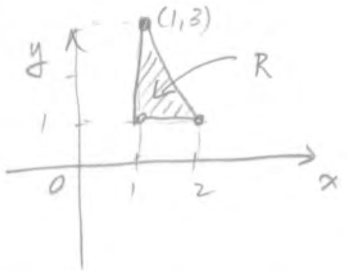
Solutions to sample/old final exam (Winter 2005)

(ENGR 233)

① We'll evaluate $\int_C x^2y\,dx + xy\,dy$ (where C is the triangle going from $(1,1)$ to $(2,1)$ to $(1,3)$ and back to $(1,1)$) using Green's therrem.



∴ The eq. of the line passing through the points $(1,3)$ and $(2,1)$ is :

$$y-3=\frac{1-3}{2-1}\cdot(x-1)$$

Hence $$\int_C x^2y\,dx+xy\,dy=\iint_R\left[\frac{\partial}{\partial x}(xy)-\frac{\partial}{\partial y}(x^2y)\right]dA=$$

$$=\iint_R (y-x^2)\,dA=\int_1^2\int_1^{-2x+5}(y-x^2)\,dy\,dx=\int_1^2\left[\left(\frac{y^2}{2}-x^2y\right)\Big|_1^{-2x+5}\right]dx$$

$$=\int_1^2\left[\frac{1}{2}(5-2x)^2-\frac{1}{2}-x^2(5-2x)+x^2\right]dx$$

$$=\int_1^2\left(\frac{1}{2}(25-20x+4x^2)-\frac{1}{2}-5x^2+10x^3+x^2\right)dx=$$

$$=\int_1^2(12-10x+2x^2-5x^2+10x^3+x^2)\,dx=\int_1^2(12-10x-2x^2+10x^3)\,dx$$

$$=\left(12x-5x^2-\frac{2}{3}x^3+\frac{5}{2}x^4\right)\Big|_1^2=\left(24-20-\frac{16}{3}+40\right)-\left(12-5-\frac{2}{3}+\frac{5}{2}\right)=\frac{179}{6}$$

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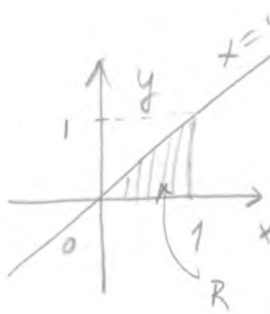
② Evaluate $\int_0^1\int_y^1 y^2\sqrt{1+x^4}\,dx\,dy$ by changing the order of integration



$$R: \begin{cases} 0 \le y \le 1 \\ y \le x \le 1 \end{cases} \qquad \begin{cases} 0 \le x \le 1 \\ 0 \le y \le x \end{cases}$$

So: $$\int_0^1\int_y^1 y^2\sqrt{1+x^4}\,dx\,dy = \int_0^1\int_0^x y^2\sqrt{1+x^4}\,dy\,dx = \int_0^1 \frac{x^3}{3}\cdot\sqrt{1+x^4}\,dx$$

$$\underset{\substack{\uparrow \\ u=1+x^4 \\ du=4x^3dx}}{=} \frac{1}{12}\int_1^2 u^{\frac{1}{2}}du = \frac{1}{12}\left(\frac{u^{3/2}}{3/2}\right)\Big|_1^2 = \frac{1}{12}\cdot\frac{2}{3}\cdot\left(2^{3/2}-1\right) = \frac{1}{18}\left(2\sqrt{2}-1\right)$$

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(3) If S is the surface $z = xy^4 + e^{2xy}$, then a normal vector to S at the point $(1,0,1)$ is:

$$\left(y^4 + e^{2xy}\cdot 2y,\ 4y^3x + e^{2xy}\cdot 2x,\ -1\right)\Big|_{(1,0,1)} =$$

$$= (0, 2, -1)$$

Thus, the eq. of the tangent plane is:

$$(x-1,\ y-0,\ z-1)\cdot(0,2,-1) = 0$$

or $$0\cdot(x-1) + 2(y-0) - (z-1) = 0$$

or $$2y - z + 1 = 0$$

The eq. of the normal line is: (in parametric form)

$$\vec{l}(t) = (1,0,1) + t\cdot(0,2,-1) \quad \# \quad \Longleftrightarrow \begin{cases} x(t) = 1 \\ y(t) = 2t \\ z(t) = 1-t \end{cases} \quad \#$$

