Portfolio Optimisation with Sequential Monte Carlo

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Abstract

As computer system size and complexity grow, formulating effective policies require more sophistication. There are many risk factors need to be considered, some of which may be in conflict. Inevitably, unpredictable circumstances that demand decisions will arise during operation. In some cases an automated response may be imperative; in other cases these may be ill-advised. Manual decisions made to override the default ones, in a sense, redefining the underlying policy. This matter is further complicated in highly dynamic operational environments like mobile ad-hoc networks, in which the risk factors may be changing continually. Thus, security policies must be able to change and adapt to the operational needs.

This study investigates the potential of evolutionary algorithms as a tool in determining the optimal security policies that suit such environments. This thesis reviews some fundamental concepts in related domains. It presents three applications of evolutionary algorithms in solving problems that are of direct relevance. These include the inference of security policies from decision examples, the dynamic adaptation of security policies and the optimisation of security policies for a specific set of missions. The results show that the inference approaches based on evolutionary algorithms are very promising. These approaches are also sufficiently generic to be used as general dynamic classification algorithms.

The thesis concludes with an evaluation on the work done, the extent to which the work justify the thesis hypothesis and some possible directions on how evolutionary algorithms can be applied to address a wider range of relevant problems on the domain of concern.

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Declaration

The work contained in this thesis is my own work unless otherwise stated.

I am the primary author of all work reported in this thesis. Advice on specific aspects of the work was provided by my supervisor, Dr. Nikolas Kantas.

Chapter 1

Introduction

Resource allocation is a common challenge for every investor. In the investment decision making process, investors decide how much resources are allocated to different investable assets to form a portfolio with the aim to optimise the performance of the overall portfolio is better off than any other according to some criterion. The criterion can be different to the investors. (Some investors may have different considerations such as tax considerations and legal restriction on investment assets or holding periods.)

Two common objectives, often contradicting, are the financial return and the investment risk. The Markowitz's modern portfolio theory [1] proposes a portfolio selection framework. In Markowitz's model, it is assumed that investor attempt to maximize a portfolio's return and minimize the risk (measured by the variance of the portfolio return. Based on this criteria, the set of non-dominance portfolio is known as the *efficient porfolios*. Using variance as a risk measure has its limitation. Variance is a symmetric measure; an out performing asset (than the expected return) is deemed to as risky as an under perfoming one. Many alternative risk measurements have been proposed, e.g. Sortino ratio, Conditional Value at Risk (CVaR), etc.. Refer [2] for details.

Surprisingly, there are some investment managers who have no interest on maximising their portfolio's return. Instead, the main objective of portfolio management for such a fund is simply track and replicate the exposure of a benchmark index as close as possible. These funds are attractive as it provides investors the exposure to the market, at the same time, minimal active management required makes the fund less vulnerable to change of management and has the advatanges of lower fees and taxes. The performance of these funds are often assessed in term of how well the fund tracks the benchmark index using some pre-defined metrics.

1.1 Technical Approach

Traditionally, portfolio optimization have been explored in an analytical fashion, adopting necessary assumption as necessary. This seems rather restrictive; there are many instances where numerical method has been used to derive an approximate or even more effective solution to the problem in question. For example, Monte Carlo technique is used to do integral, evolutionary techniques applied in engineering domains, etc..

Our approach to the problem in this thesis is a radical one. We view a portfolio optimisation as a *stochastic* control problem. We adopt the Bayesian view and treat tehse parameters as random variables. The objective is to find the sequence of control parameters that optimise the control objective defined in terms of portfolio return and financial risk. We investigate the potential of using SMCs as the means of determining the optimal strategy, or at least excellent, strategies for the portfolio optimisation problem in question. The main reason of choosing SMCs is its ability to carry out *sequential* update on the posterior distribution over time fit well with parameter inference in stochastic process. Moreoever, these techniques have achieved significant success in their applications on many domains. Of course, other heuristic search tecniques are also potentially applicable.

To investigate this approach, we first applied the technique on to a simpled determinstic reference model. This model is doubly useful. It demonstrates the concept nicely and serves as a basic model to allow us to gain further understanding on the tunable parameters. We then considered a simplied a market model with two assets: one risky asset with its price modelled as Brownian motion with drift, and one zero interest risk-free asset (constant) which has an analytical solution. Using SMCs, we search for the optimal strategy (the set of control parameters at each time point) that optimise against the optimisation criteria (a.k.a. reward) in terms of expected return over the investment period, and minimize the financial risk (variance of the return) using SMC techniques. The results are then evaluated against with the analytic solution described in [3].

1.2 Thesis hypothesis

Formally, the hypothesis of the thesis is stated as follows:

> Sequential Monte Carlo (SMCs) have the potential to be an effective means of searching the optimal strategy for portfolio problem.

We attempt to examine this hypothesis from three different perspectives:

1. Exploring the potential of SMCs in searching the optimal strategy from mean-variance criteria with constraints.

2. Exploring the sensitivity of the techniques in terms of the parameter settings.

3. Exploring the trade-off between the estimation accuracy, the complexity of the problem complexity and the computational efforts numerically and providing suggestions for real-world problem.

Given the time frame, we fully understand it is impossible to evaluate our approach on full scale strategy. The aim is to establish the plausibility or, at the very least, a greater understanding of the strengths and weaknesses of the above approach.

1.3 Thesis organisation

The subsequent chapters of this thesis are organised as follows:

- Chapter 2 reviews some fundamental concept in Monte Carlo method that are related to this thesis. It begins with a brief introduction to basic methods such as perfect Monte Carlo sampling, rejection sampling, importance sampling. It then details two common Markov Chain Monte Carlo (MCMC) techniques, namely Metropolis-Hastings and Gibbs Sampling. Lastly, it introduces the Sequential Monte Carlo (SMC) technique used in this thesis.

