Problem Set 6

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Problem 0932175

a

$x_{n+1} = \frac{2x_n s+9}{2x^2}$ by Newton's method. If $x_0 = 2$, $x_1 = \frac{25}{12}$, and $x_2 = \frac{46802}{22500}$, and $x_3 \approx 2.08008382306424$.

b

$h_0 = -\frac{2^3-9}{2*2^2}$, or $\frac{1}{12}$.$U_1 = \left[2, \frac{13}{6}\right]$.$f(x_0) = -1, f'(x_0) = 12$, and $M = 13$.

c

$\frac{|f(x_0)M|}{|f'(x_0)|^2} = \frac{13}{144}$, which is obviously less than $\frac{1}{2}$, so the sequence converges.

Problem 09321715

$$\begin{aligned} \mathbf{F}\left(\begin{pmatrix}\mathrm{y}\\ \mathrm{z}\end{pmatrix}, \lambda\right) &= 1 + a - \lambda + a^2 y = 0 \\ a + (2-\lambda)y + az &= 0 \\ -a + (3-\lambda)z &= 0 \end{aligned}$$

for $\left(\begin{pmatrix}\mathrm{y}\\ \mathrm{z}\end{pmatrix}, \lambda\right)$ starting at $\left(\begin{pmatrix}0\\ 0\end{pmatrix}, 1\right)$ with a as a parameter.

$$\mathbf{F}\left(\begin{pmatrix}\mathrm{y}\\ \mathrm{z}\end{pmatrix}, \lambda\right) = \begin{pmatrix}a\\ a\\ -a\end{pmatrix}, \text{which has a magnitude of } |a|\sqrt{3}.$$

$$\left[\mathbf{DF}\left(\begin{pmatrix}\mathrm{y}\\ \mathrm{z}\end{pmatrix}, \lambda\right)\right] = \begin{bmatrix}a^2 & 0 & -1\\ 2-\lambda & a & -y\\ 0 & 3-\lambda & -z\end{bmatrix}, \left[\mathbf{DF}\left(\begin{pmatrix}0\\ 0\end{pmatrix}, 1\right)\right] = \begin{bmatrix}a^2 & 0 & -1\\ 1 & a & 0\\ 0 & 2 & 0\end{bmatrix}$$

The inverse $\left[\mathbf{DF}\left(\begin{pmatrix}0\\ 0\end{pmatrix}, 1\right)\right]^{-1}$ is

$$\begin{bmatrix}0 & 1 & -\frac{a}{2}\\ 0 & 0 & \frac{1}{2}\\ -1 & a^2 & -\frac{a^3}{2}\end{bmatrix}$$

Squaring the magnitude of the inverse yields $\frac{9}{4} + \frac{a^2}{4} + a^4 + \frac{a^6}{4}$.

$$\left|\left[\mathbf{DF}\left(\begin{pmatrix}y_1\\ z_1\end{pmatrix}, \lambda_1\right)\right] - \left[\mathbf{DF}\left(\begin{pmatrix}y_2\\ z_2\end{pmatrix}, \lambda_2\right)\right]\right| = \left|\begin{bmatrix}0 & 0 & 0\\ -\lambda_1+\lambda_2 & 0 & y_2-y_1\\ 0 & -\lambda_1+\lambda_2 & -z_2+z_1\end{bmatrix}\right|$$

$$\text{The above matrix} = \sqrt{2(\lambda_1-\lambda_2)^2 + (y_1+y_2)^2 + (z_1-z_2)^2} \le \sqrt{2}\left|\left(\begin{pmatrix}y_1\\ z_1\end{pmatrix}, \lambda_1\right) - \left(\begin{pmatrix}y_2\\ z_2\end{pmatrix}, \lambda_2\right)\right|$$

The Kantorovich inequality will be satisfied if

$$|a|\sqrt{3}\left(\frac{9}{4} + \frac{a^2}{4} + a^4 + \frac{a^6}{4}\right)\sqrt{2} \le \frac{1}{2}$$

$|a|\sqrt{3}\left(\frac{9}{4} + \frac{a^2}{4} + a^4 + \frac{a^6}{4}\right)\sqrt{2}$ is an increasing function of $|a|$, and it will be satisfied for $|a| < a_0$, where a_0 is a positive number that satisfies the inequality.