(4)

a) $\operatorname{curl} \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y e^{xy} z + 2xy^2z^3 & x e^{xy} z + 2yx^2z^3 & e^{xy} + 3x^2y^2z^2 \end{vmatrix} =$

$$= \left[\frac{\partial}{\partial y}(e^{xy} + 3x^2y^2z^2) - \frac{\partial}{\partial z}(x e^{xy} z + 2yx^2z^3)\right]\vec{i}$$

$$- \left[\frac{\partial}{\partial x}(e^{xy} + 3x^2y^2z^2) - \frac{\partial}{\partial z}(y e^{xy} z + 2xy^2z^3)\right]\vec{j}$$

$$+ \left[\frac{\partial}{\partial x}(x e^{xy} z + 2yx^2z^3) - \frac{\partial}{\partial y}(y e^{xy} z + 2xy^2z^3)\right]\vec{k} =$$

$$= (e^{xy}\cdot x + 6x^2yz^2 - xe^{xy} - 6yx^2z^2)\vec{i}$$

$$-(e^{xy}\cdot y + 6xy^2z^2 - ye^{xy} - 6xy^2z^2)\vec{j}$$

$$+(\cancel{e^{xy}z} + \cancel{xe^{xy}yz} + 4yxz^3 - \cancel{e^{xy}z} - \cancel{ye^{xy}xz} - 4xyz^3)\vec{k}$$

$$= 0\cdot\vec{i} - 0\cdot\vec{j} + 0\cdot\vec{k} = \vec{0}$$

(i.e. the vector field $\vec{F}$ is conservative, hence question b) makes sense. A potential function for $\vec{F}$ exists if and only if $\operatorname{curl}\vec{F} = \vec{0}$)

b) Let $\varphi(x,y,z)$ be the potential fn. of $\vec{F}$

Thus $\frac{\partial \varphi}{\partial x} = y e^{xy} z + 2xy^2z^3$

$\frac{\partial \varphi}{\partial y} = x e^{xy} z + 2yx^2z^3$ and $\frac{\partial \varphi}{\partial z} = e^{xy} + 3x^2y^2z^2$

Integrate $\frac{\partial \varphi}{\partial x} = y e^{xy} z + 2xy^2 z^3$ with respect to x.

$\Rightarrow \varphi(x,y,z) = e^{xy} z + x^2 y^2 z^3 + c(y,z)$

$\Rightarrow \frac{\partial \varphi}{\partial y} = x e^{xy} z + 2x^2 y z^3 + \frac{\partial c}{\partial y} = x e^{xy} z + 2yx^2 z^3 \Rightarrow$

also equal to the r.h.s. of the second equation

$\Rightarrow \frac{\partial c}{\partial y} = 0 \Rightarrow c = C(z)$ (is a function of z only)

So: $\frac{\partial \varphi}{\partial z} = e^{xy} + 3x^2y^2z^2 + C'(z) = e^{xy} + 3x^2y^2z^2 \Rightarrow C'(z) = 0$

from 3rd equation

So C is just a constant (function).

We may take C to be zero. Thus a potential function φ to $\vec{F}$ is

$\varphi(x,y,z) = e^{xy} z + x^2 y^2 z^3$

c) As the vector field is conservative $\int_C \vec{F} \cdot d\vec{r}$ depends only on the values of φ at the endpoints of C. (φ is from part b))

These endpoints are: $\vec{r}(0)$ and $\vec{r}(1)$:

$\vec{r}(0) = (0, -1, 1)$

$\vec{r}(1) = (-1, 0, 1)$

Thus $W = \int_C \vec{F} \cdot d\vec{r} = \varphi(-1,0,1) - \varphi(0,-1,1) = (1+0) - (1-0) = 0.$

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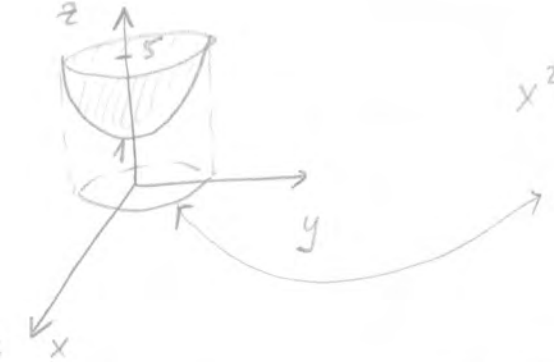
(5)



