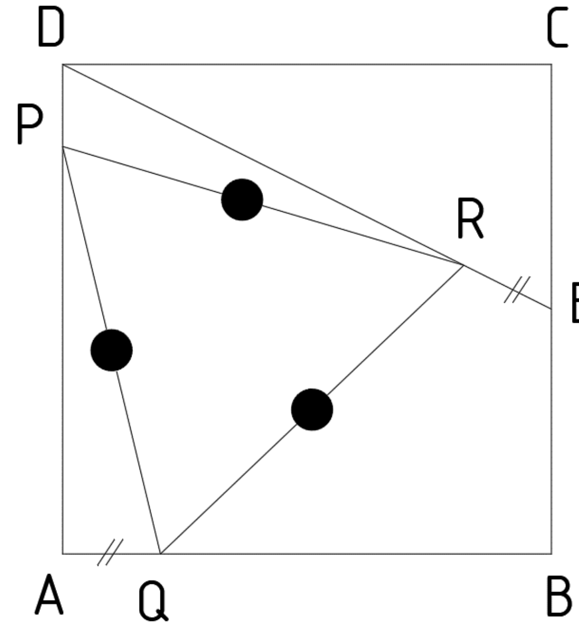
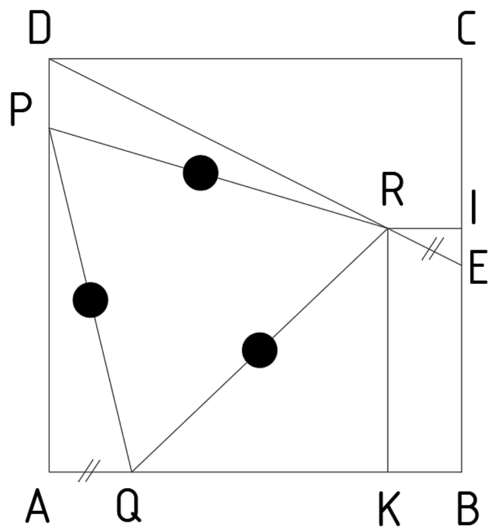


ABCD is a square with a = 12 cm, find [PQR] ?

Let $AQ = RE = x, AP = y$ $(0 < x, y < a)$. Then from R draw 2 perpendicular lines to BC, AB at I and K, like figure bellow:



- In triangle PAQ right at A: $PQ^2 = AQ^2 + AP^2 = x^2 + y^2$
- In triangle DPR:

$$PR^2 = DP^2 + DR^2 - 2DP.DRcos(PDR)$$

$$\Rightarrow PR^2 = (a-y)^2 + \left(\frac{\sqrt{5}}{2}a - x\right)^2 - 2(a-y)\left(\frac{\sqrt{5}}{2}a - x\right)\frac{\left(\frac{a}{2}\right)}{\left(\frac{\sqrt{5}}{2}a\right)}$$

$$\Rightarrow PR^2 = \frac{5}{4}a^2 - ay - \frac{3\sqrt{5}}{5}ax - \frac{2\sqrt{5}}{5}xy + x^2 + y^2$$

- With RI // CD, using Thales theorem in ECD triangle, we have:

$$EI = \frac{\sqrt{5}}{5}x, RI = \frac{2\sqrt{5}}{5}x$$

$$\Rightarrow \begin{cases} RK = IB = IE + EB = \frac{\sqrt{5}}{5}x + \frac{a}{2} = \frac{2\sqrt{5}x+5a}{10} \\ QK = AB - AQ - KB = a - x - \frac{2\sqrt{5}}{5}x = \frac{5a-(2\sqrt{5}+5)x}{5} \end{cases}$$

$$\Rightarrow QR^2 = RK^2 + QK^2 = \frac{125a^2-(60\sqrt{5}+200)ax+(200+80\sqrt{5})x^2}{100}$$

with condition that $125a^2 - (60\sqrt{5} + 200)ax + (200 + 80\sqrt{5})x^2 > 0$ **(1)**

1. $PQ = PR$ $\rightarrow PQ^2 = PR^2$

$$\Rightarrow \frac{5}{4}a^2 - ay - \frac{3\sqrt{5}}{5}ax - \frac{2\sqrt{5}}{5}xy = 0$$

$$\Rightarrow x = \frac{\frac{5}{4}a^2-ay}{\frac{3\sqrt{5}}{5}a+\frac{2\sqrt{5}}{5}y} \qquad (*)$$

