Continuous Time Mean-Variance Portfolio Selection Problem

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Abstract

This thesis is devoted to Markowitz's mean-variance portfolio selection problem in continuous time financial markets, where we aim to minimise the risk of the investment, which is expressed by the variance of the terminal wealth, with a given level of expected return. This thesis consists of an existing literature review and my original extension work.

Stochastic linear-quadratic (LQ) control approach and martingale approach are two main methods in dealing with continuous time mean-variance portfolio selection problem. Half of the thesis is allocated to the review of these approaches. The background and motivation, the development, the current status, and the open questions of both approaches are introduced and studied.

After the literature review, my extension work is done by martingale approach to find the explicit form of optimal portfolio in an incomplete market when the market parameters are random processes. Specifically, the explicit forms of optimal wealth process and optimal portfolio are obtained for an incomplete market when the market parameters are some simple kind of random processes.

Key Words: mean-variance portfolio selection, continuous time, stochastic LQ control, martingale approach, deterministic/random parameter.

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Chapter 1

Introduction

The study of financial portfolio selection theory should date back to Markowitz's Nobel-prize-winning work (1952, 1959). He worked on single-period mean-variance portfolio selection problem, which became the foundation for modern financial portfolio theory. Mean-variance portfolio selection is to allocate the total wealth among a number of assets (risk-free and risky), with the aim of maximising the expected level of return $Ex(T)$ and minimising the level of risk. In Markowitz's framework, he used the variance of the final wealth $Var(x(T))$ as the measure of risk, then the problem became how to minimise portfolio's variance subject to a prescribed level of return. Given the expected level of return z, the portfolio which achieves the minimum variance is said to be *optimal*; and the pair $(Var(x(T)), z)$ is called a *variance minimising frontier*, where z runs over the whole real axis. If this portfolio also achieves the maximum expected level of return among the portfolios which have the same variance, it is said to be *efficient*. Then we call the pair of the minimum variance and the maximum expected level of return the *efficient frontier*. Apparently, efficient frontier is a part of variance minimising frontier. Apart from Markowitz's work, analytical result for single-period problem was also derived by Merton (1973) when short selling is allowed.

Since Markowitz's pioneering work, this subject has attracted a huge amount of researches on itself. There has been significant development from single-period case to multi-period discrete-time case (see: Smith (1967); Chen, Jen and Zionts (1971); Samuelson (1969); etc.,) and continuous time case (see: Merton (1969); Cox and Huang (1989); Duffie and Richardson (1991); Karatzas and Shreve (1987); etc.,) during the last decades. However, when studying the multi-period case, instead of using mean-variance model, which includes the pair $(Ex(T), Var(x(T)))$, the expected utility of the terminal wealth $E(U(x(T)))$ is used, where U is the utility function. Then the problem becomes maximising $E(U(x(T)))$. When using this approach, the tradeoff between expected return and risk "makes an investment decision much less intuitive" (Zhou and Li, 2000). Furthermore, we are facing several practical difficulties when using expected utility approach. Because "few if any investors know their utility functions; nor do the functions which financial engineers and financial economists find analytically convenient necessarily represent a particular investor's attitude towards risk and return" (Markowitz, 2004).

Having realised these, mean-variance model seems to be a better and more reasonable way to deal with portfolio selection problem, especially when the market is less volatile. However, compared with the dominating results which were obtained by maximising the expected utility functions of the terminal wealth, mean-variance portfolio selection problem for multi-period model has not been studied further and developed until recently (Li and Ng, 2000). One of the difficulties comes from the term $[E(x(T))]^2$ arising from $Var(x(T))$. Indeed, there has been several studies on this topic (see: Samuelson (1986); Hakansson (1971); Grauer and Hakansson (1993); and Pliska (1997)); however, neither analytical nor efficient numerical results are obtained. Li and Ng's (2000) work can be viewed as a breakthrough. They extended Markowitz's single-period analytical result to multi-period, discrete time portfolio selection. They used a so-called embedding technique to combine $Ex(T)$ and $Var(x(T))$ to be a single objective $J(u(\cdot)) = -Ex(T) + \mu Var(x(T))$, where μ can be any positive number. And they obtained the optimal portfolio and efficient frontier explicitly, which extended Markowitz's work into multi-period.

Extension to continuous time became more complicated, since it can not be simply seen as the limit of multi-period model by dividing the investment period again and again and make it go to infinitesimal. Nevertheless, research on continuous-time Markowitz's mean-variance model still became more and more active and many elegant results were obtained. Up to now, two main methods have been adopted to solve these problems. They are stochastic LQ control approach and martingale approach respectively.

Zhou and Li (2000) introduced stochastic linear-quadratic (LQ) control, for the first time, as a framework to deal with continuous time mean-variance problem. Then we can use the language of stochastic control to formulate this problem, and we can apply the results from stochastic control theory to have deeper understanding of this problem. Therefore, this portfolio selection problem was generalised to more complicated situations. For example, Lim and Zhou (2002) studied the same problem with random interest rate, appreciation rate and volatility rate. As will be discussed later, the generalisation to random parameters is worth studying and challenging (Lim and Zhou, 2002). Apart from these, Zhou and Yin (2003) featured assets in a regime switching market; and Li, Zhou and Lim (2001) studied the problem where a short selling constraint is added.

Martingale approach is another important method in solving mean-variance portfolio selection problem; and compared with LQ approach, martingale approach appears more natural to deal with these problems. This method was first used to solve portfolio optimisation problem (under expected utility framework) by Harrison and Kreps (1979) and Pliska (1982, 1986) with the use of risk neutral (equivalent martingale) probability measure. Applying this approach to mean-variance framework, the problem can be divided into two sub-problems. Firstly, we should find the optimal replicable terminal wealth, which is expressed by a random variable. Then the optimal portfolio $\pi(\cdot)$ can be determined to be the portfolio which replicates the optimal terminal wealth. Compared with the relatively straightforward way to formulate the problem, the second part of the problem is difficult to solve before LQ framework was introduced. After the significant progress of the development of LQ control approach, Bielecki, Jin, Pliska and Zhou (2005) managed to solve the continuous mean-variance problem by martingale approach. The key step for handling part two is the convex optimisation technique which was used in the LQ control framework. The same as by LQ control approach, efficient portfolio and efficient frontier are given explicitly when market parameters are deterministic.

However, in view of mean-variance framework, all these papers, regardless the deterministic parameters or random parameters, assumed that the market is complete. Though the extension to incomplete markets is by no means routine and can be more involved, it is worth studying. Unfortunately, we have not seen many research and interesting results of mean-variance portfolio selection problem in incomplete markets. Indeed, there are many literature studying the continuous time portfolio selection in incomplete markets, but most of which were using expected utility framework (see: Cvitanic and Karatzas (1992); He and Pearson (1991); Karatzas, Lehoczky, Shreve and Xu (1991); Schachermayer (2001)). They "do not at all cover the mean-variance models for the main reason that the assumptions typically imposed on a utility function are not satisfied by a mean-variance model" (Jin and Zhou, 2005). To my best knowledge, the only main papers that deal with Markowitz's mean-variance problem in incomplete markets are Lim (2004), Jin (2004), and Jin and Zhou (2005). Lim (2004) studied the unconstraint mean-variance problem with random parameters in incomplete markets from the perspective of LQ control problem, with the focus on the stochastic Riccati equation. Jin (2004), and Jin and Zhou (2005) applied martingale approach to solve mean-variance problem in an incomplete market, they studied the following four cases respectively: portfolios are unconstrained, shorting is prohibited, bankruptcy is prohibited, and both short-selling and bankruptcy are prohibited.

In this thesis, I will concentrate on the continuous time mean-variance portfolio selection problem (expected utility formulation will not be discussed in details). After a review of the existing literature on both stochastic LQ control approach and martingale approach, I will concentrate on mean-variance model in an incomplete market, which has not attracted many researches until now.

In view of all the existing result, efficient portfolio and frontier are only obtained explicitly with deterministic parameters; and nobody has solved the same problem with random parameters. It is desirable to know the explicit forms of them with market parameters being random processes. Thus I will apply martingale approach and extend their existing result to the case when the some randomness is added to the appreciation rate and stochastic volatility (but not random in general sense), and try to find the explicit solution of the optimal portfolio. Specifically, I divide the whole period $[0, T]$ into two parts: $[0, \frac{T}{2}]$ and $[\frac{T}{2}, T]$, the volatility and appreciation rates in these two parts are two different values; but in contrast to the deterministic case, they are $\mathcal{F}_{\frac{T}{2}}$-adapted **random variables** in the second half. After adding such randomness to the parameters, I concentrate on $[\frac{T}{2}, T]$ first, with $t = \frac{T}{2}$ being the new 'initial' point, I use the same method as Jin and Zhou (2005), obtain the optimal terminal wealth and hence the minimum value of the objective function, which are conditional on $\mathcal{F}_{\frac{T}{2}}$. Then I focus on $[0, \frac{T}{2}]$, take the expectation of such conditional minimum value as the new objective function in $[0, \frac{T}{2}]$ and try to find the optimal portfoli. General random process can be viewed as the limit of this so-called two-period random process. Therefore, when dealing with general case, we can continue dividing each interval into two and repeat the same technique to each sub-interval for infinitely many times, and take the limit of it. I will describe this idea in details in Chapter 5.

The rest of the thesis is organised as follows. In Chapter 2 the continuous-time mean-variance portfolio selection model is formulated. In Chapter 3 we review the related work under stochastic LQ control framework on both complete and incomplete markets. Chapter 4 is allocated for the related researches on the martingale approach. Chapter 5 is my original extension work, which gives the analytical solution explicitly of the portfolio selection problem by martingale approach after adding some randomness to the volatility rate and appreciation rate. Finally, Chapter 6 concludes the whole thesis.

Chapter 2

The Model

2.1 The Market

In this thesis T is a fixed terminal time and $(\Omega, \mathcal{F}, P, \mathcal{F}_t)$ is a filtered complete probability space, on which we define a standard $\mathcal{F}_t$- adapted n-dimensional Brownian motion $W(t) \equiv (W^1(t), W^2(t), ..., W^n(t))'$, with $W(0) = 0$. We assume that $\mathcal{F}_t$ is generated by this standard Brownian motion, i.e. $\mathcal{F}_t = \sigma\{W(s) : s \leqslant t\}$.

$L^2_{\mathcal{F}}(0, T; \mathbb{R}^d)$ is denoted as the set of all $\mathbb{R}^d$-valued, $\mathcal{F}_t$-progressively measurable stochastic processes $f(\cdot) = \{f(t) : 0 \leqslant t \leqslant T\}$, with the norm $\|f(\cdot)\|_{L^2_{\mathcal{F}}(0,T;\mathbb{R}^d)} := (E \int_0^T |f(t)|^2 dt)^{\frac{1}{2}} < +\infty$. $L^\infty_{\mathcal{F}}(0, T; \mathbb{R}^d)$ is denoted as the set of all uniformly bounded stochastic processes. And $L^2_{\mathcal{F}_T}(\Omega; \mathbb{R}^d)$ is denoted as the set of all $\mathbb{R}^d$-valued, $\mathcal{F}_t$-measurable random variables η such that $\|\eta\|_{L^2_{\mathcal{F}_T}(\Omega;\mathbb{R}^d)} := (E|\eta|^2)^{\frac{1}{2}} < +\infty$.

We use the following additional notation:

- M' is the transpose of any vector or matrix M;
- $|M|^2 = \sum_{i,j} m_{ij}^2$ for any vector or matrix $M = (m_{ij})$;
- $\alpha^+ = \max\{\alpha, 0\}$ for any real number α;
- $\mathbb{R}^d_+$ is the set of d-dimensional vectors with nonnegative constraint.

Suppose we have a market in which there are $m + 1$ assets traded continuously. One of the assets is risk-free asset with price process $S_0(t)$ satisfying the following ordinary differential equation:

$$\begin{cases} dS_0(t) = r(t)S_0(t)dt, t \in [0, T] \\ S_0(0) = s_0 > 0 \end{cases} \tag{2.1}$$

where the interest rate $r(t)$ is a uniformly bounded, $\mathcal{F}_t$- adapted and scalar-valued process. The other m assets are risky assets with the price processes $S_i(t), i = 1, \cdots, m$, which satisfy the following stochastic differential equations:

$$\begin{cases} dS_i(t) = S_i(t)[a_i(t)dt + \sum_{j=1}^{n} \sigma_{ij}(t)dW^j(t)], t \in [0,T], i = 1,2,\cdots,m \\ S_i(0) = s_i > 0 \end{cases} \tag{2.2}$$

where $a_i(t)$ is the appreciation rate and $\sigma_{ij}(t)$ is the volatility rate. Also they are both uniformly bounded, $\mathcal{F}_t$- adapted and scalar-valued processes.

If we consider an agent whose total wealth at time t is $x(t)$, and the amount of money invested in the ith $(i = 1,\cdots m)$ stock is $\pi_i(t) = N_i(t)S_i(t)$, then we have

$$x(t) = \sum_{i=0}^{m} \pi_i(t) = \sum_{i=0}^{m} N_i(t)S_i(t) \tag{2.3}$$

We need the following basic assumption for the later discussion:

Definition 2.1 *The portfolio $\pi(t)$ is called self-financing if there is no withdrawal or injection of funds, and all wealth changes only arise from changes from asset prices. Mathematically, $dx(t) = \sum_{i=0}^{m} N_i(t)dS_i(t)$.*

If the agent's strategy is *self-financing*, then differentiate (2.3) with respect to t and substitute (2.1) and (2.2) into it, we have the wealth equation as follows:

$$\begin{cases} dx(t) = [r(t)x(t) + \sum_{i=1}^{m}(a_i(t) - r(t))\pi_i(t)]dt + \sum_{i=1}^{m}\sum_{j=1}^{n} \pi_i(t)\sigma_{ij}(t)dW^j(t) \\ x(0) = x_0 \end{cases} \tag{2.4}$$

Denote

$$\begin{cases} \sigma(t) := (\sigma_{ij}(t))_{m\times n} \\ B(t) := (b_1(t),\cdots,b_m(t))' := (a_1(t) - r(t),\cdots,a_m(t) - r(t))' \\ \pi(t) := (\pi_1(t),\cdots,\pi_m(t))' \end{cases}$$

Then we can write the vector form of (2.4) as:

$$\begin{cases} dx(t) = [r(t)x(t) + \pi(t)'B(t)]dt + \pi(t)'\sigma(t)dW(t) \\ x(0) = x_0 \end{cases} \tag{2.5}$$

Before further discussion, some basic definitions should be introduced.

