EMAT 233 - Solutions to Second Midterm Exam - Winter 2005

Problem 1. *The following questions refer to the function*

$$z = f(x, y) = \ln \sqrt{x^2 + y^2}, \quad (x, y) \neq (0, 0).$$

(i) Find the directional derivative of the function at the point $(1, 1)$ *in the direction of the vector* $2i + 3j$.

Answer: First we find the gradient $\vec{\nabla} f = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial x}\mathbf{j}$. Using the properties of the logarithm and the chain rule, we compute:

$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}(\frac{1}{2}\ln(x^2 + y^2)) = \frac{1}{2}\frac{1}{x^2 + y^2}(2x) = \frac{x}{x^2 + y^2}.$$

By symmetry,

$$\frac{\partial f}{\partial x} = \frac{y}{x^2 + y^2}.$$

At the point $(1, 1)$ we have

$$\vec{\nabla} f(1, 1) = \frac{1}{2}\mathbf{i} + \frac{1}{2}\mathbf{j}.$$

To find the directional derivative in the direction of $2i + 3j$, we must use a unit vector in that direction:

$$\mathbf{u} = \frac{2i + 3j}{\sqrt{2^2 + 3^2}} = \frac{2i + 3j}{\sqrt{13}}.$$

Then

$$D_{\mathbf{u}} f(1, 1) = \vec{\nabla} f(1, 1) \cdot \mathbf{u} = \frac{1}{2}\frac{2}{\sqrt{13}} + \frac{1}{2}\frac{3}{\sqrt{13}} = \frac{5}{2\sqrt{13}}.$$

(ii) *In which direction is the function increasing most rapidly at the point* $(1, 1)$*? Find the maximum value of the directional derivative at this point.*

Answer: Since $D_{\mathbf{u}} f = \vec{\nabla} f \cdot \mathbf{u} = \|\vec{\nabla} f\| \|\mathbf{u}\| \cos\theta$, where θ is the angle between $\vec{\nabla} f$ and $\mathbf{u}$, the directional derivative is maximized when $\theta = 0$, i.e. when $\mathbf{u}$ points in the same direction as $\vec{\nabla} f$. The maximum value of the directional derivative is then equal to $\|\vec{\nabla} f\|$.

In this case, at the point $(1, 1)$ the function increases most rapidly in the direction of $\vec{\nabla} f(1, 1) = \frac{1}{2}\mathbf{i} + \frac{1}{2}\mathbf{j}$, with a maximum rate of increase of

$$\|\vec{\nabla} f(1, 1)\| = \sqrt{\left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2} = \sqrt{\frac{1}{2}} = \frac{1}{\sqrt{2}}.$$

(Note that the direction of $\vec{\nabla} f$ is the direction pointing radially outward from the origin. Recall from Midterm 1 that the level curves of the function f are circles around the origin, and therefore the gradient must be perpendicular to them.)

(iii) *Find an equation of the tangent plane to the graph of the function at the point* $(1, 1)$.

Answer: The graph of the function is the surface given by the equation $z = f(x, y)$. This is the level surface $F = 0$ of the function

$$F(x, y, z) = f(x, y) - z.$$

To find the tangent place to the graph of the function, we use the normal vector to the surface given by

$$\vec{\nabla} F = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial x}\mathbf{j} - \mathbf{k},$$

which, when $x = 1, y = 1$ is, as in part (i),

$$\frac{1}{2}\mathbf{i} + \frac{1}{2}\mathbf{j} - \mathbf{k}.$$

The point on the graph of the function is given by $x = 1, y = 1$ and $z = \ln\sqrt{1+1} = \frac{\ln 2}{2}$. Therefore the equation of the plane through this point and perpendicular to $\vec{\nabla} F$ is

$$(\frac{1}{2}\mathbf{i} + \frac{1}{2}\mathbf{j} - \mathbf{k}) \cdot ((x-1)\mathbf{i} + (y-1)\mathbf{j} + (z - \frac{\ln 2}{2}) = 0.$$

