MATH 203 Winter 2020

Ramtin Sasani

Winter 2019 Final solution

1) a) Since the variable x is in the exponent, we should use log but we need some modification first:

$$3^{2x} + 2.3^{x+1} = 16 \quad \Longrightarrow (3^x)^2 + 2.3.3^x + 9 = 25$$
$$\Longrightarrow (3^x + 3)^2 = 25 \quad \Longrightarrow 3^x + 3 = \pm 5 \Longrightarrow 3^x = 2, -8$$

-8 is not acceptable since 3^x is positive so $x = \log_3 2$

b) For the inverse:

$$y = \ln(1 + e^{2x}) \quad \Longrightarrow e^y = 1 + e^{2x} \Longrightarrow e^{2x} = e^y - 1$$
$$\Longrightarrow 2x = \ln(e^y - 1) \Longrightarrow x = \frac{1}{2}\ln(e^y - 1)$$

$1 + e^{2x}$ is always positive so $D_f = \mathbb{R}$ which also the range of inverse function f^{-1}. Range of f is the domain of f^{-1}. We must have $e^y > 1$ so $y > 0$ hence $R_f = \mathbb{R}^+$

2) a) Since $x^2 - 4$ changes sign at $x = 2$, we should evaluate left and right limits separately:

$$\lim_{x \longrightarrow 2^-} \frac{|x^2 - 4|}{x^2 + x - 6} = \lim_{x \longrightarrow 2^-} \frac{4 - x^2}{x^2 + x - 6} = \lim_{x \longrightarrow 2^-} -\frac{(x-2)(x+2)}{(x-2)(x+3)}$$
$$= \lim_{x \longrightarrow 2^-} -\frac{x+2}{x+3} = -\frac{4}{5}$$

$$\lim_{x \longrightarrow 2^+} \frac{|x^2 - 4|}{x^2 + x - 6} = \lim_{x \longrightarrow 2^-} \frac{x^2 - 4}{x^2 + x - 6} = \lim_{x \longrightarrow 2^-} \frac{(x-2)(x+2)}{(x-2)(x+3)}$$
$$= \lim_{x \longrightarrow 2^-} \frac{x+2}{x+3} = \frac{4}{5}$$

Since left and right limits are not equal, the limit does not exist.

b) We multiple and divide by conjugate

$$\lim_{x\longrightarrow-\infty}\sqrt{x^2+5x+1}+x=\lim_{x\longrightarrow-\infty}(\sqrt{x^2+5x+1}+x)\frac{\sqrt{x^2+5x+1}-x}{\sqrt{x^2+5x+1}-x}$$
$$=\lim_{x\longrightarrow-\infty}\frac{x^2+5x+1-x^2}{\sqrt{x^2+5x+1}-x}=\lim_{x\longrightarrow-\infty}\frac{5x+1}{\sqrt{x^2+5x+1}-x}$$
$$=\lim_{x\longrightarrow-\infty}\frac{5+1/x}{\sqrt{1+5/x+1/x^2}-1}=-\infty$$

At $-\infty$, we have $|\frac{5}{x}|>|\frac{1}{x^2}|$ and $5/x<0$ and $1/x^2>0$ so $5/x+1/x^2<0$ so $1+5/x+1/x^2<1$ so $\sqrt{1+5/x+1/x^2}<1$ which means the denominator is negative.

c) ln is a continuous function so we can switch ln and lim

$$\lim_{x\longrightarrow\infty}\ln\frac{1+2x+2x^3}{x(3+2x+x^2)}=\ln\lim_{x\longrightarrow\infty}\frac{1+2x+2x^3}{3x+2x^2+x^3}$$
$$=\ln\lim_{x\longrightarrow\infty}\frac{1/x^3+2/x^2+2}{3/x^2+2/x+1}=\ln 2$$

3) a) We should find the limits at infinity.

$$\lim_{x\longrightarrow\infty}\frac{3^{x+1}+2.4^x}{4^x-16}=\lim_{x\longrightarrow\infty}\frac{3(3/4)^x+2}{1-16/4^x}=2$$

$$\lim_{x\longrightarrow-\infty}\frac{3^{x+1}+2.4^x}{4^x-16}=\frac{0+0}{-16}=0$$

So we have horizontal asymptotes $y=2$ at $+\infty$ and $y=0$ at $-\infty$.

b) For a function defined by a fraction, vertical asymptotes occur at zeros of denominator (if numerator is nonzero). Denominator is zero at $x=2$ and numerator is $3^2+2.4^2\neq 0$ so $\lim_{x\longrightarrow 2}f(x)=\infty$ hence vertical asymptote of $x=2$.

