MULTIPERIOD MEAN-VARIANCE PORTFOLIO OPTIMIZATION VIA MARKET CLONING

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Abstract

The problem of finding the mean variance optimal portfolio in a multiperiod model can not be solved directly by means of dynamic programming. In order to find a solution we therefore first introduce independent market clones having the same distributional properties as the original market, and we replace the portfolio mean and variance by their *empirical* counterparts. We then use dynamic programming to derive portfolios maximizing a weighted sum of the empirical mean and variance. By letting the number of market clones converge to infinity we are able to solve the original mean variance problem.

Keywords. Dynamic programming, mean variance optimization, optimal portfolios, market clones, independent returns, empirical mean.
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1 Introduction

We consider the multi-period version of the mean variance portfolio optimization problem, famously considered first in the one period case by Markowitz [5], [6]. In the multiperiod case a difficulty in solving the optimization problem stems from the fact that dynamic programming techniques may not be applied directly since the variance does not satisfy the tower property of conditional expectations. In order to circumvent this hurdle, we introduce many independent clones of the market. As an auxiliary step, we first aim at finding a Markov strategy that, simultaneously applied to *all* clones, maximizes a weighted sum of the empirical mean and empirical variance of the terminal wealth obtained in any market clone. Though dynamic programming does not apply to cost functions involving a variance, it *does* apply to cost functions involving *empirical variances*. We can thus explicitly solve the auxiliary problem by means of dynamic programming. Letting the number of clones tend to infinity, and invoking the strong law of large numbers, we then obtain the solution of the original mean variance portfolio optimization problem.

Throughout we will consider an economic agent rebalancing portfolios at discrete trading times $0 = t_0 < t_1 < \cdots < t_N = T$, with T being a finite time horizon. We allow only for self-financing portfolio strategies and denote the agent's portfolio value at time t_n by r_n. We suppose that our agent aims at maximizing the weighted sum of the terminal revenues' expectation and variance

$$E(r_N) - \lambda\, var(r_N) = E(Q_\lambda(r_N, r_N^2)), \tag{1}$$

where $\lambda > 0$ is referred to as the *risk-aversion parameter*, and $Q_\lambda(a,b) = a + \lambda a^2 - \lambda b$ as the *target function*.

In the case where the assets have independent returns, a solution of the mean variance problem (1) has been given in [2]. In order to solve (1), in [2] the authors consider first the auxiliary problem of finding the portfolio maximizing the weighted sum of the first and second moments

$$aE(r_N) - \lambda E(r_N^2), \tag{2}$$

with $a, \lambda \in \mathbb{R}$. They show that *if* z^* is a solution of (1), then there exists a parameter a^* for which z^* maximizes (2) (see Theorem 1 in [2]). With this necessary condition at hand, the authors derive for any parameter set (a, λ) an explicit representation of the portfolio $z^*(a, \lambda)$ maximizing (2). Then they choose among the portfolios $\{z^*(a, \lambda) : a \in \mathbb{R}\}$ the one that maximizes (1).

We present here a new approach that allows to determine the solution of (1). In contrast to [2], we perform a *verification*, guaranteeing that the solution obtained is *indeed* an optimal solution of (1). Besides, we consider not only the case of independent returns, but also of independent increments.

The problem of selecting an optimal portfolio has been studied in many research articles. Frequently the weighted sum of revenues' expectation and variance is replaced by other target functions, e.g. utility functions, that allow to use dynamic programming directly. For a selection of references we refer to [2]. We also draw the attention to the recent paper [1] that, in the case of the classical continuous Black and Scholes model, solves the mean-variance problem using the maximum principle.

The paper is organized as follows. In Section 2 we describe in more detail the model setup and how we approximate problem (1) by using empirical variances. We then solve the approximating problem by means of stochastic dynamic programming. In Section 3 we do this first for arithmetic price processes, i.e. price processes having independent increments. In Section 4, by letting the number of market clones converge to infinity, we are able to derive the solution of (1). Finally, we turn to geometric market models, i.e. models with price processes having independent returns. In Section 5 we first solve the approximating problem, and then, in Section 6, we derive the solution of (1).

2 A Bellman principle for the multiperiod mean-variance portfolio optimization problem

We suppose that the agent invests in a financial market consisting of $l+1$, $l \in \mathbb{N}$, assets. One of these assets is considered as non-risky and taken as numéraire. The other assets are considered as *risky*, and the price at time t_n of one share of risky asset i, in units of the numéraire, will be denoted by $s_n(i)$, $1 \leq i \leq l$.

We denote by $(z_n)_{0 \leq n \leq N-1}$ an investment strategy, interpreting $z_n(i)$, $1 \leq i \leq l$, as the number of shares of risky asset i in the agent's portfolio between trading times t_n and t_{n+1}.

The agent's starting capital or initial revenue at time t_0 will be denoted by r_0. Following an investment plan (z_n), the agent's revenues up to time t_n are given by

$$r_n = r_0 + \sum_{k=0}^{n-1} < z_k, s_{k+1} - s_k > = \; r_0 + \sum_{k=0}^{n-1} \sum_{i=1}^{l} z_k(i)(s_{k+1}(i) - s_k(i)).$$

It follows that the sequence $(r_n : n = 0, ..., N)$ satisfies the dynamics

$$r_{n+1} = r_n + < z_n, s_{n+1} - s_n >, \quad n = 0, ..., N-1.$$

We allow for the portfolio z_n to incorporate any market information revealed up to time t_n, but not beyond. Mathematically this means that we require any strategy to be adapted to the filtration generated by the asset price process $\mathcal{F}_n = \sigma(s_k : k \leq n)$, $0 \leq n \leq N$.

The strategy maximizing (1) can not be solved directly with dynamic programming techniques. The main reason being that the variance does not satisfy the tower property with respect to conditioning. In other words, the variance of a conditional variance does not coincide with the variance. Nevertheless, we will see that one can break down the problem of finding the maximizer of (1) into N similar single-step subproblems.

