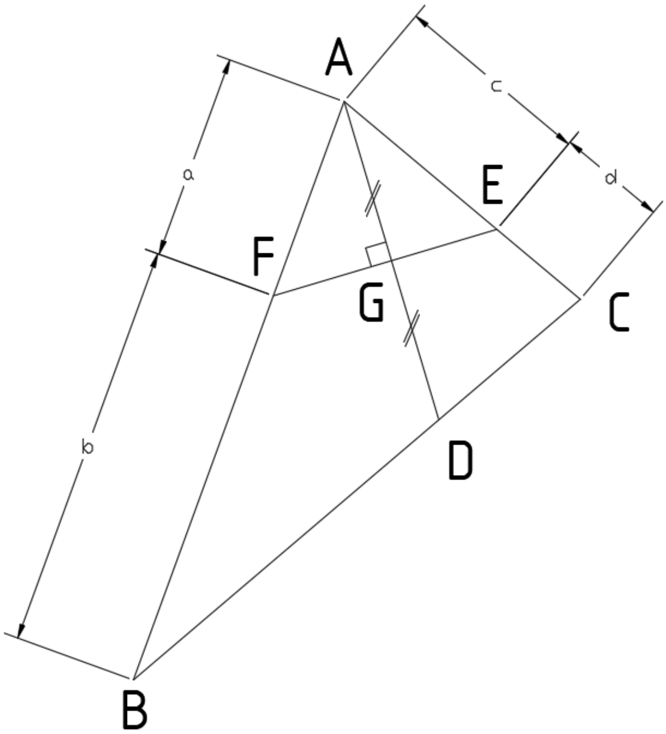


Given this figure. Find the ratio of BD/CD by value a, b, c, d?

First, we realize that if we have triangle ABC only with a, b, c, d are const, the solution is found depend on angle BAC.

Let $BAC = \alpha, BAD = \alpha_1$ $(\alpha_1 < \alpha \in (0,2\pi))$

Then, we suppose the process how to build that problem:

- Define AB line with $AB = a + b$, point E is between A, B with $AE = a$
- Define point C with $BAC = \alpha$, $AC = c + d$, point F is between A, C

that $AF = c$

- Define point G, D with AG is altitude of triangle AEF, D is mirror with A by EF line

Now we ask ourselves that, when will B, D, C are colinear?

This is what we are going to do now!

1/Condition that the figure has solution (B, D, C are colinear)

Easy to see that it's really hard if we use normal calculating to do

In general, vector is useful to solve colinear problems

Remember that, if we have 3 points A, B, C, they will colinear only if:

$$\overrightarrow{AB} = k\overrightarrow{AC}, k \in R$$

or the others relationship between vectors create from these original points

Turn back to our problem, if we want B, C, D are colinear, we just find relationship between $\overrightarrow{BC}$ and $\overrightarrow{CD}$ by perform them in the same way. Let's do it!

- $\overrightarrow{BD} = \overrightarrow{BA} + \overrightarrow{AD} = -\left(\frac{a+b}{a}\right)\overrightarrow{AF} + 2\overrightarrow{AG} = -\left(\frac{a+b}{a}\right)\overrightarrow{AF} + 2\left(\frac{GE}{EF}\overrightarrow{AF} + \frac{GF}{EF}\overrightarrow{AE}\right)$
$$= \left(\frac{2GF}{EF}\right)\overrightarrow{AE} + \left(\frac{2GE}{EF} - 1 - \frac{b}{a}\right)\overrightarrow{AF}$$
- $\overrightarrow{CD} = \overrightarrow{CA} + \overrightarrow{AD} = -\left(\frac{c+d}{c}\right)\overrightarrow{AE} + 2\overrightarrow{AG} = -\left(\frac{c+d}{d}\right)\overrightarrow{AF} + 2\left(\frac{GE}{EF}\overrightarrow{AF} + \frac{GF}{EF}\overrightarrow{AE}\right)$
$$= \left(\frac{2GF}{EF} - 1 - \frac{d}{c}\right)\overrightarrow{AE} + \left(\frac{2GE}{EF}\right)\overrightarrow{AF}$$

