PROBLEMS FOR CHAPTER 3

4. Solve the following linear system using 5 digit rounding by
   a) Gaussian Elimination with Backward Substitutions.
   b) Gaussian Elimination with Backward Substitution and using maximal column pivoting.
   c) Gaussian Elimination with Backward Substitution using scaled column pivoting.

   $3.3330\ x_1 + 15920\ x_2 - 10.333\ x_3 = 15913$

   $2.2220\ x_1 + 16.710\ x_2 + 9.6120\ x_3 = 28.544$

   $1.5611\ x_1 + 5.1791\ x_2 + 1.6852\ x_3 = 8.4254$

Solution: (a) Gaussian Elimination method

The augmented matrix is given by

$$\tilde{A}^1 = \begin{bmatrix} 3.3330 & 15920 & 10.333 & \mathrm{M} & 15913 \\ 2.2220 & 16.710 & 9.6120 & \mathrm{M} & 28.544 \\ 1.5611 & 5.1791 & 1.6852 & \mathrm{M} & 8.4254 \end{bmatrix}$$

for $\begin{matrix} i=1 \\ j=2 \end{matrix}$ $\quad \text{Row } 2 - \dfrac{2.222}{3.333}\text{Row } 1 \rightarrow \text{New Row } 2$

for $\begin{matrix} i=1 \\ j=3 \end{matrix}$ $\quad \text{Row } 3 - \dfrac{1.5611}{3.333}\text{Row } 1 \rightarrow \text{New Row } 3$

The new matrix formed is referred to as the next augmented matrix.

$$\tilde{A}^2 = \begin{bmatrix} 3.3330 & 15920 & 10.333 & \mathrm{M} & 15913 \\ 0 & -10596.62 & 2.7233 & \mathrm{M} & -10580.1 \\ 0 & -7451.38 & -3.1545 & \mathrm{M} & -7444.86 \end{bmatrix}$$

for $\begin{matrix} i=1 \\ j=3 \end{matrix}$ $\quad \text{Row3} - \dfrac{-7451.38}{-10596.62}\ \text{Row2} \rightarrow \text{New Row3}$

Since all values below the pivot element become zero, the next augmented matrix becomes:

$$\tilde{A}^3 = \begin{bmatrix} 3.3330 & 15920 & 10.333 & \mathrm{M} & 15913 \\ 0 & -10596.62 & 2.7233 & \mathrm{M} & -10580.1 \\ 0 & 0 & -5.0695 & \mathrm{M} & -5.07807 \end{bmatrix}$$

Back substitution:

$$x_3 = 1.002$$

$$x_2 = 0.9987$$

$$x_1 = 1.0016$$

(b) Gaussian Elimination method with maximal scaling it's the same as before as the maximum value is in the pivoting row

The augmented matrix is given by

$$\tilde{A}^1 = \begin{bmatrix} 3.3330 & 15920 & 10.333 & \mathrm{M} & 15913 \\ 2.2220 & 16.710 & 9.6120 & \mathrm{M} & 28.544 \\ 1.5611 & 5.1791 & 1.6852 & \mathrm{M} & 8.4254 \end{bmatrix}$$

for $\begin{matrix} i = 1 \\ j = 2 \end{matrix}$ $\quad \mathrm{Row\ 2} - \dfrac{2.222}{3.333}\mathrm{Row\ 1} \rightarrow \mathrm{New\ Row\ 2}$

for $\begin{matrix} i = 1 \\ j = 3 \end{matrix}$ $\quad \mathrm{Row\ 3} - \dfrac{1.5611}{3.333}\mathrm{Row\ 1} \rightarrow \mathrm{New\ Row\ 3}$

The new matrix formed is referred to as the next augmented matrix.

$$\tilde{A}^2 = \begin{bmatrix} 3.3330 & 15920 & 10.333 & \vdots & 15913 \\ 0 & -10596.62 & 2.7233 & \vdots & -10580.1 \\ 0 & -7451.38 & -3.1545 & \vdots & -7444.86 \end{bmatrix}$$

for $\begin{matrix} i = 1 \\ j = 3 \end{matrix}$ $\quad \text{Row3} - \dfrac{-7451.38}{-10596.62}\,\text{Row2} \rightarrow \text{New Row3}$

