PROBLEMS FOR CHAPTER 7

2. Perform 5 iterations of the following matrix using power method and obtain the eigenvalue and eigenvector.

$$\begin{bmatrix} 1 & 3 & 2 \\ 3 & -1 & 1 \\ 2 & 1 & -2 \end{bmatrix}$$

Solution:

$$A = \begin{bmatrix} 1 & 3 & 2 \\ 3 & -1 & 1 \\ 2 & 1 & -2 \end{bmatrix}$$

Initial guess $x^{(0)} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

$$y^{(1)} = Ax^{(0)} = \begin{bmatrix} 1 & 3 & 2 \\ 3 & -1 & 1 \\ 2 & 1 & -2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 6 \\ 3 \\ 1 \end{bmatrix} = 6 \begin{bmatrix} 1 \\ 0.5 \\ 0.1667 \end{bmatrix}$$

$$y^{(2)} = \begin{bmatrix} 1 & 3 & 2 \\ 3 & -1 & 1 \\ 2 & 1 & -2 \end{bmatrix} \begin{bmatrix} 1 \\ 0.5 \\ 0.1667 \end{bmatrix} = \begin{bmatrix} 2.833 \\ 2.667 \\ 2.1667 \end{bmatrix} = 2.833 \begin{bmatrix} 1 \\ 0.94118 \\ 0.7647 \end{bmatrix}$$

$$y^{(3)} = \begin{bmatrix} 1 & 3 & 2 \\ 3 & -1 & 1 \\ 2 & 1 & -2 \end{bmatrix} \begin{bmatrix} 1 \\ 0.94118 \\ 0.7647 \end{bmatrix} = \begin{bmatrix} 5.3529 \\ 2.8235 \\ 1.4118 \end{bmatrix} = 5.3529 \begin{bmatrix} 1 \\ 0.52747 \\ 0.26374 \end{bmatrix}$$

$$y^{(4)} = \begin{bmatrix} 1 & 3 & 2 \\ 3 & -1 & 1 \\ 2 & 1 & -2 \end{bmatrix} \begin{bmatrix} 1 \\ 0.52747 \\ 0.26374 \end{bmatrix} = \begin{bmatrix} 3.1099 \\ 2.7363 \\ 2 \end{bmatrix} = 3.1099 \begin{bmatrix} 1 \\ 0.87986 \\ 0.64311 \end{bmatrix}$$

$$y^{(5)} = \begin{bmatrix} 1 & 3 & 2 \\ 3 & -1 & 1 \\ 2 & 1 & -2 \end{bmatrix} \begin{bmatrix} 1 \\ 0.87986 \\ 0.64311 \end{bmatrix} = \begin{bmatrix} 4.9258 \\ 2.7633 \\ 1.5936 \end{bmatrix} = 4.9258 \begin{bmatrix} 1 \\ 0.5610 \\ 0.3235 \end{bmatrix}$$

6. Using Gerschgorin theoren give the regions of eigenvalues for the system.

$$A = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 4 & -2 \\ 0 & -2 & 2 \end{bmatrix}$$

Solution:

We use the Gerschgorin theorem to find the regions for the eigenvalues for the system

The values of A_{ii} are 1, 4 and 2, hence the circles formed are;

Center = 1 radius = $|1| + |-1|$ = 2 units

Center = 4 radius = $|-1| + |-2|$ = 3 units

Center = 2 radius = $|-2| + |0|$ = 2 units

From Gerschgorin circles we can only asses that the eigenvalues fall in the range $-1 \le \lambda \le 7$.

8. Find the largest eigenvalue and eigenvector of the following matrix using Power method within 0.001 error.

$$A = \begin{bmatrix} 4 & 0 & 0 \\ -1 & 2 & 0 \\ -2 & -1 & -3 \end{bmatrix}$$

Solution:

$$A = \begin{bmatrix} 4 & 0 & 0 \\ -1 & 2 & 0 \\ -2 & 4 & 7 \end{bmatrix} \quad \text{trail vector } x^{(0)} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$i = 1 \quad y^{(1)} = \begin{bmatrix} 4 & 0 & 0 \\ -1 & 2 & 0 \\ -2 & -1 & -3 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \\ -6 \end{bmatrix} = -6 \begin{bmatrix} -0.6667 \\ -0.1667 \\ 1 \end{bmatrix}$$

$$i = 1 \quad y^{(2)} = \begin{bmatrix} 4 & 0 & 0 \\ -1 & 2 & 0 \\ -2 & -1 & 3 \end{bmatrix} \begin{bmatrix} -0.6667 \\ -0.1667 \\ 1 \end{bmatrix} = \begin{bmatrix} -2.6667 \\ 0.3333 \\ -1.5 \end{bmatrix} = 2.6667 \begin{bmatrix} 1 \\ -0.125 \\ 0.5625 \end{bmatrix}$$

Continuing with subsequent calculations in the table form

i	x^{i-1}			y^i			λ	er
1	1	1	1	4	1	-6	-6	
2	0.6667	0.1667	1	-2.6667	0.3383	-1.5	-2.6667	0.555
3	1	-0.125	0.5625	4	-1.25	-3.5625	4	0.3333
4	1	-0.3125	0.8906	4	-1.625	0.98438	4	0

10. (a) Check whether the following matrix is positive definite

$$A = \begin{bmatrix} 4 & 1 & 2 \\ 1 & 5 & 3 \\ 2 & 3 & 7 \end{bmatrix}$$

(b) Find the largest eigenvalue of A using the Power method.

