Department of Computer Science & Software Engineering

Comp232 Mathematics for Computer Science

Fall 2020

Assignment 1. Solutions.

1. For each of the following statements use a truth table to determine whether it is a tautology, a contradiction, or a contingency.

   (a) $(\underbrace{(p \vee r)}_{\mathbf{a}} \wedge \underbrace{(q \vee r)}_{\mathbf{b}}) \leftrightarrow \underbrace{((p \wedge q) \vee r)}_{\mathbf{c}}$

   Solution: Tautology.

p	q	r	$\overbrace{p \vee r}^{\mathbf{a}}$	$\overbrace{q \vee r}^{\mathbf{b}}$	$\mathbf{a} \wedge \mathbf{b}$	$p \wedge q$	$\overbrace{(p \wedge q) \vee r}^{\mathbf{c}}$	$(\mathbf{a} \wedge \mathbf{b}) \leftrightarrow \mathbf{c}$
T	T	T	T	T	T	T	T	T
T	T	F	T	T	T	T	T	T
T	F	T	T	T	T	F	T	T
F	T	T	T	T	T	F	T	T
T	F	F	T	F	F	F	F	T
F	T	F	F	T	F	F	F	T
F	F	T	T	T	T	F	T	T
F	F	F	F	F	F	F	F	T

   (b) $\underbrace{(p \wedge \underbrace{(\neg(\neg p \vee q)))}_{\mathbf{a}}}_{\mathbf{b}} \vee \underbrace{(p \wedge q)}_{\mathbf{c}}$

   Solution: Contingency.

p	q	$\neg p$	$\neg p \vee q$	$\overbrace{\neg(\neg p \vee q)}^{\mathbf{a}}$	$\overbrace{p \wedge \mathbf{a}}^{\mathbf{b}}$	$\overbrace{p \wedge q}^{\mathbf{c}}$	$\mathbf{b} \vee \mathbf{c}$
T	T	F	T	F	F	T	T
T	F	F	F	T	T	F	T
F	T	T	T	F	F	F	F
F	F	T	T	F	F	F	F

(c) $\underbrace{(p \wedge \underbrace{(\neg q \rightarrow \neg p)}_{\mathbf{a}})}_{\mathbf{b}} \rightarrow q$

Solution: Tautology.

p	q	$\neg p$	$\neg q$	$\overbrace{\neg q \rightarrow \neg p}^{\mathbf{a}}$	$\overbrace{p \wedge \mathbf{a}}^{\mathbf{b}}$	$\mathbf{b} \rightarrow q$
T	T	F	F	T	T	T
T	F	F	T	F	F	T
F	T	T	F	T	F	T
F	F	T	T	T	F	T

(d) $(\underbrace{(p \rightarrow r)}_{\mathbf{a}} \vee \underbrace{(q \rightarrow r)}_{\mathbf{b}}) \rightarrow \underbrace{((p \vee q) \rightarrow r)}_{\mathbf{c}}$

Solution: Contingency.

p	q	r	$\overbrace{p \rightarrow r}^{\mathbf{a}}$	$\overbrace{q \rightarrow r}^{\mathbf{b}}$	$\mathbf{a} \vee \mathbf{b}$	$p \vee q$	$\overbrace{(p \vee q) \rightarrow r}^{\mathbf{c}}$	$(\mathbf{a} \vee \mathbf{b}) \rightarrow \mathbf{c}$
T	T	T	T	T	T	T	T	T
T	T	F	F	F	F	T	F	T
T	F	T	T	T	T	T	T	T
F	T	T	T	T	T	T	T	T
T	F	F	F	T	T	T	F	F
F	T	F	T	F	T	T	F	F
F	F	T	T	T	T	F	T	T
F	F	F	T	T	T	F	T	T

2. For each of the following logical equivalences state whether it is valid or invalid. If invalid then give a counterexample (*e.g.*, based on a truth table). If valid then give an algebraic proof using logical equivalences from Tables 6, 7, and 8 from Section 1.3 of textbook.