- Chapter 3 briefly review the state of the art of portfolio optisation problem. It then discusses how a portfolio problem can be transformed naturally into standard parameter estimation in Sequential Monte Carlo frameowrk.

- Chapter 4 details the toy experiment in which we attempt to use SMC to track a reference signal generated by a known synthetic model.

- Chapter 5 details the experiment in using SMC to infer the optimal control for portfolio that tracks real-world indices. In particular, we focus on the major stock indices accross the continient. It first discusses the market model that is used. It then details the problem form ulation. Lastly, it discusses the experimental results, in comparison to the theorical results.

- Chapter 6 concludes the thesis by evaluating the degree to which the hypothesis has been justfied and outlines potential work for the future.

Chapter 2

Monte Carlo Methods

Monte Carlo Methods have achieved significant success in its aplication to various domains in the last few decades. This chapter reviews some fundamental concept in Monte Carlo method that are related to this thesis. It first begins with a summary on the main concept of Bayesian inference. It then discusses some basic Monte Carlo methods such as perfect Monte Carlo sampling, rejection sampling, importance sampling. Lastly, it details the Sequential Monte Carlo (SMC) techniquesthat is used to do portfolio optimisation, along with various enhancement made to the framework.

2.1 Bayesian Inference

In Bayesian inference framework, each unknown parameter in the model is assumed to be random variable and is associated with a prior distribution that characterises the initial belief. The inference process is merely about updating the belief with new observable evidence in a systematic fashion using Bayes theorem.

Formally, let M be the Bayes model of interest, θ be the set of parameters of the model, $p(\theta \mid M)$ be the prior distribution (initial belief) and $p(x \mid \theta, M)$ be the likelihood (probability of observing an observation x given the model) then posterior distribution (updated belief) is given as follows:

$$\begin{aligned} p(\theta \mid x, M) &= \frac{p(x \mid \theta, M)\ p(\theta \mid M)}{p(x \mid M)} \\ &\propto p(x \mid \theta, M)\ p(\theta \mid M) \qquad (2.1) \\ \text{posterior} &\propto \text{likelihood} \times \text{prior} \qquad (2.2) \end{aligned}$$

This problem formulation is elegant, but there remains some subtle issues in practice. One particular issue is about the calculation of the normalisation constant

$p(x \mid M)$ in (2.1), which demands us to be able to carry out the following integral analytically:

$$p(x \mid M) = \int p(x \mid \theta, M)p(\theta \mid M)\ d\theta \tag{2.3}$$

This is often infeasible. A often way to circumvent this requirement is by making use of conjugate prior that yields posterior distributions from the same family in an analytical fashion.

Instead of a closed form solution, the Method Carlo methods offer a numerical solution in estimating the integral using sampling technique. The need of calculating integral that does not posess analytic solution also arises in the marginalisation process of nuisance parameters, calculating expectation of a function, etc.

2.2 Perfect Monte Carlo

Consider the calculation the expectation of a function, I of the following form:

$$I = \mathbb{E}_p[f(x)] = \int f(x)p(x)\ dx \tag{2.4}$$

Assuming we are able to sample N independent and identically distributed (i.i.d.) samples of x from $p(\cdot)$, denote these as $\{x^{(i)}\}$ where $i \in \{1 \dots N\}$, a Monte Carlo estimate of I using the the point masses of the samples is:

$$\hat{I} = \frac{1}{N}\sum_{i=1}^{N} f(x^{(i)}) \tag{2.5}$$

This approximation can be informally described as discretization of the continuous distribution with *random* support. This estimate has been shown to be unbiased and converge almost surely to be unbiased and converge almost surely to I as $N \to \infty$ by the Law of Large number [4].

Morever, if the variance of $f(\cdot)$ is bounded ($\sigma_f^2 < \infty$), then the following central limit theorem holds:

$$\sqrt{N}(\hat{I} - I) \Longrightarrow N(0, \sigma_f^2) \tag{2.6}$$

as $N \to \infty$, where $\Longrightarrow$ denotes convergence in distribution [5]. The key point to note here is this converengence rate of $\frac{1}{\sqrt{N}}$ is independent of the dimensions of x. This is in constrast with any determinstic method that has a rate that decreases as the integral dimension increases [4]. This is the main advantage of Monte Carlo integration.

2.3 Rejection sampling

However, it is not always possible to sample directly from the distribution $p(\cdot)$. Suppose we can find an instrumental distribution (a.k.a. proposal distribution), $q(\cdot)$, that is easy to sample from and has the property such that $cq(x)$ dominates $p(x)$ for all x, i.e., $cq(x) \geq p(x) \geq 0$ for all x, then to get a random sample from $p(\cdot)$, we can first sample from $q(\cdot)$ instead and accept the sample with acceptance probability $\alpha(x) = \frac{p(x)}{cq(x)}$. If the sample is rejected, repeat the process until success. This algorithm is summarised in Algorithm 1.

Algorithm 1 Rejection Sampling

function RejectionSampling(n)
    $\mathcal{X} = \{\ \}$
    repeat
        sample $x \sim q(\cdot)$
        sample $u \sim \mathcal{U}(0, 1)$
        if $u \leq \frac{p(x)}{cq(x)}$ **then**
            $\mathcal{X} \leftarrow \mathcal{X} \cup \{x\}$
        end if
    until len($\mathcal{X}$)=n
    return $\mathcal{X}$
end function

Looking at the acceptance ratio formula, it is not difficult to see that the optimal instrumental distribution, q^*, is the one that minimizes the space bounded by $cq(x)$ subject to the constraint that it still dominates the target density $p(x)$. As the dimension of x increases, this algorithm becomes very ineffecient because the acceptance ratio whic is essentially defined as the ratio of two embedded spaces tends towards zero. Therefore, many generated examples would be rejected.

