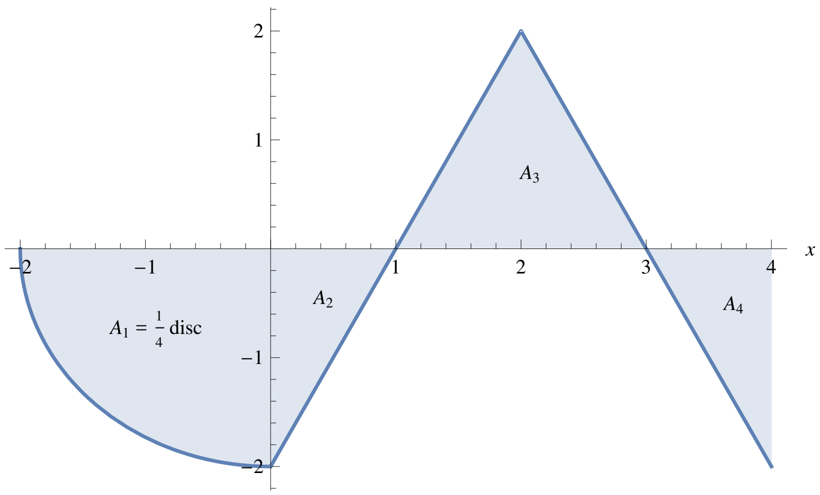
Final: Apr, 2016

Task 1 Integral in terms of areas; the Fundamental Theorem of Calculus

(a) $f(x) = \begin{cases} -\sqrt{4-x^2} & -2 \leqslant x \leqslant 0 \\ 2-2|x-2| & 0 < x \leqslant 4 \end{cases}$



$$\int_{-2}^{4} f(x)\,\mathrm{d}x = \text{net area} = -\frac{1}{4}\cdot \text{area of disc of radius 4} - \text{area of triangle with base 1 and height 2}$$
$$+ \text{area of triangle with base 2 and height 2} - \text{area of triangle with base 1 and height 2}$$
$$= -\frac{1}{4}\pi\cdot 2^2 - \frac{1}{2}\cdot 1\cdot 2 + \frac{1}{2}\cdot 2\cdot 2 - \frac{1}{2}\cdot 1\cdot 2 = -\pi.$$

(b) $F(x) = \displaystyle\int_x^{x^2} \mathrm{e}^{\sin \pi t}\mathrm{d}t$

A combination of the Chain Rule and the Fundamental Theorem of Calculus gives

$$\frac{\mathrm{d}}{\mathrm{d}x}\int_{a(x)}^{b(x)} f(t)\,\mathrm{d}t = b'(x)f\big(b(x)\big) - a'(x)f\big(a(x)\big)$$

Then

$$\frac{\mathrm{d}}{\mathrm{d}x}F(x) = 2x\mathrm{e}^{\sin \pi x^2} - \mathrm{e}^{\sin \pi x}.$$

A function $F(x)$ is increasing at $x=a$ when $F'(a)$ is positive,
and decreasing at $x=a$ when $F'(a)$ is negative.

$$\left.\frac{\mathrm{d}}{\mathrm{d}x}F(x)\right|_{x=1} = 2\mathrm{e}^{\sin \pi} - \mathrm{e}^{\sin \pi} = 2-1 = 1.$$

F is increasing at $x=1$.

Task 2 Indefinite integral with condition

$F'(x) = \dfrac{x^2+2x}{x^2+4}$, $F(0)=0$

Integration of a rational function.

The function $F'(x)$ is an improper fraction, so divide. Then we see that the denominator is an irreducible quadratic term; in the mist general case such integral is a sum of a logarithmic and an arctangent functions

$$\begin{aligned}
F(x) &= \int \frac{x^2+2x}{x^2+4}\,\mathrm{d}x = \int \left(1+\frac{2x-4}{x^2+4}\right)\mathrm{d}x = x + \int \frac{2x\,\mathrm{d}x}{x^2+4}\bigg|_{z=x^2+4} - 4\int \frac{\mathrm{d}x}{x^2+4}\bigg|_{y=x/2} \\
&= x + \int \frac{\mathrm{d}z}{z} - 2\int \frac{\mathrm{d}y}{y^2+1} = x + \ln z - 2\arctan y + C \\
&= x + \ln|x^2+4| - 2\arctan\frac{x}{2} + C,
\end{aligned}$$

