Quiz #3 - Solutions

1. Find the tangential and normal components of the acceleration vector of a particle with position function $\mathbf{r}(t) = t\mathbf{i} + 2t\mathbf{j} + t^2\mathbf{k}$.

The tangential and normal component for acceleration are

$$a_T = \frac{\vec{r}\,'(t) \cdot \vec{r}\,''(t)}{||\vec{r}\,'(t)||} \qquad \text{and} \qquad a_N = \frac{||\vec{r}\,'(t) \times \vec{r}\,''(t)||}{||\vec{r}\,'(t)||}$$

We compute

$$\begin{aligned}
&\vec{r}(t) = (t, 2t, t^2) \quad \Rightarrow \quad \vec{r}\,'(t) = (1, 2, 2t) \quad \Rightarrow \quad \vec{r}\,''(t) = (0, 0, 2),\\
&||\vec{r}\,'(t)|| = \sqrt{1 + 4 + 4t^2} = \sqrt{5 + 4t^2},\\
&\vec{r}\,' \cdot \vec{r}\,'' = (1, 2, 2t) \cdot (0, 0, 2) = 4t.\\
&\vec{r}\,' \times \vec{r}\,'' = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 2 & 2t \\ 0 & 0 & 2 \end{vmatrix} = 4\vec{i} - 2\vec{j} = (4, -2, 0)\\
&||\vec{r}\,' \times \vec{r}\,''|| = \sqrt{16 + 4} = 2\sqrt{5}
\end{aligned}$$

This gives

$$a_T = \frac{4t}{\sqrt{5 + 4t^2}} \qquad \text{and} \qquad a_N = \frac{2\sqrt{5}}{\sqrt{5 + 4t^2}}.$$

2. Show that if a particle moves with constant speed, then the velocity and acceleration vectors are orthogonal.

Let $\vec{r}(t)$ be the position vector. If the speed is constant, then $||\vec{r}\,'(t)|| = c$.

In particular, $\vec{r}\,' \cdot \vec{r}\,' = ||\vec{r}\,'||^2 = c^2 = k$ for some scalar $k \in \mathbb{R}$. Differentiating both sides of this equations yields

$$\begin{aligned}
\tfrac{d}{dt}\left(\vec{r}\,' \cdot \vec{r}\,' = k\right)&\\
\vec{r}\,'' \cdot \vec{r}\,' + \vec{r}\,' \cdot \vec{r}\,'' =& 0\\
2\vec{r}\,'' \cdot \vec{r} =& 0\\
\vec{r}\,'' \cdot \vec{r}\,' =& 0.
\end{aligned}$$

Since their dot product is zero, $\vec{r}\,'' = \vec{a}(t)$ and $\vec{r}\,' = \vec{v}(t)$ are perpendicular.

3. Find the work done by the force field $\mathbf{F}(x, y, z) = z\mathbf{i} + x\mathbf{j} + y\mathbf{k}$ in moving a particle from the point $(3, 0, 0)$ to the point $(0, \frac{\pi}{2}, 3)$ along a straight line.

The work done is given by the integral $\int_{\mathcal{C}} \vec{F} \cdot d\vec{r}$, where $\mathcal{C}$ is the line from $(3, 0, 0)$ to $(0, \frac{\pi}{2}, 3)$.

First, we choose a parametrization for the curve $\mathcal{C}$.

$$\begin{aligned}
\vec{r}(t) =& (1 - t)(3, 0, 0) + t(0, \tfrac{\pi}{2}, 3), \qquad \text{for} \qquad 0 \le t \le 1,\\
=& (3(1 - t), \tfrac{\pi}{2}t, 3t),\\
\vec{r}\,'(t) =& (-3, \tfrac{\pi}{2}, 3).
\end{aligned}$$

We compute the integral

$$\begin{aligned}\int_{\mathcal{C}} \vec{F} \cdot d\vec{r} &= \int_0^1 \vec{f}(\vec{r}(t)) \cdot \vec{r}\,'(t)\ dt \\ &= \int_0^1 (3t, 3(1-t), \tfrac{\pi}{2}t) \cdot (-3, \tfrac{\pi}{2}, 3)\ dt \\ &= \int_0^1 (-9t + \tfrac{3\pi}{2})\ dt = \tfrac{3}{2}(\pi - 3).\end{aligned}$$