2.4 Importance sampling

Instead of making a binary accept-reject decision on each sample, the key concept in important sampling is assign weighting to each sample (obtained from the instrumental distribution, $q(\cdot)$) based on how well the sample resembles the target distribution, $p(\cdot)$. More formally, assume we have an instrumental distribution, $q(\cdot)$ that has support that

includes $p(\cdot)$, we can re-write (2.4) as:

$$\begin{aligned} I &= \int f(x)\frac{p(x)}{q(x)}q(x)\ dx \\ &= \int f(x)w(x)q(x)\ dx \\ &= \mathbb{E}_q[f(x)w(x)] \end{aligned} \tag{2.7}$$

where $w(x)$ is commonly referred as the importance weight. This reformulation leads to the following Monte Carlo estimate of I:

$$\begin{aligned} \hat{I} &= \frac{\frac{1}{N}\sum_{i=1}^{N}\tilde{w}(x^{(i)})f(x^{(i)})}{\frac{1}{N}\sum_{j=1}^{N}\tilde{w}(x^{(j)})} \\ &= \sum_{i=1}^{N}\frac{\tilde{w}(x^{(i)})}{\sum_{j=1}^{N}\tilde{w}(x^{(j)})}f(x^{(i)}) \\ &= \sum_{i=1}^{N}\hat{w}(x^{(i)})f(x^{(i)}) \end{aligned} \tag{2.8}$$

where $\tilde{w}(x^{(i)}) = \dfrac{p(x^{(i)})}{q(x^{(i)})}$ and $\hat{w}(x^{(i)}) = \dfrac{\tilde{w}(x^{(i)})}{\sum_{j=1}^{N}\tilde{w}(x^{(j)})}$ are referred to as unnormalised and normalised importance weight respectively [6]. This estimate is biased as it consists of the ratio of two estimates, yet it is still asymtoptically consistent.

To obtain samples from the target distribution, $p(\cdot)$, an additional resampling step can be introduced. In the first step, we draw a set of samples $\{\tilde{x}^{(i)}\}$ from the instrumental distribution and compute their associated normalised importance weights, $\hat{w}(\tilde{x}^{(i)})$. In the resampling step, we draw the final sample set, $\{x^{(i)}\}$ from this intermediate set of samples by taking into the importance weights. This algorithm is summarised in Algorithm 2

There are many ways of implementing the resampling stage. A simple direct implementation is to select the sample from the intermediate stage according to a Multinomial distribution with the success probability parameter set to the vector of normalised weights, $\hat{w}(x^{(i)})$, i.e., the chance of a sample point being replicated is proportional to its weight. This resampling step however introduces extra variance to the estimators, yet this can be a crucial step in the sequential scheme that we shall look in the following section to avoid sampling degeneracy over time.

Algorithm 2 Importance Sampling

function ImportanceSampling(n)
  $\tilde{\mathcal{X}} = \{\ \}$
  repeat
    sample $\tilde{x} \sim q(\cdot)$
    $\tilde{\mathcal{X}} \leftarrow \tilde{X} \cup \{\tilde{x}\}$
  until len($\tilde{X}$)=n
  calculate importance weights, $\hat{w}(\tilde{x}^{(i)}) = \dfrac{\tilde{w}(\tilde{x}^{(i)})}{\sum_{j=1}^{N} \tilde{w}(\tilde{x}^{(j)})}$
  $X = \{\ \}$
  repeat
    resample x from $\tilde{\mathcal{X}}$ (taking into account the importance weights)
    $\mathcal{X} \leftarrow \mathcal{X} \cup \{x\}$
  until len($\mathcal{X}$)=n
  return $\mathcal{X}$
end function

2.5 Sequential Monte Carlo

To motivate why Sequential Monte Carlo is useful, consider a target distribution of interest $p(x_{1:n})$, and for simplicity, assuming we can sample directly from the distribution $p(x_{1:n})$, the minimal computational complexity of the sampling scheme would be at least linear in n. Sequential Monte Carlo (SMC) provides a way to obtain samples for each sequential time step in a *fixed* amount of computational time in Hidden Markov Models (HMMs). We shall begin with a brieft introduction on HMMs that is cruicial to understand SMC in the next section. Refer [6] for further details of inference techniques for HMMs in general.

2.5.1 Hidden Markov Models

HMMs can be seen as a class of models that consist of two related processes: an underlying Markov process, X_t, which is the target process of interest, and a observable process, Y_t, which its state can be measured and therefore provides some information about X_t. Moreover, it is assumed that these two processes have conditional independence properties as shown using the graphical model representation in Figure 2.1.

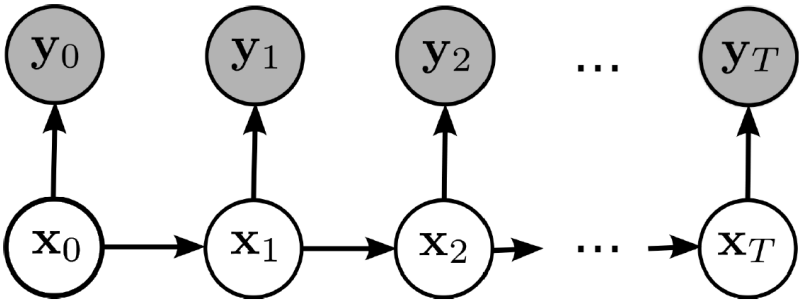

Figure 2.1: Hidden Markov Models

These properties can be summarised as follows:

$$\begin{aligned} p(x_t \mid x_{1:t-1}) &= f(x_t \mid x_{t-1}) \\ p(y_t \mid x_{1:t}, y_{1:t-1}) &= g(y_t \mid x_t) \end{aligned} \tag{2.9}$$

where $f(x_t \mid x_{t-1})$ is the transition density and $g(y_t \mid x_t)$ is the likelihood.