the condition $F(0)=0$ allows to find the constant C

$$F(0) = \ln 4 + C \qquad \Rightarrow \qquad C = -\ln 4$$

$$F(x) = x + \ln\left|\frac{x^2}{4}+1\right| - 2\arctan\frac{x}{2}.$$

Task 3 Indefinite integrals

(a) — integration of a rational function. The function is a proper fraction, a product of two linear terms in the denominator; then the integral is a sum of logarithmic functions

$$\begin{aligned}
(a)\quad &\int \frac{13-x}{x^2-x-6}\,\mathrm{d}x = \int \frac{13-x}{(x+2)(x-3)}\,\mathrm{d}x = \int \frac{2}{x-3}\,\mathrm{d}x - \int \frac{3}{x+2}\,\mathrm{d}x \\
&= 2\ln|x-3| - 3\ln|x+2| + C;
\end{aligned}$$

(b) — Integration by parts (twice), because of $\ln x$ under the integral sign

$$\begin{aligned}
(b)\quad &\int x^{3/2}\ln^2(x)\,\mathrm{d}x = \int \ln^2(x)\,\mathrm{d}\left(\frac{2}{5}x^{5/2}\right) = \frac{2}{5}x^{5/2}\ln^2(x) - \frac{2}{5}\int x^{5/2}\cdot 2\ln x\cdot\frac{1}{x}\,\mathrm{d}x \\
&= \frac{2}{5}x^{5/2}\ln^2(x) - \frac{4}{5}\int x^{3/2}\ln x\cdot\,\mathrm{d}x = \frac{2}{5}x^{5/2}\ln^2(x) - \frac{4}{5}\cdot\frac{2}{5}x^{5/2}\ln x + \frac{4}{5}\cdot\frac{2}{5}\int x^{5/2}\cdot\frac{1}{x}\,\mathrm{d}x \\
&= \frac{2}{5}x^{5/2}\ln^2(x) - \frac{8}{25}x^{5/2}\ln x + \frac{8}{25}\int x^{3/2}\,\mathrm{d}x = \frac{2}{5}x^{5/2}\ln^2(x) - \frac{8}{25}x^{5/2}\ln x + \frac{16}{125}x^{5/2} + C \\
&= \frac{2}{125}\left(25\ln^2(x) - 20\ln x + 8\right)x^{5/2} + C.
\end{aligned}$$

Task 4 Definite integrals

Never combine finite and infinite integrals in one equality chain. Do not forget to change the limits in a definite integral.

(a) — the Substitution Rule to rid of 2^x

$$(a)\quad \int_0^1 \frac{2^x\,\mathrm{d}x}{4^x+1}\quad \left|\begin{array}{l} 2^x = u \\ 2^x \ln 2\,\mathrm{d}x = \mathrm{d}u \end{array}\right| = \frac{1}{\ln 2}\int_1^2 \frac{\mathrm{d}u}{u^2+1} = \frac{1}{\ln 2}\arctan u\Big|_1^2 = \frac{1}{\ln 2}\Big(\arctan 2 - \arctan 1\Big);$$

or

$$\int \frac{2^x\,\mathrm{d}x}{4^x+1}\quad \left|\begin{array}{l} 2^x = u \\ 2^x \ln 2\,\mathrm{d}x = \mathrm{d}u \end{array}\right| = \frac{1}{\ln 2}\int \frac{\mathrm{d}u}{u^2+1} = \frac{1}{\ln 2}\arctan u = \frac{1}{\ln 2}\arctan 2^x$$

$$\int_0^1 \frac{2^x\,\mathrm{d}x}{4^x+1} = \frac{1}{\ln 2}\Big(\arctan 2^1 - \arctan 2^0\Big) = \frac{1}{\ln 2}\Big(\arctan 2 - \arctan 1\Big).$$

(b) — Trigonometric Substitution $x = \sin\theta$ because of the term $\sqrt{a^2 - x^2}$.

$$(b)\quad \int_1^2 \sqrt{4-x^2}\,\mathrm{d}x \quad \left|\begin{array}{l} x = 2\sin\theta \\ \mathrm{d}x = 2\cos\theta\,\mathrm{d}\theta \end{array}\right| = \int_{\pi/6}^{\pi/2} \sqrt{4-4\sin^2\theta}\cdot 2\cos\theta\,\mathrm{d}\theta = \int_{\pi/6}^{\pi/2} 4\cos^2\theta\,\mathrm{d}\theta =$$