B, C, D are colinear $\Leftrightarrow \overrightarrow{BC} = k\overrightarrow{CD}$ or:

$$\frac{\frac{2GF}{EF} - 1 - \frac{d}{c}}{\left(\frac{2GF}{EF}\right)} = \frac{\frac{2GE}{EF}}{\left(\frac{2GE}{EF} - 1 - \frac{b}{a}\right)}$$

$\Rightarrow 1 - \frac{1+m}{\left(\frac{2GF}{EF}\right)} = \frac{1}{1-\frac{1+n}{\frac{2GE}{EF}}} \quad \left(m = \frac{d}{c}, n = \frac{b}{a}\right)$ **(1)**

In triangle AEF:

$$EF = \sqrt{a^2 + c^2 - 2ac.\cos\alpha}$$

$$AG = \frac{AE.AF\sin(EAF)}{EF} = \frac{ac.\sin\alpha}{\sqrt{a^2 + c^2 - 2ac.\cos\alpha}}$$

Besides that:

$$AG = AE\cos EAG = AF\cos FAG$$

$\Rightarrow a\cos\alpha_1 = c.\cos(\alpha - \alpha_1)$

$\Rightarrow \tan\alpha_1 = \frac{a-c.\cos\alpha}{c.\sin\alpha}$

$\Rightarrow FG = AG\tan\alpha_1 = \frac{a^2-ac.\cos\alpha}{\sqrt{a^2+c^2-2ac.\cos\alpha}}$

$\Rightarrow \frac{GF}{EF} = \frac{a^2-ac.\cos\alpha}{a^2+c^2-2ac.\cos\alpha}$

$\Rightarrow \frac{GF}{EF} = \frac{c^2-ac.\cos\alpha}{a^2+c^2-2ac.\cos\alpha}$

Now **(1)** become:

$$1 - \frac{(1+m)(a^2+c^2-2ac.\cos\alpha)}{2(a^2-ac.\cos\alpha)} = \frac{1}{1 - \frac{(1+n)(a^2+c^2-2ac.\cos\alpha)}{2(c^2-ac.\cos\alpha)}}$$

$\Rightarrow \frac{(1-m)a^2-(m+1)c^2+(2acm)\cos\alpha}{2a^2-2ac.\cos\alpha} = \frac{2c^2-2ac.\cos\alpha}{-(1+n)a^2+(1-n)c^2+(2acn)\cos\alpha}$

$\Rightarrow (a^2 + c^2 - 2ac.cos\alpha)(a^2m - a^2n - c^2m + c^2n - a^2 - c^2 + 2ac.cos\alpha + a^2mn + c^2mn - 2acmn.cos\alpha) = 0$

Notice that $a^2 + c^2 - 2ac.cos\alpha = EF^2 > 0$

$\Rightarrow a^2m - a^2n - c^2m + c^2n - a^2 - c^2 + 2ac.cos\alpha + a^2mn + c^2mn - 2acmn.cos\alpha = 0$

$\Rightarrow cos\alpha = \frac{a^2(mn+m-n-1)+c^2(mn-m+n-1)}{2acmn-2ac}$

$\Rightarrow cos\alpha = \frac{(ad-bc)(a^2-c^2)+(bd-ac)(a^2+c^2)}{2ac(bd-ac)}$, replace $m = \frac{d}{c}, n = \frac{b}{a}$ **(2)**

Don't forget the condition: $bd - ac \neq 0$

There is one more thing that make these proof is true, this is, AD line have to lie between AB and AC line

$\Rightarrow$ angle EAG have to $< 90°$

$\Rightarrow tan\alpha_1 = \frac{a-c.cos\alpha}{c.sin\alpha} > 0 \Leftrightarrow cos\alpha < \frac{a}{c}$ $(sin\alpha > 0\ \forall\ \alpha \in (0,2\pi))$

Then I use Matlab application to find solutions for this problem:

- Function *calculateAlpha(a,b,c,d)*: calculate α from a, b, c, d
- Function *calculateAlpha1(a,c,Alpha)*: calculate α_1 from a, c, α
- Function *findAlpha*(): find solutions

