Solutions to the final exam - Fall 2005

① (a) $k(t) = \dfrac{\|\vec{v}(t) \times \vec{a}(t)\|}{\|\vec{v}(t)\|^3}$.

(b) $\vec{r}(t) = a\cos t\,\vec{j} + a\sin t\,\vec{k} \quad , 0 \le t \le 2\pi$

$\vec{r}\,'(t) = -a\sin t\,\vec{j} + a\cos t\,\vec{k}$

$\vec{r}\,''(t) = -a\cos t\,\vec{j} - a\sin t\,\vec{k}$

also $\vec{r}\,' \times \vec{r}\,''(t) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 0 & -a\sin t & a\cos t \\ 0 & -a\cos t & -a\sin t \end{vmatrix}$

$\parallel$

$\vec{v}(t) \times \vec{a}(t)$

$$= (+a^2\sin^2 t + a^2\cos^2 t)\,\vec{i} = a^2\,\vec{i}$$

Hence $k(t) = \dfrac{a^2}{\left(\sqrt{a^2\sin^2 t + a^2\cos^2 t}\right)^3} = \dfrac{a^2}{a^3} = \dfrac{1}{a}$.

(c) C = circle of radius a in the yz-plane with the center at $(0,0,0)$.
(Notice that $k(t) = \dfrac{1}{\text{radius}} = \dfrac{1}{a}$; the radius of curvature is a!)

② (a) $\operatorname{grad} f = \dfrac{\partial f}{\partial x}\vec{i} + \dfrac{\partial f}{\partial y}\vec{j} + \dfrac{\partial f}{\partial z}\vec{k}$

$\operatorname{div}(\operatorname{grad} f) = \dfrac{\partial}{\partial x}\left(\dfrac{\partial f}{\partial x}\right) + \dfrac{\partial}{\partial y}\left(\dfrac{\partial f}{\partial y}\right) + \dfrac{\partial}{\partial z}\left(\dfrac{\partial f}{\partial z}\right) = \dfrac{\partial^2 f}{\partial x^2} + \dfrac{\partial^2 f}{\partial y^2} + \dfrac{\partial^2 f}{\partial z^2}$

(b) $\dfrac{\partial f}{\partial x} = -\dfrac{1}{2}(x^2+y^2+z^2)^{-3/2} \cdot 2x = -x(x^2+y^2+z^2)^{-3/2}$

$$\frac{\partial^2 f}{\partial x^2} = -(x^2+y^2+z^2)^{-3/2} - x\cdot\left(-\frac{3}{2}\right)\cdot(x^2+y^2+z^2)^{-5/2}\cdot 2x =$$

$$= -(x^2+y^2+z^2)^{-5/2}\cdot\left[(x^2+y^2+z^2) - \frac{3}{2}\cdot 2x^2\right]$$

$$= -(x^2+y^2+z^2)^{-5/2}(-2x^2+y^2+z^2)$$

since f is symmetric in x, y, z, we'll have

$$\frac{\partial^2 f}{\partial y^2} = -(x^2+y^2+z^2)^{-5/2}\cdot(x^2-2y^2+z^2)$$

$$\frac{\partial^2 f}{\partial z^2} = -(x^2+y^2+z^2)^{-5/2}\cdot(x^2+y^2-2z^2)$$

$$\therefore \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} = -(x^2+y^2+z^2)^{-5/2}\cdot 0 = 0.$$

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(3) $z = 25 - x^2 - y^2 \Rightarrow \vec{n} = (-2x, -2y, -1)\Big|_{(3,-4,0)}$

$$= (-6, 8, -1)$$

$\therefore$ An equation of the tangent plane is

$$-6(x-3) + 8(y+4) - (z-0) = 0$$

$$-6x + 18 + 8y + 32 - z = 0$$

$$-6x + 8y - z + 50 = 0$$

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(4) (a) $\text{curl}\,\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2x+e^{-y} & 4y-xe^{-y} & 0 \end{vmatrix}$

$= 0\cdot\vec{i} + 0\,\vec{j} + (-e^{-y}+e^{-y})\,\vec{k} = \vec{0}$ (the vector field is conservative)

$$\int_C \vec{F}\cdot d\vec{r} = \int_C (2x+e^{-y})\,dx + (4y-xe^{-y})\,dy$$

$$y = x^4, \quad 0 \le x \le 1$$
$$dy = 4x^3\,dx$$

$$= \int_0^1 (2x+e^{-x^4})\,dx + \int_0^1 (4x^4 - xe^{-x^4}\cdot 4x^3)\,dx$$

The integrals $\int e^{-x^4}dx$ and $\int x^4 e^{-x^4}dx$ cannot be expressed in terms of elementary functions; hence we can find the value of $\int_C \vec{F}\cdot d\vec{r}$ only with the potential function.

