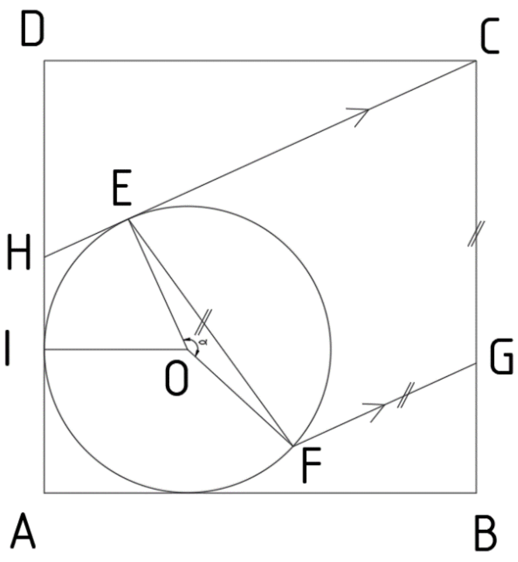
Case 1: EF is not parallel with BC



Let $EOF = \alpha$

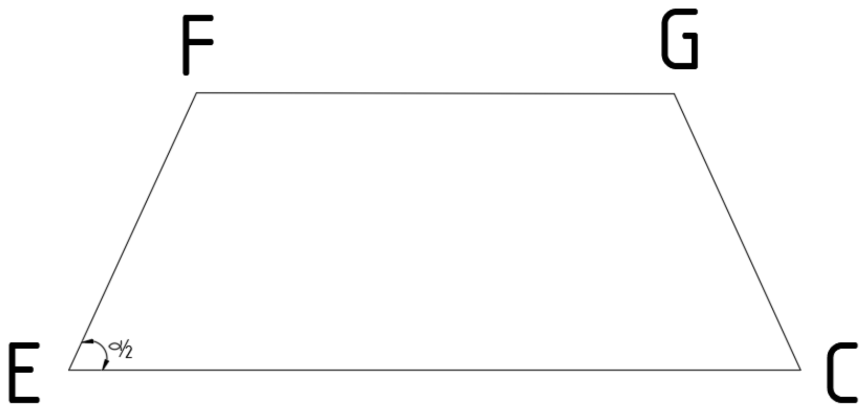
In triangle EOF, we have:

$$EF^2 = 2OE^2(1 - cosEOF) = 2r^2\left(2\sin^2\frac{\alpha}{2}\right)$$

⇨ $EF = 2rsin\frac{\alpha}{2}$

Not hard to prove that EFGH is isosceles trapezoid

In trapezoid EFGC:



⇨ $\boldsymbol{EC = FG + 2\frac{EF}{\cos\frac{\alpha}{2}} = 2r\sin\frac{\alpha}{2}\left(2\cos\frac{\alpha}{2} + 1\right)}$

Notice that:

$$DCH = OEF = 90 - FEC = 90 - \frac{\alpha}{2}$$

$$\tan(DCH + ECO) = tan45 = 1$$

$\Rightarrow \frac{tanDCH+tanECO}{1-tanDCH.tanECO} = 1$

$\Rightarrow tanECO = \frac{1-tanDCH}{1+tanDCH} = \frac{\tan\frac{\alpha}{2}-1}{\tan\frac{\alpha}{2}+1}, tanDCH = \tan\left(90-\frac{\alpha}{2}\right) = \frac{1}{\tan\frac{\alpha}{2}}$

In triangle ECO, $ECO = \frac{r}{EC}$

$\Rightarrow \boldsymbol{EC = \left(\frac{\tan\frac{\alpha}{2}+1}{\tan\frac{\alpha}{2}-1}\right)r}$, with condition $\frac{\tan\frac{\alpha}{2}+1}{\tan\frac{\alpha}{2}-1} > 0$ **(1)**

But in different ways, we also have:

$$EC = \sqrt{CO^2 - OE^2} = \sqrt{(AC-AO)^2 - OE^2}$$

$$= \sqrt{\left(\left(\sqrt{2}a\right) - \sqrt{2}r\right)^2 - r^2} = \sqrt{2a^2 + r^2 - 4ar}$$

Codition for that:

$$\begin{cases} 2a^2 + r^2 - 4ar > 0 \\ EO < EC \end{cases}$$

$\Rightarrow \begin{cases} 2a^2 + r^2 - 4ar > 0 \\ r < \frac{a}{2} \end{cases}, \boldsymbol{EC = \sqrt{2a^2 + r^2 - 4ar}}$ **(2)**

Another equation:

$$DH + HI + IA = DA$$

$\Rightarrow \frac{a}{\tan\frac{\alpha}{2}} + \frac{a}{\sin\frac{\alpha}{2}} - \sqrt{2a^2 + r^2 - 4ar} + r = a$

$\Rightarrow a\left(\frac{1}{\tan\frac{\alpha}{2}} + \frac{1}{\sin\frac{\alpha}{2}} - 1\right) + r = \sqrt{2a^2 + r^2 - 4ar}$

