EMAT 232
APRIL - 2000

14

Concordia University
Department of Mathematics and Statistics

Course	Number	Section	
EMAT	232/4	all	
Examination	**Date**	**Time**	**Pages**
Final	April 2000	3 hours	2
Instructor(s)			**Course Examiner**
Bouchard, Bracken, Cummins, Gauthier, Hayes, Keviczky, Mokhtarian, Thaine			C. David
Special Instructions	Answer all questions. **NO CALCULATORS ALLOWED.**		

Marks

[11] 1. The trajectory of a particle is given by

$$C: \mathbf{r}(t) = (1 + \sin(t))\,\mathbf{i} + \frac{\cos(t)}{\sqrt{2}}\,\mathbf{j} + \frac{\cos(t)}{\sqrt{2}}\,\mathbf{k} \qquad t \geq 0.$$

a) Find the acceleration, the tangential acceleration, and the normal acceleration at the point $P = (2, 0, 0)$.

b) What is the arclength corresponding to the time interval $0 \leq t \leq 1$?

[11] 2. Verify the divergence theorem $\iint_S \mathbf{F} \cdot \mathbf{n}\, dA = \iiint_T \text{div}\, \mathbf{F}\, dV$

where $\mathbf{F}(x, y, z) = (-z, x, x)$, T is the cylinder $x^2 + y^2 \leq 16, 0 \leq z \leq 4$, and S is the boundary of T.

Go back

~~[11]~~ 3. a) Find a potential function for the vector function $\mathbf{F}\,(x, y, z) = \left(x^2 + y^2 + z^2\right)^{-3/2} (x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k})$

b) Evaluate the line integral $\int_C \mathbf{F} \cdot d\mathbf{r}$ over the path

$$C: \mathbf{r}(t) = \cos(t)\,\mathbf{i} + \cos^{44}(t)\,\mathbf{j} + 12\sin^{3/2}(t)\,\mathbf{k} \qquad 0 \leq t \leq \frac{\pi}{2}.$$

[11] 4. Use Stocke's theorem to evaluate $\oint_C \mathbf{F} \cdot d\mathbf{r}$ where $\mathbf{F}(x, y, z) = (2y, z^2, 3)$ and C is the circle $x^2 + y^2 = 1,\ z = -1$ with a counterclockwise orientation.

[11] 5. Given the following system of equations

$$\left\{ \begin{aligned} x + y + z + w &= 0 \\ x - y + z - w &= 0 \\ 2x + 2z &= 0 \\ -2y - 2w &= 0 \\ x - y + z - w &= 0 \end{aligned} \right\}$$

a) What is the rank of the system?

b) What is the number of free variables?

c) Describe all the solutions of the system.

[11] **6.** Using the divergence theorem, evaluate the surface integral $\iint_S \mathbf{F} \cdot \mathbf{n}\, dA$, where

$$\mathbf{F}(x, y, z) = (x - ye^z,\ \ y^2 + xz,\ \ z - y^2\sqrt{1 - x^2}),$$

and S is the boundary of the cube $-1 \le x \le 1,\ -1 \le y \le 1,\ -1 \le z \le 1$ oriented with an outward normal.

[11] **7.** a) Show that a closed bounded region T with boundary surface S has volume

$$V = \frac{1}{3} \iint_S (x, y, z) \cdot \mathbf{n}\, dA.$$

b) Find a parametrization for the ellipsoid $\frac{x^2}{4} + \frac{y^2}{9} + \frac{z^2}{16} = 1$.

c) Use part a) to find the volume inside the ellipsoid $\frac{x^2}{4} + \frac{y^2}{9} + \frac{z^2}{16} = 1$.

[11] **8.** a) What are the eigenvalues and eigenvectors of

$$A = \begin{bmatrix} 1 & 2 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 10 \end{bmatrix}?$$

b) What are the eigenvalues and eigenvectors of A^{-1}?

[12] **9.** a) Find λ_1, λ_2, the eigenvalues of $A = \begin{bmatrix} 8 & 6 \\ 6 & -8 \end{bmatrix}$.

b) Find an orthogonal matrix P such that $P^T A P = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$.

c) Identify the curve $8x^2 + 12xy - 8y^2 = 16$.

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14

Concordia University
Final Examination - EMAT 232 - All sections

Date: December 1998
Time Allowed: 3 hours.
Instructors: J. McKay.
Course Examiner: C. David.
Directions: Answer all questions. NO CALCULATORS.

MARKS

(10) 1. Find the curvature $\kappa(t)$ of the circular helix $\mathbf{r}(t) = 3\cos 2t\,\mathbf{i} + 3\sin 2t\,\mathbf{j} + t\,\mathbf{k}$.

(10) 2. Show that the vector field

$$F(x, y, z) = 2x\,\mathbf{i} + 3y^2z\,\mathbf{j} + y^3\,\mathbf{k}$$

is conservative, and find a potential function. Use this to evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$ for any curve C from $P = (0, 1, 0)$ to $Q = (2, 2, 1)$.

(10) 3. Find the work done by the force $\mathbf{F}(x, y, z) = (z, x, y)$ in the displacement along the curve $y = x^2$, $z = 2$ from $(1, 1, 2)$ to $(2, 4, 2)$.

(10) 4. Use Green's Theorem to evaluate

$$\int_C (x^5 - 5y)\,dx + (y^6 + x)\,dy$$

where C is the boundary of the circle $x^2 + y^2 = 4$ traversed counterclockwise.

(10) 5. Use the Divergence Theorem to evaluate the surface integral

$$\iint_S \mathbf{F} \cdot \mathbf{n}\,dA$$

where $\mathbf{F}(x, y, z) = 3y^2x\,\mathbf{i} + (xz + 3x^2y)\,\mathbf{j} + z^3\mathbf{k}$ and S is the sphere $x^2 + y^2 + z^2 = 4$ parametrized with outside normal.

(10) 6. Let $\mathbf{F}(x, y, z) = (x^2yz, xy^2z, xyz^2)$. Find $\text{div}(\mathbf{F})$ and $\text{curl}(\mathbf{F})$.

(10) 7. Describe all solutions of the system of equations

$$\begin{aligned} -x - y + 7z + w &= -3 \\ 2x + y - 10z - 3w &= 4 \\ 5x + y - 19z - 9w &= 7. \end{aligned}$$

(10) 8. Find the inverse of the matrix $A = \begin{pmatrix} 1 & 3 & -6 \\ -1 & -2 & 8 \\ 1 & 4 & -3 \end{pmatrix}$.

(10) 9. Find a basis of eigenvectors and diagonalize

$$A = \begin{pmatrix} 6 & 8 \\ 8 & -6 \end{pmatrix}.$$

(10) 10. Let

$$A = \begin{pmatrix} i & 1 \\ -1 & 2i \end{pmatrix}.$$

(a) Is A hermitian, skewhermitian, unitary? Justify.

(b) Find the eigenvalues of A.