(c) Find the other two eigenvalues by substituting the above calculated eigenvalue in $|A - \lambda I| = 0$

Solution: (a) For matrix to be positive definite; the determinants of all the co-factor matrices must positive

$$A = \begin{bmatrix} 4 & 1 & 2 \\ 1 & 5 & 3 \\ 2 & 3 & 7 \end{bmatrix}$$

The determinants of all the co-factor matrices are

$$\text{Det } |4| > 0$$

$$\text{Det } \begin{vmatrix} 4 & 1 \\ 1 & 5 \end{vmatrix} = 19 > 0$$

$$\text{Det } \begin{vmatrix} 4 & 1 & 2 \\ 1 & 5 & 3 \\ 2 & 3 & 7 \end{vmatrix} = 89 > 0$$

Since the determinants of all the cofactor matrices are greater than 0, matrix A is positive definite

(b) Using Power method

$$A = \begin{bmatrix} 4 & 1 & 2 \\ 1 & 5 & 3 \\ 2 & 3 & 7 \end{bmatrix} \qquad \text{trail vector } x^{(0)} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$i = 1 \quad y^{(1)} = \begin{bmatrix} 4 & 1 & 2 \\ 1 & 5 & 3 \\ 2 & 3 & 7 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 7 \\ 9 \\ 12 \end{bmatrix} = 12 \begin{bmatrix} 0.5833 \\ 0.75 \\ 1 \end{bmatrix}$$

$$i = 2 \quad y^{(2)} = \begin{bmatrix} 4 & 1 & 2 \\ 1 & 5 & 3 \\ 2 & 3 & 7 \end{bmatrix} \begin{bmatrix} 0.5833 \\ 0.75 \\ 1 \end{bmatrix} = \begin{bmatrix} 5.0833 \\ 7.3333 \\ 10.4167 \end{bmatrix} = 10.4167 \begin{bmatrix} 0.488 \\ 0.7040 \\ 1 \end{bmatrix}$$

$$i = 3 \quad y^{(3)} = \begin{bmatrix} 4 & 1 & 2 \\ 1 & 5 & 3 \\ 2 & 3 & 7 \end{bmatrix} \begin{bmatrix} 0.488 \\ 0.7040 \\ 1 \end{bmatrix} = \begin{bmatrix} 4.656 \\ 7.008 \\ 10.088 \end{bmatrix} = 10.088 \begin{bmatrix} 0.4615 \\ 0.6947 \\ 1 \end{bmatrix}$$

$$i = 4 \quad y^{(4)} = \begin{bmatrix} 4 & 1 & 2 \\ 1 & 5 & 3 \\ 2 & 3 & 7 \end{bmatrix} \begin{bmatrix} 0.4615 \\ 0.6947 \\ 1 \end{bmatrix} = \begin{bmatrix} 4.5408 \\ 6.9350 \\ 10.0071 \end{bmatrix} = 10.0071 \begin{bmatrix} 0.4538 \\ 0.693 \\ 1 \end{bmatrix}$$

$$i = 5 \quad y^{(5)} = \begin{bmatrix} 4 & 1 & 2 \\ 1 & 5 & 3 \\ 2 & 3 & 7 \end{bmatrix} \begin{bmatrix} 0.4538 \\ 0.693 \\ 1 \end{bmatrix} = \begin{bmatrix} 4.5080 \\ 6.9188 \\ 9.9865 \end{bmatrix} = 9.9865 \begin{bmatrix} 0.4514 \\ 0.6928 \\ 1 \end{bmatrix}$$

$$i = 6 \quad y^{(6)} = \begin{bmatrix} 4 & 1 & 2 \\ 1 & 5 & 3 \\ 2 & 3 & 7 \end{bmatrix} \begin{bmatrix} 0.4514 \\ 0.6928 \\ 1 \end{bmatrix} = \begin{bmatrix} 4.4985 \\ 6.9155 \\ 9.9813 \end{bmatrix} = 9.9813 \begin{bmatrix} 0.4507 \\ 0.6928 \\ 1 \end{bmatrix}$$

After 6 iterations

$$\lambda = 9.9813 \qquad x = \begin{bmatrix} 0.4507 \\ 0.6928 \\ 1 \end{bmatrix}$$

(c) $(A - \lambda I)x = 0$

therefore

$$D(\lambda) = \det(\lambda) = \det(A - \lambda I) = \begin{bmatrix} 4-\lambda & 1 & 2 \\ 1 & 5-\lambda & 3 \\ 2 & 3 & 7-\lambda \end{bmatrix} = 0$$

This is expanded to get

$$P(\lambda) = \lambda^3 - 16\lambda^2 + 69\lambda - 89 = 0$$

Now that λ_i is known, P(λ) can be reduced by one degree.

$$\frac{P(\lambda)}{(\lambda-\lambda_i)} = \frac{\lambda^3 - 16\lambda^2 + 69\lambda - 89}{(\lambda - 9.9813)}$$

Now P(λ)=($\lambda - 9.9813)(\lambda^2 - 6.0187\lambda + 8.9255) = 0$

The remaining eigenvalues are evaluated by simply applying the quadratic equation.

$$\lambda_2, \lambda_3 = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{6.0187 \pm \sqrt{36.2247 - 4(8.9255)}}{2} = 3.371{,}2.648$$