This gives

$$\frac{1}{2}(x-1) + \frac{1}{2}(y-1) - (z - \frac{\ln 2}{2}) = 0.$$

Multiplying by 2 and simplifying results in the equation

$$x + y - 2z = 2 - \ln 2.$$

Problem 2. *Consider a force given by the vector field*

$$\mathbf{F}(x, y, z) = (10xe^z - y\sin x)\mathbf{i} + \cos x\mathbf{j} + 5x^2e^z\mathbf{k}$$

in the whole of 3*-dimensional space.*

(i) Compute the divergence and the curl of the vector field $\mathbf{F}$*.*

Answer:

$$\text{div}\mathbf{F} = \vec{\nabla} \cdot \mathbf{F} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z} = \frac{\partial}{\partial x}(10xe^z - y\sin x) + \frac{\partial}{\partial y}\cos x + \frac{\partial}{\partial z}5x^2e^z$$

$$= 10e^z - y\cos x + 5x^2e^z = e^z(10 + 5x^2) - y\cos x.$$

$$\text{curl } \mathbf{F} = \vec{\nabla} \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ (10xe^z - y\sin x) & \cos x & 5x^2e^z \end{vmatrix}$$

$$= (0-0)\mathbf{i} - (10xe^z - 10xe^z)\mathbf{j} + (-\sin x + \sin x)\mathbf{k} = \vec{0}.$$

(ii) *Explain why* $\mathbf{F}$ *is a conservative vector field and find a potential function* f *so that* $\vec{\nabla} f = \mathbf{F}$*.*

Answer: Since the curl of $\mathbf{F}$ is zero, it has a chance of being a conservative vector field. However, to know for sure, we must check its domain of definition. The components of this vector field are made up of functions (polynomials, sine, cosine, exponential) which are continuous and have continuous derivatives in all of 3-dimensional space. Therefore the domain does not have any "holes" and is a simply-connected domain. This means that we can use the condition $\text{curl}\mathbf{F} = \vec{0}$ to conclude that $\mathbf{F}$ is conservative.

Now we can try to find a potential function f for which $\vec{\nabla} f = \mathbf{F}$. Finding this function is another way of demonstrating that the vector field is conservative.

First, we take an anti-derivative of the first component of $\mathbf{F}$ with respect to x:

$$f(x,y,z) = \int P dx = \int (10xe^z - y\sin x)\, dx = 5x^2e^z + y\cos x + g(y,z).$$

Differentiating with respect to y and comparing to the second component of $\mathbf{F}$, we get

$$\cos x = Q = \frac{\partial f}{\partial y} = \cos x + \frac{\partial g}{\partial y}(y,z),$$

which shows that $\frac{\partial g}{\partial y}(y,z) = 0$ and therefore g does not depend on y, i.e. $g(y,z)$ is only a function of z, call it $h(z)$. Then we can write

$$f(x,y,z) = 5x^2e^z + y\cos x + h(z).$$

Finally, differentiating with respect to z and comparing to the third component of $\mathbf{F}$ gives

$$5x^2e^z = R = \frac{\partial f}{\partial z} = 5x^2e^z + h'(z),$$

so $h'(z) = 0$ and $h(z) = C$, a constant. Thus the most general form of the potential is

$$f(x,y,z) = 5x^2e^z + y\cos x + C.$$

(iii) *Find the work done by the force in moving a particle along the line from* $(\pi,0,0)$ *to* $(\pi,2,1)$.

Answer: There are several ways to solve this. Since $\mathbf{F}$ is conservative, the easiest way is to use the fact that the work done in moving a particle from point A to point B is the difference in potential between the two points, i.e.

$$W = f(B) - f(A) = 5x^2e^z + y\cos x + C\Big|_{(\pi,0,0)}^{(\pi,2,1)} = 5\pi^2e^1 + 2\cos\pi - 5\pi^2e^0 - 0 = 5\pi^2(e-1) - 2.$$

Alternatively, if we want to find the work using a line integral, i.e. $W = \int_C \mathbf{F}\cdot d\mathbf{r}$, then we need to parametrize the path C, which in this case is the line segment between $(\pi,0,0)$ to $(\pi,2,1)$. This can be done by taking the initial point plus the parameter times the vector between the two points, i.e.

