Continuous time mean-variance portfolio optimization through the mean field approach

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January 30, 2014

Abstract

A simple mean-variance portfolio optimization problem in continuous time is solved using the mean field approach. In this approach, the original optimal control problem, which is time inconsistent, is viewed as the McKean-Vlasov limit of a family of controlled many-component weakly interacting systems. The prelimit problems are solved by dynamic programming, and the solution to the original problem is obtained by passage to the limit.

2010 AMS subject classifications: 91G10, 49L20; 82C22, 60H10.

Key words and phrases: Portfolio optimization, mean-variance criterion, optimal control, time inconsistency, dynamic programming, McKean-Vlasov limit, law of large numbers.

1 Introduction

In this paper, we study and solve a mean-variance portfolio optimization problem in continuous time using the mean field approach introduced in the context of discrete time mean-variance problems by Ankirchner and Dermoune [2011].

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The problem of mean-variance portfolio optimization goes back, at least, to Markowitz [1952], who considered a single period (one time step) model. The model allows investors to be risk averse. Investment decisions are therefore made following two different objectives: to maximize expected return and to minimize financial risk. The two conflicting objectives can be combined; decisions are made so as to maximize a difference between expectation and variance of the random quantity representing wealth at terminal time. In a multi-period framework (discrete or continuous time models), this kind of optimization problem is time inconsistent in the sense that an investment strategy that is optimal over the entire time interval need not be optimal over subintervals. As a consequence, Bellman's Principle of Dynamic Programming does not hold for the underlying control problem.

Various approaches to multi-period mean-variance portfolio optimization have been explored; here, we mention three different approaches from the mathematically oriented literature.

The first approach, due to Li and Ng [2000] in a discrete time setting, works by embedding the time inconsistent mean-variance optimization problem into a one-parameter family of standard time-consistent optimal control problems. By solving these problems for each value of the parameter and then choosing the right parameter value according to a compatibility constraint, one can solve the original problem. This technique is extended to continuous time models in Zhou and Li [2000].

A different approach is based on a game theoretic interpretation of time inconsistent optimal control problems. Time inconsistency is interpreted in terms of changes in agents' preferences, and an optimal control problem is viewed as a game where the players are the future incarnations of their own preferences. Solutions of the optimization problem are then defined in terms of sub-game perfect Nash equilibria; see Björk and Murgoci [2010] and the references therein. The framework is used in Björk et al. [2014] to solve a continuous time mean-variance portfolio optimization problems with risk-aversion parameter that may depend on current wealth.

A third approach starts from the observation that dynamic mean-variance optimization problems can be seen as stochastic optimal control problems of McKean-Vlasov or mean field type [for instance Carmona et al., 2013]. For this class of control problems, the coefficients of the costs and dynamics may depend on the law of the controlled state process. As a consequence, the cost functional may be non-linear with respect to the joint law of the state and control process. In the context of mean-variance optimization, the coefficients of the controlled dynamics have standard form, while the cost (or gain) functional is a quadratic polynomial of the expected value of the state process; it is therefore non-linear as a functional of the law of the state process. The connection between mean-variance optimization and control of McKean-Vlasov type has been exploited in Andersson and Djehiche [2011]. There, the authors derive a version of the stochastic Pontryagin maximum principle for continuous time optimal control problems of McKean-Vlasov type. The maximum principle is then used to obtain the optimal control for the mean-variance problem solved in Zhou and Li [2000]. The work of Ankirchner and Dermoune [2011], who treat mean-variance portfolio optimization problem in discrete time, also builds on the connection with McKean-Vlasov control. Those authors, in order to circumvent the difficulties arising from time inconsistency, interpret the original control problem as the McKean-Vlasov limit of a family of controlled weakly interacting systems where interaction through the empirical measure appears only in the gain functional, while each component ("market clone") follows the same dynamics as that of the original controlled system. The approximate K-component control problem has the portfolio mean and portfolio variance replaced with their empirical counterparts, and dynamic programming can be used to determine the value function and an optimal strategy. The original optimization problem is then solved by letting the number of components K tend to infinity.

The aim of this paper is to show that the mean field approach as introduced by Ankirchner and Dermoune [2011] in discrete time can be successfully applied to continuous time models, as well. We consider one of the simplest situations, namely a market model with exactly one risky asset and one risk-free asset. The risk-free asset is assumed to be constant (zero interest rate), while the evolution of the price of the risky asset is modeled as an arithmetic Brownian motion (Bachelier dynamics). The case of a geometric Brownian motion (Merton-Black-Scholes dynamics) can be handled by re-interpreting the control processes: instead of controlling the (fractional, possibly negative) number of assets, control is exerted in terms of the market value of the assets. In the situation considered here, we obtain an explicit expression for the optimal investment strategy in feedback form. The result in itself is, of course, not new. It is essentially a special case of the solution obtained by Zhou and Li [2000] or Andersson and Djehiche [2011]. The result could also be derived by discretization and convergence arguments from the solutions of the discrete time problems studied in Ankirchner and Dermoune [2011]. Here, however, we derive the solution by dynamic programming in continuous time. The same approach could be applied to more complicated mean-variance optimization problems.

The rest of this paper is organized as follows. In Section 2, we formulate the mean-variance optimization problem and the corresponding finite horizon optimal control problem. In Section 3, we introduce auxiliary K-component ("clone") optimal control problems. These problems, which are of linear-quadratic type, are solved explicitly using dynamic programming. In Section 4, we pass to the limit as the number of components (or "clones" or particles) K tends to infinity. We obtain a limit feedback strategy and a limit value function, which are shown to yield the solution to the original optimization problem.

2 Original optimization problem

We consider a financial market consisting of two assets, one risky (for instance, a stock) and one non-risky (a bond). For simplicity, we assume that the interest rate for the bond is equal to zero, the bond price therefore constant and, without loss of generality, equal to one. The price of the risky asset, which we denote by S, is assumed to follow an arithmetic Brownian motion (or Brownian motion with drift). The process S thus verifies the equation

$$dS(t) = \mu\, dt + \sigma\, dW(t) \tag{1}$$

with deterministic initial condition $S(0) = s_0 > 0$. In Eq. (1), $\mu > 0$ is the average rate of return, $\sigma > 0$ the volatility, and $(W(t))_{t\geq 0}$ a standard one-dimensional Wiener process defined on a filtered probability space $(\Omega_\circ, \mathcal{F}^\circ, (\mathcal{F}^\circ_t), \mathbb{P}_\circ)$ satisfying the usual hypotheses.

Taking the point of view of a small investor, let $u(t)$ denote the (not necessarily integer) number of shares of the risky asset he or she holds at any given time t, and let $X(t)$ be the value of the corresponding self-financing portfolio, which consists of the risky and the risk-free asset. The process X

then evolves according to

$$dX(t) = \mu\, u(t)dt + \sigma\, u(t)dW(t) \tag{2}$$

with deterministic initial condition $X(0) = x_0$, where $x_0 > 0$ is the initial capital. There is an implicit constraint on the investment strategy u due to the stochastic environment of the problem, namely, u may depend on the evolution of the random processes only up to the current time. The strategy u must therefore be non-anticipative, that is, u has to be $(\mathcal{F}_t^\circ)$-adapted. Below, we assume that u belongs to $H_T^2((\mathcal{F}_t^\circ))$, the space of all real-valued $(\mathcal{F}_t^\circ)$-progressively measurable processes v such that $\mathbb{E}\left(\int_0^T |v(t)|^2 dt\right) < \infty$, where $T > 0$ denotes the finite time horizon and $\mathbb{E}$ the expectation with respect to $\mathbb{P}_\circ$. As is well known, if no boundedness or integrability conditions were placed on the investment strategies, then any distribution at terminal time could be attained.

