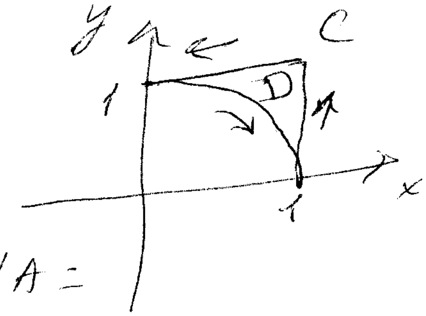
Sample exam solutions

(1) Find $\cancel{\iint}$ $\oint_C (x^2-y^2)\,dx+(x^2+y^2)\,dy$ if the contour C consists of the segment $x=1,\ 0\le y\le 1$, the segment $y=1,\ 0\le x\le 1$, and the portion of the circle $x^2+y^2=1$ in the 1-st quadrant.



Solution:

By the Green's Theorem,

$$I=\oint_C P\,dx+Q\,dy=\iint_D\left(\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\right)dA=$$

$$=\iint_D (2x+2y)\,dA = I_1 - I_2$$

$$I_1=\int_0^1\int_0^1 (2x+2y)\,dy\,dx = \int_0^1 (2x+1)\,dx = 2$$

$$I_2=\iint_{x^2+y^2\le 1}(2x+2y)\,dA=\int_0^{\pi/2}\int_0^1 (2r\cos\theta+2r\sin\theta)\,r\,dr\,d\theta$$

$$=\int_0^{\pi/2}\int_0^1 2r^2(\cos\theta+\sin\theta)\,dr\,d\theta=\frac{2}{3}\int_0^{\pi/2}(\cos\theta+\sin\theta)\,d\theta$$

$$=\frac{2}{3}\cdot 2=\frac{4}{3}$$

$$I=I_1-I_2=2-\frac{4}{3}=\frac{2}{3}.$$ (Answer)

(2) $\mathbb{F} = (2x - 2z)\,\mathbf{i} + z\,\mathbf{j} + (y - 2x)\,\mathbf{k}$.

Find $\displaystyle\int_{(1,2,3)}^{(3,2,1)} \mathbb{F} \cdot d\mathbf{r}$.

<u>Solution</u> First make sure that the integral is path independent.

$$\operatorname{curl} \mathbb{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial_x & \partial_y & \partial_z \\ (2x-2z) & z & (y-2x) \end{vmatrix} = (0, 0, 0); \text{ OK}$$

Now find the function $f(x, y, z)$ s.t.

$\mathbb{F} = \nabla f$.

$$\frac{\partial f}{\partial x} = 2x - 2z; \quad f(x,y,z) = \int (2x - 2z)\,\cancel{dx} + g(y,z) =$$

$$f(x,y,z) = \int (2x - 2z)\,dx + g(y,z) = x^2 - 2xz + g(y,z);$$

$$\frac{\partial f}{\partial y} = \frac{\partial g}{\partial y} = z; \quad g(y,z) = yz + h(z);$$

$$f(x,y,z) = x^2 - 2xz + yz + h(z)$$

$$\frac{\partial f}{\partial z} = -2x + y + h'(z) = y - 2x; \quad h'(z) = 0; \; h(z) = C$$