Problem 126355

a

Suppose a solution of the form $\begin{bmatrix} x & x \\ x & x \end{bmatrix}$ exists. Then

$$\begin{bmatrix} 2x^2 & 2x^2 \\ 2x^2 & 2x^2 \end{bmatrix} + \begin{bmatrix} x & x \\ x & x \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

This is equal to the polynomial $x^2 + x - 1$, which has solutions of -1 and $\frac{1}{2}$, yielding $\begin{bmatrix} -1 & -1 \\ -1 & -1 \end{bmatrix}$ and $\begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}$.

b

We want to solve the equation $X^2 + X - \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, so $F(X) = X^2 + X - \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$.
$|F(I)| = 2$, and since $[\mathbf{D}F(I)]\,(H) = HI + IH + H = -3H$, $[\mathbf{D}F(I)]^{-1}\,(H) = \frac{H}{3}$.
Therefore, $\left|\left|\,[\mathbf{D}F(I)]^{-1}\,\right|\right| = \frac{1}{3}$.

$$\begin{aligned}
\left|\left|\,[\mathbf{D}F(A)]\,\right|\right| - \left|\left|\,[\mathbf{D}F(B)]\,\right|\right| &= \sup_{|H|=1} \Big|(AH + HA + H) - (BH + HB + H)\Big| \\
&= \sup_{|H|=1} \Big|H(A-B) - (A-B)H\Big| \\
&\leq \Big(|H(A-B) - (A-B)H|\Big) \\
&\leq \Big(|H||(A-B)| - |(A-B)||H|\Big) \leq 2|A-B|.
\end{aligned}$$

Problem 126356

$X = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and $C = \begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix}$. We can substitute these values to get

$$f\begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \begin{bmatrix} a^2 + bc + a - C_{11} \\ b(a+d+1) - C_{12} \\ c(a+d+1) - C_{21} \\ d^2 + bc + d - C_{22} \end{bmatrix}, \text{ and the derivative is } \begin{bmatrix} 2a+1 & c & b & 0 \\ b & a+d+1 & 0 & b \\ c & 0 & a+d+1 & c \\ 0 & c & b & 2d+1 \end{bmatrix}$$

When we take second partial derivatives at 0, with M = 4, $f(C) = -C$, and $f'(C) = I$. All C such that $||C|| \leq \frac{1}{8}$ will converge. The same process starting at I yields $f = 2I - C$ $f' = 3I$. All C such that $||2I - C|| \leq \frac{1}{24}$ will converge.

Problem 126357

$$\begin{aligned}
||f(A_0)|| &= \sup_{|\tilde{\mathbf{x}}|} \left|\left|\begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right|\right| = \sup_{|\tilde{\mathbf{x}}|} \left|\left|\begin{bmatrix} x_1 - x_2 \\ x_1 - x_2 \end{bmatrix}\right|\right| \\
&= \sup_{|\tilde{\mathbf{x}}|} \sqrt{2(x_1 - x_2)^2} = \sup_{|\tilde{\mathbf{x}}|} \sqrt{2(x_1 + x_2)^2 - 4x_1x_2}
\end{aligned}$$

$x_1^2 + x_2^2 = 1$, so we can make the substitution $x_1 = cos\theta, x_2 = sin\theta$.This yields

$$= \sup_{|\tilde{\mathbf{x}}|} \sqrt{2(x_1 + x_2)^2 - 4x_1x_2} == \sup_{|\tilde{\mathbf{x}}|} \sqrt{2 - 2\sin 2\theta} = 2.$$

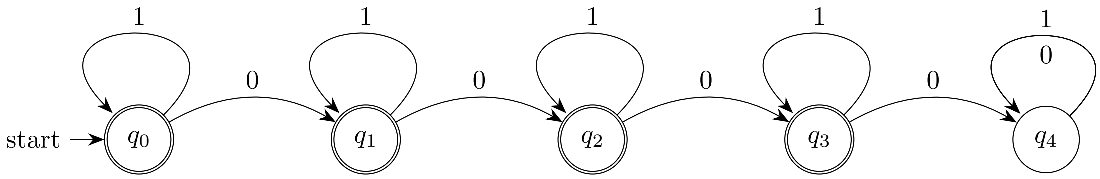
$$\begin{aligned}
\left|\left|\,[\mathbf{D}F(A_1)]\,\right|\right| - \left|\left|\,[\mathbf{D}F(A_2)]\,\right|\right| &= \sup_{|B|=1} \Big|\Big([\mathbf{D}F(A_1)] - \mathbf{D}F(A_2)]\Big)B\Big| \\
&= \sup_{|B|=1} \Big|A_1B + BA_1 + \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} - A_2B - BA_2 - \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\Big|
\end{aligned}$$