Our agent wants to choose a strategy u in order to maximize expected return over a fixed time interval $[0, T]$, while trying at the same time to minimize financial risk. Interpreting risk as the variance of the underlying process and switching from gains to be maximized to costs to be minimized, the optimization problem is therefore to

$$\begin{aligned} &\text{minimize } J(u) \doteq \lambda\, \mathbb{V}\text{ar}\,(X(T)) - \mathbb{E}\,(X(T)) \\ &\text{subject to } \begin{cases} u \in H_T^2((\mathcal{F}_t^\circ)), \\ X \text{ satisfies Eq. (2) with strategy } u \text{ and } X(0) = x_0. \end{cases} \end{aligned} \tag{3}$$

In (3), $\lambda > 0$ is a fixed parameter, the *risk aversion parameter*. A strategy $\bar{u} \in H_T^2((\mathcal{F}_t^\circ))$ is called *optimal* if $J(\bar{u}) = \inf_{u \in H_T^2((\mathcal{F}_t^\circ))} J(u)$. A function $\bar{z}\colon [0, T] \times \mathbb{R} \to \mathbb{R}$ is called an *optimal feedback control* if the equation

$$\bar{X}(t) = x_0 + \mu \int_0^t \bar{z}\left(s, \bar{X}(s)\right) dt + \sigma \int_0^t \bar{z}\left(s, \bar{X}(s)\right) dW(t), \quad t \in [0, T], \tag{4}$$

possesses a unique strong solution $\bar{X}$ such that

$$\bar{u}(t, \omega) \doteq \bar{z}\left(t, \bar{X}(t, \omega)\right), \quad t \in [0, T], \omega \in \Omega_\circ,$$

defines an optimal strategy, that is, $\bar{u} \in H_T^2((\mathcal{F}_t^\circ))$ and $J(\bar{u}) = \inf_{u \in H_T^2((\mathcal{F}_t^\circ))} J(u)$. The strategy $\bar{u}$ will be referred to as the strategy induced by $\bar{z}$, and the process $\bar{X}$ will be referred to as the portfolio process induced by $\bar{z}$. Notice that both $\bar{u}$ and $\bar{X}$ depend on $\bar{z}$ as well as the driving Wiener process.

3 Auxiliary prelimit optimization problems

Let $K \in \mathbb{N} \setminus \{1\}$, and let W_j, $j \in \{1, \ldots, K\}$, be K independent standard one-dimensional Wiener processes defined on some filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})$. The processes W_j can be interpreted as independent clones of the Wiener process in (1). For $j \in \{1, \ldots, K\}$, define S_j according to Eq. (1) with W replaced by W_j. If u_j is a real-valued $(\mathcal{F}_t)$-adapted process, then the value of the agent's portfolio in market clone j follows the dynamics

$$dX_j(t) = \mu\, u_j(t)dt + \sigma\, u_j(t)dW_j(t). \tag{5}$$

The $\mathbb{R}^K$-valued process $\boldsymbol{X} = (X_1, \ldots, X_K)^{\mathsf{T}}$ (we use boldface symbols for K-dimensional column vectors) thus obeys the stochastic differential equation

$$d\boldsymbol{X}(t) = \mu\, \boldsymbol{u}(t)dt + \sigma\, \mathrm{diag}\left(u_1(t), \ldots, u_K(t)\right) d\boldsymbol{W}(t), \tag{6}$$

where $\boldsymbol{u} = (u_1, \ldots, u_K)^{\mathsf{T}}$ is the agent's investment strategy for the K market clones and $\boldsymbol{W} = (W_1, \ldots, W_K)^{\mathsf{T}}$. Clearly, $\boldsymbol{W}$ is a K-dimensional standard Wiener.

Given a vector $\boldsymbol{v} \in \mathbb{R}^K$, the *empirical measure* associated with $\boldsymbol{v}$ is the probability measure on the Borel sets of $\mathbb{R}$ given by

$$\mu^K_{\boldsymbol{v}} \doteq \frac{1}{K} \sum_{j=1}^{K} \delta_{v_j},$$

where δ_x denotes Dirac measure concentrated in $x \in \mathbb{R}$. The *empirical mean* of a K-dimensional vector is the mean calculated with respect to its empirical measure, which equals the arithmetic mean:

$$\mathrm{em}(\boldsymbol{v}) \doteq \int_{\mathbb{R}} y\, \mu^K_{\boldsymbol{v}}(dy) = \frac{1}{K} \sum_{i=1}^{K} v_j.$$

Similarly, the empirical variance of a K-dimensional vector is the variance from the empirical mean calculated with respect to the empirical measure:

$$\begin{aligned} \mathrm{emp}\,\mathbb{V}\mathrm{ar}(\boldsymbol{v}) &\doteq \int_{\mathbb{R}} (y - \mathrm{em}(\boldsymbol{v}))^2\, \mu^K_{\boldsymbol{v}}(dy) \\ &= \frac{1}{K} \sum_{j=1}^{K} (v_j - \mathrm{em}(\boldsymbol{v}))^2 \\ &= \mathrm{em}\left(\boldsymbol{v}^2\right) - \mathrm{em}(\boldsymbol{v})^2. \end{aligned}$$

The square in $\boldsymbol{v}^2$ is to be understood component-wise, that is, $\boldsymbol{v}^2$ is the K-dimensional vector $(v_1^2, \ldots, v_K^2)^\mathsf{T}$.

Observation 1. *In statistics, there are two different standard estimators for the variance of a sample of cardinality K:*

$$\frac{\sum_{i=1}^{K}(x_i - \bar{x})^2}{K} \qquad \text{and} \qquad \frac{\sum_{i=1}^{K}(x_i - \bar{x})^2}{K-1},$$

where $\bar{x} \doteq \frac{1}{K}(x_1 + x_2 + \cdots + x_K)$ is the standard estimator for the mean. The first estimator is called the biased sample variance, the second estimator the unbiased sample variance. For simplicity, we use the first one; notice that for large values of the sample size K the difference is negligible.

Recall that T is the finite time horizon. In setting up the K-clone optimization problem, it will be convenient to allow the initial time and the initial state to vary. In addition, we adopt the weak formulation of a stochastic control problem, that is, the stochastic basis and the driving noise processes are not fixed but part of the control. To be more precise, let $\mathcal{U}_K$ be the set of all triples $((\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P}), \boldsymbol{W}, \boldsymbol{u})$ such that $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})$ is a filtered probability space satisfying the usual hypotheses, $\boldsymbol{W}$ is a K-dimensional standard $(\mathcal{F}_t)$-Wiener process, and $\boldsymbol{u} = (u_1, \ldots, u_K)^\mathsf{T}$ is such that $u_j \in H_T^2((\mathcal{F}_t))$ for every $j \in \{1, \ldots, K\}$. With a slight abuse of notation, we will occasionally write $\boldsymbol{u} \in \mathcal{U}_K$ instead of $((\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P}), \boldsymbol{W}, \boldsymbol{u}) \in \mathcal{U}_K$. For $(t, \boldsymbol{x}) \in [0, T] \times \mathbb{R}^K$, the K-clone optimization problem is to

$$\begin{aligned} &\text{minimize } J_K(t, \boldsymbol{x}; \boldsymbol{u}) \doteq \lambda\, \mathbb{E}\left(\mathrm{em}\left(\boldsymbol{X}(T)^2\right) - \mathrm{em}\left(\boldsymbol{X}(T)\right)^2\right) - \mathbb{E}\left(\mathrm{em}\left(\boldsymbol{X}(T)\right)\right) \\ &\text{subject to } \begin{cases} \boldsymbol{u} \in \mathcal{U}_K, \\ \boldsymbol{X} \text{ solves Eq. (6) with strategy } \boldsymbol{u} \text{ and } \boldsymbol{X}(t) = \boldsymbol{x}. \end{cases} \end{aligned} \tag{7}$$

