Linear Algebra Notes
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1 Rank

Given a matrix A, calculate its rank. If A is in order $n \times n = 2 \times 2$, then compute the determinant of A, $\det(A)$. If $\det(A) \neq 0$, then its rank is 2. Otherwise, its rank $< n = 2$. This rule is applicable for all $n \times n$ matrices.
To find the rank of matrices that its determinant=0 or hard to calculate determinant, follow these steps:

1. Use row operations, including row exchange, row adding, row suabtracting, row multiplication and row division, to get the row-echelon form of the matrix.

2. To check if one is in row-echelon form, check if (1) zero-rows are below non-zero-rows; (2) the elements below the first non-zero element of the matrix should be zero, which is equivalent to that the first non-zero element in the next row should occur to its right; (3) all elements below the leading diagon should be zero

3. then the rank of the matrix is the number of non-zero rows of the echelon-form matrix

4. **Note:** it is not necessary to make one into a row-echelon form if you can observe the number of independent non-zero rows before final row operations, such as row exchange.

2 Singularity

When it comes to "invertible", we are focusing on $m \times m$ square matrices. The matrices that are not invertible has singularity.
A matrix is singular if and only if its determinant is 0. A property is that $\det A^{-1} = (\det A)^{-1}$.
The inverse, if it exists, of a square matrix must be **unique** and can be found through

1. Gaussian Elimination

2. Newton's method

3. Cayley-Hamilton Method

4. Eigen Decomposition Method

3 Eigenvalues and Eigenvectors

$$|A - \lambda I| = 0$$

$$Av = \lambda v$$

- Singular Matrices have Zero Eigenvalues
- If A is a square matrix, then $\lambda = 0$ is not an eigenvalue of A
- multiplication: the eigenvalue of A is λ, then the eigenvalue of aA is $a\lambda$
- power: the eigenvalue of A is λ, then the eigenvalue of A^n is λ^n
- polynomial: the eigenvalue of A is λ, then the eigenvalue of $p(A)$ is $p(\lambda)$
- inverse: the eigenvalue of A is λ, then the eigenvalue of A^{-1} is λ^{-1}
- :transpose: the eigenvalue of A is λ, then the eigenvalue of A^T is also λ

Click **here** to see Eigenvalues and Eigenvectors visualization.