which after some manipulation yields that

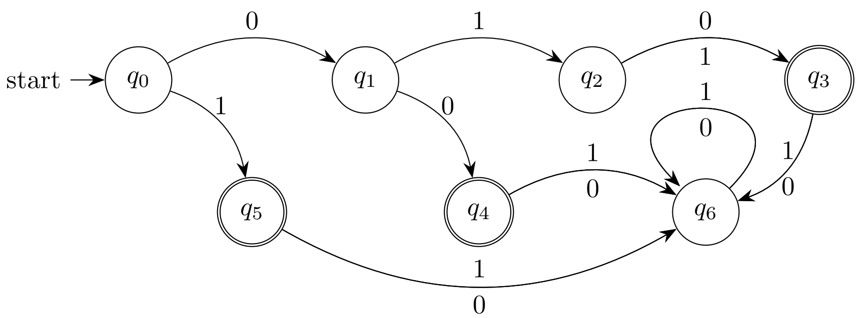
$$\left|\left|[\mathbf{D}F(A_1)]\right|\right| - \left|\left|[\mathbf{D}F(A_2)]\right|\right| = 2|A_1 - A_2|.$$

Problem Part 2

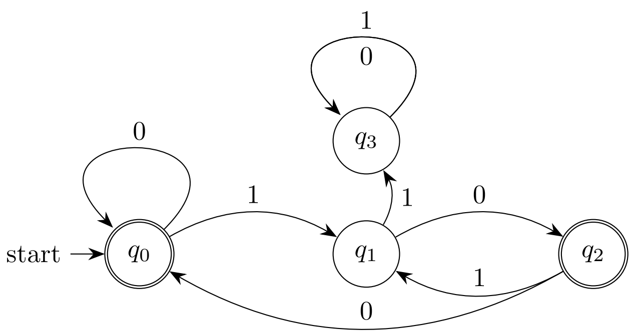
a



b



c



Problem 382749811

Computing the jth derivative of the fraction given in the hint yields

$$\frac{f^{(j)}(a+h) - \sum_{i=j}^{k} \frac{f^{(i)}(a)}{(i-j)!} h^{i-j}}{\frac{k!}{(k-j)!} h^{k-j}},$$

where $f^{(i)}$ signifies the ith derivative of f. When evaluated at $h = 0$, both the numerator and the denominator yield 0, given that $0 \leq j < k$. L'Hopital's rule is therefore satisfied until the kth derivative, when the numerator is 0 and the denominator is $k!$, making the limit 0.

Now, we need to prove the uniqueness of this limit. Take two polynomials p_a, p_b of degree $\leq k$, such that

$$\lim_{h \to 0} \frac{f(a+h) - p_a(a+h)}{h^k} = 0 \qquad \lim_{h \to 0} \frac{f(a+h) - p_b(a+h)}{h^k} = 0$$

It is clear that

$$\lim_{h \to 0} \frac{p_a(a+h) - p_a(a+h)}{h^k} = 0$$

If the two polynomials are not equal, then there will be some nonzero term in the numerator, say at a term $i \leq k$, and the limit will only be zero if $\lim_{h \to 0} \frac{1}{h^{k-i}} = 0$, which will not happen if we supposed that $i \leq k$.

Problem 382749813

Evaluate the partials and partial derivatives at $\begin{pmatrix} -2 \\ -3 \end{pmatrix}$. This yields

$$\begin{aligned}
D_{(1,0)}f\begin{pmatrix} -2 \\ -3 \end{pmatrix} &= \frac{1-3}{2\sqrt{-2-3+6}} = -1 \\
D_{(0,1)}f\begin{pmatrix} -2 \\ -3 \end{pmatrix} &= \frac{1-2}{2\sqrt{-2-3+6}} = -\frac{1}{2} \\
D_{(2,0)}f\begin{pmatrix} -2 \\ -3 \end{pmatrix} &= \frac{-(1-3)(\frac{1-3}{2\sqrt{-2-3+6}})}{2\sqrt{-2-3+6}} = -1 \\
D_{(1,1)}f\begin{pmatrix} -2 \\ -3 \end{pmatrix} &= \frac{\sqrt{-2-3+6}-(1-3)(\frac{(1-3)(1-2)}{2\sqrt{-2-3+6}})}{2\sqrt{-2-3+6}} = 0 \\
D_{(0,2)}f\begin{pmatrix} -2 \\ -3 \end{pmatrix} &= \frac{-(1-2)(\frac{1-2}{2\sqrt{-2-3+6}})}{2\sqrt{-2-3+6}} = -\frac{1}{4}
\end{aligned}$$

