Chair of Scientific Computing in Computer Science
Department of Informatics
Technical University of Munich

Eexam
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Note:

- During the attendance check a sticker containing a unique code will be put on this exam.
- This code contains a unique number that associates this exam with your registration number.
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Introduction to Quantum Computing

Exam:	IN2381 / Final Exam	**Date:**	Friday 25th February, 2022
Examiner:	Prof. Dr. Christian Mendl	**Time:**	14:15 – 15:45

	P 1	P 2	P 3
I			

Working instructions

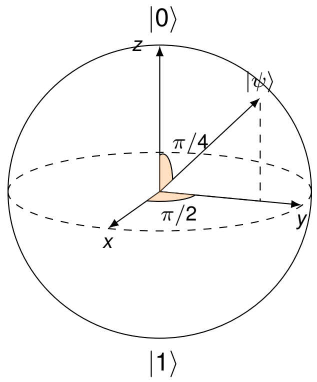
- This exam consists of **12 pages** with a total of **3 problems**.
  Please make sure now that you received a complete copy of the exam.
- The total amount of achievable credits in this exam is 60 credits.
- Detaching pages from the exam is prohibited.
- Allowed resources:
  - one **A4 sheet** (both sides) with your own notes
  - one **analog dictionary** English $\leftrightarrow$ native language
- Subproblems marked by * can be solved without results of previous subproblems.
- **Answers are only accepted if the solution approach is documented.** Give a reason for each answer unless explicitly stated otherwise in the respective subproblem.
- Do not write with red or green colors nor use pencils.
- Physically turn off all electronic devices, put them into your bag and close the bag.

Left room from __________ to __________ / Early submission at __________

Problem 1 Bloch Sphere (20 credits)

0 1 2 3 4 5

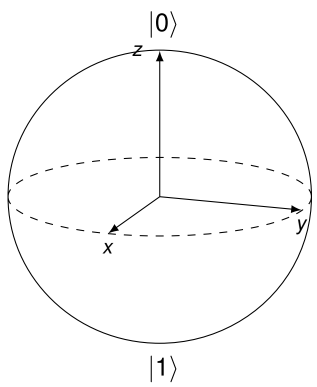
a) We consider the quantum state $|\psi\rangle$ specified by its Bloch vector as:



Determine the coefficients α and β in the standard basis representation $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$. What is the probability to measure 0 and 1, respectively?

0 1 2 3 4

b) We now apply the Pauli-Z gate to the system. **Mark** the resulting state (including angles) on the Bloch sphere below, and again provide the probabilities of measuring 0 or 1.



c) Now, apply the Hadamard gate to the state from (b). Again **mark** the resulting state (including angles) on the Bloch sphere below.

Hint: You can use the geometric interpretation of the Hadamard operation, or compute the resulting quantum state algebraically first.

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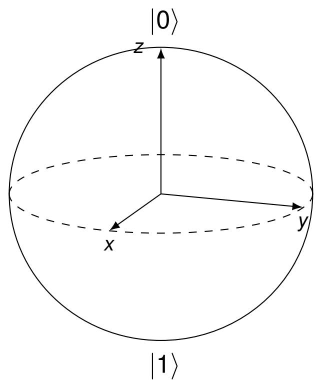
6



0 1 2 3 4 5

d)* We now consider a single-qubit quantum system described by the following density matrix:

$$\rho = \frac{1}{2}\begin{pmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & 1 \end{pmatrix}.$$

Compute the updated density matrix after applying $R_y(\pi/2)$ to the system.

