1. Is the vector field $\vec{F}(x, y, z) = \langle y + z, x + z, x + y \rangle$ conservative? If no, justify your answer. If yes, justify your answer and evaluate the line integral

$$\int_C \vec{F} \cdot d\vec{r},$$

where C is any smooth path from $(0, 0, 0)$ to $(1, 2, 3)$.

[10 marks]

This is very similar to the following problems: WeBWork Assignment 5, Problems 1, 2, 5, 9; class examples 27, 28, 29, 30, 31.

Solution. Yes, $\vec{F}$ is conservative since its domain is all of $\mathcal{R}^2$, which is simply connected, and

$$\operatorname{curl}(\vec{F}) = \begin{vmatrix} \vec{\imath} & \vec{\jmath} & \vec{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ y + z & x + z & x + y \end{vmatrix} = \langle 1 - 1, -(1 - 1), 1 - 1 \rangle \equiv \vec{0}.$$

Since $\vec{F}$ is conservative, it is independent of path and for the evaluation of the line integral we may take C to be the straight line

$$\vec{r}(t) = t\langle 1, 2, 3 \rangle, \quad 0 \le t \le 1.$$

We have

$$\vec{F}(\vec{r}(t)) = t\langle 5, 4, 3 \rangle, \quad \vec{r}'(t) = \langle 1, 2, 3 \rangle, \quad \vec{F}(\vec{r}(t)) \cdot \vec{r}'(t) = 22t.$$

Thus,

$$\int_C \vec{F} \cdot \mathrm{d}\vec{r} = \int_0^1 22t \, \mathrm{d}t = 11.$$

Alternative solution. Yes, $\vec{F}$ is conservative since it is equal to $\nabla\phi$ throughout its domain (all of $\mathcal{R}^2$), for some scalar field ϕ, as we now show.

We seek $\phi = \phi(x, y, z)$ such that

$$\phi_x = y + z, \quad \phi_y = x + z, \quad \phi_z = x + y.$$

Now, $\phi_z = x + y$ implies

$$\phi(x, y, z) = xz + yz + f(x, y).$$

Next, $\phi_y = x + z$ implies

$$z + f_y(x, y) = x + z \qquad \Longrightarrow \qquad f(x, y) = xy + g(x).$$

Combining yields

$$\phi(x, y, z) = xz + yz + g(x).$$

Finally, $\phi_x = y + z$ implies

$$z + g'(x) = y + z \qquad \Longrightarrow \qquad g(x) = xy + c,$$

c a constant. Taking $c = 0$, we obtain

$$\phi(x, y, z) = xz + yz + xy.$$

By the fundamental theorem for line integrals, we have

$$\int_C \vec{F} \cdot \mathrm{d}\vec{r} = \int_C (\nabla\phi) \cdot \mathrm{d}\vec{r} = \phi(1, 2, 3) - \phi(0, 0, 0) = 11.$$

2. Evaluate $\displaystyle\int_0^1 \int_x^1 2\cos(y^2)\,\mathrm{d}y\,\mathrm{d}x$.

[10 marks]

This is very similar to the following problems: WeBWork Assignment 6, Problems 5 and 6; class examples 33(b) and 36.

Solution. Let

$$D = \{(x, y) : 0 \leq x \leq 1, x \leq y \leq 1\}.$$

One may verify [a picture helps] that

$$D = \{(x, y) : 0 \le y \le 1, 0 \le x \le y\}.$$

Thus, applying Fubini's theorem twice,

$$\begin{aligned}
\int_0^1 \int_x^1 2\cos(y^2)\,\mathrm{d}y\,\mathrm{d}x &= \iint_D 2\cos(y^2)\,\mathrm{d}A \\
&= \int_0^1 \int_0^y 2\cos(y^2)\,\mathrm{d}x\,\mathrm{d}y \\
&= \int_0^1 \Big[2x\cos(y^2)\Big]_{x=0}^{x=y}\,\mathrm{d}y \\
&= \int_0^1 2y\cos(y^2)\,\mathrm{d}x \\
&= \sin(y^2)\Big|_0^1 \\
&= \sin(1).
\end{aligned}$$

We have $\sin(1) = 0.84147\ldots$, but it is not necessary to give a decimal approximation to your final answer.

The key here is to change order of integration. The function $2\cos(y^2)$ does not have an antiderivative that can be expressed in terms of elementary functions. Any argument purporting to lead to an evaluation of the double integral, that does not involve changing order of integration, is necessarily flawed.

3. Evaluate $\iint_S yz\,\mathrm{d}S$, where S is the part of the plane $x + y + z = 2$ that lies in the first octant.

[10 marks]

This is very similar to the following problems: Quiz 3; class example 56.

Solution. The projection of S onto the xy-plane ($z = 0$) is

$$D = \{(x, y) : x + y \le 2, x \ge 0, y \ge 0\}.$$

(If we set $z = 0$ in the equation defining S, we get the line $x + y = 2$, which is part of the boundary of D; we are also told that S lies in the first octant, whence $x \geq 0$ and $y \geq 0$.) We may write D as a type I region [a picture helps]:

$$D = \{(x, y) : 0 \leq x \leq 2, 0 \leq y \leq 2 - x\}.$$

Now, re-writing the equation for S as $z = 2 - x - y$, we have

$$yz = y(2 - x - y),$$

and

$$\sqrt{z_x^2 + z_y^2 + 1} = \sqrt{(-1)^2 + (-1)^2 + 1} = \sqrt{3}, \quad \text{i.e.} \quad \mathrm{d}S = \sqrt{3}\,\mathrm{d}A.$$