Definition 2.2 *$\pi(t)$ is called an admissible portfolio strategy if it is self-financing, measurable, $\mathcal{F}_t$-adapted and $\pi(\cdot) \in L^2_{\mathcal{F}}(0,T;\mathbb{R}^m)$.*
A wealth-portfolio pair $(x(\cdot),\pi(\cdot))$ is called admissible if it satisfies (2.5).

Definition 2.3 *An admissible portfolio is called an arbitrage opportunity on $[0,T]$ if there is an initial wealth $x_0 \leqslant 0$ and a time $t \in [0,T]$, such that the corresponding wealth process $x(\cdot)$ satisfies $P(x(t) \geqslant 0) = 1$ and $P(x(t) > 0) > 0$.*
And a market is called arbitrage-free on $[0,T]$ if there is no arbitrage opportunity on $[0,T]$.

Definition 2.4 *An $\mathcal{F}_t$-measurable random variable $\xi \in L^2_{\mathcal{F}_T}(\Omega, \mathbb{R})$ is called a European Contingent Claim.*
A contingent claim ξ is called replicable if there is an initial wealth x_0 and an admissible wealth-portfolio pair $(x(\cdot), \pi(\cdot))$ satisfying (2.5), such that the corresponding wealth process $x(\cdot)$ satisfies $x(T) = \xi$.
A market is called complete if any contingent claim $\xi \in L^2_{\mathcal{F}_T}(\Omega, \mathbb{R})$ is replicable.

We should mention that in our market model, the number of risky assets is normally not more than the dimension of the underlying Brownian motion (i.e. $m \leqslant n$); otherwise there may be arbitrage opportunity in the market. At this stage, we do not specify the conditions under which the equality holds.

Finally, throughout this article, we assume that there is $\theta(\cdot) \in L^\infty_{\mathcal{F}}(0, T; \mathbb{R}^n)$ such that $\sigma(t)\theta(t) = B(t)$ a.s., a.e.$t \in [0, T]$.

The market is *complete* when $\sigma(t)'\sigma(t)$ is uniformly positive definite (i.e. $\exists \delta > 0$, such that $\sigma(t)'\sigma(t) \geqslant \delta I_n$), under this condition $\sigma(t)$ is invertible, and therefore there is a unique $\dot{\theta}(t)$ satisfying $\sigma(t)\theta(t) = B(t)$ ($\dot{\theta}(t) = \sigma(t)^{-1}B(t)$), which is called the *pricing kernel.* If $\sigma(t)'\sigma(t)$ is not uniformly positive definite, then $\sigma(t)$ may not be invertible, and under which the market is *incomplete.* The portfolio selection problem in incomplete markets is more involved, and it is our main concern in this dissertation.

2.2 Markowitz's Mean-Variance Model

If we fix the expected level of return $Ex(T) = z$ and the initial wealth x_0, we can formulate the general continuous-time Markowitz's mean-variance portfolio selection problem as

$$Minimise \quad Var(x(T)) = E[x(T)^2] - z^2$$

$$s.t \begin{cases} Ex(T) = z \\ (x(\cdot), \pi(\cdot)) \text{ satisfies equation (2.5)} \\ (x(\cdot), \pi(\cdot)) \text{ is an admissible pair} \end{cases} \tag{2.6}$$

Denote $(x^*(t), \pi^*(t))$ is the optimal portfolio for this problem corresponding to a given z, which is called an efficient portfolio. Therefore, there is no other portfolio $\pi(\cdot)$, such that $Ex^{\pi(\cdot)}(T) \geqslant Ex^{\pi^*(\cdot)}(T)$ and $Var(x^{\pi(\cdot)}(T)) \leqslant Var(x^{\pi^*(\cdot)}(T))$, and at least one of the inequalities holds strictly. Furthermore, we call the set of all points $(Var(x^*(T)), z = Ex^*(t))$ the efficient frontier.

Chapter 3

Stochastic LQ Control Approach

Faithful study of continuous time Markowitz's mean-variance portfolio selection problem was originally done via stochastic LQ control approach. With the additional constraints added, the problem became more complicated, and after which martingale approach was adopted again to study this problem. Therefore, although I am not using stochastic LQ control approach in later chapters, it will be helpful to review the development, main ideas and key results under stochastic LQ control framework.

3.1 Complete market case

Zhou and Li (2000) proposed and studied, for the first time, the continuous time mean-variance portfolio selection problem in complete markets under the stochastic LQ control framework. After their initial work on this theory, some generalisations have been done. For example, Li, Zhou and Lim (2001) adjusted the model by introducing a short selling constraint; and Lim and Zhou (2002) extended their original work to the problem with random coefficients, which is more delicate and involved than the deterministic case. I will give a brief review of the work they have done.

3.1.1 Unconstraint portfolio model

In the paper by Zhou and Li (2000), with the assumption that the parameters are deterministic and portfolio is unconstraint, they consider the mean-variance problem as an LQ control problem, and used the embedding technique, which is similar to Li and Ng (2000), to transform the original problem into an auxiliary one. They solved the auxiliary problem by standard LQ control theory explicitly and then determined the optimal solution of the original problem. In this section, I will briefly review the work they have done.

Consider the original problem (2.6), we aim to minimise $Var(x(T)) = E[x(T)^2] - z^2$,which contains quadratic form. This is, though not the same, similar to the cost functional in the LQ problem, which reminds us to adjust this problem to make it solvable in LQ control framework. The idea in Zhou and Li's paper is inherited from Li and Ng (2000) for discrete time model. With a Lagrange multiplier, they converted the original problem into the following auxiliary problem:

$$Minimise \quad Var(x(T)) = E[x(T)^2] - 2\lambda Ex(T)$$

$$s.t. \begin{cases} (x(\cdot), \pi(\cdot)) \text{ satisfies equation (2.5)} \\ (x(\cdot), \pi(\cdot)) \text{ is an admissible pair} \end{cases} \tag{3.1}$$

We can see problem (3.1) is equivalent to the following problem with $y(t) = x(t) - \lambda$:

$$Minimise \quad E[y(T)^2]$$

$$s.t. \begin{cases} dy(t) = [r(t)y(t) + \lambda r(t) + \pi(t)'B(t)]dt + \pi(t)'\sigma(t)dW(t) \\ y(0) = x_0 - \lambda \\ (y(\cdot), \pi(\cdot)) \text{ is an admissible pair} \end{cases} \tag{3.2}$$

And thus this problem becomes solvable under the standard LQ control framework. With the routine procedure, write down the corresponding stochastic Riccati equation, the optimal portfolio $\pi^*(t)$ is obtained as (equation 5.12: Zhou and Li, 2000):

$$\pi^*(t) = -(\sigma(t)\sigma(t)')^{-1}B(t)(x^*(t) - \lambda e^{-\int_t^T r(s)ds}) \tag{3.3}$$

and λ can be found, via $Ex^*_\lambda(T) = z$, to be $\lambda(z) = \frac{z - x_0 e^{\int_0^T (r(s) - \theta(s)'\theta(s))ds}}{1 - e^{-\int_0^T \theta(s)'\theta(s)ds}}$, where θ is defined in the same way as Chapter 2.

Finally, the minimum variance can be gained as:

$$Var(x^*(T)) = (x_0 - \lambda(z)e^{-\int_0^T r(s)ds})^2 e^{\int_0^T 2r(s)ds}(e^{\int_0^T \theta(s)'\theta(s)ds} - e^{-\int_0^T \theta(s)'\theta(s)ds}) \tag{3.4}$$

Denote $\rho(t) = \exp\{-\int_0^t (r + \frac{1}{2}\theta'\theta)ds - \int_0^t \theta' dW(s)\}$, substitute $\lambda(z)$ into (3.4), then:

$$Var(x^*(T)) = \frac{e^{-\int_0^T \rho(s)ds}}{1 - e^{-\int_0^T \rho(s)ds}}(Ex^*(T) - x_0 e^{\int_0^T r(s)ds})^2, \quad z = Ex^*(T) \geqslant x_0 e^{\int_0^T r(s)ds} \tag{3.5}$$

then we get the efficient frontier as the pair $(Var(x^*(T)), Ex^*(T))$ (Theorem 6.1: Zhou and Li, 2000).

3.1.2 With the addition of short selling prohibition

Zhou and Li's work solved the mean-variance problem under the standard stochastic LQ framework, which requires the portfolio to be unconstraint. However, it is meaningful to know what the optimal portfolio is if there is a short selling constraint. Indeed there have been many researches on this problem, but they mainly used expected utility method. Li, Zhou and Lim (2001) studied such mean-variance problem under LQ control framework and obtained the optimal portfolio and the efficient frontier explicitly. We should notice that with the short selling prohibition, the problem became more difficult than the initial one, because the stochastic Riccati equation may not satisfy the control constraint. In order to cope with this, they considered the HJB equation and conjectured a continuous solution of the HJB equation via two Riccati equations, and then showed that it is the viscosity solution of the equation. Then the optimal portfolio and the efficient frontier are derived explicitly via viscosity verification theorem. Now I will introduce the main idea and steps in their paper.

When short-selling is prohibited, problem (2.6) becomes:

$$Minimise \quad \frac{1}{2}E[(x(T)-z)^2]$$

$$s.t. \begin{cases} Ex(T) = z \\ dx(t) = [r(t)x(t) + \pi(t)'B(t)]dt + \pi(t)'\sigma(t)dW(t) \\ x(0) = x_0 \\ \pi(\cdot) \in L^2_{\mathcal{F}}(0,T;\mathbb{R}^m_+) \end{cases} \tag{3.6}$$

Similarly to the previous section, with a Lagrange multiplier μ, problem (3.6) can be converted to:

$$Minimise \quad \frac{1}{2}E[((x(T)-z)+\mu)^2]$$

$$s.t. \begin{cases} dx(t) = [r(t)x(t) + \pi(t)'B(t)]dt + \pi(t)'\sigma(t)dW(t) \\ x(0) = x_0 \\ \pi(\cdot) \in L^2_{\mathcal{F}}(0,T;\mathbb{R}^m_+) \end{cases} \tag{3.7}$$

From this formulation, we can see the control is constrained. Therefore "the conventional 'completion of square' approach to the unconstrained LQ control problem, which involves the Riccati equation, does not apply" (Li, Zhou and Lim, 2001). They instead studied a general constraint LQ control problem by considering the corresponding HJB equation, and problem (3.6) can be viewed as a special case of it.

After solving the general constrained stochastic LQ control problem (see section 3: Li, Zhou and Lim, 2001), the optimal portfolio strategy and efficient frontier can be obtained by simply applying the general results. To be specific, the optimal solution of (3.7) is obtained first, and then by analysing the relation between the variance and the expected value of terminal wealth, the optimal solution for problem (3.6) is obtained as follows (Theorem 4.2: Li, Zhou and Lim, 2001):

$$\pi^*(t) = \begin{cases} -(\sigma(t)\sigma(t)')^{-1}[\bar{u}(t)+B(t)][X-(z-\mu^*)e^{-\int_t^T r(s)ds}], & X \leqslant (z-\mu^*)e^{-\int_t^T r(s)ds} \\ 0, & X \leqslant (z-\mu^*)e^{-\int_t^T r(s)ds} \end{cases}$$

and

$$Var(x(T)) = \frac{(z - x_0 e^{\int_0^T r(s)ds})^2}{e^{\int_0^T \bar{\theta}(s)'\bar{\theta}(s)ds} - 1}$$

where $\mu^* = \frac{(z-x_0 e^{\int_0^T r(s)ds})^2}{1-e^{\int_0^T \bar{\theta}(s)'\bar{\theta}(s)ds}}$, and $\bar{u}(t)$ and $\bar{\theta}(t)$ are determined by

$$\bar{u}(t) = argmin_{u(t)\in[0,\infty)^m} \frac{1}{2}|\sigma(t)^{-1}u(t) + \sigma(t)^{-1}B(t)|^2, \quad \bar{\theta}(t) = \sigma(t)^{-1}\bar{u}(t) + \sigma(t)^{-1}B(t)$$

3.1.3 With the permission of random parameters

A natural extension is to study the case in which all the market coefficients are random processes. It is worth studying mean-variance portfolio selection problem with random parameters. Lim (2004) summarised several reasons for doing so: first it enables us to use more accurate market models for asset price dynamics; the second reason comes from the parameter estimate from market data, the models used for portfolio selection may have random parameters. Lim and Zhou (2002) studied such problem by LQ control and BSDE techniques.

Compared with the deterministic parameters case, the same problem with random parameters (interest rate, appreciation rate and volatility) becomes more difficult. According to Lim and Zhou (2002), the previous embedding technique is no longer applicable. Furthermore, when applying dynamic programming, since the HJB equation becomes a BSDE with random coefficients, it is rather complicated to deal with. In addition, the LQ approach as done in Zhou and Li (2000) is also difficult to handle, because the corresponding stochastic Riccati equation, which is just a backward ODE in the deterministic parameter case, will become "a fully nonlinear, singular backward stochastic differential equation for which the usual assumptions are not satisfied" (Lim and Zhou, 2002), and the SRE may not be solvable in the general sense. To handle these difficulties, Lim and Zhou (2002) proved the global solvability of the stochastic Riccati equation, and derived the optimal strategy analytical by solving this equation.

Similarly to the deterministic case, the original problem (2.6) with the level of expected return constrained, can be converted into the auxiliary problem without such constraint (same as problem 3.2).