At this point, it is worth emphasizing here that HMMs are designed to capture systems that evolve from one state to another over time, generating observation after each state move. The inference problem is typically about estimating the state(s) in *real-time* given the observations. This imposes an implicit requirement from the computation perspective that the estimate calculation cost should remain constant over time, i.e., the calculation cost does not increase with the increasing number of states.

Arguably, the most common inference problem in HMMs is the smoothing distribution, $p(x_{1:t} \mid y_{1:t})$, that is estimating the states $x_{1:t}$ based on the sequence of observations up to time t, $y_{1:t}$. Using Bayes rules, we can write the density of the distribution of interest as follows:

$$\begin{aligned} p(x_{1:t} \mid y_{1:t}) &\propto p(x_{1:t} \mid y_{1:t-1}) g(y_t \mid x_t) \\ &= p(x_{1:t} \mid y_{1:t-1}) f(x_t \mid x_{t-1}) g(y_t \mid x_t) \end{aligned} \tag{2.10}$$

This recursion is often re-written into two separate steps: the prediction step (the estimation of distribution of t states given only $t-1$ states) and the update step (the correction of the predicted distribution taking into account the new observation) as follows:

$$\begin{aligned} p(x_{1:t} \mid y_{1:t-1}) &= p(x_{1:t-1} \mid y_{1:t-1}) f(x_t \mid x_{t-1}) \\ p(x_{1:t} \mid y_{1:t}) &= \frac{p(x_{1:t} \mid y_{1:t-1}) g(y_t \mid x_t)}{\int p(x_{1:t} \mid y_{1:t-1}) g(y_t \mid x_t)\, dx_{1:t}} \end{aligned} \tag{2.11}$$

Moreover, the estimate of any other smoothing distribution $p(x_{j:k} \mid y_{1:t})$ where $(j \leq k \leq l)$ can be obtained by integrating out x that are not interested in as follows:

$$p(x_{j:l} \mid y_{1:t}) = \int p(x_{1:t} \mid y_{1:t}) dx_{1:j,l+1:t} \tag{2.12}$$

One particular smoothing distribution of interest is the final marginal distribution $p(x_t \mid y_{1:t})$, which is often referred to as the filtering distribution.

Another distribution of interest is the prediction distribution, that is the estimation of the distribution of any unseen *future* state based on the sequence of observations up to time. If we let $j = 1$ and $l \geq n$ in (2.12), we obtain the following equation:

$$p(x_{j:l} \mid y_{1:t}) = p(x_{j:t} \mid y_{1:t}) \prod_{i=t+1}^{k} f(x_i \mid x_{i-1}) \tag{2.13}$$

Therefore, any prediction density can be obtained by simply integrating out the variables of not interest from the above equation.

While the distribution estimation problem may appear to be simple, it is in fact far from being resolved in practice. The integral appear in the above equations are often intractable and can only be estimated except in the very specific setting discussed below.

2.5.2 Kalman Filter

In the linear Gaussian setting in which the transition density and likelihood are each a Gaussian distribution with center lied at a point of a linear combination of the known conditional variables, u_t of the following form:

$$\begin{aligned} f_t(x_t \mid x_{t-1}, u_t) &= N(A_t(u_t)x_{t-1} + F_t(u_t), B_t(u_t)B_t(u_t)^T) \\ g_t(y_t \mid x_t, u_t) &= N(C_t(u_t)x_t, D_t(u_t)D_t(u_t)^T) \end{aligned} \tag{2.14}$$

where A_t, B_t, C_t, D_t are appropriate known matrix or vector operations, that may depend on the given conditional variable u_t.

Using the properties of Gaussian distribution, the integral can be resolved analytically. This leads the widely used *Kalman Filter* [7], which has the following recursive

solution as follows:

$$\mu_{t|t-1} = A_t(u_t)(\mu_{t-1|t-1})X_{t-1} + F_t(u_t) \tag{2.15}$$
$$\Sigma_{t|t-1} = A_t(u_t)\Sigma_{t-1|t-1}A_t(u_t)^T + B_t(u_t)B_t(u_t)^T \tag{2.16}$$
$$S_t = C_t(u_t)\Sigma_{t|t-1}C_t(u_t)^T + D_t(u_t)D_t(u_t)^T \tag{2.17}$$
$$y_{t|t-1} = C_t(u_t)\mu_{t|t-1} + G_t(u_t) \tag{2.18}$$
$$\mu_{t|t} = \mu_{t|t-1} + \Sigma_{t|t-1}C_t(u_t)S_t^{-1}(y_t - t_{t|t-1}) \tag{2.19}$$
$$\Sigma_{t|t} = \Sigma_{t|t-1} - \Sigma_{t|t-1}C_t(u_t)S_t^{-1}C_t(u_t)\Sigma_{t|t-1} \tag{2.20}$$

where $\mu_{t|t-1}$ and $\Sigma_{t|t-1}$ are the predicted mean and covariance of the state x_t, $y_{t|t-1}$ and S_t are the mean and covariance of the measurement at time t and $\mu_{t|t}$ and $\Sigma_{t|t}$ are the estimated mean and covariance of the state x_t after seeing the observation y_t.