The code here:

```
function Alpha = calculateAlpha(a,b,c,d)
    Alpha = acosd(((a*d-b*c)*(a^2 - c^2) + (b*d - a*c)*(a^2 + c^2))/(2*a*c*(b*d-a*c)));
end
```

```
function Alpha1 = calculateAlpha1(a,c,Alpha)
    Alpha1 = atan((a - c*cos(Alpha*pi/180))/(c*sin(Alpha*pi/180)))*180/pi;
end
```

```
function findAlpha = findAlpha()
i = 1;
X = (1 : 6);
for a = 1 : 1 : 99
    for b = 1 : 1 : 99
        for c = 1 : 1 : 99
            for d = 1 : 1 : 99
                Alpha = calculateAlpha(a,b,c,d);
                if (Alpha > 0) && (Alpha < 90)
                    Alpha1 = calculateAlpha1(a,c,Alpha);
                    if (Alpha1 > 0) && (Alpha1 < 90)
                        X(i,1) = a; X(i,2) = b; X(i,3) = c;
                        X(i,4) = d; X(i,5) = Alpha; X(i,6) = Alpha1;
                        i = i+1;
                        fprintf('a = %d, b = %d, c = %d, d = %d, alpha(degree) = %f\n',a,b,c,d,Alpha);
                    end
                end
            end
        end
    end
end
```

Some result I have:

```
a = 8, b = 8, c = 98, d = 54, alpha(degree) = 85.317583
a = 8, b = 8, c = 98, d = 55, alpha(degree) = 85.317583
a = 8, b = 8, c = 98, d = 56, alpha(degree) = 85.317583
a = 8, b = 8, c = 98, d = 57, alpha(degree) = 85.317583
a = 8, b = 8, c = 98, d = 58, alpha(degree) = 85.317583
a = 8, b = 8, c = 98, d = 59, alpha(degree) = 85.317583
a = 8, b = 8, c = 98, d = 60, alpha(degree) = 85.317583
a = 8, b = 8, c = 98, d = 61, alpha(degree) = 85.317583
a = 8, b = 8, c = 98, d = 62, alpha(degree) = 85.317583
a = 8, b = 8, c = 98, d = 63, alpha(degree) = 85.317583
a = 8, b = 8, c = 98, d = 64, alpha(degree) = 85.317583
a = 8, b = 8, c = 98, d = 65, alpha(degree) = 85.317583
a = 8, b = 8, c = 98, d = 66, alpha(degree) = 85.317583
a = 8, b = 8, c = 98, d = 67, alpha(degree) = 85.317583
a = 8, b = 8, c = 98, d = 68, alpha(degree) = 85.317583
a = 8, b = 8, c = 98, d = 69, alpha(degree) = 85.317583
a = 8, b = 8, c = 98, d = 70, alpha(degree) = 85.317583
a = 8, b = 8, c = 98, d = 71, alpha(degree) = 85.317583
a = 8, b = 8, c = 98, d = 72, alpha(degree) = 85.317583
a = 8, b = 8, c = 98, d = 73, alpha(degree) = 85.317583
a = 8, b = 8, c = 98, d = 74, alpha(degree) = 85.317583
a = 8, b = 8, c = 98, d = 75, alpha(degree) = 85.317583
a = 8, b = 8, c = 98, d = 76, alpha(degree) = 85.317583
a = 8, b = 8, c = 98, d = 77, alpha(degree) = 85.317583
a = 8, b = 8, c = 98, d = 78, alpha(degree) = 85.317583
a = 8, b = 8, c = 98, d = 79, alpha(degree) = 85.317583
a = 8, b = 8, c = 98, d = 80, alpha(degree) = 85.317583
a = 8, b = 8, c = 98, d = 81, alpha(degree) = 85.317583
a = 8, b = 8, c = 98, d = 82, alpha(degree) = 85.317583
a = 8, b = 8, c = 98, d = 83, alpha(degree) = 85.317583
a = 8, b = 8, c = 98, d = 84, alpha(degree) = 85.317583
a = 8, b = 8, c = 98, d = 85, alpha(degree) = 85.317583
a = 8, b = 8, c = 98, d = 86, alpha(degree) = 85.317583
a = 8, b = 8, c = 98, d = 87, alpha(degree) = 85.317583
a = 8, b = 8, c = 98, d = 88, alpha(degree) = 85.317583
a = 8, b = 8, c = 98, d = 89, alpha(degree) = 85.317583
a = 8, b = 8, c = 98, d = 90, alpha(degree) = 85.317583
a = 8, b = 8, c = 98, d = 91, alpha(degree) = 85.317583
a = 8, b = 8, c = 98, d = 92, alpha(degree) = 85.317583
a = 8, b = 8, c = 98, d = 93, alpha(degree) = 85.317583
```