The *value function* associated with (7) is defined as

$$V_K(t, \boldsymbol{x}) \doteq \inf_{\boldsymbol{u} \in \mathcal{U}_K} J_K(t, \boldsymbol{x}; \boldsymbol{u}), \quad (t, \boldsymbol{x}) \in [0, T] \times \mathbb{R}^K.$$

Theorem 1. *The value function for the optimization problem* (7) *is given by*

$$V_K(t, \boldsymbol{x}) = \frac{1}{4\lambda}\left(e^{\frac{\mu^2(T-t)}{\sigma^2(1-1/K)}} - 1\right) - \mathrm{em}(\boldsymbol{x}) + \lambda e^{-\frac{\mu^2(T-t)}{\sigma^2(1-1/K)}}\left(\mathrm{em}\left(\boldsymbol{x}^2\right) - \mathrm{em}(\boldsymbol{x})^2\right),$$

and the function $\bar{\boldsymbol{z}}^{(K)}\colon [0,T]\times\mathbb{R}^K \to \mathbb{R}^K$ *defined by*

$$\bar{z}_j^{(K)}(t,\boldsymbol{x}) \doteq \frac{\mu}{\sigma^2(1-1/K)}\left(\mathrm{em}(\boldsymbol{x}) - x_j + \frac{1}{2\lambda}e^{\frac{\mu^2(T-t)}{\sigma^2(1-1/K)}}\right), \quad j\in\{1,\dots,K\},$$

yields an optimal feedback control.

Proof. The cost functional J_K is linear-quadratic in the state vector $\boldsymbol{X}(T)$. In fact, denoting by $\boldsymbol{c}$ the K-dimensional vector $\frac{-1}{K}(1,\cdots,1)^{\mathsf{T}}$ and by G the $K\times K$ symmetric matrix

$$\frac{2\lambda}{K^2}\begin{bmatrix} K-1 & -1 & -1 & \cdots & -1 \\ -1 & K-1 & -1 & \cdots & -1 \\ -1 & -1 & K-1 & \cdots & -1 \\ \vdots & \vdots & \vdots & \ddots & -1 \\ -1 & -1 & -1 & -1 & K-1 \end{bmatrix},$$

we have

$$J_K(t,\boldsymbol{x},\boldsymbol{z}) = \frac{1}{2}\langle G\boldsymbol{X}(T),\boldsymbol{X}(T)\rangle + \langle \boldsymbol{c},\boldsymbol{X}(T)\rangle .$$

The controlled dynamics, which are given by Eq. (2), can be rewritten as

$$d\boldsymbol{X}(t) = \boldsymbol{b}\left(t,\boldsymbol{X}(t),\boldsymbol{z}(t)\right)dt + \Sigma\left(t,\boldsymbol{X}(t),\boldsymbol{z}(t)\right)d\boldsymbol{W}(t)$$

with functions $\boldsymbol{b}\colon [0,\infty)\times\mathbb{R}^K\times\mathbb{R}^K\to\mathbb{R}^K$, $\Sigma\colon [0,\infty)\times\mathbb{R}^K\times\mathbb{R}^K\to\mathbb{R}^{K\times K}$ defined as

$$b_j(t,\boldsymbol{x},\boldsymbol{\gamma}) \doteq \mu\,\gamma_j, \qquad \Sigma_{ij}(t,\boldsymbol{x},\boldsymbol{\gamma}) \doteq \sigma\,\gamma_j\,\delta_{ij}, \qquad i,j\in\{1,\dots,K\}.$$

The control problem (7) is thus of the linear-quadratic type (i.e., linear-affine dynamics, linear-quadratic costs); its *generalized Hamiltonian* H is given by

$$H\left(t,\boldsymbol{x},\boldsymbol{\gamma},\boldsymbol{p},P\right) \doteq \sum_{j=1}^{K}\left(\mu\,p_j\gamma_j + \frac{\sigma^2}{2}P_{jj}\gamma_j^2\right).$$

By the dynamic programming principle, the value function should solve (at least in the sense of viscosity solutions) the Hamilton-Jacobi-Bellman terminal value problem

$$\begin{cases} -\frac{\partial}{\partial t}v - \inf_{\boldsymbol{\gamma}\in\mathbb{R}^K} H\left(t,\boldsymbol{x},\boldsymbol{\gamma},\nabla_{\boldsymbol{x}}v, D^2_{\boldsymbol{xx}}v\right) = 0 & \text{if } (t,\boldsymbol{x})\in[0,T)\times\mathbb{R}^K, \\ v(T,\boldsymbol{x}) = \frac{1}{2}\langle G\boldsymbol{x},\boldsymbol{x}\rangle + \langle \boldsymbol{c},\boldsymbol{x}\rangle & \text{if } t=T, \boldsymbol{x}\in\mathbb{R}^K. \end{cases} \tag{8}$$

The static optimization problem in (8) can be solved explicitly: For $(t, \boldsymbol{x}, \boldsymbol{p}) \in [0,T] \times \mathbb{R}^K \times \mathbb{R}^K$, $P = (P_{ij})$ a symmetric $K \times K$ matrix with non-zero diagonal entries,

$$\operatorname{argmin}_{\boldsymbol{\gamma}\in\mathbb{R}^K} H\left(t, \boldsymbol{x}, \boldsymbol{\gamma}, \boldsymbol{p}, P\right) = \left(-\frac{\mu}{\sigma}\frac{p_1}{P_{11}}, \dots, -\frac{\mu}{\sigma}\frac{p_K}{P_{KK}}\right)^{\mathsf{T}}. \tag{9}$$

By a standard verification theorem [cf. Theorem III.8.1 in Fleming and Soner, 2006, p. 135], if v is a classical solution of (8), that is, v is in $C^{1,2}([0,T] \times \mathbb{R}^K)$, satisfies (8) in the sense of classical calculus, and v as well as its first and second order partial derivatives are of at most polynomial growth, then v coincides with the value function of the optimization problem (7). In view of the form of the controlled dynamics (linear in the control, no explicit state dependency), the linear-quadratic terminal costs, zero running costs, and the finite time horizon T, a good guess for $v\colon [0,T] \times \mathbb{R}^K \to \mathbb{R}^K$ is

$$v(t, \boldsymbol{x}) = \frac{e^{\alpha(T-t)}}{2} \langle G\boldsymbol{x}, \boldsymbol{x}\rangle + e^{\beta(T-t)} \langle \boldsymbol{c}, \boldsymbol{x}\rangle + f(t)$$

for some constants $\alpha, \beta \in [0, \infty)$ and some function $f \in C^1([0,T])$ such that $f(T) = 0$. With this ansatz, v is in $C^{1,2}([0,T] \times \mathbb{R}^K)$; plugging v into (8) and using (9), one finds the unknown parameters α, β, $f(\cdot)$ according to

$$\alpha = -\frac{\mu^2}{\sigma^2}\frac{K}{K-1}, \quad \beta = 0, \quad f(t) = \frac{1}{4\lambda}\left(1 - e^{\frac{\mu^2(T-t)}{\sigma^2(1-1/K)}}\right), \quad t \in [0,T].$$