let $\varphi(x,y)$ such that $\begin{cases} \frac{\partial\varphi}{\partial x} = 2x+e^{-y} \\ \frac{\partial\varphi}{\partial y} = 4y - xe^{-y} \end{cases}$

$\Rightarrow \varphi(x,y) = x^2 + xe^{-y} + c(y) \Rightarrow \frac{\partial\varphi}{\partial y} = -xe^{-y} + c'(y) \Rightarrow$

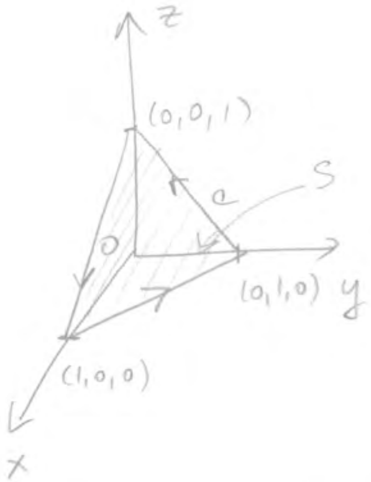
$\Rightarrow c'(y) = 4y \Rightarrow c(y) = 2y^2 + c$, c = constant

$\therefore$ A potential function for $\vec{F}$ is $\varphi(x,y) = x^2 + xe^{-y} + 2y^2$

$$\Rightarrow \int_C \vec{F}\cdot d\vec{r} = \varphi(1,1) - \varphi(0,0) = 1 + e^{-1} + 2 - 0 = 3 + \frac{1}{e}.$$

(b)

$$\operatorname{curl}\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 0 & xy & -xz \end{vmatrix} = 0\,\vec{i} - (-z)\,\vec{j} + y\cdot\vec{k} = z\vec{j} + y\vec{k}$$



Since we calculated the curl $\vec{F}$ we may as well use Stokes' Theorem to calculate $\int_C \vec{F}\cdot d\vec{r}$. Otherwise we need $\int_C \vec{F}\cdot d\vec{r} = \int_{C_1} \vec{F}\cdot d\vec{r} + \int_{C_2} \vec{F}\, d\vec{r} + \int_{C_3} \vec{F}\, d\vec{r}$

where C_1, C_2, C_3 are the sides of the triangle.

To apply Stokes' theorem we also need to know the eq. of S.

The plane passing through $(1,0,0)$, $(0,1,0)$ and $(0,0,1)$ is $x+y+z=1$.

So $\vec{n} = \dfrac{(1,1,1)}{\sqrt{3}}$. ; $dS = \sqrt{3}\, dx\, dy$

$$\therefore \int_C \vec{F}\cdot d\vec{r} = \int_S (\operatorname{curl}\vec{F})\Big|_S \cdot \vec{n}\, dS = \int_0^1\int_0^{1-y} (0, z, y)\Big|_{z=1-x-y} \cdot (1,1,1)\, dx\, dy$$

$$= \int_0^1\int_0^{1-y} (1-x-\cancel{y}+\cancel{y})\, dx\, dy = \int_0^1 \left(x - \frac{x^2}{2}\right)\Big|_0^{1-y} dy = \int_0^1 \left[1-y-\frac{(1-y)^2}{2}\right] dy$$

$$= \left[y - \frac{y^2}{2} - \frac{(y-1)^3}{6}\right]\Big|_0^1 = 1 - \frac{1}{2} - \frac{1}{6} = \frac{2}{6} = \frac{1}{3}.$$

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(5)

$$\text{mass} = \iint_R xy\,dA = \int_0^2\int_0^{\sqrt{1-(x-1)^2}} xy\,dy\,dx = \int_0^2 x\,\frac{y^2}{2}\Big|_0^{\sqrt{1-(x-1)^2}}dx$$

$$= \int_0^2 x\cdot\frac{1}{2}\left(1-(x-1)^2\right)dx = \frac{1}{2}\int_0^2 x(2x-x^2)\,dx = \frac{1}{2}\int_0^2(2x^2-x^3)\,dx$$

$$= \frac{1}{2}\left(\frac{2}{3}x^3-\frac{x^4}{4}\right)\Big|_0^2 = \frac{1}{2}\left(\frac{2}{3}\cdot 8-4\right) = \frac{8}{3}-2 = \frac{2}{3}.$$

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(6) (a) See the text book for the complete statement.