[10] **10.**

$$\begin{array}{ccccccccc} x & + & y & + & z & + & w & = & 0 \\ x & - & y & + & z & - & w & = & 0 \\ 2x & & & + & 2z & & & = & 0 \\ & - & 2y & & & - & 2w & = & 0 \\ x & + & 2y & + & 2z & & & = & 0 \end{array}$$

What is the rank of above system?
What is the general solution?
What is the dimension of solution space?
What is the geometric form of solution space?

[10] **11.** Are the following matrices orthogonal, symmetric or Hermitian?

$$\begin{bmatrix} 6 & 8 \\ 8 & 6 \end{bmatrix}, \quad \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 4 & i \\ -i & 2 \end{bmatrix}$$

[6] **12.** $\oint_C \frac{\partial w}{\partial n} dS = \iint_R \nabla^2 w dxdy$. Find $\oint \frac{\partial w}{\partial n} dS$ if $w = e^x + e^y$ and R is the square

$$\begin{array}{ccccc} 0 & \leq & x & \leq & 2 \\ 0 & \leq & y & \leq & 2 \end{array}$$

bjc/JH

Concordia University

Department of Mathematics and Statistics

Emat 232 /4 Lec S, Test #2

March 20, 2000

Instructor: F. Mokhtarian **Time: 9:00-10:30**

CALCULATORS NOT ALLOWED

1. (5 marks) Using the divergence theorem of Gauss, evaluate the surface integral $\iint_S \vec{F} \cdot \vec{n} dA$, where

$$\vec{F}(x,y,z) = [x^2 + ye^z, y^2 + ze^x, z^2 + xe^y]$$

and S is part of the cylinder $x^2 + y^2 = 1$ between the xy plane and $z = x + 1$ including top and bottom with outward pointing normal.

2. (5 marks) Use Stoke's Theorem to evaluate the line integral $\oint_C \vec{F} \cdot d\vec{r}$ where

$$\vec{F} = [e^x \sin y, e^x \cos y - z, y]$$

and $C = C_1 \cup C_2$ with C_1 and C_2 the two boundary curves of the truncated cone $S : z = \sqrt{x^2 + y^2}$ between $z = 1$ and $z = 2$ with the top curve oriented clockwise and the bottom curve oriented counterclockwise.

3. (3 marks) Find area of surface S, where S is the portion of sphere $x^2+y^2+z^2 = 1$ that lies above the cone $z = \sqrt{x^2 + y^2}$.

4. (2 marks) Suppose that f is a harmonic function in some domain D that contains a region T and its boundary surface S satisfying the assumptions in the divergence theorem. Show that

$$\iint_S f \frac{\partial f}{\partial n} dA = \iiint_T |\nabla f|^2 dV$$

(Hint: Apply divergence theorem to $\vec{F} = f \nabla f$).

5. (5 marks) Find all solutions of the following system of linear equations:

$$\begin{aligned} x_1 + 2x_2 - x_3 - 5x_4 + 2x_5 &= -3 \\ x_2 + x_3 - 2x_4 - 4x_5 &= 1 \\ 2x_1 - 3x_2 + 2x_3 + 4x_4 - x_5 &= 9 \end{aligned}$$

$x_5 = C$

$x_4 = S$

Concordia University
Final Examination - EMAT 232 - All sections

Date: April 1999.
Time Allowed: 3 hours.
Instructors: I. Chen, C. Cummins, C. David, J. Hayes, A. Keviczky, A. Rajaei, A. Zaharescu.
Course Examiner: C. David.
Directions: Answer all questions. NO CALCULATORS.

MARKS

(14) 1. (a) Find the equation of the tangent line to the curve $C : \mathbf{r}(t) = (2\cos t, 3\sin t, t)$ at the point $P = (-2, 0, \pi)$.

(b) Find the length of the cardioid $r(\theta) = 2(1 - \cos\theta)$.

(6) 2. Let $\mathbf{v}(t)$ be any differentiable vector function $\mathbf{v} : \mathbb{R} \to \mathbb{R}^3$. Show that

$$\frac{d}{dt}|\mathbf{v}| = \frac{\mathbf{v}' \cdot \mathbf{v}}{|\mathbf{v}|}.$$

(10) 3. Find the work done by the force $\mathbf{F}(x, y, z) = (z, x, y)$ in the displacement along the curve $y = x^2$, $z = 2$ from $(1, 1, 2)$ to $(2, 4, 2)$.

(10) 4. Use Green's Theorem to evaluate

$$\int_C (x^2 + 5y)\, dx + x^2\, dy$$

where C is the triangle with vertices $(0, 0)$, $(1, 0)$ and $(0, 2)$ oriented counterclockwise.

(20) 5. (a) Give a parametrization of the sphere $x^2 + y^2 + z^2 = 1$.

(b) Compute the normal of your parametrization in (a). Is it pointing inside or outside the sphere? Justify your answer.

(c) *Without using the divergence theorem*, compute the surface integral

$$\iint_S \mathbf{F} \cdot \mathbf{n}\, dA$$

where S is the sphere $x^2 + y^2 + z^2 = 1$ parametrized with outside normal and $\mathbf{F}(x, y, z) = (x, y, z)$.

(d) Use the divergence theorem to compute the surface integral in (c).

(10) 6. Use Stocke's Theorem to evaluate

$$\int_C \mathbf{F} \cdot d\mathbf{r}$$

where $\mathbf{F}(x, y, z) = (y^2, 3, 2x)$ and C is the circle $x^2 + z^2 = 1$ and $y = 4$ with counter-clockwise orientation when viewed from the origin.

(8) 7. Find all solutions of the system of equations

$$\begin{aligned} x + 2y - 3z + 2w &= 1 \\ 2x + 5y - 2z + 3w &= 3 \\ 5x + 11y - 8z + 11w &= 9. \end{aligned}$$

(8) 8. Find the inverse of the matrix $A = \begin{pmatrix} 1 & 3 & -6 \\ -1 & -2 & 8 \\ 1 & 4 & -3 \end{pmatrix}$.

(14) 9. Let

$$A = \begin{pmatrix} 6 & 8 \\ 8 & -6 \end{pmatrix}.$$

(a) Find 2 linearly independent eigenvectors of A.

(b) What kind of conic is represented by the quadratic form

$$6x_1^2 + 16x_1x_2 - 6x_2^2 = 10\ ?$$

Sketch the principal axes and the conic.
Hint: The symmetric matrix associated to the conic is the matrix A above.

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Concordia University
Department of Mathematics and Statistics

Course EMAT	**Number** 232/2	**Section** all	
Examination Final	**Date** December 1999	**Time** 3 hours	**Pages** 3
Instructor(s) Bouchard, Hayes, Keviczky, Mokhtarian, Rajaei			**Course Examiner** J. Hayes
Special Instructions	Answer all questions. Calculators are permitted. Ruled Paper Booklets.		

Marks

[8] **1.** Let the trajectory of a particle be given by

$$\vec{r}(t) = 2t\,\vec{i} + t^2\vec{j} + (t^3/3)\,\vec{k} \quad t \geq 0$$

a) Find the arclength between A = (0,0,0) and B = (2,1,1/3).

b) What is the tangential acceleration at point B ?

c) At point C = (4,4,8/3) the velocity is $\vec{v} = 2\,\vec{i} + 4\,\vec{j} + 4\,\vec{k}$ and the normal acceleration is $\vec{a}_N = -(4/3)\,\vec{i} - (2/3)\,\vec{j} + (4/3)\,\vec{k}$. What is the curvature at this point?