With this choice of the parameters,

$$v(t, \boldsymbol{x}) = \frac{1}{4\lambda}\left(e^{\frac{\mu^2(T-t)}{\sigma^2(1-1/K)}} - 1\right) - \operatorname{em}(\boldsymbol{x}) + \lambda e^{-\frac{\mu^2(T-t)}{\sigma^2(1-1/K)}} \left(\operatorname{em}\left(\boldsymbol{x}^2\right) - \operatorname{em}(\boldsymbol{x})^2\right),$$

which, by the verification theorem cited above, is equal to the value function of control problem (7). Calculating the derivatives $\nabla_{\boldsymbol{x}} v(t, \boldsymbol{x})$, $D^2_{\boldsymbol{x}\boldsymbol{x}} v(t, \boldsymbol{x})$ and plugging them into (9) yields the feedback control $\bar{\boldsymbol{z}}^{(K)}$ introduced above. Notice that $\bar{\boldsymbol{z}}^{(K)}$ is Lipschitz continuous in the state vector $\boldsymbol{x}$ so that the equation

$$\begin{aligned} d\boldsymbol{X}(t) &= \mu\, \bar{\boldsymbol{z}}^{(K)}\left(t, \boldsymbol{X}(t)\right) dt \\ &\quad + \sigma \operatorname{diag}\left(\bar{z}_1^{(K)}\left(t, \boldsymbol{X}(t)\right), \dots, \bar{z}_K^{(K)}\left(t, \boldsymbol{X}(t)\right)\right) d\boldsymbol{W}(t), \end{aligned}$$

possesses a unique (Markovian) solution given any deterministic initial condition. It follows [cf. Fleming and Soner, 2006, p. 136] that $\bar{\boldsymbol{z}}^{(K)}$ is an optimal feedback control. Since the argmin in (9) is unique, the optimal feedback control $\bar{\boldsymbol{z}}^{(K)}$ is unique. The corresponding optimal strategy, given an initial condition $\boldsymbol{X}(t_0) = \boldsymbol{x}$, are determined according to $\boldsymbol{u}(t,\omega) \doteq \bar{\boldsymbol{z}}^{(K)}(t, \boldsymbol{X}(t,\omega))$, $t \geq t_0$, $\omega \in \Omega$, and induce elements of $\mathcal{U}_K$. $\square$

4 Passage to the limit

We next show that by letting the number of clones K tend to infinity, we obtain a feedback control that is optimal for the original optimization problem (3). By Theorem 1, the optimal feedback control for the K-clone optimization problem is given by $\bar{\boldsymbol{z}}^{(K)} = \left(\bar{z}_1^{(K)}, \ldots, \bar{z}_K^{(K)}\right)^{\mathsf{T}}$ with

$$\bar{z}_j^{(K)}(t, \boldsymbol{x}) = \frac{\mu}{\sigma^2(1-1/K)} \left(\mathrm{em}(\boldsymbol{x}) - x_j + \frac{1}{2\lambda} e^{\frac{\mu^2(T-t)}{\sigma^2(1-1/K)}} \right). \tag{10}$$

In passing to the limit, it will be convenient to work with a fixed stochastic basis carrying an infinite family of independent one-dimensional standard Wiener processes. Thus, let $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})$ be a filtered probability space satisfying the usual hypotheses and carrying a sequence $(W_j)_{j\in\mathbb{N}}$ of independent one-dimensional $(\mathcal{F}_t)$-Wiener processes. Recall that x_0 is the initial state at time zero for the portfolio process of the original problem (3). For $K \in \mathbb{N} \setminus \{1\}$, set $\boldsymbol{x}_0^{(K)} \doteq (x_0, \ldots, x_0)^{\mathsf{T}} \in \mathbb{R}^K$, and let $\bar{\boldsymbol{X}}^{(K)} = \left(\bar{X}_1^{(K)}, \ldots, \bar{X}_K^{(K)}\right)^{\mathsf{T}}$ denote the unique strong solution of the system of stochastic differential equations

$$d\bar{X}_j^{(K)}(t) = \mu\, \bar{z}_j^{(K)}\left(t, \bar{\boldsymbol{X}}^{(K)}(t)\right) dt + \sigma\, \bar{z}_j^{(K)}\left(t, \bar{\boldsymbol{X}}^{(K)}(t)\right) dW_j(t), \tag{11}$$

$j \in \{1, \ldots, K\}$, with initial condition $\bar{\boldsymbol{X}}^{(K)}(0) = \boldsymbol{x}_0^{(K)}$. The process $\bar{\boldsymbol{X}}^{(K)}$ has the same distribution as the unique solution to Eq. (6) (or the system of equations determined by (5)) when feedback control $\bar{\boldsymbol{z}}^{(K)}$ is applied and the initial condition is $\boldsymbol{x}_0^{(K)}$ at time zero; for the corresponding costs we have

$$\lambda \left(\mathrm{em}\left(\bar{\boldsymbol{X}}^{(K)}(T)^2 \right) - \mathrm{em}\left(\bar{\boldsymbol{X}}^{(K)}(T) \right) \right) - \mathrm{em}\left(\bar{\boldsymbol{X}}^{(K)}(T) \right) = V_K\left(0, \boldsymbol{x}_0^{(K)}\right).$$

Define the function $\bar{z}\colon [0,T] \times \mathbb{R} \to \mathbb{R}$ by

$$\bar{z}(t,x) \doteq \frac{\mu}{\sigma^2} \left(x_0 + \frac{1}{2\lambda} e^{\frac{\mu^2}{\sigma^2} T} - x \right). \tag{12}$$

The function $\bar{z}$ will turn out to be an optimal feedback control for the optimization problem of Section 2. Notice that $\bar{z}$ is Lipschitz continuous in the space variable. For $j \in \mathbb{N}$, let $\bar{X}_j$ be the unique strong solution of

$$\bar{X}_j(t) = x_0 + \mu \int_0^t \bar{z}\left(s, \bar{X}_j(s)\right) ds + \sigma \int_0^t \bar{z}\left(s, \bar{X}_j(s)\right) dW_j(s). \tag{13}$$

Lemma 1. *Define functions $m, n\colon [0,T] \to \mathbb{R}$ by*

$$\begin{aligned}
m(t) &\doteq x_0 + \frac{1}{2\lambda}\left(e^{\frac{\mu^2}{\sigma^2}T} - e^{\frac{\mu^2}{\sigma^2}(T-t)}\right)\\
n(t) &\doteq x_0^2 + \frac{x_0}{\lambda}\left(e^{\frac{\mu^2}{\sigma^2}T} - e^{\frac{\mu^2}{\sigma^2}(T-t)}\right) + \frac{1}{4\lambda^2}\left(e^{2\frac{\mu^2}{\sigma^2}(T-t)} - e^{\frac{\mu^2}{\sigma^2}(2T-t)}\right).
\end{aligned}$$

Then the following convergences hold uniformly in $t \in [0,T]$:

a) $\lim_{K\to\infty} \mathrm{em}\left(\bar{\boldsymbol{X}}^{(K)}(t)\right) = m(t)$ in $L^2(\mathbb{P})$;

b) $\lim_{K\to\infty} \mathbb{E}\left(\mathrm{em}\left(\left(\bar{\boldsymbol{X}}^{(K)}(t)\right)^2\right)\right) = n(t)$;

c) for every $j \in \mathbb{N}$, $\lim_{K\to\infty} \bar{z}_j^{(K)}(t, \bar{\boldsymbol{X}}^{(K)}(t)) = \bar{z}(t, \bar{X}_j(t))$ and $\lim_{K\to\infty} \bar{X}_j^{(K)}(t) = \bar{X}_j(t)$ in $L^2(\mathbb{P})$.

