Solutions to the Final Exam - Winter 2008

① ⓐ $\operatorname{curl} \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2 y & xy^2 & 2xyz \end{vmatrix} = (2xz - 0)\vec{i} - (2yz)\vec{j} + (y^2 - x^2)\vec{k}$

$\operatorname{div}(\operatorname{curl} \vec{F}) = \frac{\partial}{\partial x}(2xz) + \frac{\partial}{\partial y}(-2yz) + \frac{\partial}{\partial z}(y^2 - x^2) = 2z - 2z = 0$

ⓑ curl (div $\vec{F}$) cannot be computed as div $\vec{F}$ is a function and curl is an operator for vector fields

ⓒ $\operatorname{grad}(\operatorname{div} \vec{F}) = \operatorname{grad}\left(\frac{\partial}{\partial x}(x^2 y) + \frac{\partial}{\partial y}(xy^2) + \frac{\partial}{\partial z}(2xyz) \right) =$

$= \operatorname{grad}(2xy + 2xy + 2xy) = \operatorname{grad}(6xy) = (6y)\vec{i} + (6x)\vec{j}$

ⓓ div (grad $\vec{F}$) cannot be computed as grad cannot be applied to vector fields. #

② Let $g(x,y,z) = z^3 - xyz$

$\nabla g(x,y,z) = (-yz, -xz, 3z^2 - xy)$

$\nabla g(4, \frac{1}{2}, -1) = (\frac{1}{2}, 4, 1)$

$\therefore$ An equation of the tangent plane to the surface $z^3 - xyz = 1$ at the point $(4, \frac{1}{2}, -1)$ is:

$\frac{1}{2}(x-4) + 4(y - \frac{1}{2}) + (z+1) = 0$ #

(3) $$\int_0^2\int_{y^2}^4 e^{\sqrt{x^3}}\,dx\,dy = \int_0^4\int_0^{\sqrt{x}} e^{\sqrt{x^3}}\,dy\,dx =$$

$$= \int_0^4 \sqrt{x}\, e^{\sqrt{x^3}}\,dx \underset{\uparrow}{=} \int_0^8 \frac{2}{3} e^{u}\,du =$$

$u = \sqrt{x^3}$ $\left(u = x^{3/2}\right)$

$du = \frac{3}{2} x^{\frac{1}{2}}\,dx$

$$= \frac{2}{3} e^{u}\Big|_0^8 = \frac{2}{3}\left(e^8 - 1\right).$$ #

(4) let $\vec{u} = \vec{i}$; then $D_{\vec{u}} f(2,1,3) = \nabla f(2,1,3) \cdot \vec{u} =$

$$= \left(\frac{y}{z^2}, \frac{x}{z^2}, -\frac{2xy}{z^3}\right)\Big|_{(2,1,3)} \cdot (0,1,0) =$$

$$= \left(\frac{1}{9}, \frac{2}{9}, \frac{-4}{27}\right)\cdot(0,1,0) = \frac{2}{9}$$

For $\vec{i} + 2\vec{j} - \vec{k}$, notice first that this vector does not have magnitude 1, so we need to normalize it : $\vec{u} = \frac{(1,2,-1)}{\sqrt{6}}$

Then $D_{\vec{u}} f(2,1,3) = \left(\frac{1}{9}, \frac{2}{9}, \frac{-4}{27}\right)\cdot(1,2,-1)\cdot\frac{1}{\sqrt{6}} = \frac{1}{6}\left(\frac{5}{9} + \frac{4}{27}\right) =$

$$= \frac{1}{6}\cdot\frac{19}{27} = \frac{19}{162}$$ #

(5) This question asks for $\iint_S (\text{curl}\, \vec{F}) \cdot \vec{n}\, ds$.

