Emat 233 Midterm Exam
November 17, 2005

The exam is out of 50 points. No caclulators allowed. Write all solutions in full. Correct answers with incorrect or incomplete justification will not receive full credit.

[**10 points**] **Problem 1.** Compute the curl and the divergence of the vector field

$$\vec{F}(x,y,z) = yze^x\mathbf{i} + (2x-3yz)\mathbf{j} + xy^2z^3\mathbf{k}.$$

Solution. Write $\vec{F} = P\mathbf{i} + Q\mathbf{j} + R\mathbf{k}$. Then

$$\begin{aligned}
\operatorname{div}\vec{F} = \nabla\cdot\vec{F} &= P_x + Q_y + R_z \\
&= \frac{\partial}{\partial x}(yze^x) + \frac{\partial}{\partial y}(2x-3yz) + \frac{\partial}{\partial z}(xy^2z^3) \\
&= \boxed{yze^x - 3z + 3xy^2z^2},
\end{aligned}$$

and

$$\begin{aligned}
\operatorname{curl}\vec{F} = \nabla\times\vec{F} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ P & Q & R \end{vmatrix} \\
&= \begin{vmatrix} \partial/\partial y & \partial/\partial z \\ 2x-3yz & xy^2z^3 \end{vmatrix}\mathbf{i} - \begin{vmatrix} \partial/\partial x & \partial/\partial z \\ yze^x & xy^2z^3 \end{vmatrix}\mathbf{j} + \begin{vmatrix} \partial/\partial x & \partial/\partial y \\ yze^x & 2x-3yz \end{vmatrix}\mathbf{k} \\
&= (2xyz^3 - (-3y))\mathbf{i} - (y^2z^3 - ye^x)\mathbf{j} + (2 - ze^x)\mathbf{k} \\
&= \boxed{y(2xz^3+3)\mathbf{i} - y(yz^3 - e^x)\mathbf{j} + (2-ze^x)\mathbf{k}}
\end{aligned}$$

[**10 points**] **Problem 2.** Evaluate the following integral by reversing the order of integration:

$$\int_0^4 \int_{\sqrt{y}}^2 \sqrt{x^3+1}\,\mathrm{d}x\,\mathrm{d}y.$$

Solution. The region is bounded by the curve $x = \sqrt{y}$ on the left, and the line $x = 2$ on the right, going from $y = 0$ to $y = 4$. This is the same as the region bounded by the curve $y = x^2$ on top, and the line $y = 0$ on bottom, going from $x = 0$ to $x = 2$. The resulting integral is

$$\begin{aligned}
\int_0^2 \int_0^{x^2} (x^3+1)^{\frac{1}{2}}\,\mathrm{d}y\,\mathrm{d}x &= \int_0^2 y(x^3+1)^{\frac{1}{2}}\Big|_{y=0}^{y=x^2}\,\mathrm{d}x \\
&= \int_0^2 x^2(x^3+1)^{\frac{1}{2}}\,\mathrm{d}x
\end{aligned}$$

Making the substitution $u = x^3 + 1$, $\mathrm{d}u = 3x^2\mathrm{d}x$, we obtain

$$\begin{aligned}
\int_0^2 x^2(x^3+1)^{\frac{1}{2}}\,\mathrm{d}x &= \frac{1}{3}\int_1^9 u^{\frac{1}{2}}\,\mathrm{d}u \\
&= \frac{1}{3}(\frac{2}{3}u^{\frac{3}{2}})\Big|_1^9 \\
&= \frac{1}{3}(\frac{2}{3}(27) - \frac{2}{3}(1)) = \boxed{\frac{52}{9}}
\end{aligned}$$

[10 points] Problem 3. Define a force field $\vec{F}$ by

$$\vec{F}(x, y) = (2xy)\mathbf{i} + (x^2)\mathbf{j}.$$

(a) Show that this vector field is conservative, i.e. the line integrals $\int_{\mathcal{C}} \vec{F} \cdot \mathrm{d}\vec{r}$ are independent of path, and find a function $\phi(x, y)$ such that $\vec{F} = \vec{\nabla}\phi$.
(b) Let $\mathcal{C}$ be the upper half of the circle $x^2 + y^2 = 1$ in the xy plane, going from the point $(1, 0)$ to the point $(-1, 0)$. Compute the work done by the force $\vec{F}$ along the curve $\mathcal{C}$ in two different ways:
(i) Using the function ϕ found in part (a).
(ii) Computing the line integral $\int_{\mathcal{C}} \vec{F} \cdot \mathrm{d}\vec{r}$ directly along any convenient path between the endpoints.

Solution. (a) Check if $\vec{F} = P\mathbf{i} + Q\mathbf{j}$ is conservative:

$$\begin{aligned}
\frac{\partial Q}{\partial x} &= 2x \\
\frac{\partial P}{\partial y} &= 2x.
\end{aligned}$$

Since these are equal, and the functions P and Q are defined and continuously differentiable in the whole plane, we can conclude that $\vec{F}$ is conservative (that is , $P\mathrm{d}x + Q\mathrm{d}y$ is exact). If $\vec{F} = \vec{\nabla}\phi$, then ϕ must satisfy

$$\begin{aligned}
\frac{\partial \phi}{\partial x} &= P = 2xy \\
\frac{\partial \phi}{\partial y} &= Q = x^2.
\end{aligned}$$

Antidifferentiating the first equation with respect to x, we find $\phi = x^2y + g(y)$, where g is some unknown function of y. Taking the derivative of this with respect to y, we get $\phi_y = x^2 + g'(y)$; comparing with the second equation, we see that $g'(y) = 0$ and hence $g(y) = C$ for some constant C. So we have found that $\vec{F} = \vec{\nabla}\phi$, where

$$\boxed{\phi = x^2y + C}$$

for some constant C. (Alternately, you can just *guess* that $\phi = x^2y + C$, provided you justify your guess. That is, you must write something like: "for this ϕ, $\frac{\partial\phi}{\partial x} = 2xy$ and $\frac{\partial\phi}{\partial y} = x^2$, so $\vec{F} = \vec{\nabla}\phi$ as required".)

