MATH 203 Exam Prep – Winter 2020

Student Success Center

Concordia University

1. (a) Sketch the graph of the function $f(x) = (|x| - 1)^2$. (Suggestion: start from the graph of the standard parabola and use appropriate transformations).

(b) Solve for x: $2log_2(x+2) - log_2(x^2-4) = 5^{log_5(2)}$.

(c) Given the function $f = \frac{4 \cdot 3^x}{6 + 3^x}$, find the inverse function f^{-1}, and determine the domain and the range of f^{-1}.

(d) Suppose $f(x) = \sqrt[3]{x-1}$ and $g(x) = 1 + \left(\frac{x}{1+x^3}\right)^3$. Find $f \circ g$ and $g \circ f$ and their domains.

(e) Find the range of the function $f = e^{2x} + 2e^x$, the inverse function f^{-1}, and the range of f^{-1}. (HINT: assume $e^x = u$ to see how to find f^{-1})

2. Evaluate the limit if it exists, or explain why the limit does not exist.

(a) $\lim\limits_{x \to 3} \frac{|x-3|}{x^2 - x - 6}$ (b) $\lim\limits_{x \to 1} \frac{x-1}{3 - \sqrt{x^2+8}}$ (c) $\lim\limits_{x \to \infty} ln\left(\frac{\sqrt{x^4 + 9x^6}}{(2+3x)(4+x^2)}\right)$

(d) $\lim\limits_{x \to \infty} \frac{(x^3+1)(2x-3)^2}{(x+1)^2(3x+2)^3}$

(Do not use l'Hôpital's Rule)

3.

Option 1

Find all the horizontal and vertical asymptotes of a function

(a) $f(x) = \frac{|x|\sqrt{4x^2+1} - 2x^2}{x^2 - 3}$ (b) $f(x) = \frac{3^{x+1}}{3^x - 9}$ (c) $f(x) = \frac{\sqrt{9x^4 + 2x^2 + 1}}{x^2 + 4x}$

Option 2

(a) Consider the function $f(x) = \frac{|x^2 + 4x - 5|}{x^2 - 25}$. Calculate both one-sided limits at the point(s) where the function is undefined.

(b) Find the value of a and b so that the function,

$$f(x) = \begin{cases} 5 + x^2 & if \quad x \le 0 \\ ax + b & if \quad 0 < x \le 1 \\ \dfrac{25}{x} & if \quad x > 1 \end{cases}$$

is continuous everywhere. Sketch the graph of this function.

4. Find the derivatives of the following functions (show your work for full marks)：

(a) $f(x) = \frac{2\sqrt{x} - 3\sqrt[3]{x^2} + 4\sqrt[4]{x^3}}{x^{1/12}}$

(b) $f(x) = sec^2(\arctan(2x^2))$

(c) $f(x) = \frac{\arctan(2^{x+1})}{tan(x) - x}$

(d) $f(x) = (1 + 2x)^{x^2}$

(e) $f(x) = ln\left(\frac{x^4}{x+3}\right) + e^3$

(f) $f(x) = ln(x^2 sin(x) + xcos(x^2))$

(g) $f(x) = \sqrt{xsin(x^3) + sin(x^3 - x)}$

(h) $f(x) = (x^{3/2} + 1)(x^{3/2} - 1)tan(x)$

(i) $f(x) = \frac{\arcsin^2(x)}{\sqrt{1 - x^2}}$

5.

(a) The equation of a curve is $y^4 tan(x) = xy^3 + y - 1$ and defines y implicitly as a function of x. Verify that the point $(0,1)$ belongs to this curve and find an equation of the tangent line to the curve at this point.

(b) Verify that the point $(1,2)$ belongs to the curve defined by the equation $y^3 - 2xy - 2\sqrt{3 + x^2} = 0$, and find the equation of the tangent line to the curve at this point.

(c) A particle is moving along a plane curve $2x^2 + 5y^2 = 22$. At the moment when $x = -1$ the x-coordinate changes at the rate $5\,cm/s$. If the y-coordinate is positive at this moment, is it increasing or decreasing? How fast?

(d) At 1PM, ship A is 5km strictly to the west of ship B. Ship A is sailing west at speed 20 km/hr and ship B is sailing north at 30 km/hr. How fast (in km/hr) is the distance between ships changing at 3PM?

(e) Two cars start simultaneously moving away from the intersection of two orthogonal streets at the speeds $v_1 = 12\,m/s$ going east, and $v_2 = 16\,m/s$ going north. How fast is the distance between the cars increasing at the instant $t = 5$ seconds after they start moving from the intersection?

(f) Use the l'Hôpital's rule to evaluate the following

a. $\lim_{x\to 0}\frac{e^{x^2}-1}{1-cos(2x)}$

b. $\lim_{x\to 0}\frac{e^{x}-e^{-x}-2x}{x-sin(x)}$

6.

Option 1

Let $f(x) = \frac{x}{3x-1}$; $g(x) = \frac{x+1}{x+3}$; $h(x) = 3 + x + 3x^2 - x^3$

(a) Find the slope m of the secant line joining the points:

- $(1, f(1))$ and $(3, f(3))$
- $(-1, g(-1))$ and $(2, f(2))$
- $(0, f(0))$ and $(3, f(3))$

(b) Find all points $x = c$ (if any) on the interval:

- [1,3] such that $f'(c) = m$
- [-1,2] such that $g'(c) = m$
- [0,3] such that $h'(c) = m$

Option 2

Let $f(x) = (x+3)^{-1}$.