Consider the following BSDEs (omit the argument t):

$$\begin{cases} dP = -\{[2r - B'(\sigma\sigma')^{-1}B]P - 2B'\sigma^{-1\prime}\Lambda - \frac{1}{P}\Lambda'\Lambda\}dt + \Lambda' dW \\ P(T) = 1 \\ P(t) > 0, \forall t \in [0,T] \end{cases} \tag{3.8}$$

$$\begin{cases} dh = [rh + B'(\sigma^{-1})'\eta]dt + \eta' dW \\ h(T) = \lambda \end{cases} \tag{3.9}$$

and the following forward SDE:

$$\begin{cases} dx = \{[r - B'(\sigma\sigma')^{-1}(B + \sigma\frac{\Lambda}{P})]x - B'(\sigma\sigma')^{-1}[(B + \sigma\frac{\Lambda}{P})h + \sigma\eta]\}dt \\ +\{[\sigma^{-1}B + \frac{\Lambda}{P}](x+h) + \eta\}' dW \\ x(0) = x_0 \end{cases} \tag{3.10}$$

It can be seen that if the coefficients are deterministic, then we have $\Lambda = 0$ and $\eta = 0$, so the stochastic Riccati equation (3.8) becomes an ODE and $h(t) = \lambda e^{-\int_t^T r(s)ds}$ (i.e. $h(t)$ becomes the discounted factor). And (3.10) becomes a linear SDE with bounded coefficients. Under this condition, the problem is reduced to the same as in section 2.1.1.

While with the random coefficients, the solution of (3.8) is the pair $(P, \Lambda) \in L^\infty_{\mathcal{F}}(\Omega, C(0,T)) \times L^2_{\mathcal{F}}(0, T; \mathbb{R}^m)$. When the coefficients become random, Yong and Zhou (1999) proved that (3.9) admits a unique solution pair (h, η); however, as stated by Lim and Zhou (2002), the existence and uniqueness of the solutions for (3.8) and (3.10) will be much more complicated. This is the vital difference between the deterministic case, and the breakthrough of Lim and Zhou (2002) is that they have proved the existence and uniqueness of equation (3.8), and showed that this is sufficient to the solvability of the LQ control problem. In other words, the original problem is solvable if the stochastic Riccati equation (3.8) has a unique solution.

When $B(\cdot)$ and $\sigma(\cdot)$ are not zero at the same time, Lim and Zhou (2002) derived the unique efficient portfolio corresponding to the expected level of return z as follows:

$$\pi^*(t) = -(\sigma(t)\sigma(t)')^{-1}[(B(t) + \sigma(t)\frac{\Lambda(t)}{P(t)})(x^*(t) - \tilde{\lambda}h(t)) - \tilde{\lambda}\sigma(t)\eta(t)] \tag{3.11}$$

where

$$\tilde{\lambda} = \frac{z - P(0)h(0)x_0}{1 - P(0)h(0)^2}$$

and the efficient frontier is given by:

$$Var(x^*(T)) = \frac{P(0)h(0)^2}{1 - P(0)h(0)^2}(z - \frac{x_0}{h(0)})^2, \quad z \geqslant \frac{x_0}{h(0)} \tag{3.12}$$

3.2 Incomplete market case

For many reasons the market can be incomplete, and as a natural extension, portfolio selection problem in incomplete markets should be studied. However, compared to the dominating study of expected utility framework, there are very few literature which deal with mean-variance portfolio selection in an incomplete market. To my best knowledge, the only three papers solving such problem are Lim (2004), Jin (2004), and Jin and Zhou (2005), where the first one used stochastic LQ control framework and the other two solved it by martingale approach. In this section, I will concentrate on Lim's work, review his main idea and key steps. Martingale approach will be discussed in the next Chapter.

Incompleteness of the market implies the number of risky assets is fewer than the number of driving Brownian motion ($m < n$). Lim characterised this fact by a slightly different idea. He introduced

$$\bar{W}(t) = (W(t)', B(t)')' = (W^1(t), \cdots, W^m(t), V^1(t), \cdots, V^d(t))'$$

where $V(t)$ is used to model the incompleteness. In Lim's paper, $\sigma_{ij}(\cdot)$ are assumed to be an $m \times m$ matrix with the property $\sigma(t)\sigma(t)' > \delta I$ for some $\delta > 0$, which means $\sigma(t)$ is invertible; then define $\theta(t) = \sigma(t)^{-1}B(t)'$. note that the $\sigma(t)$, $\theta(t)$ and $B(t)$ are slightly different from the ones we defined in Chapter 2 since the incompleteness is not expressed by them.

Then consider the following BSDEs (omit the argument t):

$$\begin{cases} dP = [(-2r + |\theta|^2)P + 2\theta'\Lambda_1 + \frac{1}{P}\Lambda_1'\Lambda_1]dt + \Lambda_1' dW(t) + \Lambda_2' dV(t) \\ P(T) = 1 \\ P(t) > 0, \quad \forall t \in [0, T] \end{cases} \tag{3.13}$$

$$\begin{cases} dh = [rh + \theta_1'\eta_1 - \frac{\Lambda_2'}{P}\eta_2]dt + \eta_1' dW(t) + \eta_2' dV(t) \\ h(T) = \lambda \end{cases} \tag{3.14}$$

where k is the Lagrange multiplier defined in (3.1).
We can see that if $d = 0$, equation (3.13) becomes (3.8) and equation (3.14) becomes (3.9), the problem will be reduced to complete market case (see section 3.1.3). Assume the solution of the SRE (3.13) is the pair $(P, \Lambda) \in L^\infty_{\mathcal{F}}(\Omega, C(0, T; \mathbb{R})) \times L^2_{\mathcal{F}}(0, T; \mathbb{R}^{m+d})$; and the solution of (3.14) is a pair $(h, \eta) \in L^2_{\mathcal{F}}(\Omega, C(0, T; \mathbb{R})) \times L^2_{\mathcal{F}}(0, T; \mathbb{R}^{m+d})$.

Lim showed that the original problem can be solved if (3.13) and (3.14) are solvable. Then it suffices to prove the existence and uniqueness of these equations. It is also shown that the linear BSDE (3.14) depends on the solution of (3.13) through the term $-\frac{\Lambda_2(t)}{P(t)}$. Therefore, the key issue is the existence and uniqueness of the solution of (3.13). Similar to the complete market case, since $\frac{1}{P}\Lambda_1'\Lambda$ is neither Lipschitz continuous nor linear growth, the standard SDE results is not applicable. Lim proved the existence and uniqueness of (3.13) (see section 5: Lim, 2004) with an alternative approach. Once this is done, the optimal portfolio and the efficient frontier of the mean-variance portfolio selection problem can be obtained and the optimal solutions are expressed by the solutions of these BSDEs (see Theorem 6.3: Lim, 2004).

However, Lim only solved the problem when portfolio is unconstrained, while the same problem with constrained portfolio is difficult to solve under stochastic LQ control framework because "such optimal control problem with a state constraint is extremely difficult" (Jin, 2004). Therefore, martingale approach is applied with problems with constraints, which will be discussed in the next chapter.

Chapter 4

Martingale Approach

Martingale approach is the one I will use intensively in this thesis, thus it is of great importance to introduce the essential idea of this method. With this aim, I will review the development of this approach and pay special attention to its application in solving continuous time mean-variance portfolio selection problem, its application under expected utility framework will be introduced briefly.

4.1 Development and general idea

Using the equivalent martingale measures and martingale representation theorem as a tool to study portfolio selection problem began with Harrison and Kreps (1979). Shortly after that, Pliska (1982, 1986) applied martingale methods, for the first time, in solving portfolio selection problem under the expected utility maximisation framework, with Cox and Huang (1986, 1989), Kazatzas, Lehoczky, and Shreve (1987) followed.

Pliska (1982) solved discrete time stochastic control problem by martingale method. Thereafter martingale method was applied in solving continuous time portfolio optimisation problem in complete markets, see Karatzas (1989) with unconstrained portfolio and Korn and Trautmann (1995) with constrained portfolio.

Also there have been many works dealing with continuous portfolio selection problems in incomplete markets. Pages (1987) was the first person to use martingale approach in incomplete markets, he studied a Brownian motion model where the number of stocks was fewer than the dimension of the driving Brownian motion. Thereafter there have been a number of related researches. For example, Karatzas, Lehoczky, Shreve and Xu (1991) uses local martingales rather than martingales to characterise the incompleteness in continuous time models. Schachermayer (2001) considered optimal investment in incomplete markets when wealth may become negative.

All the above work, among many others, were in the expected utility framework, and such results do not cover mean-variance models. However, the main idea of martingale approach for both utility analysis and mean-variance portfolio selection problem is essentially the same. The key idea of martingale approach on utility analysis is to decompose the problem into three stages. Firstly, martingale theory should be used to characterise the set of attainable terminal wealth, which is presented by a random variable. Secondly, convex analysis should be used to find the attainable terminal wealth maximising the expected utility; and finally we should determine the trading strategy which generates the optimal attainable wealth. While in the mean-variance problem, instead of finding the terminal wealth which maximise the expected utility, we should find one to minimise the variance of the terminal wealth.

In this thesis, I am focusing on martingale approach applied to the mean-variance problem, and my extension work in the next chapter will be directly based on Jin (2004), and Jin and Zhou (2005), which are both under mean-variance framework. Therefore I will not go into further details of the expected utility framework.

4.2 Application to mean-variance portfolio selection

4.2.1 Complete market case

Indeed continuous time mean-variance portfolio selection problem has been deeply and widely solved under stochastic LQ control framework (see the previous Chapter); however, the results are obtained with bankruptcy allowed. The reason for the possibility of bankruptcy is that under LQ control framework, instead of proportions of wealth which is conventionally adopted, the amount of money invested on each asset is used to describe portfolio strategy.

Bielecki, Jin, Pliska and Zhou (2005) studied the continuous time mean-variance problem where all the coefficients are allowed to be random processes and the bankruptcy is prohibited (i.e. $x(t) \geqslant 0$, $t \in [0, T]$). They solved the problem by martingale approach, explicit forms of optimal portfolio and efficient frontier are provided when the market coefficients are deterministic. In this section, I will give a short review of solving continuous time mean-variance portfolio selection problem in complete markets.

In this case, though mean-variance framework, instead of expected utility, is considered, the essential idea of the martingale approach is the same as before. Specifically, we can write the general scheme of this approach as:

1. Find the optimal terminal wealth $x^*(T)$, which is a random variable, by solving a static optimisation problem;

2. Find the portfolio $\pi^*(t)$ which replicate $x^*(T)$;

3. Then $\pi^*(t)$ is an optimal portfolio.

The general market model is defined in Chapter 2.
In addition, denote $\alpha(t) = \exp(-\int_0^t \frac{1}{2}\theta(s)'\theta(s)ds - \int_0^t \theta(s)'dW(s))$ $(\theta(t) = \sigma(t)^{-1}B(t))$, then we can define

$$\rho(t) = \exp(-\int_0^t r(s)ds) \cdot \alpha(t) = \exp(-\int_0^t (r(s) + \frac{1}{2}\theta(s)'\theta(s))ds - \int_0^t \theta(s)'dW(s))$$

First the condition 'bankruptcy prohibition' should be characterised. It can be defined straightforwardly as $x(t) \geqslant 0$ a.s., $\forall t \in [0, T]$. Bielecki et al (2005) established the following result, which makes the observation easier to handle.

Proposition 4.1 *Let $x(\cdot)$ be a wealth process under an admissible portfolio $\pi(\cdot)$. If $x(T) \geqslant 0$ a.s., then $x(t) \geqslant 0$ a.s. $\forall t \in [0, T]$.*

With this proposition, the original problem (2.6) can be reformulated as

$$Minimise \quad Var(x(T)) = E[x(T)^2] - z^2$$

$$s.t \begin{cases} Ex(T) = z > 0 \\ (x(\cdot), \pi(\cdot)) \text{ satisfies equation (2.5) with } x_0 > 0 \\ (x(\cdot), \pi(\cdot)) \text{ is an admissible pair} \\ x(T) \geqslant 0, a.s. \end{cases} \tag{4.1}$$

Note that the non-negativity constraint of z and x_0 is added in order to prevent trivial cases. Using an extension of the martingale approach, Bielecki et al (2005) decomposed problem (4.1) into two sub-problems, the first one is:

$$Minimise \quad E[X^2] - z^2$$

$$s.t \begin{cases} Ex(T) = z > 0 \\ E[\rho(T)X] = x_0 > 0 \\ X \in L^2_{\mathcal{F}_T}(\Omega, \mathbb{R}), \quad X \geqslant 0, a.s. \end{cases} \tag{4.2}$$

and secondly the trading strategy which replicates the optimal solution of problem (4.2) X^* can be found via:

$$\begin{cases} dx(t) = [r(t)x(t) + \pi(t)'B(t)]dt + \pi(t)'\sigma(t)dW(t) \\ x(T) = X^* \end{cases} \tag{4.3}$$

There is a strong connection between problems (4.1)-(4.3). To be specific, if $(x^*(\cdot), \pi^*(\cdot))$ is optimal for (4.1), then $x^*(T)$ is optimal for (4.2) with $(x^*(\cdot), \pi^*(\cdot))$ satisfying (4.3). Conversely, if $X^*$is optimal for (4.2), then (4.3) must have a solution $(x^*(\cdot), \pi^*(\cdot))$ which is optimal for (4.1). (Theorem 2.1: Bielecki et al, 2005)

Having formulated the problem, the first sub-problem (4.2) should be solved. Apply Lagrange multiplier method, Bielecki et al proved:

Theorem 4.1 *If (4.2) admits a solution X^*, then $X^* = (\lambda - \mu\rho(T))^+$, where (λ, μ) solves the following equations*

$$\begin{cases} E[(\lambda - \mu\rho(T))^+] = z \\ E[\rho(T)(\lambda - \mu\rho(T))^+] = x_0 \end{cases} \tag{4.4}$$

Conversely, if (λ, μ) satisfies (4.4), then $X^* = (\lambda - \mu\rho(T))^+$ must be an optimal solution of (4.2).