For a sequence $(\lambda_n : n = 0, ..., N)$ one can derive, conditional to $r_n = r$ and $s_n = s$, the optimal single step strategies z_n maximizing

$$E(Q_{\lambda_{n+1}}(r_{n+1}, r_{n+1}^2)), \tag{3}$$

over all $z \in \mathbb{R}^l$. We show that there exists a sequence $(\lambda_n : n = 0, ..., N)$ with terminal value $\lambda_N = \lambda$ such that the sequence of the optimal single step strategies $(z_n : n = 0, ..., N-1)$ maximizes the multi step problem (1).

Before giving the idea of the proof we need to introduce some notation. We consider K independent clones $(s_n^j : j = 1, \ldots, K)$ of the price process (s_n). By this we mean that every process (s_n^j) has the same distribution as (s_n). We assume that all these processes are defined on the same probability space $(\Omega, \mathcal{F}, P)$.

If $X = (x^j : j = 1, \ldots, K)$ and $Y = (y^j : j = 1, \ldots, K)$ are two finite vectors of length K, then we write XY for the Hadamard product defined by $XY = (x^j y^j : j = 1, \ldots, K)$. In particular $X^2 := XX = ((x^j)^2 : j = 1, \ldots, K)$. The *empirical mean* of a sequence of real numbers $X = (x^j : j = 1, \ldots, K)$ is denoted by

$$\mathrm{em}(X) = \frac{\sum_{j=1}^K x^j}{K}.$$

Notice that the empirical variance is given by

$$\mathrm{em}(X^2) - (\mathrm{em}(X))^2.$$

Throughout the paper we will use capital letters for the K dimensional vectors, in particular we write

$$\begin{aligned} S_n &= (s_n^1, \ldots, s_n^K), \\ z_n^j &= (z_n^j(1), \ldots, z_n^j(l)), \\ Z_n &= (z_n^1, \ldots, z_n^K), \\ R_n &= (r_n^1, \ldots, r_n^K). \end{aligned}$$

We emphasize that in general $(z_n^j, r_n^j : j = 1, ..., K)$ are not i.i.d. The revenue dynamics for clone j are given by

$$r_{n+1}^j = r_n^j + < z_n^j, s_{n+1}^j - s_n^j >,$$

with initial values

$$r_0^j = r_0, \quad s_0^j = s_0,$$

for all j. The mean-variance optimization will be made in two steps:

Step 1: Approximating the problem by using empirical means and variances.

We fix K, and we replace the weighted sum of the revenues' expectation and variance (1) by its empirical approximation

$$E(\mathrm{em}(R_N) - \lambda \mathrm{em}[(R_N - em(R_N))^2]) = E(Q_\lambda(em(R_N), em((R_N)^2))). \tag{4}$$

Notice that the square product is built *pointwise* in the sense of Hadamard products. We maximize problem (4) among the set of strategies $Z_n = (z_n^j : 1 \leq j \leq K)$ such that $z_n^j = f_n^j(R_n, S_n)$ with f_n^j belonging to the set of measurable functions $\mathbb{R}^{(1+l)\times K} \to \mathbb{R}^l$. We are looking for measurable functions

$$((f_0)^* := ((f_0^j)^*, 1 \leq j \leq K), (f_1)^* := ((f_1^j)^*, 1 \leq j \leq K), \ldots, (f_{N-1})^* := ((f_{N-1}^j)^*, 1 \leq j \leq K))$$

which maximize the problem

$$J_K(s_0) = \sup_{\{f_0,\ldots,f_{N-1}\}} E(Q_\lambda(em(R_N), em((R_N)^2))). \tag{5}$$

The latter problem can be solved following the dynamic programming principle (see Bertsekas [3], [4]). To this end define recursively

$$V_N(R_N, S_N) = Q_\lambda(em(R_N), em((R_N)^2)),$$
$$V_n(R_n, S_n) = \sup_Z E_n(V_{n+1}(R_n + Z(S_{n+1} - S_n), S_{n+1}))$$

with $n = N-1, \ldots, 0$, and E_n denoting the conditional expectation

$$E_n = E(\cdot \mid S_0, \ldots, S_n).$$

Then $V_0(R_0, S_0) = J_K(s_0)$ is the solution of (5).

Step 2: Taking the limit $K \to +\infty$.

We show that for all j the optimal revenues and strategies $(r_n^j, (f_n^j)^*(R_n, S_n))$ converge almost surely to random variables $(\bar{r}_n^j, f_n(\bar{r}_n^j, s_n^j))$, as $K \to +\infty$. The limiting sequence $(\bar{r}_n^j : j \geq 1)$ is i.i.d., and the functions f_n depend only on the distribution of s_{n+1} conditional to s_n. We will show that there exists a sequence $(\bar{\lambda}_n : n = 0, ..., N)$ with terminal value $\bar{\lambda}_N = \lambda$ such that conditional to $r_n = r$ and $s_n = s$, the function $f_n(r, s)$ maximizes

$$E(Q_{\bar{\lambda}_{n+1}}(r_{n+1}, r_{n+1}^2)), \tag{6}$$

over all $z \in \mathbb{R}^l$. Moreover if $(r_n : n = 0, ..., N-1)$ is the solution of the system

$$r_{n+1} = r_n + < f_n(r_n, s_n), s_{n+1} - s_n >,$$

with initial value r_0, then the strategy $(f_n(r_n, s_n) : n = 0, ..., N-1)$ maximizes the weighted sum (1) of the revenues' expectation and variance.

We perform this 2 step program first under the assumption that the financial assets have independent price increments.

3 Solving the approximating problem in the arithmetic case

Suppose that the process (s_n) has independent increments, and let us denote

$$\Delta s_{n+1}^j(i) = s_{n+1}^j(i) - s_n^j(i).$$

Besides, we assume that s_n is square integrable for all $n = 1, ..., N$, and denote by $m_n = (m_n(i))_{1 \le i \le l}$ the vector of conditional expectations, i.e.

$$m_n(i) = E_n(\Delta s_{n+1}(i)),$$

and by q_n the covariance matrix of the increments, i.e.

$$q_n(i,k) = E_n((\Delta s_{n+1}(i) - m_n(i))(\Delta s_{n+1}(k) - m_n(k))),$$

for $1 \le i, k \le l$.