$$f(x,y,z) = x^2 - 2xz + yz.$$

$$\int_{(1,2,3)}^{(3,2,1)} \mathbb{F}\cdot d\mathbf{r} = f(3,2,1) - f(1,2,3) =$$

$$= (9 - 2\cdot 3 + 2) - (1 - 2\cdot 3 + 2\cdot 3) =$$

$$= 5 - 1 = 4 \text{ (Answer)}.$$

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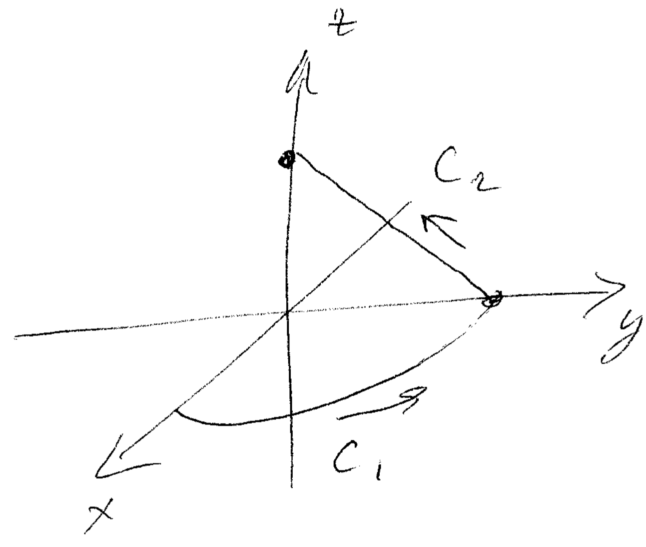
(3) Find $\int_C \mathbb{F}\cdot d\mathbf{r}$ if $\mathbb{F}(x,y,z) = (z^2, xy, 2y)$, and C consists of the piece of the circle $x^2+y^2=1$, $z=0$ in the 1-st quadrant, and the segment of the line connecting the points $(0,1,0)$ and $(0,0,1)$.

Solution.



$$C = C_1 + C_2$$

C_1: $x = \cos t$, $y = \sin t$, $z = 0$ $\left(0 \le t \le \frac{\pi}{2}\right)$

C_2: $x=0$, $y=1-t$, $z=t$ $(0 \le t \le 1)$.

$$\int_C \mathbb{F} \cdot d\mathbf{r} = \int_{C_1} \ldots + \int_{C_2} \ldots = I_1 + I_2$$

$$\cancel{I_1 = \int_{C_1} \mathbb{F} \cdot d\mathbf{r} = \int_0^{\pi/2} (0, \cos t \cdot \sin t, 2\sin t) \cdot (-\sin t, \cos t,}$$

$$I_1 = \int_{C_1} \mathbb{F} \cdot d\mathbf{r} = \int_0^{\pi/2} (0, \cos t \cdot \sin t, 2\sin t) \cdot (-\sin t, \cos t, 0) dt$$

$$= \int_0^{\pi/2} \cos^2 t \sin t \, dt =$$

$u = \cos t$
$du = -\sin t \, dt$

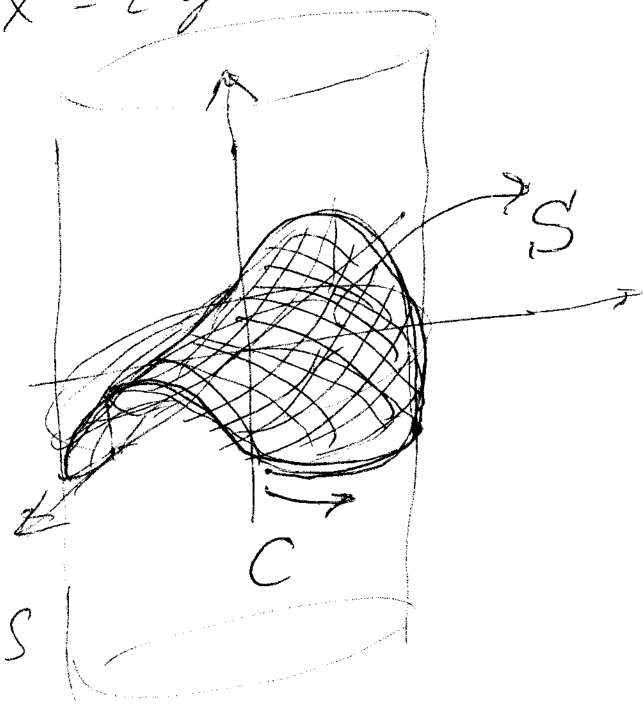
$$= \int_1^0 u^2 \cdot (-du) = \int_0^1 u^2 du = \frac{1}{3}$$

$$I_2 = \int_{C_2} \mathbb{F} \cdot d\mathbf{r} = \int_0^1 (t^2, 0, 2-2t) \cdot (0, -1, 1) \, dt =$$

$$= \int_0^1 (2-2t) \, dt = 1.$$

$$I = \frac{1}{3} + 1 = \frac{4}{3}$$ (Answer)

(4) Using the Stokes Theorem, find $\oint_C \mathbb{F}\cdot d\mathbb{r}$ if $\mathbb{F}(x,y,z) = (y, -2x+z, x)$, and the contour C is the intersection of the surface $z = x^2 - 2y^2$ and the cylinder $x^2+y^2=4$.