$$= 2\int_{\pi/6}^{\pi/2}(1+\cos 2\theta)\,\mathrm{d}\theta = 2\Big(\theta + \frac{1}{2}\sin 2\theta\Big)\Big|_{\pi/6}^{\pi/2} = 2\Big(\frac{\pi}{2} + \frac{1}{2}\sin\pi\Big) - 2\Big(\frac{\pi}{6} + \frac{1}{2}\sin\frac{\pi}{3}\Big)$$

$$= \frac{2\pi}{3} - \frac{\sqrt{3}}{2}$$

or

$$\int \sqrt{4-x^2}\,\mathrm{d}x \quad \left|\begin{array}{l} x = 2\sin\theta \\ \mathrm{d}x = 2\cos\theta\,\mathrm{d}\theta \end{array}\right| = \int \sqrt{4-4\sin^2\theta}\cdot 2\cos\theta\,\mathrm{d}\theta = \int 4\cos^2\theta\,\mathrm{d}\theta =$$

$$= 2\int(1+\cos 2\theta)\,\mathrm{d}\theta = 2\theta + \sin 2\theta + C = 2\arcsin\frac{x}{2} + 2\cdot\frac{x}{2}\cdot\frac{1}{2}\sqrt{4-x^2} + C$$

$$= 2\arcsin\frac{x}{2} + \frac{x}{2}\sqrt{4-x^2} + C,$$

$$\int_1^2 \sqrt{4-x^2}\,\mathrm{d}x = 2\arcsin\frac{x}{2} + \frac{x}{2}\sqrt{4-x^2}\Big|_1^2 = \pi - \frac{\pi}{3} - \frac{\sqrt{3}}{2}$$

Task 5 Improper integrals

Check if the integrand function is continuous at the limits and throughput the interval of integration. Replace the point of discontinuity by another number t approaching the singular point.

(a) — an integral with an infinite limit is always improper

$$(a)\quad \int_{\mathrm{e}}^{\infty} \frac{\mathrm{d}x}{x\ln x^2} = \lim_{t\to\infty}\int_{\mathrm{e}}^{t} \frac{\mathrm{d}x}{x\ln x^2} = \lim_{t\to\infty}\int_{\mathrm{e}}^{t} \frac{\mathrm{d}\ln x}{2\ln x} = \lim_{t\to\infty}\frac{1}{2}\ln(\ln x)\Big|_{\mathrm{e}}^{t} = \infty$$

or a different way of integration, these two results differ in a constant

$$\int_{\mathrm{e}}^{\infty} \frac{\mathrm{d}x}{x\ln x^2} = \lim_{t\to\infty}\int_{\mathrm{e}}^{t} \frac{x\,\mathrm{d}x}{x^2\ln x^2} = \lim_{t\to\infty}\frac{1}{2}\int_{\mathrm{e}}^{t} \frac{\mathrm{d}x^2}{x^2\ln x^2} = \lim_{t\to\infty}\frac{1}{2}\int_{\mathrm{e}}^{t} \frac{\mathrm{d}\ln x^2}{\ln x^2} = \lim_{t\to\infty}\frac{1}{2}\ln(\ln x^2)\Big|_{\mathrm{e}}^{t} = \infty$$

(b) — discontinuity is located at the root of the denominator: $x = 1$

$$(b) \quad \int_0^1 \frac{\mathrm{d}x}{(1-x)^{3/4}} = \lim_{t\to 1}\int_0^t (1-x)^{-3/4}\,\mathrm{d}x = \lim_{t\to 1}(-1)\cdot 4\cdot (1-x)^{1/4}\Big|_0^t = \lim_{t\to 1}\big(-4(1-t)^{1/4}+4\big) = 4.$$

