Kevin Camellini – 26771009 – ENGR 391- Numerical Methods In Engineering – Winter 2018 - Final Exam Crib Sheet

TAYLOR

$f(x) = f(a) + (x-a)^1 f'(a) + \frac{(x-a)^2 f''(a)}{2!} \dots + \frac{(x-a)^n f^n(a)}{n!}$

Example: $\frac{dy}{dx} = 2x \to f(1) = 1 \to a = 1$, approximate 1.2?

$f = x^2, f' = 2x, f'' = 2, f''' = 0$

$f(1) = 1, f'(1) = 2, f''(1) = 2, f(1) = 0$

$f(x) = f(1) + \frac{(x-1)^1 f'(1)}{1!} + \frac{(x-1)^2 f''(1)}{2!}$

Notes: Cannot use Taylor for $(x-1)^{0.5}$ because $f' = 0.5(x-1)^{-0.5}$ and $f'(0) = \frac{0.5}{\sqrt{-1}}$ and $f'(1) = \frac{0.5}{\sqrt{0}}$ are both not defined.

f' ∫ π

$\frac{d}{dx}\sin = \cos$
$\cos = -\sin$
$\tan = \sec^2$
$= 1/\cos^2$
$\cot = -\cos$
$\sec = \sec\tan$
$\csc = -\csc\cot$
$\tan^{-1} = \frac{1}{1+x^2}$
$\cos^{-1} \neq 1/\cos$

$\frac{d}{dx} n = 0$
$x^n = nx^{n-1}$
$e^x = e^x$
$e^{3x} = 3e^x$
$\ln x = 1/x$
$n^x = n^x \ln x$
$\log_a x = 1/x \ln a$

$\int 1 = x$
$\int \frac{1}{x} = \ln x$
$\int x^n = \frac{x^{n+1}}{n+1}$
$\int n^x = n^x/\ln x$
$\int 1 = x$
$\int \cos = \sin$
$\int \sin = -\cos$
$\int \sec^2 = \tan$

$1 = \sin^2 + \cos^2$
$1 = \sec^2 - \tan^2$
$1 = \csc^2 - \cot^2$
$\sin^2 = \sin(2x)$
$\cos^2 = 2\sin x\cos x$
$\tan^2 = 2\tan\sec^2$
$\tan = \frac{\sin}{\cos} = \frac{1}{\cot}$
csc=1/sin
sec=1/cos

ε

Absolute E_t:
$E_t = TS - NS$
TS = NS + ε_t

$\varepsilon_t = \left|\frac{TS - NS}{TS}\right|_{x100}$
stop when $\varepsilon_t < \varepsilon_s$

$\varepsilon_a = \left|\frac{current - previous}{current}\right|_{x100}$

ε_t = relative true error
ε_a = approx. relative error
ε_s = desired relative error

Truncation: 3.456=3.45 | Sig. Figures: 1.23=3,0.006=1 | Rounding: 3.456=3.46 | 4 Digit Rounding: 3.4567=3.457

ROOTS

Bisection:
$P = \frac{a+b}{2}$

False Position:
$P = \frac{af(b) - bf(a)}{f(b) - f(a)}$

1: incremental search
2: $a = lower$, b=upper
3: if p is + then p is b
elsif p is – then p is a
4: Solution = P
Stop: $\varepsilon = \left|\frac{P_{prev} - P_{curr}}{P_{prev}}\right|_{x100}$

Newton-Raphson:
$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}$
1: x_0 is closest negative
2: Find $f'(x)$ 3. get $f'(x_0)$
each step get $f'(x)$& $f'(x)$

Secant:
$x_{i+1} = x_i - \frac{f(x_i)(x_i - x_{i-1})}{f(x_i) - f(x_{i-1})}$
1: x_0 is closest negative
*use secant when you want to avoid finding $f'(x)$
*each step get x_i and $f(x_i)$

Modified Newton-R:
$x_{i+1} = x_i - \frac{f(x_i)f'(x_i)}{[f'(x_i)]^2 - f(x_i)f''(x_i)}$
*need to find f′ and f″
*use when there are multiple roots

LINEAR ALGEBRA

Pivoting:
$\begin{bmatrix}1 & 1 & 1\\7 & 1 & 1\\1 & 3 & 1\end{bmatrix} \to \begin{bmatrix}7 & 1 & 1\\1 & 1 & 1\\1 & 3 & 1\end{bmatrix}$
*avoids division by zero
≡ Maximum Scaling