   (a) $p \to (q \to r) \equiv q \to (\neg p \vee r)$

   Solution: Valid.

$$\begin{array}{lll} & p \to (q \to r) & \\ \equiv & \neg p \vee (q \to r) & \text{law for conditional} \\ \equiv & \neg p \vee (\neg q \vee r) & \text{law for conditional} \\ \equiv & (\neg p \vee \neg q) \vee r & \text{associativity} \\ \equiv & (\neg q \vee \neg p) \vee r & \text{commutativity} \\ \equiv & \neg q \vee (\neg p \vee r) & \text{associativity} \\ \equiv & q \to (\neg p \vee r) & \text{law for conditional} \end{array}$$

   (b) $(p \to r) \wedge (q \to r) \equiv ((p \wedge q) \to r)$

   Solution: Invalid.
   If $p = T$, $q = F$, and $r = F$ then the LHS is False, while the RHS is True.

   (c) $(p \to q) \wedge (p \to r) \equiv (p \to (q \wedge r))$

   Solution: Valid.

$$\begin{array}{lll} & (p \to q) \wedge (p \to r) & \\ \equiv & (\neg p \vee q) \wedge (\neg p \vee r) & \text{law for conditional} \\ \equiv & \neg p \vee (q \wedge r) & \text{distributivity} \\ \equiv & (p \to (q \wedge r)) & \text{law for conditional} \end{array}$$

   (d) $((p \vee q) \wedge (\neg p \vee r)) \equiv (q \vee r)$

   Solution: Invalid.
   If $p = T$, $q = T$, and $r = F$ then the LHS is False, while the RHS is True.

3. Write down the negations of each of the following statements *in their simplest form* (*i.e.*, do not simply state "It is not the case that..."). Below, x denotes a real number, $x \in \mathbb{R}$.

   (a) The plane is early or my watch is slow.

   Solution: *The plane is on time and my watch is on time.*
   This is of the form $e \vee s$. The negation is $\neg(e \vee s) \equiv \neg e \wedge \neg s$.

   (b) Doing the assignments is a sufficient condition for John to pass the course.

   Solution: *John does the assignments but does not pass the course.*
   This is of the form $a \to p$. The negation is $\neg(a \to p) \equiv a \wedge \neg p$.

   (c) If x is positive, then x is not negative and x is not 0.

   Solution: *x is positive, and x is negative or x is 0.*
   This is of the form $p \to (\neg n \wedge \neg z)$.
   The negation is $\neg\Big(p \to (\neg n \wedge \neg z)\Big) \equiv \neg\Big(\neg p \vee (\neg n \wedge \neg z)\Big) \equiv \Big(\neg\neg p \wedge \neg(\neg n \wedge \neg z)\Big) \equiv \Big(p \wedge \neg(\neg n \wedge \neg z)\Big) \equiv \Big(p \wedge (\neg\neg n \vee \neg\neg z)\Big) \equiv \Big(p \wedge (n \vee z)\Big)$.

   (d) $(0 < x \leq 1) \vee (-1 < x < 0)$

   Solution: $(x = 0) \vee (x \leq -1) \vee (x > 1)$.
   The negation is that x does not lie in the half-open interval $(0, 1]$ and x does not lie in the open interval $(-1, 0)$
   To think of this in propositional logic, we consider the sentence in the form

$$(x > 0 \wedge x \leq 1) \vee (x > -1 \wedge x < 0)$$

   We get:

$$
\begin{aligned}
\neg\Big((x > 0 \wedge x \leq 1) \vee (x > -1 \wedge x < 0)\Big) &\equiv \\
\neg(x > 0 \wedge x \leq 1) \wedge \neg(x > -1 \wedge x < 0) &\equiv \\
\Big(\neg(x > 0) \vee \neg(x \leq 1)\Big) \wedge \Big(\neg(x > -1) \vee \neg(x < 0)\Big) &\equiv \\
\Big(x \leq 0 \vee x > 1)\Big) \wedge \Big(x \leq -1 \vee x \geq 0\Big) &\equiv \\
\Big(x \leq 0 \wedge x \leq -1\Big) \vee \Big(x \leq 0 \wedge x \geq 0\Big) \vee \Big(x > 1 \wedge x \leq -1\Big) \vee \Big(x > 1 \wedge x \geq 0\Big) &\equiv \\
x \leq -1 \vee x = 0 \vee F \vee x > 1 &\equiv \\
x \leq -1 \vee x = 0 \vee x > 1 &
\end{aligned}
$$

4. Write the following statements in predicate form, using logical operators $\wedge$, $\vee$, $\neg$, and quantifiers $\forall$, $\exists$. Below $\mathbb{Z}^+$ denotes all positive integers $\{1, 2, 3, \ldots\}$.

