CONCORDIA UNIVERSITY
Department of Mathematics & Statistics

Course	**Number**	**Sections**
Mathematics	203	All
Examination	**Date**	**Duration**
Midterm Test	2 March, 2014	1 h 30 min
Special Instructions:	Only approved calculators are allowed **Show your work for full marks**	

1. (11 marks): **(a)** An open rectangular box (no top) with volume 3 cubic meters has a square base. If the length of each side of the base is x and the height is h, express the surface area S of the box as a function of x only (**not** x and h).

(b) Let $f(x) = \sqrt{2x-8}$ and $g(x) = x^2 - 5$ Find the composite functions $f \circ g$ and $g \circ f$, and determine their domains.

(c) Find the inverse function $f^{-1}(x)$ of $f(x) = \log_5(2x-1)$ and determine the domain and the range of $f^{-1}(x)$.

2. (8 marks) Find the limit or explain why the limit does not exist:

(a) $\displaystyle\lim_{t\to 0}\left(\frac{1}{t} - \frac{1}{t\sqrt{1+t}}\right)$

(b) $\displaystyle\lim_{x\to 3}\frac{x\,|x-3|}{x^2-9}$

3. (4 marks) Find all horizontal and vertical asymptotes of the graph

$$y = \frac{x^2 - x}{x^2 - 6x + 5}$$

4. (5 marks) Find the second derivative of

$$f(x) = x\,e^x(1 + e^{-x}).$$

(continued on the other side)

5. (16 marks) Find the derivatives of the following functions. (You don't need to simplify the final answer, but you must show how you calculate it):

(a) $f(x) = x\sqrt{x}\left(x - \frac{1}{x}\right)^2$

(b) $f(x) = (1 + x^3)\,e^{3x}$

$e^{3x} + e^{3x}x^3$

(c) $f(x) = \dfrac{\cos^2 x}{1 + \tan x}$

(d) $f(x) = \sqrt{x^2 + \cos(e^{x^3 \sin x})}$

6. (6 marks) Given the function $f(x) = \dfrac{2x}{2 + x}$,

(a) Calculate $f'(x)$ using its definition as a limit of difference quotient.

(b) Write equation of the tangent line to the curve $y = f(x)$ at the point $(2, f(2))$.

Bonus Question (3 marks). Give an example of a function $f(x)$ for which $f'(0)$ exists but $f''(0)$ does not, or explain why this is impossible.

SOLUTION

1.

(a) The surface area is $S = x^2 + 4xh = x^2 + \frac{12}{x}$, where in the last equality we have used the relationship $3 = x^2 h$.

(b)

- $f \circ g(x) = \sqrt{2(x^2-5)-8} = \sqrt{2}\sqrt{x^2-9}$ and then, $Dom_{fog} = (-\infty, -3] \cup [3, +\infty)$.
- $g \circ f(x) = (\sqrt{2x-8})^2 - 5 = 2x - 13$ and then, $Dom_{gof} = \{x \in \mathbb{R}, \quad 2x - 8 \geq 0\} = [4, +\infty)$.

Note that in general, $Dom_{gof} = \{x \in Dom_f, \quad f(x) \in Dom_g\}$.

(c) $f^{-1}(x) = \frac{5^x+1}{2}$. $Dom_{f^{-1}} = range_f = \mathbb{R}$ and $range_{f^{-1}} = Dom_f = (\frac{1}{2}, +\infty)$.

- Note that in this case, we can also use the general rule: $(f_1 \circ f_2)^{-1} = f_2^{-1} \circ f_1^{-1}$ (here, $f = f_1 \circ f_2$ with $f_1(x) = \log_5(x)$ and $f_2(x) = 2x - 1$). It turns out that $f_2^{-1}(x) = \frac{x+1}{2}$ and $f_1^{-1}(x) = 5^x$, and then $f^{-1} = f_2^{-1} \circ f_1^{-1}(x) = \frac{5^x+1}{2}$.

2.

(a) $\lim_{t\to 0}\left(\frac{1}{t} - \frac{1}{t\sqrt{1+t}}\right) = \lim_{t\to 0}\frac{\sqrt{1+t}-1}{t\sqrt{1+t}} = \lim_{t\to 0}\frac{(t+1)-1^2}{t\sqrt{1+t}\,(\sqrt{1+t}+1)}$ where in last equality we have multiplied the nominator as well as the denominator by the conjugate expression $\sqrt{1+t}+1$.

