NOTES ON LINEAR ALGEBRA DONE WRONG

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1. Chapter 1

1.1. Vector spaces.

- Axioms – familiar rules of algebra, just applied to vectors
    - Cannot mix vectors + scalars in axioms

1.2. Linear combinations, bases.

- A system of vectors $v_1, v_2, ..., v_n \in V$ is called a basis for V if any vector v admits a *unique* representation

$$v = \sum_{k=1}^{p} a_k v_k$$

- Standard basis in $\mathbb{F}^n$ (where $\mathbb{F}$ is $\mathbb{R}$ or $\mathbb{C}$)

$$e_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, e_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, e_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix}, ..., e_n = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix}$$

- Standard basis for $\mathbb{P}_n$ (the polynomials of degree at most n)

$$e_0 := 1, e_1 := t, e_2 := t^2, e_3 := t^3, ..., e_n := t^n$$

- A system of vectors $v_1, v_2, ..., v_p \in V$ is a *generating system* (also *spanning system* or *complete system*) in V if any vector $v \in V$ admits representation as a linear combination of $v_1, v_2, ..., v_p$
    - Only difference from def. of basis is that we do not assume the representation is *unique*

- A linear combination $\alpha_1 v_1 + \alpha_2 v_2 + ... + \alpha_p v_p$ is called *trivial* if $a_k = 0\ \forall k$.
    - **Trivial linear combination** is always equal to 0

- A system of vectors $v_1, v_2, ...v_p \in V$ is called *linearly independent* if only the trivial linear combination of $v_1, v_2, ..., v_p$ equals 0.
    - In other words, the system is linearly independent $\iff x_1 v_1 + x_2 v_2 + ... + x_p v_p = 0$ has only the trivial solution $x_1 = x_2 = ...x_p = 0$

- If a system is not linearly independent, it is *linearly dependent*

- A system of vectors $v_1, v_2, ..., v_p$ is called linearly dependent if

there exist scalars $\alpha_1, \alpha_2, ..., \alpha_p$, where

$$\sum_{k=1}^{p} |a_k| \neq 0$$

$$\text{such that} \sum_{k=1}^{p} a_k v_k = 0$$

- Alternatively, a system $v_1, v_2, ..., v_p$ is linearly independent $\iff$ the equation

$$x_1 v_1 + x_2 v_2 + ... + x_p v_p = 0$$

has a **non-trivial** solution

- Non-trivial meaning *at least one* x_k is different from 0, or

$$\sum_{k=1}^{p} |x_k| \neq 0$$

Remark. Another notion of linear independence or dependence (best understood in a Cartesian plane or 3-D space) is whether or not each vector affords an additional “dimension” of movement. If each vector allows access to a new dimension, then the only way to remain at 0 (the origin) is for each vector to be scaled by coefficient 0. Otherwise, if two vectors access the exact same dimensions, they can be scaled in such a way to negate each other, allowing 0 to be represented with non-zero coefficients.

Theorem 1.2.1. *A system of vectors* $v_1, v_2, ..., v_p \in V$ *is linearly dependent* $\iff$ *one of the vectors,* v_k*, can be represented as a linear combination of the other vectors*

$$v_k = \sum_{j=1 \land j \neq k}^{p} \beta_j v_j$$

Proof. Suppose that we have a system $v_1, v_2, ..., v_p$ that is linearly dependent. Then, for some indices k, we must have

$$\sum_{k=1}^{p} |\alpha_k| \neq 0$$

Then we can write the system as

$$\alpha_k v_k + \sum_{j=1 \land j \neq k}^{p} \alpha_j v_j = 0$$

Moving terms around, we have

$$v_k = \sum_{j=1 \land j \neq k}^{p} -\frac{\alpha_j}{\alpha_k} v_j$$

Which is $v_k = \sum_{j=1 \land j \neq k}^{p} \beta_j v_j$, with $\beta_j = -\frac{\alpha_j}{\alpha_k}$. $\square$

- Trivially, any basis is a linearly independent system

- Recall that a basis in V allows any vector $\in V$ a unique representation as $\sum_{k=1}^{p} a_k v_k$
- This implies 0 is given a unique representation by the basis. Since the trivial linear combination always gives zero, regardless of linear (in)dependence, the trivial linear combination must also be the *only one* giving 0, satisfying the definition of linear independence.