Recall the equality $\oint_C \vec{F}\cdot d\vec{r} = \int_R \left(\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\right)dA$

(b) By part a, $\oint_C \vec{F}\cdot d\vec{r} = \int_0^1\int_{-1}^1 (y^2-6x^2y)\,dx\,dy$

$$= \int_0^1 (xy^2-2x^3y)\Big|_{-1}^1 dy$$

$$= \int_0^1 (2y^2+4y)\,dy = \left(\frac{2}{3}y^3+2y^2\right)\Big|_0^1 = \frac{2}{3}+2 = \frac{8}{3}.$$

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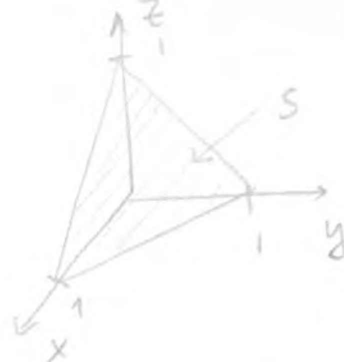
⑦



$$\vec{n} = \frac{(1,1,1)}{\sqrt{3}}$$

$$ds = \sqrt{3}\, dA$$

$$\iint_S \vec{F}\cdot\vec{n}\, ds = \int_0^1\int_0^{1-y} (0, z, y)\Big|_{z=1-x-y} \cdot (1,1,1)\, dx\, dy = \int_0^1\int_0^{1-y} (1-x)\, dx\, dy =$$

$$= \int_0^1 \left(x - \frac{x^2}{2}\right)\Big|_0^{1-y} dy = \int_0^1 \left[1-y-\frac{(1-y)^2}{2}\right] dy = \left[y - \frac{y^2}{2} - \frac{(y-1)^3}{6}\right]\Big|_0^1$$

$$= 1 - \frac{1}{2} - \frac{1}{6} = \frac{1}{3}.$$

(This is 4ⓑ all over again.) #

⑧ ⓐ See the textbook for all hypothesis.

Recall here only: $\oint_C \vec{F}\cdot d\vec{r} = \iint_S \operatorname{curl}\vec{F}\cdot\vec{n}\, ds$

ⓑ The right-hand side of the previous equality is calculated in pb. 4 part b. Its value is $1/3$.

To calculate $\oint_C \vec{F}\cdot d\vec{r}$ without Stokes' theorem, we'll parametrize each side of the triangle. (See the picture of 4b).

Let C_1 be the segment connecting $(0,1,0)$ to $(0,0,1)$ $\Rightarrow$

$$C_1: \begin{cases} x(t) = 0 \\ y(t) = 1-t \\ z(t) = t \end{cases}$$

$$0 \le t \le 1$$

$$\Rightarrow \vec{F}\Big|_{C_1} = 0\vec{i} + 0\cdot\vec{j} + 0\cdot\vec{k} \quad (\text{as } x=0) \Rightarrow \int_{C_1} \vec{F}\cdot d\vec{r} = 0$$

$$C_2: \begin{cases} x(t) = t \\ y(t) = 0 \\ z(t) = 1-t \end{cases} \qquad \text{i.e. } \vec{r}(t) = (t, 0, 1-t)$$

$$\vec{r}\,'(t) = (1, 0, -1)$$

$$0 \le t \le 1$$

$$\Rightarrow \vec{F}\Big|_{C_2} = 0\vec{i} + 0\cdot\vec{j} - t(1-t)\vec{k} \Rightarrow \int_{C_2} \vec{F}\cdot d\vec{r} = \int_0^1 (t-t^2)\,dt$$

$$= \left(\frac{t^2}{2} - \frac{t^3}{3}\right)\Big|_0^1 = \frac{1}{2} - \frac{1}{3} = \frac{1}{6}.$$

$$C_3: \begin{cases} x(t) = 1-t \\ y(t) = t \\ z(t) = 0 \end{cases} \quad \text{or} \quad \begin{matrix} \vec{r}(t) = (1-t, t, 0) \\ \vec{r}\,'(t) = (-1, 1, 0) \end{matrix}$$

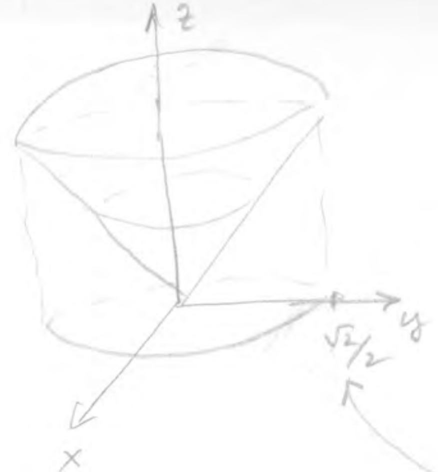
$$0 \le t \le 1$$

$$\Rightarrow \vec{F}\Big|_{C_3} = 0\cdot\vec{i} + t(1-t)\vec{j} + 0\cdot\vec{k} \Rightarrow \int_{C_3} \vec{F}\cdot d\vec{r} = \int_0^1 (t-t^2)\,dt = \frac{1}{6}$$