$[P] = \begin{bmatrix}0 & 1 & 0\\1 & 0 & 0\\0 & 0 & 1\end{bmatrix}$
1. keep track of pivots
2. $Ax = Pb$
*avoids ill condition

Determinant: 2x2
$[A] = \begin{bmatrix}a & b\\c & d\end{bmatrix}$
$\det[A] = ad - bc$

Inverse:
$[A]^{-1} = \frac{\begin{bmatrix}d & -b\\-c & a\end{bmatrix}}{ad-bc}$

$A^{-1} = \left[\begin{array}{ccc|ccc}a & b & c & 1 & 0 & 0\\d & e & f & 0 & 1 & 0\\g & h & i & 0 & 0 & 1\end{array}\right]$

Determinant: (3x3) $a\begin{vmatrix}e & f\\h & i\end{vmatrix} - b\begin{vmatrix}d & f\\g & i\end{vmatrix} + c\begin{vmatrix}d & e\\g & h\end{vmatrix}$

Matrix Transpose:
$A = \begin{bmatrix}1 & 2\\3 & 4\\5 & 6\end{bmatrix}$ $A^T = \begin{bmatrix}1 & 3 & 5\\2 & 4 & 5\end{bmatrix}$

Cramer's Rule:
$\begin{bmatrix}a & b & c\\d & e & f\\g & h & i\end{bmatrix}\begin{bmatrix}b_1\\b_2\\b_3\end{bmatrix}$ $x_1 = \frac{\begin{vmatrix}b_1 & b & c\\b_2 & e & f\\b_3 & h & i\end{vmatrix}}{\det[A]}$ $x_2 = \frac{\begin{vmatrix}a & b_1 & c\\b & b_2 & f\\c & b_3 & i\end{vmatrix}}{\det[A]}$ $x_3 = \frac{\begin{vmatrix}a & b & b_1\\b & c & b_2\\c & d & b_3\end{vmatrix}}{\det[A]}$

Euclidean-Norm: $\|A\|_E = \sqrt{\Sigma each^2}$
P-Norm: $\|A\|_p = \sqrt[p]{\Sigma each^2}$
Infinity-Norm: $\|A\|_\infty = \Sigma|row|_{max}$
Column-Norm: $\|A\|_1 = \Sigma|col|_{max}$
*absolute value of |each entry| ∞ and 1!

Condition Number: $\|A\|_\infty \times \|A^{-1}\|_\infty$
>1well Conditioned
≈ 0 ill Conditioned
* ill condition: small changes in input cause huge changes in output

System of Linear Algebraic Equations: [A]{X} = {B} → Set [A] = [L][U]

LU decomposition:
$\begin{bmatrix}1 & 0 & 0\\L_{21} & 1 & 0\\L_{31} & L_{32} & 1\end{bmatrix}\begin{bmatrix}u_{11} & u_{12} & u_{13}\\0 & u_{22} & u_{23}\\0 & 0 & u_{32}\end{bmatrix}$

Crout's decomposition:
$\begin{bmatrix}L_{11} & 0 & 0\\L_{21} & L_{22} & 0\\L_{31} & L_{32} & L_{33}\end{bmatrix}\begin{bmatrix}1 & u_{12} & u_{13}\\0 & 1 & u_{23}\\0 & 0 & 1\end{bmatrix}$

Steps: (**Doolittle**)
1. keep [L] off to the side
2. perform gauss, keep track of multipliers
3. solve [L]{d}={b} for d
4. Solve [U]{x}={d} for X

* use LU instead of Gauss when there's risk of dividing by zero

LU Doolittle Example: *cannot swap rows

$\begin{bmatrix}2 & 5 & 1\\6 & 11 & 1\\-4 & 2 & 3\end{bmatrix} \xrightarrow[L_{21}=R_2-3R_1]{} \begin{bmatrix}2 & 5 & 1\\0 & -4 & -2\\-4 & 2 & 3\end{bmatrix} \xrightarrow[L_{31}=R_3+2R_1]{} \begin{bmatrix}2 & 5 & 1\\0 & -4 & -2\\0 & 8 & 5\end{bmatrix} \xrightarrow[L_{32}=R_3+2R_2]{} \begin{bmatrix}2 & 5 & 1\\0 & -4 & -2\\0 & 0 & 1\end{bmatrix}$ then $L = \begin{bmatrix}1 & 0 & 0\\3 & 1 & 0\\-2 & 2 & 1\end{bmatrix}$, $U = \begin{bmatrix}2 & 5 & 1\\0 & -4 & -2\\0 & 0 & 1\end{bmatrix}$ and b= $\begin{Bmatrix}3\\3\\-6\end{Bmatrix}$

