CONCORDIA UNIVERSITY

DEPARTMENT OF COMPUTER SCIENCE AND SOFTWARE ENGINEERING

COMP232 MATHEMATICS FOR COMPUTER SCIENCE Winter 2019

ASSIGNMENT 3. SOLUTION.

1. Let the function $f : \mathbb{R} \longrightarrow \mathbb{R}$ be given by

$$f(x) = \frac{x+1}{x-1} \quad \text{if } x \neq 1 \ , \qquad f(x) = 1 \quad \text{if } x = 1 \ .$$

Draw the graph of f versus the values of x. Is f a bijection (*i.e.*, one-to-one and onto)? If yes then give a proof and derive a formula for f^{-1}. If no then explain why not.

SOLUTION: We'll prove that f is a bijection.

First we show that f is one-to-one. Suppose that

$$f(x_1) = f(x_2), \quad x_1, x_2 \in \mathbb{R}, x_1 \neq x_2.$$

Then

$$\begin{aligned} \frac{x_1+1}{x_1-1} &= \frac{x_2+1}{x_2-1} \\ (x_1+1)(x_2-1) &= (x_2+1)(x_1-1) \\ x_1 &= x_2. \end{aligned}$$

If $x_1 = 1$ and $x_2 \neq 1$ or $x_1 \neq 1$ and $x_2 = 1$, then $f(x_1) \neq f(x_2)$ which completes the proof of one-to-one.

Next we show that f is onto. Let $y = 1$. Then $f(x) = y$ has unique solution $x = 1$. Suppose that $y \in \mathbb{R}, x, y \neq 1$. Then equation $f(x) = y$ has the following unique solution:

$$\begin{aligned} f(x) &= y \\ \frac{x+1}{x-1} &= y \\ (x+1) = &= (x-1)y \\ x = \frac{y+1}{y-1} & \end{aligned}$$

which completes the proof of onto property of f and produces $f^{-1}(x) = \frac{x+1}{x-1}$ in the process.

We can alternatively prove that f is a bijection as follows. It is easily verified that f is its own inverse:

$$f(f(x)) = \frac{\frac{x+1}{x-1}+1}{\frac{x+1}{x-1}-1} = \frac{\frac{2x}{x-1}}{\frac{2}{x-1}} = x, \quad \text{if } x \neq 1 \ , \quad \text{and} \quad f(f(1)) = f(1) = 1 \ .$$

Since f is invertible is must also be one-to-one and onto.

2. Let $f : \mathbb{Z}^2 \longrightarrow \mathbb{Z}^2$ be defined as $f(m,n) = (m-n,n)$. Is f indeed a properly defined function from $\mathbb{Z}^2$ to $\mathbb{Z}^2$? Is f a bijection, *i.e.*, one-to-one and onto? If yes then give a proof and derive a formula for f^{-1}. If no then explain why not.

   Also derive a formula for the composite function f_k, for $k \in \mathbb{Z}^+$. Here f_2 denotes the composite function $f \circ f$, f_3 denotes the composite function $f \circ f \circ f$, *etc.* (You are asked to derive the formula for f_k for general $k \in \mathbb{Z}^+$.) Is f_k a bijection? If yes then give a proof and derive a formula for its inverse f_k^{-1}. If no then explain why not.

   SOLUTION: Clearly f is a well-defined function from $\mathbb{Z}^2$ to $\mathbb{Z}^2$: For any integer pair (m,n) the value of $f(m,n)$ is an integer pair.

   It is easily seen f has an inverse, namely, $f^{-1}(m,n) = (m+n,n)$. Thus f is a bijection.

   We have $f_1(m,n) = f(m,n) = (m-n,n)$,
   $f_2(m,n) = f(f_1(m,n)) = f(m-n,n) = (m-2n,n)$,
   $f_3(m,n) = f(f_2(m,n)) = f(m-2n,n) = f(m-3n,n)$,
   $f_4(m,n) = f(f_3(m,n)) = f(m-3n,n) = f(m-4n,n)$, *etc.*

   From this we see that $f_k(m,n) = (m-kn,n)$, which can be proved formally by induction. It is also easily checked that f_k has an inverse, namely, $f_k^{-1}(m,n) \;=\; (m+kn\;,\;n)$, so that f is a bijection.

