**PROBLEMS FOR CHAPTER 7**

1. Use Power method to find the larger eigenvalue and the corresponding eigenvector using 5 iterations.

(a) $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$

**Solution:**

a)

$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ using general form (A - λI) x = 0

Initial guess $x^{(0)} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$y^{(1)}$= A$x^{(0)}$ = $\lambda^1$ $x^1$

= $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 7 \end{bmatrix} = 7 \begin{bmatrix} 0.4286 \\ 1 \end{bmatrix}$

Where $\lambda^1 = 7$

$x^1 = \begin{bmatrix} 0.4286 \\ 1 \end{bmatrix}$

$y^2 = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 0.4286 \\ 1 \end{bmatrix} = \begin{bmatrix} 2.4286 \\ 5.2858 \end{bmatrix} = 5.2858 \begin{bmatrix} 0.4595 \\ 1 \end{bmatrix}$

We continue in the form of a table

| $i$ | $x^{i-1}$ | | $y^i$ | | λ |
|---|---|---|---|---|---|
| 1 | 1 | 1 | 3 | 7 | 7 |
| 2 | 0.2486 | 1 | 2.4286 | 5.2858 | 5.2858 |
| 3 | 0.4595 | 1 | 2.4595 | 5.3785 | 5.3785 |
| 4 | 0.4573 | 1 | 2.4573 | 5.3719 | 5.3719 |
| 5 | 0.4574 | 1 | 2.4574 | 5.3722 | 5.3722 |

After 5 iteration

λ = 5.3722 $x = \begin{bmatrix} 0.4574 \\ 1 \end{bmatrix}$

b)

$$\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

Initial guess $x^{(0)} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$y^{(1)}$ = A$x^{(0)}$ = $\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix} = 2 \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

Where λ = 2 x = $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$y^{(2)} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} = 1 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

Which brings us to the initial guess therefore performing iterations will give us the same results.

| i | $x^{i-1}$ | | $y^{i}$ | | λ |
|---|---|---|---|---|---|
| 1 | 1 | 1 | 2 | 0 | 2 |
| 2 | 1 | 0 | 1 | 1 | 1 |
| 3 | 1 | 1 | 2 | 0 | 2 |
| 4 | 1 | 0 | 1 | 1 | 1 |
| 5 | 1 | 1 | 2 | 0 | 2 |

7. Using the Power method check the largest eigenvalue of matrix A using 7 iterations.

$$A = \begin{bmatrix} 0 & 2 & 3 \\ -10 & -1 & 2 \\ -2 & 4 & 7 \end{bmatrix}$$

**Solution:**

$$A = \begin{bmatrix} 0 & 2 & 3 \\ -10 & -1 & 2 \\ -2 & 4 & 7 \end{bmatrix} \qquad \text{trail } x^{(0)} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$i = 1 \quad y^{(1)} = \begin{bmatrix} 0 & 2 & 3 \\ -10 & -1 & 2 \\ -2 & 4 & 7 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \\ 7 \end{bmatrix} = 7 \begin{bmatrix} 0.42857 \\ 0.28571 \\ 1 \end{bmatrix}$$

$$i = 2 \quad y^{(2)} = \begin{bmatrix} 0 & 2 & 3 \\ -10 & -1 & 2 \\ -2 & 4 & 7 \end{bmatrix} \begin{bmatrix} 0.42857 \\ 0.28571 \\ 1 \end{bmatrix} = \begin{bmatrix} 3.5714 \\ 2.5714 \\ 7.2857 \end{bmatrix} = 7.2857 \begin{bmatrix} 0.4902 \\ -0.35294 \\ 1 \end{bmatrix}$$

$$i = 1 \quad y^{(3)} = \begin{bmatrix} 0 & 2 & 3 \\ -10 & -1 & 2 \\ -2 & 4 & 7 \end{bmatrix} \begin{bmatrix} 0.4902 \\ -0.35294 \\ 1 \end{bmatrix} = \begin{bmatrix} 2.2941 \\ -2.549 \\ 4.6078 \end{bmatrix} = 4.6078 \begin{bmatrix} 0.49787 \\ -0.55319 \\ 1 \end{bmatrix}$$