Problem 2 Grover's Search Algorithm (20 credits)

We consider using Grover's search as a form of database search. We have determined that 6 qubits are sufficient to fully represent the search space and have encoded each item in the database into the computational basis. The search criterion is $f(\mathbf{x}) = 1$, given an input state $|\mathbf{x}\rangle = |x_1 x_2 x_3 x_4 x_5 x_6\rangle$ which represents the $\left(x_1 2^5 + x_2 2^4 + x_3 2^3 + x_4 2^2 + x_5 2^1 + x_6 2^0\right)$-th item in the database, and

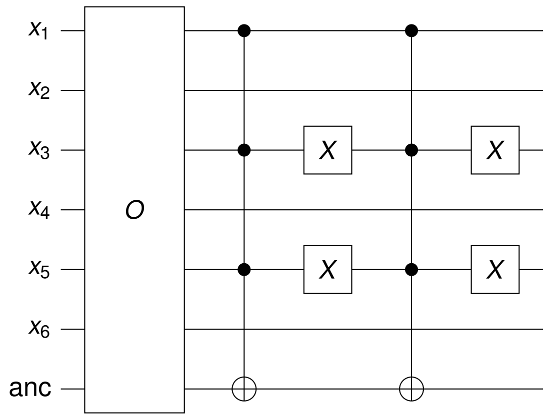
$$f(\mathbf{x}) = x_3 \oplus x_5 \oplus 1.$$

a) Find the input states that satisfy $f(\mathbf{x}) = 1$ and determine the number of viable solutions. 0 1 2

b)* State the definition of a quantum oracle used in the lecture. Design a suitable oracle O based on this definition for the search criterion, by drawing a quantum circuit implementing the oracle using **only** CNOTs (with a single control qubit), Pauli-X and Pauli-Z gates. 0 1 2 3 4 5 6

0 1 2 3 4 5

c)* The circuit below shows a modified oracle, where O is the oracle as specified in part (b).



Explain the action of this circuit. Identify the modified search criterion corresponding to the new oracle. What is the size of the new solution space?

0 1 2 3 4

d)* Determine the rotation angle per application of the Grover operator for a search space dimension $N = 64$ and $M = 2$ solutions.

Specify the optimal number of rotations. (Symbolic expressions are sufficient.) What is the resulting probability of obtaining a solution after applying these Grover rotations?

e)* Consider the scenario that we wish for multiple search criteria to be satisfied simultaneously. Given a quantum oracle for each criterion, how could one combine these into a large oracle that ensures that all criteria are satisfied?

Hint: You can use multiple ancilla qubits.

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Problem 3 **Quantum Operations (20 credits)**

Consider a single-qubit system A with the following density matrix:

$$\rho_A = \frac{1}{4}\begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix}$$

0 1 2

a)* Is this a pure or a mixed state? Clearly state your reasoning.

0 1 2 3 4

b)* Recall that there exists a process called purification, by which we can extend the system A into a larger quantum system AR, such that $\rho_A = \mathrm{tr}_R[|\psi\rangle\langle\psi|]$. Find such a state $|\psi\rangle$ on the extended system, using as the environment an additional qubit.

0 1 2 3 4

c)* Assume that originally both subsystems A and R were in state $|0\rangle$. Draw a circuit which outputs the state $|\psi\rangle$ you found in part (b). Hint: You only need a rotation gate and one two-qubit gate. The relation $\cos(\frac{\pi}{3}) = \frac{1}{2}$ might be helpful. If you didn't solve part (b), you may use: $|\psi\rangle = \frac{1}{\sqrt{5}}|00\rangle + \frac{2}{\sqrt{5}}|11\rangle$.

d)* We consider a principal quantum system in state $|\phi\rangle$ and an environment qubit initialized at $|0\rangle$. Now a unitary operator U acts on the combined system. Provide a formula for computing the resulting density matrix describing the principal system.

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e) If we ignore the environment qubit, we observe that subsystem A underwent the following transformation:

$$|0\rangle\langle 0| \mapsto \rho_A,$$

with ρ_A specified above. Starting from the circuit you have constructed in part (c), find the operation $\mathcal{E}$ acting on A. Check that it indeed gives ρ_A as output.

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Additional space for solutions–clearly mark the (sub)problem your answers are related to and strike out invalid solutions.