Solutions to Midterm Exam Emat 233

February, 2006

Problem 1. *Consider the surface given by the formula*

$$1 = \cos(zx)e^{z+y}$$

(a) *Find the equation of the tangent plane at the point* $(\pi, 0, 0)$

Solution: Let $f(x,y,z) = \cos(zx)e^{z+y}$. Then the surface given by $1 = \cos(zx)e^{z+y}$ is the level surface $\{(x,y,z) : f(x,y,z) = 1\}$. To find the tangent plane at a point we first need to find a normal vector to the surface at that point, which is given by the gradient

$$\vec{\nabla} f = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} + \frac{\partial f}{\partial z}\mathbf{k}.$$

Computing the partial derivatives (using the chain rule, the product rule and the fact that $e^{z+y} = e^z e^y$), we get:

$$\begin{aligned}
\frac{\partial f}{\partial x} &= -z\sin(zx)e^{z+y} \\
\frac{\partial f}{\partial y} &= \cos(zx)e^{z+y} \\
\frac{\partial f}{\partial z} &= -x\sin(zx)e^{z+y} + \cos(zx)e^{z+y}.
\end{aligned}$$

Plugging in $x = \pi, y = 0, z = 0$ gives:

$$\vec{\nabla} f(\pi,0,0) = -0\sin(0)e^0\mathbf{i} + \cos(0)e^0\mathbf{j} + (-\pi\sin(0)e^0 + \cos(0)e^0)\mathbf{k} = 0\mathbf{i} + 1\mathbf{j} + (-0+1)\mathbf{k} = \mathbf{j} + \mathbf{k}.$$

This is a normal vector to the plane. Therefore any vector in the plane, of the form $(x-\pi, y-0, z-0)$ must be perpendicular to this vector, i.e.

$$0 = (\mathbf{j} + \mathbf{k}) \cdot ((x-\pi)\mathbf{i} + y\mathbf{j} + z\mathbf{k}) = 0(x-\pi) + y + z.$$

This gives the equation of the plane:

$$y + z = 0.$$

(b) *Find normal line in symmetric or in parametric form passing through the point* $(\pi, 0, 0)$.

Solution: Since we already know a normal vector at that point, namely $\vec{\nabla} f(\pi,0,0) = \mathbf{j} + \mathbf{k} = (0,1,1)$, all we need to do is give the equation of the line passing through $(\pi, 0, 0)$ in the direction of this vector. The parametric equation is:

$$\vec{r}(t) = (\pi, 0, 0) + t(0, 1, 1),$$

or in terms of coordinates: $x = \pi, y = t, z = t$.

To get the symmetric equations we eliminate the t so that we have $x = \pi, y = z$.

Problem 2 *Let* $\mathbf{r}(t) = 3\cos(t)\mathbf{i} + 2\sin(t)\mathbf{j} + t\mathbf{k}$ *be the position vector of a moving particle.*

(a) *Find the velocity vector* $\mathbf{v}(t)$ *and the acceleration vector* $\mathbf{a}(t)$ *at any* t.

Solution: The velocity is the derivative of the position vector:

$$\mathbf{v}(t) = \mathbf{r}'(t) = -3\sin(t)\mathbf{i} + 2\cos(t)\mathbf{j} + 1\mathbf{k},$$

and the acceleration is the second derivative:

$$\mathbf{a}(t) = \mathbf{r}''(t) = -3\cos(t)\mathbf{i} - 2\sin(t)\mathbf{j}.$$

(b) *Find the tangential component a_T of the acceleration vector $\mathbf{a}(t)$ at any t.*
Solution: From the formula sheet, we know that

$$a_T = \frac{\mathbf{r}'(t) \cdot \mathbf{r}''(t)}{\|\mathbf{r}'(t)\|} = \frac{\mathbf{v}(t) \cdot \mathbf{a}(t)}{\|\mathbf{v}(t)\|}.$$

Computing the dot product, we get:

$$\mathbf{v}(t) \cdot \mathbf{a}(t) = (-3\sin(t))(-3\cos(t)) + (2\cos(t))(-2\sin(t)) + 0 = 5\sin(t)\cos(t).$$

For the speed we have

$$\|\mathbf{v}(t)\| = \sqrt{(-3\sin(t))^2 + (2\cos(t))^2 + 1} = \sqrt{9\sin^2(t) + 4\cos^2(t) + 1} = \sqrt{5\sin^2(t) + 4 + 1} = \sqrt{5}\sqrt{\sin^2(t) + 1}.$$

Dividing, we get

$$a_T = \frac{5\sin(t)\cos(t)}{\sqrt{5}\sqrt{\sin^2(t)+1}} = \frac{\sqrt{5}\sin(t)\cos(t)}{\sqrt{\sin^2(t)+1}}.$$

(c) *At what point(s) does the particle pass through the xy-plane?*
Solution: The xy-plane is given by the equation $z = 0$. Therefore the curve intersects it when $z = t = 0$, which gives $x = 3\cos(0) = 3$ and $y = 2\sin(0) = 0$. The point of intersection is thus $(3, 0, 0)$.