Moreover, for every $j \in \mathbb{N}$, every $t \in [0,T]$,

$$\mathbb{E}\left[\bar{X}_j(t)\right] = m(t), \qquad \mathbb{E}\left[\left(\bar{X}_j(t)\right)^2\right] = n(t).$$

Proof. For $K \in \mathbb{N} \setminus \{1\}$, $t \in [0,T]$, set

$$Y^{(K)}(t) \doteq \mathrm{em}\left(\bar{\boldsymbol{X}}^{(K)}(t)\right), \quad L^{(K)}(t) \doteq \mathrm{em}\left(\bar{\boldsymbol{X}}^{(K)}(t)^2\right), \quad C_K \doteq \frac{\mu}{\sigma^2(1-1/K)},$$

$$g_K(t) \doteq \frac{1}{2\lambda} e^{\frac{\mu^2}{\sigma^2(1-1/K)}t}, \qquad g(t) \doteq \frac{1}{2\lambda} e^{\frac{\mu^2}{\sigma^2}t}.$$

Clearly, $C_K \to \mu/\sigma^2$, $g_K(t) \to g(t)$ uniformly in $t \in [0,T]$ as $K \to \infty$. By definition, $Y^{(K)}(t) = \frac{1}{K}\sum_{j=1}^K X_j^{(K)}(t)$, $L^{(K)}(t) = \frac{1}{K}\sum_{j=1}^K X_j^{(K)}(t)^2$. Notice that

$$\begin{aligned}
m(t) &= x_0 + g(T) - g(T-t),\\
n(t) &= m(t)^2 - g(T-t/2)^2 + g(T-t)^2.
\end{aligned}$$

In view of (10) and (11), $Y^{(K)}$ solves the stochastic differential equation

$$\begin{aligned} dY^{(K)}(t) &= \mu\, C_K\, g_K(T-t)\, dt \\ &\qquad + \frac{1}{K}\sum_{j=1}^{K} \sigma\, C_K \left(Y^{(K)}(t) - X_j^{(K)}(t) + g_K(T-t)\right) dW_j(t) \end{aligned} \tag{14}$$

with initial condition $Y^{(K)}(0) = x_0$, while $L^{(K)}$ solves

$$\begin{aligned} &dL^{(K)}(t) \\ &= 2\mu\, C_K \left(Y^{(K)}(t)^2 - L^{(K)}(t) + Y^{(K)}(t)\, g_K(T-t)\right) dt \\ &\quad + \sigma^2 C_K^2 \left(L^{(K)}(t) - Y^{(K)}(t)^2 + g_K(T-t)^2\right) dt \\ &\quad + \frac{1}{K}\sum_{j=1}^{K} 2\sigma\, C_K\, X_j^{(K)}(t) \left(Y^{(K)}(t) - X_j^{(K)}(t) + g_K(T-t)\right) dW_j(t) \end{aligned} \tag{15}$$

with initial condition $L^{(K)}(0) = x_0^2$. By the independence of the Wiener processes $W_1, \ldots, W_K$, the Burkholder-Davis-Gundy inequality, and Gronwall's lemma, $X_1^{(K)}, \ldots, X_K^{(K)}$, $Y^{(K)}$, $L^{(K)}$ possess moments of any polynomial order; moreover,

$$\sup_{K\in\mathbb{N}} \sup_{s\in[0,T]} \mathbb{E}\left(L^{(K)}(s)\right) = \sup_{K\in\mathbb{N}} \sup_{s\in[0,T]} \mathbb{E}\left(\frac{1}{K}\sum_{j=1}^{K} \left|X_j^{(K)}(s)\right|^2\right) < \infty. \tag{16}$$

In order to prove the limit in a), it is enough to show that $\limsup_{K\to\infty} \sup_{t\in[0,T]} \mathbb{E}\left(\left|Y^{(K)}(t) - m(t)\right|^2\right) \leq 0$. Integration of Eq. (14) yields, for every $t \in [0,T]$,

$$\begin{aligned} Y^{(K)}(t) &= x_0 + g_K(T) - g_K(T-t) \\ &\qquad + \frac{1}{K}\sum_{j=1}^{K} \int_0^t \sigma\, C_K \left(Y^{(K)}(s) - X_j^{(K)}(s) + g_K(T-s)\right) dW_j(s). \end{aligned}$$

Using Itô's isometry and the independence of the Wiener processes, one finds that

$$\begin{aligned}
&\mathbb{E}\left(\left|Y^{(K)}(t)-m(t)\right|^2\right)\\
&\leq 2\left(g_K(T)-g(T)+g(T-t)-g_K(T-t)\right)\\
&\quad+\frac{2\mu^2}{\sigma^2(K-1)^2}\,\mathbb{E}\left(\left(\sum_{j=1}^{K}\int_0^t\left(Y^{(K)}(s)-X_j^{(K)}(s)+g_K(T-s)\right)dW_j(s)\right)^2\right)\\
&=2\left(g_K(T)-g(T)+g(T-t)-g_K(T-t)\right)\\
&\quad+\frac{2\mu^2}{\sigma^2(K-1)^2}\sum_{j=1}^{K}\int_0^t\mathbb{E}\left(\left(Y^{(K)}(s)-X_j^{(K)}(s)+g_K(T-s)\right)^2\right)ds\\
&\leq 2\left(g_K(T)-g(T)+g(T-t)-g_K(T-t)\right)\\
&\quad+\frac{16T\mu^2}{\sigma^2(K-1)^2}\left(2K\,g(T)+\sup_{s\in[0,T]}\mathbb{E}\left(\sum_{j=1}^{K}\left(X_j^{(K)}(s)\right)^2\right)\right).
\end{aligned}$$

Since $g_K(t)\to g(t)$ uniformly and thanks to (16),

$$\begin{aligned}
&\limsup_{K\to\infty}\sup_{t\in[0,T]}\mathbb{E}\left(\left|Y^{(K)}(t)-m(t)\right|^2\right)\\
&\leq\limsup_{K\to\infty}\frac{16T\mu^2K}{\sigma^2(K-1)^2}\left(2\,g(T)+\sup_{s\in[0,T]}\mathbb{E}\left(\frac{1}{K}\sum_{j=1}^{K}\left(X_j^{(K)}(s)\right)^2\right)\right)=0.
\end{aligned}$$

In order to prove b), set

$$n_K(t)\doteq\mathbb{E}\left(L^{(K)}(t)\right),\quad t\in[0,T].$$

We have to show that $\lim_{K\to\infty}n_K(t)=n(t)$ for every $t\in[0,T]$. Integration of Eq. (15) yields, for $t\in[0,T]$,

$$\begin{aligned}
&L^{(K)}(t)\\
&=x_0^2+\left(2\mu\,C_K-\sigma^2C_K^2\right)\int_0^t\left(Y^{(K)}(s)^2-L^{(K)}(s)\right)ds\\
&\quad+2\mu\,C_K\int_0^t Y^{(K)}(s)\,g_K(T-s)\,ds+\sigma^2C_K^2\int_0^t g_K(T-s)^2ds\\
&\quad+\frac{1}{K}\sum_{j=1}^{K}\int_0^t 2\sigma\,C_K\,X_j^{(K)}(s)\left(Y^{(K)}(s)-X_j^{(K)}(s)+g_K(T-s)\right)dW_j(s).
\end{aligned}$$

