Solutions to the Final Exam - Winter 2007

(1)

(A) $\dfrac{dy}{dx} = \dfrac{y^2 e^x}{1+e^{2x}}$ is a separable ODE.

$$\frac{1}{y^2}\,dy = \frac{e^x\,dx}{1+e^{2x}}$$

$$\int \frac{1}{y^2}\,dy = \int \frac{e^x}{1+(e^x)^2}\,dx \qquad u = e^x,\quad du = e^x\,dx$$

$$-\frac{1}{y} = \int \frac{1}{1+u^2}\,du$$

$$-\frac{1}{y} = \arctan(e^x) + c$$, general solution of the ODE in implicit form

c = constant

(B) $\dfrac{dy}{dx} - \dfrac{2}{x}y = x^2\cos x$ is a linear ODE

$\mu(x) = e^{-\int \frac{2}{x}dx} = e^{-2\ln x} = e^{\ln x^{-2}} = x^{-2}$ is an integrating factor

$$x^{-2}\frac{dy}{dx} - 2x^{-3}y = \cos x$$

$$\frac{d}{dx}\left(x^{-2}y\right) = \cos x \implies x^{-2}y = \int \cos x\,dx$$

$\implies y = x^2(\sin x + c)$, c = constant

Furthermore, we want the constant C such that $y(\pi)=4$.

$\therefore \quad 4 = \pi^2(\sin\pi + C)$

$4 = \pi^2 C \implies C = \frac{4}{\pi^2}$

$\therefore$ Final answer: $y = x^2(\sin x + \frac{4}{\pi^2})$ #

(2) Note that $\frac{\partial}{\partial x}(4y+9x^2) = 18x \neq \frac{\partial}{\partial y}(6xy) = 6x$

We calculate $\frac{M_y - N_x}{N} = \frac{\frac{\partial}{\partial y}(6xy) - \frac{\partial}{\partial x}(4y+9x^2)}{4y+9x^2} =$

$= \frac{6x-18x}{4y+9x^2} = \frac{-12x}{4y+9x^2}$ not a function of x only

and $\frac{N_x - M_y}{M} = \frac{18x-6x}{6xy} = \frac{12x}{6xy} = \frac{2}{y}$ a function of y

only $\Rightarrow$ Set up: $\frac{\mu'(y)}{\mu(y)} = \frac{2}{y} \implies \ln\mu(y) = 2\ln y + c$

(take $C=0$).

$\Rightarrow$ An integrating factor for this eq. is $\ln\mu(y) = 2\ln y = \ln y^2$

i.e. $\mu(y) = y^2$

We multiply by y^2 the ODE to obtain:

$6xy^3\,dx + (4y^3 + 9x^2y^2)\,dy = 0$ (This ODE is exact.)

Thus, we seek a function of two variables $f(x,y)$ such that

$$\begin{cases} \dfrac{\partial f}{\partial x} = 6xy^3 & \Rightarrow f(x,y) = 3x^2y^3 + c(y) \\ \dfrac{\partial f}{\partial y} = 4y^3 + 9x^2y^2 & \end{cases}$$

$$\Rightarrow \frac{\partial f}{\partial y} = 9x^2y^2 + c'(y) = 4y^3 + 9x^2y^2 \Rightarrow c'(y) = 4y^3$$

Thus $c(y) = y^4 + c_1$ so $f(x,y) = 3x^2y^3 + y^4 + c$

$c =$ constant

Hence the solution to the given ODE, in implicit form, is:

$$3x^2y^3 + y^4 + c = 0, \quad c = \text{constant}$$

(3) (A) $\dfrac{dy}{dx} = \dfrac{x+3y}{3x+y}$ is a homogeneous ODE solvable with the substitution $u = \dfrac{y}{x} \Leftrightarrow y = ux$

$$\frac{dy}{dx} = \frac{du}{dx} \cdot x + u$$

$$x\frac{du}{dx} + u = \frac{x + 3ux}{3x + ux}$$

$$x\frac{du}{dx} = \frac{\cancel{x}(1+3u)}{\cancel{x}(3+u)} - u$$

$$x\frac{du}{dx} = \frac{1+3u}{3+u} - u \quad \text{or} \quad x\frac{du}{dx} = \frac{1+3u-3u-u^2}{3+u}$$