There are various extensions have been developed to this approach. For example, the Extended Kalman Filter (EKF) which uses Taylor Series expansion to linearise at the conditional variables locally, Unscentend Kalman Filter, etc. Refer [8] for further details.

2.5.3 Sequential Important Sampling (SIS)

In more general setting, here is however no analytical solution for this estimation problem. Sequential Monte Carlo provides a systematic way to approximate the solution to this estimation problem. Assuming that it is possible to decompose the selected importance distribution in the following form:

$$\begin{aligned} q_n(x_{1:t}) &= q_n(x_{1:t-1})q_n(x_t \mid x_{1:t-1}) \\ &= q_1(x_1)\prod_{i=2}^{n} q_i(x_i \mid x_{1:t-1}) \end{aligned} \tag{2.21}$$

we can then obtain sample of $X_{1:n} \sim q_n(x_{1:t-1})$ at time t by first sample $X_1 \sim q_1$ at time 1 then $X_i \sim q_i(x_i \mid x_{1:t-1})$ for time i from $2 \dots n$. The corresonding weights associated to each sample $X_{1:n}$ can also calculated in a similar recursion fashion using

the following decomposition:

$$\begin{aligned} w_n(x_{1:t}) &= \frac{p_t(x_{1:t})}{q_t(x_{1:t})} \\ &= \frac{p_{t-1}(x_{1:t-1})}{q_{t-1}(x_{1:t-1})} \frac{p_t(x_{1:t})}{p_{t-1}(x_{1:t-1}) q_n(x_t \mid x_{1:t-1})} \qquad (2.22) \\ &= w_1(x_1) \prod_{i=2}^{n} \frac{p_i(x_{1:i})}{p_{i-1}(x_{1:i-1}) q + n(x_i \mid x_{1:i-1})} \\ &= w_1(x_1) \prod_{i=2}^{n} \alpha_k(x_{1:i}) \qquad (2.23) \end{aligned}$$

where $\alpha_k(x_{1:i})$ is often referred to as incremental importance weight function. This algorithm is summarised in X.

2.5.4 Optimal Proposal Distribution

While SIS is attractive, it is nothing but a specialised version of importance sampling introduced earlier in 2.4. As the state space increases with the number of time step t. Direct importance sampling on a state space with increasing size is not efficient. The weights of the samples start to degenerate quickly, in the sense that the weights start to concentrate on only a small number of samples, i.e., many of the samples have negligible weights and therefore rendered useless in estimating the expectation. It can be shown that the importance weights will increase with every iteration, and therefore the quality of the estimators will decrease over time [].

To alleviate this weight degeneracy issue, we can rewrite (2.22) as follows:

$$\begin{aligned} w_n(x_{1:t}) &= \frac{p_{t-1}(x_{1:t-1})}{q_{t-1}(x_{1:t-1})} \frac{p_t(x_{1:t})}{p_{t-1}(x_{1:t-1}) q + n(x_t \mid x_{1:t-1})} \\ &= 1 + 1 \qquad (2.24) \end{aligned}$$

Looking at (2.24), it is obvious that the proposal distribution, q_{t+1} that minimise the variance of the importance weight, $w_n(x_{1:t})$ takes the following form:

$$q_{t+1}(x + 1 \mid x_{t-1} \propto f_{t+1}(x_{t+1} \mid \qquad (2.25)$$

This is often referred to as the optimal proposal distribution.

In general, it is not always possible to sample from this optimal proposal distribution. Yet, the knowledge of its form can be used to guide the design of a reasonable good proposal distribution, which can be sampled from. Using a better proposal distribution reduces the amount of variance introduced, but does not totally eliminate degeneracy problem.

2.5.5 Sequential Importance Resampling (SIR)

The variance in importance weight accumulates over iterations. This suggests a possible solution is to "reset" the weights associated to the samples somehow during the iterations. Sequential Importance Resampling (SIR) introduces an additional resampling step to SIS step in a similar fashsion as discussed in Section 2.4. After resampling, the weight of each samples are reset to be equal, i.e., $\frac{1}{N}$. This algorithm is summarised in X.

Besides the simpliest multinomial resampling scheme, many different resampling schemes have been proposed in the literature. For example, stratified resampling [] as the name suggested splitting the samples in stratums to ensure the good coverage on the resulting sample set, residual resampling [] that has an effect in reducing the variance of the weights, etc. See [] for further details on the comparison of these sampling schemes.

However, resampling is not a silver bullet for sampling imporishment. Essentially, resampling provides a mechanism to eliminate low weight samples to give way to replicate *copies* of high weight samples. This allows all samples to participate and contribute to forma good estimation of the distribution of interest. This is obvious for the case of estimating filtering distribution and predictive distribution.

Over time, this replication reduces the number of distinct values available for previous time steps. The start of the trajectory will eventually become the same. This phenomena is known as sample impoverishment. This is a fundamental weakness of SMC, in which the history of the path is not re-written. The lose of diversity in the sample set will have a negative impact when it comes to estimating snoothing distribution.

2.5.5.1 Resample-Move Algorithm

To counteract this sample impoverishment, Resample-Move Algorithm [] is proposed to introduce some perturbance to the samples (so to diversify them) without changing the distribution they represent. This is accomplished by using MCMC steps with a Markov Kernel, K that is invariant to the target distribution.

In the original paper, this is done in a way by introducing an additional MCMC "move" step to each sample after resampling step according to a Markov kernel that is invariant to the target distribution. This algorithm is summarised in X.