The stochastic integral in the above display is a true martingale thanks to the L^p-integrability of the components of $\boldsymbol{X}^{(K)}$. Using the Fubini-Tonelli theorem, it follows that $n_K(\cdot)$ satisfies the integral equation

$$\begin{aligned}
& n_K(t) \\
& = x_0^2 + \left(2\mu\, C_K - \sigma^2 C_K^2\right) \int_0^t \left(\mathbb{E}\left(Y^{(K)}(s)^2\right) - n_K(s)\right) ds \\
& \quad + 2\mu\, C_K \int_0^t \mathbb{E}\left(Y^{(K)}(s)\right) g_K(T-s)\, ds + \sigma^2 C_K^2 \int_0^t g_K(T-s)^2 ds
\end{aligned}$$

Notice that $\lim_{K\to\infty} \sigma^2 C_K^2 = \frac{\mu^2}{\sigma^2} = \lim_{K\to\infty} \mu\, C_K$. Let $\bar{n}(\cdot)$ denote the unique solution over $[0,T]$ of the integral equation

$$\bar{n}(t) = x_0^2 + \frac{\mu^2}{\sigma^2} \int_0^t (m(s) + g(T-s))^2\, ds - \frac{\mu^2}{\sigma^2} \int_0^t \bar{n}(s) ds.$$

By part a), $\mathbb{E}\left(Y^{(K)}(s)\right) \to m(s)$ and $\mathbb{E}\left(Y^{(K)}(s)^2\right) \to m(s)^2$ as $K \to \infty$, uniformly in $s \in [0,T]$. It follows that $\lim_{K\to\infty} n_K(t) = \bar{n}(t)$, uniformly in $t \in [0,T]$. Since $m(s) + g(T-s) = x_0 + \frac{1}{2\lambda} e^{\frac{\mu^2}{\sigma^2}T}$, $\bar{n}(\cdot)$ is the unique solution over $[0,T]$ of the integral equation

$$\bar{n}(t) = x_0^2 + \frac{\mu^2}{\sigma^2} t \left(x_0 + \frac{1}{2\lambda} e^{\frac{\mu^2}{\sigma^2}T}\right)^2 - \frac{\mu^2}{\sigma^2} \int_0^t \bar{n}(s) ds$$

or, equivalently, the unique solution over $[0,T]$ of the differential equation

$$\dot{\bar{n}}(t) = \frac{\mu^2}{\sigma^2} \left(x_0 + \frac{1}{2\lambda} e^{\frac{\mu^2}{\sigma^2}T}\right)^2 - \frac{\mu^2}{\sigma^2} \bar{n}(t)$$

with initial condition $\bar{n}(0) = x_0^2$. That solution is given by

$$\begin{aligned}
\bar{n}(t) &= e^{-\frac{\mu^2}{\sigma^2}t} \left(x_0^2 + \frac{\mu^2}{\sigma^2} \left(x_0 + \frac{1}{2\lambda} e^{\frac{\mu^2}{\sigma^2}T}\right)^2 \int_0^t e^{\frac{\mu^2}{\sigma^2}s} ds \right) \\
&= x_0^2 + \frac{x_0}{\lambda} \left(e^{\frac{\mu^2}{\sigma^2}T} - e^{\frac{\mu^2}{\sigma^2}(T-t)} \right) + \frac{1}{4\lambda^2} \left(e^{2\frac{\mu^2}{\sigma^2}T} - e^{\frac{\mu^2}{\sigma^2}(2T-t)} \right).
\end{aligned}$$

Therefore, $\lim_{K\to\infty} n_K(t) = \bar{n}(t) = n(t)$, uniformly in $t \in [0,T]$.

For part c), fix $j \in \mathbb{N}$, and let $\bar{X}_j$ be the unique solution of Eq. (13) with $\bar{X}_j(0) = x_0$ and driving Wiener process W_j. In order to prove c), it is enough to show that

$\limsup_{K\to\infty} \sup_{t\in[0,T]} \mathbb{E}\left(\left|\bar{z}_j^{(K)}\left(t, \bar{\boldsymbol{X}}^{(K)}(t)\right) - \bar{z}\left(t, \bar{X}_j(t)\right)\right|^2\right) \leq 0$, and analogously for $\bar{X}_j^{(K)}(t)$ and $\bar{X}_j(t)$. For $t \in [0,T]$, $K \in \mathbb{N}$, set

$$
\begin{aligned}
R_K(t) \doteq{} & \frac{\mu}{\sigma^2(1-1/K)}\left(\operatorname{em}\left(\bar{\boldsymbol{X}}_j^K(t)\right) + g_K(T-t)\right) - \frac{\mu}{\sigma^2}\left(x_0 + g(T)\right) \\
& + \frac{\mu}{\sigma^2(K-1)} \bar{X}_j^{(K)}(t).
\end{aligned}
$$

Clearly, $\lim_{K\to\infty} \frac{\mu}{\sigma^2(1-1/K)} = \frac{\mu}{\sigma^2}$. Thanks to part a),

$$
\operatorname{em}\left(\bar{\boldsymbol{X}}^{(K)}(t)\right) g_K(T-t) \overset{K\to\infty}{\longrightarrow} m(t) + g(T-t) = x_0 + g(T)
$$

in $L^2(\mathbb{P})$, uniformly in $t \in [0,T]$. By the symmetry of Eq. (11) and the initial condition, $\bar{X}_1^{(K)}(t), \ldots, \bar{X}_K^{(K)}(t)$ have the same distribution for every $t \in [0,T]$. Estimate (16) therefore implies that

$$
\sup_{K\in\mathbb{N}} \sup_{t\in[0,T]} \mathbb{E}\left(\left|\bar{X}_j^{(K)}(t)\right|^2\right) < \infty.
$$

It follows that

$$
\sup_{t\in[0,T]} \mathbb{E}\left(\left|R_K(t)\right|^2\right) \overset{K\to\infty}{\longrightarrow} 0.
$$

Now, for every $t \in [0,T]$,

$$
\begin{aligned}
\left|\bar{z}_j^{(K)}\left(t, \bar{\boldsymbol{X}}^{(K)}(t)\right) - \bar{z}\left(t, \bar{X}_j(t)\right)\right|^2 &= \left|\frac{\mu}{\sigma^2}\left(\bar{X}_j^{(K)}(t) - \bar{X}_j(t)\right) + R_K(t)\right|^2 \\
&\leq 2\frac{\mu^2}{\sigma^4}\left|\bar{X}_j^{(K)}(t) - \bar{X}_j(t)\right|^2 + 2\left|R_K(t)\right|^2.
\end{aligned}
$$