This ODE in u and x is separable:

$$\frac{3+u}{1-u^2}\,du = \frac{1}{x}\,dx$$

Use partial fractions decomposition to evaluate $\int \frac{3+u}{1-u^2}\,du$.

$$\frac{3+u}{1-u^2} = \frac{A}{1-u} + \frac{B}{1+u}$$

$$3+u = A(1+u) + B(1-u)$$

$u=1$: $\quad 4 = 2A \;\Rightarrow\; A=2$

$u=-1$: $\quad 2 = 2B \;\Rightarrow\; B=1$

$$\therefore\quad \frac{3+u}{1-u^2} = \frac{2}{1-u} + \frac{1}{1+u}$$

and $\int \frac{3+u}{1-u^2}\,du = 2\int \frac{1}{1-u}\,du + \int \frac{1}{1+u}\,du =$

$$= 2\cdot(-1)\ln|1-u| + \ln|1+u| + c$$

$$= -2\ln|1-u| + \ln|1+u| + c$$

$\therefore\quad -2\ln\left|1-\frac{y}{x}\right| + \ln\left|1+\frac{y}{x}\right| + c = \ln x \quad$, $c=$ constant

is the general solution of the given ODE in implicit form (this form can be simplified further:

$$\ln\left(\left|1-\frac{y}{x}\right|^{+2}\cdot\left|1+\frac{y}{x}\right|^{-1}\cdot x\right) = c \iff \left|1-\frac{y}{x}\right|^{2}\cdot x = c\left|1+\frac{y}{x}\right|$$

$c=$ constant, etc.)

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(13) $\frac{dy}{dx} + y = 8xy^4$ is a Bernoulli equation with $n=4$

$\Rightarrow$ We'll use the substitution $u = y^{-3}$

$$\frac{du}{dx} = -3y^{-4}\frac{dy}{dx}$$

$$-3y^{-4}\frac{dy}{dx} - 3y^{-3} = -24x$$

$\frac{du}{dx} - 3u = -24x$ is linear in $u(x)$.

$\Rightarrow \mu(x) = e^{-3x} \Rightarrow e^{-3x}u' - 3e^{-3x}u = -24xe^{-3x}$

$\Rightarrow (e^{-3x}\cdot u)' = -24xe^{-3x}$

$\Rightarrow e^{-3x}\cdot u = -24\int xe^{-3x}dx$

We'll use integration by parts for the last integral:

$u = x \Rightarrow du = dx$
$dv = e^{-3x}dx \quad v = -\frac{1}{3}e^{-3x}$

$\Rightarrow e^{-3x}\cdot u = -24\cdot\left(-\frac{1}{3}xe^{-3x} + \int \frac{1}{3}e^{-3x}dx\right)$

$e^{-3x}\cdot u = 8xe^{-3x} + \frac{8}{3}e^{-3x} + c$

$\Rightarrow y^{-3} = 8x + \frac{8}{3} + ce^{3x} \Rightarrow y = \left(8x + \frac{8}{3} + ce^{3x}\right)^{3}$

c = constant. #

(4) Let $q(t)$ denote the quantity of salt at time t

So: $q(0) = 500$ (kg)

$$\frac{dq}{dt} = \text{Rate (in)} - \text{Rate (out)}$$

$$\frac{dq}{dt} = 10 \cdot 0{,}3 - 10\,\frac{q}{10{,}000}$$

$$\frac{dq}{dt} = 3 - \frac{q}{1{,}000}$$

$$\frac{dq}{dt} + \frac{q}{1000} = 3 \quad \text{linear ODE}$$

$$\mu(t) = e^{\int \frac{1}{1000}\,dt} = e^{\frac{t}{1000}}$$

$$\left(e^{t/1000} \cdot q\right)' = 3e^{\frac{t}{1000}}$$

$$e^{t/1{,}000} \cdot q = \frac{3}{\frac{1}{1{,}000}} \cdot e^{\frac{t}{1{,}000}} + c$$

$$q(t) = 3{,}000 + c\,e^{-t/1{,}000}$$

$$500 = q(0) = 3{,}000 + c \Rightarrow c = -2{,}500$$

So (A): $q(t) = 3{,}000 - 2{,}500 \cdot e^{-t/1{,}000}$

(B) as $t \to \infty$, $e^{-t/1{,}000} \to 0$, so $q(t) \to 3{,}000$ kg

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(5) Let $y = x^m$, $y' = mx^{m-1}$, $y'' = m(m-1)x^{m-2}$

$x^2y'' - 6xy' - 18y = 0$ is a Cauchy-Euler eq.