This does not entirely solve the smoothing distribution estimation problem. To apply Markov Kernels with invariant distribution corresponding to the smoothing dis-

tribution, the space that Markov kernel is defined has to increase at each iteration. This implies the computation time increases linearly with time. Moreover, fast mixing high dimension Markov kernel in itself is not easy to design. In practice, one could use a sliding windows approach, in which MCMC Kernels which diversify the samples of the previous n time step at each iteration. This has a *fixed* additional cost to each iteration.

2.5.6 Effective sample size (ESS)

Resampling step induces additional Monte Carlo variance to the weights. Yet, this step is necessary to avoids accumulation of estimation variance on weights over time and therefore result in a more stable approximation to the filtering and preditive distribution.

To trading off these two competing requirements, one possible way to monitor the effective sample size (ESS) which provides a measure on the quality of the weighted samples. Two possible estimation are as follows:

$$ESS \approx \frac{1}{E[w^2]} \approx \frac{\left(\sum_{i=0}^{N} w_i\right)^2}{\sum_{i=0}^{N} w_i^2} \tag{2.26}$$

A possible implementation is that resampling step is only triggered if the ESS_t faill below certain threshold at time t, say $N/2$. See [] for detail discussion on ESS.

2.5.7 Rao-blackwellised (Marginal) Important sampling

In practice, many models may not be entirely Gaussian. Some states may be linear and Gaussian, conditionally upon the other states. The naive way to model this is to use SMC to model all the states. A better approach would be making use of the Gaussian properties. The states are split into two categories: the linear Gaussian states and the non-linear states. Then, the Kalman Filter which is optimal and possess a closed form solution can be used to model the linear Gaussian states and the SMC can be to model the non-linear states. This marginalisation setting will yield estimates with smaller variances.

To illustrate this, let consider the a simple conditional linear Gaussian Model, in which we have:

$$x_t^N \sim f(x_t^N \mid x_{0:t-1}^N) x_t^L \qquad = A(x_t^N) x_{t-1}^L + u_t^L y_t = B(x_t^N) x_t^L + v_t^L \tag{2.27}$$

where A and B are appropriate matrices and $u_t^L \sim N(0, C_u)$ and $v_t^L \sim N(0, C_v)$ independently. In this model, conditioning the non-linear states $x_{0:t}^N$ and observations $y_{0:t}$, the linear states are jointly Gaussian, with its mean and covariances can be calculated using Kalman filter recursion presented earlier. To obtain the posterior distribution of the non-linear states, we marginalise the linear states as follows:

$$p(x_{0:t}^N \mid y_{0:t}) = \int p(x_{0:t}^L, x_{0:t}^N \mid y_{0:t}) \, dx_{0:t}^L \tag{2.28}$$

SMC is then run on this non-linear states. The resulting algorithm is very similar, except to allow the marginalised system is no longer Markovian and therefore the update rules are changed slightly to be the following:

$$\begin{aligned} p(x_{1:t} \mid y_{1:t-1}) &= p(x_{1:t-1} \mid y_{1:t-1}) f(x_t \mid x_{t-1}) \\ p(x_{1:t} \mid y_{1:t}) &= \frac{p(x_{1:t} \mid y_{1:t-1}) g(y_t \mid x_t)}{\int p(x_{1:t} \mid y_{1:t-1}) g(y_t \mid x_t) \, dx_{1:t}} \end{aligned} \tag{2.29}$$

and the posterior density for hte linear part can be apprximated as follows:

$$p(x_t^N \mid y_{0:t}) \approx \sum_{i=1}^{N} w_t^{(i)} p(x_t^L \mid x_{0:t}^N, y_{0:t}) \tag{2.30}$$

where the conditional densities $p(x_t^L \mid x_{0:t}^N, y_{0:t})$ is again calculated using Kalman filtering recursion. The algorithm is shown in X.

The discussion has been focused on linear Gaussian model. Some generalisation are possible for this basic model. The simplest extension to this is to allow matrices A, B to depeond on time and any elements of nonlinear stat sequence $x_{0:t}$. This does not require change of formula. There is another important class model is the discrete state-space HMM m, in which the states are discrete. The discrete state values can be marginalised using HMM forward algorithm insead of Kalman filter. See [] for further detail.

2.6 Conclusion

This chapter presents a review of Monte Carlo method, with a particular focus on Sequential Monte Carlo method that is used extensively in this thesis for portfolio optimisation. It begins to describe traditional Monte Carlo sampling techniques. It then introduce a simple SMC algorithm, and couple of extensions that have been proposed to improve the performance of the algorithm. Lastly, the chapter presents a

simple concrete example, in which SMC is used to estimate the filtering and smoothing distribution for the problem in question.

It is worth nothing the algorithm is not restricted to sequential filtering problem. For example, it has been established that it is possible to use SMC within MCMC framework (pMCMC, where p stands for particle) [9] to solve other problems. In the next chapter, we will show Sequential Monte Carlo is used as a maximiser to search for a optimal strategy for a portfolio, given a multiplicative reward function.

Chapter 3

Portfolio optimisation

3.1 Technical Approach

Traditionally, multi-period mean-variance portfolio optimization have been explired in an analytical fashion, adopting necessary assumption as necessary. This seems rather restrictive; there are many instances where numerical method has been used to derive an approximate or even more effective solution to the problem in question. For example, Monte Carlo technique is used to do integral, evolutionary techniques applied in engineering domains, etc..