Let us introduce the two sets $I_n = \{i : m_n(i) \neq 0\}$ and $J_n = \{i : m_n(i) = 0\}$. In the sequel we will need the following notations. We denote by A_n the $l \times l$ matrix with entries

$$a_n(i,k) = \begin{cases} m_n(k) + (1 - K^{-1}) \frac{q_n(i,k)}{m_n(i)}, & i \in I_n, \\ q_n(i,k), & i \in J_n, \end{cases}, \tag{7}$$

for all $1 \le k \le l$.

Let $d_n \in \mathbb{R}^l$ be the vector defined by

$$d_n(i) = \begin{cases} 1, & i \in I_n, \\ 0, & i \in J_n. \end{cases}$$

We have now everything at hand we need in order to describe the solution of the approximating problem (4) in the arithmetic case.

Theorem 3.1. *Assume that the process (s_n) is square integrable and has independent increments, and that the matrices A_n defined in (7) are invertible. For all $n \in \{0, \dots, N-1\}$ the value function is given by*

$$V_n(R) = v_n + Q_{\lambda_n}(em(R), em(R^2)). \tag{8}$$

The constants λ_n and v_n are recursively defined as follows: The terminal values are given by $\lambda_N = \lambda$ and $v_N = 0$, and for $n < N$ the constants are given by

$$\begin{aligned} v_n &= v_{n+1} + \frac{< w_n, m_n >}{2\lambda_{n+1}(1- < w_n, m_n >)} - \qquad (9) \\ &\quad (1 - K^{-1}) \frac{< q_n w_n, w_n >}{4\lambda_{n+1}(1- < w_n, m_n >)^2}, \qquad (10) \\ \lambda_n &= \lambda_{n+1}[(1- < w_n, m_n >)^2 + (1 - K^{-1}) < q_n w_n, w_n >], \qquad (11) \end{aligned}$$

where w_n is the vector in $\mathbb{R}^l$ defined by

$$w_n = A_n^{-1} d_n. \tag{12}$$

Moreover, the number of assets between t_n and t_{n+1} to be hold in the optimal portfolio is given by

$$z_n^j \;=\; \frac{1}{2\lambda_{n+1}(1- < w_n, m_n >)} w_n + (em(R) - r^j) w_n. \tag{13}$$

Proof. We proof the result via backward induction. The terminal condition is

$$V_N(R) \;=\; Q_\lambda(em(R), em(R^2)).$$

Suppose that for $n+1 \le N$ the value function V_{n+1} is given by (8). A priori we do not know that V_n only depends on R, but not on S. Then

$$\begin{aligned} V_n(R,S) &= \sup_{Z \in \mathbb{R}^{K \times l}} E_n(V_{n+1}(R+ < Z, \Delta S_{n+1} >)) \\ &= v_{n+1} + \sup_{Z \in \mathbb{R}^{K \times l}} E_n(Q_{\lambda_{n+1}}(\mathrm{em}(R+ < Z, \Delta S_{n+1} >), \mathrm{em}((R+ < Z, \Delta S_{n+1} >)^2)). \end{aligned}$$

Notice that $< Z, \Delta S_{n+1} >= (< z^j, \Delta s_{n+1}^j >: 1 \le j \le K)$ is a K-dimensional vector of conditionally independent random variables. Lemma A.1 implies

$$V_n(R,S) \;=\; v_{n+1} + \sup_Z \varphi_n(Z), \tag{14}$$

where

$$\begin{aligned} \varphi_n(Z) &= Q_{\lambda_{n+1}}[\mathrm{em}(R+ < Z, m_n >), \mathrm{em}((R+ < Z, m_n >)^2)] \\ &\quad -\lambda_{n+1}(1 - K^{-1})\mathrm{em}(< q_n Z, Z >). \end{aligned}$$

We next derive the first order conditions for the maximizer of φ_n. To this end let

$$a = \mathrm{em}(R+ < Z, m_n >), \quad b = \mathrm{em}((R+ < Z, m_n >)^2),$$

and denote by $\partial_a Q_\lambda$ and $\partial_b Q_\lambda$ the partial derivative of Q_λ with respect to its first resp. second variable. Let e_i be the vector of R^l such that $e_i(k) = 1$ if $k = i$, and $e_i(k) = 0$ if $i \neq k$. Notice that

$$\partial_{z^j(i)}(< q_n z^j, z^j >) = \partial_{z^j(i)} \sum_{l=1}^d \sum_{k=1}^d q_n(l,k) z^j(l) z^j(k) = 2 \sum_{l=1}^d q_n(l,i) z^j(l) = 2 < q_n e_i, z^j > .$$

Therefore, the partial derivatives of φ_n satisfy

$$\begin{aligned} \partial_{z^j(i)}\varphi_n(Z) &= \partial_a Q_{\lambda_{n+1}} \partial_{z^j(i)} a + \partial_b Q_{\lambda_{n+1}} \partial_{z^j(i)} b - 2\lambda_{n+1}(1 - K^{-1})K^{-1} < q_n e_i, z^j > \\ &= K^{-1} m_n(i)(1 + 2\lambda_{n+1} a) - 2K^{-1}\lambda_{n+1} m_n(i)(r^j + < z^j, m_n >) \\ &\quad -2\lambda_{n+1}(1 - K^{-1})K^{-1} < q_n e_i, z^j > \\ &= K^{-1} m_n(i) - 2K^{-1}\lambda_{n+1} m_n(i)[r^j + < z^j, m_n > -a] \\ &\quad -2\lambda_{n+1}(1 - K^{-1})K^{-1} < q_n e_i, z^j > . \end{aligned}$$

If $i \in I_n$, then

$$\begin{aligned} < m_n, z > +(1-K^{-1})(m_n(i))^{-1} < q_n e_i, z > &= \sum_{k=1}^{l} \left(m_n(k) + (1-K^{-1}) \frac{q_n(i,k)}{m_n(i)} \right) z(k) \\ &= < e_i, A_n z >, \end{aligned}$$

for all $z \in R^l$, where A_n is $l \times l$ matrix defined in (7). If $i \in J_n$, then we have

$$< q_n e_i, z > = < e_i, A_n z >,$$

for all $z \in R^l$. It follows that the maximizer Z of (14) solves the first order conditions

$$\begin{aligned} \frac{1}{2\lambda_{n+1}} &= [r^j - em(R) + < e_i, A_n z^j > - em(< Z, m_n >)], \quad i \in I_n, j = 1, \dots, K, \\ 0 &= < e_i, A_n z^j >, \quad i \in J_n, j = 1, \dots, K. \end{aligned}$$