Get [A] = [L][U] :
$\begin{bmatrix}2 & 5 & 1\\6 & 11 & 1\\-4 & 2 & 3\end{bmatrix} = \begin{bmatrix}1 & 0 & 0\\3 & 1 & 0\\-2 & 2 & 1\end{bmatrix}\begin{bmatrix}2 & 5 & 1\\0 & -4 & -2\\0 & 0 & 1\end{bmatrix}$

[L]{d} = {b} for d: * pivoting [L]{d} = [P]{b}
$\begin{bmatrix}1 & 0 & 0\\3 & 1 & 0\\-2 & 2 & 1\end{bmatrix}\begin{bmatrix}d_1\\d_2\\d_3\end{bmatrix} = \begin{bmatrix}3\\3\\-6\end{bmatrix} \xrightarrow[solve]{} \begin{bmatrix}d_1\\d_2\\d_3\end{bmatrix} = \begin{bmatrix}3\\-6\\-12\end{bmatrix}$

Solve [U]{x} = {d} for x:
$\begin{bmatrix}2 & 5 & 1\\0 & -4 & -2\\0 & 0 & 1\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} = \begin{bmatrix}3\\-6\\-12\end{bmatrix} \xrightarrow[solve]{} \begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} = \begin{bmatrix}105/4\\-15/2\\-12\end{bmatrix}$

Jacobi Iteration:
1. $x_{i+1} = \frac{b_1 - y_i}{coefx}$ and $y_{i+1} = \frac{b_2 - x_i}{coefy}$
2. initial guess (0,0)
3. create a table with $x_0, x_1 \dots y_0 \dots$

Example:
$\left[\begin{array}{cc|c}3 & 1 & 5\\1 & 2 & 5\end{array}\right]$ $x_{i+1} = \frac{5-y_i}{3}$ $y_{i+1} = \frac{5-x_i}{2}$

x_i	0	1.6	0.83	1.1
y_i	0	2.5	1.6	2.083

*converging towards TS

Gauss Seidel Iteration (3x3): *better for memory than Jacobi
1. $x_1 = \frac{b_1 - a_{12}x_2 - a_{13}x_3}{a_{11}}$ $x_2 = \frac{b_2 - a_{21}x_1 - a_{23}x_3}{a_{12}}$ $x_3 = \frac{b_3 - a_{31}x_1 - a_{32}x_2}{a_{23}}$
2. Initial guess $x_1 = (0,0,0)$ obtain new x_1
3. Plug new x_1 into $x_2 = (x_1, 0, 0)$ obtain new x_2
4. x_2 into $x_3 = (x_1, x_2, 0)$ obtain new x_3 5. Repeat until desired $\mathcal{E}$

INTERPOLATION (Curve Fitting)

n data points → max polynomial = n-1 (4 data points = 3rd order polynomial)→ interpolation = inside, extrapolation = outside (x_0 to x_n)

Direct Method:
$\begin{bmatrix}1 & x_0 & x_0^2\\1 & x_1 & x_1^2\\1 & x_2 & x_2^2\end{bmatrix}\begin{Bmatrix}a_0\\a_1\\a_2\end{Bmatrix} = \begin{bmatrix}y_0\\y_1\\y_2\end{bmatrix}$

Least Squares Regression:
$\begin{bmatrix}n & \Sigma x & \Sigma x^2\\\Sigma x & \Sigma x^2 & \Sigma x^3\\\Sigma x^2 & \Sigma x^3 & \Sigma x^4\end{bmatrix}\begin{Bmatrix}a_0\\a_1\\a_2\end{Bmatrix} = \begin{bmatrix}\Sigma y\\\Sigma xy\\\Sigma x^2 y\end{bmatrix}$
* where $\Sigma = \Sigma_0^{n-1}$
* 00 = 1 apparently
* $y = a_0 + a_1x + a_2x^2$

