PROBLEMS FOR CHAPTER 1

2. Given a differential equation $\frac{dy}{dx}$ = 2x, 0<x<5 and with y (1) = 1.Obtain the approximate value of y (1.2) using Taylor series expansion.

Solution:

$\frac{dy}{dx}$= 2x   0<x<5   y(1)=1

Taylor Series:

$$f(x) = f(a) + (x-a)f'(a) + \frac{(x-a)^2\ f''(a)}{2!} + \frac{(x-a)^n\ f''(a)}{n!}$$

$$f(x) = f(1) + (x-1)f'(1) + \frac{(x-1)^2\ f''(a)}{2!} + \frac{(x-1)^3\ f''(a)}{3!}$$

f(1) =y(1) = 1

$$f'(1) = \frac{dy}{dx} = 2x = 2 * 1 = 2$$

$$f''(1) = \frac{d^2y}{dx^2} = 2$$

$$f'''(1) = \frac{d^3y}{dx^3} = 0$$

$$f(x) = 1 + (x-1)2 + \frac{(x-1)^2 * 2}{2!} + 0$$

Therefore, $y(1.2) = f(1.2) = 1 + (1.2 - 1)2 + \frac{(1.2-1)^2 * 2}{2*1}$

= 1.44

3, Obtain the Taylor series expansion of the polynomial $P(x) = a_0 + a_1x+a_2x^2 + \dots + a_nx^n$ Comment on the result.

Solution:

solved assuming about 0

$P(x) = a_0 + a_1x + a_2x^2 \dots\dots + a_nx^n$

$P'(x) = a_1 +2 a_2x + 3a_3x^2 \dots\dots + na_nx^{n-1}$

$P''(x) = 2a_2 + 3*2 a_3x \dots\dots + n(n-1) a_nx^{n-p2}$

$P^n(x) = n(n-1)(n-2)\dots\dots[n-(n+1)]a_n$

Therefore Taylor Series expansion (Assuming about 0) is

$f(x) = f(a) + (x-a)f'(a) + \frac{(x-a)^2 f''(a)}{2!} \dots\dots + \frac{(x-n)^2 f^n(a)}{n!}$

$P(0) = a_0 + 0 \dots\dots +0$

$P'(0) = a_1 + 0 \dots\dots +0$

$P''(0) = 2a_2 + 0 \dots\dots +0 + 0$

$P'''(0) = 3 * 2a_3 + 0 \dots\dots +0$

$P^n(0) = n(n-1)(n-2)\dots\dots 3 * 2 * 1 a_n$

Taylor Series expansion about0 is

$P(x) = P'(a) + (x-a)P'(a) + (x-a)^2 \frac{p''(a)}{2!} \dots\dots + (x-a)^n \frac{p^n a}{n!}$

$= a_0 + (x-0)a_1 + (x-0)^2 \frac{2a_2}{2!} + (x-0)^3 * \frac{3*2a_3}{3!} \dots + (x-0)^n \frac{n(n-1)-1a_n}{n!}$

$P(x) = a_0 + a_1x + a_2x^2 + a_3x^3 \dots\dots + a_nx^n$

Therefore the Taylor Series of a polynomial about 0 is the polynomial itself.

8. Show that $f(x) = (x-1)^{0.5}$ cannot be expanded in Taylor series about x = 0 or x = 1, but can be expanded about x = 2. (Hint: f(0), f'(0), f''(0), etc. need evaluation of $(-1)0^{.5}$ which does not have real values).

Solution:

$f(x) = (x-1)^{0.5}$

Taylor Series

$f(x) = f(a) + (x-a)f'(a) + \frac{(x-a)^2 f''(a)}{2!} + \frac{(x-a)^3 f^3(a)}{3!}$......

$f'(x) = 0.5 * (x - 1)^{-0.5}$

$f''(x) = 0.25 * (x - 1)^{-1.5}$

$f'''(x) = 0.375 * (x - 1)^{-2.5}$

therefore, f(x) about 0 is

$f(a) = \sqrt{-1}$

$f'(a) = \frac{1}{2\sqrt{-1}}$

$f''(a) = 1 / (4 - 1^{3/2})$

since these do not have real values f(x) cannot be expanded about f(x) = 0

f(x) about 1

f(a) = 0

$f'(a) = \frac{1}{2\sqrt{0}}$ i.e. not defined

$f''(a) = \frac{1}{4-0^{3/2}}$

since these values are not defined f(x) cannot be expanded about x = 1

f(x) about 2

$f(a) = \sqrt{1} = 1$

$f'(a) = \frac{1}{2\sqrt{1}} = 0.5$

$f''(a) = \frac{1}{4-1^{3/2}} = 0.25$

since all these values are all real values f(x) can be expanded about x = 2.

10 Find the integral

$$I = \int_0^1 e^x \, dx$$

using Taylor series expansion of I about x = 0, to an accuracy of 3 digits.

Solution:

$$I = \int_0^1 e^x \, dx$$

e^x = 1 + x + $\frac{x^2}{2!}$ + $\frac{x^3}{3!}$ + $\frac{x^4}{4!}$ + $\frac{x^5}{5!}$ +.... + $\frac{x^n}{n!}$

$$I = \int_0^1 \sum_0^n \frac{x^n}{n!} \, dx$$

$$I = \left[\sum_0^n \frac{x^{n+1}}{(n+1)!}\right]$$

Getting e^x = 1 + x + $\frac{x^2}{2!}$ + $\frac{x^3}{3!}$ + $\frac{x^4}{4!}$ + $\frac{x^5}{5!}$ +.... + $\frac{x^n}{n!}$

From the previous questions

I = 1 + $\frac{1^2}{2!}$ + $\frac{1^3}{3!}$+ $\frac{1^4}{4!}$+ $\frac{1^5}{5!}$+......+ $\frac{1^{n+1}}{(n+1)!}$
= 1.718