10.102. $p(y_i | p) = p^{y_i}(1-p)^{1-y_i}$ $y_i = 0, 1$.

a) $H_0: p = p_0$ vs $H_a: p = p_a$, $p_0 < p_a$.

i) $\dfrac{L(p_0)}{L(p_a)} = \dfrac{p_0^{\Sigma y_i}(1-p_0)^{n-\Sigma y_i}}{p_a^{\Sigma y_i}(1-p_a)^{n-\Sigma y_i}} = \left[\dfrac{p_0(1-p_a)}{p_a(1-p_0)}\right]^{\Sigma y_i}\left(\dfrac{1-p_0}{1-p_a}\right)^n$.

ii) test rejects when

$$\sum_{i=1}^{n} y_i \ln\left(\frac{p_0(1-p_a)}{p_a(1-p_0)}\right) + n\ln\left(\frac{1-p_0}{1-p_a}\right) < \ln k.$$

$$\sum_{i=1}^{n} y_i > \left[\ln k - n\ln\left(\frac{1-p_0}{1-p_a}\right)\right] \times \left[\ln \frac{p_0(1-p_a)}{p_a(1-p_0)}\right]^{-1} = k^*$$

iii) $RR = \{\sum_{i=1}^{n} y_i > k^*\}$

b) $\alpha = p(\sum_{i=1}^{n} y_i > k^* | p_0)$. Under H_0 true, $\sum_{i=1}^{n} y_i \rightarrow$ binomial (n, p_0).

c) Because the critical value can be specified without with any value of p_a, we can extend it to UMP test.

10-62 $P_1 = 0.6$ $P_2 = 0.7$ Sample $\hat{p}_1 = 55/70$ $\hat{p}_2 = 23/70$.

$H_0: p_1 - p_2 = 0$

$H_a: p_1 - p_2 > 0$

$$z = \frac{.786 - .329}{\sqrt{(.567)(.443) \cdot 2/70}}, \text{ where } 55+23/140 = .567$$

$$= 5.493 \quad (> z_{.05} = 1.645)$$

So we reject H_0.

known → unkown.

Date. No.

EX 10.94 $Y_1, \ldots, Y_n \sim N(\mu, \sigma^2)$

MP test $H_0: \sigma^2 = \sigma_0^2$ vs $H_1: \sigma^2 = \sigma_1^2$ $\quad \sigma_1^2 < \sigma_0^2$.

MP test : $\dfrac{L(\theta_0)}{L(\theta_1)} = \dfrac{L(\theta_0^2)}{L(\theta_1^2)}$

$$L(\theta) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{\frac{-\Sigma(y_i-\mu)^2}{\sigma^2}}$$

so, MP test $= \left\{ \dfrac{\left(\frac{1}{\sqrt{2\pi\sigma_0^2}}\right)^n e^{\frac{-\Sigma(y_i-\mu)^2}{\sigma_0^2}}}{\left(\frac{1}{\sqrt{2\pi\sigma_1^2}}\right)^n \cdot e^{\frac{-\Sigma(y_i-\mu)^2}{\sigma_1^2}}} < k \right\}$

$$= \left\{ \left(\frac{\sqrt{\sigma_1^2}}{\sqrt{\sigma_0^2}}\right)^n e^{-\Sigma(y_i-\mu)^2\left(\frac{1}{\sigma_0^2}-\frac{1}{\sigma_1^2}\right)} < k \right\}$$

$$= \left\{ e^{-\Sigma(y_i-\mu)^2\left(\frac{1}{\sigma_0^2}-\frac{1}{\sigma_1^2}\right)} < \underbrace{k \cdot \left(\frac{\sqrt{\sigma_0^2}}{\sqrt{\sigma_1^2}}\right)}_{\text{constant} = k'} \right\}$$

$$= \left\{ -\Sigma(y_i-\mu)^2\left(\frac{1}{\sigma_0^2}-\frac{1}{\sigma_1^2}\right) < \underbrace{\log k \left(\frac{\sqrt{\sigma_0^2}}{\sqrt{\sigma_1^2}}\right)}_{\text{constant} = k''} \right\}$$

$$= \left\{ \Sigma(y_i-\mu)^2 \underbrace{\left(\frac{1}{\sigma_0^2}-\frac{1}{\sigma_1^2}\right)}_{= \text{negative } (w/\ \sigma_1^2 < \sigma_0^2)} > -k'' \Rightarrow \underline{k'''} \right\}$$

$$= \left\{ \Sigma(y_i-\mu)^2 < \underbrace{k''' \cdot \frac{1}{\left(\frac{1}{\sigma_0^2}-\frac{1}{\sigma_1^2}\right)}}_{= k''''} \right\}$$

$$= \left\{ \Sigma(y_i-\mu)^2 < k'''' \right\}$$

* If one asks $H_0: \sigma^2 = \sigma_0^2$ vs $H_1: \sigma^2 > \sigma_0^2$

→ For any $\sigma_3^2 > \sigma_0^2$,

MP test for $H_0: \sigma^2 = \sigma_0^2$ vs $H_1: \sigma^2 = \sigma_3^2$. $\{\Sigma(y_i-\mu)^2 < k\}$

So we can extend ump test.