(a) Find the slope m of the secant line joining the points $(0, f(0))$ and $(4, f(4))$.

(b) Show there is a point $x = c$ on the interval $(0,\ 4)$ such that $f'(c) = m$. Why does this not contradict the Mean Value Theorem?

7.

Option 1

Consider the function $y = 3x + x^{-1}$

(a) Use the **definition of the derivative** to find the formula for dy/dx.

(b) Find the linearization $L(x)$ of the function $y(x)$ at $a = 2$.

(c) Find the differential dy and evaluate it for the values $x = 2$ and $dx = 0.1$.

Option 2

Consider the function $f(x) = \sqrt{x^2+8}$

(a) Use the **appropriate differentiation rules** to find the derivative $f'(x)$.

(b) Use the **definition of the derivative** to verify the answer in part a.

(c) Find the differential of the function.

(d) Use the differential above, or (equivalently) use the linear approximation at $a = 1$ (with the approximate choice of Δx) to find the approximate value of $\sqrt{8.49}$. Check the approximation with your calculator.

$\boxed{\text{Option 3}}$

The volume of a sphere with radius r is given by the formula $V(r) = \frac{4}{3}\pi r^3$.

(a) Use the **definition of the derivative** to show that $\frac{dV}{dr} = 4\pi r^2$.

(b) If a is given fixed value of r, write the formula for the linearization of the volume function $V(r)$ at a.

(c) Use this linearization to calculate the thickness Δr (in cm) of a layer of paint on the surface of a spherical ball with radius $r = 52$ cm if the total volume of paint used is 340 cm^3.

8.

(a) Find the absolute extrema of:

- $f(x) = xe^{-x^2}$ on the interval $\left[-\frac{1}{2}, 1\right]$.
- $g(x) = \frac{2x}{x^2 + x + 1}$ on the interval $[0, 3]$.
- $h(x) = (3x - 4)^4(4x - 3)^3$ on the interval $[0, 1]$.

(b) Find the point (x_0, y_0) on the curve $y = 2\sqrt{x}$ that is closest to the point $(3,0)$.

(c) A rectangle is inscribed with its base on the x-axis and its upper corners on the parabola $y = 12 - x^2$. Find the dimensions of such rectangle with the maximum possible area.

(d) Find the radius r and the height h of a cylinder can that is open at the top and has a volume 1000 cm^3, but has the smallest possible surface area.

(e) A box with a square base is to be constructed with a volume of 50 m^3. The material for the bottom and the sides of the box costs \$2/m^2, and the material for the top costs \$5/m^2. Find the dimensions that minimize the cost of the box.

(f) A rectangle *ABCD* has sides parallel to the coordinate axes and the point *A* is located at the origin. Point *B* is on the positive x-axis and point *C* is on the graph of the function $y = e^{-2x}$ and has positive x and y coordinates. Find the coordinates of the point *C* that maximizes the area of the rectangle.

9. Given the functions, execute parts (a) to (d) for each of them:
   - $f(x) = 2x^2 - x^4$
   - $g(x) = \frac{2x}{x^2 + 9}$
   - $h(x) = xe^{-2x^2}$
   - $y(x) = \frac{4(1-x)}{x^2}$

   (a) Find the domain and check for symmetry. Find all asymptotes (if any).
   (b) Find the interval(s) where the function is increasing, interval(s) where the function is decreasing, and local extrema (if any).
   (c) Find the interval(s) where the function is concave upward, interval(s) where the function is concave downward and inflection point(s) (if any).
   (d) Sketch the graph of the function using the information above.

Bonus Question

- Let $f(x)$ be a cubic of the form $f(x) = x^3 + ax^2 + bx + c$. Prove that f is increasing on $(-\infty, \infty)$ if $b > \frac{a^2}{3}$.
- Let $y = f(x)$ and $u = g(x)$ be twice differentiable functions. Use the Chain rule to derive the following formula for the second derivative of the composite function $h(x) = f(g(x))$:
$$h''(x) = f''(u)\big(g'(x)\big)^2 + f'(u)g''(x)$$
- Given the equation $x^5 + 5x = 5$,
  (a) Use the Intermediate value Theorem to show that there is a solution between 0 and 1.
  (b) Use the Mean Value Theorem to show that there cannot be more than one solution between 0 and 1.
- We know that a function f is differentiable on the interval $[0, 2]$ and has values $f(0) = 0$, $f(1) = 1$ and $f(2) = -1$. Is this information sufficient to claim, using the Mean Value Theorem, that the tangent line to the graph of $f(x)$ must be horizontal at least at one point x in the interval $(0, 2)$? Explain why yes or why not.
- Let $p(x) = x^4 + a^2x^2 - 2a^2x$, where a is any real number. Prove that the graph $y = p(x)$ has at least one point of local minimum on the interval (-1, 1).

Extra ☺:

- Calculate the second derivative $f''(x)$ of the function

$f(x) = x^{1/12}\left(\sqrt{x} - x^{-1/2}\right)x^{ax}$, where a is a parameter, and find $f''(0)$.

- Let $f(x) = \frac{4-3x^7}{x^5}$. Find $f'''(x)$.

REFERENCES:

The questions in this document have been selected from the following final exams at Concordia University:

- December 2011, 2014, 2016
- April 2015, 2016, 2018