CONCORDIA UNIVERSITY DEPARTMENT OF COMPUTER SCIENCE AND SOFTWARE ENGINEERING.

COMP 233.

Midterm Exam.

1. (25 marks)

A. Suppose the sample space S consists of all lowercase five-letter words having distinct alphabetic characters. Assuming the outcomes in S to be equally likely, what is the probability of randomly drawing a word in which only the second and the third character is a vowel, i.e.: one of: a, e, i, o, u, y?

B. Determine how many nonnegative integer solutions there are to:

$$x1 + x2 + x3 + x4 = 20$$

C. Each of 2 cabinets identical in appearance has 2 drawers. Cabinet A contains a silver coin in each drawer, and cabinet B contains a silver coin in one of its drawers and a gold coin in the other. A cabinet is randomly selected, one of its drawers is opened, and a silver coin is found. What is the probability that there is a silver coin in the other drawer?

D. A satellite system consists of 4 components, and can function adequately if at least 2 of the 4 components are in working condition. Each component works independently of the others. The probability of a component working is 0.6. What is the probability that the system functions adequately?

E. Defects in a certain wire occur at the rate of one per 10 meters. Assume the defects have a Poisson distribution. Find the probability that a 20 meter wire has at most 1 defect.

2. (20 marks) Consider the following table representing a joint probability mass function $P_{XY}(x, y)$.

	Y=6	Y=8	Y=10	$P_X(\cdot)$
X=1	$\frac{1}{5}$	0	$\frac{1}{5}$	$\frac{2}{5}$
X=2	0	$\frac{1}{5}$	0	$\frac{1}{5}$
X=3	$\frac{1}{5}$	0	$\frac{1}{5}$	$\frac{2}{5}$
$P_Y(\cdot)$	$\frac{2}{5}$	$\frac{1}{5}$	$\frac{2}{5}$	1

A. Compute E[X].
B. Compute E[XY].
C. Compute Cov(X, Y).
D. Are X and Y independent random variables? Prove your answer.

3. (30 marks)

$$f(x) = \begin{cases} x + 1 & -1 < x \leq 0 \\ 1 - x & 0 < x \leq 1 \\ 0 & otherwise \end{cases}$$

A. Draw the graph of f(x).
B. Determine the distribution function F(x).
C. Draw the graph of F(x).
D. Determine E[X].
E. Find P(|X|< 1/2).
F. Compute Var(X).

4. (15 marks) Consider random variables X, Y with the joint density:

$$f_{X,Y}(x,y) = \begin{cases} ce^{-x}e^{-2y} & 0 < x, 0 < y \\ 0 & otherwise. \end{cases}$$

A. Find the value of c.

B. Determine the marginal densities $f_X(x)$ and $f_Y(y)$.

C. Determine whether X and Y are independent.

5. A. **(5 marks)** Assume we have a large body of text, for example, articles from a publication. Assume we know that the articles are on average 1000 characters long with a standard deviation of 200 characters. Use Chebyshev's Inequality to find the lower bound on the percentage of articles having length between 600 and 1400 characters?

B. **(5 marks)** The time that a skier takes on a downhill course has a Normal Distribution with a mean of 12.3 minutes and standard deviation of 0.4 minutes. Find the probability that on a random run the skier takes between 12.1 and 12.5 minutes.

MIDTERM

SOLUTIONS.

1. a) The number of words satisfying the requirements is:

$$(20)\cdot(6)\ (5)\ (19)(18) = 205{,}200.$$

The number of elements in S is:

$$(26)(25)(24)(23)(22) = \frac{26!}{21!} = 7{,}893{,}600.$$

So the required probability is $205{,}200/7{,}893{,}600 = 0.026$.

b)

$$C(20 + 4 - 1, 4 - 1) = C(23,3) = \frac{23!}{20!\,3!} = 1{,}771.$$

c)

P{SC in the second drawer|SC in the first drawer}

= P{SC 2 ∈ II |SC ∈ I|SC ∈ }

= P{SC|SC ∈ I and SC ∈ II}/P{SC ∈ I}

= P{A}/[P{SC ∈ I/A}P{A} + P{SC ∈ I/B}P{B}]

= 1/2/[1 · 1/2 + 1/2 · 1/2] = (1/2)/(3/4) = 2/3.

