PROBLEMS FOR CHAPTER 4

2. Find the best fit in the least squares sense, to the data.

x_i	0	1	2	3	4	5	6	7	8	9
f_i	0	2	2	5	5	6	7	7	7	10

by a polynomial of degree at most 3.

Solution: Lets start with order 1

$$P(x) = a_0 + a_1x^1$$

The matrix has the form:

$$\begin{bmatrix} \sum_{i=0}^{3} x_i^0 & \sum_{i=0}^{3} x_i^1 \\ \sum_{i=0}^{3} x_i^1 & \sum_{i=0}^{3} x_i^2 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \end{bmatrix} = \begin{bmatrix} \sum_{i=0}^{3} f(x_i) \\ \sum_{i=0}^{3} f(x_i)x_i \end{bmatrix}$$

substituting the data gives

$$\begin{bmatrix} 10 & 45 \\ 45 & 285 \end{bmatrix} \begin{bmatrix} a_o \\ a_1 \end{bmatrix} = \begin{bmatrix} 51 \\ 308 \end{bmatrix}$$

Solving the matrix gives us

$$a_1 = 0.9515$$

$$a_0 = 0.8182$$

Hence $P(x) = 0.9515x + 0.8182$

x	0	1	2	3	4	5	6	7	8	9
fx	0	2	2	5	5	6	7	7	7	10
Px	0.8182	1.7697	2.7212	3.6727	4.6242	5.5757	6.5272	7.4787	8.4302	9.3817
E	0.6695	0.0530	0.5201	1.7617	0.1412	0.1800	0.2235	0.2292	2.0455	0.3823

$$E = \sum_{i=0}^{n} |P(x_i) - f(x_i)|^2$$

E= 6.206

Order 2

$$P(x) = a_0 + a_1x^1 + a_2x^2$$

The Matrix is

$$\begin{bmatrix} 10 & 45 & 285 \\ 45 & 285 & 2025 \\ 285 & 2025 & 15333 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 51 \\ 308 \\ 2138 \end{bmatrix}$$

Solving the matrix gives us

$$a_2 = -0.0417$$

$$a_1 = 1.3265$$

$$a_0 = 0.3182$$

Hence $P(x) = -0.0417x^2 + 1.3265x + 0.3182$

x	0	1	2	3	4	5	6	7	8	9
fx	0	2	2	5	5	6	7	7	7	10
Px	0.3182	1.6030	2.8044	3.9224	4.9570	5.9082	6.7760	7.5604	8.2614	8.8790
E	0.1013	0.1576	0.6471	1.1612	0.0018	0.0084	0.0502	0.3140	1.5911	1.2566

$$E = \sum_{i=0}^{n} |P(x_i) - f(x_i)|^2$$

E= 5.289

Order 3

$$P(x) = a_0 + a_1x^1 + a_2x^2 + a_3x^3$$

The matrix is of the form

$$\begin{bmatrix} 10 & 45 & 285 & 2025 \\ 45 & 285 & 2025 & 15333 \\ 285 & 2025 & 15333 & 120825 \\ 2025 & 15333 & 120825 & 978405 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 51 \\ 308 \\ 2138 \\ 16010 \end{bmatrix}$$

Solving the matrix gives us

$$a_3 = 0.0196$$

$$a_2 = -0.3065$$

$$a_1 = 2.2310$$

$$a_0 = -0.1762$$

Hence $P(x) = 0.0196x^3 - 0.3065x^2 + 2.2310x – 0.1762$

x	0	1	2	3	4	5	6	7	8	9
fx	0	2	2	5	5	6	7	7	7	10
Px	-0.1762	1.7679	3.2166	4.2875	5.0982	5.7663	6.4094	7.1451	8.0910	9.3647
E	0.0310	0.0539	1.4801	0.5077	0.0096	0.0546	0.3488	0.0211	1.1903	0.4036

$$E = \sum_{i=0}^{n} |P(x_i)-f(x_i)|^2$$

E= 4.101

7. From the following set of data construct a function of the type $f(x) = a e^{x} + b e^{-x}$ using the principle of least squares.

