# ALE

ALE solves the problem of mixing effects from different features. As with the function $M(x_1)$, ALE uses the conditional distribution to average over other features, but instead of averaging the predictions directly, it averages ***differences in predictions*** to block the effect of correlated features. The ALE function is defined as follows:

$$\begin{aligned}\text{ALE}(x_1) &= \int_{\min(x_1)}^{x_1} \mathbb{E}\left[\frac{\partial f(X_1, X_2)}{\partial X_1}\Big| X_1 = z_1\right] dz_1 - c_1 \\ &= \underbrace{\int_{\min(x_1)}^{x_1} \int p(x_2|z_1) \frac{\partial f(z_1, x_2)}{\partial z_1} dx_2 dz_1}_{\text{uncentered ALE}} - c_1,\end{aligned}$$

where the constant $c_1$ is chosen such that the resulting ALE values are independent of the point $\min(x_1)$ and have zero mean over the distribution $p(x_1)$.

常量 $c_1$ 的选择是为了使
与 $\min(x_1)$ 点无关，并
在 $p(x_1)$ 上的平均值

The term $\dfrac{\partial f(x_1, x_2)}{\partial x_1}$ is called the ***local effect*** of $x_1$ on $f$. Averaging the local effect over the conditional distribution $p(x_2|x_1)$ allows us to isolate the effect of $x_1$ from the effects of other correlated features avoiding the issue of $M$ plots which directly average the predictor $f$. Finally, note that the local effects are integrated over the range of $x_1$, this corresponds to the ***accumulated*** in ALE. This is done as a means of visualizing the ***global*** effect of the feature by "piecing together" the calculated local effects.

In practice, we calculate the local effects by finite differences so the predictor $f$ need not be differentiable. Thus, to estimate the ALE from data, we compute the following:

$$\widehat{\text{ALE}}(x_1) = \underbrace{\sum_{k=1}^{k(x_1)} \frac{1}{n(k)} \sum_{i:x_1^{(i)} \in N(k)} \left[f(z_k, x_{\setminus 1}^{(i)}) - f(z_{k-1}, x_{\setminus 1}^{(i)})\right]}_{\text{uncentered ALE}} - c_1.$$

$n(k)$ 代表这个区间的
$k(x_1)$ 表示 $x_1$ 落入的

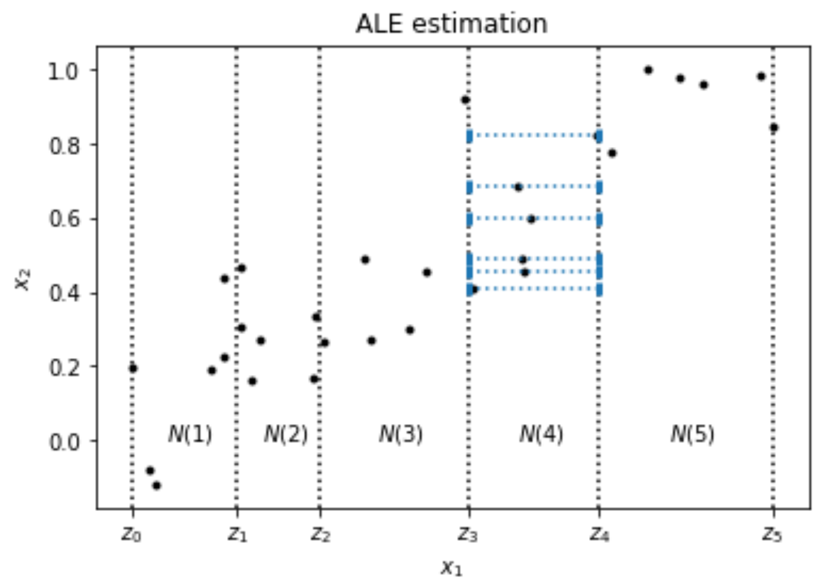
Here $z_0, z_1, \ldots$ is a sufficiently fine grid of the feature $x_1$ (typically quantiles so that each resulting interval contains a similar number of points), $N(k)$ denotes the interval $[z_{k-1}, z_k)$, $n(k)$ denotes the number of points falling into interval $N(k)$ and $k(x_1)$ denotes the index of the interval into which $x_1$ falls into, i.e. $x_1 \in [z_{k(x_1)-1}, z_{k(x_1)})$.

Finally, the notation $f(z_k, x_{\setminus 1}^{(i)})$ means that for instance $i$ we replace $x_1$ with the value of the right interval end-point $z_k$ (likewise for the left interval end-point using $z_{k-1}$), leaving the rest of the features unchanged, and evaluate the difference of predictions at these points.

区间内的 instant 的 fe
$x_1$ 值，都换成区间的
然后计算 difference.

The following plot illustrates the ALE estimation process. We have subdivided the feature range of $x_1$ into $5$ bins with roughly the same number of points indexed by $N(k)$. Focusing on bin $N(4)$, for each point falling into this bin, we replace their $x_1$ feature value by the left and right end-points of the interval, $z_3$ and $z_4$. Then we evaluate the difference of the predictions of these points and calculate the average by dividing by the number of points in this interval $n(4)$. We do this for every interval and sum up (accumulate) the results. Finally, to calculate the constant $c_1$, we subtract the expectation over $p(x_1)$ of the calculated uncentered ALE so that the resulting ALE values have mean zero over the distribution $p(x_1)$.

要计算常量 $c_1$，要减去
非中心的 ALE 的 $p(x_1)$
期望值.（也就是平均
这样 ALE 在 $p(x_1)$ 上
平均值为 0.

