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Midterm test Math 203/2-2014

1. (a) If $f(x) = \sqrt{3-x}$ and $g(x) = x^2 - 2$ evaluate and determine the domain of
   i. $(f \circ g)(x)$ and
   ii. $(g \circ f)(x)$.

   (b) Determine the inverse function $f^{-1}(x)$ if $f(x) = \dfrac{2}{e^x + 1}$ and determine the domains $\mathbf{D}_f$ and $\mathbf{D}_{f^{-1}}$ and the ranges $\mathbf{R}_f$ and $\mathbf{R}_{f^{-1}}$.

2. Calculate the following limits, or explain why they do not exist:

   (a) $\displaystyle\lim_{x\to\infty} \frac{(10+x)\sqrt{x^6+4x^3}}{1+4x^2+2x^4}$;

   (b) $\displaystyle\lim_{x\to 2} \frac{6x-12}{|x-2|}$.

3. Let $y = \dfrac{3x^2-3}{x^2-2x-3}$, write the equations of the

   (a) vertical asymptotes and
   (b) horizontal asymptotes.

4. Calculate the second derivative $f''(x)$ of $f(x) = \sin\left(x^2-1\right)$.

5. Calculate the derivatives of (*please, do not simplify*):

   (a) $f(x) = x^{5/2}x^{-2}\tan x$;
   (b) $f(x) = \left(x^3 - 3x\right)\cos x + \sin^2 x$;
   (c) $f(x) = \dfrac{e^{2x}}{e^{-2x}+1} + \sec x$;
   (d) $f(x) = \cos\left(x\sqrt{x^3+5}\right)$.

6. For $f(x) = \sqrt{2x+5}$

   (a) use the definition of derivative (no rules) to calculate $f'(x)$;
   (b) write an equation of the tangent line to $y = f(x)$ at the point $\mathbf{A}(2, ?)$.

Bonus Consider $f(x) = \begin{cases} x+1 & if \quad x \le -1 \\ ax^2 - 1 & if \quad x > 1 \end{cases}$. Determine the value of a that makes $f(x)$ differentiable everywhere, or explain why it is impossible.

Solutions for the Midterm test Math 203/2-2014

1. (a) If $f(x) = \sqrt{3-x}$ and $g(x) = x^2 - 2$ evaluate and determine the domain of

   i. $(f \circ g)(x) = f\left(x^2-2\right) = \sqrt{3-(x^2-2)} = \sqrt{5-x^2}$ and with domain $\mathbf{D}_{f\circ g} = \left[-\sqrt{5}, \sqrt{5}\right]$;

   ii. $(g \circ f)(x) = g\left(\sqrt{3-x}\right) = \left(\sqrt{3-x}^{\,2} - 2\right) \overset{x\leq 3}{\equiv} 1 - x$ and with domain $\mathbf{D}_{g\circ f} = (-\infty, 3]$.

   (b) Determine the inverse function $f^{-1}(x)$ if $f(x) = \dfrac{2}{e^x+1} \to x = \dfrac{2}{e^y+1} \to xe^y + x = 2 \to e^y = \dfrac{2-x}{x} \to y = \ln\left(\dfrac{2}{x} - 1\right)$;
   and determine the: $\mathbf{D}_f = (-\infty, \infty) = \mathbf{R}_{f^{-1}}$ and $\mathbf{D}_{f^{-1}} = (0,2) = \mathbf{R}_f$
   .

2. Calculate the following limits, or explain why they do not exist:

   (a) $$\lim_{x\to\infty} \frac{(10+x)\sqrt{x^6+4x^3}}{1+4x^2+2x^4} = \lim_{x\to\infty} \frac{x^4\left(\dfrac{10}{x}+1\right)\sqrt{1+\dfrac{4}{x^3}}}{x^4\left(\dfrac{1}{x^4}+\dfrac{4}{x^2}+2\right)} = \frac{1}{2};$$

