PROBLEMS FOR CHAPTER 9

1. Obtain the solution of the differential equation

$$\dot{y} = -0.1\,y + \sin(2\pi\, t) \text{ with } y(0) = 1.0$$

Tabulate the solution for $0 \le t \le 1$ at 0.1 interval. Use Euler's method with h = 0.1. Compare the solution with that obtained by Midpoint method.

Solution:

Euler's Equation

$y(x_{i+1}) = y(i) + hg(x_i, y(x_i))$

$g(t,y) = y' = 0.1y + \sin(2\pi t)$ and $y(0) = 1.0$

Where h = 0.1

Using Euler's Equation => $y(t_1) = y(t_0) + hg(t_0, t(t_0))$

$y(0.1) = y(0) + hg(0,1) = 1 + 0.1[0.1\ (1) + \sin(2\pi*0)] = 1.01$
$y(0.2) = y(0.1) + hg(0.1,1.01) = 1.01 + 0.1[0.1\ (1.01) + \sin(2\pi*0.1)] = 1.0789$
$y(0.3) = y(0.2) + hg(0.2,1.0789) = 1.0789 + 0.1[0.1\ (1.0789) + \sin(2\pi*0.2)] = 1.1848$
$y(0.4) = y(0.3) + hg(0.3,1.1848) = 1.1848 + 0.1[0.1\ (1.1848) + \sin(2\pi*0.3)] = 1.2917$
$y(0.5) = y(0.4) + hg(0.4,1.2917) = 1.2917 + 0.1[0.1\ (1.2917) + \sin(2\pi*0.4)] = 1.3634$
$y(0.6) = y(0.5) + hg(0.5,1.3634) = 1.3634 + 0.1[0.1\ (1.3634) + \sin(2\pi*0.5)] = 1.3771$
$y(0.7) = y(0.6) + hg(0.6,1.3771) = 1.3771 + 0.1[0.1\ (1.3771) + \sin(2\pi*0.6)] = 1.3320$
$y(0.8) = y(0.7) + hg(0.7,1.3320) = 1.3320 + 0.1[0.1\ (1.3320) + \sin(2\pi*0.7)] = 1.2503$
$y(0.9) = y(0.8) + hg(0.8,1.2503) = 1.2503 + 0.1[0.1\ (1.2503) + \sin(2\pi*0.8)] = 1.1677$
$y(1) = y(0.9) + hg(0.9,1.1677) = 1.1677 + 0.1[0.1\ (1.1677) + \sin(2\pi*0.9)] = 1.1206$

using mid-point method or Range-kutta method

$y(t_1) = y(t_0) + hg[(t_0 + \frac{h}{2}), y(t_0) + \frac{h}{2}\, g(t_0, y(t_0)]$
$y(0.1) = y(0) + h\ \{0.1[y(t_0) + \frac{h}{2}\,[0.1xy(t_0) + \sin(2\pi t_0)] + \sin(2\pi(t_0 + \frac{h}{2}))]\}$
$= 1 + 0.1\{0.1\ [1 + \frac{0.1}{2}\,[0.1 * 1 + \sin(2\pi 0)]] + \sin(2\pi\frac{0.1}{2})\}$
$= 1.04056$

Similarly, calculating and forming a comparison table we get

	Euler Method	Mid-point or R.K. Method
y(0.1)	1.01	1.04056
y(0.2)	1.0789	1.13095
y(0.3)	1.1848	1.24116
y(0.4)	1.2917	1.33362
y(0.5)	1.3634	1.37761
y(0.6)	1.3771	1.36095
y(0.7)	1.3320	1.29468
y(0.8)	1.2503	1.20885
y(0.9)	1.1677	1.14102
y(1)	1.1206	1.12190

3. Try Euler's method with step size h = 0.5 on the following problem.

$$y' = t \sin t + e^{-t}, \quad 0 \le t \le 1 \qquad y(0) = 0$$

<u>Solution:</u>

$g(t, y) = y' = t \sin t + e^{-t}$; $h = 0.5$, $y(0) = 0$

$y(t_1) = y(t_0) + hg(t_0, y(t_0))$

$y(0.5) = y(0) + 0.5 (t_0 \sin(t_0) + e^{-t_0})$

$= 0 + 0.5 (0 * \sin(0) + e^0)$

$= 0.5$

$y(1) = 0.5 + 0.5 (0.5 * \sin(0.5) + e^{-0.5})$

$= 0.9231$

7. The function y(x) is defined by the following problem

$$\frac{dy}{dx} = x^2 - y^2 \qquad y(0) = 1$$

Find y(0.2) using

(a) Euler's method: $h = 0.1$

(b) Runge-Kutta method of order 4 with $h = 0.1$

Solution:

$g(x, y) = \frac{dy}{dx} = x^2 - y^2$; $y(0) = 1$

a) Euler Method, $h = 0.1$

$y(x_1) = y(x_0) + hg(x_0, y(x_0))$

$y(0.1) = y(0) + 0.1\,(x_0^2 - y(0)^2)$
$= 1 + 0.1\,(0^2 - 1^2)$
$= 0.9$

$y(0.2) = 0.9 + 0.1\,(0.1^2 - 0.9^2)$
$= 0.82$

b) Range-Kutta order 4 method , $h = 0.1$

$y(x_1) = y(x_0) + \frac{h}{6}(k_1 + 2k_2 + 2k_3 + k_4)$

$k_1 = g\,(x_0, y(x_0)) = x_0^2 - y(0)^2 = 0^2 - 1^2 = -1$

$k_2 = g\,(x_0 + \frac{h}{2}, y(0) + k_1 \frac{h}{2}) = (0.05)^2 - (1 - 1 * \frac{0.1}{2})^2 = -0.9$

