Department of Mathematics & Statistics

Course	**Number**	**Section(s)**
Mathematics	203	All
Examination	**Date**	**Pages**
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Instructors	**Course Examiners**
D. Dryanov, A. Fenwick, Y. Guo, I. Gorelyshev, H. Kisilevsky, J. Lewis, Y. Ren	A. Atoyan & H. Proppe

Special Instructions

▷ **Only Sharp EL 531 or Casio FX 300 MS calculators are allowed.**

MARKS

[11] **1.** (a) Sketch the graph of the function $f(x) = (|x| - 1)^2$. (Suggestion: start from the graph of the standard parabola and use appropriate transformations).

(b) Suppose $f(x) = x + \dfrac{1}{x}$ and $g(x) = \dfrac{x+1}{x+2}$. Find $f \circ g$ and $f \circ f$.

(c) Find the inverse of the function $f(x) = \ln(e^x + 1)$. Determine the domain and range of f and f^{-1}.

[8] **2.** Evaluate the limits:

(a) $\displaystyle\lim_{x \to 1} \frac{\sqrt{x^2+48} - 7}{x - 1}$ (b) $\displaystyle\lim_{x \to -\infty} \frac{\sqrt{17x^5 + 4x^{12}}}{13x^5 + 8x^6}$

Do not use l'Hôpital's rule.

[11] **3.** (a) Consider the function $f(x) = \dfrac{|x-1|}{|x+1|} \cdot \dfrac{x+1}{x-1}$.

Calculate both one-sided limits at the point(s) where the function is undefined.

(b) Find the value(s) of x where the following function is discontinuous. For each such value, clearly state why $f(x)$ is discontinuous by using the definition of continuity at a point.

$$f(x) = \begin{cases} \dfrac{1}{x} & \text{if} \quad x < 1 \\ x & \text{if} \quad 1 \le x < 2 \\ 3x^2 - 4x & \text{if} \quad x \ge 2 \end{cases}$$

[15] **4.** Find derivatives of the functions (you don't have to simplify the answers):

(a) $f(x) = \dfrac{2\sqrt{x} - 3\sqrt[3]{x^2} + 4\sqrt[4]{x^3}}{x^{1/12}}$;

(b) $f(x) = e^{\sin x} + e^x \sin x$;

(c) $f(x) = \dfrac{\ln(x^2 + x + 1)}{x^2 + x + 1}$;

(d) $f(x) = \sec^2(\arctan(2x^2))$;

(e) $f(x) = (\sin 3x)^{x^2}$ (use logarithmic differentiation).

[10] **5.** (a) Use the definition of the derivative as the limit of the difference quotient to find $f'(x)$ if $f(x) = \dfrac{1}{x}$.

(c) Find the linear approximation of $g(x) = \tan x$ at $a = \pi/4$.

(d) Use this linear approximation to approximate $\tan x$ if $x = \dfrac{\pi}{4} - \dfrac{1}{10}$.

[9] **6.** (a) Verify that the point (3,1) belongs to the curve defined implicitly by the equation $x^2y - xy^2 = 6$, and find an equation of the tangent line to the curve at this point.

(b) Use l'Hôpital's rule to evaluate $\lim\limits_{x \to 0} \dfrac{e^x - e^{-x} - 2x}{x - \sin(x)}$.

[6] **7.** Let $f(x) = (x - 3)^{-1}$.

(a) Find the slope m of the secant line joining the points $(0, f(0))$ and $(4, f(4))$.

(b) Show there is no point $x = c$ in the interval $(0, 4)$ such that $f'(c) = m$. Why does this not contradict the Mean Value Theorem?

[12] **8.** (a) Boyle's Law states that the product of the pressure P of a gas, and its volume V, are related by the equation $PV = k$, where k is a constant. At an instant when the volume of a gas is 2400 cm^3, the pressure it exerts is 400 kPa. Find the rate of change of the pressure if the volume is decreasing at the rate of 300 cm^3/min.

(b) A box with a square base is to be constructed with a volume of 20 m^3. The material for the box costs $\$0.30/\text{m}^2$, and the material for the top costs $\$0.20/\text{m}^2$. Find the dimensions that minimize the cost of the box.

[8] **9.** Let $g(x) = (3x-4)^4(4x-3)^3$.

(a) Find $g'(x)$.

(b) Find the absolute maximum and minimum values of $g(x)$ on the interval $[0, 1]$.

[10] **10.** Consider the following function and its derivatives:

$$f(x) = \frac{4(1-x)}{x^2} \qquad f'(x) = \frac{4(x-2)}{x^3} \qquad f''(x) = \frac{-8(x-3)}{x^4}$$

(a) Find the domain and check for symmetry. Find all horizontal and vertical asymptotes (if any).

(b) Find the interval(s) where the function is increasing, interval(s) where the function is decreasing, and local maxima and minima (if any).

(c) Find the interval(s) where the function is concave upward, interval(s) where the function is concave downward and inflection point(s) (if any).

(d) Sketch the graph of the function.

[5] **Bonus Question**

Let $f(x)$ be a cubic of the form $f(x) = x^3 + ax^2 + bx + c$. Prove that f is increasing on $(-\infty, \infty)$ if $b > \dfrac{a^2}{3}$.