Let $\frac{1}{\tan\frac{\alpha}{2}} + \frac{1}{\sin\frac{\alpha}{2}} - 1 = m$

$\Rightarrow am + r = \sqrt{2a^2 + r^2 - 4ar}$

$\Rightarrow a^2m^2 + 2amr + r^2 = 2a^2 + r^2 - 4ar$

$\Rightarrow r = \frac{a(2-m^2)}{2(m+2)}$, with $m = \frac{1}{\tan\frac{\alpha}{2}} + \frac{1}{\sin\frac{\alpha}{2}} - 1$

Besides that:

$EC = \left(\frac{\tan\frac{\alpha}{2}+1}{\tan\frac{\alpha}{2}-1}\right)r = \sqrt{2a^2 + r^2 - 4ar}$ **(3)**

Then we have:

$$\left(\frac{\tan\frac{\alpha}{2}+1}{\tan\frac{\alpha}{2}-1}\right)r = 2r\sin\frac{\alpha}{2}\left(2\cos\frac{\alpha}{2}+1\right), r > 0$$

Let $\cos\frac{\alpha}{2} = k,\ k \in [-1,1]$

Case 1.1: $\sin\frac{\alpha}{2} > 0$

⇨ $\frac{\sqrt{1-k^2}+k}{\sqrt{1-k^2}-k} = 4k\sqrt{1-k^2} + 2\sqrt{1-k^2}$

⇨ $k_1 \approx -0{,}956515 < 0; k_2 \approx -0{,}39008 < 0; k_3 \approx 0{,}481697 > 0$

⇨ $tan\frac{\alpha_{1,2}}{2} < 0;\ tan\frac{\alpha_3}{2} > 0$

⇨ $tan\frac{\alpha_1}{2} = -\sqrt{\frac{1}{cos^2\frac{\alpha_1}{2}} - 1} = -\sqrt{\frac{1}{k_1^2} - 1} = -0{,}30494$ (rejected by condition (1)), $sin\frac{\alpha_1}{2} = 0{,}29168$

⇨ $tan\frac{\alpha_2}{2} = -2{,}36049$ (accepted by condition (1)), $sin\frac{\alpha_2}{2} = 0{,}92078$

⇨ $tan\frac{\alpha_3}{2} = 1{,}81927$ (accepted by condition (1)), $sin\frac{\alpha_3}{2} = 0{,}87634$

⇨ $m_2 = -0{,}33761; m_3 = 0{,}69078$

⇨ $r_2 = 0{,}56726a = 6{,}80712$ (rejected by condition (2)),

$r_3 = 0{,}28297a = 3{,}39564$ (accepted by condition (2) and (3))

⇨ $S = r^2 = r_3^2 = 11{,}53037\ cm^2$

Case 1.2: $\sin\frac{\alpha}{2} < 0$

⇨ $\frac{-\sqrt{1-k^2}+k}{-\sqrt{1-k^2}-k} = -4k\sqrt{1-k^2} - 2\sqrt{1-k^2}$

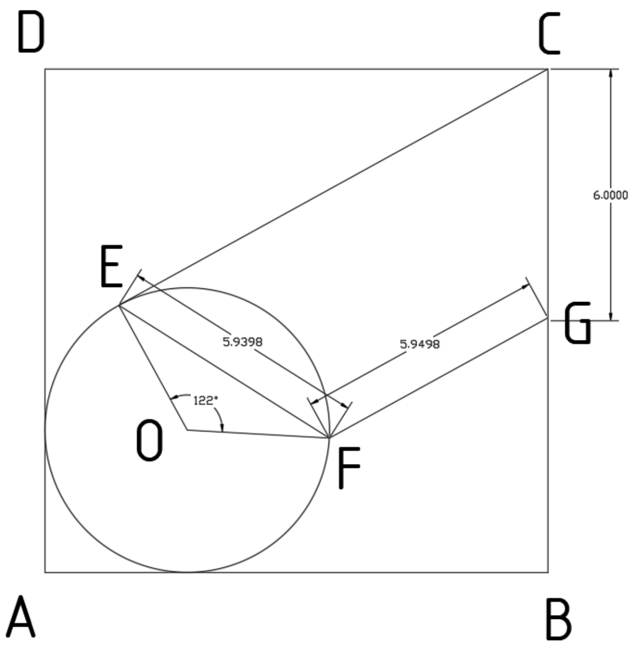
⇨ $k \approx 0{,}991643 > 0$

⇨ $tan\frac{\alpha}{2} < 0$

⇨ $tan\frac{\alpha}{2} = -\sqrt{\frac{1}{cos^2\frac{\alpha}{2}} - 1} = -\sqrt{\frac{1}{k^2} - 1} = -0{,}130099$ (rejected by condition (1))