$x^2+y^2+1=5$

$x^2+y^2=4$

We want only S in the first octant

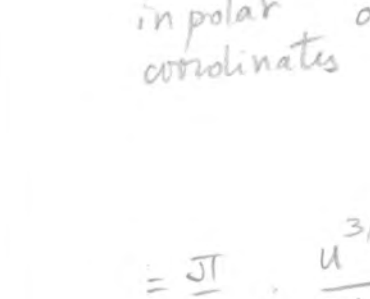
$$4.\ \text{Area}(S) = \iint_S dS = \iint_{\substack{D=\text{disk of}\\ \text{radius } 2\\ \text{centered at}\\ (0,0)}} \sqrt{1+\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2}\, dA = \iint_D \sqrt{1+4x^2+4y^2}\, dA$$

$$\underset{\substack{\uparrow\\ \text{in polar}\\ \text{coordinates}}}{=} \int_0^{2\pi}\int_0^2 \sqrt{1+4r^2}\cdot r\,dr\,d\theta \underset{\substack{\uparrow\\ u=1+4r^2\\ du=8r\,dr}}{=} 2\pi \int_1^{17} \cdot \frac{1}{8}\, u^{1/2}\, du$$

$$= \frac{\pi}{4} \cdot \frac{u^{3/2}}{3/2}\Big|_1^{17} = \frac{\pi}{6}\left(17^{3/2}-1\right) \Rightarrow \text{Area}(S) = \frac{\pi}{24}\left(17^{3/2}-1\right)$$

(We could have obtained $A(S)$ as $A(S) = \int_0^{\pi/2}\int_0^2 r\sqrt{1+4r^2}\, dr\, d\theta$.) #

(6) $$\text{mass} = \iint_R x^2\, dA = \int_{-2}^{2}\int_{x^2}^{4} x^2\, dy\, dx = \int_{-2}^{2} x^2 y\Big|_{x^2}^{4}\, dx =$$



$$= \int_{-2}^{2} (4x^2-x^4)\, dx = 2\int_0^2 (4x^2-x^4)\, dx = 2\left(\frac{4}{3}x^3-\frac{x^5}{5}\right)\Big|_0^2$$

$$= 2\left(\frac{32}{3}-\frac{32}{5}\right) = \frac{128}{15}.$$

Because R is symmetric with respect to the x-axis and the density is the same at points equally far away from the x-axis, we can conclude that $\bar{x} = 0$

(We may also see this from $\bar{x} = \frac{1}{m}\iint_R x(x^2)\,dA$.)

$$\bar{y} = \frac{1}{m}\iint_R y\cdot x^2\,dA = \frac{15}{128}\cdot\int_{-2}^{2}\int_{x^2}^{4} yx^2\,dy\,dx = \frac{15}{128}\int_{-2}^{2} \frac{x^2y^2}{2}\Big|_{x^2}^{4}\,dx =$$

$$= \frac{15}{128}\int_{-2}^{2}\frac{x^2}{2}(16-x^4)\,dx = \frac{15}{256}\cdot 2\int_0^2 x^2(16-x^4)\,dx = \frac{15}{128}\left(\frac{16}{3}x^3 - \frac{1}{7}x^7\right)\Big|_0^2$$

$$= \frac{15}{128}\left(\frac{16}{3}\cdot 8 - \frac{128}{7}\right) = 15\left(\frac{1}{3}-\frac{1}{7}\right) = \frac{15\cdot 4}{21} = \frac{60}{21} = \frac{20}{7}.$$

Hence the center of mass is at $(0, \frac{20}{7})$.