Let $x = -2 + u, y = -3 + v$.
$\vec{h} = \begin{pmatrix} u \\ v \end{pmatrix}$, making the second degree Taylor polynomial at $\begin{pmatrix} -2 \\ -3 \end{pmatrix}$ $1 - u - \frac{1}{2}v + \frac{1}{2}(-u^2 - \frac{1}{4}v^2)$.
Therefore, $P^2_{f,\begin{pmatrix} -2 \\ -3 \end{pmatrix}}\begin{pmatrix} -2+u \\ -3+v \end{pmatrix} = 1 - u - \frac{1}{2}v - \frac{1}{2}u^2 - \frac{1}{8}v^2$.

Problem 875125

Rewrite $f(x)$ as $f(x) = A + Bx + Cx^2 + R(x)$, $R(x) \in o(h^2)$. Then

$$\begin{aligned}
h(af(0)+bf(h)+cf(2h)) &= (aA + b(A + Bh + Ch^2) + c(A + 2Bh + 4Ch^2)) + h(bR(h) + cR(2h)) \\
&= hA(a+b+c) + h^2B(b+2c) + h^3C(b+4c) + h(bR(h)+cR(2h)).
\end{aligned}$$

Moreover,

$$\int_0^{2h} f(t)dt = 2Ah + 4B\frac{h^2}{2} + 8C\frac{h^3}{3} + \int_0^{2h} R(t)dt.$$

Both $h(bR(h) + cR(2h)), \int_0^{2h} R(t)dt \in o(h^3)$. We see that

$$\begin{aligned}
a + b + c &= 2 \\
b + 2c &= 2 \\
b + 4c &= \frac{8}{3}
\end{aligned}$$

From the error terms, we see that for all functions f of class C^3, $h(af(0) + bf(h) + cf(2h)) - \int_0^h f(t)dt \in o(h^3)$.

Problem 875127

Suppose that $f\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \sin(xyz) - z = 0$, $\mathbf{a} = \begin{pmatrix} \frac{\pi}{2} \\ 1 \\ 1 \end{pmatrix}$. We see that

$$\begin{aligned}
D_3 f\begin{pmatrix} x \\ y \\ z \end{pmatrix} &= xy\cos(xyz) - 1 \\
f(\mathbf{a}) &= 0 \\
D_3 f(\mathbf{a}) &= -1
\end{aligned}$$

. Since $D_3f(\mathbf{a})$ is nonzero, $[\mathbf{D}f(\mathbf{a})]$ is onto, and we can use the Implicit Function Theorem to assert that $f = 0$ expresses one variable in terms of the other two. $D_3f(\mathbf{a}) = -1$ is invertible, so z is the one variable expressed in terms of the other two; that is to say, there exists a $g\begin{pmatrix} y \\ y \end{pmatrix} = z$ in some neighborhood of $\mathbf{p} = \begin{pmatrix} \frac{\pi}{2} \\ 1 \end{pmatrix}$, $g(\mathbf{p}) = 1$.
By Theorem 875127, we see that

$$P^2_{g,(\mathbf{p})} = 1 + D_1g(\mathbf{p})\dot{x} + D_2g(\mathbf{p})\dot{y} + \frac{1}{2}D_1^2g(\mathbf{p})\dot{x}^2 + D_1D_2(\mathbf{p})\dot{x}\dot{y} + \frac{1}{2}D_2^2g(\mathbf{p})\dot{y}^2,$$

where dotted variables symbolize increments in their respective directions. Note that, from Theorem 2.10.14,

$$[\mathbf{Dg}(\mathbf{p})] = -[D_3f(\mathbf{a}]^{-1}[D_1f(\mathbf{a}), D_2f(\mathbf{a})] = -[-1]^{-1}[0\ 0]$$

We see now that the previous equation becomes

$$1 + \dot{z} = P^2_{g,(\mathbf{p})}\begin{pmatrix} \frac{\pi}{2} + \dot{x} \\ 1 + \dot{y} \end{pmatrix} = 1 + \frac{1}{2}D_1^2g(\mathbf{p})\dot{x}^2 + D_1D_2(\mathbf{p})\dot{x}\dot{y} + \frac{1}{2}D_2^2g(\mathbf{p})\dot{y}^2.$$