In view of Eq. (11) and Eq. (13), respectively, using Hölder's inequality, Itô's isometry, and the Fubini-Tonelli theorem, we have

$$
\begin{aligned}
& \mathbb{E}\left(\left|\bar{X}_j^{(K)}(t) - \bar{X}_j(t)\right|^2\right) \\
& \leq 2\mu^2\, \mathbb{E}\left(\left(\int_0^t \left(\bar{z}_j^{(K)}\left(s, \bar{\boldsymbol{X}}^{(K)}(s)\right) - \bar{z}\left(s, \bar{X}_j(s)\right)\right) ds\right)^2\right) \\
& \quad + 2\sigma^2\, \mathbb{E}\left(\left(\int_0^t \left(\bar{z}_j^{(K)}\left(s, \bar{\boldsymbol{X}}^{(K)}(s)\right) - \bar{z}\left(s, \bar{X}_j(s)\right)\right) dW_j(s)\right)^2\right) \\
& \leq 2\left(\mu^2 T + \sigma^2\right) \int_0^t \mathbb{E}\left(\left|\bar{z}_j^{(K)}\left(s, \bar{\boldsymbol{X}}^{(K)}(s)\right) - \bar{z}\left(s, \bar{X}_j(s)\right)\right|^2\right) ds.
\end{aligned}
$$

It follows that, for every $t \in [0,T]$,

$$
\begin{aligned}
&\mathbb{E}\left(\left|\bar{z}_j^{(K)}\left(t, \bar{\boldsymbol{X}}^{(K)}(t)\right) - \bar{z}\left(t, \bar{X}_j(t)\right)\right|^2\right) \\
&\leq 4\left(\frac{\mu^4}{\sigma^4}T + \frac{\mu^2}{\sigma^2}\right)\int_0^t \mathbb{E}\left(\left|\bar{z}_j^{(K)}\left(s, \bar{\boldsymbol{X}}^{(K)}(s)\right) - \bar{z}\left(s, \bar{X}_j(s)\right)\right|^2\right) ds \\
&\quad + 2\sup_{s\in[0,T]} \mathbb{E}\left(\left|R_K(s)\right|^2\right).
\end{aligned}
$$

Therefore, by Gronwall's lemma,

$$
\begin{aligned}
&\sup_{t\in[0,T]} \mathbb{E}\left(\left|\bar{z}_j^{(K)}\left(t, \bar{\boldsymbol{X}}^{(K)}(t)\right) - \bar{z}\left(t, \bar{X}_j(t)\right)\right|^2\right) \\
&\qquad\qquad\qquad\qquad \leq 2\sup_{t\in[0,T]} \mathbb{E}\left(\left|R_K(t)\right|^2\right) e^{4\left(\frac{\mu^4}{\sigma^4}T+\frac{\mu^2}{\sigma^2}\right)T}.
\end{aligned}
$$

Since $\sup_{t\in[0,T]} \mathbb{E}\left(\left|R_K(t)\right|^2\right) \to 0$ as $K \to \infty$, we have

$$
\limsup_{K\to\infty} \sup_{t\in[0,T]} \mathbb{E}\left(\left|\bar{z}_j^{(K)}\left(t, \bar{\boldsymbol{X}}^{(K)}(t)\right) - \bar{z}\left(t, \bar{X}_j(t)\right)\right|^2\right) \leq 0.
$$

Similarly, for every $t \in [0,T]$,

$$
\begin{aligned}
\mathbb{E}\left(\left|\bar{X}_j^{(K)}(t) - \bar{X}_j(t)\right|^2\right) &\leq 4\left(\frac{\mu^4}{\sigma^4}T + \frac{\mu^2}{\sigma^2}\right)\int_0^t \mathbb{E}\left(\left|\bar{X}^{(K)}(s) - \bar{X}_j(s)\right|^2\right) ds \\
&\quad + 4T\left(\mu^2 T + \sigma^2\right)\sup_{s\in[0,T]} \mathbb{E}\left(\left|R_K(s)\right|^2\right).
\end{aligned}
$$

Thus, by Gronwall's lemma and since $\sup_{t\in[0,T]} \mathbb{E}\left(\left|R_K(t)\right|^2\right) \to 0$,

$$
\limsup_{K\to\infty} \sup_{t\in[0,T]} \mathbb{E}\left(\left|\bar{X}_j^{(K)}(t) - \bar{X}_j(t)\right|^2\right) \leq 0.
$$

The last part of the assertion is now a consequence of parts a), b), c, and the fact that $\boldsymbol{X}^{(K)}$ has identically distributed components for every $K \in \mathbb{N}$. Alternatively, the first and second moments of $\bar{X}_j(t)$ can be calculated directly from Eq. (13) and (12), the definition of $\bar{z}$. $\square$

As Theorem 2 below shows, $\bar{z}$ as defined by (12) yields an optimal feedback control for the original mean-variance optimization problem. Notice that $\bar{z}$ is affine-linear in the current state x, does not depend on the current time t, while it depends on the time horizon T as well as the initial state x_0. This last dependence is due to the non-linear nature of the cost functional, which makes the optimization problem inconsistent in time.

Theorem 2. *The function*

$$\bar{z}(t,x) \doteq \frac{\mu}{\sigma^2}\left(x_0 + \frac{1}{2\lambda} e^{\frac{\mu^2}{\sigma^2}T} - x\right), \quad (t,x) \in [0,T]\times\mathbb{R},$$

yields an optimal feedback control for the original optimization problem (3), *and the minimal costs are given by*

$$\lambda\, \mathbb{V}\mathrm{ar}\left(\bar{X}(T)\right) - \mathbb{E}\left(\bar{X}(T)\right) = \frac{1}{4\lambda}\left(e^{\frac{\mu^2}{\sigma^2}T} - 1\right) - x_0,$$

where $\bar{X}$ is the portfolio process induced by $\bar{z}$.

Proof. The strategies and portfolio processes of the original optimization problem (3) are defined on the filtered probability space $(\Omega_\circ, \mathcal{F}^\circ, (\mathcal{F}^\circ_t), \mathbb{P}_\circ)$ with W an $(\mathcal{F}^\circ_t)$-Wiener process. Let $\bar{X}$ be the portfolio process induced by $\bar{z}$, that is, $\bar{X}$ is the unique strong solution of Eq. (4):

$$\bar{X}(t) = x_0 + \mu\int_0^t \bar{z}\left(s, \bar{X}(s)\right) ds + \sigma\int_0^t \bar{z}\left(s,\bar{X}(s)\right) dW(s), \quad t\in[0,T].$$

Let $\bar{u}$ be the strategy induced by $\bar{z}$:

$$\bar{u}(t,\omega) \doteq \bar{z}\left(t, \bar{X}(t,\omega)\right), \quad t\in[0,T],\ \omega\in\Omega.$$

Then $\bar{u}$ is real-valued square-integrable process with continuous trajectories, and $\bar{u}$ is adapted to the filtration generated by the Wiener process W. It follows that $\bar{u}\in H^2_T((\mathcal{F}^\circ_t))$. Moreover, $\bar{X}$ coincides with the unique solution of Eq. (2) with strategy $u = \bar{u}$ and initial condition $\bar{X}(0) = x_0$. By definition,

$$J(\bar{u}) = \lambda\left(\mathbb{E}\left(\bar{X}(T)^2\right) - \mathbb{E}\left(\bar{X}(T)\right)^2\right) - \mathbb{E}\left(\bar{X}(T)\right).$$