Solution

By the Stokes Theorem,

$$\oint_C \mathbb{F}\cdot d\mathbb{r} = \iint_S \operatorname{curl}\mathbb{F}\cdot \mathbb{n}\, dS$$

~~where~~ $$\operatorname{Curl}\mathbb{F} = \begin{vmatrix} \mathbb{i} & \mathbb{j} & \mathbb{k} \\ \partial_x & \partial_y & \partial_z \\ y & -2x+z & x \end{vmatrix} =$$

$$= (-1, -1, -1).$$

The surface S: $z = x^2 - 2y^2$, $x^2 + y^2 \le 4$.

$$\oint \mathbf{F}\cdot d\mathbf{r} = \iint_S \operatorname{curl} \mathbf{F} \cdot \mathbf{n}\, dS =$$

$$\cancel{= \iint_D (-1)} \qquad \boxed{(D:\ 0 \le x^2 + y^2 \le 4)}$$

$$= \iint_D (-1, -1, -1)\cdot(-2x,\ 4y,\ 1)\, dA_{xy}$$

$$= \iint_D (2x - 4y - 1)\, dA_{xy} =$$

$$= \int_0^{2\pi}\int_0^{2} (2r\cos\theta - 4r\sin\theta - 1)\, r\, dr\, d\theta$$

$$\cancel{= \int_0^{2\pi} \left(\frac{2}{3}\cos\theta - \frac{4}{3}\sin\theta - \frac{1}{2}\right) d\theta = -\pi}$$

(Answer)

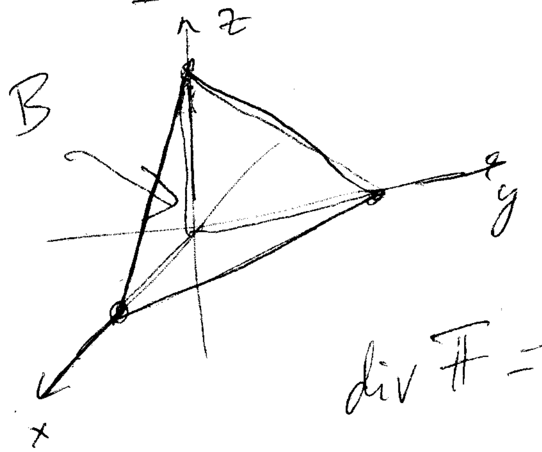
$$= \int_0^{2\pi} \left(\frac{16}{3}\cos\theta - \frac{32}{3}\sin\theta - 2\right) d\theta = -4\pi.$$

(5) Using the Divergence Theorem, find the flux of the field

$\mathbf{F}(x,y,z) = -xy\,\mathbf{i} + 2yz\,\mathbf{j} + 3xz\,\mathbf{k}$ through the surface of the body

$B: x \geq 0, \quad y \geq 0, \quad z \geq 0,$
$x + y + z \leq 1.$

Solution: $I = \iint_S \mathbf{F} \cdot \mathbf{n}\, dS = \iiint_B \operatorname{div} \mathbf{F}\, dV$



$\operatorname{div} \mathbf{F} = -y + 2z + 3x = 3x - y + 2z.$

$I = \iiint_B (3x - y + 2z)\, dV =$

$= \int_0^1 \int_0^{1-x} \int_0^{1-x-y} (3x - y + 2z)\, dz\, dy\, dx =$

$$= \int_0^1 \int_0^{1-x} \left[3x(1-x-y) - y(1-x-y) + (1-x-y)^2\right] dy\, dx$$

$$= \int_0^1 \int_0^{1-x} \left[3x - 3x^2 - 3xy - y + xy + y^2 + 1 + x^2 + y^2 - 2x - 2y + 2xy\right] dy\, dx =$$