With the auxiliary equation:

$$m(m-1) - 6m - 18 = 0$$

$m^2 - 7m - 18 = 0$. Its roots are $\dfrac{7 \pm \sqrt{49+72}}{2} = \dfrac{7 \pm 11}{2} \begin{cases} 9 \\ -2 \end{cases}$

Hence $y_{gen}(x) = c_1 x^{-2} + c_2 x^9$

$$\left.\begin{aligned} y(1) = 2 &\Rightarrow c_1 + c_2 = 2 \\ y'(1) = 16 &\Rightarrow -2c_1 + 9c_2 = 16 \end{aligned}\right| \cdot 2$$

$$\Rightarrow \begin{cases} 2c_1 + 2c_2 = 4 \\ -2c_1 + 9c_2 = 16 \end{cases}$$

$$/ \quad 11c_2 = 20 \Rightarrow c_2 = \frac{20}{11}$$

$$c_1 = 2 - c_2 = 2 - \frac{20}{11} = \frac{2}{11}$$

<u>Answer:</u> $y(x) = \frac{2}{11}x^{-2} + \frac{20}{11}x^9$ #

(6) Consider first: $y'' - 4y = 0$ with $r^2 - 4 = 0 \Rightarrow r = \pm 2$

$\Rightarrow y_c(x) = c_1 e^{2x} + c_2 e^{-2x}$

$$W(e^{2x}, e^{-2x}) = \begin{vmatrix} e^{2x} & e^{-2x} \\ 2e^{2x} & -2e^{-2x} \end{vmatrix} = -4.$$

$$y_p(x) = u_1(x)\cdot y_1(x) + u_2(x)\cdot y_2(x) = u_1\cdot e^{2x} + u_2 e^{-2x}$$

$$u_1' = \frac{\begin{vmatrix} 0 & e^{-2x} \\ x^{-1}e^{2x} & -2e^{-2x} \end{vmatrix}}{W} = \frac{1}{4}\cdot x^{-1}$$

$$u_1 = \frac{1}{4}\int \frac{1}{x}\,dx = \frac{1}{4}\ln x$$, we take $c=0$.

$$u_2' = \frac{\begin{vmatrix} e^{2x} & 0 \\ 2e^{2x} & x^{-1}e^{2x} \end{vmatrix}}{W} = -\frac{1}{4}\cdot x^{-1}e^{4x}$$

$$u_2 = -\frac{1}{4}\int \frac{1}{x}e^{4x}\,dx$$ (and this is the best one can do!)

$$\Rightarrow y_p(x) = \left(\frac{1}{4}\ln x\right)\cdot e^{2x} + \left(-\frac{1}{4}\int \frac{1}{x}e^{4x}\,dx\right)e^{-2x}.$$

Answer: $y_{gen}(x) = y_c(x) + y_p(x)$, c_1, c_2 = constants

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(7)

$$mx'' = -kx - 24x'$$

$$4x'' + 24x' + 36x = 0$$

$x(0) = 2$ (m)

$x'(0) = -12$ (m/sec)

$4r^2 + 24r + 36 = 0 \Leftrightarrow r^2 + 6r + 9 = 0$ has the double root $r = -3$

So $x(t) = c_1 e^{-3t} + c_2 t e^{-3t}$

$x(0) = 2$: $c_1 + 0 = 2 \Rightarrow c_1 = 2$

$x'(t) = -3c_1 e^{-3t} - 3c_2 t e^{-3t} + c_2 e^{-3t}$

$x'(0) = -3c_1 + c_2 = -12$

$-6 + c_2 = -12, \quad c_2 = -6$

$\therefore \; x(t) = 2e^{-3t} - 6te^{-3t}$ #

(8) $10Q'' + 40Q' + \frac{100}{2} Q = 150\cos(10t)$

$10Q'' + 40Q' + 50Q = 0 \Rightarrow r^2 + 4r + 5 = 0$

with roots $\frac{-4 \pm \sqrt{16-25}}{2} = \frac{-4 \pm 3i}{2}$

$\therefore \; Q_c(t) = c_1 e^{-2t}\cos\left(\frac{3t}{2}\right) + c_2 e^{-2t}\sin\left(\frac{3t}{2}\right)$

We seek $Q_p(t)$ of the form $Q_p(t) = A\cos(10t) + B\sin(10t)$.