Then, $\lim_{t\to 0}\left(\frac{1}{t} - \frac{1}{t\sqrt{1+t}}\right) = \lim_{t\to 0}\frac{t}{t\sqrt{1+t}\,(\sqrt{1+t}+1)} = \lim_{t\to 0}\frac{1}{\sqrt{1+t}\,(\sqrt{1+t}+1)} = 1/2.$

(b) $\lim_{x\to 3^+}\frac{x\,|x-3|}{x^2-9} = \lim_{x\to 3^+}\frac{x\,(x-3)}{x^2-9} = \lim_{x\to 3^+}\frac{x}{x+3} = 1/2$ and

$\lim_{x\to 3^-}\frac{x\,|x-3|}{x^2-9} = \lim_{x\to 3^+}\frac{-x\,(x-3)}{x^2-9} = \lim_{x\to 3^+}\frac{-x}{x+3} = -1/2$. The limit from the right is different from the left hand limit and then, $\lim_{x\to 3}\frac{x\,|x-3|}{x^2-9}$ does not exist.

3.

First of all, the domain of $y = \frac{x^2-x}{x^2-6x+5}$ is $\mathbb{R}\backslash\{1,5\}$.

$\lim_{x\to\infty(or -\infty)}\frac{x^2-x}{x^2-6x+5} = \lim_{x\to\infty}\frac{x(x-1)}{(x-1)(x-5)} = \lim_{x\to\infty}\frac{x}{(x-5)} = \lim_{x\to\infty}\frac{1}{(1-\frac{5}{x})} = 1$. Then the graph has an horizontal asymptote of equation $y = 1$.

$\lim_{x\to 5}\frac{x^2-x}{x^2-6x+5} = \lim_{x\to 5}\frac{x}{(x-5)} = \infty$ (Indeterminate form $5/0$) and then, The line $x = 5$ is a vertical asymptote.

However, the line of equation $x = 1$ cannot be a vertical asymptote. In fact, $\lim_{x\to 1} \dfrac{x^2 - x}{x^2 - 6x + 5}$ $= \lim_{x\to 1} \dfrac{x}{(x-5)} = -1/4$ is finite.

5.

Note that $f(x) = x\,(e^x + e^{x-x}) = x(e^x + e^0) = xe^x + x$. Using derivatives rules, $f'(x) = e^x + xe^x + 1 = e^x(x+1) + 1$ and then, $f''(x) = e^x(x+1) + e^x = e^x(x+2)$.

4. Using derivatives rules and Chain rule:

(a) $f'(x) = \sqrt{x}\left(x - \dfrac{1}{x}\right)^2 + x\dfrac{1}{2\sqrt{x}}\left(x - \dfrac{1}{x}\right)^2 + x\sqrt{x}\,2\left(x - \dfrac{1}{x}\right)\left(1 + \dfrac{1}{x^2}\right).$

(b) $f'(x) = 3x^2\, e^{3x} + 3(1+x^3)\, e^{3x}$.

(c) $f'(x) = \dfrac{-2\sin x(1+\tan x) - \cos^2 x\,(\dfrac{1}{\cos^2 x})}{(1+\tan x)^2} = \dfrac{-2\sin x(1+\tan x) - 1}{(1+\tan x)^2}.$

(d) $f'(x) = \dfrac{1}{2}[x^2 + \cos(e^{x^3 \sin x})]^{-1/2}\,[2x - \sin(e^{x^3 \sin x})\, e^{x^3 \sin x}\,(3x^2\, \sin x + x^3\, \cos x)].$

6.

(a) By definition, for any $x \neq -2$,

$$f'(x) = \lim_{h\to 0} \frac{\dfrac{2(x+h)}{2+(x+h)} - \dfrac{2x}{2+x}}{h} = \lim_{h\to 0} \frac{2(x+h)(2+x) - 2x(x+2+h)}{h(x+2+h)(2+x)}$$
$$= \lim_{h\to 0} \frac{4h}{h(x+2+h)(2+x)} = \lim_{h\to 0} \frac{4}{(x+2+h)(2+x)}$$
$$= \frac{4}{(x+2)^2}.$$

(b) Using (a), $f'(2) = \dfrac{4}{(2+2)^2} = 1/4$ which is the slope of the tangent line at $(2, f(2))$. Then, the equation of the tangent line is of the form $y = \dfrac{1}{4}x + b$ (b is constant to be determined by the relation $f(2) = \dfrac{1}{4}2 + b$). Finally, since $f(2) = 1$, $b = 1/2$ and then, $y = \dfrac{1}{4}x + \dfrac{1}{2}$ is the equation of the tangent line to the curve $y = f(x)$ at the point $(2, 1)$.

Bonus Question

Given $f(x) = x^{4/3}$ for example, $f'(x) = \dfrac{4}{3}x^{1/3}$ is defined at 0 and $f'(0) = 0$. However, $f''(x) = \dfrac{4}{9}x^{-2/3} = \dfrac{4}{9\,x^{2/3}}$ is not defined at 0, and then $f''(0)$ does not exist.