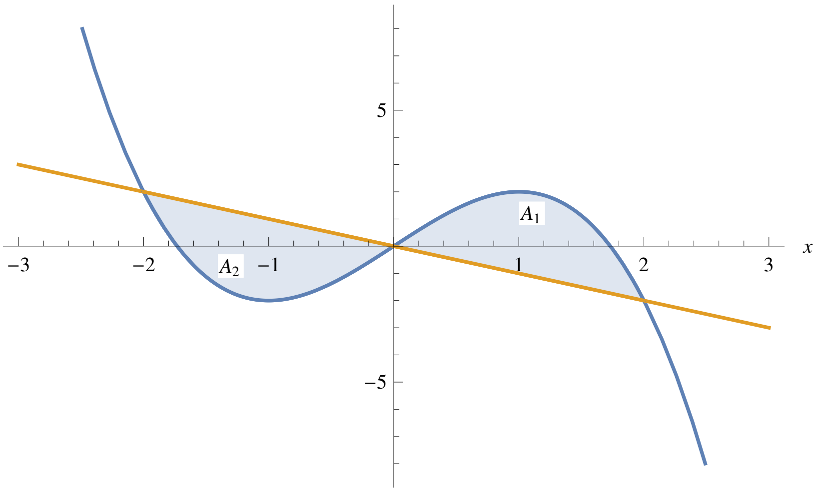
Task 6 Area, volume, average

(a) Area

Find the points of intersection $y = x(3-x^2)$ and $y = -x$

$$\begin{gathered} x(3-x^2) = -x \\ x^3 - 4x = 0 \\ x(x^2-4) = 0 \\ x = -2,\ 0,\ 2. \end{gathered}$$

The both functions are odd. Taking into account the symmetry property, the area is 2 times the integral on a half-interval of integration: $A = A_1 + A_2 = 2A_1$.



$$A = 2\int_0^2 (4x - x^3)\,\mathrm{d}x = 2\left(2x^2 - \frac{1}{4}x^4\right)\Big|_0^2 = 2\left(8 - \frac{16}{4}\right) = 8.$$

Without taking into account the symmetry property the two region should be considered separately, because they have different top and bottom functions.

(b) Volume

$y = \cos(2x)$ and x-axis on $[0, \pi/2]$. Volume of revolution about $y = -1$.

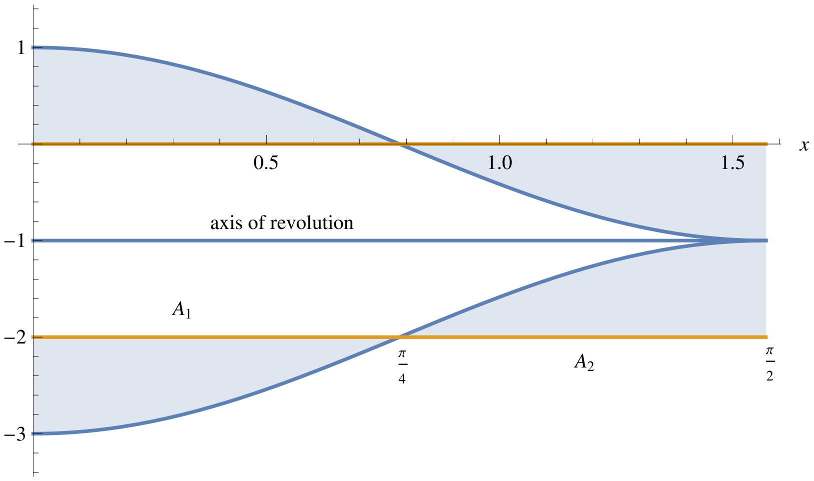


Fig. The middle section

$$\begin{aligned} A_1(x) &= \pi\big((1+\cos(2x))^2 - 1^2\big), && x \in [0, \pi/4] \\ A_2(x) &= \pi\big(1^2 - (1+\cos(2x))^2\big), && x \in [\pi/4, \pi/2] \end{aligned}$$

$$\begin{aligned} V &= \int_0^{\pi/4} A_1(x)\,\mathrm{d}x + \int_{\pi/4}^{\pi/2} A_2(x)\,\mathrm{d}x \\ &= \pi \int_0^{\pi/4} \big((1+\cos(2x))^2 - 1^2\big)\mathrm{d}x + \pi \int_{\pi/4}^{\pi/2} \big(1^2 - (1+\cos(2x))^2\big)\mathrm{d}x \\ &= \pi \int_0^{\pi/4} \big(2\cos(2x) + \cos(2x)^2\big)\mathrm{d}x - \pi \int_{\pi/4}^{\pi/2} \big(2\cos(2x) + \cos(2x)^2\big)\mathrm{d}x \\ &= \pi \sin(2x)\big|_0^{\pi/4} + \frac{\pi}{2}\int_0^{\pi/4} \big(1+\cos(4x)\big)\mathrm{d}x - \pi\sin(2x)\big|_{\pi/4}^{\pi/2} - \frac{\pi}{2}\int_{\pi/4}^{\pi/2}\big(1+\cos(4x)\big)\mathrm{d}x \\ &= \pi + \frac{\pi}{2}\left(x + \frac{1}{4}\sin(4x)\right)\Bigg|_0^{\pi/4} + \pi - \frac{\pi}{2}\left(x + \frac{1}{4}\sin(4x)\right)\Bigg|_{\pi/4}^{\pi/2} = 2\pi + \frac{\pi^2}{8} - \frac{\pi^2}{4} + \frac{\pi^2}{8} = 2\pi. \end{aligned}$$