Our approach to the problem in this thesis is a radical one. We view a portfolio optimisation as a *stochastic* control problem. We adopt the Bayesian view and treat tehse parameters as random variables. The objective is to find the sequence of control parameters that minimize the control objective defined in terms of portfolio return and financial risk. We investigate the potential of using SMCs as the means of determining the optimal strategy, or at least excellent, strategies for multi-period mean-variance portfolio optimisation problem. The main reason of choosing SMCs is its ability to carry out *sequential* update on the posterior distribution over time fit well with parameter inference in stochastic process. Moreoever, these techniques have achieved significant success in their applications on many domains. Of course, other heuristic search tecniques are also potentially applicable.

3.2 Portfolio Optimisation

Resource allocation is a common challenge for every investor. In the investment decision making process, investors decide how much resources are allocated to different investable assets to form a portfolio with the aim to optimise the performance of the

overall portfolio is better off than any other according to some criterion. The criterion can be different to the investors. (Some investors may have different considerations such as tax considerations and legal restriction on investment assets or holding periods.)

Two common objectives, often contradicting, are the financial return and the investment risk. The Markowitz's modern portfolio theory [1] proposes a portfolio selection framework. In Markowitz's model, it is assumed that investor attempt to maximize a portfolio's return and minimize the risk (measured by the variance of the portfolio re turn. Based on this criteria, the set of non-dominance portfolio is known as the *efficient porfolios*. Using variance as a ris measure has its limitation. Variance is a symmetric measure; an out performing asset (than the expected return) is deemed to as risky as an under perfoming one. Many alternatve risk measurements have been proposed, e.g., Sortino ratio, Conditional Value at Risk (CVaR), etc.. Refer [2] for details.

In the original Marowitz model, the investment decision problem is viewed and solved as a single time-step problem. In practice, investment often span across multi time-step period and adjustments may be made to the allocation periodically to achive better performance as needed. This problem is much more difficult to deal with because it is time inconsistent in the sense that an investment strategy that is optimal over the whole period may not be the optimal one over a sub-interval of the period. This violates the Bellman's Priciple of Optimality [10]. Consequently, dynamic programming approach is not applicable here.

For index tracker fund manager, the main objective of portfolio management is to track and replicate the exposure of a benchmark index[1]. Different investors have different risk appetite and goals, yet it is safe assumed that investor attempt to maximize a portfolio's return and minimize the risk (measured by the variance of the portfolio return. However, there are other constraint an investor need to consider in practice, e.g., asset type, holiding periods, etc.

Different metrics have been introduced to quantify the mismatch between the performance of a fund and its benchmark. For example, tracking difference is the sum of absolute difference in returns between of a fund and its benchmark. Here, we adopt the tracking error as our metric, which is defined to be the standard deviation of the absolute difference of the returns of the fund and the benchmark defined in [11]. Formally, tracking error is defined to be:

$$\epsilon = \sqrt{(E[(r_p - r_b)^2])} \tag{3.1}$$

[1]The lack of active management generally makes the fund less vulnerable to change of management and has the advatanges of lower fees and taxes

The tracking error can be caused by differnet factors: some of which can be summarised as follows:

1. Benchmark index rebalance — the benchmark index is re-weighting its constitutents periodically to reflex the changes on the market based on its methodology. To track the index, the fund has to adjust its portfolio accordingly. This will incur some trasaction costs. During the index rebalance period, cash drag can happen between the liquidation of the constituents that have weights reduced/dropped and the addition of the constituents that have weights increased/added. This cash is essentially not participating in the market and therefore do not reflex the changes on the benchmark index.

2. different assumption on dividend trinvestment and taxation — This is best illustrate with examples. For example, the benchmark index calculation may assume immediate reinvestment of dividends on ex-dividend dates but the fund may only able to reinvest the dividend after receiving it. The tax treatment may on the dividends amy also different too.

3. Replication and Sampling techniques — funds may choose too replicate the benchmakr index by selecting a subset of the constituents (often the ones with larger weights and more liquid) in an effort to minimize the transaction costs. This exclusion of the smaller, less liquid constitutents may introduce another source od tracking error, especially under stressed market.

4. Total Expense Ratio — the average annual expense that is charged to the fund on daily basis to cover the management cost.

This list is by no mean exclusive. See [] for further detail.

3.3 Portfolio Optimisation as a Stochastic Control Problem

Traditionally, the state space models used in portfolio management are deterministic. For example, forcasting and predict f[29, 30], valuing electricity contracts for hedging in deregulated electricity markets, the short term available wind power and the temperature driven consumer demand (see [28, 29, 30] and the references within), or when examining the power transfer fluctuations across transmission lines [27]. TeIt seems

there is a pressing interest for stochastic modelling from computational statistics when analytical solution is not available.

We focus here the problem in minimizing the tracking error between a portfolio and its benchmark index using a stochastic control modelling approach. Our aim is to determine what investment actions (buy or sell) on the necessary constituents a portfolio manager has to do on a daily basis across the investment horizon to minimize the tracking error of the fund managed. We will proceed by presenting the stochastic state space model that we assume throughout this thesis. This model is by no mean to compete the state of the art model in realistic portfolio optimisation, but to motivate further work in this direction.

MOdel and objective function

3.4 Technical Approach

The technical approach is to

Portfolio managers is facing constant challenge

3.5 Objective in portfolio optimisation

We adopt the conditional linear Gaussian model with the following form:

$$X_t = A_t(U_t)X_{t-1} + B_tU_tW_t + F_tU_t, W_t \sim N(0, I) \tag{3.2}$$

$$Y_t = C_t(U_t)X_{t-1} + D_tU_tV_t + G_tU_t, N_t \sim N(0, I) \tag{3.3}$$

where A_n,B_n, C_n, D_n, F_n, and G_n are appropriate matrix/vector functions, $U_{tt\geq 0}$ is a determinstic control input sequence that is used regulate the hidden states, $X_{tt\geq 0}$, which are assumed to be a discrete time hidden Markov process that has an initial value x_0 and admit Gaussian transition density $f_t(x_t \mid x_{t_1}, u_t$ and $Y_{tt\geq 0}$ is the only observable process which has a Gaussian conditional likelihood density $g_n(y_n \mid x_n, u_n)$.