Note that if there is no non-negativity constraint, then the optimal solution to (4.2) can be simplified to $X^* = \lambda - \mu\rho(T)$, and λ and μ can be determined via

$$\begin{cases} E[\lambda - \mu\rho(T] = z \\ E[\rho(T)(\lambda - \mu\rho(T))] = x_0 \end{cases} \tag{4.5}$$

Before going into further details on the bankruptcy prohibited problem, let us consider the simplest model–unconstraint portfolio and deterministic parameters. In this case λ and μ can be determined directly from (4.5):

$$\lambda = \frac{z - x_0 e^{\int_0^T (r(s) - \theta(s)'\theta(s))ds}}{1 - e^{-\int_0^T \theta(s)'\theta(s)ds}}, \qquad \mu = \frac{z - x_0 e^{\int_0^T r(s)ds}}{e^{-\int_0^T r(s)ds}(e^{\int_0^T \theta(s)'\theta(s)ds} - 1)}$$

then $x^*(t)$ can be determined from $x^*(t) = \rho(t)^{-1}E[X^*\rho(T)|\mathcal{F}_t]$ (since $x(t)\rho(t)$ is a martingale):

$$x^*(t) = \lambda \exp(-\int_t^T r(s)ds) - \mu \exp(-\int_t^T (2r(s) - \theta(s)'\theta(s))ds)\rho(t)$$

Differentiate $x^*(t)$ with respect to t, we have:

$$dx^*(t) = r(t)x^*(t)dt + [(\lambda e^{-\int_t^T r(s)ds} - x^*(t))(\sigma(t)')^{-1}\theta(t)]'[B(t)dt + \sigma(t)dW(t)]$$

According to the definition of *replicating*, comparing this equation with (2.5), we have the efficient portfolio as

$$\pi^*(t) = (\lambda \exp(-\int_t^T r(s)ds) - x^*(t))(\sigma(t)\sigma(t)')^{-1}B(t) \tag{4.6}$$

Comparing this result and (3.3), which is the optimal portfolio derived via stochastic LQ approach (Zhou and Li, 2000), we can see that they are exactly the same.

Now we can go back to the bankruptcy prohibited problem (4.1). Compared with solving the problem, the existence and uniqueness of these two Lagrange multipliers is a more complicated problem. The following theorem describes the sufficient condition for their existence (Theorem 5.1 Bielecki, et al, 2005)

Theorem 4.2 *Equations (4.4) admit a unique solution* (λ, μ) *for any* $x_0 > 0$ *and* $z > 0$ *which satisfy* $a < \frac{x_0}{z} < b$*, where*

$$a = \inf_{Y \in L^2_{\mathcal{F}_T}(\Omega, \mathbb{R}), Y \geqslant 0, EY > 0} \frac{E[\rho(T)Y]}{EY}, \quad b = \sup_{Y \in L^2_{\mathcal{F}_T}(\Omega, \mathbb{R}), Y \geqslant 0, EY > 0} \frac{E[\rho(T)Y]}{EY}$$

Now the efficient portfolio and efficient frontier should be considered. As discussed in previous part, efficient frontier is a part of variance minimising frontier, it is natural to get variance minimising frontier first.

Theorem 4.3 *(Theorem 6.1, Bielecki, et al, 2005) The unique variance minimising portfolio for (4.1) corresponding to* $z > 0$ *and* $a < \frac{x_0}{z} < b$ *is given by*

$$\pi^*(t) = (\sigma(t)')^{-1} z^*(t)$$

where $(x^*(\cdot), z^*(\cdot))$ *is the unique solution to the BSDE*

$$\begin{cases} dx(t) = [r(t)x(t) + \theta(t)z(t)]dt + z(t)'dW(t) \\ x(T) = (\lambda - \mu\rho(T))^+ \end{cases}$$

and (λ, μ) *is determined by (4.7).*

Having shown that the efficient frontier is exactly the portion of the variance minimising frontier corresponding to $z \in [\frac{x_0}{E[\rho(T)]}, \frac{x_0}{a})$, Bielecki et al. (2005) stated that the efficient frontier is determined by

$$Var(x^*(T)) = \lambda(z)z - \mu(z)x_0 - z^2, \quad z \in [\frac{x_0}{E[\rho(T)]}, \frac{x_0}{a}) \tag{4.7}$$

where $z = Ex^*(T)$ and $(\lambda(z), \mu(z))$ is determined by (4.4).

In an special case where $r(\cdot)$ and $\theta(\cdot)$ are deterministic functions, note that a variance minimising portfolio replicates the time-T payoff of the contingent claim $(\lambda - \mu\rho(T))^+$, and the efficient portfolio is the variance minimising portfolio with $z \geqslant x_0 e^{\int_0^T r(t)dt}$. The efficient portfolio is thus derived via Black-Scholes equation, the following theorem gives the explicit form of it (Theorem 7.2: Bielecki, et al, 2005):

Theorem 4.4 *Assume* $\int_0^T |\theta(t)|^2 dt > 0$*, then there exists a unique efficient portfolio for (4.1) corresponding to any given* $z \geqslant x_0 e^{\int_0^T r(t)dt}$*. Moreover, the efficient frontier is:*

$$\begin{cases} E(x^*(T)) = \frac{\eta e^{\int_0^T r(t)dt} N_1(\eta) - N_2(\eta)}{\eta N_2(\eta) - e^{-\int_0^T [r(t) - \theta(t)'\theta(t)]dt} N_3(\eta)} x_0 \\ Var(x^*(T)) = [\frac{\eta}{\eta N_1(\eta) - e^{-\int_0^T r(t)dt} N_2(\eta)} - 1][Ex^*(T)]^2 - \frac{x_0}{\eta N_1(\eta) - e^{-\int_0^T r(t)dt} N_2(\eta)} Ex^*(T), \quad \eta > 0 \end{cases}$$

where $N_{1,2}(\eta) = N(\frac{\ln \eta + \int_0^T [r(t) \pm \frac{1}{2}\theta(t)'\theta(t)]dt}{\sqrt{\int_0^T \theta(t)'\theta(t)dt}})$, $\qquad N_3(\eta) = N(\frac{\ln \eta + \int_0^T [r(t) \frac{3}{2}\theta(t)'\theta(t)]dt}{\sqrt{\int_0^T \theta(t)'\theta(t)dt}})$

Again, comparing this result with (3.3) and (4.6), we can see although there is no closed analytical form of efficient frontier when bankruptcy is prohibited, there are high resemblance of these results.

4.2.2 Incomplete Market Case

In an incomplete market, not all the terminal wealth can be replicated by an admissible portfolio, therefore the main difficulty in solving the problem is to find the terminal wealth which are replicable. Jin (2004) and Jin and Zhou (2005)'s main achievement is that they characterised the set of replicable terminal wealth with some equivalent conditions which are easier to handle. And then the problem can be formulated in a easier way, which is followed by the routine technique as in the complete market case. As will be seen, the advantages of martingale approach are significant when portfolio is constrained. In this section, I will go over the main idea in these papers, state the key steps of solving the problems and list their main results for efficient portfolio and frontier. The content in the section has a very strong relation with my extension work, which will be discussed in the next Chapter.

4.2.2.1 Problem formulation

Firstly, denote C as the constraint set for $(x(\cdot), \pi(\cdot))$, and the replicable (attainable) set of terminal wealth is defined as:

A_C:=$\{X \in L^2_{\mathcal{F}_T}(\Omega;\mathbb{R})$: there is $x \in \mathbb{R}$ and $\pi \in \Pi$ such that $(x(\cdot), \pi(\cdot)) \in C$ satisfies (2.5) with $x(0) = x$ and $x(T) = X\}$

For $\forall\theta \in L^\infty_{\mathcal{F}}(0, T, \mathbb{R}^n)$, we can define

$$H_\theta(t) := \exp\{-\int_0^t [r(s) + \frac{1}{2}|\theta(s)|^2]ds - \int_0^t \theta(s)'dW(s)\}$$

or equivalently,

$$\begin{cases} dH_\theta(t) = -r(t)H_\theta(t)dt - H_\theta(t)\theta(t)'dW(t) \\ H_\theta(0) = 1 \end{cases}$$

According to the assumption about θ in Chapter 2, we define:

$$\Theta := \{\theta \in L^\infty_{\mathcal{F}}(0, T, \mathbb{R}^n) : \sigma(t)\theta(t) = B(t), a.s., a.e.t \in [0, T]\}$$

$$\hat{\Theta} := \{\theta \in L^\infty_{\mathcal{F}}(0, T, \mathbb{R}^n) : \sigma(t)\theta(t) \geqslant B(t), a.s., a.e.t \in [0, T]\}$$

$$\theta^*(t) := argmin_{\theta\in\{\theta\in\mathbb{R}^n:\sigma(t)\theta=B(t)\}}|\theta|^2, \quad \hat{\theta}(t) := argmin_{\theta\in\{\theta\in\mathbb{R}^n:\sigma(t)\theta\geqslant B(t)\}}|\theta|^2$$

. Then we have (Lemma 2.2: Jin and Zhou, 2005):

- There exists an $\mathbb{R}^m$-valued, $\mathcal{F}_t$-progressively measurable process $u(\cdot)$ such that $\sigma(t)'u(t) = \theta^*(t)$, a.s., a.e.$t \in [0, T]$.
- There exists an $\mathbb{R}^m_+$-valued, $\mathcal{F}_t$-progressively measurable process $v(\cdot)$ such that $\sigma(t)'v(t) = \hat{\theta}(t)$, a.s., a.e.$t \in [0, T]$.
- For any $\theta \in \Theta$, $\theta^*(t)\theta(t) = |\theta^*(t)|^2$, a.s., a.e.$t \in [0, T]$.

Then given an admissible wealth-portfolio pair $(x(\cdot), \pi(\cdot))$ and $\theta \in \Theta$, we know

$$x(t) = H_\theta(t)^{-1} E(x(T)H_\theta(T)|\mathcal{F}_t), \quad a.s., \forall t \in [0, T] \tag{4.8}$$

We will see that the above proposition and (4.8) play a central role in locating the optimal wealth process $x^*(t)$.

Recall the steps of martingale approach, firstly we should consider a static optimisation problem, which is crucial for the following discussion. It can be formulated as follows:

$$Minimise \quad E[X^2] - z^2$$

$$s.t. \begin{cases} EX = z \\ E[XH_{\theta^*}(T)] = x_0 \\ X \in A_C \end{cases} \tag{4.9}$$

Similar to the complete market case, note the fact that set C is convex, apply a Lagrange multiplier approach, Jin and Zhou (2005) showed that if X^* is an optimal solution to problem (4.9), then there exists a pair (λ, μ), such that X^* is also optimal for

$$Minimise \quad E[X - (\lambda - \mu H_{\theta^*}(T))]^2$$

$$s.t. \quad X \in A_C \tag{4.10}$$

And conversely, if there is a pair (λ, μ) such that the optimal solution of (4.10) X^* satisfies

$$\begin{cases} EX^* = z \\ E[X^* H_{\theta^*}(T)] = x_0 \end{cases} \tag{4.11}$$

then X^* is an optimal solution for problem (4.9).

Having got these, we can firstly solve (4.10) to get $X^* = X^*(\lambda, \mu)$ and determine (λ, μ) via (4.11); then any admissible portfolio that replicates $X^*(\lambda, \mu)$ is an efficient portfolio to the original problem (2.6). And how to deal with $X \in A_C$ in (4.10) is the key issue. Jin (2004), and Jin and Zhou (2005) characterised A_C in four cases, I will review the main ideas and results respectively.

4.2.2.2 Unconstrained portfolio

In the first case, $C = L^2_{\mathcal{F}}(0, T, \mathbb{R}) \times \Pi$, the following theorem states the equivalent conditions of $X \in A_C$ (Theorem 4.1: Jin and Zhou, 2005):

Theorem 4.5 *Given $X \in L^2_{\mathcal{F}_T}(\Omega, \mathbb{R})$, then the following are equivalent:*

1. *$X \in A_C$;*

2. *$E[XH_\theta(T)]$ is independent of $\theta \in \Theta$;*

3. $E[XH_\theta(T)]$ *is independent of* $\theta \in \Theta_1$*, where* $\Theta_1 = \{\theta \in \Theta : |\theta - \theta^*| \leqslant 1\}$*.*

Then straightforwardly, problem (4.10) can be written as:

$$Minimise \quad E[X - (\lambda - \mu H_{\theta^*}(T))]^2$$

$$s.t. \begin{cases} X \in L^2_{\mathcal{F}_T}(\Omega; \mathbb{R}) \\ E[X(H_\theta(T) - H_{\theta^*}(T))] = 0 \quad \forall \theta \in \Theta \end{cases} \tag{4.12}$$

Note that problem (4.12) is still not readily solvable since it contains infinitely many constraints, and we should find an equivalent form of (4.12) which is much convenient to solve. Notice that (4.12) is a convex optimisation problem, denote $L := span\{H_\theta(T) - H_{\theta^*}(T) : \theta \in \Theta\}$, then for any given (λ, μ), consider the following problem

$$Minimise_{Y \in \bar{L}} \quad E[\lambda - \mu H_{\theta^*}(T) - Y]^2 \tag{4.13}$$

Jin and Zhou (2005) proved that (4.13) admits a unique optimal solution Y^*, and Y^* is the optimal solution if and only if $\lambda - \mu H_{\theta^*}(T) - Y^* \in A_C$; and the unique optimal solution to (4.12) is $X^* = \lambda - \mu H_{\theta^*}(T) - Y^*$.

Now we can summarise the procedure of solving such problem: Firstly solve (4.13) to get Y^*, then automatically we have X^*, which is the optimal solution to (4.12) and (4.10). And then determine (λ, μ) via (4.11) to get the optimal solution of (4.9) as $X^* = X^*(\lambda, \mu)$. Finally the admissible portfolio replicating $X^* = X^*(\lambda, \mu)$ is an efficient portfolio to the original problem (2.6).