We next apply Lemma A.2 with $A = A_n$, G the identity matrix,

$$c_i^j = \left(\frac{1}{2\lambda_{n+1}} - r^j + \mathrm{em}(R) \right) d_n(i)$$

and $b_i^j = m_n(i)$. The constant α, defined in (33), is given by

$$\alpha = \frac{< A_n^{-1} d_n, m_n >}{2\lambda_{n+1}(1 - < A_n^{-1} d_n, m_n >)}.$$

Let w_n be defined as in (12) and let $\beta_n = < w_n, m_n >$. Observe that $\alpha = \frac{\beta_n}{2\lambda_{n+1}(1-\beta_n)}$. The maximizer Z is given by

$$\begin{aligned} z^j &= \alpha A_n^{-1} d_n + (\frac{1}{2\lambda_{n+1}} + em(R) - r^j) A_n^{-1} d_n \\ &= \frac{1}{2\lambda_{n+1}(1-\beta_n)} w_n + (em(R) - r^j) w_n. \end{aligned}$$

It follows that

$$r^j + < z_n^j, m_n > = \frac{\beta_n}{2\lambda_{n+1}(1-\beta_n)} + \mathrm{em}(R)\beta_n + (1-\beta_n) r^j. \tag{15}$$

Using (15) and the fact that $\mathrm{em}((R + < Z_n, m_n >)^2) - [\mathrm{em}(R + < Z_n, m_n >)]^2$ is the empirical variance of the sequence $(r^j + < z_n^j, m_n >)$, $j = 1, \dots, K$, we obtain

$$\mathrm{em}((R + < Z_n, m_n >)^2) - [\mathrm{em}(R + < Z_n, m_n >)]^2 = (1-\beta_n)^2 \left(\mathrm{em}(R^2) - (\mathrm{em}(R))^2 \right),$$

and hence

$$\begin{aligned} & Q_{\lambda_{n+1}}(\mathrm{em}(R + < Z_n, m_n >), \mathrm{em}((R + < Z_n, m_n >)^2)) \\ = & \frac{\beta_n}{2\lambda_{n+1}(1-\beta_n)} + \mathrm{em}(R) - \lambda_{n+1}(1-\beta_n)^2 \left(\mathrm{em}(R^2) - (\mathrm{em}(R))^2 \right). \end{aligned}$$

Moreover we have

$$\mathrm{em}(< q_n Z_n, Z_n >) \;=\; \left(\frac{1}{4\lambda_{n+1}^2(1-\beta_n)^2} + [em(R^2) - (em(R))^2]\right) < q_n w_n, w_n > .$$

Therefore, we may derive from (14) that

$$\begin{aligned} V_n(R,S) &= v_{n+1} + \varphi_n(Z_n) \\ &= v_n + Q_{\lambda_n}(\mathrm{em}(R), \mathrm{em}(R^2)), \end{aligned}$$

where

$$\begin{aligned} v_n &= v_{n+1} + \frac{\beta_n}{2\lambda_{n+1}(1-\beta_n)} - (1-K^{-1}) < q_n w_n, w_n > \frac{1}{4\lambda_{n+1}(1-\beta_n)^2}, \\ \lambda_n &= \lambda_{n+1}[(1-\beta_n)^2 + (1-K^{-1}) < q_n w_n, w_n >]. \end{aligned}$$

□

Remark 3.2. If $l = 1$, then the recursion formula (9) can be considerably simplified. To this end let $a_n = \frac{q_n}{m_n^2} \in (0, \infty]$. In this case, for $n \leq N-1$, the constants λ_n and v_n satisfy

$$\lambda_n = \lambda \prod_{k=n}^{N-1} \frac{(1-K^{-1})a_k}{1+(1-K^{-1})a_k}, \tag{16}$$

$$v_n = v_{n+1} + \frac{1}{2}h_n, \tag{17}$$

where

$$h_n = \frac{1}{2\lambda_{n+1}} \frac{1}{(1-K^{-1})a_n}. \tag{18}$$

In Equation (16) we use the convention that $\frac{\infty}{\infty} = 1$. As $K \to +\infty$, $\lambda_n \to \bar{\lambda}_n$, $v_n \to \bar{v}_n$, with

$$\bar{\lambda}_n = \lambda \prod_{k=n}^{N-1} \frac{a_k}{1+a_k}, \tag{19}$$

$$\bar{v}_n = \bar{v}_{n+1} + \frac{1}{4\bar{\lambda}_{n+1} a_n} \tag{20}$$

for $n \leq N-1$.

4 Taking limits in the arithmetic case

We next show that by letting the number of clones $K \to \infty$, we obtain a strategy solving our original problem (1) for arithmetic price processes.

Equation (13) and the initial conditions $r_0^j = r_0$, for $j = 1, ..., K$, imply

$$\begin{aligned} z_0^j &= \frac{1}{2\lambda_1(1- < w_0, m_0 >)} w_0 + (em(R_0) - r_0^j) w_n \\ &= \frac{1}{2\lambda_1(1- < w_0, m_0 >)} w_0. \end{aligned}$$

It follows that

$$\begin{aligned} r_1^j &= r_0^j + < z_0^j, s_1^j - s_0 >, \\ z_1^j &= \frac{1}{2\lambda_2(1- < w_1, m_1 >)} w_1 + (em(R_1) - r_1^j) w_1, \\ &\vdots \\ r_n^j &= r_{n-1}^j + < z_{n-1}^j, s_n^j - s_{n-1}^j >, \\ z_n^j &= \frac{1}{2\lambda_{n+1}(1- < w_n, m_n >)} w_n + (em(R_n) - r_n^j) w_n, \end{aligned}$$

with $n = 1, \ldots, N$. The pair (w_n, λ_n) possesses a limit $(\bar{w}_n, \bar{\lambda}_n)$ as $K \to +\infty$, for any $n = 0, \ldots, N$. Note that $\bar{w}_n = \bar{A}_n^{-1} d_n$, where $\bar{A}_n$ is the $l \times l$ matrix with entries

$$\bar{a}_n(i,k) = \begin{cases} m_n(k) + \frac{q_n(i,k)}{m_n(i)}, & i \in I_n, \\ q_n(i,k), & i \in J_n, \end{cases}$$

for all $1 \le k \le l$, and $\bar{\lambda}_n$ satisfies the backward recursion formula

$$\bar{\lambda}_n \;=\; \bar{\lambda}_{n+1}[(1- < \bar{w}_n, m_n >)^2 + < q_n \bar{w}_n, \bar{w}_n >],$$

with terminal value $\bar{\lambda}_N = \lambda$.