   We assume that the Universe of Discourse is all numbers
   (natural $\mathbb{N}$, integers $\mathbb{Z}$, rational $\mathbb{Q}$, irrational $\mathbb{R}$, complex $\mathbb{C}$, ...).

   (a) The square of a positive integer is always bigger than the integer.

   Solution: $\forall x \left(x \in \mathbb{Z}^+ \ \rightarrow \ x^2 > x \right)$

   (b) There is no integer solution to the equation $x = x + 1$.

   Solution: $\forall x \left(x \in \mathbb{Z} \ \rightarrow \ x \neq x + 1 \right)$ or $\neg \left[\ \exists x \left(x \in \mathbb{Z} \ \wedge \ x = x + 1 \right) \ \right]$

   (c) The absolute value of an integer is not necessarily positive.

   Solution: $\exists x \left(x \in \mathbb{Z} \ \wedge \ |x| \not> 0 \right)$

   (d) The absolute value of the sum of two integers does not exceed the sum of the absolute values of those integers.

   Solution: $\forall x \, \forall y \left((x \in \mathbb{Z} \wedge y \in \mathbb{Z}) \ \rightarrow \ |x + y| \leq |x| + |y| \right)$

5. Let P and Q be predicates on the set S, where S has two elements, say, $S = \{a, b\}$. Then the statement $\forall x P(x)$ can also be written in full detail as $P(a) \wedge P(b)$. Rewrite each of the statements below in a similar fashion, using P, Q, and logical operators, but without using quantifiers.

(a) $\forall x, y(P(x) \vee Q(y)) \equiv \forall x\Big(\forall y\big(P(x) \vee Q(y)\big)\Big)$

Solution:

$$\begin{aligned}
& \forall x\Big(\forall y\big(P(x) \vee Q(y)\big)\Big) \\
\equiv\ & \forall y\Big(P(a) \vee Q(y)\Big) \bigwedge \forall y\Big(P(b) \vee Q(y)\Big) \\
\equiv\ & \Big[\Big(P(a) \vee Q(a)\Big) \wedge \Big(P(a) \vee Q(b)\Big)\Big] \\
\bigwedge\ & \Big[\Big(P(b) \vee Q(a)\Big) \wedge \Big(P(b) \vee Q(b)\Big)\Big] \\
\equiv\ & \Big[P(a) \vee \Big(Q(a) \wedge Q(b)\Big)\Big] \\
\bigwedge\ & \Big[P(b) \vee \Big(Q(a) \wedge Q(b)\Big)\Big]
\end{aligned}$$

(b) $\exists x P(x) \vee \exists x Q(x)$

Solution:

$$\begin{aligned}
& \exists x P(x) \vee \exists x Q(x) \\
\equiv\ & \Big(P(a) \vee P(b)\Big) \vee \Big(\exists x Q(x)\Big) \\
\equiv\ & \Big(P(a) \vee P(b)\Big) \vee \Big(Q(a) \vee Q(b)\Big)
\end{aligned}$$

(c) $\exists x P(x) \wedge \exists x Q(x)$

Solution:

$$\begin{aligned}
& \exists x P(x) \wedge \exists x Q(x) \\
\equiv\ & \Big(P(a) \vee P(b)\Big) \wedge \Big(\exists x Q(x)\Big) \\
\equiv\ & \Big(P(a) \vee P(b)\Big) \wedge \Big(Q(a) \vee Q(b)\Big)
\end{aligned}$$

(d) $\exists x, y(P(x) \wedge Q(y)) \equiv \exists x\Big(\exists y\big(P(x) \wedge Q(y)\big)\Big)$

