Christian B. Mendl, Pedro Hack, Keefe Huang, Irene López Gutiérrez

Exercise 2.2 (Basic single qubit gates)
Imagine you are playing a game against an adversary. The game consists of multiple trials through which the adversary performs one of the following with equal probability:

1. They flip a coin and send you $|0\rangle$ or $|1\rangle$ depending on the outcome.

OR

2. They send you the state $\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$.

Your goal is to decide which of the two they performed, and you win if you can decide correctly for $\frac{3}{4}$ of the trials on average.

(a) Before you make your guess (based on a quantum measurement on the qubit), you are allowed to perform **one** of the gates X, Y, Z or H. Compute the outputs you would obtain in each situation with each of these gates.

(b) Which of the gates would allow you to win the game? Explain your strategy.

Solution

(a) If we apply X then the outcome for each scenario will be:

1. $|1\rangle$ or $|0\rangle$.
2. $\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$.

If we apply Y:

1. $i\,|1\rangle$ or $-i\,|0\rangle$.
2. $\frac{i}{\sqrt{2}}(-|0\rangle + |1\rangle)$.

If we apply Z:

1. $|0\rangle$ or $-|1\rangle$.
2. $\frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$.

If we apply H:

1. $\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$ or $\frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$.
2. $|0\rangle$.

(b) Applying a Hadamard gate allows us to win the game. In the second scenario we would always measure 0, so if we measure 1 we know with certainty that we are in the first scenario. Therefore, our strategy would work as follows:

- Apply a Hadamard gate.
- Measure in the standard basis to obtain either 0 or 1.
- If we obtain 1 say we are in scenario 1.
- If we obtain 0 say we are in scenario 2.

Note that we will obtain 1 in $\frac{1}{4}$ of the trials, for which we will always be correct. For the remaining $\frac{3}{4}$ we will be correct in $\frac{2}{3}$ of the trials. Therefore, we will be correct $1 \cdot \frac{1}{4} + \frac{2}{3} \cdot \frac{3}{4} = \frac{3}{4}$ of the times, as required.