Example (2nd order):
x = [5, 10, 15] y = [19.4, 18.7, 18.2]
1. $\begin{bmatrix}3 & 30 & 350\\30 & 350 & 4500\\350 & 2025 & 61250\end{bmatrix}\begin{Bmatrix}a_0\\a_1\\a_2\end{Bmatrix} = \begin{bmatrix}56.3\\557\\64500\end{bmatrix}$
2. $a_0 = 20.3, a_1 = -0.2, a_2 = 0.004$
3. Calculate P(x_i), E for each, ΣE total
* perfect fit S_r=0 and r^2=1
* S_r should 1 decrease w/each order

Error:
$\bar{y} = \frac{\Sigma y_i}{n}$ = avergage
$E = S_r = |P(x_i) - f(x_i)|^2$
$S_t = \Sigma(y_i - \bar{y})^2$
$r^2 = \left|\frac{S_t - S_r}{S_t}\right|$ coeff. Of determination
$S_{y/x} = \sqrt{\frac{S_r}{n-1}}$ standard error of estimate

Newton Divided Difference:
$b_0 = y_0$ $b_1 = y_{10} = \left(\frac{y_1 - y_0}{x_1 - x_0}\right)$
$b_2 = y_{3210} = \frac{\frac{y_2 - y_1}{x_1 - x_0} - \frac{y_1 - y_0}{x_1 - x_0}}{x_2 - x_0}$
$b_n = y_{m\dots0} = \frac{y_{m\dots1} - y_{m-1\dots0}}{x_m - x_0}$

	0th (b_0)	1st (b_1)	2nd (b_2)	3rd (b_1)
x_0	y_0	y_{10}	y_{210}	y_{3210}
x_1	y_1	y_{21}		
x_2	y_2		y_{321}	
x_3	y_3	y_{32}		

$f_n(x) = b_0 + b_1(x - x_0) + b_2(x - x_0)(x - x_1) + \dots + b_n(x - x_0)\dots(x - x_{n-1})$

Example (Newton div. dif): approximate f(16)?
x = [10, 15, 20, 22.5, 30]
y = [227.04, 362.78, 517.35, 602.97, 901.67]
0th: $b_0 = y_0 \dots b_3 = y_3$
1st: $b_1 = y_{10} = \frac{y_1 - y_0}{x_1 - x_0} = \frac{362.78 - 227.04}{15 - 10} \dots y_{20} \dots y_{32}$
* at step 1 use results from 0, @2 use results from 1 & 0
* construct f_n using top most values

LAGRANGE

$$P(x) = \sum_{i=0}^{n} f(x_i) L_{i,k}$$

$$L_{i,k} = \prod_{\substack{j=0 \\ j\neq k}}^{n} \frac{(x - x_j)}{(x_k - x_j)}$$

Example:
x = [5, 10, 15] y = [19.4, 18.7, 18.2]

$L_{2,0} \prod_{\substack{j=0 \\ j\neq 0}}^{n} = \frac{(x-x_1)}{(x_0-x_1)}\frac{(x-x_2)}{(x_0-x_2)} = \frac{x^2-35x+150}{50}$

$L_{2,1} \prod_{\substack{j=0 \\ j\neq 1}}^{n} = \frac{(x-5)(x-15)}{(10-5)(10-15)} = \frac{x^2-25x+75}{-25}$ ...

$P(x) =$
$19.2\left(\frac{x^2-35x+150}{150}\right) + \cdots +$
$18.2\left(\frac{x^2-15x+150}{-50}\right)$
$P(x) = 0.004x^2 - 0.34x + 20.3$

Normal Equations Method:
$y = a_0 + a_1 x + a_2 x^2$ (6 equations given 3 unknowns)
1: Set up matrix [3 cols, 6 rows]{a's}={y's}
2: Calculate A^T, then A^TA
3: A^Tb solve for $A^TAx \equiv A^Tb$ **4**: Do regular gauss

NUMERICAL INTEGRATION $\int_a^b f(x)dx$

$h = step\ size = \frac{b-a}{n}$ & n= segments = #steps * ex: (a=30) (b=8) (n=2) → (h=11)

Trapezoid:
$$I = (b-a)\left[\frac{f(a)+f(b)}{2}\right]$$
$\int_a^b f(x)dx \approx f_1(x) = a_0 + a_1 x$