(b) (i) By the fundamental theorem of calculus for line integrals,

$$\int_{\mathcal{C}} \vec{F}\cdot \mathrm{d}\vec{r} = \phi(-1,0) - \phi(1,0) = (-1)^2(0) - (1)^2(0) = \boxed{0}.$$

(ii) Let $\mathcal{C}'$ be the straight line from the point $(1,0)$ to the point $(-1,0)$. This line segment has the equation $y = 0$, $-1 \le x \le 1$. Consequently, from the path independence shown in (a),

$$\begin{aligned}\int_{\mathcal{C}} \vec{F}\cdot \mathrm{d}\vec{r} &= \int_{\mathcal{C}'} \vec{F}\cdot \mathrm{d}\vec{r}\\ &= \int_{\mathcal{C}'} (2xy)\,\mathrm{d}x + x^2\,\mathrm{d}y\\ &= \int_1^{-1} (2x(0))\,\mathrm{d}x + x^2(0) = \boxed{0}\end{aligned}$$

If you want to do it the long way, you can parametrize the semicircular curve $\mathcal{C}$ by $x(t) = \cos t$, $y(t) = \sin t$, $0 \le t \le \pi$. Then $dx = -\sin t dt$, $dy = \cos t dt$, and the line integral becomes

$$\begin{aligned}\int_{\mathcal{C}} (2xy)\,\mathrm{d}x + x^2\,\mathrm{d}y &= \int_0^{\pi} (2\cos t\sin t)(-\sin t) + (\cos t)^2(\cos t)\,\mathrm{d}t\\ &= \int_0^{\pi} (\cos^2 t - 2\sin^2 t)\cos t\,\mathrm{d}t\\ &= \int_0^{\pi} (1 - 3\sin^2 t)\cos t\,\mathrm{d}t\\ &= (\sin t - \sin^3 t)\Big|_0^{\pi} = \boxed{0}\end{aligned}$$

[10 points] Problem 4. By using the appropriate theorem (which must be named) compute the circulation $\oint_{\mathcal{C}} \vec{F}\cdot \mathrm{d}\vec{r}$ of the vector field

$$\vec{F}(x,y) = (3xy - e^{-x^4})\mathbf{i} + (5xy + \cos^2(y^{45}))\mathbf{j}$$

around the curve $\mathcal{C}$ given by the boundary of the rectangle of vertices $(1,1)$, $(3,1)$, $(3,2)$, $(1,2)$ oriented counterclockwise.

Solution. The integral is too complicated to do directly. But we can apply $\boxed{\text{Green's theorem}}$:

$$\oint_{\mathcal{C}} \vec{F}\cdot \mathrm{d}\vec{r} = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\mathrm{d}A,$$

where $\vec{F} = P\mathbf{i} + Q\mathbf{j}$ and C is the (oriented) boundary of R, provided P, Q, P_y and Q_x are defined and continuous in R. This gives

$$\begin{aligned}
\oint_{\mathcal{C}} \vec{F} \cdot \mathrm{d}\vec{r} &= \int_1^2 \int_1^3 (5y - 3x)\,\mathrm{d}x\,\mathrm{d}y \\
&= \int_1^2 (5xy - \frac{3}{2}x^2)\Big|_{x=1}^{x=3} \mathrm{d}y \\
&= \int_1^2 (10y - 12)\,\mathrm{d}y \\
&= (5y^2 - 12y)\Big|_1^2 = \boxed{3}.
\end{aligned}$$

[10 points] Problem 5. Compute the double integral

$$\iint_{\mathcal{R}} \frac{y}{\sqrt{x^2+y^2}} \sin(x^2+y^2)\,\mathrm{d}A,$$

where $\mathcal{R}$ is the region above the x-axis, bounded by the x-axis and the circles $x^2+y^2=1$ and $x^2+y^2=4$.

Solution. Because of the shape of the region, we try converting the integral to polar coordinates: $x = r\cos\theta$, $y = r\sin\theta$, $\mathrm{d}A = r\mathrm{d}r\mathrm{d}\theta$, and

$$\begin{aligned}
\iint_{\mathcal{R}} \frac{y}{\sqrt{x^2+y^2}} \sin(x^2+y^2)\,\mathrm{d}A &= \int_0^\pi \int_1^2 \frac{r\sin\theta}{r} \sin(r^2) r\,\mathrm{d}r\,\mathrm{d}\theta \\
&= \int_0^\pi \int_1^2 r\sin(r^2)\sin\theta\,\mathrm{d}r\,\mathrm{d}\theta \\
&= \int_0^\pi (-\frac{1}{2}\cos(r^2))\Big|_{r=1}^{r=2} \sin\theta\,\mathrm{d}\theta \\
&= -\frac{1}{2}(\cos(4) - \cos(1)) \int_0^\pi \sin\theta\,\mathrm{d}\theta \\
&= -\frac{1}{2}(\cos(4) - \cos(1))(-\cos\theta)\Big|_0^\pi \\
&= -\frac{1}{2}(\cos(4) - \cos(1))(-(-1) + 1) \\
&= \boxed{\cos(1) - \cos(4)}.
\end{aligned}$$