3. If A and B are sets and $f : A \longrightarrow B$, then for any subset S of A we define

$$f(S) \;=\; \{b \in B : b = f(a) \text{ for some } a \in S\}\ .$$

   Similarly, for any subset T of B we define the *pre-image* of T as

$$f^{-1}(T) \;=\; \{a \in A : f(a) \in T\}\ .$$

   Note that $f^{-1}(T)$ is well defined even if f does not have an inverse !

   Now let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be defined as $f(x) = x^2$. Let S_1 denote the closed interval $[-2,1]$, that is all $x \in \mathbb{R}$ that satisfy $-2 \le x \le 1$, and let S_2 be the open interval $(-1,2)$, that is all $x \in \mathbb{R}$ that satisfy $-1 < x < 2$. Also let $T_1 = S_1$ and $T_2 = S_2$.

   Determine

$$f(S_1 \cup S_2)\ ,\ f(S_1) \cup f(S_2)\ ,\ f(S_1 \cap S_2)\ ,\ f(S_1) \cap f(S_2)\ ,$$

   and

$$f^{-1}(T_1 \cup T_2)\ ,\ f^{-1}(T_1) \cup f^{-1}(T_2)\ ,\ f^{-1}(T_1 \cap T_2)\ ,\ \text{and } f^{-1}(T_1) \cap f^{-1}(T_2)\ .$$

   SOLUTION:

   We see that

$$S_1 \cup S_2 = T_1 \cup T_2 = \big\{x \in \mathbb{R} : -2 \le x < 2\big\} \qquad S_1 \cap S_2 = T_1 \cap T_2 = \big\{x \in \mathbb{R} : -1 < x \le 1\big\}$$

$$f(S_1) = \big\{x \in \mathbb{R} : 0 \le x \le 4\big\} \qquad f(S_2) = \big\{x \in \mathbb{R} : 0 \le x < 4\big\}$$

$$f^{-1}(T_1) = \big\{x \in \mathbb{R} : -1 \le x \le 1\big\} \qquad f^{-1}(T_2) = \big\{x \in \mathbb{R} : -\sqrt{2} < x < \sqrt{2}\big\}$$

so that

$$f(S_1 \cup S_2) = \big\{x \in \mathbb{R} : 0 \le x \le 4\big\} \qquad f(S_1) \cup f(S_2) = \big\{x \in \mathbb{R} : 0 \le x \le 4\big\}$$

$$f(S_1 \cap S_2) = \big\{x \in \mathbb{R} : 0 \le x \le 1\big\} \qquad f(S_1) \cap f(S_2) = \big\{x \in \mathbb{R} : 0 \le x < 4\big\}$$

$$f^{-1}(T_1 \cup T_2) = \big\{x \in \mathbb{R} : -\sqrt{2} < x < \sqrt{2}\big\} \qquad f^{-1}(T_1) \cup f^{-1}(T_2) = \big\{x \in \mathbb{R} : -\sqrt{2} < x < \sqrt{2}\big\}$$

$$f^{-1}(T_1 \cap T_2) = \big\{x \in \mathbb{R} : -1 \le x \le 1\big\} \qquad f^{-1}(T_1) \cap f^{-1}(T_2) = \big\{x \in \mathbb{R} : -1 \le x \le 1\big\}$$

4. (a) Prove that $\lfloor -x \rfloor = -\lceil x \rceil$ and $\lceil -x \rceil = -\lfloor x \rfloor$.

PROOF: If x is an integer then

$$\lfloor -x \rfloor = -x, \quad \lceil x \rceil = x \ , \quad \lceil -x \rceil = -x \ , \quad \text{and} \quad \lfloor x \rfloor = x \ ,$$

so that the two identities are clearly satisfied.