[8] **2.** Find the inverse of $A = \begin{bmatrix} 3 & -1 & 1 \\ -15 & 6 & -5 \\ 5 & -2 & 2 \end{bmatrix}$.

Solve AX = B, $B = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, $X = \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix}$

What is $|A|$?

[7] **3.** Consider the following vector field: $\vec{F}(x,y,z) = (x^2+y^2+z^2)^{-3/2}[x\,\vec{i} + y\,\vec{i} + z\,\vec{k}]$; and the surface = $\{(x,y,z) : z = 1\}$ oriented with $\vec{n}$ (the unit normal) going up. Compute the flux integral $\displaystyle\iint_S \vec{F}\cdot\vec{n}\,dA$

Hints : 1) At some point you need polar coordiantes.

2) $\displaystyle\int_0^\infty \frac{r}{(1+r^2)^{3/2}}\,dr = 1$

[8] **4.** Consider the following torus surface (x,y,z coordinates are in meters)

$$S : \vec{r}(u,v) = (3+cos(v))cos(u)\,\vec{i} + (3+cos(v))sin(u)\,\vec{j} + sin(v)\,\vec{k}$$

Find the surface area.

[12] **5.** a) Give a parameterization of the cone $S : z = \sqrt{x^2 + y^2}$, $x^2 + y^2 \leq 1$ using polar coordinates.

b) Compute the normal of your parameterization in a). Do it pointing up or down? Justify your answer.

c) Without using the divergence theorem compute the flux integral $\displaystyle\iint_S \overrightarrow{F} \cdot \overrightarrow{n}\, dA$

where $S : z = \sqrt{x^2 + y^2}$, $x^2 + y^2 \leq 1$ parameterized with upward normal and $\overrightarrow{F} = \frac{1}{\sqrt{2}} \left(\frac{-x}{\sqrt{x^2+y^2}}, \frac{-y}{\sqrt{x^2+y^2}}, 1 \right)$.

[8] **6.** Evaluate $\displaystyle\oint_C \overrightarrow{F} \cdot d\overrightarrow{r}$ where $\overrightarrow{F}(x, y, z) = -4z\,\overrightarrow{i} + 2x\,\overrightarrow{j} - 2x\,\overrightarrow{k}$ and C is the ellipse given by $\overrightarrow{C} : \overrightarrow{r}(t) = cos(t)\,\overrightarrow{i} + sin(t)\,\overrightarrow{j} + [2sin(t) + 1]\,\overrightarrow{k} \quad t \in [0.2\pi]$. or $C = \{(x, y, z) | x^2 + y^2 = 1, z = 2y + 1\}$

[8] **7.** By Green's theorem for $\overrightarrow{F} = (y^2 - y, 2xy + 2x)$ and $C = \{(x, y) | x^2 + y^2 = 1\}$, find $\displaystyle\oint_C F \cdot dR =$

[8] **8.** By the divergent theorem, find $\displaystyle\iint_S F \cdot n dA$ for $\overrightarrow{F} = (x^3, y^3, z^3)$ and S is the surface of the sphere $x^2 + y^2 + z^2 = 4$.

[10] **9.** Given the conic section

$$17x_1^2 - 30x, x_2 + 17x_2^2 = 128$$

Transform to a principal axis where the new equation has no X_1X_2 term. Graph and identify conic section.

Concordia University
Department of Mathematics and Statistics

Course	**Number**	**Section**	
EMAT	232/4	all	
Examination	**Date**	**Time**	**Pages**
Final	April 2000	3 hours	2
Instructor(s)			**Course Examiner**
Bouchard, Bracken, Cummins, Gauthier, Hayes, Keviczky, Mokhtarian, Thaine			C. David
Special Instructions	Answer all questions. **NO CALCULATORS ALLOWED.**		

Marks

[11] √**1.** The trajectory of a particle is given by

$$C: \ \mathbf{r}(t) = (1 + \sin(t))\ \mathbf{i} + \frac{\cos(t)}{\sqrt{2}}\ \mathbf{j} + \frac{\cos(t)}{\sqrt{2}}\ \mathbf{k} \qquad t \geq 0.$$

√a) Find the acceleration, the tangential acceleration, and the normal acceleration at the point $P = (2, 0, 0)$.

√b) What is the arclength corresponding to the time interval $0 \leq t \leq 1$?

[11] **2.** Verify the divergence theorem $\displaystyle\iint_S \mathbf{F} \cdot \mathbf{n}\, dA = \iiint_T \operatorname{div} \mathbf{F}\, dV$

where $\mathbf{F}(x, y, z) = (-z, x, x)$, T is the cylinder $x^2 + y^2 \leq 16, 0 \leq z \leq 4$, and S is the boundary of T.

[11] √**3.** a) Find a potential for the vector function $\mathbf{F}\,(x, y, z) = \left(x^2 + y^2 + z^2\right)^{-3/2} (x\ \mathbf{i} + y\ \mathbf{j} + z\ \mathbf{k})$

b) Evaluate the line integral $\displaystyle\int_C \mathbf{F} \cdot\ d\mathbf{r}$ over the path

$$C: \ \mathbf{r}(t) = \cos(t)\ \mathbf{i} + \cos^{44}(t)\ \mathbf{j} + 12 \sin^{3/2}(t)\ \mathbf{k} \qquad 0 \leq t \leq \frac{\pi}{2}.$$

[11] **4.** Use Stocke's theorem to evaluate $\displaystyle\oint_C \mathbf{F} \cdot d\mathbf{r}$ where $\mathbf{F}(x, y, z) = (2y, z^2, 3)$ and C is the circle $x^2 + y^2 = 1,\ z = -1$ with a counterclockwise orientation.

[11] **5.** Given the following system of equations

$$\left\{ \begin{aligned} x + y + z + w &= 0 \\ x - y + z - w &= 0 \\ 2x + 2z &= 0 \\ -2y - 2w &= 0 \\ x - y + z - w &= 0 \end{aligned} \right\}$$

a) What is the rank of the system?

b) What is the number of free variables?

c) Describe all the solutions of the system.

[11] **6.** Using the divergence theorem, evaluate the surface integral $\iint_S \mathbf{F} \cdot \mathbf{n}\, dA$, where

$$\mathbf{F}(x, y, z) = (x - ye^z,\ \ y^2 + xz,\ \ z - y^2\sqrt{1 - x^2}),$$

and S is the boundary of the cube $-1 \le x \le 1,\ -1 \le y \le 1,\ -1 \le z \le 1$ oriented with an outward normal.

[11] **7.** a) Show that a closed bounded region T with boundary surface S has volume

$$V = \frac{1}{3}\iint_S (x, y, z) \cdot \mathbf{n}\, dA.$$

b) Find a parametrization for the ellipsoid $\dfrac{x^2}{4} + \dfrac{y^2}{9} + \dfrac{z^2}{16} = 1.$

c) Use part a) to find the volume inside the ellipsoid $\dfrac{x^2}{4} + \dfrac{y^2}{9} + \dfrac{z^2}{16} = 1.$

[11] **8.** a) What are the eigenvalues and eigenvectors of

$$A = \begin{bmatrix} 1 & 2 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 10 \end{bmatrix}?$$

b) What are the eigenvalues and eigenvectors of A^{-1}?