x	0.2	0.3	0.4	0.5
f(x)	2.0	5.0	3.5	3.0

Solution: The equation type

$$f(x) = a e^{x} + b e^{-x}$$

Can be transformed into

$$f(x) = a v + b /v$$

where $v = e^{x}$

Which can then be transformed into

$$w = a u + b$$

where $u=v^2$ and $w= f(x)v$

Now just continue as if approximating a polynomial of degree (1),

For the given data,

$$\begin{bmatrix} \sum_{i=0}^{n} u^0 & \sum_{i=0}^{n} u^1 \\ \sum_{i=0}^{n} u^1 & \sum_{i=0}^{n} u^2 \end{bmatrix} \begin{bmatrix} b \\ a \end{bmatrix} = \begin{bmatrix} \sum_{i=0}^{n} w \\ \sum_{i=0}^{n} u.w \end{bmatrix}$$

i	x	f(x)	$u=v^2$	w=f(x)*v	u.w
0	0.2	2	1.4918	2.4428	3.6442
1	0.3	5	1.8221	6.7439	12.298
2	0.4	3.5	2.2255	5.2214	11.6204
3	0.5	3	2.7183	4.9462	13.4451
$\sum_{1=0}^{3} =$	4200	11405	6,534950 5	3744000	4094,68995

Substituting the above values we get:

$$\begin{bmatrix} 4 & 8.2578 \\ 8.2578 & 17.8877 \end{bmatrix} \begin{bmatrix} b \\ a \end{bmatrix} = \begin{bmatrix} 19.36 \\ 41.01 \end{bmatrix}$$

Solving the matrix gives us

b= 2.278

a= 1.241

w= 1.241 u + 2.278

Transforming back to the original equation we get

$$f(x) = 1.241\, e^{x} + 2.278\, e^{-x}$$

8. The stress and strain are known to follow a relation of the type

$\sigma = k_1\, \varepsilon \exp(-k_2\varepsilon)$

Obtain the least squares fit using the below data.

Stress (σ)	Strain (ε)
1030 psi	$260\text{x}10^{-6}$ in/in
1410	410
1720	510
2060	710
2435	960
2750	1350

Solution: The equation

$$\sigma = k_1\ \varepsilon \exp(-k_2\varepsilon).$$

Can be transformed into

$$(\sigma/\varepsilon) = k_1\, e^{-k2\varepsilon}$$

Which can then be transformed into

$$\ln(\sigma/\varepsilon) = \ln k_1 - k_2\varepsilon$$

Now just continue as if approximating a polynomial of degree (1),

For the given data,

$$\begin{bmatrix} \sum_{i=0}^{n} \varepsilon_i^0 & \sum_{i=0}^{n} \varepsilon_i^1 \\ \sum_{i=0}^{n} \varepsilon_i^1 & \sum_{i=0}^{n} \varepsilon_i^2 \end{bmatrix} \begin{bmatrix} \ln k_1 \\ k_2 \end{bmatrix} = \begin{bmatrix} \sum_{i=0}^{n} \ln(\sigma/\varepsilon) \\ \sum_{i=0}^{n} \varepsilon . \ln(\sigma/\varepsilon) \end{bmatrix}$$

i	ε	σ	ln (σ/ ε)	(ε_i^2)	ε_i ln (σ/ ε)
0	260	1030	1,376632	67600	357,92444
1	410	1410	1,235188	168100	506,42701
2	510	1720	1,215669	260100	619,99111
3	710	2060	1,065196	504100	756,28937
4	960	2435	0,930769	921600	893,538
5	1350	2750	0,711496	1822500	960,52003
$\sum_{1=0}^{3}=$	4200	11405	6,534950 5	3744000	4094,68995

Substituting the above values we get:

$$\begin{bmatrix} 6 & 4200 \\ 4200 & 3744000 \end{bmatrix} \begin{bmatrix} \ln k_1 \\ k_2 \end{bmatrix} = \begin{bmatrix} 6.5349 \\ 4094.69 \end{bmatrix}$$

Solving the matrix gives us

$$\ln k_1 = 1.5068$$

$$k_2 = -0.0006$$

$$\ln(\sigma/\varepsilon) = 1.5068 - 0.0006\ \varepsilon$$

Transforming back to the original equation we get

$$P(\varepsilon) = 4.5123\ \varepsilon\ e^{-0.0006\varepsilon}$$

ε	260	410	510	710	960	1350
σ	1030	1410	1720	2060	2435	2750
P(ε)	1003,74	1446,59	1694,6266	2092,41	2435,093	2709,9

9. The relationship between resistance R, velocity v and time t is given by

$$t = \int_{v_0}^{v_1} \frac{m}{R(v)} dv$$

where $R(v) = -v^{3/2}$ and m=1kg, v_0 = 10 m/sec, v_1 =5 m/sec Evaluate the integrand f(v) = m /R(v) at 6 equally spaced velocities between 5 and 10 m/sec and fit the best least-squares polynomial fit.