$Q_p'(t) = -10A\sin(10t) + 10B\cos(10t)$

$Q_p''(t) = -100A\cos(10t) - 100B\sin(10t)$

$\Rightarrow \; -100A\cos(10t) - 100B\sin(10t) - 40A\sin(10t) + 40B\cos(10t)$

$+ 5A\cos(10t) + 5B\sin(10t) = 15\cos(10t)$

$(-100A + 40B + 5A)\cos(10t) + (-100B - 40A + 5B)\sin(10t) = 15\cos(10t)$

$\therefore \begin{cases} -95A + 40B = 15 \\ -40A - 95B = 0 \Rightarrow A = -\frac{95B}{40} \end{cases}$

$\frac{95^2}{40}B + 40B = 15 \Rightarrow B\left(\frac{95^2 + 40^2}{40}\right) = 15 \Rightarrow B = \frac{15 \cdot 40}{95^2 + 40^2} = \frac{600}{10{,}625}$

$A = -\frac{95}{40} \cdot \frac{15 \cdot 40}{95^2 + 40^2} = \frac{-95 \cdot 15}{95^2 + 40^2} = -\frac{1425}{10{,}625}$

$\therefore$ (A) $Q_{gen}(t) = c_1 e^{-2t} \cos\frac{3t}{2} + c_2 e^{-2t} \sin\frac{3t}{2} - \frac{1{,}425}{10{,}625}\cos(10t) + \frac{600}{10{,}625}\sin(10t)$

$$Q(0) = c_1 - \frac{1{,}425}{10{,}625} = 1{,}500 \implies c_1 = 1{,}500 + \frac{1{,}425}{10{,}625}$$

$$Q'(t) = c_1\left(-2e^{-2t}\cos\frac{3t}{2} + e^{-2t}\cdot\left(-\sin\frac{3t}{2}\right)\cdot\frac{3}{2}\right)$$

$$+ c_2\left(-2e^{-2t}\sin\frac{3t}{2} + e^{-2t}\cdot\cos\frac{3t}{2}\cdot\frac{3}{2}\right)$$

$$+ \frac{14{,}250}{10{,}625}\sin(10t) + \frac{6{,}000}{10{,}625}\cos(10t)$$

$$Q'(0) = -2c_1 + \frac{3}{2}c_2 + \frac{6{,}000}{10{,}625} = 0$$

$$c_2 = \frac{2}{3}\left(2c_1 - \frac{6{,}000}{10{,}625}\right) = \frac{2}{3}\left(3{,}000 + \frac{2{,}850 - 6{,}000}{10{,}625}\right)$$

$$= \frac{2}{3}\left(3{,}000 - \frac{3{,}150}{10{,}625}\right)$$

$$= 2{,}000 - \frac{2{,}100}{10{,}625}$$

Finally

$$Q(t) = \left(1{,}500 + \frac{1{,}425}{10{,}625}\right)e^{-2t}\cos\frac{3t}{2} + \left(2{,}000 - \frac{2{,}100}{10{,}625}\right)e^{-2t}\sin\frac{3t}{2} - \frac{1425}{10{,}625}\cos(10t)$$

$$+ \frac{600}{10{,}625}\sin(10t)$$

(B) The transient terms are the 1st two (containing e^{-2t} which approaches zero as $t \to \infty$)