$$\therefore \oint_C \vec{F}\cdot d\vec{r} = \int_{C_1} \vec{F}\cdot d\vec{r} + \int_{C_2} \vec{F}\cdot d\vec{r} + \int_{C_3} \vec{F}\cdot d\vec{r} = 0 + \frac{1}{6} + \frac{1}{6} = \frac{1}{3}.$$

and Stokes' Thm. is checked. #

(9)



$$\sqrt{1-x^2-y^2} = \sqrt{x^2+y^2}$$

$$1-x^2-y^2 = x^2+y^2$$

$$1 = 2x^2+2y^2$$

$$\frac{1}{2} = x^2+y^2$$

Due to the symmetry of the solid, the center of mass is on the z-axis.

Hence $\bar{x} = \bar{y} = 0$.

$$\bar{z} = \frac{1}{\text{mass}} \cdot \iiint_D \rho z \, dV = \frac{1}{\rho \cdot \text{Vol}(D)} \cdot \iiint_D \rho z \, dV = \frac{1}{\text{Vol}(D)} \iiint_D z \, dV.$$

Using cylindrical coordinates $z=\sqrt{x^2+y^2}$ becomes $z=r$,

while $z=\sqrt{1-x^2-y^2}$ becomes $z=\sqrt{1-r^2}$.

So: $$\bar{z} = \frac{1}{\frac{\pi(2-\sqrt{2})}{3}} \cdot \int_0^{\sqrt{2}/2} \int_0^{2\pi} \int_r^{\sqrt{1-r^2}} z \cdot r \, dz \, d\theta \, dr = \frac{3}{\pi(2-\sqrt{2})} \int_0^{\sqrt{2}/2} \int_0^{2\pi} r \frac{z^2}{2} \Big|_r^{\sqrt{1-r^2}} d\theta \, dr$$

$$= \frac{3}{\pi(2-\sqrt{2})} \cdot \int_0^{\sqrt{2}/2} \int_0^{2\pi} \frac{r}{2}\left(1-r^2-r^2\right) d\theta \, dr = \frac{3}{\pi(2-\sqrt{2})} \cdot 2\pi \int_0^{\sqrt{2}/2} \frac{1}{2}\left(1-2r^2\right) r \, dr$$

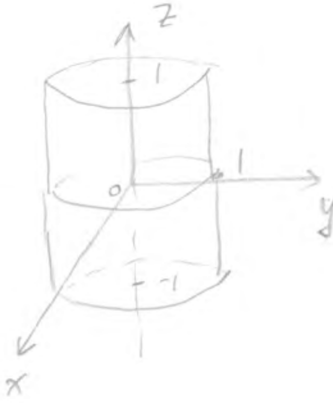
$$= \frac{3}{2-\sqrt{2}} \cdot \int_0^{\sqrt{2}/2} (r-2r^3) \, dr = \frac{3}{2-\sqrt{2}} \cdot \left(\frac{r^2}{2} - \frac{r^4}{2}\right) \Big|_0^{\sqrt{2}/2} = \frac{3}{2-\sqrt{2}}\left(\frac{1}{4} - \frac{1}{8}\right)$$

$$= \frac{3}{2-\sqrt{2}} \cdot \frac{1}{8} = \frac{3}{8(2-\sqrt{2})}.$$ Final answer : $\left(0, 0, \frac{3}{8(2-\sqrt{2})}\right)$ #

(10) (a) Refer to the textbook for the complete statement.

Recall here the main formula: $\displaystyle\iint_S \vec{F}\cdot\vec{n}\,ds = \iiint_D \operatorname{div}\vec{F}\,dV$.

(b)



$\operatorname{div}\vec{F} = z + 0 + 1 = z+1$

Hence $\displaystyle\iint_S \vec{F}\cdot\vec{n}\,ds = \int_0^1\int_0^{2\pi}\int_{-1}^{1}(z+1)\,r\,dz\,d\theta\,dr = \int_0^1\int_0^{2\pi} r\left(\frac{z^2}{2}+z\right)\Big|_{-1}^{1} d\theta\,dr$

(We used again cylindrical coordinates)

$$= \int_0^1\int_0^{2\pi} r\cdot 2\,d\theta\,dr = \int_0^1 2r\theta\,\Big|_0^{2\pi} dr = \int_0^1 4r\pi\,dr = 2\pi r^2\Big|_0^1$$

$= 2\pi$.

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