(c) Average: $\quad f_{\mathbf{av}} = \dfrac{1}{b-a}\displaystyle\int_a^b f(x)\,\mathrm{d}x$

Average of $f(x) = \sec^4 x$ on $-\pi/4, \pi/4$

A trigonometric integral with even power of sec, then substitution $u = \tan x$.
Here a new variable was not introduced, so the limits remain unchanged

$$\begin{aligned} f_{\text{av}} &= \frac{1}{\pi/2}\int_{-\pi/4}^{\pi/4} \sec^4 x\,\mathrm{d}x = \frac{2}{\pi}\cdot 2\int_0^{\pi/4} \sec^4 x\,\mathrm{d}x = \frac{4}{\pi}\int_0^{\pi/4} \sec^2 x\,\mathrm{d}\tan x \\ &= \frac{4}{\pi}\int_0^{\pi/4} (1+\tan^2 x)\mathrm{d}\tan x = \frac{4}{\pi}\left(\tan x + \frac{1}{3}\tan^3 x\right)\Bigg|_0^{\pi/4} = \frac{4}{\pi}\left(1 + \frac{1}{3}\right) = \frac{16}{3\pi}. \end{aligned}$$

Task 7 Sequences

$$(a) \quad \lim_{n\to\infty} \frac{e^n - n^3}{3^n} < \lim_{n\to\infty} \left(\frac{\mathrm{e}}{3}\right)^n = 0$$

the latter is a geometric series with the common ratio $r = \mathrm{e}/3 \approx 0.9$.

$$(b) \quad \lim_{n\to\infty} \frac{(-1)^n n}{\sqrt{1+4n^2}} = \lim_{n\to\infty} (-1)^n \lim_{n\to\infty} \frac{n}{\sqrt{1+4n^2}} = \lim_{n\to\infty} (-1)^n \lim_{n\to\infty} \frac{1/n}{\sqrt{(1/n^2)+4}} = \frac{1}{2} \lim_{n\to\infty} (-1)^n,$$

the limit does not exist.

$$(c) \quad \lim_{n\to\infty} \left(\ln(n+2n^2) - \ln(2n+n^2)\right) = \lim_{n\to\infty} \ln \frac{n+2n^2}{2n+n^2} = \ln\left(\lim_{n\to\infty} \frac{n+2n^2}{2n+n^2}\right)$$

$$= \ln\left(\lim_{n\to\infty} \frac{1/n+2}{2/n+1}\right) = \ln 2,$$

we use here the continuity of the function $\ln$, thus $\lim_{n\to\infty} \ln a_n = \ln\left(\lim_{n\to\infty} a_n\right)$.

Task 8 Series

$$(a) \quad \sum_{n=1}^{\infty} \frac{n^{2/3}}{1+2n} \quad \text{compare with } p\text{-series} \quad \sum_{n=1}^{\infty} \frac{1}{n^{1/3}}, \quad p = \frac{1}{3}, \quad \text{the latter is divergent.}$$

By the Limit Comparison Test $\lim_{n\to\infty} \frac{n^{2/3}}{1+2n} \cdot n^{1/3} = \lim_{n\to\infty} \frac{n}{1+2n} = \lim_{n\to\infty} \frac{1}{1/n+2} = \frac{1}{2}$.

Therefore, the given series is divergent.

$$(b) \quad \sum_{n=1}^{\infty} (-1)^{n+1} \sin\frac{1}{n}.$$

By Alternating Series Test the given series is convergent.

Indeed, $\lim_{n\to\infty} \sin\frac{1}{n} = \sin\left(\lim_{n\to\infty} \frac{1}{n}\right) = 0$, and $\forall n \quad \sin\frac{1}{n+1} < \sin\frac{1}{n}$.

At the same time, the series $\sum_{n=1}^{\infty} \sin\frac{1}{n}$ can be compared with a divergent series $\sum_{n=1}^{\infty} \frac{1}{n}$.

So by the Limit Comparison Test $\lim_{n\to\infty} \frac{\sin 1/n}{1/n} = \lim_{x\to 0} \frac{\sin x}{x} = 1$,

this means that the corresponding series of absolute values is divergent.