It looks too hard with integer solutions. Then I reversed the problem to check:

Drawing triangle ABC with $AB = m,\ AC = n, BAC = \alpha\ \left(m, n \in R \text{ and } \alpha \in\ (0,2\pi)\right)$. Point D is lies random on BC. The middle line of AD will intersect with AB, AC at F, E. We get the value of $AF = a, FB = b, AE = c, EC = d$ and also angle $EAG = \alpha_1$. We will use the function above to check the angle (α, α_1).

But first, we have to turn back to our main problem: ratio BD/CD

2/Ratio BD/CD

- Triangle BAD: $\frac{BD}{\sin BAD} = \frac{AD}{\sin ABC}$

⇨ $\frac{BD}{\sin(\alpha_1)} = \frac{AD}{\sin ABC}$

- Triangle CAD: $\frac{CD}{\sin CAD} = \frac{AD}{\sin ACB}$

⇨ $\frac{CD}{\sin(\alpha_2)} = \frac{AD}{\sin ACB}$ $(\alpha_2 = \alpha - \alpha_1)$

- Triangle ABC: $\frac{AB}{\sin ACB} = \frac{AC}{\sin ABC}$

⇨ $\frac{a+b}{\sin ACB} = \frac{c+d}{\sin ABC}$

Then:

$$\frac{BD}{CD} = \left(\frac{a+b}{c+d}\right).\frac{sin\alpha_1}{sin\alpha_2}, with\ tan\alpha_1 = \frac{a - c.cos\alpha}{c.sin\alpha}, \alpha_2 = \alpha - \alpha_1$$

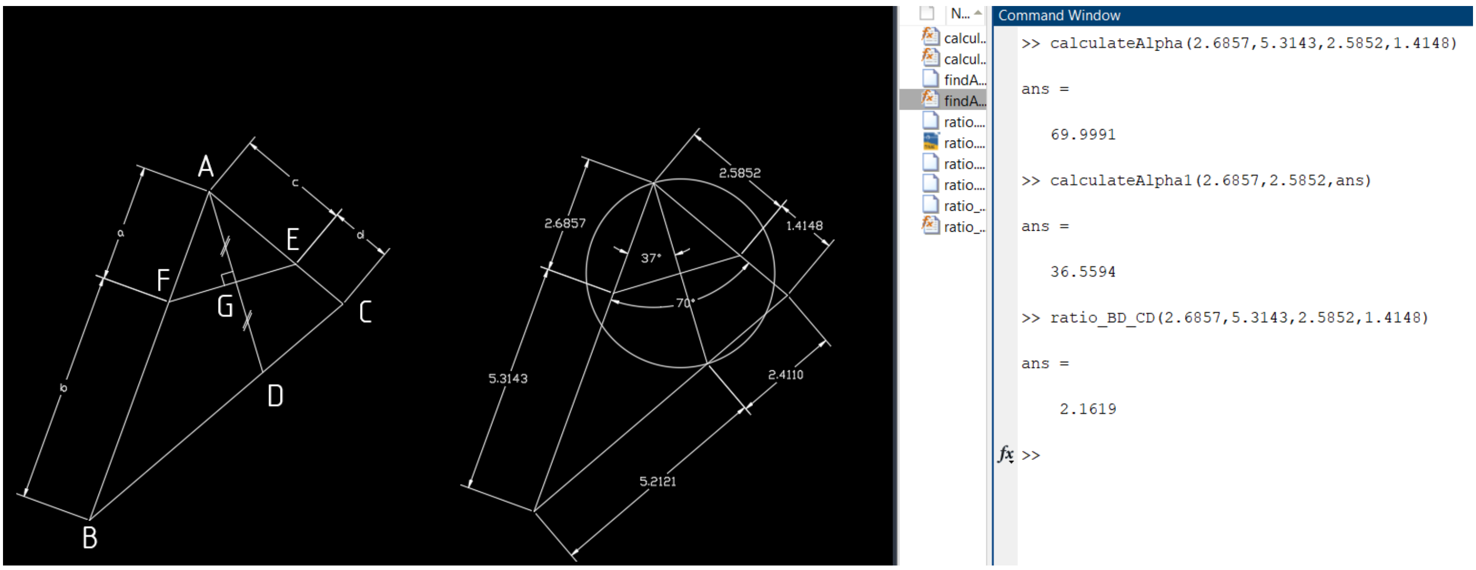
From that. I write a function to get ratio $\frac{BD}{CD}$:

```
function ratio = ratio_BD_CD(a,b,c,d)
    Alpha = calculateAlpha(a,b,c,d);
    Alpha1 = calculateAlpha1(a,c,Alpha);
    Alpha2 = Alpha - Alpha1;
    ratio = ((a+b)/(c+d))*sin(Alpha1*pi/180)/sin(Alpha2*pi/180);
end
```

