ENGR-233 MOCK FINAL EXAM

Problem 1. Find the equation of the plane passing through the points $A(1,3,-2),\ B(3,-4,1),\ C(-1,2,1)$.

Answer: $9x+6y+8z-11=0$.

Problem 2. A planet of mass m moves around a star of mass M. The planet orbit is assumed to be a circle, the star being at its center.

(a) Suppose the orbit radius is R. Find the period of the planet (the duration of its "year").

Answer: The period $T=\frac{2\pi R^{3/2}}{G^{1/2}M^{1/2}}$.

(b) Suppose the speed of the planet is v. Find the radius R of the orbit.

Answer: $R=\frac{GM}{v^2}$.

(Hint: The force acting on the planet $\mathbf{F}=-GmM\frac{\mathbf{r}}{||\mathbf{r}||^3}$ where $\mathbf{r}$ is the vector connecting the star and the planet, and G is the gravity constant; the planet acceleration is defined by the 2-d Newton's Law $\mathbf{F}=m\mathbf{r}''$.)

Problem 3. (a) Find the divergence of the field $\mathbf{F}=(x^2-y^2)\mathbf{i}+xyz\mathbf{j}+(z^2-x^2)\mathbf{k}$ at the point $(1,2,3)$.

Answer: div $\mathbf{F}(1,2,3)=11$.

(b) Find the curvature of the curve defined by the parametric equations $x=e^t\cos t,\ y=e^t\sin t,\ z=e^t$ at the point $(1,0,1)$.

Answer: The point $(1,0,1)$ corresponds to the value of parameter $t=0$; the curvature at this point, $\kappa(0)=\sqrt{2}/3$.

Problem 4. Consider the plane velocity field

$$\mathbf{u}=\left(\frac{y}{(x-1)^2+y^2}+\frac{y}{(x+1)^2+y^2}\right)\mathbf{i}-\left(\frac{x-1}{(x-1)^2+y^2}+\frac{x+1}{(x+1)^2+y^2}\right)\mathbf{j}.$$

(a) Find div $\mathbf{u}$;

Answer: Outside the points $(-1,0)$ and $(1,0)$, $\operatorname{div}\mathbf{u}=0$.

(b) Find curl $\mathbf{u}$.

Answer: Outside the same points, $\operatorname{curl}\mathbf{u}=0$.

Problem 5. Find $\int_C \sin y\,dx+\cos x\,dy$ where C is a union of the line segments from $(0,0)$ to $(0,\pi/2)$ and from $(0,\pi/2)$ to $(\pi/2,\pi/2)$.

Answer: $\int_C \sin y\,dx+\cos x\,dy=\pi$.

Problem 6. (a) Find $\int\limits_C x^2y^2ds$ where C is the line $x = 2\cos t, y = 2\sin t,\ 0 \le t \le \pi/3$.

Answer: $4\pi/3 + \sqrt{3}/2$ (Sorry, there was a misprint in the contour: it is $x = 2\cos t, y = 2\sin t$, and not $x = 2\cos t, y = \sin t$ as it was printed.)

(b) Find the flux of the field $\mathbf{F} = (x^2 - y^2)\mathbf{i} + (y^2 - z^2)\mathbf{j} + (z^2 - x^2)\mathbf{k}$ through the surface of the sphere $x^2 + y^2 + z^2 = 4$ (use the Divergence Theorem).

Answer: The flux is zero.

Problem 7. (a) Find $\oint\limits_C \mathbf{F}\cdot ds$ if $\mathbf{F} = e^x\cos y\mathbf{i} - e^x\sin y\mathbf{j}$, and C is the circle $x^2 + (y-\pi)^2 = \pi^2$.

Answer: $\oint\limits_C \mathbf{F}\cdot ds = 0$;

(b) Find the work done by the force $\mathbf{F} = y\mathbf{i} - x\mathbf{j}$ along the circle $(x-1)^2 + y^2 = 1$ (use the Green's Theorem).

Answer: The work is equal to zero.

Problem 8. Evaluate $\iiint\limits_R xyzdV$ where R is a polyhedron bounded by the planes $x = 0,\ y = 0,\ z = 0,\ x + y + z = 1$.

Answer: $\iiint\limits_R xyzdV = 5/16 - 11/30 + 1/18 \approx 0.00139$.

Problem 9. (a) For the vector field $\mathbf{u} = \frac{x\mathbf{i}+y\mathbf{j}+z\mathbf{k}}{(x^2+y^2+z^2)^{3/2}}$, find div $\mathbf{u}$.

Answer: For $x^2 + y^2 + z^2 > 0$, $\quad \operatorname{div} \mathbf{u} = 0$.

(b) For the same field, find $\iint\limits_S \mathbf{u}\cdot\mathbf{n}ds$ where S is the sphere $x^2 + y^2 + z^2 = 1$, and $\mathbf{n}$ is the unit outer normal vector to S.

Answer: $\iint\limits_S \mathbf{u}\cdot\mathbf{n}ds = 4\pi$.

(c) Explain why the results of (a) and (b) don't contradict the Divergence Theorem.

Answer: The field $\mathbf{u}$ is not bounded near the origin $x = y = z = 0$, and therefore, the Divergence Theorem is not applicable.

Problem 10. Using cylindrical coordinates, find the volume of the body of revolution formed by rotation of the disk $(x-1)^2 + z^2 < 1$ around the z-axis (draw a picture).

Answer: $V = 2\pi^2$.