[12] **9.** a) Find λ_1, λ_2, the eigenvalues of $A = \begin{bmatrix} 8 & 6 \\ 6 & -8 \end{bmatrix}$.

b) Find an orthogonal matrix P such that $P^T AP = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$.

c) Identify the curve $8x^2 + 12xy - 8y^2 = 16$.

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Concordia University

Final Examination - EMAT 232 - All sections

Date: May 1998

Time Allowed: 3 hours.

Instructors: A. Akbary, C. David, G. Haché, J. Hayes, A. Keviczky, A. Schweizer, C. Simons.

Course Examiner: Chantal David.

Directions: Answer all questions. NO CALCULATORS.

MARKS

(10) 1. Let C be the circular helix $\mathbf{r}(t) = 2\cos t\,\mathbf{i} + 2\sin t\,\mathbf{j} + 5t\,\mathbf{k}$ from $(2,0,0)$ to $(-2,0,5\pi)$.

(a) Find the length of C.

(b) Find the curvature $\kappa(t)$.

(10) 2. Compute $\int_C \mathbf{F} \cdot d\mathbf{r}$ where C is the boundary curve of the rectangle $0 \leq x \leq 1$, $0 \leq y \leq 2$ oriented counterclockwise, and $\mathbf{F}(x,y) = 5xy\,\mathbf{i} + (x^2+1)\,\mathbf{j}$.

(10) 3. Show that the vector field

$$F(x,y,z) = 2x\,\mathbf{i} + 3y^2z\,\mathbf{j} + y^3\,\mathbf{k}$$

is conservative, and find a potential function. Use this to evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$ for any curve C from $P = (0,1,0)$ to $Q = (2,2,1)$.

(10) 4. Let S be the cone $z^2 = x^2 + y^2$, $0 \leq z \leq 4$.

(a) Give a parametrization $\mathbf{r}(u,v)$ of the surface S (including the bounds for u and v).

(b) Find the normal $\mathbf{N}(u,v)$. Does the normal point inside or outside S? Justify.

(c) Describe the parameter curves $u = k$ and $v = k$ on the surface S.

(10) 5. Compute the surface integral

$$\iint_S \mathbf{F} \cdot \mathbf{n}\, dA$$

where $\mathbf{F}(x, y, x) = xy^2\,\mathbf{i} + x^2y\,\mathbf{j} + \mathbf{k}$ and S is the surface consisting of the paraboloid $z = x^2 + y^2$, $0 \le z \le 4$ and of the circular disk $x^2 + y^2 \le 4$, $z = 4$. Assume also that S is parametrized with outside normal.

(10) 6. Use Stoke's Theorem to evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$ where $\mathbf{F}(x, y, z) = xyz^2\,\mathbf{k}$ and C is the circle $x^2 + z^2 = 1$ and $y = 4$ with counterclockwise orientation when viewed from the origin.

(10) 7. Describe all solutions of the system of equations

$$\begin{aligned} -x - y + 7z + w &= -3 \\ 2x + y - 10z - 3w &= 4 \\ 5x + y - 19z - 9w &= 7. \end{aligned}$$

Let A be the 3×4 matrix of coefficients of the system of linear equations above. What is the rank of A?

(10) 8. (a) Find the determinant of the matrix

$$A = \begin{pmatrix} 8 & -2 & 3 & 19 \\ 1 & 2 & 0 & 4 \\ 1 & 1 & 0 & 2 \\ -3 & 9 & 0 & 13 \end{pmatrix}.$$

(b) Find the inverse of the matrix $A = \begin{pmatrix} 1 & 2 & 1 \\ 0 & 2 & -3 \\ 1 & 0 & 2 \end{pmatrix}$.

(20) 9. Let

$$A = \begin{pmatrix} 6 & 8 \\ 8 & -6 \end{pmatrix}.$$

(a) Find the eigenvalues of A.

(b) Find all the eigenvectors of A.

(c) Find a matrix T such that $D = T^{-1}AT$ is a diagonal matrix.

(d) What type of conic section is represented by the quadratic form

$$6x_1^2 + 16x_1x_2 - 6x_2^2 = 10.$$

Sketch the principle axes and the conic.
Hint: The symmetric matrix associated to the conic is the matrix A given above.

c 14

Concordia University
Final Examination - EMAT 232 - All sections

Date: April 1999.
Time Allowed: 3 hours.
Instructors: I. Chen, C. Cummins, C. David, J. Hayes, A. Keviczky, A. Rajaei, A. Zaharescu.
Course Examiner: C. David.
Directions: Answer all questions. NO CALCULATORS.

MARKS

(14) 1. (a) Find the equation of the tangent line to the curve $C : \mathbf{r}(t) = (2\cos t, 3\sin t, t)$ at the point $P = (-2, 0, \pi)$.

(b) Find the length of the cardioid $r(\theta) = 2(1 - \cos\theta)$.

(6) 2. Let $\mathbf{v}(t)$ be any differentiable vector function $\mathbf{v} : \mathbb{R} \to \mathbb{R}^3$. Show that

$$\frac{d}{dt}|\mathbf{v}| = \frac{\mathbf{v}' \cdot \mathbf{v}}{|\mathbf{v}|}.$$

(10) 3. Find the work done by the force $\mathbf{F}(x, y, z) = (z, x, y)$ in the displacement along the curve $y = x^2$, $z = 2$ from $(1, 1, 2)$ to $(2, 4, 2)$.

(10) 4. Use Green's Theorem to evaluate

$$\int_C (x^2 + 5y)\, dx + x^2\, dy$$

where C is the triangle with vertices $(0, 0)$, $(1, 0)$ and $(0, 2)$ oriented counterclockwise.

(20) 5. (a) Give a parametrization of the sphere $x^2 + y^2 + z^2 = 1$.

(b) Compute the normal of your parametrization in (a). Is it pointing inside or outside the sphere? Justify your answer.

(c) *Without using the divergence theorem*, compute the surface integral

$$\iint_S \mathbf{F} \cdot \mathbf{n}\, dA$$

where S is the sphere $x^2 + y^2 + z^2 = 1$ parametrized with outside normal and $\mathbf{F}(x, y, z) = (x, y, z)$.

Department of Mathematics and Statistics

(d) Use the divergence theorem to compute the surface integral in (c).

(10) 6. Use Stocke's Theorem to evaluate

$$\int_C \mathbf{F} \cdot d\mathbf{r}$$

where $\mathbf{F}(x, y, z) = (y^2, 3, 2x)$ and C is the circle $x^2 + z^2 = 1$ and $y = 4$ with counter-clockwise orientation when viewed from the origin.

(8) 7. Find all solutions of the system of equations

$$\begin{aligned} x + 2y - 3z + 2w &= 1 \\ 2x + 5y - 2z + 3w &= 3 \\ 5x + 11y - 8z + 11w &= 9. \end{aligned}$$

(8) 8. Find the inverse of the matrix $A = \begin{pmatrix} 1 & 3 & -6 \\ -1 & -2 & 8 \\ 1 & 4 & -3 \end{pmatrix}$.

