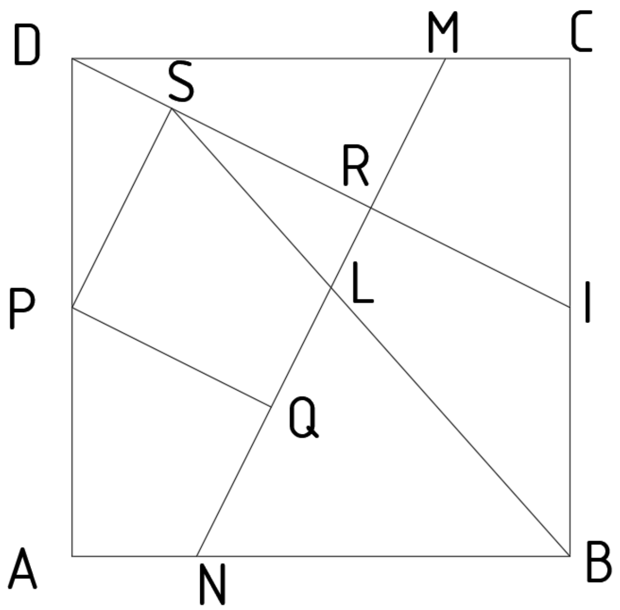
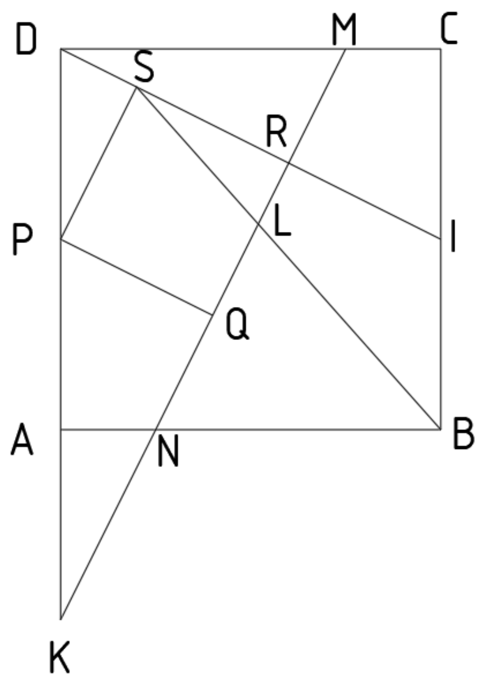


ABCD and PQRS are squares. Given that the length of square ABCD is $a = 12\ cm$ and $CM = AN = x = 5\ cm$. Find [BLRI] ?

First, extend MN at meets AD at K, like figure bellow:



Using Thales theorem in triangle DKM with AN // DM, we have:

$$\frac{AN}{DM} = \frac{AK}{DK}$$

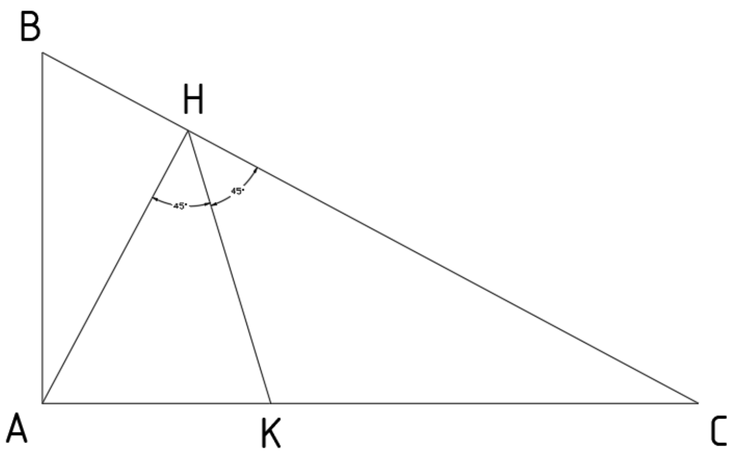
⇨ $\frac{x}{a-x} = \frac{AK}{a+AK}$

⇨ $AK = \frac{ax}{a-2x}, 0 < x < a/2$

⇨ $DK = AK + AD = \frac{ax}{a-2x} + a = \frac{a^2-ax}{a-2x}$

Not hard to see that, if PQRS is a square, RP has to be bisector of angle DRK

Now we have a small problem like that:



Given that triangle ABC is right at A (with $AB = a, AC = b$), AH is altitude and HK is bisector. Calculate HK by a, b?

