CONCORDIA UNIVERSITY DEPARTMENT OF COMPUTER SCIENCE AND SOFTWARE ENGINEERING

COMP/MATH 339 Combinatorics FALL 2020

Final Exam (December 21, 9:00-12:00)

If doubt exists as to the interpretation of any question, the student is urged to make a clear statement of any assumptions made.
Show all the steps of your calculations and justify your answers (except Questions 1 and 2).
Submit your solutions as a PDF file (maximum 10-12 pages) on Moodle, under Final Exam Exam
Every page of your solution file must contain your name and student ID
Total marks 40

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1. (2.5 points) True or False? No justification is required.

   i) If graph G is connected and planar then any subgraph of G is connected and planar.
   ii) Consider any bipartite graph $G = (V, E)$ (V is partitioned as $V = X \cup Y$) with 62 edges, and $|X| = 7$, $|Y| = 9$. Then there is a complete matching in G.
   iii) If the maxdegree in graph G is 7 then G is 7 colorable.
   iv) The complete bipartite graph $K_{7,10}$ contains an Euler trail.
   v) There is a vertex of degree at most 5 in any planar graph.

2. (2 points) Short answer questions. Just give your answer, no explanation is required.

   a) Give the chromatic number of the cycle graph C_n on n vertices.
   b) Give the chromatic number of the wheel graph W_n on n vertices ($n-1$ vertices on the cycle and one vertex connected to all $n-1$ vertices of the cycle).
   c) Give the chromatic number of the complete graph K_n on n vertices.
   d) Give the chromatic number of the n-dimensional hypercube graph Q_n on n vertices.

3. a) (4 points) Evaluate (Show your work): $\sum_{k=0}^{30} \binom{60}{2k}$

   b) (4 points) Give a combinatorial proof for $\binom{m}{r}\binom{m-r}{n} = \binom{n+r}{r}\binom{m}{n+r}$

4. (4 points) Define the integer sequence $a_0, a_1, a_2, \ldots$ recursively by

   1) $a_0 = 1$, $a_1 = 2$; and

   2) For $n \geq 2$, $a_n = 2a_{n-1} + a_{n-2}$

   Prove by induction that $a_n \leq (\sqrt{6})^n$ for all $n \geq 0$.

5. (3 points) Let $G = (V, E)$ be a loop-free connected unicyclic graph (contains one single cycle). If G has five vertices of degree 2, two vertices of degree 3, and three vertices of degree 4, how many pendant vertices (leaves) does it have. (Assuming that maximum degree in G is 4).

6. (4 points) Find all the nonisomorphic connected loop-free unicyclic (containing one single cycle) graphs on 6 vertices. Draw all of them.

7. a) (2 points) Let $G = (V, E)$ be a loop-free connected planar graph with $|V| \geq 12$. Prove that its complement must be nonplanar.

   b) (0.5 point) Give a counterexample to part a) for $|V| = 5$. In other words give an example of a loop-free connected planar graph G with $|V| = 5$ such that its complement is also planar

8. a) (2 points) Let $G = (V, E)$ be a loop-free undirected graph with $|V| = n \geq 3$, and $\deg(x) + \deg(y) \geq n - 1$ for all nonadjacent vertices x, $y \in V$. Prove that there is a path of length at most 2 between each pair of vertices of G.

   b) (3 points) Prove that any undirected, loop-free graph on $n > 1$ vertices with at least $\binom{n-1}{2} + 1$ edges is connected. For any $n > 1$ give an example of a disconnected graph on n vertices and $\binom{n-1}{2}$ edges.

9. (4 points) Let $G = (V, E)$ be a loop-free undirected connected graph and $|V| = |E|$. Prove that G contains exactly one cycle.

10. (5 points) Prove that any tree $T = (V, E)$ has $2 + \sum_{u \in V, deg(u) \geq 3} (\deg(u) - 2)$ leaves.