

Homework Solutions: Lambda Calculus

Learning Objectives:

1. Understand evaluation order
2. Understand church encoding
3. Learn to perform β -reduction

Instructions:

- Total points: 47 pt
- Early deadline: Mar 31 (Wed) at 11:59 PM; Regular deadline: Apr 2 (Fri) at 11:59 PM (you can continue working on the homework till TA starts to grade the homework).
- Submit one pdf file on Canvas under Assignments, Homework 6 submission. You are encouraged to use latex. But we will accept a scanned copy as well.

Questions:

1. (9 pt) [β -reduction] Perform β -reduction for the following λ expressions.

- (a) (3 pt) $((\lambda(x)(\lambda(y)(x y)))((\lambda(a) a) a))((\lambda(b) b) b)$
- (b) (3 pt) $((\lambda(x)(\lambda(y)(y)))((\lambda(z) z) a)) b$
- (c) (3 pt) $((\lambda(x)(x x))(\lambda(y) y)) x$

Sol.

(a) (3pt)

$$\begin{aligned}
 & (((\lambda(x)(\lambda(y)(x y)))((\lambda(a) a) a))((\lambda(b) b) b)) & (1) \\
 = & (((\lambda(x)(\lambda(y)(x y))) a) b) & (2) \\
 = & (((\lambda(y)(a y))) b) & (3) \\
 = & (a b) & (4)
 \end{aligned}$$

(b) (3pt)

$$\begin{aligned}
 & (((\lambda(x)(\lambda(y)(y)))((\lambda(z) z) a)) b) & (1) \\
 = & (((\lambda(x)(\lambda(y)(y)))a) b) & (2) \\
 = & (((\lambda(x)(\lambda(y)(y)))a) b) & (3) \\
 = & (((\lambda(y)(y))) b) & (4) \\
 = & b & (5)
 \end{aligned}$$

(c) (3pt)

$$\begin{aligned}
 & (((\lambda(x)(x\ x))(\lambda(y)\ y))\ x) && (1) \\
 = & (((\lambda(y)y)(\lambda(y)\ y))\ x) && (2) \\
 = & ((\lambda(y)\ y)\ x) && (3) \\
 = & x && (4)
 \end{aligned}$$

2. (4 pt) [Evaluation order] The goal of this problem is to help you understand the evaluation order of lambda calculus.

In the following, show the steps of β -reduction for the lambda expression using two types of evaluation orders

$$((\lambda(x)\ p)((\lambda(y)(y\ y))(\lambda(z)(z\ z))))$$

Sol.

(a) (3pt)

$$\begin{aligned}
 & ((\lambda(x)p)((\lambda(y)(y\ y))(\lambda(z)(z\ z)))) && (1) \\
 = & p && (2)
 \end{aligned}$$

(b) (3pt)

$$\begin{aligned}
 & ((\lambda(x)p)((\lambda(y)(y\ y))(\lambda(z)(z\ z)))) && (1) \\
 = & ((\lambda(x)p)((\lambda(z)(z\ z))(\lambda(z)(z\ z)))) && (2) \\
 = & ((\lambda(x)p)((\lambda(z)(z\ z))(\lambda(z)(z\ z)))) && (3) \\
 = & ((\lambda(x)p)((\lambda(z)(z\ z))(\lambda(z)(z\ z)))) && (4) \\
 = & \dots && (5)
 \end{aligned}$$

3. (7 pt) [Church Encoding] Encode the following logic Boolean operations using the encoding of *true*, *false*, *ite*, *not* and *or* provided in the lecture.

(a) (3 pt) *and* $a\ b$ (b) (4 pt) $a \rightarrow b$ **Sol.**

```
(a)      (ite a
          (ite b  true  false)
          false
        )
```

(b)

```
(ite a
    (ite b true false)
    true
)
```

4. (16 pt) [Church Encoding and understanding the semantics of lambda expressions] Using *zero*, *one* and *two* as well as *succ*, *true* and *false* provided in the lecture, answer the following two questions:

- (a) (4 pt) What is the result of $((\lambda(z)((one\ f)\ z))\ (succ\ zero))$?
- (b) Suppose we define *unknown*: $(\lambda(x)(\lambda(y)(\lambda(z)\ z)))$ and *g*: $(\lambda(n)((n\ unknown)\ true))$, what is the result of:
 - i. (4 pt) $(g\ zero)$
 - ii. (3 pt) $(g\ one)$
 - iii. (2 pt) $(g\ two)$
 - iv. (3 pt) What mathematical/logical operation is computed by g?

Sol.