Lemma 4.1. *The following assertions hold true for all $n \in \{0, \ldots, N\}$:*

$$r_n^j \text{ converges as } K \to \infty, \text{ say to } \bar{r}_n^j, \qquad \text{(H1)}$$

a.s. and in L^2; the sequence of the limiting random variables $(\bar{r}_n^j : j \ge 1)$ is i.i.d. and square integrable. We have

$$\lim_{K \to +\infty} em(R_n Y) = E(\bar{r}_n^1) E(y^1) \qquad \text{(H2)},$$

a.s. and in L^2, and

$$\lim_{K \to \infty} \mathrm{em}(R_n^2 Y) = E((\bar{r}_n^1)^2) E(y^1) \qquad \text{(H3)},$$

a.s. and in L^1 for all i.i.d. sequences $Y = (y^j : j \ge 1)$ having finite second moments and being $\sigma(\Delta S_{n+k} : k \ge 1)$-measurable.

Proof. We prove the statement via induction on n. It is easy to show that (H1) - (H3) hold true for $n = 0$. Next suppose that (H1) - (H3) is true for $n < N$. We show that it is true also for $n + 1$. Let us denote

$$\begin{aligned} c_n &= \frac{1}{2\lambda_{n+1}(1-<w_n, m_n>)} \\ \bar{c}_n &= \frac{1}{2\bar{\lambda}_{n+1}(1-<\bar{w}_n, m_n>)}. \end{aligned}$$

Note that

$$\begin{aligned} r^j_{n+1} &= r^j_n + <z^j_n, \Delta s^j_{n+1}> \\ &= r^j_n + (c_n + \mathrm{em}(R_n) - r^j_n) <w_n, \Delta s^j_{n+1}>, \end{aligned}$$

and from the induction hypothesis we derive that a.s. and in L^2

$$\begin{aligned} r^j_{n+1} &\to \bar{r}^j_n + (\bar{c}_n + E(\bar{r}^1_n) - \bar{r}^j_n) <\bar{w}_n, \Delta s^j_{n+1}> =: \bar{r}^j_{n+1}, \\ \mathrm{em}(R_{n+1}Y) &\to E(\bar{r}^1_{n+1})E(y^1), \end{aligned}$$

for all sequences $Y = (y^j : j \geq 1)$ of i.i.d. random variables having finite second moments and being $\sigma(\Delta S_{n+1+k} : k \geq 1)$-measurable. Thus we have shown (H1) and (H2) for $n+1$. It remains to show (H3), i.e.

$$\mathrm{em}(R^2_{n+1}Y) \to E((\bar{r}^1_{n+1})^2)E(y^1)$$

a.s. and in L^1. Again, from the recursion formula

$$r^j_{n+1} = r^j_n + (c_n + \mathrm{em}(R_n) - r^j_n) <w_n, \Delta s^j_{n+1}>$$

and from the induction hypothesis we can deduce that

$$\mathrm{em}(R^2_{n+1}Y) \to E((\bar{r}^1_{n+1})^2)E(y^1)$$

a.s. and in L^1, which completes the proof.

□

The previous lemma implies that the sequence $(\bar{z}^j_n, \bar{r}^j_n : j \geq 1)$ has the same distribution as (z_n, r_n), recursively defined by

$$\begin{aligned} r_n &= r_{n-1} + <z_{n-1}, s_n - s_{n-1}>, \\ z_n &= (\bar{c}_n + E(r_n) - r_n)\bar{w}_n, \end{aligned}$$

with $n = 1, ..., N$. From that we can show that

$$\begin{aligned} E(r_n) &= \sum_{k=0}^{n-1} \bar{c}_k <\bar{w}_k, m_{k+1}>, \\ z_n &= (\bar{c}_n + \sum_{k=0}^{n-1} \bar{c}_k <\bar{w}_k, m_{k+1}> - r_n)\bar{w}_n, \end{aligned}$$

for all $n = 1, ..., N-1$. It turns out that (z_n) is the strategy solving (1), and (r_n) is its associated value process.

Theorem 4.2. *Suppose that s_n has moments of order 2 for all n, and has independent increments. The strategy $(z_n : n = 0, \ldots, N-1)$ solves the maximization of the weighted sum of the revenues' expectation and variance (1) among all $(\mathcal{F}_n)$-adapted strategies.*

Proof. Note that r_N are terminal revenues the investor gets if she or he follows the strategy (z_n). Denote by $R_N = (r_N^j)_{1 \le j \le K}$ the vector of K "clones" of the terminal revenues when following the optimal strategy $(z_n^j)_{1 \le j \le K}$ given by (13). The empirical mean of R_N converges to the expectation $E(r_N)$, and the empirical mean of R_N^2 to $E(r_N^2)$ as $K \to \infty$. Thus,

$$\lim_{K \to \infty} Q_\lambda(\mathrm{em}(R_N), \mathrm{em}(R_N^2)) = Q_\lambda(E(r_N), E(r_N^2)), \text{ a.s.}$$

Denote by $r_n(g)$ the revenues obtained when following an adapted strategy $g_n = g_n(s_0, \ldots, s_n)$. Moreover, let $g_n^j = g_n(s_0^j, \ldots, s_n^j)$ and $G = (g_n^j : 0 \le n \le N-1, 1 \le j \le K)$. Denote by R_N^G the vector of revenues when following the strategy G. Obviously, G is suboptimal and we have

$$EQ_\lambda(\mathrm{em}(R_N^G), \mathrm{em}((R_N^G)^2)) \le EQ_\lambda(\mathrm{em}(R_N), \mathrm{em}(R_N^2)).$$