Please refer to the end of this document for an alternate solution to problem 1.C.

d)

$C(4,2)0.6^2(1-0.6)^2 + C(4,3).6^3(1-0.6) + C(4,4).6^4(1-0.6)^0$

=0.8208 .

e)

$$P\{N = k\} = e^{\lambda}\frac{\lambda^k}{k!}, \qquad \lambda = 2 \text{ defects/20-meters.}$$

$$P\{N \le 1\} = e^{-2}\left(\frac{2^0}{0!} + \frac{2^1}{1!}\right) = 3e^{-2} = 0.41.$$

2. a)

$$EX = 1\cdot\frac{2}{5} + 2\cdot\frac{1}{5} + 3\cdot\frac{2}{5} = 2.$$

b)

$$E_{XY} = \sum_{i=1}^{3}\sum_{j=1}^{3} x_i y_j P_{XY}(x_i, x_j)$$

$$= 1\cdot 6\cdot\frac{1}{5} + 1\cdot 8\cdot 0 + 1\cdot 10\cdot\frac{1}{5} + 2\cdot 6\cdot 0 + 2\cdot 8\cdot\frac{1}{5} + 2\cdot 10\cdot 0 + 3\cdot 6\cdot\frac{1}{5} + 3\cdot 8\cdot 0 + 3\cdot 10.\frac{1}{5} = 16.$$

c)

$$Cov(X,Y) = E_{XY} - E_X E_Y$$

$$E_Y = 6\cdot\frac{2}{5} + 8\cdot\frac{1}{5} + 10\cdot\frac{2}{5} = 8$$

$$Cov(X,Y) = 16 - 2\cdot 8 = 0.$$

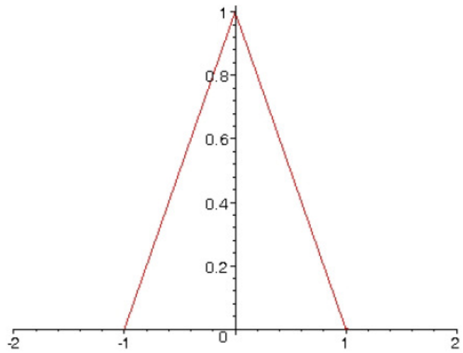
d) X and Y are not independent as

$$P_{X,Y}(x,y) \ne P_X(x)P_Y(y).$$

Indeed, e. g.,

$$P_{X,Y}(1,8) = 0 \ne P_X(1)P_Y(8) = \frac{2}{5}\cdot\frac{1}{5} = \frac{2}{25}.$$

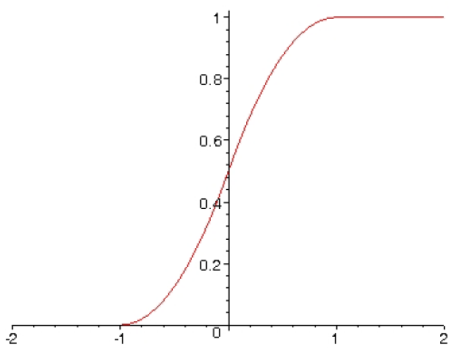
3. a)



b)

$$F(x) = \int_{-\infty}^{x} f(y)dy = \begin{cases} 0, & x \leq 1 \\ \int_{-1}^{x} (y + 1)dy = \frac{1}{2}x^2 + x + \frac{1}{2}, & -1 < x \leq 0 \\ \frac{1}{2} + \int_{0}^{x} (1 - y)dy = -\frac{1}{2}x^2 + x + \frac{1}{2}, & 0 < x \leq 1 \\ 1, & x > 1 \end{cases}$$

c)



d) E[X] = 0 since f(x) is even function.

e)

$$\begin{aligned} & P\left\{|X| < \tfrac{1}{2}\right\} \\ &= 1 - P\{|X| \geq \tfrac{1}{2}\} \\ &= 1 - P\{X \geq \tfrac{1}{2}\} + P\{X \leq -\tfrac{1}{2}\} \\ &= 1 - 2P\{X \geq \tfrac{1}{2}\} \\ &= 1 - 2(1 - P\left\{X < \tfrac{1}{2}\right\}) \\ &= 1 - 2\left(1 - F\left(\tfrac{1}{2}\right)\right) \\ &= 1 - 2(-(-\tfrac{1}{8} + \tfrac{1}{2} + \tfrac{1}{2})) \\ &= 1 - 2(1 - \tfrac{7}{8}) \\ &= 1 - 2 \cdot \tfrac{1}{8} \\ &= \tfrac{3}{4}. \end{aligned}$$

ALTERNATE SOLUTION:
P(|X| < 0.5) = P(-0.5 < X < 0.5)
= F(0.5) - F(-0.5) = 7/8 - 1/8 = 6/8 = 3/4.