1. a) $V = 3\ m^3$

$$3 = x^2 \cdot h \Rightarrow \frac{3}{x^2} = h$$

$$S = x^2 + 4xh$$

$$S = x^2 + 4x \cdot \left(\frac{3}{x^2}\right)$$

$$\boxed{S = x^2 + 12x^{-1}}$$

b) $f(x) = \sqrt{2x-8}$ $\quad g(x) = x^2 - 5$

$2x^2 - 18 \geqslant 0$

$2x^2 \geqslant 18$

$x^2 \geqslant 9$

$x \geqslant \pm 3$

$$f \circ g(x) = \sqrt{2(x^2-5)-8}$$
$$\text{"} = \sqrt{2x^2 - 10 - 8}$$
$$\text{"} = \sqrt{2x^2 - 18}$$

Domain: $2x^2 - 18 \geqslant 0$

$x \geqslant \pm 3$

$(-\infty, -3] \cup [3, +\infty)$

$$g \circ f(x) = \left(\sqrt{2x-8}\right)^2 - 5$$
$$\text{"} = 2x - 8 - 5$$
$$\text{"} = 2x - 13$$

Domain: $2x - 8 \geqslant 0$

$2x \geqslant 8$

$x \geqslant 4$

$[4, +\infty)$

$x^2 \geqslant 9$

$x \geqslant \pm 3$ Domain: $2x^2 - 18 \geqslant 0$

$x \geqslant \pm 3$

$(-\infty, -3] \cup [3, +\infty)$

Domain: $2x - 8 \geqslant 0$

$2x \geqslant 8$

$x \geqslant 4$

$[-4, +\infty)$

c) $f(x) = \log_5(2x-1)$

$f^{-1}(x) \Rightarrow x = \log_5(2y-1)$

$5^x = 5^{\log_5(2y-1)}$

$5^x = 2y-1$

$5^x + 1 = 2y$

$f^{-1}(x) \Rightarrow y = (5^x+1)/2$

Domain of $f(x) = \log_5(2x-1)$

$2x-1 > 0$

$2x > 1$

$x > 1/2$

Domam $f^{-1}(x) = x \in \mathbb{R}$

Range $f^{-1}(x) = x > 1/2$

2. a) $\lim_{t\to 0} \left(\frac{1}{t} - \frac{1}{t\cdot\sqrt{1+t}}\right)$

$\lim_{t\to 0} \left(\frac{\sqrt{1+t} - 1}{t\cdot\sqrt{1+t}}\right)\cdot\left(\frac{\sqrt{1+t}+1}{\sqrt{1+t}+1}\right)$

$\lim_{t\to 0} \left(\frac{\cancel{1}+t-\cancel{1}}{t(1+t)+t\sqrt{1+t}}\right)$

$\lim_{t\to 0} \left(\frac{\cancel{t}}{\cancel{t}((1+t)+(\sqrt{1+t}))}\right)$

$\lim_{t\to 0} \left(\frac{1}{1+t+\sqrt{1+t}}\right)$

$\lim_{t\to 0} \left(\frac{1}{1+0+\sqrt{1+0}}\right)$

$\lim_{t\to 0} \left(\frac{1}{2}\right)$

b) $\lim_{x\to 3} \frac{x|x-3|}{x^2-9}$

$\lim_{x\to 3} \frac{x(x-3)}{(x+3)(x-3)}$	$\lim_{x\to 3} \frac{x\cdot -1(\cancel{x-3})}{(x+3)(\cancel{x-3})}$
$\lim_{x\to 3} \frac{x}{x+3}$	$\lim_{x\to 3} \frac{-x}{x+3}$
$\lim_{x\to 3} \frac{3}{3+3}$	$\lim_{x\to 3} \frac{-3}{3+3}$
$\lim_{x\to 3} \frac{3}{6} = \boxed{\frac{1}{2}}$	$\lim_{x\to 3} \frac{-3}{6} = \boxed{-\frac{1}{2}}$

$\frac{1}{2} \neq -\frac{1}{2}$

$\therefore \lim_{x\to 3}$ DNE

3. $y = \dfrac{x^2 - x}{x^2 - 6x + 5}$

V-A: $x^2 - 6x + 5 = 0$ $\rightarrow$ $\dfrac{x^2 - x}{(x-1)(x-5)}$

$(x-1)(x-5) = 0$

$x \neq 1 \quad x \neq 5$

$\dfrac{x(\cancel{x-1})}{(\cancel{x-1})(x-5)}$

$\dfrac{x}{x-5}$ $\boxed{V\text{-}A = 5}$

H-A: $\lim_{x \to \infty} \dfrac{1x^2 - \cancel{x}}{1x^2 - \cancel{6x + 5}}$ $\dfrac{1}{1} = \boxed{1}$