If x is not an integer then it can be uniquely written as $x = n + r$, with $n \in \mathbb{Z}$ and where r is not zero, but $0 < r < 1$.

Then

$$\lfloor -x \rfloor = \lfloor -(n+r) \rfloor = \lfloor -n - r \rfloor = -n + \lfloor -r \rfloor = -n - 1 \ ,$$

$$\lceil x \rceil = \lceil n + r \rceil = n + \lceil r \rceil = n + 1 \ ,$$

$$\lceil -x \rceil = \lceil -(n+r) \rceil = -n + \lceil -r \rceil = -n \ ,$$

$$\lfloor x \rfloor = \lfloor n + r \rfloor = n + \lfloor r \rfloor = n \ ,$$

so that both identities are again satisfied.

(b) Give a proof by cases that $\lfloor 4x \rfloor = \lfloor x \rfloor + \lfloor x + \frac{1}{4} \rfloor + \lfloor x + \frac{1}{2} \rfloor + \lfloor x + \frac{3}{4} \rfloor$.

PROOF: We can uniquely write x as $x = n + r$, with $n \in \mathbb{Z}$ and $0 \le r < 1$. It is then sufficient to consider the following four cases:

i. $0 \le r < \frac{1}{4}$: Then

$$0 \le 4r < 1, \qquad \frac{1}{4} \le r + \frac{1}{4} < \frac{2}{4}, \qquad \frac{2}{4} \le r + \frac{2}{4} < \frac{3}{4}, \qquad \frac{3}{4} \le r + \frac{3}{4} < 1,$$

so that

$$\lfloor 4x \rfloor = \lfloor 4n + 4r \rfloor = 4n, \qquad \lfloor x \rfloor = \lfloor n + r \rfloor = n,$$

and

$$\lfloor x + \frac{1}{4} \rfloor = \lfloor n + r + \frac{1}{4} \rfloor = n, \quad \lfloor x + \frac{2}{4} \rfloor = \lfloor n + r + \frac{2}{4} \rfloor = n, \quad \lfloor x + \frac{3}{4} \rfloor = \lfloor n + r + \frac{3}{4} \rfloor = n.$$

Hence $\lfloor x \rfloor + \lfloor x + \frac{1}{4} \rfloor + \lfloor x + \frac{1}{2} \rfloor + \lfloor x + \frac{3}{4} \rfloor \ = \ n + n + n + n \ = \ 4n \ = \ \lfloor 4x \rfloor$.

ii. $\frac{1}{4} \le r < \frac{2}{4}$: Then

$$1 \le 4r < 2, \qquad \frac{2}{4} \le r + \frac{1}{4} < \frac{3}{4}, \qquad \frac{3}{4} \le r + \frac{2}{4} < 1, \qquad 1 \le r + \frac{3}{4} < \frac{5}{4},$$

so that

$$\lfloor 4x \rfloor = \lfloor 4n + 4r \rfloor = 4n + 1, \qquad \lfloor x \rfloor = \lfloor n + r \rfloor = n,$$

and

$$\lfloor x + \frac{1}{4} \rfloor = \lfloor n + r + \frac{1}{4} \rfloor = n, \quad \lfloor x + \frac{2}{4} \rfloor = \lfloor n + r + \frac{2}{4} \rfloor = n, \quad \lfloor x + \frac{3}{4} \rfloor = \lfloor n + r + \frac{3}{4} \rfloor = n+1.$$

Hence $\lfloor x \rfloor + \lfloor x + \frac{1}{4} \rfloor + \lfloor x + \frac{1}{2} \rfloor + \lfloor x + \frac{3}{4} \rfloor \ = \ n + n + n + (n+1) \ = \ 4n+1 \ = \ \lfloor 4x \rfloor$.