2. $PQ = QR$ $\rightarrow PQ^2 = QR^2$

$$\Rightarrow 125a^2 - (60\sqrt{5} + 200)ax + (200 + 80\sqrt{5})x^2 = 100(x^2 + y^2)$$

$$\Rightarrow (20 + 16\sqrt{5})x^2 - (40 + 12\sqrt{5})ax + 25a^2 - 20y^2 = 0 \qquad (**)$$

For simpler, let $m = 20 + 16\sqrt{5}, n = 40 + 12\sqrt{5}$ and replace x in (**) with (*), we have:

$$m\left(\frac{\frac{5}{4}a^2 - ay}{\frac{3\sqrt{5}}{5}a + \frac{2\sqrt{5}}{5}y}\right)^2 - na\left(\frac{\frac{5}{4}a^2 - ay}{\frac{3\sqrt{5}}{5}a + \frac{2\sqrt{5}}{5}y}\right) + 25a^2 - 20y^2 = 0$$

Multiple both side with $\left(\frac{3\sqrt{5}}{5}a + \frac{2\sqrt{5}}{5}y\right)^2$ and simplified that, at last we have:

$$-16y^4 - (48a)y^3 + \left(a^2m + \frac{2\sqrt{5}}{5}a^2n - 16a^2\right)y^2 + \left(-\frac{5}{2}a^3m - \frac{\sqrt{5}}{2}a^3n + \frac{3}{\sqrt{5}}a^3n + 60a^3\right)y + \left(45 + \frac{25}{16}m - \frac{3\sqrt{5}}{4}n\right)a^4 = 0 \qquad (***)$$

with $m = 20 + 16\sqrt{5}, n = 40 + 12\sqrt{5}$ and a is the length of square ABCD

That means y is the root of equation (***) and x from y (equation (**)) is accpected by condition (1), then the solution is right.

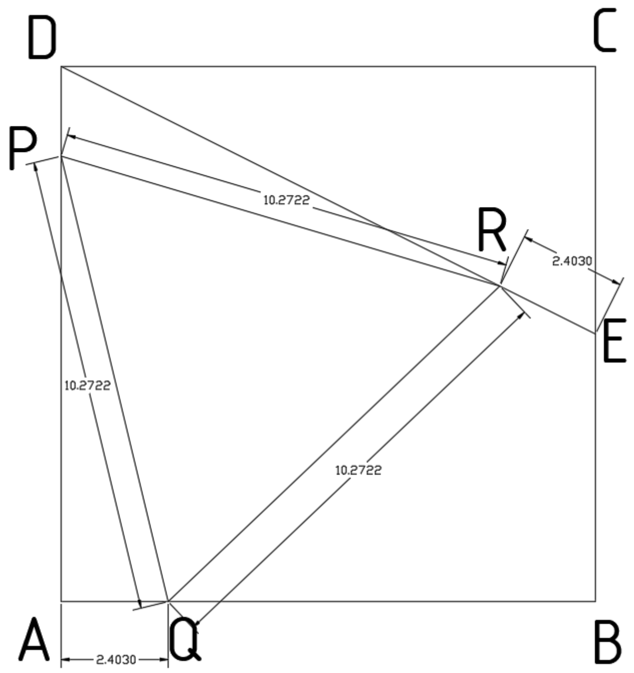
$$\rightarrow [PQR] = \frac{\sqrt{3}}{4}PQ^2 = \frac{\sqrt{3}}{4}(x^2 + y^2)$$

In this figure, we have $a = 12\ cm$, using the result in (***), we can have:

$y = 9{,}98715 \rightarrow x = 2{,}403$ (accepted by condition (1)).

$$\Rightarrow [PQR] = \frac{\sqrt{3}}{4}(x^2 + y^2) = \frac{\sqrt{3}}{4}(2{,}403^2 + 9{,}98715^2) \approx 45{,}69\ cm^2$$

Check with drawing:


D
C
P
10.2722
R
2.4030
E
10.2722
10.2722
A
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Q
B