Letting $K \to \infty$, the strong law of large numbers implies

$$Q_\lambda(E(r_N(g)), E(r_N^2(g))) \le Q_\lambda(E(r_N), E(r_N^2)),$$

which shows that (z_n) is indeed the optimal strategy. $\square$

5 Solving the approximating problem in the geometric case

Suppose that the process (s_n) has independent returns $\frac{s_{n+1}-s_n}{s_n}$. We suppose that the returns $\frac{s_{n+1}-s_n}{s_n}$ are square integrable for all $n = 0, ..., N-1$. We denote by $p_n \in \mathbb{R}^l$ the vector of expected returns between time t_n and t_{n+1}, i.e.

$$p_n(i) = E\left(\frac{s_{n+1}(i) - s_n(i)}{s_n(i)}\right),$$

$1 \le i \le l$, and by $u_n \in \mathbb{R}^{l \times l}$ the covariance matrix of the returns, i.e.

$$u_n(i,k) = E\left[\left(\frac{s_{n+1}(i) - s_n(i)}{s_n(i)} - p_n(i)\right)\left(\frac{s_{n+1}(k) - s_n(k)}{s_n(k)} - p_n(k)\right)\right],$$

for all $i, k = 1, \ldots, l$. The value function of our approximating problem has the same representation as in the arithmetic case, the only difference being that in the recursion formulas (9) and

(11) the expectation and covariance of the *increments* have to be replaced by the expectation and the covariance of the *returns*. Hence the entries of l by l matrix A_n become

$$a_n(i,k) = \begin{cases} p_n(k) + (1-K^{-1})\frac{u_n(i,k)}{m_n(i)}, & i \in I_n, \\ u_n(i,k), & i \in J_n, \end{cases}$$

for all $1 \leq k \leq l$.

Theorem 5.1. *Assume that the process (s_n) has square integrable and independent returns. For all $n \in \{0, \ldots, N-1\}$ the value function is given by*

$$V_n(R) = v_n + Q_{\lambda_n}(em(R), em(R^2)). \tag{21}$$

Here $\lambda_N = \lambda$ and $v_N = 0$, and for $n < N$, the constants λ_n and v_n are recursively defined by

$$\begin{aligned} v_n &= v_{n+1} + \frac{< w_n, p_n >}{2\lambda_{n+1}(1- < w_n, p_n >)} - (1-K^{-1})\frac{< u_n w_n, w_n >}{4\lambda_{n+1}}, \\ \lambda_n &= \lambda_{n+1}[(1- < w_n, p_n >)^2 + (1-K^{-1}) < u_n w_n, w_n >], \end{aligned}$$

where w_n is the vector in $\mathbb{R}^l$ defined by

$$w_n = A_n^{-1} d_n. \tag{22}$$

Finally, the optimal amount of money to be invested in each share between time t_n and t_{n+1} is given by

$$s_n^j z^j = \frac{1}{2\lambda_{n+1}(1- < w_n, p_n >)} w_n + (em(R) - r^j) w_n.$$

Proof. The proof is based on a backward induction almost identical to one of Theorem 3.1. We therefore summarize only the basic steps. The terminal condition is

$$V_N(R) = Q_\lambda(em(R), em(R^2)).$$

Suppose that for $n+1 \leq N$ the value function V_{n+1} is given by (21).

Let $X_n = (x_n^j(i))$ where $x_n^j(i) = \frac{s_{n+1}^j(i) - s_n^j(i)}{s_n^j(i)}$. Then

$$\begin{aligned} V_n(R, S_n) &= \sup_{Z \in \mathbb{R}^{K \times l}} E_n(V_{n+1}(R+ < Z, \Delta S_{n+1} >)) \\ &= v_{n+1} + \sup_{Z \in \mathbb{R}^{K \times l}} E_n(Q_{\lambda_{n+1}}(\mathrm{em}(R+ < S_n Z, X_n >), \mathrm{em}((R+ < S_n Z, X_n >)^2)). \end{aligned}$$

Lemma A.1 implies

$$V_n(R, S_n) = v_{n+1} + \sup_Z \varphi_n(Z), \tag{23}$$

where

$$\begin{aligned}\varphi_n(Z) &= Q_{\lambda_{n+1}}[\mathrm{em}(R+ < S_n Z, p_n >), \mathrm{em}((R+ < S_n Z, p_n >)^2)] \\ &\quad -\lambda_{n+1}(1-K^{-1})\mathrm{em}(< u_n S_n Z, S_n Z >).\end{aligned}$$

The maximizer Z of φ_n satisfies the first order conditions

$$\begin{aligned} 0 &= 1 - 2\lambda_{n+1}[r^j - em(R) + < e_i, A_n(s_n^j z^j) > -em(< S_n Z, p_n >)], \quad i \in I(n), j = 1, \dots, K, \\ 0 &= < e_i, A_n(s_n^j z^j) >, \quad i \in J(n), j = 1, \dots, K. \end{aligned}$$

Applying Lemma A.2 with $A = A_n$, G the identity matrix,

$$c_i^j = \left(\frac{1}{2\lambda_{n+1}} - r^j + \mathrm{em}(R) \right) d_n(i),$$

and $b_i^j = p_n(i)$, gives us the maximizer Z. Proceeding as in the proof of Theorem 3.1 we get that the value function V_n satisfies (21). □

6 Taking limits in the geometric case

Similarly to the arithmetic case, one can show for geometric models that the optimal solution of the approximating problem (4) converges to the solution of (1) if the number of clones K converges to infinity. Throughout this section we will make the same assumptions and use the same notation as in Section 5. The convergence results are summarized in the following theorem, the proof of which is very similar to the corresponding result in the arithmetic case and therefore omitted.

Theorem 6.1. *1) Sequence $(\lambda_n, v_n, A_n, w_n)$ converges as $K \to +\infty$ to a limit $(\bar\lambda_n, \bar v_n, \bar A_n, \bar w_n)$ given by*

$$\begin{aligned} \bar\lambda_n &= \lambda\, \Pi_{k=n+1}^{N-1}(1- < \bar w_n, p_n >), && (24) \\ \bar v_n &= \bar v_{n+1} + \frac{< \bar w_n, p_n >}{2\bar\lambda_{n+1}(1- < \bar w_n, p_n >)} - \frac{< u_n \bar w_n, \bar w_n >}{4\bar\lambda_{n+1}}, && (25) \end{aligned}$$

with $n < N$, $\bar\lambda_N = \lambda$ and $\bar v_N = 0$.