$$\mathbf{r}(t) = \pi\mathbf{i} + t(2\mathbf{j} + \mathbf{k}) = \pi\mathbf{i} + 2t\mathbf{j} + t\mathbf{k}, \quad 0 \le t \le 1.$$

These parametrizations for x, y, z need to be plugged into $\mathbf{F}$ to get

$$\mathbf{F}(\pi, 2t, t) = (10\pi e^t - 2t\sin\pi)\mathbf{i} + \cos\pi\mathbf{j} + 5\pi^2e^t\mathbf{k} = 10\pi e^t\mathbf{i} - \mathbf{j} + 5\pi^2e^t\mathbf{k}.$$

Using the velocity vector $\mathbf{r}'(t) = 2\mathbf{j} + \mathbf{k}$, we can now compute

$$\int \mathbf{F} \cdot d\mathbf{r} = \int_0^1 \mathbf{F}(\pi, 2t, t) \cdot \mathbf{r}'(t)dt = \int_0^1 (-2 + 5\pi^2 e^t)dt = -2t + 5\pi^2 e^t\Big|_0^1 = 5\pi^2(e-1) - 2.$$

A third possibility is to compute the line integral using a different path, since in the case of a conservative vector field we have path independence. We may choose the two line segments $(\pi, 0, 0)$ to $(\pi, 2, 0)$, parallel to the y-axis, and $(\pi, 2, 0)$ to $(\pi, 2, 1)$, parallel to z-axis. This allows us to parametrize the first segment by y, with $x = \pi$, $z = 0$, and $dx = dz = 0$, and the second segment by z, with $x = \pi, y = 2$, and $dx = dy = 0$. Now we can write

$$W = \int_{C_1+C_2} Pdx + Qdy + Rdz = \int_0^2 \cos\pi dy + \int_0^1 5\pi^2 e^z dz = -2 + 5\pi^2 e^z\Big|_0^1 = 5\pi^2(e-1) - 2.$$

Problem 3(i). *(i) Let C be the closed curve in the xy-plane made up of the line segment from $(0,0)$ to $(1,0)$, the arc of the circle $x^2 + y^2 = 1$ from $(1,0)$ to $(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$, and the line segment from $(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$ to $(0,0)$, traversed in the counterclockwise direction. Compute the line integral*

$$\int_C x^2 dy - dx.$$

Answer: Write $C = C_1 + C_2 + C_3$. The line segment C_1 lies along the x axis and can be parametrize by x, $0 \le x \le 1$, with $y = 0$ and $dy = 0$. This gives

$$\int_{C_1} x^2 dy - dx = \int_0^1 (-dx) = -1.$$

For C_2, we can use the standard parametrization of the circle $x = \cos t$, $y = \sin t$, $dx = -\sin t dt$, $dy = \cos t dt$. Here $0 \le t \le \frac{\pi}{4}$. Thus

$$\begin{aligned}
\int_{C_2} x^2 dy - dx &= \int_0^{\frac{\pi}{4}} (\cos^3 t + \sin t)dt \\
&= \int_0^{\frac{\pi}{4}} (1 - \sin^2 t)\cos t dt - \cos t\Big|_0^{\frac{\pi}{4}} \\
&= \left(\sin t - \frac{1}{3}\sin^3 t\right)\Big|_0^{\frac{\pi}{4}} - \frac{\sqrt{2}}{2} + 1 \\
&= \frac{\sqrt{2}}{2} - \frac{1}{3}\frac{\sqrt{2}}{4} - \frac{\sqrt{2}}{2} + 1 \\
&= 1 - \frac{\sqrt{2}}{12}.
\end{aligned}$$

Alternatively, we could have also written $x = \sqrt{1 - y^2}$ to get

$$\int_{C_2} x^2 dy - dx = \int_0^{\frac{1}{\sqrt{2}}} (1 - y^2)dy - \int_1^{\frac{1}{\sqrt{2}}} dx = \left(y - \frac{y^3}{3}\right)\Big|_0^{\frac{1}{\sqrt{2}}} - (\frac{1}{\sqrt{2}} - 1) = 1 - \frac{\sqrt{2}}{12}.$$