iii. $\frac{2}{4} \le r < \frac{3}{4}$: Then

$$2 \le 4r < 3, \qquad \frac{3}{4} \le r + \frac{1}{4} < 1, \qquad 1 \le r + \frac{2}{4} < \frac{5}{4}, \qquad \frac{5}{4} \le r + \frac{3}{4} < \frac{6}{4},$$

so that

$$\lfloor 4x \rfloor = \lfloor 4n + 4r \rfloor = 4n + 2, \qquad \lfloor x \rfloor = \lfloor n + r \rfloor = n,$$

and

$$\lfloor x + \frac{1}{4} \rfloor = \lfloor n + r + \frac{1}{4} \rfloor = n, \quad \lfloor x + \frac{2}{4} \rfloor = \lfloor n + r + \frac{2}{4} \rfloor = n+1, \quad \lfloor x + \frac{3}{4} \rfloor = \lfloor n + r + \frac{3}{4} \rfloor = n+1.$$

Hence $\lfloor x \rfloor + \lfloor x + \frac{1}{4} \rfloor + \lfloor x + \frac{1}{2} \rfloor + \lfloor x + \frac{3}{4} \rfloor \ = \ n + n + (n+1) + (n+1) \ = \ 4n+2 \ = \ \lfloor 4x \rfloor$.

iv. $\frac{3}{4} \le r < 1$: Then

$$3 \le 4r < 4, \qquad 1 \le r + \frac{1}{4} < \frac{5}{4}, \qquad \frac{5}{4} \le r + \frac{2}{4} < \frac{6}{4}, \qquad \frac{6}{4} \le r + \frac{3}{4} < \frac{7}{4},$$

so that

$$\lfloor 4x \rfloor = \lfloor 4n + 4r \rfloor = 4n + 3, \qquad \lfloor x \rfloor = \lfloor n + r \rfloor = n,$$

and

$$\lfloor x + \frac{1}{4} \rfloor = \lfloor n + r + \frac{1}{4} \rfloor = n+1, \quad \lfloor x + \frac{2}{4} \rfloor = \lfloor n + r + \frac{2}{4} \rfloor = n+1, \quad \lfloor x + \frac{3}{4} \rfloor = \lfloor n + r + \frac{3}{4} \rfloor = n+1.$$

Hence

$$\lfloor x \rfloor + \lfloor x + \frac{1}{4} \rfloor + \lfloor x + \frac{1}{2} \rfloor + \lfloor x + \frac{3}{4} \rfloor \ = \ n + (n+1) + (n+1) + (n+1) \ = \ 4n+3 \ = \ \lfloor 4x \rfloor \ .$$

5. (a) Use the Euclidean algorithm to determine whether or not the years 1812 and 2013 are relatively prime.

   SOLUTION:

   $\gcd(2013, 1812) = \gcd(1812, 2013 \bmod 1812) = \gcd(1812, 201) = \gcd(201, 1812 \bmod 201)$ $= \gcd(201, 3) = 3$. Thus 1812 and 2013 are not relatively prime.

   (b) Let $k, m, n \in \mathbb{Z}^+$, where k and m are relatively prime. Prove that if $k | mn$ then $k | n$.

   PROOF: Note that k does not divide m because k and m are relatively prime. For the same reason none of the factors of k (except 1) divide m. Thus, since $k|mn$, k must divide n.

   Here is an alternative proof by contradiction. Suppose that k and m are relatively prime, $k|mn$, but $k \not| \, n$. The latter two imply that $k|m$, but that contradicts the assumption that k and m are relatively prime.

6. (a) Prove that if $n \in \mathbb{Z}^+$ is odd then $n^2 \equiv 1 \pmod 8$.

   PROOF: We have $n = 2k + 1$, so that $n^2 = 4k^2 + 4k + 1 = 4k(k + 1) + 1$. Since either k or $k + 1$ is even, it follows that $8 | 4k(k + 1)$. Hence $n^2 \bmod 8 = 1$, so that $n^2 \equiv 1 \pmod 8$.