We know that by Stokes' Theorem (all hypothesis are satisfied so the theorem can be applied) we have:

$$\iint_S (\text{curl}\, \vec{F}) \cdot \vec{n}\, ds = \int_C \vec{F} \cdot d\vec{r}$$

C can be parametrized by

$\vec{r}(t) = (2\cos t,\ 2\sin t,\ 4),\ 0 \le t \le 2\pi$

$\therefore\ \vec{r}\,'(t) = (-2\sin t,\ 2\cos t,\ 0)$

and $\vec{F}\big|_C = (6 \cdot 2\sin t \cdot 4)\vec{i} - 48\cos t\, \vec{j} + 2\sin t \cdot 4 \cdot e^{4\cos^2 t + \arctan 4}\, \vec{k}$

hence

$$\int_C \vec{F} \cdot d\vec{r} = \int_0^{2\pi} (-96\sin^2 t - 96\cos^2 t)\, dt = -96 \int_0^{2\pi} dt = -192\pi .$$

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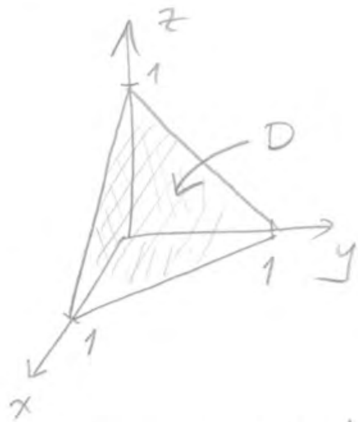
(6)



$$M = \int_0^1 \int_0^{1-x} \int_0^{1-x-y} (x+2y)\, dz\, dy\, dx =$$

$$= \int_0^1 \int_0^{1-x} (x+2y)(1-x-y)\, dy\, dx$$

$$D: \begin{cases} 0 \le x \le 1 \\ 0 \le y \le 1-x \\ 0 \le z \le 1-x-y \end{cases}$$

$$= \int_0^1 \int_0^{1-x} \left[x(1-x) + 2y(1-x) - xy - 2y^2\right] dy\, dx$$

$$= \int_0^1 \left(x(1-x)y + y^2(1-x) - \frac{xy^2}{2} - \frac{2y^3}{3} \right) \Big|_0^{1-x} dx$$

$$= \int_0^1 \left[x(1-x)^2 + (1-x)^3 - \frac{1}{2} x(1-x)^2 - \frac{2}{3}(1-x)^3 \right] dx$$

$$= \int_0^1 \left[\frac{1}{2} x(1-x)^2 + \frac{1}{3}(1-x)^3 \right] dx = \frac{1}{2} \int_0^1 (x - 2x^2 + x^3) dx$$

$$- \frac{1}{3} \cdot \frac{(1-x)^4}{4} \Big|_0^1 = \frac{1}{2} \left(\frac{x^2}{2} - \frac{2}{3}x^3 + \frac{x^4}{4} \right) \Big|_0^1 + \frac{1}{3} \cdot \frac{1}{4} =$$

$$= \frac{1}{2} \left(\frac{1}{2} - \frac{2}{3} + \frac{1}{4} \right) + \frac{1}{12} = \frac{1}{8}.$$

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(7) We seek $\varphi(x,y)$ s.t. $\begin{cases} \dfrac{\partial \varphi}{\partial x} = y e^{xy} \\ \dfrac{\partial \varphi}{\partial y} = x e^{xy} + 2y \end{cases}$

$\Rightarrow \varphi(x,y) = e^{xy} + c(y) \Rightarrow x e^{xy} + c'(y) = x e^{xy} + 2y$

$\therefore c'(y) = 2y \Rightarrow c(y) = y^2 + C$, C = constant

We may choose $C = 0$, hence $\varphi(x,y) = y^2 + e^{xy}$

So: $W = \int_C \vec{F} d\vec{r} = \varphi(1,2) - \varphi(0,0) = 4 + e^2 - 1 = 3 + e^2$.