We have equation about bisector:

$$HK^2 = AH.HC - AK.KC$$

- $AH = \frac{AB.AC}{BC} = \frac{ab}{\sqrt{a^2+b^2}}$
- $HC = \frac{AC^2}{BC} = \frac{b^2}{\sqrt{a^2+b^2}}$
- $\frac{AK}{KC} = \frac{AH}{HC} = \frac{AB}{AC} = \frac{a}{b}, AK + KC = AC = b$

➔$AK = \frac{ab}{a+b}, KC = \frac{b^2}{a+b}$

➔$HK^2 = \frac{2a^2b^4}{(a+b)^2(a^2+b^2)}$

➔$HK = \frac{\sqrt{2}ab^2}{(a+b)\sqrt{a^2+b^2}}$

Now apply that result to triangle MDK right at K, DR is altitude, RP is bisector of angle DRK and replace:

$$m = DM = a - x, n = DK = \frac{a^2 - ax}{a - 2x}$$

⇨ $RP = \frac{\sqrt{2}mn^2}{(m+n)\sqrt{m^2+n^2}}$

⇨ $SR = \frac{RP}{\sqrt{2}} = \frac{mn^2}{(m+n)\sqrt{m^2+n^2}} = u$

Triangle DCI is similar to triangle KDM

⇨ $\begin{cases} \frac{DI}{MK} = \frac{DC}{DK} \\ \frac{CI}{DC} = \frac{DM}{DK} \end{cases}$

⇨ $\begin{cases} DI = \left(\frac{a}{n}\right)\sqrt{m^2+n^2} \\ CI = \left(\frac{m}{n}\right)a, BI = BC - CI = a - \left(\frac{m}{n}\right)a = a\left(1-\frac{m}{n}\right) = p \end{cases}$

$$DS = DR - SR = \frac{mn}{\sqrt{m^2+n^2}} - \frac{mn^2}{(m+n)\sqrt{m^2+n^2}} = \frac{m^2n}{(m+n)\sqrt{m^2+n^2}} = q$$

⇨ $SI = DI - DS = \left(\frac{a}{n}\right)\sqrt{m^2+n^2} - \frac{m^2n}{(m+n)\sqrt{m^2+n^2}} = r$

In triangle DBI, we have:

- $DB = \sqrt{2}a, BI = p$
- $DS = q, SI = r$

Using Stewart theorem, we have:

$$SB^2 = \frac{BD^2SI - BI^2DS}{SI + DS} - SI.DS = \frac{2a^2r - p^2q}{q+r} - qr = k^2$$

⇨ $SB = k$

⇨ $cosBSI = \frac{SB^2+SI^2-IB^2}{2SB.SI} = \frac{k^2+r^2-p^2}{2kr} = t$

⇨ $\begin{cases} [SRL] = \frac{1}{2}SR.RL = \frac{1}{2}SR(SRtanBSI) = \frac{1}{2}u^2\sqrt{\frac{1}{t^2}-1} \\ [BSI] = \frac{1}{2}SB.SI.\sin(BSI) = \frac{1}{2}kr\sqrt{1-t^2} \end{cases}$

⇨ $[BLRI] = [BSI] - [SRL] = \frac{1}{2}\left(kr\sqrt{1-t^2} - u^2\sqrt{\frac{1}{t^2}-1}\right)\ cm^2$

In this problem, we have:

$m = 7;\ n = 42;\ u = 5{,}91836; p = 10; q = 0{,}98639; r = 11{,}17913;$

$k = 15{,}66888; t = 0{,}77209$

⇨ $[BLRI] \approx 41{,}24417\ cm^2$