Since all values below the pivot element become zero, the next augmented matrix becomes:

$$\tilde{A}^3 = \begin{bmatrix} 3.3330 & 15920 & 10.333 & \vdots & 15913 \\ 0 & -10596.62 & 2.7233 & \vdots & -10580.1 \\ 0 & 0 & -5.0695 & \vdots & -5.07807 \end{bmatrix}$$

Back substitution:

$x_3 = 1.002$

$x_2 = 0.9987$

$x_1 = 1.0016$

5. Solve the following linear system using Gaussian Elimination method, and determine whether row interchanges are necessary.

$$x_1 - x_2 + 3x_3 = 2$$
$$3x_1 - 3x_2 + x_3 = -1$$
$$x_1 + x_2 = 3$$

Solution: Gaussian Elimination

The augmented matrix is given by

$$\tilde{A}^1 = \begin{bmatrix} 1 & -1 & 3 & \vdots & 2 \\ 3 & -3 & 1 & \vdots & -1 \\ 1 & 1 & 0 & \vdots & 3 \end{bmatrix}$$

$$\text{Row } 2 - \frac{3}{1}\text{Row } 1 \to \text{New Row } 2$$

$A_{21} - \frac{A_{21}}{A_{11}} A_{11} \to A_{21}(\text{New})$ $\qquad$ $3 - \frac{3}{1}(1) \to 0$ $\qquad$ A_{21} (New)

$A_{22} - \frac{A_{21}}{A_{11}} A_{12} \to A_{22}(\text{New})$ $\qquad$ $-3 - \frac{3}{1}(-1) \to 0$ $\qquad$ A_{22} (New)

$A_{23} - \frac{A_{21}}{A_{11}} A_{13} \to A_{23}(\text{New})$ $\qquad$ $1 - \frac{3}{1}(3) \to -8$ $\qquad$ A_{23} (New)

$A_{24} - \frac{A_{21}}{A_{11}} A_{14} \to A_{24}(\text{New})$ $\qquad$ $-1 - \frac{3}{1}(2) \to -7$ $\qquad$ A_{24} (New)

$$\text{Row3} - \frac{A_{32}}{A_{22}}\ \text{Row2} \to \text{New Row3}$$

$A_{31} - \frac{A_{31}}{A_{11}} A_{11} \to A_{31}(\text{New})$ $\qquad$ $1 - \frac{1}{1}(1) \to 0$ $\qquad$ A_{31} (New)

$A_{32} - \frac{A_{31}}{A_{11}} A_{12} \to A_{32}(\text{New})$ $\qquad$ $1 - \frac{1}{1}(-1) \to 2$ $\qquad$ A_{32} (New)

$A_{33} - \frac{A_{31}}{A_{11}} A_{13} \to A_{33}(\text{New})$ $\qquad$ $0 - \frac{1}{1}(-3) \to -3$ $\qquad$ A_{33} (New)

$A_{34} - \frac{A_{31}}{A_{11}} A_{14} \to A_{34}(\text{New})$ $\qquad$ $3 - \frac{1}{1}(2) \to 1$ $\qquad$ A_{34} (New)

$$\tilde{A}^2 = \begin{bmatrix} 1 & -1 & 3 & \vdots & 2 \\ 0 & 0 & -8 & \vdots & -7 \\ 0 & 2 & -3 & \vdots & 1 \end{bmatrix}$$

Since the second pivot element is 0, row interchange is required

$$\tilde{A}^3 = \begin{bmatrix} 1 & -1 & 3 & \vdots & 2 \\ 0 & 2 & -3 & \vdots & 1 \\ 0 & 0 & -8 & \vdots & -7 \end{bmatrix}$$

After Back substitution we get:

$$x_3 = 0.875$$

$$x_2 = 1.8125$$

$$x_1 = 1.1875$$

13. Find $\|A\|_1, \|A\|_E$ and $\|A\|_\infty$

$$[A] = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 3 & 2 \\ 1 & 1 & 2 \end{bmatrix}$$