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(8) $\vec{r}(t) = (t+1)\vec{i} + (t^2-t)\vec{j} + e^{-t}\vec{k}$

$\vec{r}\,'(t) = \vec{i} + (2t-1)\vec{j} - e^{-t}\vec{k} = \vec{v}(t)$

$\vec{r}\,''(t) = 2\vec{j} + e^{-t}\vec{k} = \vec{a}(t)$

$$a_T = \frac{|\vec{v}\cdot\vec{a}|}{\|\vec{v}\|} = \frac{|(2t-1)2 - e^{-2t}|}{\sqrt{1+(2t-1)^2+e^{-2t}}}$$

$$a_N = \frac{\|\vec{v}\times\vec{a}\|}{\|\vec{v}\|}$$

$$\vec{v}\times\vec{a} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & (2t-1) & -e^{-t} \\ 0 & 2 & e^{-t} \end{vmatrix} = (2t-1+2)e^{-t}\vec{i} - e^{-t}\vec{j} + 2\vec{k}$$

$$\therefore\ a_N = \frac{\sqrt{(2t+1)^2e^{-2t}+e^{-2t}+1}}{\sqrt{1+(2t-1)^2+e^{-2t}}}$$

and $$k(t) = \frac{\|\vec{r}\,'\times\vec{r}\,''\|}{\|\vec{r}\,'\|^3} = \frac{\|\vec{v}\times\vec{a}\|}{\|\vec{v}\|^3} = \frac{\sqrt{(2t+1)^2e^{-2t}+e^{-2t}+1}}{\sqrt{(1+(2t-1)^2+e^{-2t})^3}}$$

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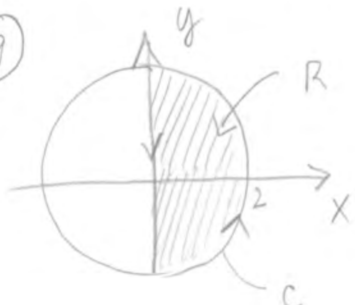

$$\oint_C y^2dx + x\,dy = \iint_R (1-2y)\,dA \ =$$

in polar coordinates

$$= \int_0^2\int_{-\pi/2}^{\pi/2} (1-2r\sin\theta)\,r\,d\theta\,dr = \int_0^2 \left(r\theta + 2r^2\cos\theta\right)\Big|_{-\pi/2}^{\pi/2} dr$$

$$= \pi\int_0^2 r\,dr = \frac{\pi}{2}\cdot r^2\Big|_0^2 = 2\pi$$

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(10)

$$\iint_S \vec{F}\cdot\vec{n}\, dS = \iiint (\operatorname{div} \vec{F})\, dV$$

$$D = \{(x,y,z) \mid x^2+y^2+z^2 \le a^2\}$$

$$= \iiint_D (3x^2+3y^2+3z^2+2z)\, dV =$$

in spherical coordinates

$$= \int_0^{2\pi}\int_0^{\pi}\int_0^{a} (3\rho^2 + 2\rho\cos\varphi)\,\rho^2 \sin\varphi \, d\rho\, d\varphi\, d\theta$$

$$= \int_0^{2\pi}\int_0^{\pi}\int_0^{a} (3\rho^4 + 2\rho^3\cos\varphi)\sin\varphi \, d\rho\, d\varphi\, d\theta$$

$$= \int_0^{2\pi}\int_0^{\pi} \left[\left(\frac{3\rho^5}{5} + \frac{2\rho^4}{4}\cos\varphi\right)\sin\varphi \,\Big|_{\rho=0}^{\rho=a}\right] d\varphi\, d\theta$$

$$= \int_0^{2\pi}\int_0^{\pi} \left[\frac{3a^5}{5}\sin\varphi + \frac{a^4}{4}\sin(2\varphi)\right] d\varphi =$$

$$= \int_0^{2\pi} \left\{\frac{3a^5}{5}(-\cos\varphi) + \frac{a^4}{4}\left[\frac{-\cos(2\varphi)}{2}\right]\right\}\Big|_{\varphi=0}^{\varphi=\pi} d\theta$$

$$= 2\pi\left[\frac{3a^5}{5}\cdot 2 + \frac{a^4}{4}\cdot 0\right] = 2\pi\,\frac{6a^5}{5} = \frac{12a^5}{5}\pi$$

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