   (b) Prove that for any $m, n \in \mathbb{Z}^+$ the number $\gcd(m + n, mn) - \gcd(m, n)$ is even.
   Hint: Consider the cases that arise depending on whether n and m are both even, both odd, or one is even and the other odd.

   PROOF: If both m and n are even then both are divisible by 2, and hence both $m+n$ and mn are divisible by 2. Thus the greatest common divisor of $m+n$ and mn must contain the factor 2, *i.e.*, it must be even. Similarly, $\gcd(m, n)$ is even. Hence $\gcd(m + n, mn) - \gcd(m, n)$ is even.

   If both m and n are odd then $m + n$ is even, while mn is odd. Thus $\gcd(m + n, mn)$ cannot contain the factor 2, *i.e.*, it must be odd. Similarly, $\gcd(m, n)$ is odd. Hence $\gcd(m + n, mn) - \gcd(m, n)$ is even.

   If m is odd and n even then $m + n$ is odd, while mn is even. Thus $\gcd(m + n, mn)$ cannot contain the factor 2, *i.e.*, it must be odd. Similarly, $\gcd(m, n)$ is odd. Hence $\gcd(m + n, mn) - \gcd(m, n)$ is even.

The case m is even and n odd follows similarly.

7. (a) Without computing the value of 100! , determine how many zeros are at the end of this number when it is written in decimal form. Justify your answer.

   SOLUTION: Consider the prime factorization of 100! Since $\sqrt{100} = 10$, all numbers between 2 and 100 can be factorized using only the prime numbers 2,3,5, and 7. Thus the prime factorization of 100! is of the form $2^k \cdot 3^\ell \cdot 5^m \cdot 7^n$, for some $\{k, \ell, m, n\} \subseteq \mathbb{N}$. The only way a zero is at the end of 100! is by a product $2 \cdot 5$ in the prime factorization. Counting the number of times that the prime number 5 appears in the prime factorization of each of the integers $1, 2, 3, \cdots, 100$ we find that it equals 24. Thus the prime factorization of 100! contains the factor 5^{24}. Similarly, counting the number of times that the prime number 2 appears, we find that it appears more that 24 times. You can thus group 24 5's and 24 2's and their product yields 24 factors 10 in factorization of 100! It follows that the decimal representation of 100! contains 24 zeroes.

   (b) Find all solutions to $m^2 - n^2 = 105$, for which both m and n are integers.

   SOLUTION: Write the equation as $p \cdot q = 3 \cdot 5 \cdot 7$, where $p = m + n$ and $q = m - n$. The possible integer solutions are

$$(p, q) = (1\ ,\ 3{\cdot}5{\cdot}7), (3\ ,\ 5{\cdot}7), (5\ ,\ 3{\cdot}7), (7\ ,\ 3{\cdot}5), (3{\cdot}5\ ,\ 7), (3{\cdot}7\ ,\ 5), (5{\cdot}7\ ,\ 3), (3{\cdot}5{\cdot}7\ ,\ 1)\ ,$$

   that is
$$(p, q) = (1, 105), (3, 35), (5, 21), (7, 15), (15, 7), (21, 5), (35, 3), (105, 1)\ .$$

   Solving for m and n we find $m = \frac{p+q}{2}$ and $n = \frac{p-q}{2}$. Thus the possible solutions are

$$(m, n) = (53, 52), (19, 16), (13, 8), (11, 4), (11, -4), (13, -8), (19, -16), (53, -52)\ .$$

8. (a) Suppose that Hilbert's Grand Hotel is fully occupied, but the hotel closes all the even numbered rooms for maintenance. Show that all guests can remain in the hotel.

   SOLUTION: We want a one-to-one function from the set of positive integers to the set of odd positive integers. The simplest one to use is $f(n) = 2n - 1$. We put the guest currently in Room n into Room $(2n - 1)$. Thus the guest in Room 1 stays put, the guest in Room 2 moves to Room 3, the guest in Room 3 moves to Room 5, and so on.