Composite Trapezoid:
$$I = \frac{h}{2}\left[f(a) + 2\sum f(x_{a+h}) + f(b)\right]$$
Ex: $I = \frac{11}{2}[\,f(8) + 2f(19) + f(30)]$

Simpsons 1/3:
$$I = \frac{h}{3}[f(a) + 4f(c) + f(b)]$$
where $c = \frac{a+b}{2}$ and $h = \frac{b-a}{n}$
$\int_a^b f(x)dx \approx f_2(x)$
* min. 4 data points

Composite Simpsons 1/3: * n = min.4 = even
$$I = \frac{h}{3}\left[f(a) + 4\sum_{1,3,5\ldots}^{n-1} f(x_i) + 2\sum_{2,4,6\ldots}^{n-2} f(x_i) + f(b)\right]$$
ex: 4 segments and 5 data points
$$I = \frac{a-b}{3*n}[f(a) + 4[f(x_1) + f(x_{13})] + 2f(x_2) + f(b)]$$

Simpsons 3/8:
$$\frac{3h}{8}[f(a) + 3f(x_1) + 3f(x_2) + f(b)]$$
* n = min. 3

Composite Simpsons 3/8:
$$I = \frac{3h}{8}\left[f(a) + 3\sum_{1,4,7}^{n-2} f(x_i) + 3\sum_{2,5,8}^{n-1} f(x_i) + 2\sum_{3,6,9}^{n-3} f(x_i) + f(b)\right]$$
* n = segments = min. 6 = multiple of 3

Gauss Quadrature:
$I = \sum_{i=1}^{n} c_i f(x_i) = c_1 f(x_1) \ldots + c_n f(x_n)$
$\int_a^b f(x)dx \approx f_3(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3$
* table is -1 to 1 need change bounds from [a,b] to [-1,1]→ (x = mt + r) → $\left(m = dx = \frac{b-a}{2}\right) \to \left(r = \frac{b+a}{2}\right)$
ex: [8,30] $(x = 11t + 19)$(dx=11dt)
$= \int_{-1}^{1}(11t+19)(11)dt = 11\sum_{i=1}^{n} c_i f(11x_i + 19)$

n	i	c_i	x_i
1	1	2	0
2	1,2	1	$-/+\ 1/\sqrt{3}$
3	1	8/9	0
	2,3	5/9	$-/+\sqrt{3/5}$
4	1	$\frac{\sqrt{18+\sqrt{30}}}{36}$	$-/+\sqrt{\frac{3}{7}-\frac{2}{7}\sqrt{6/5}}$
	2		
	3	$\frac{\sqrt{18-\sqrt{30}}}{36}$	$-/+\sqrt{\frac{3}{7}+\frac{2}{7}\sqrt{6/5}}$
	4		

Errors: $\overline{f'} = max\ |f'(x)|$
* truncation error single trapezoid, use LaGrange instead

Simpsons 1/3: $\varepsilon_a = \frac{-(b-a)h^5}{90}\overline{f^5(x)} = O(n^4)$
Composite 1/3: $\varepsilon_a = \frac{-(b-a)h^5}{180}\overline{f^5(x)} = O(n^4)$
Simpsons 3/8: $\varepsilon_a = \frac{-(b-a)h^4}{80}\overline{f^4(x)} = O(n^4)$
Composite 3/8: $\varepsilon_a = \frac{-(b-a)h^4}{180}\overline{f^4(x)} = O(n^4)$

NUMERICAL DIFFERENTIATION

x_{i-1}-----x_i-----x_{i+1}

	Forward:	Backward:	ε	Central:	ε
f'	$\frac{-f(x_i)+f(x_{i+1})}{h}$	$\frac{-f(x_{i-1})+f(x_i)}{h}$	h	$\frac{-f(x_{i-1})+f(x_{i+1})}{2h}$	h^2
f'	$\frac{-3f(x_i)+4f(x_{i+1})-f(x_{i+2})}{2h}$	$\frac{f(x_{i-2})-4f(x_{i-1})+3f(x_i)}{2h}$	h^2	$\frac{f(x_{i-2})-8f(x_{i-1})+8f(x_{i+1})-f(x_{i+2})}{12h}$	h^4
f''	$\frac{f(x_i)-2f(x_{i+1})+f(x_{i+2})}{h^2}$	$\frac{f(x_{i-2})-2f(x_{i-1})+f(x_i)}{h^2}$	h	$\frac{f(x_i)-2f(x_i)+f(x_{i+1})}{h^2}$	h^2