Thus,

$$\begin{aligned}
\iint_S yz\,\mathrm{d}S &= \iint_D y(2 - x - y)\sqrt{3}\,\mathrm{d}A \\
&= \sqrt{3}\int_0^2 \int_0^{2-x} y(2 - x - y)\,\mathrm{d}y\,\mathrm{d}x \\
&= \sqrt{3}\int_0^2 \left[(2 - x)\frac{y^2}{2} - \frac{y^3}{3}\right]_{y=0}^{y=2-x} \mathrm{d}x \\
&= \sqrt{3}\int_0^2 (2 - x)^3\left(\frac{1}{2} - \frac{1}{3}\right)\mathrm{d}x \\
&= \frac{\sqrt{3}}{6}\left[-\frac{(2 - x)^4}{4}\right]_0^2 \\
&= \frac{2\sqrt{3}}{3}.
\end{aligned}$$

We have $2\sqrt{3}/3 = 1.15470\ldots$, but it is not necessary to give a decimal approximation to your final answer. Equivalent final answers such as $2/\sqrt{3}$ and $4\sqrt{3}/6$ are acceptable.

Alternative solution. The projection of S onto the xy-plane ($z = 0$) is

$$D = \{(x, y) : x + y \leq 2, x \geq 0, y \geq 0\}.$$

(If we set $z = 0$ in the equation defining S, we get the line $x + y = 2$, which is part of the boundary of D; we are also told that S lies in the first octant, whence $x \geq 0$ and $y \geq 0$.) We may write D as a type I region [a picture helps]:

$$D = \{(x, y) : 0 \leq x \leq 2, 0 \leq y \leq 2 - x\}.$$

We may parameterize S by

$$\vec{r}(x,y) = \langle x, y, 2-x-y \rangle \quad (x,y) \in D.$$

Now,

$$\vec{r}_x = \langle 1, 0, -1 \rangle, \quad \vec{r}_y = \langle 0, 1, -1 \rangle, \quad \vec{r}_x \times \vec{r}_y = \langle 1, 1, 1 \rangle, \quad \|\vec{r}_x \times \vec{r}_y\| = \sqrt{3}.$$

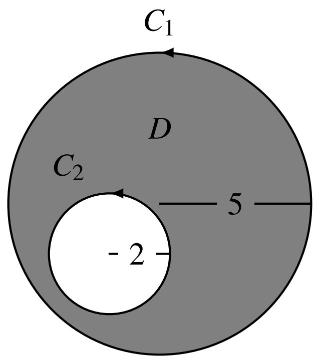
Thus,

$$\begin{aligned}
\iint_S yz \, \mathrm{d}S &= \iint_D y(2-x-y)\|\vec{r}_x \times \vec{r}_y\| \, \mathrm{d}A \\
&= \sqrt{3} \int_0^2 \int_0^{2-x} y(2-x-y) \, \mathrm{d}y \, \mathrm{d}x \\
&= \sqrt{3} \int_0^2 \left[(2-x)\frac{y^2}{2} - \frac{y^3}{3} \right]_{y=0}^{y=2-x} \mathrm{d}x \\
&= \sqrt{3} \int_0^2 (2-x)^3 \left(\frac{1}{2} - \frac{1}{3} \right) \mathrm{d}x \\
&= \frac{\sqrt{3}}{6} \left[-\frac{(2-x)^4}{4} \right]_0^2 \\
&= \frac{2\sqrt{3}}{3}.
\end{aligned}$$

We have $2\sqrt{3}/3 = 1.15470\ldots$, but it is not necessary to give a decimal approximation to your final answer. Equivalent final answers such as $2/\sqrt{3}$ and $4\sqrt{3}/6$ are acceptable.

Alternative solution. One could of course project S onto the yz-plane or the xz-plane: different solutions are tantamount to a change in notation.

4. Let D be a region in the xy-plane consisting of a disk of radius 5 with a disk of radius 2 removed. Let C_1 be the boundary of the larger disk, oriented counterclockwise, and let C_2 be the boundary of the smaller disk, also oriented counterclockwise.



Let $\vec{F}(x, y) = \langle P(x, y), Q(x, y)\rangle$ be a vector field whose component functions P and Q have continuous partial derivatives on an open region containing D.

Suppose that

$$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 10$$

throughout D, and that

$$\oint_{C_2} \vec{F} \cdot \, \mathrm{d}\vec{r} = 13\pi.$$

Determine $\oint_{C_1} \vec{F} \cdot \, \mathrm{d}\vec{r}$.

[10 marks]

This is very similar to the following problems: WeBWork Assignment 7, Problem 15; class example 49.

Solution. Let $C = C_1 \cup (-C_2)$. Then C is the boundary of D and is positively oriented. We have

$$\int_C \vec{F} \cdot d\vec{r} = \int_{C_1} \vec{F} \cdot d\vec{r} + \int_{-C_2} \vec{F} \cdot d\vec{r} = \int_{C_1} \vec{F} \cdot d\vec{r} - \int_{C_2} \vec{F} \cdot d\vec{r},$$

and so

$$\int_{C_1} \vec{F} \cdot d\vec{r} = \int_{C} \vec{F} \cdot d\vec{r} + \int_{C_2} \vec{F} \cdot d\vec{r} = \int_{C} \vec{F} \cdot d\vec{r} + 13\pi. \tag{1}$$

By Green's theorem,

$$\int_C \vec{F} \cdot d\vec{r} = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \mathrm{d}A = \iint_D 10 \, \mathrm{d}A = 10 \iint_D \mathrm{d}A = 210\pi, \tag{2}$$

since the last integral is the area of D, namely $\pi(5^2 - 2^2) = 21\pi$. Combining (1) and (2) yields

$$\int_C \vec{F} \cdot d\vec{r} = 13\pi + 210\pi = 223\pi.$$

We have $223\pi = 700.57516\ldots$, but it is not necessary to give a decimal approximation to your final answer.