$$= \int_0^1 \int_0^{1-x} \left[-2x^2 + 2y^2 + x - 3y + 1\right] dy\, dx =$$

$$= \int_0^1 \left[-2x^2(1-x) + \frac{2}{3}(1-x)^3 + x(1-x) - \frac{3}{2}(1-x)^2 + 1 - x\right] dx$$

$$= \int_0^1 \left[-2x^2 + 2x^3 + \frac{2}{3} - 2x + 2x^2 - \frac{2}{3}x^3 + x - x^2 - \frac{3}{2} + 3x - \frac{3}{2}x^2 + 1 - x\right] dx =$$

$$\cancel{= \int_0^1 \left[\frac{1}{3}x^3\right.}$$

$$= \int_0^1 \left[\frac{4}{3}x^3 - \frac{5}{2}x^2 + x + \frac{1}{6}\right] dx =$$

$$= \frac{4}{9}$$

$$= \frac{1}{3} - \frac{5}{6} + \frac{1}{2} + \frac{1}{6} = \frac{1}{6}$$

Answer

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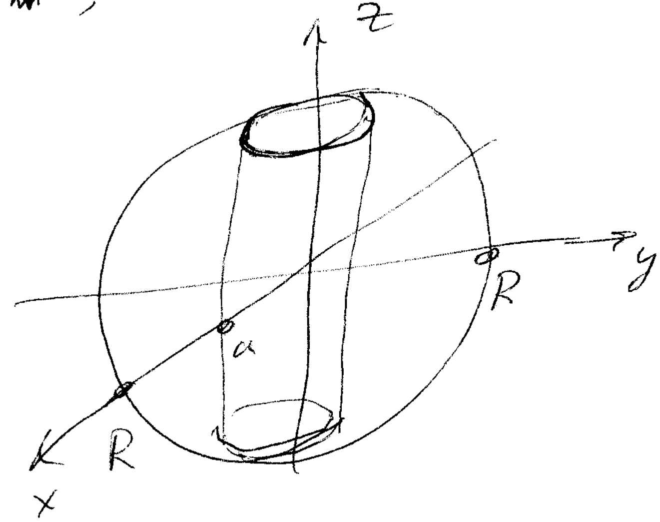
⑥ Find the moment of inertia I_z of the ball of radius R with a cylindrical hole of radius a (along the z-axis); the density is δ.

Solution

$$I_z = \iiint_B \delta (x^2+y^2)\, d\cancel{V};$$

in the cylindrical coordinates,



$$I_z = \int_0^{2\pi}\int_a^R$$

$$I_z = \int_0^{2\pi}\int_a^R\int_{-\sqrt{R^2-r^2}}^{\sqrt{R^2-a^2}} \delta r^2 \cdot r\, dz\, dr\, d\theta =$$

$$= 2\pi \int_a^R 2\delta\sqrt{R^2-r^2}\; r^3\, dr =$$

$$= 4\pi\delta \int_a^R \sqrt{R^2-r^2}\; r^3\, dr;$$

$$r^2 = s; \quad 2r\,dr = ds; \quad r\,dr = \tfrac{1}{2}\, ds$$

$$\int_a^R \sqrt{R^2-r^2}\; r^3\, dr = \frac{1}{2}\int_{a^2}^{R^2} s\sqrt{R^2-s}\; ds =$$

$$\frac{1}{2}\int_{a^2}^{R^2} \sqrt{R^2-s}\; \cdot s$$

$$= \frac{1}{2}\left[-s\cdot\frac{2}{3}(R^2-s)^{\frac{3}{2}}\Big|_{a^2}^{R^2} + \int_{a^2}^{R^2} \frac{2}{3}(R^2-s)^{\frac{3}{2}}\, ds\right] =$$

(integration by parts) $= \frac{1}{3}\left\{a^2(R^2-a^2)^{\frac{3}{2}} + \frac{1}{3}\cdot\frac{2}{5}(R^2-a^2)^{\frac{5}{2}}\right\};$

Hence,

$$I_z = \frac{4\pi\delta}{3} a^2 (R^2-a^2)^{\frac{3}{2}} + \frac{8\pi\delta}{15}(R^2-a^2)^{\frac{5}{2}}.$$

(answer)

