# NOTES ON LINEAR ALGEBRA DONE WRONG

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## 1. Chapter 1

### 1.1. Vector spaces.

- Axioms – familiar rules of algebra, just applied to vectors
  - Cannot mix vectors + scalars in axioms

### 1.2. Linear combinations, bases.

- A system of vectors $v_1, v_2, ..., v_n \in V$ is called a basis for $V$ if any vector $v$ admits a *unique* representation

$$v = \sum_{k=1}^{p} a_k v_k$$

- Standard basis in $\mathbb{F}^n$ (where $\mathbb{F}$ is $\mathbb{R}$ or $\mathbb{C}$)

$$e_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, e_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, e_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix}, ..., e_n = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix}$$

- Standard basis for $\mathbb{P}_n$ (the polynomials of degree at most $n$)

$$e_0 := 1, e_1 := t, e_2 := t^2, e_3 := t^3, ..., e_n := t^n$$

- A system of vectors $v_1, v_2, ..., v_p \in V$ is a *generating system* (also *spanning system* or *complete system*) in $V$ if any vector $v \in V$ admits representation as a linear combination of $v_1, v_2, ..., v_p$
  - Only difference from def. of basis is that we do not assume the representation is *unique*
- A linear combination $\alpha_1 v_1 + \alpha_2 v_2 + ... + \alpha_p v_p$ is called *trivial* if $a_k = 0\ \forall k$.
  - **Trivial linear combination** is always equal to 0
- A system of vectors $v_1, v_2, ...v_p \in V$ is called *linearly independent* if only the trivial linear combination of $v_1, v_2, ..., v_p$ equals 0.
  - In other words, the system is linearly independent $\iff$ $x_1 v_1 + x_2 v_2 + ... + x_p v_p = 0$ has only the trivial solution $x_1 = x_2 = ...x_p = 0$
- If a system is not linearly independent, it is *linearly dependent*
- A system of vectors $v_1, v_2, ..., v_p$ is called linearly dependent if

there exist scalars $\alpha_1, \alpha_2, ..., \alpha_p$, where

$$\sum_{k=1}^{p} |a_k| \neq 0$$

$$\text{such that} \sum_{k=1}^{p} a_k v_k = 0$$

- Alternatively, a system $v_1, v_2, ..., v_p$ is linearly independent $\iff$ the equation

$$x_1 v_1 + x_2 v_2 + ... + x_p v_p = 0$$

has a **non-trivial** solution

- Non-trivial meaning *at least one* $x_k$ is different from 0, or

$$\sum_{k=1}^{p} |x_k| \neq 0$$

*Remark.* Another notion of linear independence or dependence (best understood in a Cartesian plane or 3-D space) is whether or not each vector affords an additional "dimension" of movement. If each vector allows access to a new dimension, then the only way to remain at 0 (the origin) is for each vector to be scaled by coefficient 0. Otherwise, if two vectors access the exact same dimensions, they can be scaled in such a way to negate each other, allowing 0 to be represented with non-zero coefficients.

**Theorem 1.2.1.** *A system of vectors $v_1, v_2, ..., v_p \in V$ is linearly dependent $\iff$ one of the vectors, $v_k$, can be represented as a linear combination of the other vectors*

$$v_k = \sum_{j=1 \land j \neq k}^{p} \beta_j v_j$$

*Proof.* Suppose that we have a system $v_1, v_2, ..., v_p$ that is linearly dependent. Then, for some indices $k$, we must have

$$\sum_{k=1}^{p} |\alpha_k| \neq 0$$

Then we can write the system as

$$\alpha_k v_k + \sum_{j=1 \land j \neq k}^{p} \alpha_j v_j = 0$$

Moving terms around, we have

$$v_k = \sum_{j=1 \land j \neq k}^{p} -\frac{\alpha_j}{\alpha_k} v_j$$

Which is $v_k = \sum_{j=1 \land j \neq k}^{p} \beta_j v_j$, with $\beta_j = -\frac{\alpha_j}{\alpha_k}$. □

- Trivially, any basis is a linearly independent system

- Recall that a basis in $V$ allows any vector $\in V$ a unique representation as $\sum_{k=1}^{p} a_k v_k$
- This implies 0 is given a unique representation by the basis. Since the trivial linear combination always gives zero, regardless of linear (in)dependence, the trivial linear combination must also be the *only one* giving 0, satisfying the definition of linear independence.

Conversely,

**Theorem 1.2.2.** *A system of vectors $v_1, v_2, ..., v_n \in V$ is a basis $\iff$ it is linearly independent and complete.*

*Proof.* We seek to show that if a system of vectors is linearly independent and complete, then it is a basis. Suppose we have a system $v_1, v_2, v_3, ..., v_p \in V$, which is linearly independent and complete.

Consider any vector $v \in V$. Since the system is complete, $v$ can be represented as a linear combination

$$v = \sum_{k=1}^{p} \alpha_k v_k$$

This satifies part of the definition of a basis. We need only show that this representation is *unique*. Say we have another $\tilde{\alpha}_k$ such that admits representation as a linear combination

$$v = \sum_{k=1}^{p} \tilde{\alpha}_k v_k$$

Then,

$$v = \sum_{k=1}^{p} (\alpha_k - \tilde{\alpha}_k) v_k = \sum_{k=1}^{p} \alpha_k v_k - \sum_{k=1}^{p} \tilde{\alpha}_k v_k = v - v = 0$$

Because it's linearly independent, only the trivial linear combination can equal 0, and

$$\sum_{k=1}^{p} |a_k| = 0$$

So $\alpha_k - \tilde{\alpha}_k = 0 \ \forall k$, $a_k = \tilde{\alpha}_k$ implying that $\tilde{\alpha}_k$ does not admit a separate representation. Thus,

$$v = \sum_{k=1}^{p} \alpha_k v_k$$

is unique. □

**Theorem 1.2.3.** *Any (finite) generating system contains a basis.*

*Proof.* Suppose $v_1, v_2, v_3, v_p \in V$ is a generating set. If it's also linearly independent, then it's a basis and we are done. Otherwise, it's linearly dependent, and there is at least one vector $v_k$ which can be represented as a linear combination of the other vectors.

Thus, we can remove $v_k$ from our set, and it remains complete. If the set is still linearly dependent, then repeat the process. If we remove all items, then we have $\emptyset$ and it's not a generating system. Therefore, we can eliminate all $v_k$ which can be represented as the linear combination of other vectors $v_j$, and we are left with a linearly independent and complete set of vectors, or a basis. $\square$

### 1.3. **Linear Transformations. Matrix-vector multiplication.**

- *Transformation* $T : X \to Y$ is a rule that for each $x \in X$, $y = T(x) \in Y$
  - The *set* $X$ is called the *domain* of $T$, and the set $Y$ is called the *target space* or *codomain* of $T$.

**Definition.** Let $V, W$ be vector spaces (over the same field $\mathbb{F}$). A transformation $T : V \to W$ is called linear if

1. $T(u + v) = T(u) + T(v)\ \forall u, v \in V$
2. $T(\alpha v) = \alpha T(v)\ \forall v \in V$ and $\forall$ scalars $\alpha \in \mathbb{F}$

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