$f)$

$$E[X^2] = \int_{-1}^{0} x2(x + 1)dx + \int_{0}^{1} x^2(x + 1)dx$$
$$= \left(\frac{x^4}{4} + \frac{x^3}{3}\right)|_0^1 + \left(\frac{x^3}{3} - \frac{x^4}{4}\right)|_0^1 = \frac{1}{6}.$$
$$Var(X) = E[X^2] - E^2[X] = \frac{1}{6} - 0 = \frac{1}{6}.$$

4. a) $1 = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{X,Y}(x,y)dxdy = c\int_0^{\infty} e^{-x}dx \int_0^{\infty} e^{-2y}\, dy$
$= c[-e^{-x}]\Big|_0^{\infty} \cdot \left[-\frac{1}{2}e^{-2y}\right]\Big|_0^{\infty} = c.\frac{1}{2} \Rightarrow c = 2.$

b)

$$f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)dy = \int_0^{\infty} 2e^{-x}e^{(-2y)}dy = 2e^{-x}\left[-\frac{1}{2}e^{-2y}\right]\Big|_0^{\infty} = \begin{cases} e^{-x} & 0 < x \\ 0, & otherwise. \end{cases}$$

$$f_Y(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)dx = \int_0^{\infty} 2e^{-x}e^{(-2y)}dx = 2e^{-2y}[-e^{-x}]\Big|_0^{\infty} = \begin{cases} 2e^{-2y} & 0 < y \\ 0, & otherwise. \end{cases}$$

c) Note

$$f_{X,Y}(x,y) = \begin{cases} 2e^{-x}e^{-2y}, & 0 < x,\ 0 < y \\ 0, & otherwise \end{cases} = \begin{cases} e^{-x}.2e^{-2y}, & 0 < x, 0 < y \\ 0, & otherwise \end{cases} = f_X(x).f_Y(y)$$

Thus X and Y are independent.

5. a)

$P\{600 \le X \le 1400\}$
$= P\{600 - 1000 \le X - 1000 \le 1400 - 1000\}$
$= P\{-400 \le X - 1000 \le 400\}$
$= P\{|X - 1000| > 400\}$
$= 1 - P\{|X - \mu| > \epsilon\}$
$\ge 1 - \frac{\sigma^2}{\epsilon^2}$
$= 1 - \frac{200\text{^}2}{400^2}$
$= 1 - 0.25$
$= 0.75$

PLEASE REFER TO THE END OF THIS DOCUMENT FOR AN EDITED SOLUTION TO PROBLEM 5.A.

b) In this problem $X \sim N(12.3, 0.4^2)$.

$P\{12.1 < X < 12.5\}$
$=P\left\{\frac{12.1-12.3}{0.4} < \frac{X-12.3}{0.4} < \frac{12.5-12.3}{0.4}\right\}$
$= P\{-0.5 < Z < 0.5\}$
$= 1 - 2(1 - \Phi(0.5))$
$= 1 - 2(0.6915)$
$= 0.383$

ALTERNATE SOLUTION TO PROBLEM 1.C.

Let F: denote event of finding first silver coin.
S: denote event of finding second silver coin.
A: denote event of choosing cabinet A.
B: denote event of choosing cabinet B.

P(S|F)?

$$P(S|F) = \frac{P(S \cap F)}{P(F)}$$

$$= \frac{P(S \cap F)}{P((F \cap A) \cup (F \cap B))}$$

$$= \frac{P(S \cap F)}{P(F \cap A) + P(F \cap B)}$$

$$= \frac{P(S \cap F)}{P(F \mid A)P(A) + P(F \mid B)P(B)}$$

$$= \frac{\frac{1}{2}}{(1)\left(\frac{1}{2}\right) + \left(\frac{1}{2}\right)\left(\frac{1}{2}\right)}$$

$$P(S|F) = \frac{2}{3}.$$

EDITED SOLUTION TO PROBLEM 5.A.

Let R.V. X: denote length of articles.

$\mu = 1000.$

$\sigma = 200.$

P(600 ≤ X ≤ 1400)?

P(600 ≤ X ≤ 1400) = P(600 - 1000 ≤ X - 1000 ≤ 1400 - 1000)

= P(-400 ≤ X - 1000 ≤ 400)

= P(|X – 1000| ≤ 400)

= 1 - P(|X – 1000| > 400)

= 1 - P(|X – μ| > k) Apply Chebyshev's Inequality here

$\geq 1 - \frac{\sigma^2}{k^2}$

$\geq 1 - \frac{200^2}{400^2}$

P(600 ≤ X ≤ 1400) ≥ 0.75.