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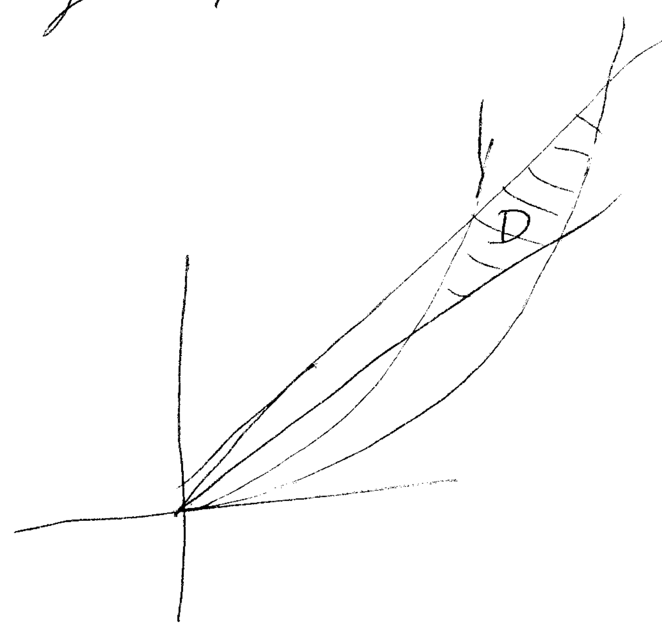
(7) Using the change of variables, find $\iint_D xy\,dA$ if D is the domain bounded by the curves $y = a_1 x^2$, $y = a_2 x^2$, $y = b_1 x$, $y = b_2 x$ $(a_1 < a_2,\ b_1 < b_2)$.



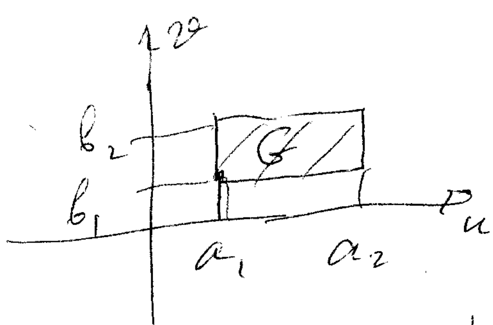
Solution

Let us introduce new variables

$$u = \frac{y}{x^2},\quad v = \frac{y}{x} \qquad (x, y > 0)$$

Then the variables u, v change in the domain

$G: a_1 \le u \le a_2,\ b_1 \le v \le a_2$:



The map $(u,v) \to (x,y)$ is determined from the system of equations

$$\begin{cases} \dfrac{y}{x^2} = u \\[2ex] \dfrac{y}{x} = v \end{cases}$$

$$x = \frac{v}{u};\quad y = vx = \frac{v^2}{u}$$

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} -\dfrac{v}{u^2} & -\dfrac{v^2}{u^2} \\[2ex] \dfrac{1}{u} & \dfrac{2v}{u} \end{vmatrix} = -\frac{v^2}{u^3}$$

Then, by the formula of the change of variables in double integral,

$$\iint_D x\,y\,dA = \iint_G x(u,v)\cdot y(u,v)\cdot \left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\,dv$$

$$= \int_{a_1}^{a_2}\int_{b_1}^{b_2} \frac{v}{u}\cdot\frac{v^2}{u}\cdot\frac{v^2}{u^3}\,dv\,du =$$

$$= \int_{a_1}^{a_2}\int_{b_1}^{b_2} v^5 u^{-5}\,dv\,du =$$

$$= \int_{a_1}^{a_2} \frac{1}{6}\left(b_2^6 - b_1^6\right) u^{-5}\,du =$$

$$= \frac{1}{6}\cdot\frac{1}{4}\left(b_2^6 - b_1^6\right)\left(a_1^{-4} - a_2^{-4}\right) =$$

$$= \frac{1}{24}\left(b_2^6 - b_1^6\right)\left(a_1^{-4} - a_2^{-4}\right).$$

(Answer)