$$i = 1 \quad y^{(4)} = \begin{bmatrix} 0 & 2 & 3 \\ -10 & -1 & 2 \\ -2 & 4 & 7 \end{bmatrix} \begin{bmatrix} 0.49787 \\ -055319 \\ 1 \end{bmatrix} = \begin{bmatrix} 1.8936 \\ -2.4255 \\ 3.7915 \end{bmatrix} = 3.7915 \begin{bmatrix} 0.49944 \\ -0.63973 \\ 1 \end{bmatrix}$$

$$i = 1 \quad y^{(5)} = \begin{bmatrix} 0 & 2 & 3 \\ -10 & -1 & 2 \\ -2 & 4 & 7 \end{bmatrix} \begin{bmatrix} 0.49944 \\ -0.63973 \\ 1 \end{bmatrix} = \begin{bmatrix} 1.7205 \\ -2.3547 \\ 3.4422 \end{bmatrix} = 3.4422 \begin{bmatrix} 0.4999 \\ -0.68406 \\ 1 \end{bmatrix}$$

$$i = 1 \quad y^{(6)} = \begin{bmatrix} 0 & 2 & 3 \\ -10 & -1 & 2 \\ -2 & 4 & 7 \end{bmatrix} \begin{bmatrix} 0.4999 \\ -0.68406 \\ 1 \end{bmatrix} = \begin{bmatrix} 1.6319 \\ -2.3143 \\ 3.2641 \end{bmatrix} = 3.2641 \begin{bmatrix} 0.5 \\ -0.70902 \\ 1 \end{bmatrix}$$

$$i = 1 \quad y^{(7)} = \begin{bmatrix} 0 & 2 & 3 \\ -10 & -1 & 2 \\ -2 & 4 & 7 \end{bmatrix} \begin{bmatrix} 0.5 \\ -0.70902 \\ 1 \end{bmatrix} = \begin{bmatrix} 1.582 \\ -2.2905 \\ 3.164 \end{bmatrix} = 3.164 \begin{bmatrix} 0.5 \\ -0.7239 \\ 1 \end{bmatrix}$$

after 7 iterations

$$\lambda = 3.164 \qquad x = \begin{bmatrix} 0.5 \\ -0.7239 \\ 1 \end{bmatrix}$$

9. Evaluate the dominant eigenvalue and the eigenvector of the given matrix by Power method. Use 4 iterations.

$$A = \begin{bmatrix} 2 & -2 & 0 \\ -1 & 2 & -1 \\ 0 & -4 & 2 \end{bmatrix}$$

**Solution:**

$$A = \begin{bmatrix} 2 & -2 & 0 \\ -1 & 2 & -1 \\ 0 & -4 & 2 \end{bmatrix} \qquad \text{trail } x^{(0)} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$i = 1 \quad y^{(1)} = \begin{bmatrix} 2 & -2 & 0 \\ -1 & 2 & -1 \\ 0 & -4 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ -2 \end{bmatrix} = -2\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$i = 2 \quad y^{(2)} = \begin{bmatrix} 2 & -2 & 0 \\ -1 & 2 & -1 \\ 0 & -4 & 2 \end{bmatrix}\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ 2 \end{bmatrix} = 2\begin{bmatrix} 0 \\ -0.5 \\ 1 \end{bmatrix}$$

$$i = 3 \quad y^{(3)} = \begin{bmatrix} 2 & -2 & 0 \\ -1 & 2 & -1 \\ 0 & -4 & 2 \end{bmatrix}\begin{bmatrix} 0 \\ -0.5 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \\ 4 \end{bmatrix} = 4\begin{bmatrix} 0.25 \\ -0.5 \\ 1 \end{bmatrix}$$

$$i = 4 \quad y^{(4)} = \begin{bmatrix} 2 & -2 & 0 \\ -1 & 2 & -1 \\ 0 & -4 & 2 \end{bmatrix}\begin{bmatrix} 0.25 \\ -0.5 \\ 1 \end{bmatrix} = \begin{bmatrix} 1.5 \\ -2.25 \\ 4.0 \end{bmatrix} = 4\begin{bmatrix} 0.375 \\ -0.5625 \\ 1 \end{bmatrix}$$

After 4 iterations

$$\lambda = 4 \qquad x = \begin{bmatrix} 0.375 \\ -0.5625 \\ 1 \end{bmatrix}$$

11. The vibration of a system of 3 masses connected by springs can be described by the following 3 equations.

$$\begin{bmatrix} (2-\lambda) & -1 & 0 \\ 1 & (2-\lambda) & -1 \\ 0 & -1 & (2-\lambda) \end{bmatrix}$$

Apply Power method with the characteristic equation to find the eigenvalues and eigenvectors of the system.