Solution:

$$
\begin{aligned}
& \exists x\Big(\exists y\big(P(x) \wedge Q(y)\big)\Big) \\
\equiv \quad & \exists y\Big(P(a) \wedge Q(y)\Big) \bigvee \exists y\Big(P(b) \wedge Q(y)\Big) \\
\equiv \quad & \Big[\Big(P(a) \wedge Q(a)\Big) \vee \Big(P(a) \wedge Q(b)\Big)\Big] \\
\bigvee \quad & \Big[\Big(P(b) \wedge Q(a)\Big) \vee \Big(P(b) \wedge Q(b)\Big)\Big] \\
\equiv \quad & \Big[P(a) \wedge \Big(Q(a) \vee Q(b)\Big)\Big] \\
\bigvee \quad & \Big[P(b) \wedge \Big(Q(a) \vee Q(b)\Big)\Big]
\end{aligned}
$$

(e) $\forall x \exists y(P(x) \wedge Q(y)) \equiv \forall x\Big(\exists y\big(P(x) \wedge Q(y)\big)\Big)$

Solution:

$$
\begin{aligned}
& \forall x\Big(\exists y\big(P(x) \wedge Q(y)\big)\Big) \\
\equiv \quad & \exists y\Big(P(a) \wedge Q(y)\Big) \bigwedge \exists y\Big(P(b) \wedge Q(y)\Big) \\
\equiv \quad & \Big[\Big(P(a) \wedge Q(a)\Big) \vee \Big(P(a) \wedge Q(b)\Big)\Big] \\
\bigwedge \quad & \Big[\Big(P(b) \wedge Q(a)\Big) \vee \Big(P(b) \wedge Q(b)\Big] \\
\equiv \quad & \Big[P(a) \wedge \Big(Q(a) \vee Q(b)\Big)\Big] \\
\bigwedge \quad & \Big[P(b) \wedge \Big(Q(a) \vee Q(b)\Big)\Big]
\end{aligned}
$$

6. Let the domain for x be the set of all students in this class and the domain for y be the set of all countries in the world. Let $P(x, y)$ denote student x has visited country y and $Q(x, y)$ denote student x has a friend in country y. Express each of the following using logical operations and quantifiers, and the propositional functions $P(x, y)$ and $Q(x, y)$.

   (a) Carlos has visited Bulgaria.

   Solution: $P(Carlos, Bulgaria)$

   (b) Every student in this class has visited the United States.

   Solution: $\forall x\ P(x, UnitedStates)$

   (c) Every student in this class has visited some country in the world.

   Solution: $\forall x\ \exists y\ P(x, y)$

   (d) There is no country that every student in this class has visited.

   Solution: $\forall y\ \exists x\ \neg P(x, y)$

   (e) There are two students in this class, who between them, have a friend in every country in the world.

   Solution: $\exists x\ \exists y\ \Big(x \neq y \land \forall z[Q(x, z) \lor Q(y, z)]\Big)$

7. For each part in the previous question, form the negation of the statement so that all negation symbols occur immediately in front of predicates. For example:

$$\neg\Big(\forall x\big(P(x)\wedge Q(x)\big)\Big) \equiv \exists x\Big(\neg\big((P(x)\wedge Q(x)\big)\Big) \equiv \exists x\Big(\big(\neg P(x)\big)\vee\big(\neg Q(x)\big)\Big)$$

(a) **Solution:** $\neg P(Carlos, Bulgaria)$

Carlos has not visited Bulgaria

(b) **Solution:** $\neg\Big(\forall x\ P(x, UnitedStates)\Big) \equiv \exists x\ \Big(\neg P(x, UnitedStates)\Big)$

There is a student in this class who has not visited the United States

(c) **Solution:** $\neg\ \Big(\forall x\ \Big[\exists y\ P(x,y)\Big]\Big) \equiv \exists x\ \neg\Big[\exists y\ P(x,y)\Big] \equiv \exists x\ \forall y\ \Big[\neg P(x,y)\Big]$

There is a student in this class who has not visited any country

(d) **Solution:**

$$\begin{aligned}
&\neg\ \Big(\forall y\ \Big[\exists x\ \neg P(x,y)\Big]\Big) &\equiv\\
&\exists y\ \neg\Big[\exists x \neg P(x,y)\Big] &\equiv\\
&\exists y\ \forall x\ \neg\Big[\neg P(x,y)\Big] &\equiv\\
&\exists y\ \forall x\ \Big[\neg\neg P(x,y)\Big] &\equiv\\
&\exists y\ \forall x\ P(x,y)
\end{aligned}$$