Inverse 3x3: $[cofactors(A)]/\det[A]$

$$\begin{bmatrix} a\begin{bmatrix} e & f \\ h & i\end{bmatrix} & b\begin{bmatrix} d & f \\ g & i\end{bmatrix} & c\begin{bmatrix} d & e \\ g & h\end{bmatrix} \\ d\begin{bmatrix} b & c \\ h & i\end{bmatrix} & e\begin{bmatrix} a & c \\ g & i\end{bmatrix} & f\begin{bmatrix} a & b \\ g & h\end{bmatrix} \\ g\begin{bmatrix} b & c \\ e & f\end{bmatrix} & h\begin{bmatrix} a & c \\ d & f\end{bmatrix} & i\begin{bmatrix} a & b \\ d & e\end{bmatrix}\end{bmatrix} \Big/ \det[A]$$

EIGENVALUE

Power Method:
1: Initial guess x^0 2: $y^{(1)} = [A]x^0$ result $= \lambda[U]$
$$y^{(1)} = \begin{bmatrix} 1 & 3 & 2 \\ 3 & -1 & 1 \\ 2 & 1 & -2\end{bmatrix}\begin{bmatrix}1\\1\\1\end{bmatrix} = \begin{bmatrix}6\\3\\1\end{bmatrix} = 6\begin{bmatrix}1\\1/2\\1/6\end{bmatrix}$$
3: $y^{(2)}$ =[A][U_1] ←leave out λ 4: repeat
$$\begin{bmatrix} 1 & 3 & 2 \\ 3 & -1 & 1 \\ 2 & 1 & -2\end{bmatrix}\begin{bmatrix}1\\1/2\\1/6\end{bmatrix} = \begin{bmatrix}17/6\\8/3\\13/6\end{bmatrix} = \frac{17}{6}\begin{bmatrix}1\\16/17\\13/17\end{bmatrix}$$
λ = eigenvalue → [U]=eigenvector

Gerschgorin Circle Theorem:
$$\begin{bmatrix} 2 & 8 & 10 \\ 8 & 4 & 5 \\ 10 & 5 & 7\end{bmatrix}$$
Approximate the **λ** of a matrix.
example creates 3 circles →
$C_1 = center(2\,)radius\ (8 + 10 = 18)$
$C_2 = center(4\,)radius\ (8 + 5 = 13)$
$C_3 = center(7\,)radius\ (10 + 5 = 15)$
* The union of all of the circles is the area where the Eigen values can be found.

Characteristic Equation:
$$\begin{bmatrix} a-\lambda & b & c \\ d & e-\lambda & f \\ g & h & i-\lambda\end{bmatrix}$$
1: $create[A - \lambda I]$
2: Get det(A-λI) of form
$(\lambda^2 + \lambda + 1)$ = characteristic polynomial
3: Eigen values = roots λ_1, λ_2 & λ_3
* ∃ a [U] for each $[A]\begin{bmatrix}u_1\\u_2\\u_3\end{bmatrix} = \lambda_1\begin{bmatrix}u_1\\u_2\\u_3\end{bmatrix}$
* solve for each [u] for each λ to get all [U]

***Inverse Power Method**: use A^{-1} instead and λ's are = 1/λ

Systems of Non-Linear Equations

Newton's Method:
1: Create the Jacobian
$$J_{ij} = \frac{\partial f_i(x)}{\partial x_j} \qquad J = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \frac{\partial f_1}{\partial x_n} \\ \frac{\partial f_2}{\partial x_1} & \ddots & \frac{\partial f_2}{\partial x_n} \\ \frac{\partial f_n}{\partial x_1} & \frac{\partial f_n}{\partial x_2} & \frac{\partial f_n}{\partial x_n}\end{bmatrix}$$
2. $f(x_0)$: *plug* initial guess x_0 *into* f $\quad x_0 = \begin{bmatrix}1\\1\\1\end{bmatrix}$
3. $J(x_0)$: plug initial guess x_0 *into* J
4. find $[J(x_0)]^{-1}$
5. $[x_{k+1}] = [x_k] - [J(x_k)]^{-1} f(x_k)$