   (b) Show that a countably infinite number of guests arriving at Hilbert's fully occupied Grand Hotel can be given rooms without evicting any current guest.

   SOLUTION: First we can make the move explained in part (a), which frees up all the even-numbered rooms. The new guests can go into those rooms (the first into Room 2, the second into Room 4, and so on).

9. Determine whether each of these sets is countable or uncountable. For those that are countably infinite, exhibit a one-to-one correspondence between the set of positive integers and that set.

   (a) integers not divisible by 3

   SOLUTION: This set is countable. The integers in the set are ± 1, ± 2, ± 4, ± 5, ± 7, and so on. We can list these numbers in the order $1, -1, 2, -2, 4, -4, 5, -5, 7, -7, \ldots$, thereby establishing the desired correspondence. In other words, the correspondence is given by 1 $\leftrightarrow$ 1, 2 $\leftrightarrow$ -1, 3 $\leftrightarrow$ 2, 4 $\leftrightarrow$ -2, 5 $\leftrightarrow$ 4, and so on.

   (b) integers divisible by 5 but not by 7

   SOLUTION: This is similar to part (a); we can simply list the elements of the set in order of increasing absolute value,listing each positive term before its corresponding negative: $5, -5, 10, -10, 15, -15, 20, -20, 25, -25, 30, -30, 40, -40, 45, -45, 50, -50, \ldots.$

   (c) the real numbers with decimal representations consisting of all 1s

   SOLUTION: This set is countable but a little tricky. We can arrange the numbers in a 2-dimensional table as follows:

$.\overline{1}$	.1	.11	.111	.1111	.11111	.111111	...
$1.\overline{1}$	1	1.1	1.11	1.111	1.1111	1.11111	...
$11.\overline{1}$	11	11.1	11.11	11.111	11.1111	11.11111	...
$111.\overline{1}$	111	111.1	111.11	111.111	111.1111	111.11111	...
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	

   where $\overline{1}$ is an infinite sequence of 1's. Thus we have shown that our set is the countable union of countable sets (each of the countable sets is one row of this table), therefore the entire set is countable. For an explicit correspondence with the positive integers, we can zigzag along the positive-sloping diagonals as in Figure 3, p. 173 of the textbook: 1 $\leftrightarrow$ $.\overline{1}$, 2 $\leftrightarrow$ $1.\overline{1}$, 3 $\leftrightarrow$ .1, 4 $\leftrightarrow$ .11, 5 $\leftrightarrow$ 1, and so on.

   (d) the real numbers with decimal representations of all 1s or 9s

   SOLUTION: This set is not countable. We can prove it by the same diagonalization argument as was used to prove that the set of all reals is uncountable. Assume to the contrary that the set is countable. List all the members in the order $r_i, i = 1, \ldots$ as follows:

$$\begin{aligned} r_1 &= 0.d_{11}d_{12}d_{13}d_{14}d_{15}d_{16}\ldots \\ r_2 &= 0.d_{21}d_{22}d_{23}d_{24}d_{25}d_{26}\ldots \\ r_3 &= 0.d_{31}d_{32}d_{33}d_{34}d_{35}d_{36}\ldots \\ &\vdots \end{aligned}$$

   Form a new real number with the decimal expansion $r = 0.d_1d_2d_3\ldots$, where

$$d_i = \begin{cases} 1 & \text{if } d_{ii} = 9 \\ 9 & \text{if } d_{ii} = 1 \text{ or } d_{ii} \text{ is blank.} \end{cases}$$

Thus r is not equal to any of the $r_1, r_2, r_3, ...,$ because it differs from r_i in its i-th position after the decimal point. Therefore there is a real number with decimal representations of all 1s or 9s that is not on the list since every real number with decimal representations of all 1s or 9s has a unique decimal expansion. Hence, all the real numbers with decimal representations of all 1s or 9s cannot be listed, so the set of real numbers with decimal representations of all 1s or 9s is uncountable.