Solution: f(v) = m /R(v) at 6 equally spaced velocities between 5 and 10 m/sec give us

n	0	1	2	3	4	5	6
v	10	9,166667	8,333333	7,5	6,6666667	5,833333	5
f(v)	-0,03162	-0,03603	-0,04157	-0,048686	-0,058095	-0,070978	-0,08944

Let's start with order 2

$$P(x) = a_0 + a_1x^1 + a_2x^2$$

The Matrix is of the form

$$\begin{bmatrix} \sum_{i=0}^{n} x_i^0 & \sum_{i=0}^{n} x_i^0 & \sum_{i=0}^{n} x_i^0 \\ \sum_{i=0}^{n} x_i^0 & \sum_{i=0}^{n} x_i^0 & \sum_{i=0}^{n} x_i^0 \\ \sum_{i=0}^{n} x_i^0 & \sum_{i=0}^{n} x_i^0 & \sum_{i=0}^{n} x_i^0 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} \sum_{i=0}^{n} f(x_i) \\ \sum_{i=0}^{n} f(x_i)x_i \\ \sum_{i=0}^{n} f(x_i)x^2{}_i \end{bmatrix}$$

$$\begin{bmatrix} 7 & 52.5 & 403.19 \\ 52.5 & 403.19 & 3390.625 \\ 413.19 & 3390.625 & 28805.46 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} -0.37643 \\ -2.6066 \\ -19.0486 \end{bmatrix}$$

Solving the matrix gives us

$a_2 = 0.00056$

$a_1 = 0.00131$

$a_0 = -0.09569$

Hence $P(x) = 0.00056\, x^2 + 0.00131x - 0.09569$

n	0	1	2	3	4	5	6
v	10	9,166667	8,333333	7,5	6,6666667	5,833333	5
f(v)	-0,03162	-0,03603	-0,04157	-0,048686	-0,058095	-0,070978	-0,08944
P(v)	-0,0266	-0,0366	-0,0459	-0,0544	-0,0621	-0,0690	-0,0751

$$E = \sum_{i=0}^{n} \left| P(x_i) - f(x_i) \right|^2$$

E= 0.0003

Therefore the above second order polynomial seems to be a good fit.

16. Use the Lagrange interpolating polynomial to approximate cos (0.750) using the following values

Cos (0.698) = 0.7661

Cos (0.733) = 0.7432

Cos (0.768) = 0.7193

Solution: Lagrange Polynomial is

$$P(x) = \sum_{k=0}^{n} f(x_k) L_{n,k}$$

$$L_{n,k} = \prod_{\substack{i=0 \\ i \neq k}}^{n} \frac{(x - x_i)}{(x_k - x_i)}$$

k=0

$$L_{2,0} = \prod_{\substack{i=0 \\ i \neq 0}}^{n} \frac{(x - x_i)}{(x_k - x_i)} = \frac{(x - x_1)}{(x_0 - x_1)} * \frac{(x - x_2)}{(x_0 - x_2)} =$$

$$\frac{(x - 0.733)(x - 0.768)}{(0.698 - 0.733)(0.698 - 0.768)} = \frac{x^2 - 1.501x + 0.562944}{0.00245}$$

k=1

$$L_{2,1} = \prod_{\substack{i=0 \\ i \neq 0}}^{n} \frac{(x - x_i)}{(x_k - x_i)} = \frac{(x - x_1)}{(x_0 - x_1)} * \frac{(x - x_2)}{(x_0 - x_2)} =$$

$$\frac{(x - 0.698)(x - 0.768)}{(0.733 - 0.698)(0.733 - 0.768)} = \frac{x^2 - 1.466x + 0.536064}{-0.001225}$$

k=2

$$L_{2,2} = \prod_{\substack{i=0 \\ i \neq 0}}^{n} \frac{(x - x_i)}{(x_k - x_i)} = \frac{(x - x_1)}{(x_0 - x_1)} * \frac{(x - x_2)}{(x_0 - x_2)} =$$

$$\frac{(x - 0.698)(x - 0.733)}{(0.768 - 0.698)(0.768 - 0.733)} = \frac{x^2 - 1.431x + 0.511634}{0.00245}$$

$$P(x) = 0.7661\left[\frac{x^2 - 1.501x + 0.562944}{0.00245}\right] + 0.7432\left[\frac{x^2 - 1.466x + 0.536064}{-0.001225}\right] + 0.7193\left[\frac{x^2 - 1.431x + 0.511634}{0.00245}\right]$$

$$P(x) = -0.40816x^2 - 0.0702x + 1.013961$$

$$P(0.75) = -0.40816(0.75)^2 - 0.0702(0.75) + 1.013961$$

$P(0.75) = 0.731716$