2) The following assertions hold true for all $n \in \{0, \dots, N\}$:

$$r_n^j \text{ converges as } K \to \infty, \text{ say to } \bar r_n^j, \qquad (H1)$$

a.s. and in L^2; the sequence of the limiting random variables $(\bar r_n^j : j \geq 1)$ is i.i.d. and square integrable. We have

$$\lim_{K \to +\infty} em(R_n Y) = E(\bar r_n^1) E(y^1) \qquad (H2),$$

a.s. and in L^2, and

$$\lim_{K\to\infty} \mathrm{em}(R_n^2 Y) = E((\bar{r}_n^1)^2)E(y^1) \qquad (H3),$$

a.s. and in L^1 for all i.i.d. sequences $Y = (y^j : j \geq 1)$ having finite second moments and being $\sigma(\frac{\Delta S_{n+k}}{S_{n+k-1}} : k \geq 1)$-measurable. The sequence $(\bar{r}_n^j, j \geq 1)$ (respectively $(\bar{z}_n^j : j \geq 1)$) satisfy for all j the dynamic

$$\begin{aligned}
\bar{r}_{n+1}^j &= \bar{r}_n^j + < s_n^j \bar{z}_n^j, \frac{s_{n+1}^j - s_n^j}{s_n^j} >, \\
s_n^j \bar{z}_n^j &= \left(\frac{1}{2\bar{\lambda}_{n+1}(1- < \bar{w}_n, p_n >)} + E(\bar{r}_n^j) - \bar{r}_n^j\right)\bar{w}_n,
\end{aligned}$$

with initial values

$$\begin{aligned}
s_0 \bar{z}_0^j &= \frac{1}{2\bar{\lambda}_1(1- < \bar{w}_0, p_0 >)}\bar{w}_0, \\
\bar{r}_0^j &= r_0.
\end{aligned}$$

3) Let us consider the sequences $(z_n : n = 0, \ldots, N-1)$, and $(r_n : n = 0, \ldots, N)$ solution of the following system:

$$\begin{aligned}
r_{n+1} &= r_n + < s_n z_n, \frac{s_{n+1} - s_n}{s_n} >, & (26)\\
s_n z_n &= \frac{1}{2\bar{\lambda}_{n+1}(1- < \bar{w}_n, p_n >)}\bar{w}_n + (E(r_n) - r_n)\bar{w}_n & (27)\\
E(r_n) &= \sum_{k=0}^{n-1} \frac{< \bar{w}_k, p_k >}{2\bar{\lambda}_{k+1}(1- < \bar{w}_k, p_k >)}, & (28)
\end{aligned}$$

with initial values r_0 and

$$s_0 z_0 = \frac{1}{2\bar{\lambda}_1(1- < \bar{w}_0, p_0 >)}\bar{w}_0.$$

If the prices are such that $\frac{\Delta s_{n+1}}{s_n}$ is square integrable for all n, then the strategy $(z_n : n = 0, \ldots, N-1)$ solves the maximization of the weighted sum of the revenues' expectation and variance (1) among all $(\mathcal{F}_n)$-adapted strategies. The corresponding revenue process is given by $(r_n : n = 0, \ldots, N-1)$.

6.1 The case $l = 1$

If $l = 1$, then the recursion formulas (24) and (25) can be considerably simplified. To this end let $b_n = \frac{u_n}{p_n^2} \in (0, \infty]$. In this case, for $n \leq N-1$, the constants $\bar{\lambda}_n$ and $\bar{v}_n$ satisfy

$$\begin{aligned}
\bar{\lambda}_n &= \lambda \prod_{k=n}^{N-1} \frac{b_k}{1+b_k}, & (29)\\
\bar{v}_n &= \bar{v}_{n+1} + \frac{1}{2}\bar{h}_n, & (30)
\end{aligned}$$

for $n \le N-1$, where

$$\bar{h}_n = \frac{1}{2\bar{\lambda}_{n+1}} \frac{1}{b_n}. \tag{31}$$

In the definition of $\bar{\lambda}_n$ we use the convention that $\frac{\infty}{\infty} = 1$.

Besides, the optimal amount of money invested in the risky asset is given by

$$\begin{aligned} s_n z_n &= \frac{1}{2\bar{\lambda}_{n+1} b_n p_n} + (E(r_n) - r_n)\frac{1}{p_n(1+b_n)} \\ &= (\sum_{k=0}^{n-1} \frac{p_{k+1}}{2\bar{\lambda}_{k+1} b_k p_k} - r_n)\frac{1}{p_n(1+b_n)} + \frac{1}{2\bar{\lambda}_{n+1} b_n p_n} \end{aligned}$$

6.2 The case $l \ge 1$ and the link with Li and Ng

Finally, we discuss the link to the article [2], and show that their results are the same as ours for geometric processes. To this end let

$$x_n = \left(\frac{s_{n+1}(1) - s_n(1)}{s_n(1)}, \ldots, \frac{s_{n+1}(l) - s_n(l)}{s_n(l)} \right)',$$

where A' denotes the transpose of a matrix A. We have

$$E(x_n x_n') = u_n + E(x_n)E(x_n').$$

Let y be a vector such that $E(x_n x_n')y = E(x_n)$, then

$$\begin{aligned} \sum_{k=1}^{l} u(i,k) y_k + \sum_{k \in I_n} p_n(i) p_n(k) y_k &= p_n(i), \quad \forall\, i \in I_n, \\ \sum_{k=1}^{l} u(i,k) y_k &= 0, \quad \forall\, i \in J_n, \end{aligned}$$

or equivalently

$$\begin{aligned} \sum_{k=1}^{l} \frac{u(i,k)}{p_n(i)} y_k + \sum_{k \in I_n} p_n(k) y_k &= 1, \quad \forall\, i \in I_n, \\ \sum_{k=1}^{l} u(i,k) y_k &= 0, \quad \forall\, i \in J_n. \end{aligned}$$

It follows that $y = A_n^{-1} d_n$ or equivalently

$$E^{-1}(x_n x_n') E(x_n) = A_n^{-1} d_n.$$

Now we can show that our result is consistent with Li and Ng:

$$\begin{aligned} u_t^* &= s_t z_t, \\ P_t &= x_t, \\ E(P_t) &= p_t, \\ E^{-1}(P_t P_t')E(P_t) &= A_n^{-1} d_n. \end{aligned}$$

Our invertibility condition on A_n is equivalent to invertibility condition on $E(P_t P_t')$ in their paper.