4) a) $f(x)=x^{1/2}(x^{1/2}-x^{-3/2})2^x=(x-x^{-1})2^x$

$$\Longrightarrow f'(x)=(1+x^{-2})2^x+(x-x^{-1})2^x\ln 2$$

b) $f(x)=\ln(\sqrt{x^9+3x^8})+\ln e^2$

$$\Longrightarrow\ f'(x)=(\sqrt{x^9+3x^8})'\frac{1}{\sqrt{x^9+3x^8}}=\frac{9x^8+24x^7}{2\sqrt{x^9+3x^8}}\frac{1}{\sqrt{x^9+3x^8}}$$
$$=\frac{9x^8+24x^7}{2(x^9+3x^8)}=\frac{9x+24}{2x^2+6x}$$

c)

$$f'(x) = \frac{(\arctan(2x))'(1+\tan(x)) - (1+\tan(x))'(\arctan(2x))}{(1+\tan(x))^2}$$
$$= \frac{\frac{2}{1+(2x)^2}(1+\tan(x)) - \sec^2(x)(\arctan(2x))}{(1+\tan(x))^2}$$

d) $g(x) = \sqrt{x^2+1}.\cos(e^x) \Longrightarrow f'(x) = g'(x)\cos(g(x))$

$$g'(x) = (\sqrt{x^2+1})'cos(e^x) + \sqrt{x^2+1}(\cos(e^x))'$$
$$= \frac{2x}{2\sqrt{x^2+1}}\cos(e^x) - \sqrt{x^2+1}.e^x\sin(e^x)$$

e) $f(x) = (e^{\ln(1+2x)})^{x^2} = e^{x^2\ln(1+2x)} = e^{g(x)} \Longrightarrow f'(x) = g'(x)e^{g(x)}$

$$g'(x) = (x^2)'\ln(1+2x) + x^2(\ln(2x+1))' = 2x\ln(2x+1) + \frac{2x^2}{2x+1}$$

5) a) We should plug the coordinates into the equation and see if it holds.

$$2.1 + 2\sqrt{3+1} = 2 + 2.2 = 6 = 2^3 - 2$$

We should find y' to get the slope of tangent line. Since we do not have an explicit equation for y, we should use implicit differentiation.

$$xy' + y + 2.2y'y\frac{1}{2\sqrt{3+y^2}} = 3x^2$$
$$\Longrightarrow y' = \frac{3x^2-y}{x+\frac{2y}{\sqrt{3+y^2}}} = \frac{3.2^2-1}{2+(2.1)/2} = \frac{11}{3}$$

So the tangent line is $y - 1 = \frac{11}{3}(x-2)$.

b) The area of rectangle as a function of length x and width y is $S(x,y) = xy$ so $S' = x'y + xy'$. Notice that here x and y are both dependant variables (on time) so given the problem data $x' = 8$ and $y' = 5$. Hence when $x = 20$ and $y = 12$ we have $S' = 8.12 + 20.5 = 196$

c) At $x = 0$ we have the indeterminate form $\frac{0}{0}$ and both top and bottom are differentiable functions.

$$\lim_{x\longrightarrow 0}\frac{e^{x^2}-1}{\cos(2x)-1} = \lim_{x\longrightarrow 0}\frac{2x.e^{x^2}}{-2\sin(2x)} = \lim_{x\longrightarrow 0}\frac{2e^{x^2}+2x(2xe^{x^2})}{-4\cos(2x)} = \frac{2}{-4} = -\frac{1}{2}$$

We need to apply l'Hopital's rule twice since the second limit is also an indeterminate form.

6) a)

$$m = \frac{f(3)-f(0)}{3-0} = \frac{6-0}{3-0} = 2$$

b) f is a continuous, differentiable function on $[0,3]$ so by mean value theorem, such c exists.

$$f'(c) = 1 + 6c - 3c^2 = 2 \quad \Longrightarrow c = 0.183\ ,\ 1.816$$

both are in $[0,3]$.

7) a)

$$\begin{aligned} f'(x) &= \lim_{h\longrightarrow 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\longrightarrow 0} \frac{\sqrt{2(x+h)+1}-\sqrt{2x+1}}{h} \\ &= \lim_{h\longrightarrow 0} \frac{\sqrt{2x+2h+1}-\sqrt{2x+1}}{h} \frac{\sqrt{2x+2h+1}+\sqrt{2x+1}}{\sqrt{2x+2h+1}+\sqrt{2x+1}} \\ &= \lim_{h\longrightarrow 0} \frac{(2x+2h+1-(2x+1)}{h(\sqrt{2x+2h+1}+\sqrt{2x+1})} = \lim_{h\longrightarrow 0} \frac{2h}{h(\sqrt{2x+2h+1}+\sqrt{2x+1})} \\ &= \lim_{h\longrightarrow 0} \frac{2}{\sqrt{2x+2h+1}+\sqrt{2x+1}} = \frac{1}{\sqrt{2x+1}} \end{aligned}$$

b) We should just write the equation of tangent line at x=4. We have $f(4) = \sqrt{2.4+1} = 3$ and $f'(4) = \frac{1}{3}$ so

$$L(x) = \tfrac{1}{3}(x-4) + 3 = \tfrac{1}{3}x + \tfrac{5}{3}$$

c) We use the linearized function:

$$f(3) \simeq L(3) = \tfrac{1}{3}3 + \tfrac{5}{3} = \tfrac{8}{3} \simeq 2.666$$

8) a) We should check the values of function at endpoints and critical points. We have

$$f'(x) = \frac{1-x^2}{(x^2-x+1)^2} = 0 \quad \Longrightarrow x = \pm 1$$

The denominator is always nonzero (why?) so f' exists everywhere on $(0,3)$. Thus, the only critical point in $(0,3)$ is at $x=1$.