(14) 9. Let

$$A = \begin{pmatrix} 6 & 8 \\ 8 & -6 \end{pmatrix}.$$

(a) Find 2 linearly independent eigenvectors of A.

(b) What kind of conic is represented by the quadratic form

$$6x_1^2 + 16x_1x_2 - 6x_2^2 = 10\ ?$$

Sketch the principal axes and the conic.
Hint: The symmetric matrix associated to the conic is the matrix A above.

A

MIDETRM EXAMINATION

EMAT 223/PRASAD

Time: 1 hour. Maximum points: 25 (5 points for each problem). Calculators are not permitted on this test. You need not evaluate in decimals expressions such as $\sqrt{3+4\pi^2}$. Just leave them as they are if they are a part of your answer. All the best!
Please show all the steps in your solution and circle the final answer.

(1) Find the angle between the vectors
$$\mathbf{i}+\mathbf{j} \text{ and } -\mathbf{i}+\mathbf{j}.$$

(2) Find the length of the curve
$$\mathbf{r}(t) = 2\cos t\mathbf{i} + 2\sin t\mathbf{j} + t\mathbf{k} \text{ for } 0 \le t \le 4\pi.$$

(3) Find $\frac{dz}{dt}$ by the chain rule when
$$z = e^{xy};\ x = 3t+1;\ y = t^2.$$

(4) Find the direction of most rapid increase of
$$f(x,y,z) = x^2 + yz \text{ at } (1,1,1)$$
and give the rate of increase in this direction.

(5) Find the unit normal to
$$xyz = -3 \text{ at } (-1,3,1).$$

$f(x,y,z) = xyz$ $\quad P(-1,3,1)$

$\nabla f(x,y,z) = yz\vec{i} + xz\vec{j} + xy\vec{k}$

$\nabla f(x,y,z) = 3\vec{i} - \vec{j} - 3\vec{k}$

Date: February 25, 2002.

$\vec{n} = \frac{\nabla f(p)}{\|\nabla f(p)\|} = \frac{3\vec{i}}{\sqrt{19}} - \frac{1}{\sqrt{19}}\vec{j} - \frac{3}{\sqrt{19}}\vec{k}$

21

Concordia University
Department of Mathematics and Statistics

Course	Number	Section	
EMAT	232/2	all	
Examination	**Date**	**Time**	**Pages**
Final	December 1999	3 hours	3
Instructor(s)			**Course Examiner**
Bouchard, Hayes, Keviczky, Mokhtarian, Rajaei			J. Hayes
Special Instructions	Answer all questions. Calculators are permitted. Ruled Paper Booklets.		

Marks

[8] **1.** Let the trajectory of a particle be given by

$$\vec{r}(t) = 2t\,\vec{i} + t^2\,\vec{j} + (t^3/3)\,\vec{k} \quad t \geq 0$$

a) Find the arclength between A = (0,0,0) and B = (2,1,1/3).

b) What is the tangential acceleration at point B ?

c) At point C = (4,4,8/3) the velocity is $\vec{v} = 2\,\vec{i} + 4\,\vec{j} + 4\,\vec{k}$ and the normal acceleration is $\vec{a}_N = -(4/3)\,\vec{i} - (2/3)\,\vec{j} + (4/3)\,\vec{k}$. What is the curvature at this point?

[8] **2.** Find the inverse of $A = \begin{bmatrix} 3 & -1 & 1 \\ -15 & 6 & -5 \\ 5 & -2 & 2 \end{bmatrix}$.

Solve AX = B, $B = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, $X = \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix}$

What is $|A|$?

[7] **3.** Consider the following vector field: $\vec{F}(x,y,z) = (x^2+y^2+z^2)^{-3/2}[x\,\vec{i} + y\,\vec{i} + z\,\vec{k}]$; and the surface = $\{(x,y,z) : z = 1\}$ oriented with $\vec{n}$ (the unit normal) going up. Compute the flux integral $\iint_S \vec{F} \cdot \vec{n}\, dA$

Hints : 1) At some point you need polar coordiantes.

2) $\int_0^\infty \frac{r}{(1+r^2)^{3/2}} dr = 1$

$0 \leq \theta \leq 2\pi$

[8] **4.** Consider the following torus surface (x,y,z coordinates are in meters)

$$S : \vec{r}(u,v) = (3+\cos(v))\cos(u)\,\vec{i} + (3+\cos(v))\sin(u)\,\vec{j} + \sin(v)\,\vec{k}$$

Find the surface area.

[10] **10.**

$$\begin{array}{rcrcrcrcl} x & + & y & + & z & + & w & = & 0 \\ x & - & y & + & z & - & w & = & 0 \\ 2x & & & + & 2z & & & = & 0 \\ & - & 2y & & & - & 2w & = & 0 \\ x & + & 2y & + & 2z & & & = & 0 \end{array}$$

What is the rank of above system?
What is the general solution?
What is the dimension of solution space?
What is the geometric form of solution space?

[10] **11.** Are the following matrices orthogonal, symmetric or Hermitian?

$$\begin{bmatrix} 6 & 8 \\ 8 & 6 \end{bmatrix}, \quad \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 4 & i \\ -i & 2 \end{bmatrix}$$

[6] **12.** $\oint\limits_C \frac{\partial w}{\partial n} dS = \iint\limits_R \nabla^2 w dxdy$. Find $\oint \frac{\partial w}{\partial n} dS$ if $w = e^x + e^y$ and R is the square

$$\begin{array}{ccccc} 0 & \leq & x & \leq & 2 \\ 0 & \leq & y & \leq & 2 \end{array}$$

bjc/JH

[12] **5.** a) Give a parameterization of the cone $S : z = \sqrt{x^2 + y^2}\ , x^2 + y^2 \leq 1$ using polar coordinates.

b) Compute the normal of your parameterization in a). Do it pointing up or down? Justify your answer.

c) Without using the divergence theorem compute the flux integral $\iint\limits_S \vec{F} \cdot \vec{n}\ dA$ where $S : z = \sqrt{x^2 + y^2}\ ,\ x^2 + y^2 \leq 1$ parameterized with upward normal and $\vec{F} = \frac{1}{\sqrt{2}} \left(\frac{-x}{\sqrt{x^2+y^2}}, \frac{-y}{\sqrt{x^2+y^2}}, 1 \right)$.

[8] **6.** Evaluate $\oint\limits_C \vec{F} \cdot d\vec{r}$ where $\vec{F}(x, y, z) = -4z\,\vec{i} + 2x\,\vec{j} - 2x\,\vec{k}$ and C is the ellipse given by $\vec{C} : \vec{r}(t) = cos(t)\,\vec{i} + sin(t)\,\vec{j} + [2sin(t) + 1]\,\vec{k}.\ \ t \in [0, 2\pi].$ or $C = \{(x, y, z) | x^2 + y^2 = 1, z = 2y + 1\}$

[8] **7.** By Green's theorem for $\vec{F} = (y^2 - y, 2xy + 2x)$ and $C = \{(x, y) | x^2 + y^2 = 1\}$, find $\oint\limits_C F \cdot dR =$

[8] **8.** By the divergent theorem, find $\iint\limits_S F \cdot n dA$ for $\vec{F} = (x^3, y^3, z^3)$ and S is the surface of the sphere $x^2 + y^2 + z^2 = 4$.