There is a country that every student in this class has visited

(e) **Solution:**

$$\begin{aligned}
&\neg\Big[\exists x\ \Big(\exists y\ \Big(x\neq y\wedge\forall z\Big[Q(x,z)\vee Q(y,z)\Big]\Big)\Big)\Big] &\equiv\\
&\forall x\ \Big[\neg\Big(\exists y\ \Big(x\neq y\wedge\forall z\Big[Q(x,z)\vee Q(y,z)\Big]\Big)\Big)\Big] &\equiv\\
&\forall x\ \Big[\forall y\ \neg\Big(x\neq y\wedge\forall z\Big[Q(x,z)\vee Q(y,z)\Big]\Big)\Big] &\equiv\\
&\forall x\ \Big[\forall y\ \Big(x=y\vee\neg\Big(\forall z\Big[Q(x,z)\vee Q(y,z)\Big]\Big)\Big)\Big] &\equiv\\
&\forall x\ \Big[\forall y\ \Big(x=y\vee\exists z\Big(\neg\Big[Q(x,z)\vee Q(y,z)\Big]\Big)\Big)\Big] &\equiv\\
&\forall x\ \Big[\forall y\ \Big(x=y\vee\exists z\Big(\neg Q(x,z)\wedge\neg Q(y,z)\Big)\Big)\Big]
\end{aligned}$$

For every pair of distinct students in this class, there is a country where neither one of them has a friend

8. Negate the following statements and transform the negation so that negation symbols immediately precede predicates. (See example in Question 7.)

   (a) $\exists x \exists y \Big(P(x,y)\Big) \;\vee\; \forall x \forall y \Big(Q(x,y)\Big)$

   Solution:

$$\begin{aligned} & \neg\Big(\exists x \exists y P(x,y) \vee \forall x \forall y Q(x,y)\Big) \\ \equiv\ & \neg\Big(\exists x \exists y P(x,y)\Big) \wedge \neg\Big(\forall x \forall y Q(x,y)\Big) \\ \equiv\ & \Big(\forall x \forall y \neg P(x,y)\Big) \wedge \Big(\exists x \exists y \neg Q(x,y)\Big) \end{aligned}$$

   (b) $\forall x \forall y \Big(Q(x,y) \leftrightarrow Q(y,x)\Big)$

   Solution:

$$\begin{aligned} & \neg\Big(\forall x \forall y \big(Q(x,y) \leftrightarrow Q(y,x)\big)\Big) \\ \equiv\ & \exists x \exists y \Big(\neg\big(Q(x,y) \leftrightarrow Q(y,x)\big)\Big) \\ \equiv\ & \exists x \exists y \neg\Big(\big(Q(x,y) \to Q(y,x)\big) \wedge \big(Q(y,x) \to Q(x,y)\big)\Big) \\ \equiv\ & \exists x \exists y \Big(\neg\big(Q(x,y) \to Q(y,x)\big) \vee \neg\big(Q(y,x) \to Q(x,y)\big)\Big) \\ \equiv\ & \exists x \exists y \Big(\neg\big(\neg Q(x,y) \vee Q(y,x)\big) \vee \neg\big(\neg Q(y,x) \vee Q(x,y)\big)\Big) \\ \equiv\ & \exists x \exists y \Big(\big(Q(x,y) \wedge \neg Q(y,x)\big) \vee \big(Q(y,x) \wedge \neg Q(x,y)\big)\Big) \end{aligned}$$

   (c) $\forall y \exists x \exists z \Big(T(x,y,z) \wedge Q(x,y)\Big)$

   Solution:

$$\begin{aligned} & \neg\Big(\forall y \exists x \exists z \big(T(x,y,z) \wedge Q(x,y)\big)\Big) \\ \equiv\ & \exists y \forall x \forall z \neg\big(T(x,y,z) \wedge Q(x,y)\big) \\ \equiv\ & \exists y \forall x \forall z \big(\neg T(x,y,z) \vee \neg Q(x,y)\big) \end{aligned}$$