**<u>Solution:</u>**

Using Power method

A = $\begin{bmatrix} 2 & -1 & 0 \\ 1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$ trail vector $x^{(0)} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

i = 1 $y^{(1)} = \begin{bmatrix} 2 & -1 & 0 \\ 1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} = 2 \begin{bmatrix} 0.5 \\ 1 \\ 0.5 \end{bmatrix}$

i = 2 $y^{(2)} = \begin{bmatrix} 2 & -1 & 0 \\ 1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} \begin{bmatrix} 0.5 \\ 1 \\ 0.5 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix} = 2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$

i = 3 $y^{(3)} = \begin{bmatrix} 2 & -1 & 0 \\ 1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \\ -1 \end{bmatrix} = 2 \begin{bmatrix} -0.5 \\ 1 \\ -0.5 \end{bmatrix}$

After 3 iterations

$\lambda = 2$ x = $\begin{bmatrix} -0.5 \\ 1 \\ -0.5 \end{bmatrix}$

$$(A-\lambda I)x = 0$$

therefore

$$D(\lambda) = \det(\lambda) = \det(A-\lambda I) = \begin{bmatrix} (2-\lambda) & -1 & 0 \\ 1 & (2-\lambda) & -1 \\ 0 & -1 & (2-\lambda) \end{bmatrix} = 0$$

This is expanded to get

$$P(\lambda) = \lambda^3 - 6\lambda^2 + 12\lambda - 8 = 0$$

Now that $\lambda_i$ is known, P($\lambda$) can be reduced by one degree.

$$\frac{P(\lambda)}{(\lambda-\lambda_i)} = \frac{(\lambda-2)^3}{(\lambda-2)}$$

Now P( $\lambda$ )=( $\lambda-2)(\lambda-2)^2 = 0) = 0$

The remaining eigenvalues are evaluated by simply applying the quadratic equation.

$$\lambda_2, \lambda_3 = 2,2$$

14. Determine the smallest eigenvalue of the given system in problem 13 using inverse Power method.

**Solution:**

$$A = \begin{bmatrix} 3 & -1 \\ -0.5 & 0.5 \end{bmatrix}$$

Therefore

$$A^{-1} = \begin{bmatrix} 0.5 & 1 \\ 0.5 & 3 \end{bmatrix}$$

Let trail vector $x^{(0)} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

i = 1 $y^{(1)} = \begin{bmatrix} 0.5 & 1 \\ 0.5 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1.5 \\ 3.5 \end{bmatrix} = 3.5 \begin{bmatrix} 0.4286 \\ 1 \end{bmatrix}$

i = 2 $y^{(2)} = \begin{bmatrix} 0.5 & 1 \\ 0.5 & 3 \end{bmatrix} \begin{bmatrix} 0.4286 \\ 1 \end{bmatrix} = \begin{bmatrix} 1.2143 \\ 3.2143 \end{bmatrix} = 3.2143 \begin{bmatrix} 0.3778 \\ 1 \end{bmatrix}$

i = 3 $y^{(3)} = \begin{bmatrix} 0.5 & 1 \\ 0.5 & 3 \end{bmatrix} \begin{bmatrix} 0.3778 \\ 1 \end{bmatrix} = \begin{bmatrix} 1.1889 \\ 3.1889 \end{bmatrix} = 3.1889 \begin{bmatrix} 0.3728 \\ 1 \end{bmatrix}$

i = 4 $y^{(4)} = \begin{bmatrix} 0.5 & 1 \\ 0.5 & 3 \end{bmatrix} \begin{bmatrix} 0.3728 \\ 1 \end{bmatrix} = \begin{bmatrix} 1.1864 \\ 3.1864 \end{bmatrix} = 3.1864 \begin{bmatrix} 0.3723 \\ 1 \end{bmatrix}$

Therefore

$\lambda = \frac{1}{3.1864} = 0.3138$ and $x = \begin{bmatrix} 0.3723 \\ 1 \end{bmatrix}$