After that, with deterministic parameters, Jin and Zhou (2005) derived the explicit forms of the efficient portfolio and efficient frontier by using the same scheme as in the complete market case. Note that when all the coefficients are deterministic, $Y^* = 0$ is an optimal solution to (4.13), hence it is the unique optimal solution. Use the fact that $\lambda - \mu H_{\theta^*}(T) \in A_C$ for deterministic coefficients, then $X^* = \lambda - \mu H_{\theta^*}(T)$ is the unique optimal solution to (4.12). According to (4.8) and the definition of replicating, we have

$$x^*(t) = H_{\theta^*}(t)^{-1} E(x(T) H^*_\theta(T)|\mathcal{F}_t) = \lambda e^{-\int_t^T r(s)ds} - \mu e^{-\int_t^T (2r(s) - |\theta^*(s)|^2)ds} H_{\theta^*}(t)$$

Apply Ito's formula to write the dynamics of $x(t)$ and compare with (2.5), the efficient portfolio and efficient frontier are derived as:

$$\pi^*(t) = [\lambda e^{-\int_t^T r(s)ds} - x^*(t)]u(t)$$

$$Var(x^*(T)) = \frac{1}{e^{\int_0^T |\theta^*(t)|^2 dt} - 1}[z - x_0 e^{\int_0^T r(t)dt}]^2, \quad z \geqslant x_0 e^{\int_0^T r(t)dt}.$$

4.2.2.3 With short-selling prohibition

In the second case when short selling is prohibited, we have

$$C = \{(x(\cdot), \pi(\cdot)) \in L^2_{\mathcal{F}}(0, T, \mathbb{R}) \times \Pi : \pi(t) \geqslant 0, a.s., a.e.t \in [0, T]\}$$

Then A_C is characterised as follows (Theorem 5.1: Jin and Zhou, 2005):

Theorem 4.6 $\forall X \in L^2_{\mathcal{F}_T}(\Omega, \mathbb{R})$, $X \in A_C$ *if and only if* $\exists \bar{\theta} \in \hat{\Theta}$ *such that* $\sup_{\theta \in \hat{\Theta}} E[XH_\theta(T)] = E[XH_{\bar{\theta}}(T)]$. *Furthermore,* $\sup_{\theta \in \hat{\Theta}} E[XH_\theta(T)] = E[XH_{\theta^*}(T)]$ *if* $X \in A_C$.

And then problem (4.10) can be written as the following equivalent form:

$$Minimise \quad E[X - (\lambda - \mu H_{\theta^*}(T))]^2$$

$$s.t. \begin{cases} X \in L^2_{\mathcal{F}_T}(\Omega; \mathbb{R}) \\ \max_{\theta \in \hat{\Theta}} E[XH_\theta(T)] = E[XH_{\theta^*}(T)] \end{cases} \tag{4.14}$$

Similar to the unconstrained case, denote $M := \{k(H_\theta(T) - H_{\theta^*}(T)) \in L^2_{\mathcal{F}_T}(\Omega, \mathbb{R}) : k \geqslant 0, \theta \in \hat{\Theta}\}$, which is a convex set. For any given (λ, μ), consider

$$Minimise_{Y \in \bar{M}} \quad E[\lambda - \mu H_{\theta^*}(T) - Y]^2 \tag{4.15}$$

Then the same as unconstraint portfolio case, (4.15) admits a unique optimal solution Y^*, and Y^* is the optimal solution if and only if $E[(\lambda - \mu H_{\theta^*}(T) - Y^*)Y^*] = 0$ and $\lambda - \mu H_{\theta^*}(T) - Y^* \in A_C$; and the unique optimal solution to (4.14) is $X^* = \lambda - \mu H_{\theta^*}(T) - Y^*$.

As with unconstraint problem, when the market coefficients are deterministic, recall $\hat{\theta}(t) := argmin_{\theta \in \{\theta \in \mathbb{R}^n : \sigma(t)\theta \geqslant B(t)\}} |\theta|^2$, Jin and Zhou (2005) showed that $\lambda - \mu H_{\hat{\theta}}(T) \in A_C$ and $Y^* = \mu(H_{\hat{\theta}}(T) - H_{\theta^*}(T))$ is the unique optimal solution to (4.15). Therefore $X^* = \lambda - \mu H_{\hat{\theta}}(T)$ is the unique optimal solution to problem (4.14), and with the same procedure, efficient portfolio and frontier can be derived as:

$$\pi^*(t) = [\lambda e^{-\int_t^T r(s)ds} - x^*(t)]v(t)$$

$$Var(x^*(T)) = \frac{1}{e^{\int_0^T |\hat{\theta}(t)|^2 dt} - 1}[z - x_0 e^{\int_0^T r(t)dt}]^2, \quad z \geqslant x_0 e^{\int_0^T r(t)dt}.$$

4.2.2.4 With bankruptcy prohibition

Bankruptcy prohibition means $x(t) \geqslant 0$, $\forall t \in [0, T]$; Bielekie, Jin, Pliska and Zhou (2005) proved that in a complete market, $x(t) \geqslant 0$ if $x(T) \geqslant 0$. This is also true for incomplete markets, so the constraint set is:

$$C = \{(x(\cdot), \pi(\cdot)) \in L^2_{\mathcal{F}}(\Omega, T; \mathbb{R}) \times \Pi : x(T) \geqslant 0\}$$

In this case, the equivalent condition for $X \in A_C$ can be expressed as (Theorem 4.5.1: Jin, 2004):

Theorem 4.7 *For any $X \in L^2_{\mathcal{F}_T}(\Omega, \mathbb{R})$, $X \in A_C$ if and only if $X \geqslant 0$ and $E[XH_\theta(T)]$ is independent of $\theta \in \Theta_1 = \{\theta \in \Theta : |\theta - \theta^*| \leqslant 1\}$.*

And with this theorem, problem (4.10) can be written as:

$$Minimise \quad E[X - (\lambda - \mu H_{\theta^*}(T))]^2$$
$$s.t. \begin{cases} X \in L^2_{\mathcal{F}_T}(\Omega; \mathbb{R}) \\ \sup_{\tilde{\theta}_1, \tilde{\theta}_2 \in \Theta_1} E[X(H_{\tilde{\theta}_1}(T) - H_{\tilde{\theta}_2}(T))] \leqslant 0 \\ X \geqslant 0 \end{cases} \tag{4.16}$$

Denote $M_1 = \{H_{\tilde{\theta}_1}(T) - H_{\tilde{\theta}_2}(T) : \tilde{\theta}_1, \tilde{\theta}_2 \in \Theta_1\}$, then $\bar{M_1} \subset L^2_{\mathcal{F}_T}(\Omega, \mathbb{R})$, which is a convex set. Thus problem (4.16) has the following equivalent form:

$$Minimise \quad E[X - (\lambda - \mu H_{\theta^*}(T))]^2$$
$$s.t. \begin{cases} X \in L^2_{\mathcal{F}_T}(\Omega; \mathbb{R}) \\ \sup_{Y \in \bar{M_1}} E[XY] \leqslant 0 \\ X \geqslant 0 \end{cases} \tag{4.17}$$

which admits a unique solution X^*.

Then it is shown that $\exists k_0 \geqslant 0$, such that $\forall k \geqslant k_0$, X^* is also the unique solution for the following problem

$$min_{X \in L^2_{\mathcal{F}_T}(\Omega; \mathbb{R}^+)}[E(X - (\lambda - \mu H_{\theta^*}(T)))^2 + 2k \sup_{Y \in M_1} E(XY)] \tag{4.18}$$

and its optimal value is $E(X^* - (\lambda - \mu H_{\theta^*}(T)))^2$. Furthermore, the unique optimal solution for (4.18) is $X^*_k = (\lambda - \mu H_{\theta^*}(T) - kY^*_k)^+$, where Y^*_k is the optimal solution for

$$min_{Y \in M_1} E[(\lambda - \mu H_{\theta^*}(T) - kY)^+]^2 \tag{4.19}$$

After that, consider problem (4.20):

$$min_{Y \in cone\{M_1\}} E[(\lambda - \mu H_{\theta^*}(T) - Y)^+]^2 \tag{4.20}$$

Problem (4.20) admits more than one optimal solutions, and Y^* is optimal for (4.20) if and only if $X^* = (\lambda - \mu H_{\theta^*}(T) - Y)^+ \in A_C$. Note that for different Y^*, X^* is the same, then the unique optimal solution for (4.16) and (4.17) is $X^* = (\lambda - \mu H_{\theta^*}(T) - Y)^+$.

When the coefficients are deterministic, Jin (2004) showed that $(\lambda - \mu H_{\theta^*}(T))^+ \in A_C$. In the same way as the complete markets, define

$$a = \inf\{\eta \in \mathbb{R} : P(H_{\theta^*}(T) < \eta) > 0\}, \quad a = \inf\{\eta \in \mathbb{R} : P(H_{\theta^*}(T) < \eta) > 0\}$$

And if $a < \frac{x_0}{z} < b$ (which ensures the existence and uniqueness of λ and μ) and $\int_0^T \theta^*(t)'\theta^*(t)dt > 0$, then the efficient portfolios and frontier can be derived in exactly the same way as in the complete markets, except for replacing $\rho(T)$ by $H_{\theta^*}(T)$. Note that as in complete markets, closed form of analytical solution cannot be obtained.

4.2.2.5 With short-selling and bankruptcy prohibition

In this case, the constraint set can be expressed as

$$C = \{(x(\cdot), \pi(\cdot)) \in L^2_{\mathcal{F}}(\Omega, T; \mathbb{R}) \times \Pi : \pi(t) \geqslant 0, a.s., a.e.t \in [0,T], x(T) \geqslant 0\}$$

Let $K \in \mathbb{R}$ satisfies $|\theta^*(t)| \leqslant K, a.s., a.e.t \in [0,T]$. Define $\hat{\Theta}_1 = \{\theta \in L^{+\infty}_{\mathcal{F}}(0,T;\mathbb{R}) : \sigma(\cdot)\theta(\cdot) \geqslant B, \exists N < K+1, s.t. |\theta(s,\omega)| \leqslant N, a.s., a.e.t \in [0,T]\}$, then the equivalent condition for $X \in A_C$ is (Theorem 4.6.1: Jin, 2004):

Theorem 4.8 *For any $X \in L^2_{\mathcal{F}_T}(\Omega, \mathbb{R})$, $X \in A_C$ if and only if $X \geqslant 0$ and there exists $\hat{\theta} \in \hat{\Theta}_1$ such that $\sup_{\theta \in \hat{\Theta}_1} E[XH_\theta(T)] = E[XH_{\hat{\theta}}(T)]$. Furthermore, $\sup_{\theta \in \hat{\Theta}_1} E[XH_\theta(T)] = E[XH_{\theta^*}(T)]$ if $X \in A_C$.*

And with this theorem, problem (4.10) can be written as:

$$\begin{aligned} & Minimise \quad E[X - (\lambda - \mu H_{\theta^*}(T))]^2 \\ & s.t. \begin{cases} X \in L^2_{\mathcal{F}_T}(\Omega; \mathbb{R}) \\ \sup_{\theta \in \hat{\Theta}_1} E[X(H_\theta(T) - H_{\theta^*}(T))] \leqslant 0 \\ X \geqslant 0 \end{cases} \end{aligned} \tag{4.21}$$

Denote $\hat{M} = \{H_\theta(T) - H_{\theta^*}(T) : \theta \in \hat{\Theta}_1\}$, then similar to section 4.2.2.3, $\bar{\hat{M}} \in L^2_{\mathcal{F}_T}(\Omega, \mathbb{R})$, which is a bounded convex set. Thus problem (4.21) has the following equivalent form:

$$\begin{aligned} & Minimise \quad E[X - (\lambda - \mu H_{\theta^*}(T))]^2 \\ & s.t. \begin{cases} X \in L^2_{\mathcal{F}_T}(\Omega; \mathbb{R}) \\ \sup_{Y \in \bar{\hat{M}}} E[XY] \leqslant 0 \\ X \geqslant 0 \end{cases} \end{aligned} \tag{4.22}$$

which admits a unique solution X^*.

Consider the following problem

$$min_{Y \in cone\{\bar{\hat{M}}\}} E[(\lambda - \mu H_{\theta^*}(T) - Y)^+]^2 \tag{4.23}$$

It is shown that for different optimal solutions of (4.23) Y^*, $X^* = (\lambda - \mu H_{\theta^*}(T) - Y)^+$ is the same, hence it is the unique optimal solution for (4.21) and (4.22). Furthermore, Y^* is optimal for (4.23) if and only if X^* is no-shorting and $E[X^*Y^*] = 0$.

In the deterministic parameter case, Jin (2004) showed that $\forall \lambda \in \mathbb{R}$ and $\forall \mu \in \mathbb{R}^+$, $Y^* = \mu(H_{\hat{\theta}}(T) - H_{\theta^*}(T))$ is optimal for (4.23), therefore $X^* = (\lambda - \mu H_{\hat{\theta}}(T))^+$ is the unique optimal solution to (4.21). With the same procedure as before, the optimal portfolio and efficient frontier are obtained explicitly (see: Theorem 4.6.4, Jin (2004)).

Chapter 5

My extension work

In Jin (2004), and Jin and Zhou (2005), explicit forms of optimal portfolio and efficient frontier are obtained only when the market parameters are deterministic. It is very difficult to solve them explicitly when the parameters are random processes. To be specific, in section 4.2.2, we have seen that it is required to get Y^* explicitly before getting X^* explicitly; however, we do not know the explicit form of Y^* if the parameters are random, hence the optimal terminal wealth X^* cannot be derived explicitly. To best of my knowledge, there is no existing literature which has dealt with this problem or has found other ways to obtain X^* explicitly.

