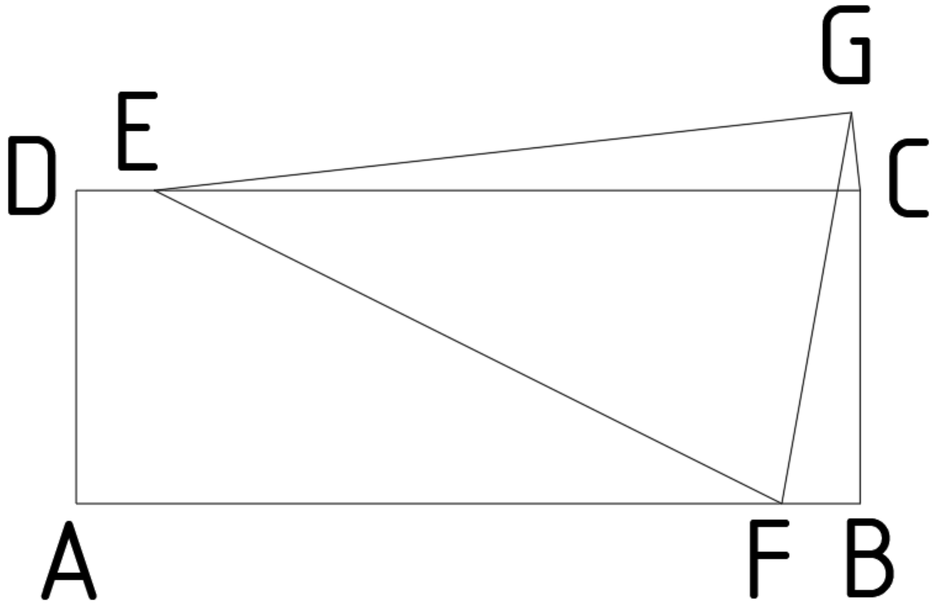


Given that:

- ABCD is a rectangle with $AD = a, AB = b\ (a, b > 0)$
- $DE = BF = CG = x, x \in (0, b)$

Find [ABCD] when trianlge EFG is equilateral

First, drawing EH that EH is perpendicular to AB. Let $HF = y, y > 0$

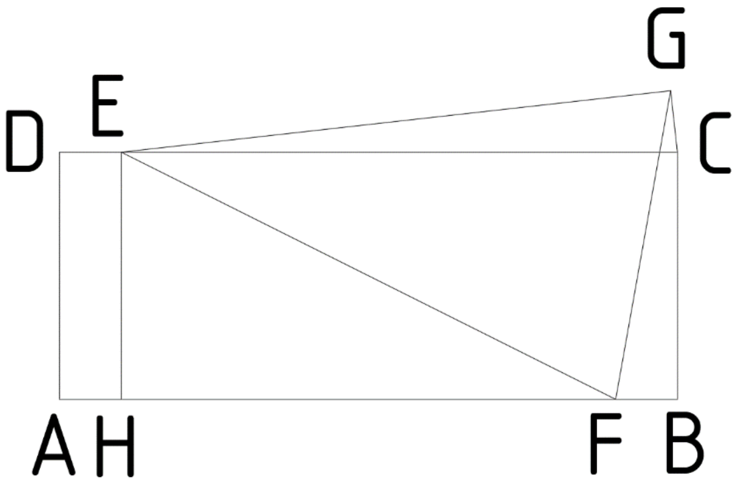
⇨ $AB = b = 2x + y$

Triangle EFG is equilateral

⇔$EF = EG$ and $GEF = 60°$

We have 2 case about that problem: Point G is inside / out of rectangle ABCD

Case 1: Point G is out of rectangle ABCD



⇨ $CEF = EFH = arctan\left(\frac{a}{y}\right) < GEF = 60°$

⇨ $HF = y = \frac{EH}{tan(CEF)} > \frac{a}{\sqrt{3}}$ **(*)**

Not hard to see that:

$$EG = EF = \sqrt{EH^2 + HF^2} = \sqrt{a^2 + y^2}$$

In triangle EGC:

$$CG^2 = EG^2 + EC^2 - 2EG.EC.cos(60° - CEF)$$

⇨ $x^2 = (a^2 + y^2) + (x + y)^2 - 2\sqrt{a^2 + y^2}(x + y)cos\left[60° - arctan\left(\frac{a}{y}\right)\right]$

⇨ $a^2 + 2xy + 2y^2 = 2\sqrt{a^2 + y^2}(x + y)[\frac{1}{2}cos\left(arctan\left(\frac{a}{y}\right)\right) + \frac{\sqrt{3}}{2}sin\left(arctan\left(\frac{a}{y}\right)\right)]$

⇨ $a^2 + 2xy + 2y^2 = \sqrt{a^2 + y^2}(x + y)[cos\left(arctan\left(\frac{a}{y}\right)\right) + \sqrt{3}\,sin\left(arctan\left(\frac{a}{y}\right)\right)]$ **(1)**

How can we handle $cos\left(arctan\left(\frac{a}{y}\right)\right), sin\left(arctan\left(\frac{a}{y}\right)\right)$?

