PROBLEMS FOR CHAPTER 6

1. Tabulate values of $y = e^x$ for $0 \le x \le 2$ at intervals of $h = 0.1$ and obtain.

   (i) y' and y'' at $x = 0.0$

   (ii) y' and y'' at $x = 0.1$

   (iii) y' and y'' at $x = 0.9$

   (iv) y' and y'' at $x = 2.0$

   using appropriate formulas.

Solution: The Tabulated values of $y = e^x$.

x	0.000	0.100	0.200	0.300	0.400	0.500	0.600	0.700	0.800	0.900	1.000
y	1.000	1.105	1.221	1.350	1.492	1.649	1.822	2.014	2.226	2.460	2.718
x	1.100	1.200	1.300	1.400	1.500	1.600	1.700	1.800	1.900	2.000	
y	3.004	3.320	3.669	4.055	4.482	4.953	5.474	6.050	6.686	7.389	

The interval size is h=0.1 and the numerical difference equations are used to get the approximations for the derivatives.

(i) y' at $x_i = 0.0$

Using the forward difference equation for $f'(x_i)$;

$$f'(x_i) \approx \frac{f(x_{i+1}) - f(x_i)}{h}$$

where;

$$x_{i+1} = 0.1;\ f(x_{i+1}) = 1.105$$

$$x_i = 0;\ f(x_i) = 1.0$$

$$y' \approx 1.0517$$

Using the forward difference equation for y′′

$$f''(x_i) \approx \frac{f(x_{i+2}) - 2f(x_{i+1}) + f(x_i)}{h^2}$$

where for $x_i = 0.0$; $f(x_i) = 1.0$

$x_{i+1} = 0.1$; $f(x_{i+1}) = 1.105$

$x_{i+2} = 0.2$; $f(x_{i+2}) = 1.221$

$$f''(0) \approx \frac{1.221 - 2(1.105) + 1}{0.01} = 1.106$$

(ii) y′ at $x_i = 0.1$

Using the central difference equation for f′ (x_i);

$$f'(x_i) \approx \frac{f(x_{i+1}) - f(x_{i-1})}{2h}$$

where;

$x_{i+1} = 0.2$; $f(x_{i+1}) = 1.221$

$x_{i-1} = 0.0$; $f(x_{i+1}) = 1.0$

$$y' \approx 1.107$$

Using the central difference equation for y′′

$$f''(x_i) = \frac{f(x_{i+1}) - 2f(x_i) + f(x_{i-1})}{h^2}$$

where for $x_i = 0.1$; $f(x_i) = 1.105$

$x_{i-1} = 0.0$; $f(x_{i+1}) = 1.0$

$x_{i+1} = 0.2$; $f(x_{i+2}) = 1.221$

$$f''(0.1) \approx \frac{1.221 - 2(1.105) + 1}{0.01} = 1.106$$

(iii) y' at $x_i = 0.9$

Using the central difference equation for $f'(x_i)$;

$$f'(x_i) \approx \frac{f(x_{i+1}) - f(x_{i-1})}{2h}$$

where;

$$x_{i+1} = 1.0; \ f(x_{i+1}) = 2.718$$

$$x_{i-1} = 0.8 \ ; \ f(x_{i+1}) = 2.226$$

$$y' \approx 2.4637$$

Using the central difference equation for y''

$$f''(x_i) = \frac{f(x_{i+1}) - 2f(x_i) + f(x_{i-1})}{h^2}$$

where for $x_i = 0.9$; $f(x_i) = 2.46$

$x_{i-1} = 0.8$; $f(x_{i+1}) = 2.226$

$x_{i+1} = 1.0$; $f(x_{i+2}) = 2.718$

$$f''(0.1) \approx \frac{2.718 - 2(2.46) + 2.226}{0.01} = 2.4616$$

(iv) y' at $x_i = 2.0$

Using the backward difference equation for $f'(x_i)$;

$$f'(x_i) \approx \frac{f(x_i) - f(x_{i-1})}{h}$$

where;

