]$
4. $\forall x\ \neg H(x) \vee S(x) \vee Q(x)$
5. $\exists x\ \ H(x) \wedge \neg S(x) \wedge \neg Q(x)$
6. *None of These*

a) All good hockey players are strong or quick.

$\forall x\ \neg H(x) \vee S(x) \vee Q(x)$, Number: (4)

b) All good hockey players are neither strong nor quick.

$\neg\exists x\ [H(x) \wedge S(x)] \vee [H(x) \wedge Q(x)]$ Number: (3)

c) Some person who is neither strong nor quick is not a good hockey player.

$\exists x\ \neg H(x) \wedge \neg[S(x) \vee Q(x)]$, Number: (1)

d) No one who is not a good hockey player is strong or quick.

None of these, Number: (6) $\rightarrow$ (this actually: $\neg\exists x\ \neg H(x) \wedge (S(x) \vee Q(x))$
$\equiv \forall x\ H(x) \vee \neg[S(x) \vee Q(x)]$
NOT on list)

e) The negation of: All good hockey players are strong or quick.

$\exists x\ H(x) \wedge \neg S(x) \wedge \neg Q(x)$, Number: (5)

[06] **6.** Theorem: $\forall n \epsilon Z$, If $n^3 + 7n + 3$ is an odd integer then n is an even integer.
Prove using the Contraposition method. (No marks if Contraposition method is not used)

Contrapositive form of Th$^{\underline{m}}$

If n is an odd integer then n^3+7n+3 is an even integer

PROOF:

$n = 2k+1,\ k \in Z$ — DEF, odd

$$n^3+7n+3 = (2k+1)^3+7(2k+1)+3$$
$$= (2k+1)(4k^2+4k+1)+7(2k+1)+3$$
$$= 8k^3+8k^2+2k+4k^2+4k+1+14k+10$$
$$= 8k^3+12k^2+20k+11$$
$$= 2(4k^3+6k^2+10k+5)+1$$

Alg. to get Form of n^3+7n+3

Since $(4k^3+6k^2+10k+5) \in Z$

$\Rightarrow n^3+7n+3$ is an odd integer

Closure of Add in Z
DEF of odd integer

Show statement is False, Could also show by PROOF (Counter Ex)

let $n=1$

$\Rightarrow n^3+7n+3 = 1^3+7(1)+3 = 11$

n odd $\wedge$ (n^3+7n+3) odd

$\Rightarrow$ STATEMENT is False

[06] **7.** Theorem: $\forall x \in R,\ x^4 + \frac{49}{x^4} \geq 14$, $x \neq 0$

Start with Backward reasoning then prove using Direct method. (No marks if these methods are not used)

Backward Reasoning:

If we want to End up with $x^4 + \frac{49}{x^4} \geq 14$

$$\Rightarrow \quad x^4\left(x^4 + \frac{49}{x^4}\right) \geq 14x^4$$

$$x^8 + 49 \geq 14x^4$$

$$x^8 - 14x^4 + 49 \geq 0$$

$$(x^4 - 7)(x^4 - 7) \geq 0$$

$$(x^4 - 7)^2 \geq 0$$

Proof (Direct) $\forall x \in R,\ (x^4 - 7)^2 \geq 0$ | the square of all $x \in R \geq 0$

$$x^8 - 14x^4 + 49 \geq 0$$

$$x^8 + 49 \geq 14x^4$$

$$\frac{x^8}{x^4} + \frac{49}{x^4} \geq \frac{14x^4}{x^4}$$

$$x^4 + \frac{49}{x^4} \geq 14$$

Algebra to get desired expression

QED

[10] **8.** Theorem: If (xy) is an Irrational number then either x is Irrational or y is Irrational

a) Prove the Theorem using the Contradiction method. (No marks if Contradiction method is not used)

PROOF: Either x is Irrational or y is Irrational is

OR $\neg$[x is Irr or y is Irr]

Assume $\neg$[x is Irr or y is Irr]

$\Rightarrow$ Assume $\neg$ x is Irr AND $\neg y$ is Irr

$\Rightarrow$ Assume x is Rational AND y is Rational

$\Rightarrow x = \frac{a}{b}$, $a, b \in \mathbb{Z}$, $b \neq 0$

$y = \frac{c}{d}$, $c, d \in \mathbb{Z}$, $d = 0$

then $xy = \frac{a}{b} * \frac{c}{d}$

$xy = \frac{ac}{bd}$

$ac \in \mathbb{Z}$, $bd \in \mathbb{Z}$

Also $b \neq 0, c \neq 0 \Rightarrow bd \neq 0$

$\Rightarrow xy$ is RATIONAL

Contradiction to given

$\Rightarrow$ x is Irrational OR y is Irrational

QED.

b) State the Converse of the above Theorem, state whether this Converse is true or false and prove your conjecture using any valid method.

Converse If x is Irrational or y is Irrational then xy is Irrational.

Converse is False

PROOF

Counter Example

$\left.\begin{array}{l}\sqrt{2} \text{ is Irrational} \\ \sqrt{8} \text{ is Irrational}\end{array}\right\} \Rightarrow \sqrt{2}$ or $\sqrt{8}$ is Irr.

But $\sqrt{2} * \sqrt{8} = \sqrt{16} = 4$ is Not Irrational

[03] **Bonus.** Consider the decision table whose input specifications are Boolean variables x, y, z
(Write all answers using Boolean Algebra notation for $\neg$, $\wedge$, $\vee$, True, False)

x	y	z	
1	1	1	xyz
0	1	1	$\bar{x}yz$
1	0	1	$x\bar{y}z$

a) In the blanks above write the Conjunction of each row in terms of x, y, z.
b) Write the Disjunction of the three conjunctions in a)

$xyz + \bar{x}yz + x\bar{y}z$

c) Show the complete simplification of the Disjunction in b)

$xyz + \bar{x}yz + x\bar{y}z$	GIVEN
$yz(x+\bar{x}) + x\bar{y}z$	Distributive
$yz(1) + x\bar{y}z$	$x+\bar{x} \equiv x \vee \neg x \equiv 1 \equiv 1$
$yz + x\bar{y}z$	$(yz)1 \equiv yz \wedge T \equiv yz$
$z(y+x\bar{y})$	