Finally, the given series is conditionally convergent.

$$(c) \quad \sum_{n=1}^{\infty} (-1)^{n+1} \frac{2^{n+1}}{n!}$$

By the Ratio Test $\lim_{n\to\infty} \frac{2^{n+2}}{(n+1)!} \frac{n!}{2^{n+1}} = \lim_{n\to\infty} \frac{2}{n+1} = 0$.

Therefore, the given series is absolutely convergent.

Task 9 Radius and interval of convergence

$\displaystyle\sum_{n=1}^{\infty} \frac{(x+2)^{3n}}{n^2 8^n}$. By the Ratio Test the given series is absolutely convergent if

$$\lim_{n\to\infty}\left|\frac{(x+2)^{3(n+1)}}{(n+1)^2 8^{n+1}}\frac{n^2 8^n}{(x+2)^{3n}}\right| = \lim_{n\to\infty}\left|\frac{(x+2)^3 n^2}{8(n+1)^2}\right| = \lim_{n\to\infty}\left|\frac{(x+2)^3}{8(1+1/n)^2}\right| = \frac{|x+2|^3}{8} < 1$$
$$\Leftrightarrow \quad |x+2|^3 < 8 \quad \Leftrightarrow \quad |x+2| < 2.$$

The endpoints of the interval of convergence are -4 and 0

$$x=-4 \qquad \sum_{n=1}^{\infty}\frac{(-2)^{3n}}{n^2 8^n} = \sum_{n=1}^{\infty}\frac{(-1)^{3n}(2^3)^n}{n^2 8^n} = \sum_{n=1}^{\infty}\frac{(-1)^{3n}}{n^2}$$
$$x=0 \qquad \sum_{n=1}^{\infty}\frac{2^{3n}}{n^2 8^n} = \sum_{n=1}^{\infty}\frac{(2^3)^n}{n^2 8^n} = \sum_{n=1}^{\infty}\frac{1}{n^2}$$

The series at $x=0$ is the series of absolute values corresponding to the series at $x=-4$, and a p-series with $p=2$, which is convergent. Thus, the given power series is absolutely convergent at $x=-4$, 0.

The radius of convergence is 2, the interval of convergence is $[-4,0]$.

Task 10 Power series

(a) Maclaurin series for $f(x)=x^2\mathrm{e}^{3x}$, that is the power series about $x=0$

$$x^2\mathrm{e}^{3x} = x^2\sum_{n=0}^{\infty}\frac{(3x)^n}{n!} = \sum_{n=0}^{\infty}\frac{3^n x^{n+2}}{n!} = x^2+3x^3+\frac{3^2}{2!}x^4+\frac{3^3}{3!}x^5+\cdots = \sum_{n=2}^{\infty}\frac{3^{n-2}x^n}{(n-2)!}.$$

(b) $\displaystyle\sum_{n=0}^{\infty}\frac{(x^2+1)^n}{2^{n+1}}$

By the Ratio Test

$$\lim_{n\to\infty}\left|\frac{(x^2+1)^{n+1}2^{n+1}}{2^{n+2}(x^2+1)^n}\right| = \lim_{n\to\infty}\frac{x^2+1}{2} < 1 \qquad \Rightarrow \qquad x^2<1 \qquad \Leftrightarrow \qquad |x|<1$$

The radius of convergence is 1. Check the convergence at endpoints.

$$x=\pm 1 \qquad \sum_{n=0}^{\infty}\frac{2^n}{2^{n+1}} = \sum_{n=0}^{\infty}\frac{1}{2} = \infty \qquad \text{by the Test for Divergence: } \lim_{n\to\infty}|a_n| \neq 0.$$

Thus, on the interval $x\in(-1,1)$ the series is absolutely convergent, otherwise the series is divergent.

Bonus question

Apply the Integral Test.

$$F(x) = \int\frac{\mathrm{d}x}{x\ln x\big(\ln(\ln x)\big)^p} = \int\frac{\mathrm{d}\ln x}{\ln x\big(\ln(\ln x)\big)^p} = \int\frac{\mathrm{d}\ln(\ln x)}{\big(\ln(\ln x)\big)^p} = \frac{1}{1-p}\big(\ln(\ln x)\big)^{1-p}.$$

$F(\infty)$ is convergent when $p>1$, and approaches 0.

$$\sum_{n=5}^{\infty}\frac{\mathrm{d}n}{n\ln n\big(\ln(\ln n)\big)^p} < \int_5^{\infty}\frac{\mathrm{d}\ln(\ln x)}{\big(\ln(\ln x)\big)^p} = F(\infty)-F(5) = \frac{\big(\ln(\ln 5)\big)^{1-p}}{p-1} < \infty$$