The last two terms are steady-state terms. #

(9)

$$\begin{cases} (D+1)x + (D-1)y = 2 & \Big| \cdot -3 \\ 3x + (D+2)y = -1 & \Big| \, (D+1) \end{cases}$$

$$\begin{cases} -3(D+1)x - 3(D-1)y = -6 \\ 3(D+1)x + (D+1)(D+2)y = -1 \end{cases}$$

$$(D^2 + \cancel{3D} + 2 - \cancel{3D} + 3)\,y = -7$$

$$(D^2+5)\,y = -7$$

$$y'' + 5y = -7$$

$$y'' + 5y = 0 \Rightarrow r^2 + 5 = 0 \Rightarrow r = \pm i\sqrt{5}$$

$$y_c(t) = c_1\cos(\sqrt{5}\,t) + c_2\sin(\sqrt{5}\,t)$$

$$\left.\begin{aligned} y_p(t) &= A \\ y_p'(t) &= 0 \\ y_p''(t) &= 0 \end{aligned}\right\} \Rightarrow \begin{aligned} 5A &= -7 \\ A &= -7/5 \end{aligned}$$

$$\therefore\ y_{gen}(t) = c_1\cos(\sqrt{5}\,t) + c_2\sin(\sqrt{5}\,t) - \frac{7}{5}$$

$$x_{gen} = \frac{1}{3}\left(-1 - (D+2)y\right) = -\frac{1}{3} - \frac{1}{3}\left(y' + 2y\right) =$$

$$= -\frac{1}{3} - \frac{1}{3}\cdot\Big(-\sqrt{5}\,c_1\sin(\sqrt{5}\,t) + c_2\sqrt{5}\cos(\sqrt{5}\,t) + 2c_1\cos(\sqrt{5}\,t)$$

$$+ 2c_2\sin(\sqrt{5}\,t) - \frac{14}{5}\Big)$$

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Answer:

$$\begin{cases} x(t) = \left(-\frac{\sqrt{5}}{3} c_2 - \frac{2c_1}{3}\right)\cos(\sqrt{5}t) + \left(\frac{\sqrt{5}}{3} c_1 - \frac{2}{3} c_2\right)\sin(\sqrt{5}t) + \frac{3}{5} \\ y(t) = c_1 \cos(\sqrt{5}t) + c_2 \sin(\sqrt{5}t) - \frac{7}{5} \end{cases}$$

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(10) let $y(x) = \sum_{n=0}^{\infty} A_n x^n$

$$y'(x) = \sum_{n=0}^{\infty} n A_n x^{n-1} = \sum_{n=1}^{\infty} n A_n x^{n-1}$$

$$y'' = \sum_{n=0}^{\infty} n(n-1) A_n x^{n-2} = \sum_{n=2}^{\infty} n(n-1) A_n x^{n-2}$$

$$\therefore \sum_{n=2}^{\infty} n(n-1) A_n x^{n-2} + 2\sum_{n=1}^{\infty} n A_n x^n + \sum_{n=0}^{\infty} A_n x^n = 0$$

$$\sum_{n=0}^{\infty} (n+2)(n+1) A_{n+2} x^n + 2\sum_{n=0}^{\infty} n A_n x^n + \sum_{n=0}^{\infty} A_n x^n = 0$$

$$\sum_{n=0}^{\infty} \left[(n+2)(n+1) A_{n+2} + 2n A_n + A_n\right] x^n = 0$$

$$\therefore A_{n+2} = -\frac{(2n+1)}{(n+2)(n+1)} A_n$$

$n=0$: $A_2 = -\frac{1}{2} \cdot A_0$

$n=2$: $A_4 = -\frac{5}{4 \cdot 3} A_2 = \frac{5}{4 \cdot 3 \cdot 2} A_0$

$n=1$: $A_3 = -\frac{\cancel{3}}{\cancel{3} \cdot 2} A_1 = -\frac{1}{2} A_1$

$n=3$: $A_5 = -\frac{7}{5 \cdot 4} A_3 = \frac{7}{5 \cdot 4 \cdot 2} A_1$

$$\Rightarrow y(x) = A_0\left(1 - \frac{1}{2}x^2 + \frac{5}{24}x^4 - \dots\right) + A_1\left(x - \frac{x^3}{2} + \frac{7}{40}x^5 - \dots\right)$$

So: $\varphi_1(x) = 1 - \frac{x^2}{2} + \frac{5}{24}x^4 - \dots$

and $\varphi_2(x) = x - \frac{x^3}{2} + \frac{7}{40}x^5 - \dots$

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