Concluding remark

We solved the mean variance portfolio optimization problem when the prices are such that the increment $s_{n+1} - s_n$ (resp. the return $\frac{s_{n+1}-s_n}{s_n}$) is independent of the past $(s_0, ..., s_n)$ for $n = 0, ..., N-1$. We can extend our solution to any sequence of prices $(s_0, ..., s_N)$ such that $(s_{n+1}(1) - s_n(1), ..., s_{n+1}(l) - s_n(l))' = f_n(s_0, ..., s_n)\varepsilon_{n+1}$, where ε_{n+1} is a square integrable random vector which is independent of the past $(s_0, ..., s_n)$, and f_n is a measurable map from $\mathbb{R}^{ln}$ to the set of $l \times l$ matrices. Here $(s_{n+1}(1) - s_n(1), ..., s_{n+1}(l) - s_n(l))'$ denotes an $l \times 1$ matrix. In the arithmetic case we simply have $f_n = 1$, and in the geometric case $f_n(s_0, ..., s_n) = s_n$.

A Appendix

Lemma A.1. *Let (X_k) be an independent family of random variables. Let*

$$\begin{aligned} X &= (X_k : k = 1, \dots, K) \\ X^2 &= (X_k^2 : k = 1, \dots, K) \\ E(X) &= (E(X_k) : k = 1, \dots, K) \\ (E(X))^2 &= ((E(X_k))^2 : k = 1, \dots, K) \\ E(X^2) &= (E(X_k^2) : k = 1, \dots, K) \\ var(X) &= (E(X_k^2) - (E(X_k))^2 : k = 1, \dots, K). \end{aligned}$$

Then

$$E[Q_\lambda(\mathrm{em}(X), \mathrm{em}(X^2))] = Q_\lambda(\mathrm{em}(E(X)), \mathrm{em}(E(X^2))) - \lambda(1 - K^{-1})em(\mathrm{var}(X)).$$

Proof. First note that $E(\mathrm{em}(X)) = \mathrm{em}(E(X))$. Second,

$$\begin{aligned} E(\mathrm{em}(X^2)) &= \frac{1}{K}\sum_k E(X_k)^2 \\ &= \frac{1}{K}\sum_k (var(X_k) + (E(X_k))^2) \\ &= \mathrm{em}((E(X))^2) + \mathrm{em}(\mathrm{var}(X)). \end{aligned}$$

Moreover,

$$\begin{aligned} E((\mathrm{em}(X))^2) &= var(\mathrm{em}(X)) + (\mathrm{em}(E(X)))^2 \\ &= \frac{1}{K^2}\sum_{1\le k\le K} var(X_k) + (\mathrm{em}(E(X)))^2 \\ &= K^{-1}\mathrm{em}(var(X)) + (\mathrm{em}(E(X)))^2. \end{aligned}$$

Altogether we obtain

$$\begin{aligned} E[Q_\lambda(\mathrm{em}(X), \mathrm{em}(X^2))] &= E\mathrm{em}(X) + \lambda E(\mathrm{em}(X))^2 - \lambda E\mathrm{em}(X^2) \\ &= Q_\lambda(\mathrm{em}(E(X)), \mathrm{em}((E(X))^2)) - \lambda(1-K^{-1})\mathrm{em}(\mathrm{var}(X)). \end{aligned}$$

$\square$

Lemma A.2. *Let $(a(i,k), g(i,k), b_i^j, c_i^j : j = 1, \dots, K, i, k = 1, \dots, l)$ be real numbers, and let $A := (a(i,k) : i, k = 1, \dots, l)$ and $G = (g(i,k) : i, k = 1, ..., l)$ be two $l \times l$ matrices. Consider the system of equations*

$$< e_i, Az^j > -\mathrm{em}(< GZ, B >)d_n(i) = c_i^j, \tag{32}$$

where $(z^j(i) : i = 1, \dots, l, j = 1, \dots, K)$ are unknown. Suppose that A is invertible and let α be the constant

$$\alpha = \frac{em(< GA^{-1}C, B >)}{1 - em(< GA^{-1}d_n, B >)}. \tag{33}$$

Then the solution of (32) *is given by*

$$z^j = A^{-1}(\alpha d_n + c^j).$$

Proof. Taking the empirical mean of System (32) with respect to j implies

$$\mathrm{em}(< e_i, AZ >) - \mathrm{em}(< GZ, B >)d_n(i) = \mathrm{em}(C_i)$$

for all $1 \le i \le l$. Now we plug the expression $\mathrm{em}(< GZ, B >)d_n(i) = \mathrm{em}(< e_i, AZ >) - \mathrm{em}(c_i)$ into System (32), and we obtain

$$< e_i, Az^j > -c_i^j - em(< e_i, AZ > -C_i) = 0. \tag{34}$$

Notice that if Z is a solution of (34), then

$$\gamma_i =< e_i, Az^j > -c_i^j$$

does not depend on j. Let $\Gamma = (\gamma_i)_{1\le i\le l} \in \mathbb{R}^l$. If A is invertible, then we get

$$z^j = A^{-1}(\Gamma + c^j).$$

Finally, if we plug the latter expression of z^j into (32), we obtain

$$< e_i, \Gamma + c^j > -\mathrm{em}(< GA^{-1}(\Gamma + C), B >) d_n(i) = c_i^j.$$

which yields

$$\gamma_i = \begin{cases} \mathrm{em}(< GA^{-1}(\Gamma + C), B >), & \text{if } i \in I_n, \\ 0, & \text{if } i \in J_n. \end{cases}$$

This further implies that

$$\gamma_i = \begin{cases} \alpha, & \text{if } i \in I_n, \\ 0, & \text{if } i \in J_n, \end{cases}$$

with α defined as in (33). $\square$

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