Conversely,

Theorem 1.2.2. *A system of vectors $v_1, v_2, ..., v_n \in V$ is a basis $\iff$ it is linearly independent and complete.*

Proof. We seek to show that if a system of vectors is linearly independent and complete, then it is a basis. Suppose we have a system $v_1, v_2, v_3, ..., v_p \in V$, which is linearly independent and complete.

Consider any vector $v \in V$. Since the system is complete, v can be represented as a linear combination

$$v = \sum_{k=1}^{p} \alpha_k v_k$$

This satifies part of the definition of a basis. We need only show that this representation is *unique*. Say we have another $\tilde{\alpha}_k$ such that admits representation as a linear combination

$$v = \sum_{k=1}^{p} \tilde{\alpha}_k v_k$$

Then,

$$v = \sum_{k=1}^{p} (\alpha_k - \tilde{\alpha}_k) v_k = \sum_{k=1}^{p} \alpha_k v_k - \sum_{k=1}^{p} \tilde{\alpha}_k v_k = v - v = 0$$

Because it's linearly independent, only the trivial linear combination can equal 0, and

$$\sum_{k=1}^{p} |a_k| = 0$$

So $\alpha_k - \tilde{\alpha}_k = 0 \; \forall k$, $a_k = \tilde{\alpha}_k$ implying that $\tilde{\alpha}_k$ does not admit a separate representation. Thus,

$$v = \sum_{k=1}^{p} \alpha_k v_k$$

is unique. □

Theorem 1.2.3. *Any (finite) generating system contains a basis.*

Proof. Suppose $v_1, v_2, v_3, v_p \in V$ is a generating set. If it's also linearly independent, then it's a basis and we are done. Otherwise, it's linearly dependent, and there is at least one vector v_k which can be represented as a linear combination of the other vectors.

Thus, we can remove v_k from our set, and it remains complete. If the set is still linearly dependent, then repeat the process. If we remove all items, then we have $\emptyset$ and it's not a generating system. Therefore, we can eliminate all v_k which can be represented as the linear combination of other vectors v_j, and we are left with a linearly independent and complete set of vectors, or a basis. $\square$

1.3. **Linear Transformations. Matrix-vector multiplication.**

- *Transformation* $T : X \to Y$ is a rule that for each $x \in X$, $y = T(x) \in Y$
  - The *set* X is called the *domain* of T, and the set Y is called the *target space* or *codomain* of T.

Definition. Let V, W be vector spaces (over the same field $\mathbb{F}$). A transformation $T : V \to W$ is called linear if

1. $T(u + v) = T(u) + T(v)\ \forall u, v \in V$
2. $T(\alpha v) = \alpha T(v)\ \forall v \in V$ and $\forall$ scalars $\alpha \in \mathbb{F}$

- Linear transformations $T : \mathbb{R} \to \mathbb{R}$ can be given by

  $$T(x) = ax \text{ where } a = T(1)$$

  Any linear transformation of $\mathbb{R}$ is just multiplication by a constant
- A linear transformation $T : \mathbb{F}^n \to \mathbb{F}^m$ can also be represented as multiplication, but by matrix, not scalar
- For $T : \mathbb{F}^n \to \mathbb{F}^m$, it is sufficient to know how T acts on the standard basis of $\mathbb{F}^n$ to compute $T(x)$ for all vectors $x \in \mathbb{F}^n$.
- If you want $Ax = T(x)$, then you have the *column by coordinate* rule

$$Ax = \sum_{k=1}^{n} x_k a_k = x_1 \begin{pmatrix} a_{1,1} \\ a_{2,1} \\ \vdots \\ a_{m,1} \end{pmatrix} + x_2 \begin{pmatrix} a_{1,2} \\ a_{2,2} \\ \vdots \\ a_{m,2} \end{pmatrix} + \ldots + x_n \begin{pmatrix} a_{1,n} \\ a_{2,n} \\ \vdots \\ a_{m,n} \end{pmatrix}$$

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