[10] **9.** Given the conic section

$$17x_1^2 - 30x_1x_2 + 17x_2^2 = 128$$

Transform to a principal axis where the new equation has no X_1X_2 term. Graph and identify conic section.

Concordia University **Solutions and Comments**

EMath 232	First Class Test	February 2001
Instructor: C Cummins	Time: 75 mins	40 marks (20% of final grade)

Q1 (8 Marks) Find the tangential and the normal acceleration of the motion of a particle given by

$$\mathbf{r}(t) = t\mathbf{i} - t^2\mathbf{j}$$

Solution $\mathbf{a}_t = (\frac{4t}{1+4t^2}, \frac{-8t^2}{1+4t^2})$, $\mathbf{a}_n = (\frac{-4t}{1+4t^2}, \frac{-2}{1+4t^2})$.

Comments Why did so few people simplify $\mathbf{a}_n$? Mostly the question was done well, except for a few people who were confused between scalars and vectors.

Remember that you can do a partial check on your answer by calculating the dot product $\mathbf{a}_t \cdot \mathbf{a}_n$. This should be zero, since $\mathbf{a}_t$ and $\mathbf{a}_n$ are perpendicular.

Q2 (8 Marks) Show that the form under the integral sign is exact (i.e. show there is a potential function) and evaluate:

$$\int_{(1,2,3)}^{(2,-1,1)} (2xe^y + ze^x + 2)\,dx + x^2e^y\,dy + e^x\,dz$$

Solution A potential is $f = x^2e^y + ze^x + 2x$ (+ constant). This gives the value of the line integral as $4e^{-1} - 3e + 2$.

Comments As explained in class and the text book, there are examples of vector functions whose curl is zero, but for which there is no potential. So $\mathbf{curl}(\mathbf{F}) = 0$ is **not** a test for exactness of $F_1\,dx + F_2\,dy + f_3\,dz$. As explained in Theorem 3 of Kreyszig on p475 and also example 4 of p476 there is a crucial property of simple connectedness that has to hold. Also the question **tells** you to find the potential function. If there is a potential then the integrand is exact - see Kreyszig p 474, particularly equation $5'$ and the paragraph before it.

For these two reasons there were few marks for only computing the curl. To get the marks you also have to explain why simple connectedness holds in this example. But since you can show exactness directly by finding a potential and since you have to do this anyway for the second part of the question, there really isn't any point in computing the curl.

When solving the system of partial differential equations almost everyone used the prime ($'$) notation for differentiation of C. This is a bad idea since you have to compute a partial derivative and the prime does not indicate which variable is being differentiated.

A lot of people seemed to be confused about how to solve the system: We start with

$$\frac{\partial f}{\partial x} = 2xe^y + ze^x + 2$$
$$\frac{\partial f}{\partial y} = x^2e^y$$
$$\frac{\partial f}{\partial z} = e^x$$

Integrating the first equation gives $f = x^2e^y + ze^x + 2x + C(y,z)$. The "constant" can depend on y and z. Substitute this expression for f in the second equation and we get: $x^2e^y + \frac{\partial C(y,z)}{\partial y} = x^2e^y$. So $\frac{\partial C(y,z)}{\partial y} = 0$. This means that $C(y,z)$ does not depend on y, but it may depend on z, so $C(y,z) = D(z)$, say.
Substituting in the last equation gives $\frac{\partial D}{\partial z} = 0$ (we could use total derivatives, since there is only one variable here). So D is a constant.
Once you have computed the potential, don't forget to check that its gradient is the original vector function ($\nabla f = \mathbf{F}$).

Q3 (8 Marks) Use Green's Theorem to evaluate

$$\oint_C (x^3 + x^6 - \frac{y}{2})\,dx + (\frac{x}{2} + y^3 + \cos(y))\,dy$$

where C is the boundary of the region $x^2 + y^2 \leq 2$, $y \geq 0$, $x \geq 0$ traversed clockwise.

Solution Green's Theorem shows that the line integral is -Area(D) where D is the part of the disc of radius $\sqrt{2}$ centre (0,0) which is in the first quadrant. Since the area of the full disc is $\pi(\sqrt{2})^2 = 2\pi$ the solution is $-\frac{\pi}{2}$.

Comments Don't forget to check the orientation.

Q4 (8 Marks) Granted sufficient differentiability of the vector field $\mathbf{F} = F_1\mathbf{i} + F_2\mathbf{j} + F_3\mathbf{k}$, show that

$$\text{div}(\mathbf{curl}(\mathbf{F})) = 0$$

Comments A lot of people did not explain which pairs of second derivatives cancel. The point is that if the second partial derivatives are continuous then $\frac{\partial^2 F}{\partial x \partial y} = \frac{\partial^2 F}{\partial y \partial x}$ etc.
Note that it is **not** true in general that $\frac{\partial}{\partial x}\left(\frac{\partial F_2}{\partial z} - \frac{\partial F_3}{\partial y}\right)$ is zero. Similarly for the other two terms you get from calculating the divergence.

Q5 (8 Marks) Calculate the line integral $\int_C \sqrt{x^2 + y^2}\,dx + \sqrt{1 - y^2}\,dy$ where C is the upper half of the circle $x^2 + y^2 = 1$, traversed from $(1,0)$ to $(-1,0)$.

Solutions After substituting $(x, y) = (\cos(t), \sin(t))$, $0 \le t \le \pi$ the integral evaluates to -2.

Comments Mostly done well. But remember that you cannot use Green's Theorem when the curve is not the boundary of a region (and hence closed). Also, if you are confused as to how many variables should be in your substitution ask yourself what is the dimension of the thing you are parameterizing. In this case you are doing a line integral along a curve. A curve is one dimensional, so the substitution should involve one variable (which we called t in this example).

Concordia University

Engineering Math 232	Second Class Test	March 2001
Instructor: C Cummins	Time: 1 Hour 15 min	Weight: 40 marks (20%)

Q1 8 marks Find the area of the surface $z = x^2 + y^2$ lying below the plane $z = 1$.
+8

Q2 8 marks Evaluate the line integral $\oint_C \mathbf{F} \cdot d\mathbf{r}$ using Stokes's theorem where $\mathbf{F} = (x, 0, x^2 y)$ and C is the intersection of the surface $x^2 + y^2 = 1$ with the plane $z = x + 1$, traversed **clockwise** when seen from above.

Q3 8 marks Use the divergence theorem to evaluate

$$\iint_S (2x + xy^2)\, dy\, dz + (-y + z^2 y)\, dz\, dx + (-z + zx^2)\, dx\, dy$$

where S is the surface $x^2 + y^2 + z^2 = 2$.