- (a) (4 pt)

$$\begin{aligned} ((\lambda(z)((one\ f)\ z))\ (succ\ zero)) &= \\ (((\lambda(z)((\lambda(f)(\lambda(x)(f\ x))))\ f)\ z))\ (succ\ zero) &= \\ (((\lambda(z)((\lambda(x)(f\ x)))\ z))\ (succ\ zero)) &= \\ (((\lambda(z)(f\ z))\ (succ\ zero)) &= \\ (((\lambda(z)(f\ z))\ ((\lambda(n)(\lambda(f)(\lambda(x)(f((n\ f)\ x)))))\ zero)) &= \\ (((\lambda(z)(f\ z))\ (\lambda(f)(\lambda(x)(f((zero\ f)\ x)))))) &= \\ (((\lambda(z)(f\ z))\ (\lambda(f)(\lambda(x)(f(((\lambda(f)(\lambda(x)x))\ f)\ x)))))) &= \\ (((\lambda(z)(f\ z))\ (\lambda(f)(\lambda(x)(f((\lambda(x)x)\ x)))))) &= \\ (((\lambda(z)(f\ z))\ (\lambda(f)(\lambda(x)(f\ x)))))) &= \\ (((\lambda(z)(f\ z))\ one)) &= \\ (f\ one) \end{aligned}$$
- (b) i. (4 pt)

$$\begin{aligned} (g\ zero) &= \\ (((\lambda(n)((n\ unknown)\ true))\ zero) &= \\ (((zero\ unknown)\ true) &= \\ (((\lambda(f)(\lambda(x)x))\ unknown)\ true) &= \\ (((\lambda(x)x)true) &= \\ true \end{aligned}$$
- ii. (3 pt)

$$\begin{aligned} (g\ one) &= \\ (((\lambda(n)((n\ unknown)\ true))\ one) &= \\ (((one\ unknown)\ true) &= \\ (((\lambda(f)(\lambda(x)(f\ x))))\ unknown)\ true) &= \\ (((\lambda(x)(unknown\ x))\ true) &= \end{aligned}$$

$$\begin{aligned} ((\lambda(x)((\lambda(x)(\lambda(y)(\lambda(z) z))) x)) \text{ true}) &= \\ ((\lambda(x)(\lambda(y)(\lambda(z) z))) \text{ true}) &= \\ ((\lambda(x)\text{false}) \text{ true}) &= \\ \text{false} & \end{aligned}$$

iii. (2 pt)

$$\begin{aligned} (g \text{ two}) &= \\ ((\lambda(n)((n \text{ unknown}) \text{ true})) \text{ two}) &= \\ ((\text{two unknown}) \text{ true}) &= \\ (((\lambda(f)(\lambda(x)(f(f x)))) \text{ unknown}) \text{ true}) &= \\ ((\lambda(x)(\text{unknown (unknown }x\text{)})) \text{ true}) &= \\ (\text{unknown (unknown true)}) &= \\ (\text{unknown ((}\lambda(x)(\lambda(y)(\lambda(z) z))\text{)) true}) &= \\ (\text{unknown }(\lambda(y)(\lambda(z) z))) &= \\ (\text{unknown false}) &= \\ ((\lambda(x)(\lambda(y)(\lambda(z) z))) \text{ false}) &= \\ (\lambda(y)(\lambda(z) z)) &= \\ \text{false} & \end{aligned}$$

iv. (3 pt) The computation performed is: testing if g is 0.

5. (11 pt) Given:

$$\begin{aligned} \text{data: } & (\lambda(x) (\lambda(y) (\lambda(z) ((z x) y)))) \\ \text{op1: } & (\lambda(p) (p (\lambda(x) (\lambda(y) x)))) \\ \text{op2: } & (\lambda(p) (p (\lambda(x) (\lambda(y) y)))) \\ \text{true: } & (\lambda(x) (\lambda(y) x)) \\ \text{false: } & (\lambda(x) (\lambda(y) y)) \end{aligned}$$

- (a) (4 pt) What is the result of $(\text{op1 } ((\text{data } a) b))$?
- (b) (4 pt) What is the result of $(\text{op2 } ((\text{data } a) b))$?
- (c) (3 pt) What computation do op1 and op2 perform?

Sol.

$$((\text{data } a) b) = (((\lambda(x) (\lambda(y) (\lambda(z) ((z x) y)))) a) b) = (\lambda(z) ((z a) b))$$

$$\begin{aligned} \text{op1: } & (\lambda(p) (p (\lambda(x) (\lambda(y) x)))) = (\lambda(p) (p \text{ true})) \\ \text{op2: } & (\lambda(p) (p (\lambda(x) (\lambda(y) y)))) = (\lambda(p) (p \text{ false})) \end{aligned}$$

- (a) $(\text{op1 } ((\text{data } a) b)) = ((\lambda(p) (p \text{ true})) (\lambda(z) ((z a) b))) =$
 $((\lambda(z) ((z a) b)) \text{ true}) =$
 $((\text{true } a) b) =$
 $((((\lambda(x) (\lambda(y) x)) a) b) =$
 a

(b) $(op2 ((data\ a)\ b)) = ((\lambda(p)\ (p\ false))(\lambda(z)\ ((z\ a)\ b))) =$
 $((\lambda(z)\ ((z\ a)\ b))false) =$
 $((false\ a)\ b) =$
 $(((\lambda(x)\ (\lambda(y)\ y))\ a)\ b) =$
 b

(c) op1 and op2 return the first and second element respectively.