We're all know that:

$tan^2\,\alpha + 1 = \frac{1}{cos^2\,\alpha}\ \forall\,\alpha \neq \frac{\pi}{2} + k\pi$

⇨ $cos\alpha = \left|\frac{1}{\sqrt{\tan^2\alpha+1}}\right|$

With $\alpha \in (0, \frac{\pi}{2})$, $\cos\alpha = \frac{1}{\sqrt{\tan^2\alpha+1}}$

Then we have:

$$cos\left(arctan\left(\frac{a}{y}\right)\right) = \frac{1}{\sqrt{\tan^2 arctan\left(\frac{a}{y}\right) + 1}} = \frac{1}{\sqrt{\left(\frac{a}{y}\right)^2 + 1}} = \frac{y}{\sqrt{a^2 + y^2}}$$

⇨ $sin\left(arctan\left(\frac{a}{y}\right)\right) = \frac{a}{\sqrt{a^2+y^2}}$

Replace $cos\left(arctan\left(\frac{a}{y}\right)\right) = \frac{y}{\sqrt{a^2+y^2}}, sin\left(arctan\left(\frac{a}{y}\right)\right) = \frac{a}{\sqrt{a^2+y^2}}$ to equation **(1)**, we have:

$$a^2 + 2xy + 2y^2 = \sqrt{a^2 + y^2}(x + y)\frac{1}{\sqrt{a^2 + y^2}}(y + \sqrt{3}a)$$

⇨ $y^2 + (x - \sqrt{3}a)y + (a^2 - \sqrt{3}ax) = 0$ **(2)**

Equation **(2)** has solution, mean that:

$\Delta = (x - \sqrt{3}a)^2 - 4(a^2 - \sqrt{3}ax) = x^2 + (2\sqrt{3}a)x - a^2 \geq 0$

⇔ $x \geq (2 - \sqrt{3})a$ **(**)**

Then we have $y = \frac{\sqrt{3}a - x + \sqrt{\Delta}}{2}$, with $\Delta = x^2 + (2\sqrt{3}a)x - a^2$

But don't forget condition (*) also: $y > \frac{a}{\sqrt{3}}$

- ⇨ $\frac{\sqrt{3}a - x + \sqrt{\Delta}}{2} > \frac{a}{\sqrt{3}}$
- ⇨ $3a - \sqrt{3}x + \sqrt{3\Delta} - 2a > 0$
- ⇨ $a - \sqrt{3}x > \sqrt{3\Delta}$, with condition $x < \frac{a}{\sqrt{3}}$ ➔ $x \in [(2 - \sqrt{3})a, \frac{a}{\sqrt{3}}]$
- ⇨ $3x^2 - (2\sqrt{3}a)x + a^2 > 3x^2 + (6\sqrt{3}a)x - 3a^2$
- ⇨ $(8\sqrt{3}a)x + 4a^2 > 0$ (always right with every value of a and x in condition)
- ⇨ Only **Case 1** can happen, point G is out of rectangle ABCD

In conclusion, the figure has solution when:

$$y = \frac{\sqrt{3}a - x + \sqrt{\Delta}}{2}, \text{with } \Delta = x^2 + (2\sqrt{3}a)x - a^2 \text{ and } x \geq (2 - \sqrt{3})a$$

The area of rectangle ABCD:

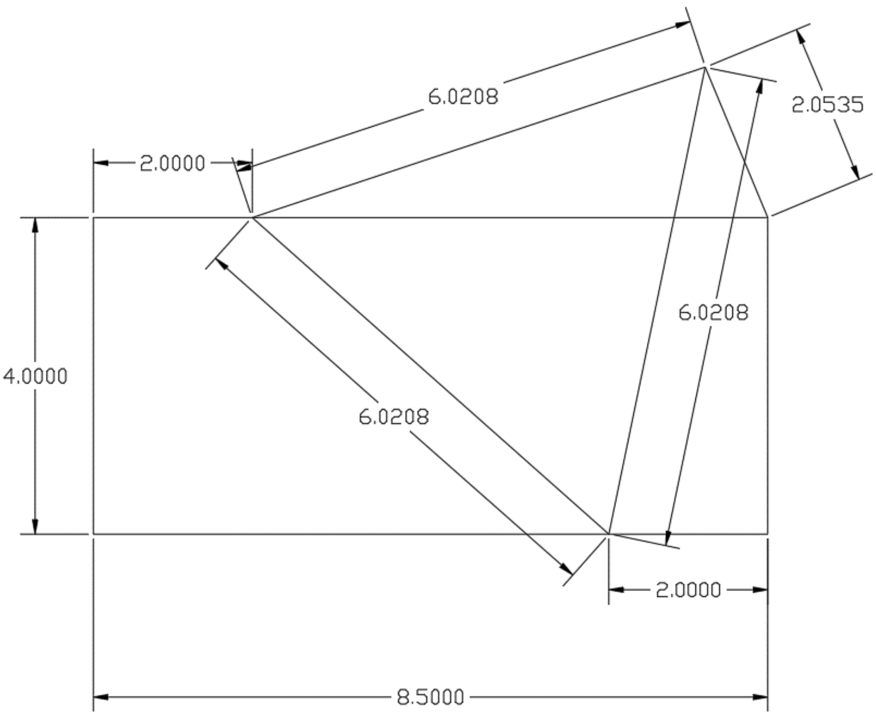
$$[ABCD] = ab = a(2x + y)$$

Check with drawing:

- $a = 4$ ➔ $x \geq 4(2 - \sqrt{3}) \approx 1{,}072$; choose $x = 2$ ➔ $y \approx 4{,}446$; choose $y = 4{,}5$

➔ $b = 2x + y = 2.2 + 4{,}5 = 8{,}5$

➔ $S = ab = 4.8{,}5 = 34$



- $a = 12$ ➔ $x \geq 12(2-\sqrt{3}) \approx 3{,}215$; choose $x = 4$ ➔ $y \approx 11{,}485$;

choose $y = 11{,}5$