Let $\tilde{u}\in H^2_T((\mathcal{F}^\circ_t))$ be any admissible strategy, and let $\tilde{X}$ be the unique solution of Eq. (2) with strategy $u = \tilde{u}$ and initial condition $\tilde{X}(0) = x_0$. To prove the statement, we have to show that $J(\bar{u}) = \frac{1}{4\lambda}\left(e^{\frac{\mu^2}{\sigma^2}T} - 1\right) - x_0$ and that $J(\bar{u}) \leq J(\tilde{u})$. Set

$$\Omega \doteq \times_{i=1}^{\infty}\Omega_{\circ}, \quad \mathcal{F} \doteq \otimes_{i=1}^{\infty}\mathcal{F}^{\circ}, \quad \mathcal{F}_t \doteq \otimes_{i=1}^{\infty}\mathcal{F}_t^{\circ},\ t \geq 0, \quad \mathbb{P} = \otimes_{i=1}^{\infty}\mathbb{P}_{\circ},$$

Then $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})$ is a filtered probability space satisfying the usual hypotheses. For every $j \in \mathbb{N}$, define processes W_j, $\bar{u}_j$, $\bar{X}_j$, $\tilde{u}_j$, $\tilde{X}_j$ by setting, for $t \geq 0$, $\omega = (\omega_i)_{i\in\mathbb{N}} \in \Omega$,

$$\begin{aligned} W_j(t,\omega) \doteq W(t,\omega_j), \quad & \bar{u}_j(t,\omega) \doteq \bar{u}(t,\omega_j), \quad && \bar{X}_j(t,\omega) \doteq \bar{X}(t,\omega_j), \\ & \tilde{u}_j(t,\omega) \doteq \tilde{u}(t,\omega_j), \quad && \tilde{X}_j(t,\omega) \doteq \tilde{X}(t,\omega_j). \end{aligned}$$

All processes thus defined are $(\mathcal{F}_t)$-progressively measurable, and $W_1, W_2, \ldots$ are independent standard $(\mathcal{F}_t)$-Wiener processes. The processes W_j, $\bar{u}_j$, $\bar{X}_j$, $\tilde{u}_j$, $\tilde{X}_j$, $j \in \mathbb{N}$, are i.i.d. copies of the processes W, $\bar{u}$, $\bar{X}$, $\tilde{u}$, $\tilde{X}$. More precisely, $(W_j, \bar{u}_j, \bar{X}_j, \tilde{u}_j, \tilde{X}_j)_{j\in\mathbb{N}}$ is a family of independent and identically distributed $\mathbb{R}^5$-valued processes living on $(\Omega, \mathcal{F}, \mathbb{P})$, and for every $j \in \mathbb{N}$,

$$\mathbb{P} \circ \left(W_j, \bar{u}_j, \bar{X}_j, \tilde{u}_j, \tilde{X}_j\right)^{-1} = \mathbb{P}_{\circ} \circ \left(W, \bar{u}, \bar{X}, \tilde{u}, \tilde{X}\right)^{-1}.$$

In fact, $\bar{X}_j$, $\tilde{X}_j$ solve Eq. (2) with Wiener process W_j in place of W, initial condition x_0 at time zero, and strategy $u = \bar{u}$ and $u = \tilde{u}$, respectively. Moreover, $\bar{X}_j$ solves Eq. (13). By Lemma 1, it follows that

$$\begin{aligned} J(\bar{u}) &= \lambda\left(\mathbb{E}\left(\bar{X}_1(T)^2\right) - \mathbb{E}\left(\bar{X}_1(T)\right)^2\right) - \mathbb{E}\left(\bar{X}_1(T)\right) \\ &= \lambda\left(n(T) - m(T)^2\right) - m(T) \\ &= \frac{1}{4\lambda}\left(e^{\frac{\mu^2}{\sigma^2}T} - 1\right) - x_0. \end{aligned}$$

It remains to show that $J(\tilde{u}) \geq J(\bar{u})$. Observe that

$$J(\bar{u}) = \lim_{K\to\infty} V_K\left(0, \boldsymbol{x}_0^{(K)}\right),$$

where V_K is the value function of Theorem 1 and $\boldsymbol{x}_0^{(K)} = (x_0, \ldots, x_0)^{\mathsf{T}} \in \mathbb{R}^K$ as above. By construction,

$$J(\tilde{u}) = \lambda\left(\mathbb{E}\left(\tilde{X}_1(T)^2\right) - \mathbb{E}\left(\tilde{X}_1(T)\right)^2\right) - \mathbb{E}\left(\tilde{X}_1(T)\right).$$

For $K \in \mathbb{N} \setminus \{1\}$, set $\tilde{\boldsymbol{X}}^{(K)} \doteq (\tilde{X}_1, \ldots, \tilde{X}_K)^\mathsf{T}$ and $\tilde{\boldsymbol{u}}^{(K)} \doteq (\tilde{u}_1, \ldots, \tilde{u}_K)^\mathsf{T}$. Then

$$\left((\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P}), (W_1, \ldots, W_K)^\mathsf{T}, \tilde{\boldsymbol{u}}^{(K)}\right) \in \mathcal{U}_K$$

and $\tilde{\boldsymbol{X}}^{(K)}$ solves Eq. (6) with strategy $\boldsymbol{u} = \tilde{\boldsymbol{u}}^{(K)}$ and initial condition $\tilde{\boldsymbol{X}}^{(K)}(0) = \boldsymbol{x}_0^{(K)}$. Recalling the definition of the cost functional in (7), we have

$$\begin{aligned} &J_K\left(0, \boldsymbol{x}_0^{(K)}; \tilde{\boldsymbol{u}}^{(K)}\right) \\ &\quad = \lambda\, \mathbb{E}\left(\mathrm{em}\left(\tilde{\boldsymbol{X}}^{(K)}(T)^2\right) - \mathrm{em}\left(\tilde{\boldsymbol{X}}^{(K)}(T)\right)^2\right) - \mathbb{E}\left(\mathrm{em}\left(\tilde{\boldsymbol{X}}^{(K)}(T)\right)\right). \end{aligned}$$

By construction, $(\tilde{X}_j(T))_{j\in\mathbb{N}}$ is an i.i.d. sequence of real-valued square-integrable random variables on $(\Omega, \mathcal{F}, \mathbb{P})$. Therefore,

$$\mathbb{E}\left(\mathrm{em}\left(\tilde{\boldsymbol{X}}^{(K)}(T)\right)\right) = \mathbb{E}\left(\tilde{X}_1(T)\right), \quad \mathbb{E}\left(\mathrm{em}\left(\tilde{\boldsymbol{X}}^{(K)}(T)^2\right)\right) = \mathbb{E}\left(\tilde{X}_1(T)^2\right),$$

while

$$\mathbb{E}\left(\mathrm{em}\left(\tilde{\boldsymbol{X}}^{(K)}(T)\right)^2\right) = \frac{K(K-1)}{K^2}\, \mathbb{E}\left(\tilde{X}_1(T)\right)^2 + \frac{K}{K^2}\, \mathbb{E}\left(\tilde{X}_1(T)^2\right).$$

It follows that

$$\lim_{K\to\infty} J_K\left(0, \boldsymbol{x}_0^{(K)}; \tilde{\boldsymbol{u}}^{(K)}\right) = J(\tilde{u}).$$

Since $J_K\left(0, \boldsymbol{x}_0^{(K)}; \tilde{\boldsymbol{u}}^{(K)}\right) \geq V_K\left(0, \boldsymbol{x}_0^{(K)}\right)$ for every $K \in \mathbb{N} \setminus \{1\}$ by definition of the value function and $\lim_{K\to\infty} V_K\left(0, \boldsymbol{x}_0^{(K)}\right) = J(\bar{u})$, we conclude that $J(\tilde{u}) \geq J(\bar{u})$. □

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