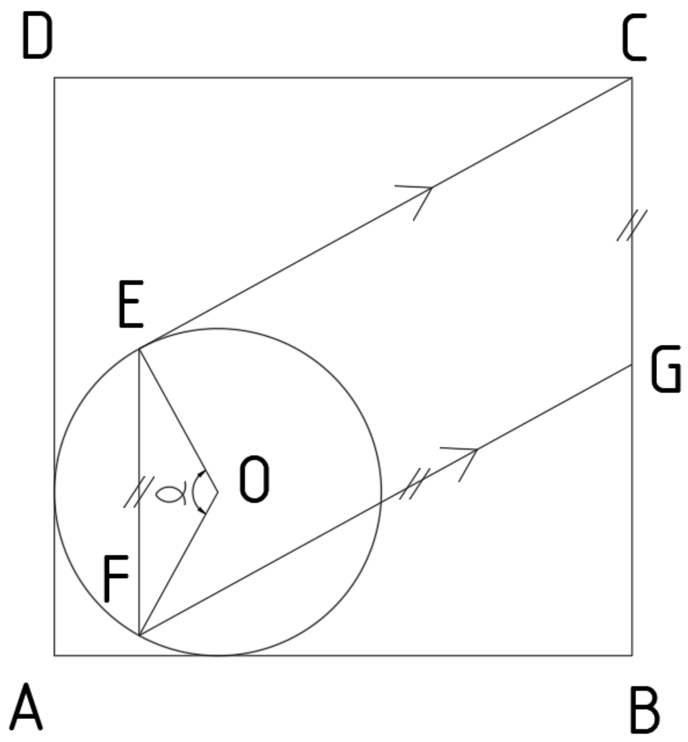
Then, in **Case 1** the answer is $S = 11{,}53037\ cm^2$ with $r = 3{,}39564\ cm$

Check with drawing:



The max error from drawing is $e_{max} = \left|\frac{6-5{,}9498}{6}\right| \approx 0{,}837\%$

Case 2: EF parallel with BC



EF parallel with BC

⇨ EF perpendicular with AB

Now it not hard to see that ECGF is prallelogram, with $EF = FG = CG = EC$

⇨ ECGF is diamond

Similar to **Case 1**, we have:

$$EF = 2rsin\frac{\alpha}{2}$$

$$CE = \sqrt{2a^2 + r^2 - 4ar}, \text{ with } \begin{cases} 2a^2 + r^2 - 4ar > 0 \\ r < \frac{a}{2} \end{cases} \quad (*)$$

With $EF = CE$:

$$2rsin\frac{\alpha}{2} = \sqrt{2a^2 + r^2 - 4ar}, with\ condition\ sin\frac{\alpha}{2} > 0 \quad (**)$$

In triangle OEF:

$$\alpha = 180 - 2OEF = 180 - 2(CEF - 90) = 360 - 2CEF = 360 - 2(180 - ECB) = 2ECB$$

⇨ $DCE = 90 - ECB = 90 - \frac{\alpha}{2}$

Using the result in case 1, we have:

$$\boldsymbol{EC} = \left(\frac{\tan\frac{\alpha}{2}+1}{\tan\frac{\alpha}{2}-1}\right)\boldsymbol{r}\text{ , with condition } \frac{\tan\frac{\alpha}{2}+1}{\tan\frac{\alpha}{2}-1} > 0 \quad (***)$$

From (**) we have:

$$4r^2\sin^2\frac{\alpha}{2} = 2a^2 + r^2 - 4ar$$

⇨ $\left(4\sin^2\frac{\alpha}{2} - 1\right) r^2 + 4ar - 2a^2 = 0$

⇨ $\Delta' = a^2\left(8\sin^2\frac{\alpha}{2} + 2\right) > 0$

⇨ $r_1 = \frac{-2a+a\sqrt{8\sin^2\frac{\alpha}{2}+2}}{4\sin^2\frac{\alpha}{2}-1} = a\left(\frac{\sqrt{8\sin^2\frac{\alpha}{2}+2}-2}{4\sin^2\frac{\alpha}{2}-1}\right) > 0, r_2 = a\left(\frac{-\sqrt{8\sin^2\frac{\alpha}{2}+2}-2}{4\sin^2\frac{\alpha}{2}-1}\right) > 0$ **(****)**

That's easy to see:

$$\frac{\tan\frac{\alpha}{2} + 1}{\tan\frac{\alpha}{2} - 1} = 2\sin\frac{\alpha}{2}$$

Let $\sin\frac{\alpha}{2} = k,\ k \in (0,1]$

Case 2.1: $\boldsymbol{\cos\frac{\alpha}{2} > 0}$

$\frac{k+\sqrt{1-k^2}}{k-\sqrt{1-k^2}} = 2k$

⇨ $k_1 \approx -0,27654 < 0$ (rejected by condition (**)); $k_2 \approx 0,95449 > 0$ (accepted by condition (**))

⇨ $\tan\frac{\alpha}{2} > 0$

⇨ $\tan\frac{\alpha}{2} = \sqrt{\frac{1}{\cos^2\frac{\alpha}{2}} - 1} = \sqrt{\frac{1}{k^2} - 1} = 0,31246$ (rejected by condition (***))

Case 2.2: $\boldsymbol{\cos\frac{\alpha}{2} < 0}$

⇨ $\frac{k-\sqrt{1-k^2}}{k+\sqrt{1-k^2}} = 2k$

⇨ **Don't have real roots**

Then, in **Case 2** there is no solution

After all, the answer is $S = 11,53037\ cm^2$ with $r = 3,39564\ cm$