Concordia University

Department of Computer Science & Software Engineering

COMP 232/4 Introduction to Discrete Mathematics Winter 2019

Assignment 1 — Solutions

1. (a)

		A $\equiv$	B $\equiv$	C $\equiv$	D $\equiv$	
p	q	$p \leftrightarrow q$	$q \leftrightarrow T$	A $\leftrightarrow T$	$p \leftrightarrow$ B	C $\equiv$ D
T	T	T	T	T	T	T
T	F	F	F	F	F	T
F	T	F	T	F	F	T
F	F	T	F	T	T	T

   (b) using transformations

$p \leftrightarrow (q \leftrightarrow T)$

$\equiv p \leftrightarrow ((q \wedge T) \vee (\neg q \wedge \neg T))$

$\equiv (p \wedge ((q \wedge T) \vee (\neg q \wedge \neg T))) \vee (\neg p \wedge \neg((q \wedge T) \vee (\neg q \wedge \neg T)))$

$\equiv (p \wedge q \wedge T) \vee (p \wedge \neg q \wedge \neg T) \vee (\neg p \wedge (\neg q \vee \neg T) \wedge (q \vee T))$ distributive law, de Morgan

$\equiv (p \wedge q) \vee F \vee (\neg p \wedge \neg q \wedge T)$ domination

$\equiv (p \wedge q) \vee (\neg p \wedge \neg q)$

$\equiv p \leftrightarrow q$

$\equiv (p \leftrightarrow q) \wedge T$ identity

$\equiv ((p \leftrightarrow q) \wedge T) \vee (\neg(p \leftrightarrow q) \wedge \neg T)$ addition

$\equiv (p \leftrightarrow q) \leftrightarrow T$

2. Suppose the universe of discourse of the propositional function $P(x, y)$ consists of pairs x and y, where x is 1,2, or 3 and y is 1,2, or 3. Write out the following propositions using disjunctions and conjunctions.

   (a) $P(1,3) \vee P(2,3) \vee P(3,3)$

   (b) $P(1,1) \wedge P(1,2) \wedge P(1,3)$

   (c) $\neg P(2,1) \vee \neg P(2,2) \vee \neg P(2,3)$

   (d) $\neg P(1,2) \wedge \neg P(2,2) \wedge P(3,2)$

3. Express the negation of the following propositions using quantifiers and then express this negation in English.

   (a) $\neg \exists x Driver(x) \wedge \neg Obey(x, Speedlimit)$
   $\forall x(Driver(x) \rightarrow Obey(x, speedlimit))$
   It is not true that some drivers do not obey the speed limit.
   All drivers obey the speed limit.

   (b) $\neg \forall x Movie(x) \rightarrow (Swedish(x) \rightarrow Serious(x))$
   Not all Swedish movies are serious.
   Some Swedish movies are not serious.

   (c) $\neg\neg \exists x Person(x) \wedge CanKeepSecret(x) \equiv \exists x Person(x) \wedge CanKeepSecret(x)$
   Some people can keep secrets.

(d) There is someone in this class who does not have a good attitude.
$\neg\exists x Person(x) \wedge Takes(x, COMP232) \wedge Attitude(x, Bad)$
$\equiv \forall x(Person(x) \wedge Takes(x, COMP232)) \rightarrow Attitude(x, Good)$
Everybody in this class has a good attitude.

4. Express each of the following statements using only quantifiers, logical connectives, and as predicates use only the mathematical operators $\times$ (multiplication), $>, <, \geq, \leq, /$ (division), $-$ (negative numbers), $-$ (difference), $+$ or any of the ten digits.

   (a) $\forall x \forall y(Integer(x) \wedge Integer(y) \wedge (x < 0) \wedge (y < 0)) \rightarrow (xy > 0)$
   (b) $\forall x \forall y(Integer(x) \wedge Integer(y) \wedge (x > 0) \wedge (y > 0)) \rightarrow (\frac{x+y}{2} > 0)$
   (c) $\neg\forall x \forall y(Integer(x) \wedge Integer(y) \wedge (x < 0) \wedge (y < 0)) \rightarrow ((x - y < 0) \wedge (y - x < 0))$

5. Determine the truth value of each of the following statements if the universe of discourse of each variable consists of all real numbers.

