Exercises for Chapter 7

7.1 We want to predict the sales figures of a startup online shop from the sales of the previous three months: 5000 Euros, 10000 Euros, 15000 Euros.

a) Compute the μ-σ-standardized time series.

The mean is

$$\mu = (5000 + 10000 + 15000)/3 = 10000$$

The standard deviation is

$$\begin{aligned} \sigma &= \sqrt{\frac{1}{3-1}((5000-10000)^2 + (10000-10000)^2 + (15000-10000)^2)} \\ &= \sqrt{\frac{1}{2}(5000^2 + 0 + 5000^2)} = 5000 \end{aligned}$$

So, the standardized time series is

$$\begin{aligned} x &= \left(\frac{5000-10000}{5000}, \frac{10000-10000}{5000}, \frac{15000-10000}{5000}\right) \\ &= (-1, 0, 1) \end{aligned}$$

b) Construct the (standardized) regression data set for a forecasting model with time horizon 1 and find the optimal linear autoregressive forecasting model with offset ($f(0) \neq 0$) and time horizon 1 for these data.

The forecasting model is $x_t = ax_{t-1} + b$. For the given data this yields
$x_2 = ax_1 + b \Rightarrow 0 = -a + b$
$x_3 = ax_2 + b \Rightarrow 1 = b \Rightarrow a = 1$
$\Rightarrow x_t = x_{t-1} + 1$

c) Using this forecasting model compute the (unstandardized) sales forecasts for the next two months.

We compute the standardized forecasts as $x_4 = x_3 + 1 = 1 + 1 = 2$ and $x_5 = x_4 + 1 = 2 + 1 = 3$, which corresponds to the unstandardized forecasts $y_4 = 2 \cdot 5000 + 10000 = 20000$ and $y_5 = 3 \cdot 5000 + 10000 = 25000$.

d) Which value will this linear forecasting model yield if time goes to infinity?

$y_\infty \to \infty$

7.2 We construct a *single* layer perceptron (SLP) using the linear forecasting model from Exercise 7.1 followed by a single neuron with hyperbolic tangent activation function. If necessary use the following approximations: $\tanh(3/4) = 5/8$, $\tanh(1) = 3/4$, $\tanh(3/2) = 29/32$, $\tanh(7/4) = 15/16$, $\tanh(2) = 1$.

a) Initialize the SLP with (the standardized equivalent of) 5000 Euros and compute the (unstandardized) sales forecasts for the following 3 months.

Applying to hyperbolic tangent yields the forecasting model

$$x_t = \tanh(x_{t-1} + 1)$$

The unstandardized initialization $y_1 = 5000$ corresponds to the standardized initialization $x_1 = (5000 - 10000)/5000 = -1$. The standardized forecasts are then computed as
$x_2 = \tanh(-1 + 1) = 0$
$x_3 = \tanh(0 + 1) \approx \frac{3}{4}$
$x_4 = \tanh(\frac{3}{4} + 1) \approx \frac{15}{16}$
This corresponds to the unstandardized forecasts

$$\begin{aligned} y &= (-1 \cdot 5000 + 10000, 0 \cdot 5000 + 10000, \frac{3}{4} \cdot 5000 + 10000, \frac{15}{16} \cdot 5000 + 1000) \\ &= (5000, 10000, 13750, 14687.50) \end{aligned}$$

b) Which value will this SLP forecasting model yield if time goes to infinity?

As time goes to infinity we have $x_\infty \approx \tanh(x_\infty + 1)$. With the approximation $\tanh(2) = 1$ we can write $1 \approx \tanh(1 + 1) \Rightarrow x_\infty = 1 \Rightarrow y_\infty = 1 \cdot 5000 + 10000 = 15000$