Q4 8 marks Let
4

$$A = \begin{pmatrix} t & 1 & -1 \\ 2t & t+1 & 0 \\ 2t & 2 & t-1 \end{pmatrix}.$$

i) Calculate $\det(A)$.
ii) Find all the values of t for which the rank of A is less than 3.

Q5 8 marks 5-8

a Determine whether or not the following vectors in $\mathbb{R}^3$ are linearly dependent:

$$(1, 2, 3), \quad (1, -1, 2), \quad (1, 8, 5).$$

b Using part a, find the rank of the matrix

$$\begin{pmatrix} 1 & 1 & 1 \\ -1 & 8 & 2 \\ 2 & 5 & 3 \end{pmatrix}.$$

Explain your reasoning carefully.

Concordia University

Engineering Math 232	Second Class Test	Solutions
Instructor: C Cummins	Weight: 20%	

Q1 One possible parameterization of the surface is $\mathbf{r}(u, v) = (u, v, u^2 + v^2)$, $u^2 + v^2 \le 1$. Computing we find that $|\mathbf{r}_u \times \mathbf{r}_v| = \sqrt{1 + 4(u^2 + v^2)}$. Let $D = \{(u, v) \mid u^2 + v^2 \le 1\}$. Using the area formula gives:

$$\begin{aligned}
\text{Area} &= \iint 1 dS \\
&= \iint_D |\mathbf{r}_u \times \mathbf{r}_v| dudv \\
&= \iint_D \sqrt{1 + 4(u^2 + v^2)}\, dudv
\end{aligned}$$

changing to polar coordinates in the u, v plane converts this last integral to

$$\int_{\theta=0}^{2\pi} \int_{r=0}^{1} (\sqrt{1 + 4r^2}) r dr d\theta = \frac{\pi}{6}\left(5^{3/2} - 1\right)$$

Q2 Let S be the part of the plane $z = x + 1$ which is inside the cylinder $x^2 + y^2 = 1$. By Stokes's Theorem:

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n}\, dS.$$

The surface S can be parametrized by $\mathbf{r}(u, v) = (u\cos(v), u\sin(v), u\cos(v) + 1)$, $0 \le u \le 1$, $0 \le v \le 2\pi$. Computing gives $\mathbf{r}_u \times \mathbf{r}_v = (-u, 0, u)$. This is upward pointing, but by the right-hand rule, the direction we need is downward pointing, so we take $(u, 0, -u)$. We can also check that this proportional to $\nabla(-x + z - 1)$. (This step is not necessary, but is a good check on the calculation). Also $\nabla \times \mathbf{F} = (x^2, -2xy, 0) = (u^2\cos^2(v), -u^2\cos(v)\sin(v), 0)$ after substitution. Finally substituting into Stokes Theorem gives:

$$\begin{aligned}
\iint_S \nabla \times \mathbf{F} \cdot \mathbf{n} dS &= \int_{u=0}^{1} \int_{v=0}^{2\pi} (u^2\cos^2(v), -u^2\cos(v)\sin(v), 0).(u, 0, -u) dvdu \\
&= \int_{u=0}^{1} \int_{v=0}^{2\pi} u^3 \cos(v)\, dvdu \\
&= \frac{\pi}{4}
\end{aligned}$$

Q3 Let $D = \{(x,y,z) \mid x^2+y^2+z^2 \le 2\}$. Applying the divergence theorem gives:

$$\iint_S (2x+xy^2)\,dy\,dz + (-y+z^2y)\,dz\,dx + (-z+zx^2)\,dx\,dy =$$

$$\iiint_D (y^2+z^2+x^2)\,dx\,dy\,dz$$

Changing to spherical coordinates gives

$$\iiint_D x^2+y^2+z^2\,dx\,dy\,dz = \int_{\theta=0}^{2\pi}\int_{\phi=0}^{\pi}\int_{\rho=0}^{\sqrt{2}} (\rho^2)\rho^2\sin(\phi)\,d\rho\,d\phi\,d\theta$$
$$= \frac{16\sqrt{2}}{5}\pi.$$

Q4

a Computing the determinant gives $t^3 - t$.

b Since the rank is less than 3 if and only if the determinant is zero, the rank is less than 3 if and only if $t = 1, -1$ or 0.

Q5

a To determine whether or not these vectors are linearly dependent we must determine whether or not there are solutions to the equation

$$c_1(1,2,3) + c_2(1,-1,2) + c_3(1,8,5) = (0,0,0)$$

other than $c_1 = c_2 = c_3 = 0$. This yields the linear system

$$\begin{aligned} c_1 + c_2 + c_3 &= 0 \\ 2c_1 - c_2 + 8c_3 &= 0 \\ 3c_1 + 2c_2 + 5c_3 &= 0 \end{aligned}$$

Solving this system we find that it has infinitely many solutions and so the three vectors are linearly dependent. One form of the solution set is $c_1 = -3c_3, c_2 = 2c_3$ and so an example of a dependency relation is:

$$3(1,2,3) - 2(1,-1,2) - (1,8,5) = (0,0,0)$$

b By part a the three columns are linear dependent and so the rank is not 3 (since row rank = column rank = rank). However the first two columns are linearly independent, since one is not a multiple of the other and neither is zero, so the rank is 2 (or find a non-zero 2×2 submatrix with non-zero determinant and use rank=det rank).

Concordia University

Engineering Math 232	Second Class Test	March 2000
Instructor: C Cummins	Time: 1 Hour 15 min	Weight: 20%

Q1 Use the divergence theorem to evaluate

$$\iint_S x^3\,dy\,dz + y^3\,dz\,dx + z^3\,dx\,dy$$

where S is the surface $x^2+y^2+z^2=2$.

Q2

a Use the divergence theorem to show that a region T with boundary S has volume:

$$\frac{1}{3}\iint_S x\,dy\,dz + y\,dz\,dx + z\,dx\,dy$$

b Use the formula of part **a** to calculate the volume of the cube: $-5 \le x \le 5$, $-5 \le y \le 5$, $-5 \le z \le 5$. (Note: there are no marks for just giving the volume - which is very easy to find. You have to show how the formula is used to find the volume).

Q3 Evaluate the line integral $\oint_C \mathbf{F}\cdot d\mathbf{r}$ using Stokes's theorem where $\mathbf{F} = (y, 0, xy)$ and C is the boundary of the triangle with vertices $(1,0,0)$, $(0,1,0)$ and $(0,0,2)$ traversed counterclockwise when seen from the origin.

Q4 Find all the solutions to the following linear system, or determine that no solutions exist:

$$\begin{aligned} x + 7y + 2z &= 1 \\ -x + 2y - 3z &= 0 \\ 2x + 5y + 3z &= 1 \end{aligned}$$

$x = 2/9$
$y = 1/9$
$z = 0$

(No marks will be given for an incorrect answer, so check your answer very carefully!)

Q5

a Determine whether or not the following vectors in $I\!R^3$ are linearly dependent:

$$(2,2,-1),\quad (2,-1,-2),\quad (2,8,1).$$

b Find the rank of the matrix

$$\begin{pmatrix} 2 & 2 & 2 \\ 2 & -1 & 8 \\ -1 & -2 & 1 \end{pmatrix}.$$

Explain your reasoning carefully. (Marks will be removed if you do not justify each step of your calculation).