$$x_i = 2.0; \ f(x_i) = 7.389$$

$$x_{i-1} = 1.9; \ f(x_{i-1}) = 6.686$$

$$y' \approx 7.031617$$

Using the forward difference equation for y′′

$$f''(x_i) \approx \frac{f(x_{i-2}) - 2f(x_{i-1}) + f(x_i)}{h^2}$$

where for $x_i = 2.0$; $f(x_i) = 7.389$

$x_{i-1} = 1.9$; $f(x_{i+1}) = 6.686$

$x_{i-2} = 1.8$; $f(x_{i+2}) = 6.050$

$$f''(0) \approx \frac{6.050 - 2(6.686) + 7.389}{0.01} = 6.6915$$

2. In a circuit with impressed voltage E(t) and inductance L, Kirchoff's first law gives the relationship.

$$E(t) = L\,\frac{di}{dt} + Ri$$

Where R is the resistance in the circuit and i is the current. Suppose we measure the current for several values of t and obtain:

t	1.00	1.01	1.02	1.03	1.04
i	3.10	3.12	3.14	3.18	3.24

where t is measured in seconds, i in amperes, the inductance L is a constant 0.98 henries, and the resistance R is 0.142 ohms. Approximate the voltage E at the values t = 1, 1.01, 1.02, 1.03, and 1.04, using the appropriate three – point formulas.

Solution: The interval size is h=0.01 and the numerical difference equations are used to get the approximations for the derivatives

1. $E_{(t)} = L\,\frac{di}{dt} + Ri$

   To find ($\frac{di}{dt}$) approximations for the above data set using the numerical difference three point formulae;

   h = 0.01

   for t = 1 and i = 3.10

   using the three point forward difference formula

$$f'(x_i) = \frac{-f(Xi+2) + 4\,f\,(Xi+1) - 3\,f(Xi)}{2h}$$

   where $x_i = 1$ $f(x_i) = 3.10$

   $x_{i+1} = 1.01$ $f(x_{i+1}) = 3.12$

   $x_{i+2} = 1.02$ $f(x_{i+2}) = 3.14$

   Therefore,

$$\frac{di}{dt} = \frac{-3.14 + 4\,(3.12) - 3\,(3.10)}{2*0.01}$$
$$= 2$$

Therefore,
E(1) = 0.98 (2) + 0.142 (3.12)
= 2.4002

For $t = 1.01$ and $i = 3.12$
Using the above three point forward difference formula
Using the three points after $t = 1.01$ from the data set

$x_i = 1.01$ $f(x_i) = 3.12$
$x_{i+1} = 1.02$ $f(x_{i+1}) = 3.14$
$x_{i+2} = 1.03$ $f(x_{i+2}) = 3.18$

$$\frac{di}{dt} = \frac{-3.18 + 4\,(3.14) - 3\,(3.12)}{2*0.01} = 1$$

Therefore,
E (1.01) = 0.98 (1) + 0.142 (3.12)
= 1.42304
For $t = 1.02$ and $i = 3.14$
Since there are no three points on either side of the data point we use four point central difference equation

$$f'(x_i) = \frac{-f(Xi+2) + 8\,f\,(Xi+1) - 8\,f\,(Xi-1) + f(Xi-2)}{12h}$$

where $x_{i-2} = 1$ $f(x_{i-2}) = 3.10$
$x_{i-1} = 1.01$ $f(x_{i-1}) = 3.12$
$x_{i+1} = 1.03$ $f(x_{i+1}) = 3.18$
$x_{i+2} = 1.04$ $f(x_{i+2}) = 3.24$
Therefore,

$$\frac{di}{dt} = \frac{-3.24 + 8\,(3.18) - 8\,(3.12) + 3.10}{12\,(0.01)}$$
$$= 2.833$$

E (1.02) = 0.98 (2.833) + 0.142 (3.14)
= 3.2225

For $t = 1.03$ and $i = 3.18$
We use the three point backward difference formula

$$f'(x_i) = \frac{3f(Xi) - 4\,f\,(Xi-1) + 3\,f(Xi-2)}{2h}$$

where $x_{i-2} = 1.01$ $f(x_{i-2}) = 3.12$
$x_{i-1} = 1.02$ $f(x_{i-1}) = 3.14$
$x_i = 1.03$ $f(x_i) = 3.18$