$$\begin{Bmatrix} 12x_1 - 3x_2^2 - 4x_3 = 7.17 \\ x_1^2 + 10x_2 - x_3 = 11.54 \\ x_2^3 + 7x_3 = 7.631\end{Bmatrix}$$
$$f = \begin{bmatrix} 12x_1 - 3x_2^2 - 4x_3 - 7.17 \\ x_1^2 + 10x_2 - x_3 - 11.54 \\ x_2^3 + 7x_3 - 7.631\end{bmatrix}$$
$$J = \begin{bmatrix} 12 & -6x_2 & -4 \\ 2x_1 & 10 & -1 \\ 0 & 3x_2^2 & 7\end{bmatrix}$$

$$f(x_0) = \begin{bmatrix}-2.17\\-1.54\\0.369\end{bmatrix} \quad J(x_0) = \begin{bmatrix} 12 & -6 & -4 \\ 2 & 10 & -1 \\ 0 & 3 & 7\end{bmatrix}$$
$$[J(x_0)]^{-1} = \begin{bmatrix} 73/396 & 5/156 & 23/468 \\ -7/468 & 7/78 & 1/234 \\ 1/156 & -1/26 & 11/78\end{bmatrix}$$
$k = 0 \to x_1 =$
$$\begin{bmatrix}1\\1\\1\end{bmatrix} - \begin{bmatrix} 73/396 & 5/156 & 23/468 \\ -7/468 & 7/78 & 1/234 \\ 1/156 & -1/26 & 11/78\end{bmatrix}\begin{bmatrix}-2.17\\-1.54\\0.638\end{bmatrix}$$

$$x_1 = \begin{bmatrix}1.2005\\1.1042\\0.9026\end{bmatrix}$$

ODE

IVP: Solve
$$\begin{Bmatrix} \frac{dy}{dx} = f(x,y) \\ y(x_0) = x_0\end{Bmatrix}$$

Euler: (1st RK)
$y_{i+1} = y_i + k_1 h$
$k_1 = f(x_i, y_i)$
h=step size

Heuns: (2nd RK)
$y_i + \frac{k_1 + k_2}{2}h$
$k_1 = f(x_i, y_i)$
$k_2 = f(x_{i+1}, y_{i+1})$

Midpoint: (2nd RK)
$y_{i+1} = y_i + k_2 h$
$k_1 = f(x_i, y_i)$
$k_2 = f\left(x_i + \frac{h}{2},\ y_i + \frac{h}{2}k_1\right)$
*predictor corrector

Ralston: (2nd Order RK)
$y_i + f\left(\frac{1}{3}k_1 + \frac{2}{3}k_2\right)h$
$k_1 = f(x_i, y_i)$
$k_2 = f\left(x_i + \frac{3}{4}h, y_i + \frac{3}{4}h\right)$

Classical: (4th order RK)
$y_i + f(k_1 + 2k_2 + 2k_3 + k_4)h$
$k_1 = f(x_i, y_i)$
$k_2 = f\left(x_i + \frac{h}{2}, y_i + \frac{h}{2}k_1\right)$
$k_3 = f\left(x_i + \frac{h}{2}, y_i + \frac{h}{2}k_2\right)$
$k_4 = f(x_i + h, y_i + k_3 h)$

* RK is not iterative, at each step approximate deteriorates, not refining it.
* using RK4 on 3rd order polynomial yields100% accuracy

Example: (midpoint)
$\frac{dy}{dx} = yx^3 - 1.5y \to y(0) = 1 \to h = 0.5$

$\underline{i = 0}$
$f(x_0, y_0) = f(0,1)$
$k_1 = y_0 x_0^3 - 1.5y_0 = -1.5$
$k_2 = f(x_0 + \frac{h}{2}, y_0 + \frac{h}{2}k_1) = -0.927$
$y_1 = y_0 + k_2 h = 0.5361$

$\underline{i = 1}$
$f(x_1, y_1) = f(0.5, 0.5361)$
$k_1 = y_1 x_1^3 - 1.5y_1 = -0.7372$
$k_2 = f(x_1 + \frac{h}{2}, y_1 + \frac{h}{2}k_1) = -0.3793$
$y_2 = y_1 + k_2 h = 0.3465$