Finally, for C_3, we can use $x = y$ as the parameter and get

$$\int_{C_3} x^2 dy - dx = \int_{\frac{1}{\sqrt{2}}}^0 (x^2 - 1)dx = \left(\frac{x^3}{3} - x\right)\Big|_{\frac{1}{\sqrt{2}}}^0 = -\frac{\sqrt{2}}{12} + \frac{\sqrt{2}}{2} = \frac{5\sqrt{2}}{12}.$$

Combining gives

$$\int_C x^2dy - dx = \frac{4\sqrt{2}}{12} = \frac{\sqrt{2}}{3}.$$

(ii) *Let R be the region bounded by the curve C, namely a $\frac{1}{8}$-th sector of the disk of radius 1 centered at the origin. Denote by $(\overline{x}, \overline{y})$ the coordinates of the center of mass (centroid) of the lamina that corresponds to the region R, assuming constant density ρ. SET UP THE DOUBLE INTEGRALS corresponding to $\overline{x}$, $\overline{y}$ in RECTANGULAR (x and y) coordinates (with whichever order of integration you choose) and POLAR coordinates. The limits of integration and all other ingredients in the integrals must be given explicitly. You may use the fact that the area of the sector is $\frac{1}{8}$-th the area of the disk of radius 1, i.e. $A = \frac{\pi}{8}$.*

Answer: Using rectangular coordinates, it is easiest to think of R as a type II region, with $0 \le y \le \frac{1}{\sqrt{2}}$ and $y \le x \le \sqrt{1-y^2}$. Recall that the mass, given constant density ρ, is just $m = \rho A$, where A is the area. Therefore we can cancel ρ out of the integral to get

$$\overline{x} = \frac{1}{m}\iint x\rho dA = \frac{1}{A}\iint xdA = \frac{8}{\pi}\int_0^{\frac{1}{\sqrt{2}}}\int_y^{\sqrt{1-y^2}} xdxdy.$$

Similarly,

$$\overline{y} = \frac{1}{m}\iint y\rho dA = \frac{8}{\pi}\int_0^{\frac{1}{\sqrt{2}}}\int_y^{\sqrt{1-y^2}} ydxdy.$$

In polar coordinates the limits of integration are just $0 \le \theta \le \pi/4$ and $0 \le r \le 1$, with $x = r\cos\theta$ and $y = r\sin\theta$. This gives

$$\overline{x} = \frac{1}{m}\iint x\rho dA = = \frac{8}{\pi}\int_0^{\frac{\pi}{4}}\int_0^1 r\cos\theta r dr d\theta$$

and

$$\overline{y} = \frac{1}{m}\iint y\rho dA = \frac{8}{\pi}\int_0^{\frac{\pi}{4}}\int_0^1 r\sin\theta r dr d\theta.$$

(iii) *Find the center of mass $(\overline{x}, \overline{y})$ in part (ii) by evaluating whichever integrals you choose.*

Answer: Since the limits of integration are simpler, let's evaluate the integrals in polar coordinates:

$$\overline{x} = \frac{8}{\pi}\int_0^{\frac{\pi}{4}}\int_0^1 r^2dr\ \cos\theta\ d\theta = \frac{8}{\pi}\int_0^{\frac{\pi}{4}} \frac{r^3}{3}\Big|_0^1 \cos\theta\ d\theta = \frac{8}{\pi}\frac{1}{3}\sin\theta\Big|_0^{\frac{\pi}{4}} = \frac{8}{\pi}\frac{\sqrt{2}}{6} = \frac{4\sqrt{2}}{3\pi}$$

and

$$\overline{y} = \frac{8}{\pi}\int_0^{\frac{\pi}{4}}\int_0^1 r^2dr\ \sin\theta\ d\theta = \frac{8}{\pi}\frac{1}{3}(-\cos\theta)\Big|_0^{\frac{\pi}{4}} = \frac{8}{3\pi}(1 - \frac{\sqrt{2}}{2}) = \frac{8-4\sqrt{2}}{3\pi}.$$

The corresponding approximate numerical values are $\overline{x} \approx 0.6, \overline{y} \approx 0.25$.