Therefore,

$$\frac{di}{dt} = \frac{3\,(3.18) - 4\,(3.14) + 3.12}{2*0.01} = 5$$

E (1.03) = 0.98 (5) + 0.142 (3.18)
= 5.35156

For $t = 1.04$ and $i = 3.24$

We use the three point backward difference equation

where $x_{i-2} = 1.02$ $\quad f(x_{i-2}) = 3.14$

$x_{i-1} = 1.03$ $\quad f(x_{i-1}) = 3.18$

$x_i = 1.04$ $\quad f(x_i) = 3.24$

Therefore,

$$\frac{di}{dt} = \frac{3\,(3.24) - 4\,(3.18) + 3.14}{2*0.01} = 7$$

E (1.04) = 0.98 (7) + 0.142 (3.24)
= 7.32

3. The following data have been experimentally obtained.

x	1.00	1.01	1.02	1.03	1.04	1.05
f(x)	1.27	1.32	1.38	1.41	1.47	1.52

(i) Approximate $f'(1)$, $f''(1)$, $f'''(1)$

(ii) Obtain a Taylor series expansion for the function using the above values, at x = 1.

Solution: (i) To obtain f'(1) we use forward difference equation

$$f'(x_i) = \frac{-f(Xi+2) + 4\,f\,(Xi+1) - 3\,f(Xi)}{2h}$$

where $x_i = 1$ $\quad f(x_i) = 1.27$

$x_{i+1} = 1.01$ $\quad f(x_{i+1}) = 1.32$

$x_{i+2} = 1.02$ $\quad f(x_{i+2}) = 1.38$

$$f'(1) = \frac{-1.38 + 4\,(1.32) - 3\,(1.27)}{2*0.01} = 4.5$$

To obtain f''(1) we use second derivative forward difference equation

$$f''(x_i) = \frac{-f(Xi+3) + 4\,f\,(Xi+2) - 5\,f\,(Xi+1) + 2\,f(Xi)}{h^2}$$

where $x_i = 1$ $f(x_i) = 1.27$

$x_{i+1} = 1.01$ $f(x_{i+1}) = 1.32$

$x_{i+2} = 1.02$ $f(x_{i+2}) = 1.38$

$x_{i+3} = 1.03$ $f(x_{i+3}) = 1.41$

$$f''(x_i) = \frac{-1.41 + 4\,(1.38) - 5\,(1.32) + 2\,(1.27)}{(0.01)^2} = 500$$

To obtain f'''(1) we use third derivative forward difference equation

$$f'''(x_i) = \frac{-3f(Xi+4) + 14\,f\,(Xi+3) - 24\,f\,(Xi+2) + 18\,f(Xi+1) - 5\,f(Xi)}{2h^3}$$

where $x_i = 1$ $f(x_i) = 1.27$

$x_{i+1} = 1.01$ $f(x_{i+1}) = 1.32$

$x_{i+2} = 1.02$ $f(x_{i+2}) = 1.38$

$x_{i+3} = 1.03$ $f(x_{i+3}) = 1.41$

$x_{i+4} = 1.04$ $f(x_{i+4}) = 1.47$

$$f'''(x_i) = \frac{-3\,(1.47) + 14\,(1.41) - 24(1.38) + 18\,(1.32) - 5\,(1.27)}{2h^3} = -190{,}000$$

(ii) Taylor series expansion about 1

$$f(x) = f(a) + \frac{(x-a)f'(a)}{1!} + \frac{(x-a)^2 f''(a)}{2!} + \frac{(x-a)^3 f'''(a)}{3!}$$

$$f(x) = 1.27 + \frac{(x-1)\,4.5}{1} + \frac{(x-1)^2\,(500)}{2!} + \frac{(x-1)^3(-190{,}000)}{3!}$$

4. Given the set of data

x	1	1.3	1.6	1.9	2.2
y	0.765	0.6201	0.4554	0.2818	0.1104

Obtain a Taylor's series expansion of the function about x =1. Obtain the necessary derivatives at 1, up to $f^{(4)}$ through numerical differentiation with error of order h.