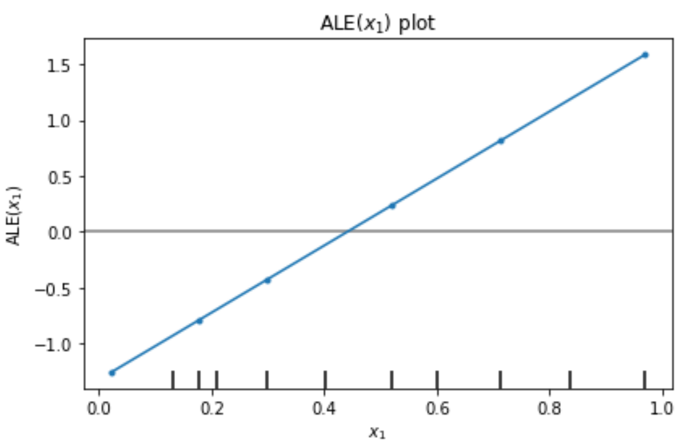


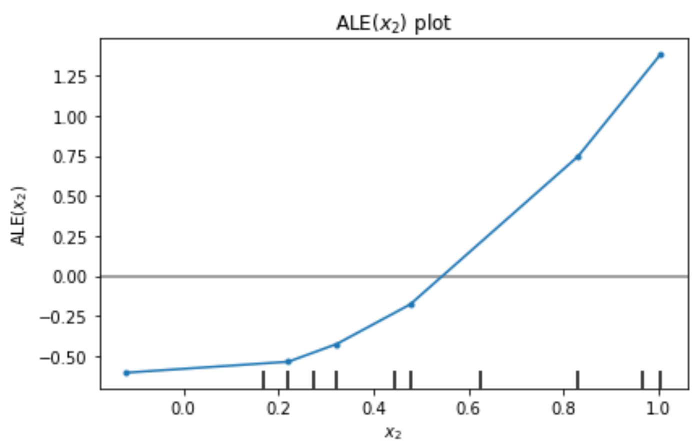

以 bin N(4) 来说，所
个区间的数据，都用
的 feature $x_1$ 值来
计算差值，然后所
求平均值。
对每个区间都这么
加起来。

We show the results of ALE calculation for a model $f(x_1, x_2) = 3x_1 + 2x_2^2$. The resulting plots correctly recover the linear effect of $x_1$ and the quadratic effect of $x_2$ on $f$. Note that the ALE is estimated for each interval edge and linearly interpolated in between, for real applications it is important to have a sufficiently fine grid but also one that has enough points into each interval for accurate estimates. The x-axis also shows feature deciles of the feature to help judge in which parts of the feature space the ALE plot is interpolating more and the estimate might be less trustworthy.

deciles 十分位数。

The value of $\mathrm{ALE}(x_i)$ is the main effect of feature $x_i$ as compared to the average prediction for the data. For example, the value of $\mathrm{ALE}(x_1) = 0.75$ at $x_1 = 0.7$, if we sample data from the joint distribution $p(x_1, x_2)$ (i.e. realistic data points) and $x_1 = 0.7$, then we would expect the first order effect of feature $x_1$ to be $0.75$ higher than the ***average*** first order effect of this feature. Seeing that the $\mathrm{ALE}(x_1)$ plot crosses zero at $x_1 \approx 0.45$, realistic data points with $x_1 \approx 0.45$ will have effect on $f$ similar to the average first order effect of $x_1$. For realistic data points with smaller $x_1$, the effect will become negative with respect to the average effect.




中心化之后的 ALE

x轴到底是什么？

Because the model $f(x_1, x_2) = 3x_1 + 2x_2^2$ is explicit and differentiable, we can calculate the ALE functions analytically which gives us even more insight. The partial derivatives are given by $(3, 4x_2)$. Assuming that the conditional distributions $p(x_2|x_1)$ and $p(x_1|x_2)$ are uniform, the expectations over the conditional distributions are equal to the partial derivatives. Next, we integrate over the range of the features to obtain the ***uncentered*** ALE functions:

$$
\begin{aligned}
\mathrm{ALE}_u(x_1) &= \int_{\min(x_1)}^{x_1} 3dz_1 = 3x_1 - 3\min(x_1) \\
\mathrm{ALE}_u(x_2) &= \int_{\min(x_2)}^{x_2} 4z_2dz_2 = 2x_2^2 - 2\min(x_2)^2.
\end{aligned}
$$

Finally, to obtaine the ALE functions, we center by setting $c_i = \mathbb{E}(\mathrm{ALE}_u(x_i))$ where the expectation is over the marginal distribution $p(x_i)$:

$$
\begin{aligned}
\mathrm{ALE}(x_1) &= 3x_1 - 3\min(x_1) - \mathbb{E}(3x_1 - 3\min(x_1)) = 3x_1 - 3\mathbb{E}(x_1) \\
\mathrm{ALE}(x_2) &= 2x_2^2 - 2\min(x_2)^2 - \mathbb{E}(2x_2^2 - 2\min(x_2)^2) = 2x_2^2 - 2\mathbb{E}(x_2^2).
\end{aligned}
$$

This calculation verifies that the ALE curves are the desired feature effects (linear for $x_1$ and quadratic for $x_2$) relative to the mean feature effects across the dataset. In fact if $f$ is additive in the individual features like our toy model, then the ALE main effects recover the correct additive components (Apley and Zhu (2016)).

Furthermore, for additive models we have the decomposition $f(x) = \mathbb{E}(f(x)) + \sum_{i=1}^{d} \mathrm{ALE}(x_i)$, here the first term which is the average prediction across the dataset $X$ can be thought of as zeroth order effects.