However, when we add some simple form of randomness to the appreciation rate and volatility rate, it is not so difficult as we imagine to obtain the optimal portfolio explicitly. For example, I can make the parameters two different values at $[0, \frac{T}{2}]$ and $[\frac{T}{2}, T]$, and the latter ones are $\mathcal{F}_{\frac{T}{2}}$-adapted random variables, rather than deterministic functions. And then the problem is solved within $[0, \frac{T}{2}]$ and $[\frac{T}{2}, T]$ respectively. Once this two-period problem is solved, it is possible to know the solution where parameters are general random processes because the general random process can be viewed as the limit of such two-period simple random process. To do this, we can use the same trick to each interval. In other words, we divide $[0, \frac{T}{2}]$ into $[0, \frac{T}{4})$ and $[\frac{T}{4}, \frac{T}{2}]$; and $[\frac{T}{2}, T]$ into $[\frac{T}{2}, \frac{3T}{4}]$ and $[\frac{3T}{4}, T]$. And then solve the problem in each half with exactly the same method as in the whole period. We can continue this process for infinitely many times, and make the length of each interval go to infinitesimal. That is to say we take the limit of the solution process, in which case the parameters will become general random processes. If this can be done, we will be able to obtain the complete solution to the problem with the market parameters being random processes. In this chapter, I will devote myself to the so-called two-period problem with unconstrained portfolio. Firstly I will describe the model. Then I will solve the two-period problem. The problem with general random parameters is left as an open question.

5.1 Problem formulation

The market model is described in this section, which is slightly different from the one in Chapter 2. The riskless asset S_0 satisfies the following ordinary differential equation for the whole period:

$$\begin{cases} dS_0(t) = r(t)S_0(t)dt, t \in [0, T] \\ S_0(0) = s_0 > 0 \end{cases} \tag{5.1}$$

Without losing generality, the interest rate $r(t)$ is assumed as a constant r throughout $[0, T]$. The other m assets are risky assets with the price processes $S_i(t), i = 1, \cdots, m$, which satisfy the following stochastic differential equations:

$$\begin{cases} dS_i(t) = S_i(t)[a_i(0)dt + \sum_{j=1}^n \sigma_{ij}(0)dW^j(t)], t \in [0, \frac{T}{2}], i = 1, 2, \cdots, m \\ S_i(0) = s_i > 0 \end{cases} \tag{5.2}$$

$$\begin{cases} dS_i(t) = S_i(t)[a_i(\frac{T}{2})dt + \sum_{j=1}^n \sigma_{ij}(\frac{T}{2})dW^j(t)], t \in [\frac{T}{2}, T], i = 1, 2, \cdots, m \\ S_i(\frac{T}{2}) = \tilde{s}_i > 0 \end{cases} \tag{5.3}$$

where $a_i(0)$ and $\sigma_{ij}(0)$ are determined at $t = 0$ and remain constant during $[0, \frac{T}{2}]$. $a_i(\frac{T}{2})$ and $\sigma_{ij}(\frac{T}{2})$ are determined at $t = \frac{T}{2}$, and remain the same during $[\frac{T}{2}, T]$, namely they are $\mathcal{F}_{\frac{T}{2}}$ -adatped **random variables**. This is vitally different from the case where the parameters are deterministic.

Denote

$$\sigma(t) = (\sigma_{ij}(t))_{m\times n} = \begin{cases} \sigma_1(0) = (\sigma_{ij}(0))_{m\times n}, & t \in [0, \frac{T}{2}] \\ \sigma_2(\frac{T}{2}) = (\sigma_{ij}(\frac{T}{2}))_{m\times n}, & t \in [\frac{T}{2}, T] \end{cases}$$

$$a(t) = (a_1(t), \cdots, a_m(t))' = \begin{cases} (a_1(0), \cdots, a_m(0))', & t \in [0, \frac{T}{2}] \\ (a_1(\frac{T}{2}), \cdots, a_m(\frac{T}{2}), & t \in [\frac{T}{2}, T] \end{cases}$$

$$B(t) = \begin{cases} B_1(0) = (b_1(0), \cdots, b_m(0))' := (a_1(0) - r, \cdots, a_m(0) - r)', & t \in [0, \frac{T}{2}] \\ B_2(\frac{T}{2}) = (b_1(\frac{T}{2}), \cdots, b_m(\frac{T}{2}))' := (a_1(\frac{T}{2}) - r, \cdots, a_m(\frac{T}{2}) - r)', & t \in [\frac{T}{2}, T] \end{cases}$$

$$\pi(t) := (\pi_1(t), \cdots, \pi_m(t))'$$

Consider an agent whose total wealth at time t is $x(t)$, and the amount of money invested in the i-th stock $(i = 1, \cdots, m)$ is $\pi_i(t) = N_i(t)S_i(t)$, then we have

$$x(t) = \sum_{i=0}^m \pi_i(t) = \sum_{i=0}^m N_i(t)S_i(t)$$

If the agent strategy is *self-financing*, $x(t)$ satisfies:

$$\begin{cases} dx(t) = [r(t)x(t) + \sum_{i=1}^{m}(a_i(t) - r(t))\pi_i(t)]dt + \sum_{i=1}^{m}\sum_{j=1}^{n}\pi_i(t)\sigma_{ij}(t)dW^j(t) \\ x(0) = x_0 \end{cases} \tag{5.4}$$

Then we can write the vector form of (5.4) as:

$$\begin{cases} dx(t) = [rx(t) + \pi(t)'B_1(0)]dt + \pi(t)'\sigma_1(0)dW(t), \quad t \in [0, \frac{T}{2}] \\ x(0) = x_0 \end{cases} \tag{5.5}$$

$$\begin{cases} dx(t) = [rx(t) + \pi(t)'B_2(\frac{T}{2})]dt + \pi(t)'\sigma_2(\frac{T}{2})dW(t), \quad t \in [\frac{T}{2}, T] \\ x(\frac{T}{2}) = y_0 \end{cases} \tag{5.6}$$

Before formulating the problem, some notation should be made:
Assume $\exists\theta_1(0) \in L^{\infty}_{\mathcal{F}}(0, \frac{T}{2}; \mathbb{R}^n)$ such that $\sigma_1(0)\theta_1(0) = B_1(0)$, a.s.,a.e.$t \in [0, \frac{T}{2}]$; and $\exists\theta_2(\frac{T}{2}) \in L^{\infty}_{\mathcal{F}}(\frac{T}{2}, T; \mathbb{R}^n)$ such that $\sigma_2(\frac{T}{2})\theta_2(\frac{T}{2}) = B_2(\frac{T}{2})$, a.s.,a.e.$t \in [\frac{T}{2}, T]$.
Then $\theta(t)$ defined in Chapter 2 can be written as $\theta(t) = \begin{cases} \theta_1(0), & t \in [0, \frac{T}{2}] \\ \theta_2(\frac{T}{2}), & t \in [\frac{T}{2}, T] \end{cases}$

Define

$$\theta_1^*(0) := argmin_{\theta \in \{\theta_1 \in \mathbb{R}^n : \sigma_1(0)\theta = B_1(0)\}}|\theta|^2, \quad \theta_2^*(\frac{T}{2}) := argmin_{\theta \in \{\theta_2 \in \mathbb{R}^n : \sigma_2(\frac{T}{2})\theta = B_2(\frac{T}{2})\}}|\theta|^2,$$

and

$$\hat{H}_{\theta_1}(t) := \exp\{-t(r + \frac{1}{2}|\theta_1(0)|^2) - \theta_1(0)'W(t)\}, \quad t \in [0, \frac{T}{2}]$$

$$\tilde{H}_{\theta_2}(t) := \exp\{-(t - \frac{T}{2})(r + \frac{1}{2}|\theta_2(\frac{T}{2})|^2) - \theta_2(\frac{T}{2})'(W(t) - W(\frac{T}{2})\}), \quad t \in [\frac{T}{2}, T]$$

Thus

$$H_\theta(t) = \exp\{-\int_0^t [r + \frac{1}{2}|\theta(s)|^2]ds - \int_0^t \theta(s)'dW(s)\} = \begin{cases} \hat{H}_{\theta_1}(t), & t \in [0, \frac{T}{2}] \\ \hat{H}_{\theta_1}(\frac{T}{2})\tilde{H}_{\theta_2}(t), & t \in [\frac{T}{2}, T] \end{cases}$$

Denote C be the constraint set, in the unconstraint portfolio case, $C = L^2_{\mathcal{F}}(0, T, \mathbb{R}) \times \Pi$, and define the sets of attainable terminal wealth in both sub-intervals:

$\hat{A}_C$:=$\{\hat{X} \in L^2_{\mathcal{F}_{\frac{T}{2}}}(\Omega; \mathbb{R})$: there is $x \in \mathbb{R}$ and $\pi \in \Pi$ such that $(x(\cdot), \pi(\cdot)) \in C$ satisfies (5.5) with $x(0) = x$ and $x(\frac{T}{2}) = \hat{X}\}$

$\tilde{A}_C$:=$\{\tilde{X} \in L^2_{\mathcal{F}_T}(\Omega; \mathbb{R})$: there is $y \in L^2_{\mathcal{F}_{\frac{T}{2}}}(\Omega; \mathbb{R})$ and $\pi \in \Pi$ such that $(x(\cdot), \pi(\cdot)) \in C$ satisfies (5.6) with $x(\frac{T}{2}) = y$ and $x(T) = \tilde{X}\}$

and in the whole interval $[0, T]$:

A_C:=$\{X \in L^2_{\mathcal{F}_T}(\Omega; \mathbb{R})$: there is $x \in \mathbb{R}$ and $\pi \in \Pi$ such that $(x(\cdot), \pi(\cdot)) \in C$ satisfies (5.4) with $x(0) = x$ and $x(T) = X\}$

After such preparing work, I am in the position to formulate the portfolio selection model. Recall the initial problem (2.6) is

$$Minimise \quad Var(x(T)) = E[x(T)^2] - z^2$$

$$s.t \begin{cases} E[x(T)] = z, \quad \pi \in \Pi \\ (x(\cdot), \pi(\cdot)) \text{ satisfies equation (5.4)} \\ (x(\cdot), \pi(\cdot)) \in C \end{cases} \tag{5.7}$$

Then consider the following static optimisation problem:

$$Minimise \quad E[X^2]$$

$$s.t \begin{cases} EX = z \\ E[XH_{\theta^*}(T)] = x_0 \\ X \in A_C \end{cases} \tag{5.8}$$

It has been proved that in order to solve (5.7), it suffices to solve (5.8). And to solve (5.8), we use the routine Lagrange multiplier approach and transfer it into the following problem:

$$Minimise \quad E[(X-\lambda)^2]$$

$$s.t \begin{cases} E[XH_{\theta^*}(T)] = x_0 \\ X \in A_C \end{cases} \tag{5.9}$$

Now we consider problem (5.9) within the interval $[\frac{T}{2}, T]$, the formulation is almost the same except that we suppose $t = \frac{T}{2}$ is the new 'initial' point, then the (5.9) becomes:

$$Minimise \quad E[(\tilde{X}-\lambda)^2 | \mathcal{F}_{\frac{T}{2}}]$$

$$s.t \begin{cases} E[\tilde{X}\tilde{H}_{\theta_2^*}(T) | \mathcal{F}_{\frac{T}{2}}] = y_0 \\ \tilde{X} \in \tilde{A}_C \end{cases} \tag{5.10}$$

It can be seen that the minimum value of the objective function in problem (5.10) is a function of $y_0 = x(\frac{T}{2})$ and $\theta_2^*(\frac{T}{2})$ (where $\theta_2^*(\frac{T}{2})$ is determined by $\sigma_2(\frac{T}{2})\theta_2^*(\frac{T}{2}) = B_2(\frac{T}{2})$), namely $minE[(\tilde{X}-\lambda)^2 | \mathcal{F}_{\frac{T}{2}}] = f(x(\frac{T}{2}), \theta_2^*(\frac{T}{2}))$. And with the property $E\{E[(\tilde{X}-\lambda)^2 | \mathcal{F}_{\frac{T}{2}}]\} = E[(\tilde{X}-\lambda)^2]$, we can formulate the problem within the interval $[0, \frac{T}{2}]$ as follows:

$$Minimise \quad E[f(x(\frac{T}{2}), \theta_2^*(\frac{T}{2}))]$$

$$s.t \begin{cases} E[\hat{X}\hat{H}_{\theta_1^*}(\frac{T}{2})] = x_0 \\ \hat{X} \in \hat{A}_C \end{cases} \tag{5.11}$$

Let me summarise the main idea and procedure of solving the whole problem. First divide the time period into two intervals of equal length, solve the problem (5.10) within $[\frac{T}{2}, T]$ with the 'initial' point $t = \frac{T}{2}$. Once we obtain the optimal solution of (5.10), substitute it into its objective function and get the minimum value, which is denoted by $f(x(\frac{T}{2}), \theta_2^*(\frac{T}{2}))$. Having done this, we can obtain the optimal attainable terminal wealth which is conditional on the value of $x(\frac{T}{2})$. Thereafter, we should solve problem (5.11), which is in the period $[0, \frac{T}{2}]$, so that the optimal attainable terminal wealth can be obtained with the starting point $t = 0$. Once these done, the difficulty caused by the randomness of the parameters can be eliminated and the optimal attainable terminal wealth can be obtained explicitly. And then the optimal wealth process $x^*(t)$ and the optimal portfolio $\pi^*(t)$ can be obtained by the same procedure as in Jin and Zhou (2005).