4. $f(x) = x \cdot e^x (1 + e^{-x})$

$f(x) = x \cdot e^x + x$

$f'(x) = x \cdot e^x + e^x + 1$

$f''(x) = 1 \cdot e^x + x \cdot e^x + e^x$

$f''(x) = xe^x + 2e^x$

4. $f(x) = x \cdot e^x (1 + e^{-x})$
$f(x) = x \cdot e^x + x$
$f'(x) = x \cdot e^x + e^x + 1$
$f''(x) = 1 \cdot e^x + x \cdot e^x + e^x$
$f''(x) = xe^x + 2e^x$

5. a) $f(x) = x\sqrt{x}\,(x - \frac{1}{x})^2$
$f(x) = x \cdot x^{1/2} \cdot (x - x^{-1})^2$
$f(x) = x^{3/2} \cdot (x - x^{-1})^2$
$f(x) = x^{3/2} \cdot (x^2 - 2 - x^{-2})$
$f(x) = x^{7/2} - 2x^{3/2} - x^{-1/2}$
$f'(x) = \frac{7}{2}x^{5/2} - 3x^{1/2} + \frac{1}{2}x^{-3/2}$

b) $f(x) = (1 + x^3)\, e^{3x}$
$f'(x) = 3x^2 \cdot e^{3x} + 3e^{3x} \cdot (1 + x^3)$

c) $f(x) = \dfrac{\cos^2 x}{1 + \tan x} = \dfrac{(\cos(x))^2}{1 + \tan(x)}$ $\dfrac{f' \cdot g - f \cdot g'}{g^2}$

$$f'(x) = \frac{[(\cos x \cdot \cos x)' \cdot 1 + \tan] - [\cos^2 x \cdot (1 + \tan x)']}{(1 + \tan x)^2}$$

$$= \frac{[(-\sin x \cdot \cos + \cos - \sin x) \cdot (1 + \tan x)] - [\cos^2 x \cdot \frac{1}{\cos^2 x}]}{(1 + \tan x)^2}$$

$$= \frac{-2\sin x \cancel{(1 + \tan x)} (\cos^2 x \cdot \frac{1}{\cos^2 x})}{(1 + \tan x)^2}$$

$-\sin \cos x + -\sin \cos x$

c) $f(x) = \dfrac{\cos^2 x}{1 + \tan x}$

$f(x) = \dfrac{(\cos x)^2}{1 + \tan x}$

c) $f(x) = \dfrac{\cos^2 x}{1 + \tan x}$

$$f(x) = \frac{(\cos x)^2}{1 + \tan x}$$

$$f'(x) = \frac{2(\cos x)^1}{(1 + \tan x)^2}$$

$$= \frac{-2\sin x(1 + \tan x) - (\sec^2 x)\cdot(\cos^2 x)}{(1 + \tan x)^2}$$

d) $f(x) = \sqrt{x^2 + \cos(e^{x^3 \sin x})}$

$f'(x) = \frac{1}{2}\left[\left(x^2 + \cos(e^{x^3 \sin x})\right)\right]^{-1/2} \cdot \left[2x - \sin(e^{x^3 \sin x})\right] \cdot \left[e^{x^3 \sin x} (3x^2 \sin x + x^3 \cdot \cos x)\right]$

6. a) $f(x) = \dfrac{2x}{2+x}$

$$f'(x) = \lim_{h \to 0} \frac{\frac{2(x+h)}{2+x+h} - \frac{2x}{2+x}}{h}$$

$$\lim_{h \to 0} \frac{\frac{2(x+h)(2+x)}{(2+x+h)(2+x)} - \frac{2x(2+x+h)}{(2+x+h)(2+x)}}{h}$$

$$\lim_{h \to 0} \frac{\frac{2(x+h)(2+x) - 2x(2+x+h)}{(2+x+h)(2+x)}}{h} \cdot \frac{1}{h}$$

$$\lim_{h \to 0} \frac{\overset{2x+2h}{2(x+h)}(2+x) - 2x(2+x+h)}{h(2+x+h)(2+x)}$$

$$\lim_{h \to 0} \frac{(\cancel{4x} + \cancel{2x^2} + 4h + \cancel{2xh}) - (\cancel{4x} + \cancel{2x^2} + \cancel{2xh})}{h(2+x+h)(2+x)}$$

$$\lim_{h \to 0} \frac{4h}{h(2+x+h)(2+x)}$$

$$\lim_{h \to 0} \frac{4}{(2+x)(2+x)} = \boxed{\frac{4}{(2+x)^2}}$$

b) $\frac{4}{(2+x)^2}$ @ $x=2$

$$\frac{4}{(2+2)^2} = \frac{1}{4}$$

$y = \frac{1}{4}x + b$
$1 = \frac{1}{4}(2) + b$
$b = \frac{1}{2}$

$$\boxed{y = \frac{1}{4}x + \frac{1}{2}}$$