$$f(0) = 0\ ,\ f(1) = 1\ ,\ f(3) = \tfrac{3}{7}$$

So the absolute maximum is 1 at $x = 1$ and absolute minimum is 0 at $x = 0$.

b) Assume the square base has the side x and the height of the box is h. The volume of the box ix $54 = hx^2$ so $h = \frac{54}{x^2}$. The domain of x is $0 < x$ Now we should write cost as a function of x. The area of the box without the top is the side area $4hx = 216/x$ plus the base area x^2 so it costs $432/x + 2x^2$ dollars to make it. It costs $6x^2$ to make the top so the cost function is

$$C(x) = \tfrac{432}{x} + 2x^2 + 6x^2 = \tfrac{432}{x} + 8x^2$$

The domain of x does not have endpoints so we should check the critical points.

$$C'(x) = \tfrac{-432}{x^2} + 16x = 0 \quad \Longrightarrow 16x^3 = 432$$

So $x = 3$. $C''(x) = \frac{864}{x^3} + 16$ is positive at $x = 3$ so by Second Derivative Test, we have local minimum at $x = 3$ which is also the absolute minimum. So the cost is minimized at $x = 3$ and $h = 54/9 = 6$.

c) For more simplicity let's write

$$f(x) = x^{-3}(x^4 + 2x^2a^2 + a^4) = x + 2a^2x^{-1} + a^4x^{-3}$$

So

$$\begin{aligned}
f'(x) &= 1 + (-1)2a^2x^{-2} + (-3)a^4x^{-4} \\
f''(x) &= (-2)(-1)2a^2x^{-3} + (-4)(-3)a^4x^{-5} \\
f'''(x) &= (-3)(-2)(-1)2a^2x^{-4} + (-5)(-4)(-3)a^4x^{-6}
\end{aligned}$$

Thus

$$f'''(1) = -12a^2 - 60a^4$$

9) a) $f'(x) = 4x - 4x^3 = 4x(1 - x^2) = 4x(1 - x)(1 + x)$ so its zeros are $x = 0, \pm 1$. We have

- For $x < -1$, $f' > 0$ so f is increasing.
- For $-1 < x < 0$, $f' < 0$ so f is decreasing.
- For $0 < x < 1$, $f' > 0$ so f is increasing.

- For $1 < x$, $f' < 0$ so f is decreasing.

By First Derivative test, $x = -1$ is local maximum, $x = 0$ is local minimum and $x = 1$ is local maximum.

b) $f''(x) = 4 - 12x^2$ so its zeros are $x = \pm 1/\sqrt{3}$. We have

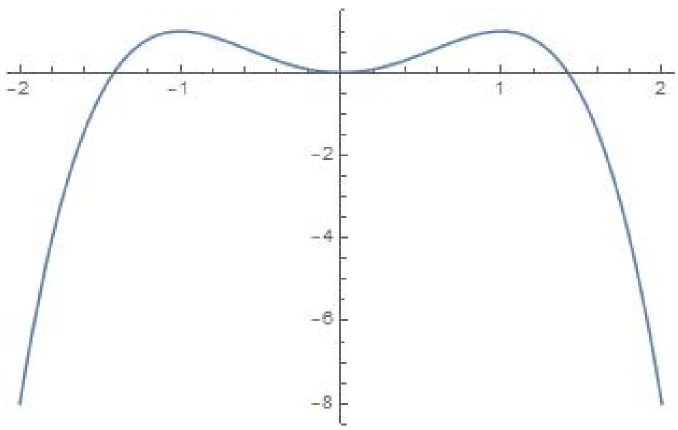
- For $x < -1/\sqrt{3}$, $f'' < 0$ so f is concave down.

- For $-1/\sqrt{3} < x < 1/\sqrt{3}$, $f'' > 0$ so f is concave up.

- For $1/\sqrt{3} < x$, $f'' < 0$ so f is concave down.

At $x = \pm 1/\sqrt{3}$ we have a change of concavity so they are inflection points.

c) The points $(\pm\sqrt{2}, 0)$ and $(0, 0)$ are on the curve. Using the information we have so far, we can plot the graph:



Bonus Question: $f \circ g$ is differentiable and

$$h' = (f \circ g)' = g'(x)f'(g(x)) = 2xf'(g(x))$$

Since f' is always (strictly) negative, then $h' > 0$ for $x < 0$ and $h' < 0$ for $x > 0$ and $h'(0) = 0$ so it only has one critical point. h' goes from positive to negative at $x = 0$ so by First Derivative Test, we have a local maximum at $x = 0$.