SELECTED PROBLEMS FROM LINEAR ALGEBRA DONE WRONG, TREIL 2017

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ABSTRACT. The textbook "Linear Algebra Done Wrong", by Sergei Treil of Brown University, has been available for free online and is an excellent resource for beginners at linear algebra. However, the lack of an official solution manual makes it hard for self-learners to check their solutions. I'll be sharing some of my selected solutions here as I work through the book.

1. Basic Notions

1.1. Vector Spaces.

Problem 1.1.4: Prove that a zero vector $\mathbf{0}$ of a vector space V is unique.

Proof. Suppose that there exists at least two distinct zero vectors, 0_1 and 0_2, both satisfying Axiom 3:

$$\begin{gathered} \forall v \in V, \\ v + 0_1 = v \\ v + 0_2 = v \end{gathered}$$

Then,

$$\begin{aligned} 0_1 &= 0_1 + 0_2 \\ 0_1 &= 0_2 + 0_1 && \text{(commutativity)} \\ 0_1 &= 0_2 && \text{(zero vector)} \end{aligned}$$

But we said 0_1 and 0_2 were distinct, or in other words, $0_1 \neq 0_2$. Therefore, by contradiction, there cannot exist more than one zero vector satisfying Axiom 3. □

Problem 1.1.6: Prove that the additive inverse, defined in Axiom 4 of a vector space is unique.

Proof. Suppose there exists at least two distinct additive inverses, w_1 and w_2, both satisfying

$$\begin{gathered} v + w_1 = 0 \\ v + w_2 = 0 \end{gathered}$$

This implies

$$\begin{aligned} v + w_1 + w_2 &= 0 + w_1 \\ w_2 &= w_1 && \text{(by additive inverse and zero vector axioms)} \end{aligned}$$

But we said that w_1 and w_2 were distinct ($w_1 \neq w_2$). Therefore, by contradiction, there cannot exist more than one unique additive inverse w such that $v + w = 0$ □

Problem 1.1.7: Prove that $0\boldsymbol{v} = \mathbf{0}$ for any vector $\boldsymbol{v} \in V$.

Proof. Note the following:

$$0v = (0+0)v = 0v + 0v \text{ by axiom 8 and 3}$$

The additive inverse of $0v$ is $-0v$. Thus:

$$\begin{aligned} 0v + 0v - 0v &= 0v - 0v \\ 0v &= 0 \qquad \text{by axiom 4} \end{aligned}$$

□

Problem 1.1.8: Prove that for any vector $\boldsymbol{v}$ its additive inverse $-\boldsymbol{v}$ is given by $(-1)\boldsymbol{v}$.

Proof.

$$\begin{aligned} v + (-1)v = 1v + (-1)v &= v(1-1) \text{ using axioms 5 and 7} \\ &= 0v = 0 \text{ as shown in 1.1.7} \end{aligned}$$

We've shown that $v + (-1)v = 0$. But we also showed in 1.1.6 that the additive inverse $-v$, defined as $v + (-v) = 0$, must be unique for a vector space. Therefore, $(-1)v$ is the same as the unique additive inverse $-v$, or $-v = (-1)v$. □

1.2. **Linear combinations, bases.**

2.1. Find a basis in the space of 3×2 matrices $M_{3\times 2}$.

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix}$$

2.2 True or false:

a) Any set containing a zero vector is linearly dependent.
True. Given any set $\{v_1, v_2, v_3, ..., 0, ..., v_p\}$, we have

$$0v_1 + 0v_2 + 0v_3 + ... + \alpha_k \cdot 0 + ... + v_p$$

where $\alpha_k \in \mathbb{F}$, which shows that the set is linearly dependent, since we have infinite non-trivial linear combinations which equal 0.

b) A basis must contain 0.
False. If a set contains 0, it's linearly dependent, as shown above. Therefore, it cannot be a basis.