EX 10.129 $Y_1, \dots, Y_n$ from

$$f(y \mid \theta_1, \theta_2) = \begin{cases} \left(\frac{1}{\theta_1}\right) e^{-(y-\theta_2)/\theta_1}, & y > \theta_2 \\ 0 \end{cases}$$

Find LRT for $H_0 : \theta_1 = \theta_{1,0}$ vs $H_a : \theta_1 > \theta_{1,0}$.

parameter space

with θ_2 unknown.

null ① $\Omega_0 = \{(\theta_1, \theta_2) : \theta_1 = \theta_{1,0}, -\infty < \theta_2 < \infty\}$

~~alternative~~ $\Omega_1 = \{(\theta_1, \theta_2) : \theta_1 > \theta_{1,0}, -\infty < \theta_2 < \infty\}$ ⇝ ignore.

null + alternative $\Omega = \Omega_0 \cup \Omega_1 = \{(\theta_1, \theta_2) : \theta_1 \geq \theta_{1,0}, -\infty < \theta_2 < \infty\}$

Likelihood Ratio Test $= \left\{ \dfrac{\max_{\theta \in \Omega_0} L(\theta)}{\max_{\theta \in \Omega} L(\theta)} < k \right\}$

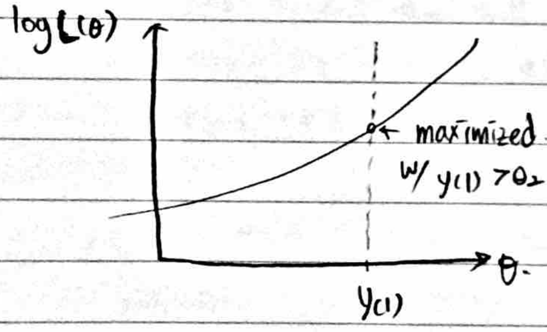
② $L(\theta) = \left(\frac{1}{\theta_1}\right)^n e^{-\frac{\sum(y_i - \theta_2)}{\theta_1}}$, $y_i > \theta_2$ → 이므로 $y_{(1)} > \theta_2$

$L(\theta), \theta \in \Omega_0 = \left(\frac{1}{\theta_{1,0}}\right)^n e^{-\frac{\sum(y_i - \theta_2)}{\theta_{1,0}}}$, $y_{(1)} > \theta_2$

#compute mle

$\max_{\theta \in \Omega_0} L(\theta) \Rightarrow \log L(\theta) = n \log \theta_{1,0} - \sum \frac{(y_i - \theta_2)}{\theta_{1,0}}$, $y_{(1)} > \theta_2$

$$\frac{d}{d\theta_2} \log L(\theta) = \frac{n}{\theta_{1,0}} > 0, \quad y_{(1)} > \theta_2$$



so $\theta_2^{MLE} = y_{(1)}$

so $\max_{\theta \in \Omega_0} L(\theta) = \left(\frac{1}{\theta_{1,0}}\right)^n e^{-\frac{\sum(y_i - \theta_2)}{\theta_{1,0}}}$.

$L(\theta) = \left(\frac{1}{\theta_1}\right)^n e^{-\frac{\sum(y_i - \theta_2)}{\theta_2}}$

$\log L(\theta) = -n \log \theta_1 - \sum \frac{(y_i - \theta_2)}{\theta_1}$, $y_{(1)} > \theta_2$

$$\frac{d}{d\theta_1} \log L(\theta) = \frac{-n}{\theta_1} + \frac{\sum(y_i - \theta_2)}{\theta_1^2} = 0$$

$$\Rightarrow -n\theta_1 + \sum(y_i - \theta_2) = 0$$

$$\Rightarrow \hat{\theta}_1 = \frac{\sum(y_i - \theta_2)}{n} = \bar{y} - \theta_2 ..$$

$$\left(\text{so } \max_{\theta \in \Omega} L(\theta) = \left(\frac{1}{\bar{y}-\theta_2}\right)^n e^{-\frac{\Sigma(y_i - y_{(1)})}{\bar{y}-\theta_2}} = \left(\frac{1}{\bar{y}-\theta_2}\right)^n e^{-n}, \right.$$

when $\theta_2 = y_{(1)}$.

Therefore,

$$LRT = \frac{\left(\frac{1}{\theta_{1,0}}\right)^n e^{-\frac{\Sigma(y_i - y_{(1)})}{y_{1,0}}}}{\left(\frac{1}{\bar{y}-y_{(1)}}\right)^n e^{-n}} < k, \quad \text{when } \theta_2 = y_{(1)}$$

$$= \left\{ \left(\frac{\bar{y}-y_{(1)}}{\theta_{1,0}}\right)^n \frac{e^{-\frac{\Sigma(y_i - y_{(1)})}{y_{1,0}}}}{e^{-n}} < k \right\}$$

$$= \left\{ (\bar{y}-y_{(1)})^n \cdot e^{-\frac{n}{y_{1,0}}(\bar{y}-y_{(1)})} < \underbrace{k \cdot \theta_{1,0}^n \cdot e^n}_{=k'} \right\}$$