With this model, the objective is to search for a sequence of controls $u_{1:t}$ that would result in a sequence of observations $y_{1:t}$ is closed to a reference signal $y^{ref}_{1:T}$. This problem is often known as stochastic regulation problem. We adopt here the following finite horizon multiplicative reward function:

$$1 + 1 \tag{3.4}$$

where the expectation is take with respect to the whole path of the Markov Chain $X_{0:T}$, i.e., $E_{x_0}[f(X_{1:t})] = \int f(X_{1:T}) \prod f_t(x_t \mid x_{t-1})\, dx_{1:t}$, with Q_n, L_n are assumed to be known. The corresponding optimal open loop policy is:

$$u^*_{1:T} = \arg\max_{u_{1:T}} J(u_{1:T}; y^{ref}_{1:t}; x_0) \tag{3.5}$$

3.6 Technical approach

Under the Bayesian inference framework, we treat the control inputs as random variables that admit a prior distribution. Assuming the sequenc

The key point to note is that for the model becomes a linear Gussian model for a given u_t, which allows us to solve x_t analytically using Kalman Filter algorithm.

Under the realm of Bayesian inference framework, we treat U as a random variable and admit a probability distribution. The objective is to compute the marginal posterior distribution density $p(u_{0:t} \mid y_{0:t}$.

This desntiy function can be derived as follows:

$$-1 + 1 \tag{3.6}$$

We can solve this equation using SMC by consider the pair etc.

However, a more efficient algorithm can be derived by considering the following factorisation:

$$p(x_{0:t}, u_{0:t} \mid y_{1:t}) = p(x_{0:t} \mid u_{0:t}, y_{1:t}) p(u_{0:t} \mid y_{1:t}) \tag{3.7}$$

Note that density $P(x_{0:t} \mid u_{0:t}, y_{1:t}$ is Gaussian mixture model, which can be computed analytically usign Kalman Filfter given the density $p(u_{0:t} \mid y_{1:t}$, which has the following recursion form:

$$1 + 1 \tag{3.8}$$

This can be resolved using SMC by using particles to do this.

3.7 Problem formulation

Assume for the time being that the control inputs are set as U1:n = u1:n and remain fixed. Recall Xnn0 and Ynn1 are assumed to be stochastic processes obeying a Markov transition density fn(xn—xn1,un) and a conditionally independent likelihood density gn(yn—xn,un) respectively. Given any observed y1:n realisation, inference about the states X1:n may be based on the following posterior density

show the algorithm

3.8 Numerical example on stationary oscillating wave

We will consider a simple linear Gaussian state space model as presented earlier as (3.3), with $A_t = B_t = C_t = D_t = I$, $F_t u_t = u_t$, $G_t u_t = 0$, $Xi = 0$, $u = 0$. This model can be re-written as follows:

$$X_t = X_{t-1} + W_t + U_t, W_t \sim N(0, I) \tag{3.9}$$

$$Y_t = X_{t-1} + V_t, N_t \sim N(0, I) \tag{3.10}$$

with the target reference is set to be an oscillating wave: $y_t = \cos(0.2\pi t + 0.3)$. This toy example is first introduced in [12]. It serves two purposes here. Firstly, it provides a simple example to verify our implementation[1] Secondly, it severs as benchmark for the following experiments in which we attempt to use more complicated reference signals and models.

Setting the maximum time period, $T = 50$,

We proceed by examining the algorithm for the following different implementations: (a) qn = fn without using the MCMC move (Step 2(d)), (b) qn = fn with the MCMC move and (c) qn being the optimal importance density of [13] without the MCMC case. In the last case the MCMC step was omitted because when the optimal importance density is used the improvement in performance was marginal. In the MCMC move we will use a random walk proposal. For = 100, 1000 and N = 200, 500, 1000, 5000, 10000 we present box plots for logT(UI1:T)inFigure1after30independentrunsofthealgorithm,whereUI1:T istheestimatorofu1:T ineach run. Similarly, in Figure 2 we plot U I1:T and the particle population nU i1:T oNi=1 taken from one run of the each of the same cases, but this time we show results only for N = 10000. Simulations took roughly 3, 70 and 4 seconds per 1000 particles for (a), (b), (c) respectively when implemented in Matlab using a 2.4 GHz processor. The algorithm seems to perform quite well in most settings and very well when the optimal importance distribution is used. For the case where qn = fn, MCMC seems to improve the performance of the algorithm. The improvement is more evident when = 1000 both in the box plots and when plotting nUi1:T oNi=1, for which the degeneracy is apparent without the MCMC step.

3.8.1 Period length performance

We extend the time step to be 90 and 250 steps and looks at the corresponding performance. Due to the increase of time step, it makes sense to have more particles to track

[1] Strictly speaking, testing increases confidence but does not prove no bug, which is almost impossible in practice.

them and also look at the performance of performance of different order of γ. Based on the same metric, the results are summarised in Figure X.

3.8.2 Increasing the power

3.9 Different reference signals

Given the initial result looks promising, we attempt to investigate with the following more complicated reference signals:

1. reference signal treding osccilating wave —
2. two un-correlated bi-variate signals —
3. two correlated bi-variate signals —
4. ten different signals —

3.10 Discussion and possible extension

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