c) Subsets of linearly dependent sets are linearly dependent.
False. We showed earlier that any linearly dependent (finite) and complete set of vectors also contains a linearly independent subset, namely, the basis.

d) Subsets of linearly independent sets are linearly independent.
True. We know that no vector in the set can be the linear combination of the other vectors of the set. Therefore, any subset of this set also has no vectors which are the linear combination of other vectors in the subset.

e) If $\alpha_1 v_1 + \alpha_2 v_2 + ... + \alpha_n v_n = 0$ then all scalars α_k are zero;

False. This is only true for linearly independent sets. Linearly dependent sets, by definition, have coefficients a_k where $a_k \neq 0$ and the linear combination equals 0.

2.3 Recall, that a matrix is called *symmetric* if $A^T = A$. Write down a basis in the space of *symmetric* 2×2 matrices (there are many possible answers). How many elements are in the basis?

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

2.4 Write down a basis for the space of

a) 3×3 symmetric matrices;

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

b) $n \times n$ symmetric matrices;

2.5 (Question not reproduced)
It is known that the system is not generating, so not all vectors $v \in V$ can be represented by linear combination of v_1 through v_r.
Let $v_{r+1} \in V$ be one such vector. If it cannot be represented as a linear combination of v_1 through v_r, it cannot be represented by

$$\sum_{j=1}^{r} a_j v_j$$

and therefore the system $v_1, v_2, ..., v_r, v_{r+1}$ is still linearly independent.

2.6. Is it possible that vectors v_1, v_2, v_3 are linearly dependent, but the vectors $w_1 = v_1 + v_2$, $w_2 = v_2 + v_3$ and $w_3 = v_3 + v_1$ are linearly independent?

No. Consider $\alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3 = 0$. Then, there exists some $\alpha_1, \alpha_2, \alpha_3$ such that

$$|\alpha_1| + |\alpha_2| + |\alpha_3| \neq 0$$

Now consider

$$\beta_1 w_1 + \beta_2 w_2 + \beta_3 w_3 = 0$$

Substituting in values:

$$\begin{gathered}
\beta_1(v_1 + v_2) + \beta_2(v_2 + v_3) + \beta_3(v_1 + v_3) = 0 \\
\beta_1 v_1 + \beta_1 v_2 + \beta_2 v_2 + \beta_2 v_3 + \beta_3 v_1 + \beta_3 v_3 = 0 \\
(\beta_1 + \beta_3)v_1 + (\beta_2 + \beta_1)v_2 + (\beta_2 + \beta_3)v_3 = 0
\end{gathered}$$

For what coefficients is this satisfied? There's the trivial linear combination, where all coefficients equal 0. However, we have $\alpha_1, \alpha_2, \alpha_3$, which are coefficients also satisfying this equation, and at least one of which is non-zero.
We have

$$\begin{gathered}
\beta_1 + \beta_3 = \alpha_1 \\
\beta_2 + \beta_1 = \alpha_2 \\
\beta_2 + \beta_3 = \alpha_3
\end{gathered}$$

Eliminating variables, we obtain

$$\begin{aligned}
\beta_3 - \beta_2 &= \alpha_1 - \alpha_2 \\
\beta_3 &= \frac{\alpha_1 - \alpha_2 + \alpha_3}{2} \\
\beta_1 &= \frac{\alpha_1 + \alpha_2 - \alpha_3}{2} \\
\beta_2 &= \frac{3\alpha_2 - \alpha_1 - \alpha_3}{2}
\end{aligned}$$

Since $\alpha_1, \alpha_2, \alpha_3$ are nonzero, at least one of $\beta_1, \beta_2, \beta_3$ are also nonzero, therefore w_1, w_2, w_3 cannot be linearly independent as there exists scalars $\beta_1, \beta_2, \beta_3$ such that

$$\begin{gathered}
\beta_1 w_1 + \beta_2 w_2 + \beta_3 w_3 = 0 \\
|\beta_1| + |\beta_2| + |\beta_3| \neq 0
\end{gathered}$$

References

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