   (a) true
   (b) false
   (c) false
   (d) true

6. (a) $\neg(\exists x \exists y P(x, y) \wedge \forall x \forall y Q(x, y))$
   $\equiv \forall x \forall y \neg P(x, y) \vee \exists x \exists y \neg Q(x, y)$
   (b) $\neg(\exists x \exists y(Q(x, y) \leftrightarrow Q(y, x)))$
   $\equiv \forall x \forall y \neg(Q(x, y) \leftrightarrow Q(y, x))$
   $\equiv \forall x \forall y(Q(x, y) \wedge \neg Q(y, x)) \vee (\neg Q(x, y) \wedge Q(y, x))$
   (c) $\neg\forall y \exists x \exists z(T(x, y, z) \vee Q(x, y))$
   $\equiv \exists y \forall x \forall z \neg(T(x, y, z) \vee Q(x, y))$
   $\equiv \exists y \forall x \forall z(\neg T(x, y, z) \wedge \neg Q(x, y))$

7. (a)
   - It is not the case that there is a student in this class who has never seen a computer.
   - It is not the case that there exists a person who is a student in this class who has never seen a computer.
   - It is not the case that there exists a person who is a student in this class and who has never seen a computer.
   - It is not the case that there exists a person such that that person is a student in this class and that person has never seen a computer.
   - It is not the case that there exists a person x in this class such that for all entities c, if x is a student in this class and c is a computer, then it is not the case that x has seen c.
   - $\neg\exists x \forall c\, Student(x, COMP232) \wedge (Computer(c) \rightarrow \neg HasSeen(x, c))$
     $\equiv \forall x \exists c\, (Student(x, COMP232) \rightarrow (Computer(c)) \wedge HasSeen(x, c))$
   - Every student in this class has seen a computer.

   (b)
   - It is not the case that there is a student in this class who has taken every mathematics course offered by Concordia.
   - It is not the case that there is a person who is a student in this class who has taken every mathematics course offered by Concordia.
   - It is not the case that there is a person who is a student in this class and who has taken every mathematics course offered by Concordia.

- It is not the case that there is a person who is a student in this class and it is true that for every mathematics course offered by Concordia, that person has taken it.
- It is not the case that there is a person p who is a student in this class and it is true that for all courses c, if c is a mathematics course offered by Concordia, then p has taken c.
- $\neg\exists p \forall c\, Student(p, COMP232) \wedge (MathCourseConcordia(c) \rightarrow HasTaken(p, c))$
  $\equiv \forall p \exists c\, Student(p, COMP232) \rightarrow (MathCourseConcordia(c) \wedge \neg HasTaken(p, c))$
- No student in this class has taken every math course offered by Concordia

8. (a) By definition. Truth table:

p	q	$p \wedge q$	$\neg(p \wedge q)$	$p \vert q$	$\neg(p \wedge q) \equiv p \vert q$
T	T	T	F	F	T
T	F	F	T	T	T
F	T	F	T	T	T
F	F	F	T	T	T

(b) $(p|q)|(p|q)$

(c) $p|p$

(d) We have shown that we can express $\neg p$ and $q \wedge q$ using only the Sheffer stroke.
We can thus argue that the Sheffer stroke is sufficient to express any logical operator if we can show that $\neg$ and $\wedge$ form a functionally complete set of logical operators (i.e. that all other logical operators can be expressed using only negation and conjunction).
We use a proof by cases:

Case $\vee$: $p \vee q \equiv \neg\neg(p \vee q) \equiv \neg(\neg p \wedge \neg q)$ by double negation and de Morgan

Case "other operators" : Constructive proof:

All other logical operators can be expressed in a truth table.
We can comprehensively describe any logic operator O using the disjunction of the conjunctions of basic variables or their negation in the following way:

- form a disjunction with one term for each row of the truth table where the column for pOq shows the truth value true

- each disjunct is itself a conjunction with one conjunct corresponding to
  - a basic variable if the value for that variable is true in this row or
  - the negation of the basic variable if its truth value in this row was false

  To illustrate: for the truth table for $p \rightarrow q$ you will construct the disjunction $(p \wedge q) \vee (p \wedge \neg q) \vee (\neg p \wedge q)$. The conditional has thus been expressed using negation, conjunction, and disjunction only.

  Since we have shown that negation, conjunction, and disjunction can be expressed using the Sheffer stroke only, the proof is complete.