   (b) $$\lim_{x\to 2} \frac{6x-12}{|x-2|} = \lim_{x\to 2} \begin{cases} \displaystyle\lim_{x\to 2^+} \frac{6(x-2)}{x-2} = 6 \\ \displaystyle\lim_{x\to 2^-} \frac{6(x-2)}{-(x-2)} = -6 \end{cases} \to \lim_{x\to 2} \frac{6x-12}{|x-2|} \text{ does}$$
   not exist

3. Let $y = \dfrac{3x^2-3}{x^2-2x-3} = \dfrac{3(x-1)(x+1)}{(x-3)(x+1)} = \dfrac{3(x-1)}{(x-3)}$ if $x \neq -1$, therefore,

   (a) vertical asymptote is $x = 3$ as $\displaystyle\lim_{x\to 3^+} \frac{3(x-1)}{(x-3)} = \infty$ and $\displaystyle\lim_{x\to 3^-} \frac{3(x-1)}{(x-3)} =$
   $-\infty$

   (b) horizontal asymptotes is $y = 3$ as $\displaystyle\lim_{x\to\infty} \frac{3x\left(1-\frac{1}{x}\right)}{x\left(1-\frac{3}{x}\right)} = \lim_{x\to-\infty} \frac{3x\left(1-\frac{1}{x}\right)}{x\left(1-\frac{3}{x}\right)} =$
   3

4. Calculate the second derivative $f''(x)$ of $f(x) = \sin\left(x^2-1\right)$:
   $f'(x) = 2x\cos\left(x^2-1\right)$ and $f''(x) = 2\cos\left(x^2-1\right) - 4x^2\sin\left(x^2-1\right)$

5. Calculate the derivatives of (*please, do not simplify*):

(a) $f(x) = x^{5/2}x^{-2}\tan x = x^{1/2}\tan x \rightarrow f'(x) = \sqrt{x}\left(\tan^2 x + 1\right) + \frac{1}{2\sqrt{x}}\tan x$

(b) $f(x) = \left(x^3 - 3x\right)\cos x + \sin^2 x \rightarrow f'(x) = \cos x\left(3x^2 - 3\right) + \sin x\left(3x - x^3\right) + 2\cos x \sin x$

(c) $f(x) = \dfrac{e^{2x}}{e^{-2x}+1} + \sec x \rightarrow f'(x) = \dfrac{\sin x}{\cos^2 x} + 2\dfrac{e^{2x}\left(e^{-2x}+1\right)+1}{\left(e^{-2x}+1\right)^2}$

(d) $f(x) = \cos\left(x\sqrt{x^3+5}\right) \rightarrow f'(x) = -\left(\sin x\sqrt{x^3+5}\right)\left(\dfrac{3x^3}{2\sqrt{x^3+5}} + \sqrt{x^3+5}\right)$
.

6. For $f(x) = \sqrt{2x+5}$

(a) use the definition of derivative (no rules) to calculate $f'(x) = \lim\limits_{h\to 0}\dfrac{\sqrt{2(x+h)+5} - \sqrt{2x+5}}{h} \times \dfrac{\sqrt{2(x+h)+5} + \sqrt{2x+5}}{\sqrt{2(x+h)+5} + \sqrt{2x+5}} = \lim\limits_{h\to 0}\dfrac{2\cancel{h}}{\cancel{h}\left(\sqrt{2(x+h)+5} + \sqrt{2x+5}\right)} = \dfrac{1}{\sqrt{2x+5}}$

(b) write an equation of the tangent line to $y = f(x)$ at the point $\mathbf{A}(2, f(2)) = \mathbf{A}(2,3)$ wit slope $m = \dfrac{1}{3} \rightarrow$ equation of the tangent line is $y = m(x-a) + f(a) = \dfrac{x-2}{3} + 3$.

Bonus Consider $f(x) = \begin{cases} x+1 & if \quad x \le -1 \\ ax^2 - 1 & if \quad x > 1 \end{cases}$. Determine the value of a that makes $f(x)$ differentiable everywhere, or explain why it is impossible.

(a) to make it continuous: $f(-1) = 0 = a - 1 \rightarrow a = 1$

(b) to make it differentiable: $f'(-1) = 1 = -2 \rightarrow$ that is impossible, therefore it cannot be made differentiable.