5.2 Solving the static optimisation problem

In this section, we carry on to solve the optimal attainable terminal wealth. Now consider solving problem (5.10), clearly it can be transformed into the following problem by adding a Lagrange multiplier μ:

$$\begin{aligned} &Minimise \quad E[(\tilde{X}-\lambda)^2|\mathcal{F}_{\frac{T}{2}}] + 2\mu\{E[\tilde{X}\tilde{H}_{\theta_2^*}(T)|\mathcal{F}_{\frac{T}{2}}] - y_0\} \\ &\qquad\qquad s.t \quad \tilde{X} \in \tilde{A}_C \end{aligned} \tag{5.12}$$

Since $\theta_2^*(\frac{T}{2})$ is $\mathcal{F}_{\frac{T}{2}}$-adapted, problem (5.12) becomes problem (4.10), except the 'initial' point becomes $t = \frac{T}{2}$. We can use exactly the same method as in Jin and Zhou (2005) to deal with this problem. Hence the optimal solution to problem (5.12) is

$$\tilde{X}^*_{\lambda,\mu} = \lambda - \mu\tilde{H}_{\theta_2^*}(T) \tag{5.13}$$

Clearly $\tilde{X}^*_{\lambda,\mu}$ is an $\mathcal{F}_{\frac{T}{2}}$-adapted random variable. Substitute this into the objective function and constraint of problem (5.10), note that

$$E[\tilde{H}_{\theta_2^*}(T)|\mathcal{F}_{\frac{T}{2}}] = \exp(-\frac{rT}{2}), \quad E[\tilde{H}_{\theta_2^*}(T)^2|\mathcal{F}_{\frac{T}{2}}] = \exp(\frac{T}{2}(|\theta_2^*(\frac{T}{2})|^2 - 2r))$$

we have:

$$\mu = \frac{\lambda\exp(-\frac{rT}{2}) - y_0}{\exp(\frac{T}{2}(|\theta_2^*(\frac{T}{2})|^2 - 2r))} \tag{5.14}$$

and

$$f(x(\frac{T}{2}), \theta_2^*(\frac{T}{2})) = E[(\tilde{X}^*_{\lambda,\mu} - \lambda)^2|\mathcal{F}_{\frac{T}{2}}] = \frac{[\lambda\exp(-\frac{rT}{2}) - y_0]^2}{\exp(\frac{T}{2}(|\theta_2^*(\frac{T}{2})|^2 - 2r))}$$

Therefore problem (5.11) becomes

$$\begin{aligned} &Minimise \quad E\{\frac{[\lambda\exp(-\frac{rT}{2}) - \hat{X}]^2}{\exp(\frac{T}{2}(|\theta_2^*(\frac{T}{2})|^2 - 2r))}\} \\ &s.t \begin{cases} E[\hat{X}\hat{H}_{\theta_1^*}(\frac{T}{2})] = x_0 \\ \hat{X} \in \hat{A}_C \end{cases} \end{aligned} \tag{5.15}$$

Here I first state my counterpart of Theorem 4.1 in Jin and Zhou's paper (2005):

1. $\hat{X} \in \hat{A}_C$;
2. $E[\hat{X}\hat{H}_{\theta_1}(\frac{T}{2})]$ is independent of $\theta_1 \in \Theta$,
   where $\Theta = \{\theta \in L^\infty_{\mathcal{F}}(0, \frac{T}{2}, \mathbb{R}^n) : \sigma(t)\theta(t) = B(t), a.s., a.e.t \in [0, \frac{T}{2}]\}$;
3. $E[\hat{X}\hat{H}_{\theta_1}(\frac{T}{2})]$ is independent of $\theta_1 \in \Theta_1$, where $\Theta_1 = \{\theta \in \Theta : \|\theta - \theta^*\| \leqslant 1\}$.

then directly problem (5.15) has the following form:

$$Minimise \quad E\{\frac{[\lambda\exp(-\frac{rT}{2})-\hat{X}]^2}{\exp(\frac{T}{2}(|\theta_2^*(\frac{T}{2})|^2-2r))}\}$$

$$s.t \begin{cases} E[\hat{X}\hat{H}_{\theta_1}(\frac{T}{2})]=x_0 \\ \hat{X}\in\hat{A}_C \end{cases}$$

Using a Lagrange multiplier, in order to solve (5.15), it suffices to solve problem (5.16)

$$Minimise \quad E\{\frac{[\lambda\exp(-\frac{rT}{2})-\hat{X}]^2}{\exp(\frac{T}{2}(|\theta_2^*(\frac{T}{2})|^2-2r))}\}-\kappa\{E[\hat{X}\hat{H}_{\theta_1}(\frac{T}{2})]-x_0\}$$

$$s.t \quad \hat{X}\in\hat{A}_C \tag{5.16}$$

Denote $\gamma=|\theta_2^*(\frac{T}{2})|^2-2r$, then problem (5.16) can be simplified to be:

$$Minimise \quad E\{[\lambda\exp(-\frac{rT}{2})-\hat{X}]^2\exp(-\frac{T}{2}\gamma)-\kappa[\hat{X}\hat{H}_{\theta_1}(\frac{T}{2})-x_0]\}$$

$$s.t \quad \hat{X}\in\hat{A}_C \tag{5.17}$$

So far I have done the first part of this problem, which is in the period $[\frac{T}{2},T]$. From now on we are ready to solve problem (5.11), which has been reformulated as problem (5.17). Firstly I solve this problem without the constraint $\hat{X}\in\hat{A}_C$, which is just a simple calculation. After that I choose properly γ and $\theta_1(0)$ to make $\hat{X}$ attainable (i.e. $\hat{X}\in\hat{A}_C$).

For the first step, consider the objective function in (5.17), denote

$$g(\hat{X})=[\lambda\exp(-\frac{rT}{2})-\hat{X}]^2\exp(-\frac{T}{2}\gamma)-\kappa[\hat{X}\hat{H}_{\theta_1}(\frac{T}{2})-x_0]$$

then differentiate g with respect to $\hat{X}$, we know $g(\hat{X})$ reaches its minimum value at

$$\hat{X}^*=\lambda\exp(-\frac{rT}{2})+\frac{1}{2}\kappa\hat{H}_{\theta_1}(\frac{T}{2})\exp(\frac{T}{2}\gamma) \tag{5.18}$$

and the minimum value is

$$g(\hat{X}^*)=-\frac{1}{4}\kappa^2[\hat{H}_{\theta_1}(\frac{T}{2})]^2\exp(\frac{T}{2}\gamma)-\kappa[\lambda\hat{H}_{\theta_1}(\frac{T}{2})\exp(-\frac{rT}{2})-x_0] \tag{5.19}$$

Therefore (5.18) is the optimal solution for problem (5.17) without constraint, now proper relation between $\theta_1(0)$ and γ should be constructed so that $\hat{X}^*\in\hat{A}_C$. Once this has been done, problem (5.17) is completely solved.

Consider (5.18), since the constant $\lambda\exp(-\frac{rT}{2})$ is already in $\hat{A}_C$. It suffices to show

$$\frac{1}{2}\kappa\hat{H}_{\theta_1}(\frac{T}{2})\exp(\frac{T}{2}\gamma)\in\hat{A}_C \tag{5.20}$$

A natural idea comes from Lemma 4.1 in Jin and Zhou (2005), which says $\hat{H}_{\theta_1^*}(\frac{T}{2})\in\hat{A}_C$. Obviously $1\in\hat{A}_C$, consequently we may think that if $\hat{H}_{\theta_1}(\frac{T}{2})\exp(\frac{T}{2}\gamma)=p+q\hat{H}_{\theta_1^*}(\frac{T}{2})$ then $\hat{X}^*\in\hat{A}_C$, where p and q are any real numbers.

However, this logic can lead to a serious problem. Firstly, we know in our model, $\gamma = |\theta_2^*(\frac{T}{2})|^2 - 2r$, then γ has the lower bound $-2r$ (i.e. $\gamma \geqslant -2r$).

Notice that

$$\hat{H}_{\theta_1}(\frac{T}{2}) \exp(\frac{T}{2}\gamma) = p \Leftrightarrow \frac{T}{2}\gamma = [\frac{T}{2}(r + \frac{1}{2}|\theta_1(0)|^2) + \ln p] + \theta_1(0)' W(\frac{T}{2}) \tag{5.21}$$

and

$$\hat{H}_{\theta_1}(\frac{T}{2}) \exp(\frac{T}{2}\gamma) = qH_{\theta_1^*}(\frac{T}{2}) \Leftrightarrow \frac{T}{2}\gamma = [\frac{T}{4}(|\theta_1(0)|^2 - |\theta_1^*(0)|^2) + \ln q] + [\theta_1(0) - \theta_1^*(0)]' W(\frac{T}{2}) \tag{5.22}$$

Recall section 5.1, we know $\theta_1(0)$ and $\theta_1^*(0)$ are constants since $\sigma_1(0)$ and $B_1(0)$ are constants which are determined at $t = 0$. Nevertheless, from (5.21) and (5.22) we know, except for trivial cases (i.e. if $\theta_1(0) = 0$ in (5.21) or $\theta_1(0) = \theta_1^*(0)$ in (5.22)), γ does not have lower bound since $W(\frac{T}{2})$ is a normally distributed random variable with mean 0 and variance $\frac{T}{2}$. Therefore we can conclude that this idea does not work in this case and an alternative technique should be considered.

Note that we aim to find the relation between γ and $\theta_1(0)$ such that (5.20) holds. We know that if we can construct a random variable ξ such that $\xi \in \hat{A}_C$ and such serious problem can be avoided, then by making

$$\hat{H}_{\theta_1}(\frac{T}{2}) \exp(\frac{T}{2}\gamma) = \xi \tag{5.23}$$

the relation between γ and $\theta_1(0)$ can be found, and problem can be solved. Now the problem becomes how to find such a ξ satisfying these.

5.3 Getting the optimal portfolio

5.3.1 Period One: $t \in [0, \frac{T}{2}]$

From the definition of $\xi \in \hat{A}_C$, we know that if $\exists \hat{x} \in \mathbb{R}$ and $\pi \in \Pi$ such that $(\hat{x}(\cdot), \pi(\cdot)) \in C$ satisfies

$$d\hat{x}(t) = [r\hat{x}(t) + \pi(t)' B_1(0)]dt + \pi(t)' \sigma_1(0) dW(t), \quad t \in [0, \frac{T}{2}] \tag{5.24}$$

with $\hat{x}(0) = \hat{x}$, $\hat{x}(\frac{T}{2}) = \xi$ and $\hat{x}(\cdot)$ is a wealth process which is different from $x(\cdot)$ in (5.5); and then we can conclude that $\xi \in \hat{A}_C$. Observe that in order to make (5.23) holds, we should make $\hat{x}(\frac{T}{2}) = \xi$, hence $\hat{x}(t)$, of exponential form, then conjecture $\pi(t)$ satisfies

$$\pi(t) = \hat{x}(t) \cdot \nu(t) \tag{5.25}$$

where $\nu(\cdot)$ is a m-dimensional process (not constant). Substitute this into (5.24), we have:

$$d\hat{x}(t) = \hat{x}(t)[(r + \nu(t)' B_1(0))dt + \nu(t)' \sigma_1(0) dW(t)], \quad t \in [0, \frac{T}{2}] \tag{5.26}$$

Then directly

$$\hat{x}(t) = \hat{x} \cdot \exp\{\int_0^t (r + \nu(s)' B_1(0) - \frac{1}{2}|\nu(t)'\sigma_1(0)|^2) ds + \int_0^t \nu(s)'\sigma_1(0) dW(s)\} \tag{5.27}$$

and

$$\xi = \hat{x}(\frac{T}{2}) = \hat{x} \cdot \exp\{\int_0^{\frac{T}{2}} (r + \nu(s)' B_1(0) - \frac{1}{2}|\nu(t)'\sigma_1(0)|^2) ds + \int_0^{\frac{T}{2}} \nu(s)'\sigma_1(0) dW(s)\} \tag{5.28}$$

Note that $\sigma_1(0)\theta_1(0) = B_1(0)$, then $\hat{H}_{\theta_1}(\frac{T}{2})\exp(\frac{T}{2}\gamma) = \xi$ is equivalent to:

$$\frac{T}{2}\gamma = [\int_0^{\frac{T}{2}} (2r + (\sigma_1(0)'\nu(s))'\theta_1(0) - \frac{1}{2}|\sigma_1(0)'\nu(s)|^2 + \frac{1}{2}|\theta_1(0)|^2) ds + \ln \hat{x}] + \int_0^{\frac{T}{2}} [\sigma_1(0)'\nu(s) + \theta_1(0)]' dW(s) \tag{5.29}$$

Before further discussion, the existence of $\nu(\cdot)$ should be considered first. It can be proved that there exists a process $\nu(\cdot)$ such that (5.25) and (5.29) make sense. That is to say, $\hat{x}(t)$ satisfying (5.25) can be found; and $\nu(\cdot)$ can be constructed such that γ which is determined by (5.29) has lower bound. The proof is beyond the scope of this thesis, now let us simply assume $\nu(\cdot)$ has been found and view it as known. Then the relation between $\theta_1(0)$ and γ which makes $\hat{X}^* \in \hat{A}_C$ is characterised by (5.29).