Solution: Using the above data set where h = 0.3 to obtain f'(1), f''(1), f'''(1), $f^{IV}(1)$ where the equations are

$$f'(x_i) = \frac{f(Xi+1) - f(Xi)}{h}$$

$$f''(x_i) = \frac{f(Xi+2) - 2\,f\,(Xi+1) + f(Xi)}{h^2}$$

$$f'''(x_i) = \frac{f(Xi+3) - 3\,f\,(Xi+2) + 3\,f\,(Xi+1) - f(Xi)}{h^3}$$

$$f^{IV}(x_i) = \frac{f(Xi+4) - 4\,f\,(Xi+3) + 6\,f\,(Xi+2) - 4\,f\,(Xi+1) + f(Xi)}{h^4}$$

where $x_i = 1$ $\quad f(x_i) = 0.765$
$x_{i+1} = 1.3$ $\quad f(x_{i+1}) = 0.6201$
$x_{i+2} = 1.6$ $\quad f(x_{i+2}) = 0.4554$
$x_{i+3} = 1.9$ $\quad f(x_{i+3}) = 0.2818$
$x_{i+4} = 2.2$ $\quad f(x_{i+4}) = 0.1104$

Therefore,

$$f'(1) = \frac{0.6201 - 0.765}{0.3} = -0.483$$

$$f''(1) = \frac{0.4554 - 2\,(0.6201) + 0.765}{0.3^2} = -0.22$$

$$f'''(1) = \frac{0.2818 - 3\,(0.4554) + 3(0.6201) - 0.765}{0.3^3} = 0.4037$$

$$f^{IV}(1) = \frac{0.1104 - 4(0.2818) + 6(0.4554) - 4(0.6201) + 0.765}{0.3^4} = 0.024691$$

Therefore Taylor series about x = 1 is

$$f(x) = 0.765 + \frac{(x-1)(-0.483)}{1!} + \frac{(x-1)^2(-0.22)}{2!} + \frac{(x-1)^3(0.4037)}{3!} + \frac{(x-1)^4(0.0246)}{4!}$$

$$f(x) = 0.765 - 0.483\,(x-1) - 0.11\,(x-1)^2 + 0.0673\,(x-1)^3 + 0.0010288\,(x-1)^4$$

5. Given the data below obtain $f'(1.30)$, $f''(1.30)$ and $f'''(1.30)$ and obtain a Taylor series expansion about $x = 1.30$. Approximate f(1.357) using the expansion.

x	1.2	1.3	1.4	1.5	1.6
f(x)	11.5901	13.7818	14.0428	14.3074	16.8619

Solution: To obtain f'(1.3), f''(1.3) and f'''(1.3) we will again use forward difference because there aren't enough point on the other sider to use central or backward difference equations

$$f'(x_i) = \frac{f(Xi{+}1) - f(Xi)}{h}$$

$$f''(x_i) = \frac{f(Xi{+}2) - 2f(Xi{+}1) + f(Xi)}{h^2}$$

$$f'''(x_i) = \frac{f(Xi{+}3) - 3f(Xi{+}2) + 3f(Xi{+}1) - f(Xi)}{h^3}$$

where $x_i = 1.3$ $\quad f(x_i) = 13.7818$

$x_{i+1} = 1.4$ $\quad f(x_{i+1}) = 14.0428$

$x_{i+2} = 1.5$ $\quad f(x_{i+2}) = 14.3074$

$x_{i+3} = 1.6$ $\quad f(x_{i+3}) = 16.8619$

Therefore,

$$f'(1.3) = \frac{14.0428 - 13.7818}{0.1} = 2.61$$

$$f''(1.3) = \frac{14.3074 - 2(14.0428) + 13.7818}{0.1^2} = 0.36$$

$$f'''(1.3) = \frac{16.8619 - 3(14.3074) + 3(14.0428) - 13.7818}{0.1^3} = 2286.3$$

Therefore Taylor series about x = 1 is

$$f(x) = 13.7818 + \frac{(x-1.3)(2.61)}{1!} + \frac{(x-1.3)^2(0.36)}{2!} + \frac{(x-1.3)^3(2286.3)}{3!}$$

$$f(1.357) = 13.7818 + \frac{(1.357-1.3)(2.61)}{1!} + \frac{(1.357-1.3)^2(0.36)}{2!} + \frac{(1.357-1.3)^3(2286.3)}{3!}$$

$$= 14.0017$$