Concordia University

Engineering Math 232	Second Class Test	Solutions
Instructor: C Cummins	Weight: 20%	

Q1 Let $D = \{(x, y, z) \mid x^2 + y^2 + z^2 \leq 2\}$. Applying the divergence theorem gives:

$$\iint_S x^3\, dy\, dz + y^3\, dz\, dx + z^3\, dx\, dy = \iiint_D 3(x^2 + y^2 + z^2)\, dx\, dy\, dz$$

Changing to spherical coordinates gives

$$\begin{aligned} 3\iiint_D x^2 + y^2 + z^2\, dx\, dy\, dz &= 3\int_{\theta=0}^{2\pi}\int_{\phi=0}^{\pi}\int_{\rho=0}^{\sqrt{2}} (\rho^2)\rho^2 \sin(\phi)\, d\rho, d\phi, d\theta \\ &= \frac{48}{5}\sqrt{2}\pi \end{aligned}$$

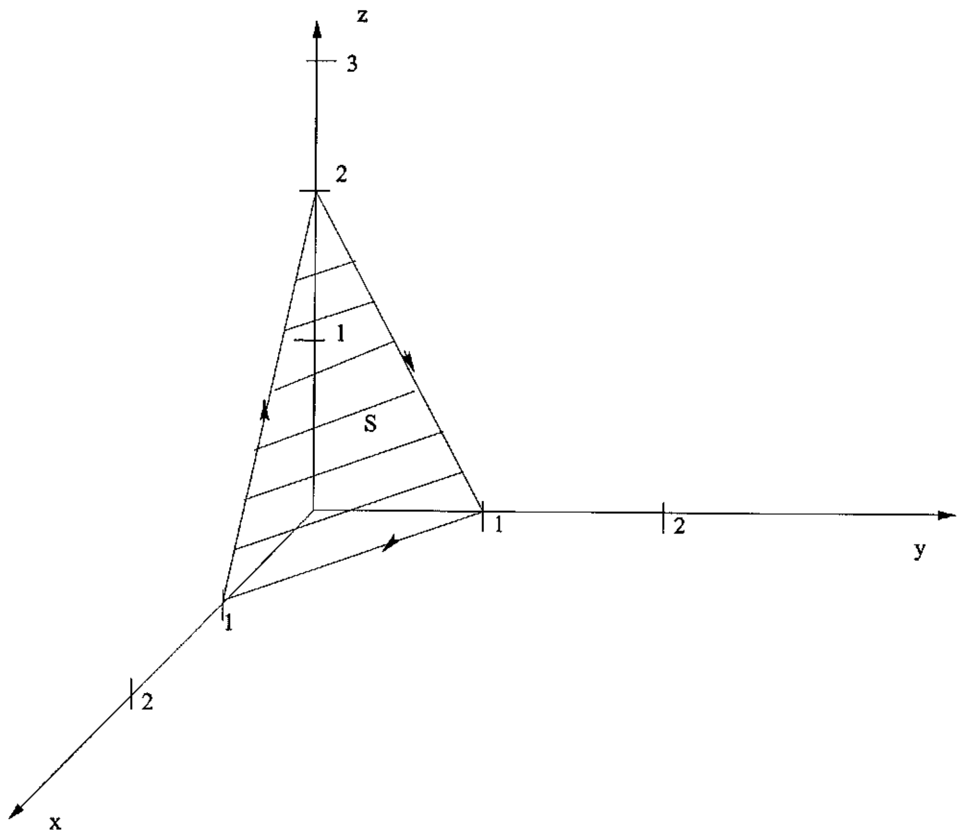
Q2

a

$$\begin{aligned} \frac{1}{3}\iint_S x\, dy\, dz + y\, dz\, dx + z\, dx\, dy &= \frac{1}{3}\iiint_T 1 + 1 + 1\, dx\, dy\, dz \\ &= \frac{1}{3}\iiint_T 3\, dx\, dy\, dz \\ &= \iiint_T 1\, dx\, dy\, dz \\ &= \text{Volume}(T). \end{aligned}$$

b By part a the volume is the sum of 6 surface integral over each of the 6 faces of the cube. By symmetry each face makes an equal contribution. So we can evaluate one of the surface integral and multiply by six.
Consider the integral over the face with $x = 5$. For this face: $\frac{1}{3}\iint_S x\, dy\, dz + y\, dz\, dx + z\, dx\, dy = \frac{1}{3}\int_{z=-5}^{5}\int_{y=-5}^{5} x\, dy\, dz = \frac{1}{3}\int_{z=-5}^{5}\int_{y=-5}^{5} 5\, dy\, dz = \frac{1}{3}500$.
Multiplying this by 6 gives 1000, which is the expected volume.

Q3 Let S be the part of the plane $x + y + \frac{1}{2}z = 1$ which is inside the triangle.



By Stokes's Theorem:

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n}\, dS.$$

The surface S can be parametrized by $\mathbf{r}(u,v) = (u, v, 2(1-u-v))$, $0 \le u \le 1$, $0 \le v \le 1-u$. Computing gives $\mathbf{r}_u \times \mathbf{r}_v = (2,2,1)$. However, by the right hand rule we require a downward pointing normal, so we multiply this by -1. Also $\nabla \times \mathbf{F} = (x, -y, -1) = (u, -v, -1)$ after substitution. Finally substituting into Stokes Theorem gives:

$$\begin{aligned}\iint_S \nabla \times \mathbf{F} \cdot \mathbf{n} dS &= \int_{u=0}^{1} \int_{v=0}^{1-u} (u, -v, -1).(-2,-2,-1) dv du \\ &= \int_{u=0}^{1} \int_{v=0}^{1-u} -2u + 2v + 1\, dv du \\ &= \frac{1}{2}\end{aligned}$$

Q4 Applying Gaussian elimination gives the solution $x = \frac{2}{9}, y = \frac{1}{9}, z = 0$.

Q5

a To determine whether or not these vectors are linearly dependent we must determine whether or not there are solutions to the equation

$$c_1(2,2,-1) + c_2(2,-1,-2) + c_3(2,8,1) = (0,0,0)$$

other than $c_1 = c_2 = c_3 = 0$. This yields the linear system

$$\begin{aligned} 2c_1 + 2c_2 + 2c_3 &= 0 \\ 2c_1 - c_2 + 8c_3 &= 0 \\ -c_1 - 2c_2 + c_3 &= 0 \end{aligned}$$

Solving this system we find that it has infinitely many solutions and so the three vectors are linearly dependent. One form of the solution set is $c_1 = -3c_3, c_2 = 2c_3$ and so and example of a dependency relation is:

$$-3(2,2,-1) + 2(2,-1,2) + (2,8,1) = (0,0,0)$$

b By part a the three columns are linear dependent and so the rank is not 3 (since row rank=column rank). However, the rank is not zero since then the matrix would have to be the zero matrix. It is also not 1 since then every row would have to be a multiple of every other row. Hence the rank is 2.
Note: if you want to use the method of row reduction to calculate the rank you **must** explain that the rank does not change when elementary row operations are performed. So the rank of the final matrix is the same as the rank of the initial matrix