Substitute (5.28) into (5.18), we have

$$\hat{X}^* = \lambda e^{-\frac{rT}{2}} + \frac{1}{2}\kappa\xi \tag{5.30}$$

Recall that $E[\hat{X}^*\hat{H}_{\theta_1^*}(\frac{T}{2})] = x_0$, we can determine κ as

$$\kappa = 2\frac{x_0 - \lambda e^{-rT}}{\hat{x}}$$

Recall (5.14), here we have $y_0 = \hat{X}^*$, then

$$\mu = -\frac{1}{2}\kappa\hat{H}_{\theta_1}(\frac{T}{2}) = \frac{\lambda e^{-rT} - x_0}{\hat{x}} e^{-\frac{T}{2}(r + \frac{1}{2}|\theta_1(0)|^2) - \theta_1(0)'W(\frac{T}{2})} \tag{5.31}$$

Recall $EX^* = E[E(\tilde{X}^*_{\lambda,\mu}|\mathcal{F}_{\frac{T}{2}})] = E[E(\lambda - \mu\tilde{H}_{\theta_2^*}(T)|\mathcal{F}_{\frac{T}{2}})] = z$, we have

$$\lambda = \frac{z\hat{x} - x_0\hat{x}e^{-rT}}{\hat{x} - e^{-2rT}}$$

Note that $\gamma = |\theta_2^*(\frac{T}{2})|^2 - 2r$, then the optimal terminal wealth with initial point $t = 0$ can be determined by (5.13) and (5.29). According to (4.8) and (5.13), substitute (5.29) into $\tilde{H}_{\theta_2^*}(T)$, the optimal wealth process with $t \in [0, \frac{T}{2}]$ is determined as:

$$\begin{aligned} x^*(t) &= \hat{H}_{\theta_1^*}(t)^{-1} E[(\lambda - \mu\tilde{H}_{\theta_2^*}(T)) H_{\theta_1^*}(\frac{T}{2})\tilde{H}_{\theta_2^*}(T)|\mathcal{F}_t] \\ &= \hat{H}_{\theta_1^*}(t)^{-1} E[E[(\lambda - \mu\tilde{H}_{\theta_2^*}(T))\hat{H}_{\theta_1^*}(\frac{T}{2})\tilde{H}_{\theta_2^*}(T)|\mathcal{F}_{\frac{T}{2}}]|\mathcal{F}_t] \\ &= \lambda e^{-r(T-t)} + (x_0 - \lambda e^{-rT}) \cdot e^{\int_0^t [r + \nu(s)'\sigma_1(0)\theta_1^*(0) - \frac{1}{2}|\nu(s)'\sigma_1(0)|^2] ds + \int_0^t \nu(s)'\sigma_1(0) dW(s)} \end{aligned}$$

Applying Ito's formula, we have

$$dx(t) = [rx(t) + \pi^*(t)'\sigma_1(0)\theta_1^*(0)]dt + \pi^*(t)'\sigma_1(0)dW(t)$$

where

$$\pi^*(t) = (x_0 - \lambda e^{-rT})\nu(t) \cdot e^{\int_0^t [r+\nu(s)'\sigma_1(0)\theta_1^*(0) - \frac{1}{2}|\nu(s)'\sigma_1(0)|^2]ds + \int_0^t \nu(s)'\sigma_1(0)dW(s)} \tag{5.32}$$

Compare the dynamics of $x(t)$ with (5.5), note that $\sigma_1(0)\theta_1^*(0) = B_1(0)$, we can conclude that the optimal portfolio is given by (5.32).

5.3.2 Period Two: $t \in [\frac{T}{2}, T]$

If $t \in [\frac{T}{2}, T]$, then the problem is exactly the same as the one in Jin and Zhou's (2005) paper, since both θ_1 and θ_2 are known at $t \in [\frac{T}{2}, T]$, then directly, we have

$$\begin{aligned}
x^*(t) &= [H_{\theta_1^*}(\frac{T}{2})\tilde{H}_{\theta_2^*}(t)]^{-1}E[(\lambda - \mu\tilde{H}_{\theta_2^*}(T))H_{\theta_1^*}(\frac{T}{2})\tilde{H}_{\theta_2^*}(T)|\mathcal{F}_t] \\
&= \tilde{H}_{\theta_2^*}(t)^{-1}E[(\lambda - \mu\tilde{H}_{\theta_2^*}(T))\tilde{H}_{\theta_2^*}(T)|\mathcal{F}_t] \\
&= \lambda e^{-r(T-t)} - \mu e^{-(T-t)(2r - |\theta_2^*(\frac{T}{2})|^2)}\tilde{H}_{\theta_2^*}(t) \\
&= \lambda e^{-r(T-t)} - \mu e^{-\frac{T}{4}(6r - 5|\theta_2^*(\frac{T}{2})|^2)} \cdot e^{(r - \frac{3}{2}|\theta_2^*(\frac{T}{2})|^2)t - \theta_2^*(\frac{T}{2})'(W(t) - W(\frac{T}{2}))}
\end{aligned}$$

where μ is given by (5.31). Then the optimal portfolio is

$$\pi^*(t) = [\lambda e^{-r(T-t)} - x^*(t)]u_2$$

where, u_2 is an m-dimensional vector such that $\sigma_2(\frac{T}{2})'u_2 = \theta_2^*(\frac{T}{2})$.

To summarise, the optimal wealth process is

$$x^*(t) = \begin{cases} \lambda e^{-r(T-t)} + (x_0 - \lambda e^{-rT}) \cdot e^{\int_0^t [r+\nu(s)'\sigma_1(0)\theta_1^*(0) - \frac{1}{2}|\nu(s)'\sigma_1(0)|^2]ds + \int_0^t \nu(s)'\sigma_1(0)dW(s)}, t \in [0, \frac{T}{2}] \\ \lambda e^{-r(T-t)} - \mu e^{-\frac{T}{4}(6r - 5|\theta_2^*(\frac{T}{2})|^2)} \cdot e^{(r - \frac{3}{2}|\theta_2^*(\frac{T}{2})|^2)t - \theta_2^*(\frac{T}{2})'(W(t) - W(\frac{T}{2}))}, \quad t \in [\frac{T}{2}, T] \end{cases} \tag{5.33}$$

and the optimal portfolio is

$$\pi^*(t) = \begin{cases} (x_0 - \lambda e^{-rT})\nu(t) \cdot e^{\int_0^t [r+\nu(s)'\sigma_1(0)\theta_1^*(0) - \frac{1}{2}|\nu(s)'\sigma_1(0)|^2]ds + \int_0^t \nu(s)'\sigma_1(0)dW(s)}, t \in [0, \frac{T}{2}] \\ (\lambda e^{-r(T-t)} - x^*(t))u_2, \quad t \in [\frac{T}{2}, T] \end{cases} \tag{5.34}$$

Remark: Up to now, I have obtained the optimal wealth processes and optimal portfolios explicitly for the original problem in [0,T]. The optimal wealth process and optimal terminal wealth are piecewise expressed because of such "two-period" randomness of the parameters.

5.4 Open questions

In this chapter, simple randomness is added to the volatility rate and appreciation rate. Specifically, we make $\sigma(t)$ and $a(t)$ piecewise defined functions. The values at $[0, \frac{T}{2}]$ are determined at $t = 0$ and the values at $[\frac{T}{2}, T]$ are determined at $t = \frac{T}{2}$, namely they are $\mathcal{F}_{\frac{T}{2}}$-adapted random variables, and the original problem becomes a two-period problem. I have solved this two-period problem and obtained the optimal wealth processes and optimal portfolios explicitly. However, this is only the first step for the problem of getting the optimal portfolio with the parameters being general random processes. On the basis of my work, further study should be carried on. Specifically, we should continue repeating the solving process in each interval for infinitely many times. In other words we view the general solution the limit of the current result, and we should find the value of such limit if possible.

We should notice that even for this two-period case, the problem is solved only when portfolios are unconstrained. Recall in section 4.2.2, four cases are studied and explicit forms of optimal portfolio are obtained with deterministic parameters. Thus such extension work of the other three cases are still open questions, however they should not be a very tough task.

In this chapter, the interest rate r is assumed to be a constant (when $r(t)$ is a time-variant deterministic function, the solutions are the same). While the interest rate itself can be a random process, and there has been a number of researches on interest rate modelling. Whether optimal portfolio can be obtained explicitly with random interest rate is still unknown. While applying the results in interest rate modelling to portfolio selection problem is a challenging topic.

Chapter 6

Conclusion Remark

Markowitz's pioneering work in single-period mean-variance portfolio selection problem has become the foundation for modern portfolio optimisation problem, although it looks somewhat simple nowadays.

Compared with a large number of related researches under the expected utility framework, study on mean-variance portfolio selection problem in multi-period and continuous time has not become active until Li and Ng's (2000) work. Thereafter, during this decade, many fruitful results have been obtained by Zhou and his colleagues, both in complete and incomplete markets. In Chapter 3 and Chapter 4, I reviewed the development of continuous time mean-variance portfolio selection problem, while stochastic LQ control approach and martingale approach were introduced respectively.

In view of all the existing literature, the optimal portfolio and efficient frontier have been derived explicitly only when the market parameters are deterministic functions, while it is difficult to obtain the explicit forms of them with random parameters. In Chapter 5 I solved a simplified model in which only simple form of randomness is considered, with the similar technique as in Jin and Zhou (2005). I derived the optimal wealth process and optimal portfolio explicitly. As I mentioned previously, considerable amount of work is left to be done to extend my result to the case where parameters are general random processes.

Bibliography

[1] Bielecki, T.R., Pliska, S.R., Jin, H. and Zhou, X.Y. (2005). "Continuous-time mean-variance portfolio selection with bankruptcy prohibition", *Mathematical Finance 15, 213-244.*

[2] Chen, A., Jen, C. and Zionts, S. (1971). "The optimal portfolio revision policy", *Journal of Business 44, 51-61.*

[3] Cox, J.C. and Huang, C.F. (1986). "A variational problem arising in financial economics", *MIT mimeo, Sloan School of Management, Massachusetts Institute of Technology, Cambridge, MA.*

[4] Cox, J.C. and Huang, C.F. (1989). "Optimal consumption and portfolio policies when asset prices follow a diffusion process", *Journal of Economic Theory 49, 33-83.*

[5] Cvitanic, J. and Karatzas, I. (1992). "Convex duality in constrained portfolio optimization", *Annals of Applied Probability 2, 767-818.*

[6] Duffie, D. and Richardson, H.R. (1991). "Mean variance hedging in continuous time", *Annals of Applied Probability 1, 1-15.*

[7] Grauer, R.R. and Hakansson, N.H. (1993). "On the use of mean-variance and quadratic approximations in implementing dynamic investment strategies: A comparison of the returns and investment policies", *Management Science 39, 856-871.*

[8] Hakansson, N.H. (1971). "Capital growth and the mean-variance approach to portfolio management", *Journal of Financial Quantitative Analysis 6, 517-557.*

[9] Harrison, J.M. and Kreps, D. (1979). "Martingales and arbitrage in multiperiod securities market", *Journal of Economic Theory 20, 381-408.*

[10] He, H. and Pearson, N. (1991). "Consumption and portfolio policies with incomplete markets and short-sale constraints: The finite-dimensional case", *Mathematical Finance 1, 1-10.*

[11] Jin, H. (2004). “Continuous-time portfolio optimization”, *PhD Thesis submitted to the Chinese University of Hong Kong.*

[12] Jin, H. and Zhou, X.Y. (2005). “Continuous-time Markowitz’s problems in an incomplete market, with no-shorting portfolios”, *Stochastic Analysis and Applications, The Abel Symposium.*

[13] Karatzas, I. (1989). “Optimization problems in the theory of continuous trading”, *SIAM Journal on Control and Optimization 27, 1221-1259.*

[14] Karatzas, I., Lehoczky, J. and Shreve, S. (1987). “Optimal portfolio and consumption decisions for a ’small investor’ on a finite horizon”, *SIAM Journal on Control and Optimization 25, 1557-1586.*

[15] Karatzas, I., Lehoczky, J. and Shreve, S. and Xu, G. (1991). “Martingale and duality methods for utility maximization in an incomplete market”, *SIAM Journal on Control and Optimization 29, 702-730.*

[16] Korn, R. and Trautmann, S. (1995). “Continuous-time portfolio optimization under terminal wealth constraints”, *ZOR 42, 69-93.*

[17] Li, D. and Ng, W.L. (2000). “Optimal dynamic portfolio selection: multiperiod mean-variance formulation”, *Mathematical Finance 10, 387-406.*

[18] Li, X., Zhou, X.Y. and Lim, A.E.B. (2001). “Dynamic mean-variance portfolio selection with no-shorting constraints”, *SIAM Journal on Control and Optimization 40, 1540-1555.*

[19] Lim, A.E.B. (2004). “Quadratic hedging and mean-variance portfolio selection with random parameters in an incomplete market”, *Mathematics of Operations Research 29, 132-161.*

[20] Lim, A.E.B. and Zhou, X.Y. (2002). “Mean-variance portfolio seelection with random parameters in a complete market”, *Mathematics of Operations Research 27, 101-120.*

[21] Markowitz, H. (1952). “Portfolio selection”, *Journal of Finance 7, 77-91.*

[22] Markowitz, H. (1959). “Portfolio selection: sfficient diversification of investment”, *John Wiley and Sons, New York.*

[23] Markowitz, H. (2004). Private Communication.

[24] Merton, R. (1969). "Lifetime portfolio selection under uncertainty: the continuous-time case", *Review of Economics and Statistics 51, 247-256.*

[25] Merton, R. (1973). "Optimum consumption and portfolio rules in a continuous time model", *Erratum 6, 213-214.*

[26] Pages, H. (1987). "Optimal consumption and portfolio policies when markets are incomplete", *MIT mimeo, Massachusetts Institute of Technology, Cambridge, MA.*

[27] Pliska, S.R. (1982). "A discrete time stochastic decision model", *Advances in Filtering and Optimal Stochastic Control, edited by Fleming, W.H. and Gorostiza, L.G., Lecture Notes in Control and Information Sciences, 42 Springer-Verlag, New York, 290-304.*

[28] Pliska, S.R. (1986). "A stochastic calculus model of continuous trading: optimal portfolios", *Mathematics of Operations Research 11, 371-384.*

[29] Pliska, S.R. (1997). "Introduction to Mathematical Finance", *Blackwell Publishers, Oxford.*

[30] Samuelson, P. (1969). "Lifetime portfolio selection by dynamic stochastic programming", *The Review of Economics and Statistics 50, 239-246.*

[31] Samuelson, P. (1986). "Lifetime portfolio selection by dynamic stochastic programming", *Review on Economic Statistics 51, 371-382.*

[32] Schachermayer, W. (2001). "Optimal investment in incomplete markets when wealth may become negative", *Annals of Applied Probability 11, 694-734.*

[33] Smith, K. (1967). "A transition model for portfolio revision", *Journal of Finance 22, 425-439.*

[34] Yong, J. and Zhou, X.Y. (1999). "Stochastic controls: Hamiltonian systems and HJB equations", *springer, New York.*

[35] Zhou, X.Y. and Li, D. (2000). "Continuous time portfolio selection: a stochastic LQ framework", *Applied Mathematics and Optimization. 42, 19-33.*

[36] Zhou, X.Y. and Jin, H. (2008). "Stochastic control and dynamic asset allocation I, II", *Lecture Notes for MSc Mathematical and Computational Finance at the University of Oxford.*

[37] Zhou, X.Y. and Yin, G. (2003). "Markowitz's mean-variance portfolio selection with regime switching: A continuous-time model", *SIAM Journal on Control and Optimization 42, 1466-1482.*