$k_3 = g\,(x_0 + \frac{h}{2}, y(0) + k_2 \frac{h}{2}) = (0.05)^{-2} - (1 - 0.9 * \frac{0.1}{2})^2 = -0.9095$

$k_4 = g\,(x_0 + h, y(0) + k_3 h) = (0.1)^2 - (1 - 0.9095 * 0.1)^2 = -0.8164$

$y(0.1) = 1 + \frac{0.1}{6}[-1 + 2(-0.9) + 2(-0.9095 + (-0.8164)]$
$= 0.9094$

$k_1 = g\,(x_{(0.1)}, y_{(0.1)}) = 0.1^2 - (0.9094)^2 = -0.8170$

$k_2 = g\,(x_{0.1} + \frac{h}{2}, y(0.1) + k_1 \frac{h}{2}) = (0.15)^2 - (0.9094 - 0.817 * \frac{0.1}{2})^2 = -0.7319$

$k_3 = g\,(x_{0.1} + \frac{h}{2}, y(0.1) + k_2 \frac{h}{2}) = (0.15)^{-2} - (0.9094 - 0.7319 * \frac{0.1}{2})^2 = -0.7393$

$k_4 = g\,(x_{0.1} + h, y(0.1) + k_3 h) = (0.2)^2 - (0.9094 - 0.7393 * 0.1)^2 = -0.658$

$y(0.2) = 0.9094 + \frac{0.1}{6}[-0.817 + 2(-0.7319) + 2(-0.73930) - 0.658]$

$y(0.2) = 0.83578$

11. Given the following equation

$$\dot{y} = \frac{y}{t} + te^t$$

with $1 \leq t \leq 2$ $y(1) = 0$ $h = 0.1$

(a) solve by Euler's method

(b) solve by Runge-Kutta method of order 2.

Solution:

$g(t, y) = y' = \frac{y}{t} + te^t$; $y(1) = 0$, $h = 0.1$

a) Euler's Method

$y(t_1) = y(t_0) + hg(t_0, y(t_0))$

$$y(1.1) = y(1) + 0.1\left(\frac{y(1)}{to} + t_0 e^{to}\right)$$
$$= 0 + 0.1\left(\frac{0}{1} + 1 e^1\right) = 0.2718$$

$$y(1.2) = 0.2718 + 0.1\left(\frac{0.2718}{1.1} + 1.1 * e^{1.1}\right) = 0.6270$$

Subsequent calculators in table formed yield

i	t	y
0	1	0
1	1.1	0.2718
2	1.2	0.6270
3	1.3	1.0777
4	1.4	1.6376
5	1.5	2.3223
6	1.6	3.1493
7	1.7	4.1387
8	1.8	5.3127
9	1.9	6.6968
10	2	8.3195

b) Range – Kutta Method Order 2

$y(t_1) = y(t_0) + hk_2$

$k_2 = g\,[(t_0 + \frac{h}{2}), (y(t_0) + k_1 \frac{h}{2})]$

$k_1 = g\,(t_0, y(t_0))$

$k_1 = \frac{y(1)}{to} + t_0 e^{to} = \frac{0}{1} * 1e^1 = 2.718$

$k_2 = \frac{(y(t0) + k1\frac{h}{2})]}{to+\frac{h}{2}} + (t_0 + \frac{h}{2})\, e^{(t0+\frac{h}{2})}$

$= \frac{0.1359}{1.05} + 1.05\, e^{1.05} = 3.12996$

$Y(1.1) = 0 + 0.1 * 3.12996 = 0.313$

Subsequent calculators in table form yield

t	y
1	0
1.1	0.313
1.2	0.7194
1.3	1.2318
1.4	1.8653
1.5	2.6366
1.6	3.5648
1.7	4.9714
1.8	5.9804
1.9	7.5194
2	9.3192

12. Solve the third order differential equation using Euler's method and obtain y(0.2).

$$y''' + xy' - 2y = x$$

$$y(0) = y''(0) = 0$$

$$y'(0) = 1$$

Solution:

$y''' + xy' - 2y = x$; $y(0) = y''(0) = 0$ and $y'(0) = 1$

we convert this second order ODE in 3 first order ODEs

g(v, y) ;	$y' = v$	$v(0) = y'(0) = 1$
	$v' = u$	$u(0) = y''(0) = 0$
	$u' = x - xv + 2y$	$y(0) = 0$

we take h = 0.1

$y(0.1) = y(0) + h(v(0))$
$v(0.1) = v(0) + h(u(0))$
$u(0.1) = u(0) + h\,(x_0 – x_0v(0) – 2y(0))$

$y(0.1) = 0 + 0.1\,(1) = 0.1$
$v(0.1) = 1 + 0.1\,(0) = 1$
$u(0.1) = 0 + 0.1\,(0 – 0*1– 2*0) = 0$

$y(0.2) = 0.1 + 0.1\,(1) = 0.2$
$v(0.2) = 1 + 0.1\,(0) = 1$
$u(0.2) = 0 + 0.1\,(0.1 – 0.1*1– 2*0.1) = 0.02$