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7. $$k(t) = \frac{\|\vec{r}\,'(t)\times\vec{r}\,''(t)\|}{\|\vec{r}\,'(t)\|^3}$$

$\vec{r}\,'(t) = (-12\sin(3t),\ 12\cos(3t),\ 2)$, $\|\vec{r}\,'(t)\| = \sqrt{144+4} = \sqrt{148}$

$\vec{r}\,''(t) = (-36\cos(3t),\ -36\sin(3t),\ 0)$

$\vec{r}\,'(t)\times\vec{r}\,''(t) = (72\sin(3t),\ 72\cos(3t),\ 432)$

$\|\vec{r}\,'(t)\times\vec{r}\,''(t)\| = \sqrt{72^2+432^2} = \sqrt{191{,}808}$

So, $$k(t) = \frac{\sqrt{191{,}808}}{148\cdot\sqrt{148}} = \frac{\sqrt{1296}}{148} = \frac{36}{148} = \frac{9}{37}$$

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(8) $$W = \int_C \vec{F}\cdot d\vec{r} = \int_0^1 \underbrace{(t+t^6, t^3, t^4)}_{\vec{F} \text{ along } C} \cdot \underbrace{(1, 2t, 3t^2)}_{\vec{r}\,'(t)}\, dt =$$

$$= \int_0^1 (t+t^6+2t^4+3t^6)\,dt = \int_0^1 (4t^6+2t^4+t)\,dt$$

$$= \left(\frac{4}{7}t^7 + \frac{2}{5}t^5 + \frac{t^2}{2}\right)\Big|_0^1 = \frac{4}{7} + \frac{2}{5} + \frac{1}{2} = \frac{103}{70}$$

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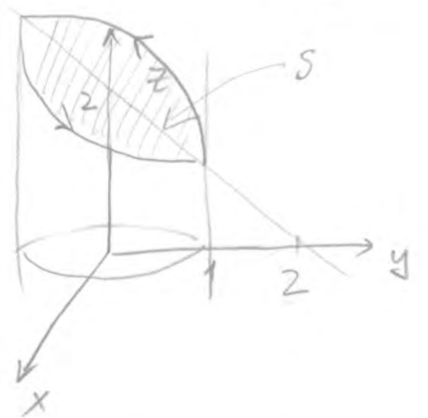
(9)



$$\int_C \vec{F}\, d\vec{r} = \iint_S \text{curl}\,\vec{F}\cdot\vec{n}\, dS \quad \text{by Stokes' Theorem.}$$

Here S is the part of the plane $y+z=2$ inside the cylinder $x^2+y^2=1$.

Hence $\vec{n} = \dfrac{(0,1,1)}{\sqrt{2}}$, $dS = \sqrt{2}\,dx\,dy$

Since $\text{curl}\,\vec{F} = 0\vec{i} + 0\vec{j} + (1+2y)\vec{k} = (1+2y)\vec{k}$, we obtain

$$\int_C \vec{F}\,d\vec{r} = \iint_D (1+2y)\,dx\,dy = \int_0^1\int_0^{2\pi} (1+2r\sin\theta)\,r\,d\theta\,dr =$$

D = disk of radius 1 in the xy-plane centered at (0,0)

polar coordinates

$$= \int_0^1 \left[(r\theta + 2r^2(-\cos\theta)) \Big|_0^{2\pi} \right] dr = \int_0^1 2\pi r \, dr = \pi r^2 \Big|_0^1 = \pi$$

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(10) $\operatorname{div} \vec{F} = \frac{\partial}{\partial x}(2x) + \frac{\partial}{\partial y}(3y) + \frac{\partial}{\partial z}(z) = 6$

$$D: \begin{cases} 0 \le z \le x+2 \\ -\sqrt{1-x^2} \le y \le \sqrt{1-x^2} \\ -1 \le x \le 1 \end{cases}$$

$$\iint_S \vec{F} \cdot \vec{n} \, ds = \iiint_D (\operatorname{div} \vec{F}) \, dV = \iiint_D 6 \, dV \underset{\uparrow}{=} 6 \int_0^1 \int_0^{2\pi} \int_0^{r\cos\theta + 2} r \, dz \, d\theta \, dr$$

cylindrical coordinates

$$= 6 \int_0^1 \int_0^{2\pi} r(r\cos\theta + 2) \, d\theta \, dr = 6 \int_0^1 \left[(r^2 \sin\theta + 2r\theta) \Big|_0^{2\pi} \right] dr$$

$$= 6 \cdot 4\pi \int_0^1 r \, dr = 12\pi$$

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