Problem 3. *Answer the following two questions for the function*

$$f(x, y) = e^{x^2+y^2}.$$

(a) *Find the directional derivative of the function at the point $(1, 1)$ in the direction of the vector $-\mathbf{i} + \mathbf{j}$.*
Solution: The directional derivative of the function f in the direction of the unit vector $\mathbf{u}$ is given by

$$\mathrm{D}_{\mathbf{u}} f = \vec{\nabla} f \cdot \mathbf{u}.$$

Computing the gradient (again using the chain rule) we get:

$$\begin{aligned} \frac{\partial f}{\partial x} &= 2xe^{x^2+y^2}, \\ \frac{\partial f}{\partial y} &= 2ye^{x^2+y^2}, \end{aligned}$$

so at the point $(1, 1)$, $\vec{\nabla} f(1, 1) = 2e^2\mathbf{i} + 2e^2\mathbf{j}$.

Now we find the unit vector $\mathbf{u}$ in the direction of $-\mathbf{i} + \mathbf{j}$:

$$\mathbf{u} = \frac{-\mathbf{i} + \mathbf{j}}{\sqrt{(-1)^2 + 1^2}} = -\frac{1}{\sqrt{2}}\mathbf{i} + \frac{1}{\sqrt{2}}\mathbf{j}.$$

Taking the dot product gives:

$$\mathrm{D}_{\mathbf{u}} f = (2e^2\mathbf{i} + 2e^2\mathbf{j}) \cdot (-\frac{1}{\sqrt{2}}\mathbf{i} + \frac{1}{\sqrt{2}}\mathbf{j}) = -\frac{2e^2}{\sqrt{2}} + \frac{2e^2}{\sqrt{2}} = 0.$$

Note that this means that **u** is perpendicular to the gradient at that point, hence is tangent to the level curve of f (which is a circle centered at the origin).

(b) *Find the direction along which the function f increases most rapidly at the point $(1,1)$, and find the maximum rate.*

Solution: We know that the function increases most rapidly in the direction of the gradient, i.e. when **u** is parallel to $\vec{\nabla} f$, and in that case the maximum rate of increase is

$$\mathrm{D}_{\mathbf{u}} f = \vec{\nabla} f \cdot \frac{\vec{\nabla} f}{\|\vec{\nabla} f\|} = \|\vec{\nabla} f\|.$$

At the point $(1,1)$, the direction of maximum increase is given by $\vec{\nabla} f(1,1) = 2e^2\mathbf{i} + 2e^2\mathbf{j}$ (or if we want a unit vector,

$$\mathbf{u} = \frac{2e^2\mathbf{i} + 2e^2\mathbf{j}}{\|2e^2\mathbf{i} + 2e^2\mathbf{j}\|} = \frac{2e^2\mathbf{i} + 2e^2\mathbf{j}}{\sqrt{4e^4 + 4e^4}} = \frac{1}{\sqrt{2}}\mathbf{i} + \frac{1}{\sqrt{2}}\mathbf{j},$$

which is at an angle of $\pi/4$ from the horizontal. The maximum rate of increase is

$$\|2e^2\mathbf{i} + 2e^2\mathbf{j}\| = \sqrt{4e^4 + 4e^4} = 2e^2\sqrt{2}.$$

Problem 4. (a) *Let $f(x,y,z) = x^2 - 2y + zx$; find* $\mathrm{grad}(f)$ *and* $\mathrm{curl}(\mathrm{grad}(f))$. *Can you find also* $\mathrm{curl}(f)$ *and* $\mathrm{div}(f)$*?* **Explain.**

Solution:

$$\mathrm{grad}(f) = \vec{\nabla} f = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} + \frac{\partial f}{\partial z}\mathbf{k} = (2x+z)\mathbf{i} + (-2)\mathbf{j} + x\mathbf{k}.$$

$$\mathrm{curl}(\mathrm{grad}(f)) = \vec{\nabla} \times \vec{\nabla} f = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ (2x+z) & -2 & x \end{vmatrix} = (\frac{\partial}{\partial y}x - \frac{\partial}{\partial z}(-2))\mathbf{i} - (\frac{\partial}{\partial x}x - \frac{\partial}{\partial z}(2x+z))\mathbf{j} + (\frac{\partial}{\partial y}x - \frac{\partial}{\partial z}(-2))\mathbf{k}$$

$$= (0-0)\mathbf{i} - (1-1)\mathbf{j} + (0-0)\mathbf{k} = \mathbf{0}$$

In fact, this will be true (i.e. $\mathrm{curl}(\mathrm{grad}(f) = \mathbf{0})$ for any function f with continuous second-order partial derivatives.

Since f is a scalar function, not a vector field, we cannot define $\mathrm{curl}(f)$ and $\mathrm{div}(f)$. The curl and divergence are only defined for vector fields.

(b) *Let $\mathbf{F}(x,y,z) = xz\mathbf{i} + y^2\mathbf{j} + \mathrm{e}^z\mathbf{k}$; find* $\mathrm{div}(\mathbf{F})$. *Can you find* $\mathrm{grad}(\mathbf{F})$ *?* **Explain.**

Solution:

$$\mathrm{div}(\mathbf{F}) = \vec{\nabla} \cdot \mathbf{F} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z} = \frac{\partial(xz)}{\partial x} + \frac{\partial(y^2)}{\partial y} + \frac{\partial(\mathrm{e}^z)}{\partial z} = z + 2y + \mathrm{e}^z.$$

Since **F** is a vector field, we cannot define $\mathrm{grad}(\mathbf{F})$. The gradient is only defined for a scalar function.