(8) The cycloid:

$$\begin{cases} x = t - R\sin\left(\frac{t}{R}\right) \\ y = R\left(1 - \cos\left(\frac{t}{R}\right)\right) \\ z = 0 \end{cases}$$

$$\mathbf{r}(t) = \left(t - R\sin\left(\frac{t}{R}\right),\ R - R\cos\left(\frac{t}{R}\right),\ 0\right)$$

$$v(t) = \left(1 - \cos\left(\frac{t}{R}\right),\ \sin\left(\frac{t}{R}\right),\ 0\right)$$

$$a(t) = \left(\frac{1}{R}\sin\left(\frac{t}{R}\right),\ \frac{1}{R}\cos\left(\frac{t}{R}\right),\ 0\right).$$

$$\cancel{\varkappa(t) = \frac{\|(v \times a) \times v\|}{(v \cdot v)^{3/2}}} \qquad \varkappa(t) = \frac{\|v \times a\|}{(v \cdot v)^{3/2}}$$

$$v \cdot v = \left(1 - \cos\left(\frac{t}{R}\right)\right)^2 + \sin^2\left(\frac{t}{R}\right) =$$

$$= 1 - 2\cos\left(\frac{t}{R}\right) + \cos^2\left(\frac{t}{R}\right) + \sin^2\left(\frac{t}{R}\right) =$$

$$= 2\left(1 - \cos\left(\frac{t}{R}\right)\right);$$

$$v \times a = \begin{vmatrix} i & j & k \\ 1-\cos\left(\frac{t}{R}\right) & \sin\left(\frac{t}{R}\right) & 0 \\ \frac{1}{R}\sin\left(\frac{t}{R}\right) & \frac{1}{R}\cos\left(\frac{t}{R}\right) & 0 \end{vmatrix}$$

$$= \left(0, 0, \frac{1}{R}\left[\cos\left(\frac{t}{R}\right) - \cos^2\left(\frac{t}{R}\right) - \sin^2\left(\frac{t}{R}\right)\right]\right) =$$

$$= \left(0, 0, \frac{1}{R}\left(\cos\left(\frac{t}{R}\right) - 1\right)\right).$$

$$\|v \times a\| = \frac{1}{R}\left(1 - \cos\left(\frac{t}{R}\right)\right)$$

So,

$$\varkappa(t) = \frac{1}{R} \frac{1-\cos\left(\frac{t}{R}\right)}{2\sqrt{2}\left(1-\cos\frac{t}{R}\right)^{3/2}} =$$

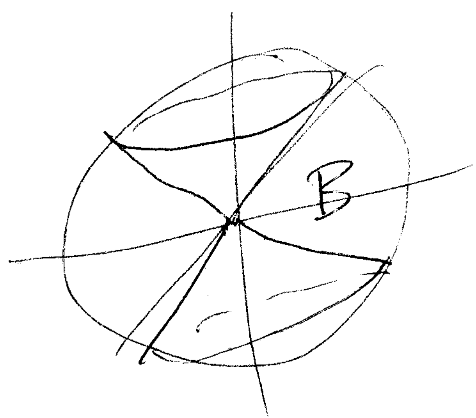
$$= \frac{1}{2\sqrt{2}R}\left(1-\cos\left(\frac{t}{R}\right)\right)^{-\frac{1}{2}}.$$

$$\boxed{\varkappa(t) \to \infty \quad \text{as } t \to 0.}$$

(9) Find the volume of the body B defined by the inequalities

$$B: x^2+y^2+z^2 \le 1, \quad x^2+y^2-z^2 \le 0.$$

Solution.



The body B is the unit ball with the removed parts $\varphi < \frac{\pi}{4}$ and $\varphi > \frac{3\pi}{4}$; hence,

$$V(B) = \int_0^{2\pi} \int_{\frac{\pi}{4}}^{3\pi/4} \int_0^1 r^2 \sin\varphi \, dr \, d\varphi \, d\theta =$$

$$= \frac{2\pi}{3} \int_{\frac{\pi}{4}}^{3\pi/4} \sin\varphi \, d\varphi = \frac{2\pi}{3}\left(\cos\frac{\pi}{4} - \cos\frac{3\pi}{4}\right)$$

$$= \frac{2\pi}{3} \cdot 2 \cdot \frac{1}{\sqrt{2}} = \frac{2\sqrt{2}}{3}\pi.$$

(Answer)