Function *ratio_BD_CD(a,b,c,d):* calculate BD/CD from a, b, c, d

3/ Check with drawing

- Check angle when we know the value of a, b, c, d

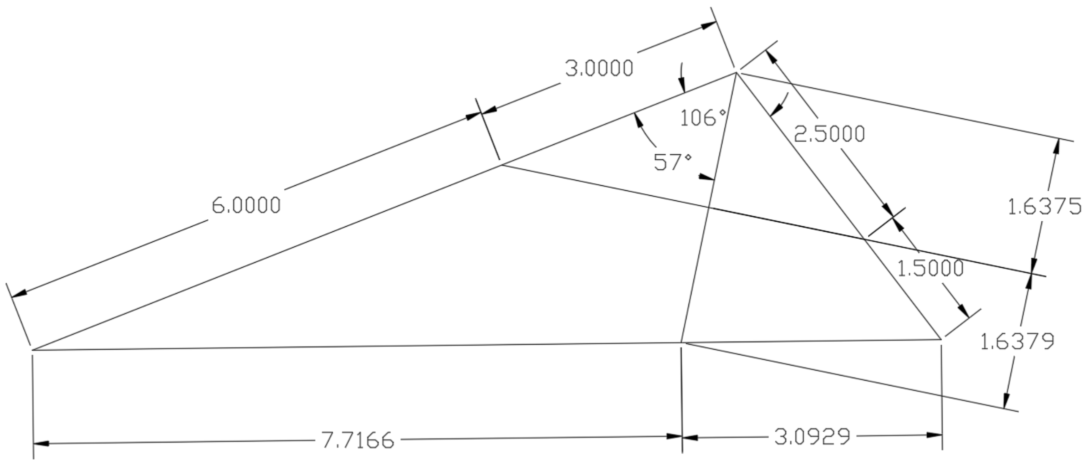


- Check solution when we have the value of a, b, c, d

```
>> calculateAlpha(3,6,2.5,1.5)

ans =

  105.4660

>> calculateAlpha1(3,2.5,ans)

ans =

   56.6899

>> ratio_BD_CD(3,6,2.5,1.5)

ans =

    2.5000
```

Using function to predict

Here is the drawing:



The precision of the solution depends on the value of angle BAC we take. In this drawing we checked the difference between the length of AG and GD is:

$$d = |AG - GD| = |1.6375 - 1.6379| = 4\text{x}10^{-4}$$

The ratio BD/CD in reality is:

$$\frac{7.7166}{3.0929} = 2.4949$$

While the prediction is 2.500, the difference is: $|2.4949 - 2.5| = 5.1\text{x}10^{-3}$