Compute

$$f\begin{pmatrix} \frac{\pi}{2} + \dot{x} \\ 1 + \dot{y} \\ 1 + \dot{z} \end{pmatrix} = -(1 + \dot{z}) + \sin((\frac{\pi}{2} + \dot{x})(1 + \dot{y})(1 + \dot{z})) = -1 - \dot{z} + 1 - \frac{1}{2}(\dot{x} + \frac{\pi}{2}\dot{y} + \frac{\pi}{2}\dot{z} + \dot{x}\dot{y} + \dot{x}\dot{z} + \frac{\pi}{2}\dot{y}\dot{z} + \dot{x}\dot{y}\dot{z})$$

($\uparrow$ the Taylor expansion of cosine)

This yields

$$P^2_{f,(\mathbf{a})}\begin{pmatrix} \frac{\pi}{2} + \dot{x} \\ 1 + \dot{y} \\ 1 + \dot{z} \end{pmatrix} = -\dot{z} - \frac{1}{2}\dot{x}^2 - \frac{\pi^2}{8}\dot{y}^2 - \frac{\pi^2}{8}\dot{z}^2 - \frac{\pi}{2}\dot{x}\dot{y} - \frac{\pi}{2}\dot{x}\dot{z} - \frac{\pi^2}{4}\dot{y}\dot{z}$$

Substituting what we found $\dot{z}$ to be into the above equation, we see

$$P^2_{f,(\mathbf{a})}\begin{pmatrix} \frac{\pi}{2} + \dot{x} \\ 1 + \dot{y} \\ P^2_{g,(\mathbf{p})} \end{pmatrix} = -(\frac{1}{2}D_1^2g(\mathbf{p})\dot{x}^2 + D_1D_2(\mathbf{p})\dot{x}\dot{y} + \frac{1}{2}D_2^2g(\mathbf{p})\dot{y}^2) - \frac{1}{2}\dot{x}^2 - \frac{\pi^2}{8}\dot{y}^2 - \frac{\pi}{2}\dot{x}\dot{y}$$

$f\begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0$ for all x, y, z, so $P^2_{f,(\mathbf{a})}\begin{pmatrix} \frac{\pi}{2} + \dot{x} \\ 1 + \dot{y} \\ P^2_{g,(\mathbf{p})} \end{pmatrix} = 0$. Making all coefficients 0 yields

$$P^2_{g,(\mathbf{p})}(\mathbf{p} + \dot{\mathbf{x}}) = 1 - \frac{1}{2}\dot{x}^2 - \frac{\pi}{2}\dot{x}\dot{y} - \frac{\pi^2}{8}\dot{y}^2,$$

or, alternatively,

$$P^2_{g,(\mathbf{p})}(\mathbf{x}) = 1 - \frac{1}{2}(x - \frac{\pi}{2})^2 - \frac{\pi}{2}(x - \frac{\pi}{2})(y - 1) - \frac{\pi}{2}(y - 1)^2.$$

Problem 12361

$$\begin{aligned}
\lim_{t\to 0}\frac{g'(c_t)}{t} &= \lim_{t\to 0}\frac{D_if(\mathbf{a} + c_t\vec{\mathbf{e}_i} + t\vec{\mathbf{e}_j}) - D_if(\mathbf{a} + c_t\vec{\mathbf{e}_i})}{t} \\
&= \lim_{t\to 0}\frac{D_if(\mathbf{a} + c_t\vec{\mathbf{e}_i} + t\vec{\mathbf{e}_j}) - D_if(\mathbf{a} + c_t\vec{\mathbf{e}_i})}{t} \\
&\quad - \left(\lim_{t\to 0}\frac{D_if(\mathbf{a} + c_t\vec{\mathbf{e}_i} + t\vec{\mathbf{e}_j}) - D_if(\mathbf{a}) + c_tD_i{}^2f(\mathbf{a}) - tD_jD_if(\mathbf{a})}{t}\right) && \text{(a)} \\
&\quad + \left(\lim_{t\to 0}\frac{D_if(\mathbf{a} + c_t\vec{\mathbf{e}_i}) - D_if(\mathbf{a}) + c_tD_i{}^2f(\mathbf{a})}{t}\right) && \text{(b)} \\
&= \lim_{t\to 0}\frac{D_if(\mathbf{a}) + c_tD_i{}^2f(\mathbf{a}) - tD_jD_if(\mathbf{a}) - D_if(\mathbf{a}) - c_tD_i{}^2f(\mathbf{a})}{t} \\
&= D_jD_if(\mathbf{a}).
\end{aligned}$$