Solution

$l_1 = \|A\|_1$ = largest absolute column sum

$$(+) \quad \begin{array}{ccc} |2| & |1| & |1| \\ |1| & |3| & |2| \\ \underline{|1|} & \underline{|1|} & \underline{|2|} \\ 4 & 5 & 5 \leftarrow \end{array}$$

maximum value resulting

$$l_1 = \|A\|_1 = 5$$

$l_\infty = \|A\|_\infty =$ largest absolute row sum

$$\begin{array}{ccccccc} |2| & + & |1| & + & |1| & = & 4 \\ |1| & + & |3| & + & |2| & = & 6 \leftarrow \\ |1| & + & |1| & + & |2| & = & 4 \end{array}$$

maximum value

$$l_\infty = \|A\|_\infty = 6$$

$$l_E = \|A\|_E = \{2^2 + 1^2 + 1^2 + 3^2 + 1^2 + 2^2 + 1^2 + 2^2 + 1^2\}^{1/2}$$

$$l_E = \|A\|_E = \{26\}^{1/2} = 5.099$$

17. Check the condition of matrix in problem 2 using the infinity norm of matrices.

Solution

$$A = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 4 & -2 \\ 0 & -2 & 2 \end{bmatrix}$$

The first step is to find A^{-1}. This is done by using Gauss Jordan elimination

$$\tilde{A}^1 = \begin{bmatrix} 1 & -2 & 0 & M & 1 & 0 & 0 \\ -1 & 4 & -2 & M & 0 & 1 & 0 \\ 0 & -2 & 2 & M & 0 & 0 & 1 \end{bmatrix}$$

$$\text{Row j} - \frac{A_{ij}}{A_{ii}} \text{Row i} \rightarrow \text{New Row j}$$

i = 1;

j =2; j = 3
Gives new row 2

$$\tilde{A}^2 = \begin{bmatrix} 1 & -1 & 0 & : & 1 & 0 & 0 \\ 0 & 3 & -2 & : & 1 & 1 & 0 \\ 0 & -2 & 2 & : & 0 & 0 & 1 \end{bmatrix}$$

i = 2

j = 3,1

$$\tilde{A}^3 = \begin{bmatrix} 1.0000 & 0 & -0.6667 & : & 1.3333 & 0.3333 & 0 \\ 0 & 3.0000 & -2.0000 & : & 1.0000 & 1.0000 & 0 \\ 0 & 0 & 0.6667 & : & 0.6667 & 0.6667 & 1.0000 \end{bmatrix}$$

i = 3

j = 2; j = 1

$$\tilde{A}^4 = \begin{bmatrix} 1.0000 & 0 & 0 & : & 2.0000 & 1.0000 & 1.0000 \\ 0 & 3.0000 & 0 & : & 3.0000 & 3.0000 & 3.0000 \\ 0 & 0 & 0.6667 & : & 0.6667 & 0.6667 & 1.0000 \end{bmatrix}$$

Divide row 3 by 0.6667 and row 2 by 3 to get identity matrix.

$$\tilde{A}^{4} = \begin{bmatrix} 1 & 0 & 0 & M\,2 & 1 & 1 \\ 0 & 1 & 0 & M\,1 & 1 & 1 \\ 0 & 0 & 1 & M\,1 & 1 & 1.5 \end{bmatrix}$$

Hence $\tilde{A}^{-1} = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1.5 \end{bmatrix}$

Now evaluate $\|A\|_{\infty}$ and $\|A\sim^{1}\|$

For $\|A\|_{\infty}$

$$A = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 4 & -2 \\ 0 & -2 & 2 \end{bmatrix} \begin{matrix} \rightarrow & |1| + |-1| + |0| = 2 \\ \rightarrow & |-1| + |4| + |-2| = 7 \\ \rightarrow & |0| + |-2| + |2| = 4 \end{matrix}$$

Maximum absolute value = $\|A\|_{\infty}$ = 7

$$\tilde{A}^{-1} = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1.5 \end{bmatrix} \begin{matrix} \rightarrow & |2| + |1| + |1| = 4 \\ \rightarrow & |1| + |1| + |1| = 3 \\ \rightarrow & |1| + |1| + |1.5| = 3.5 \end{matrix}$$

Maximum absolute value = $\|A^{-1}\|_{\infty}$=4

Now the condition number is evaluated;

$$\|A\|_{\infty} * \|A^{-1}\|_{\infty} = 7 \times 4 = 28$$