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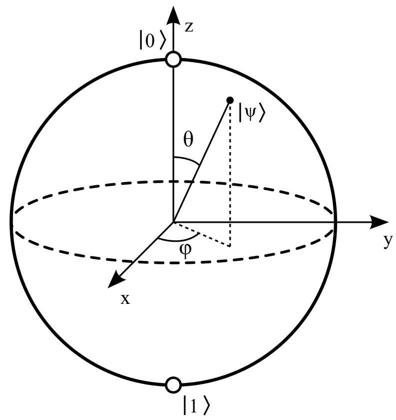
Exercise 2.1 (Bloch sphere and single qubit rotation gates)
Recall from the lecture that an arbitrary single qubit quantum state can be parametrized as

$$|\psi\rangle = \mathrm{e}^{i\gamma}\left(\cos\frac{\theta}{2}|0\rangle + \mathrm{e}^{i\varphi}\sin\frac{\theta}{2}|1\rangle\right)$$

where θ, φ and γ are real numbers, which can be chosen such that $0 \le \theta \le \pi$ and $0 \le \varphi \le 2\pi$. The angles θ and φ define the Bloch sphere representation of $|\psi\rangle$, as shown on the right.



https://commons.wikimedia.org/wiki/File:Bloch_sphere.svg

(a) Find the Bloch angles θ and φ of $|\psi\rangle = \frac{i}{2}|0\rangle - \frac{\sqrt{3}}{2}|1\rangle$, and the corresponding Bloch vector

$$\vec{r} = (\cos(\varphi)\sin(\theta), \sin(\varphi)\sin(\theta), \cos(\theta)).$$

For a real unit vector $\vec{v} \in \mathbb{R}^3$, the rotation by an angle ω about the $\vec{v}$ axis is defined as

$$R_{\vec{v}}(\omega) = \exp(-i\omega\,\vec{v}\cdot\vec{\sigma}/2) = \cos(\omega/2)I - i\sin(\omega/2)(\vec{v}\cdot\vec{\sigma}),$$

where $\vec{\sigma} = (\sigma_1, \sigma_2, \sigma_3)$ is the Pauli vector. The rotations R_x, R_y, R_z about the standard axes correspond to the special cases $\vec{v} = (1,0,0)$, $\vec{v} = (0,1,0)$ and $\vec{v} = (0,0,1)$, respectively.

(b) Compute $R_x(\frac{2\pi}{3})|\psi\rangle$ for the state $|\psi\rangle$ defined in (a), and visualize this operation on the Bloch sphere.
Hint: $\cos(\frac{\pi}{3}) = \frac{1}{2}$ and $\sin(\frac{\pi}{3}) = \frac{\sqrt{3}}{2}$.

(c) The *Z-Y decomposition* theorem states the following: given any unitary 2×2 matrix U, there exist real numbers $\alpha, \beta, \gamma, \delta \in \mathbb{R}$ such that

$$U = \mathrm{e}^{i\alpha}R_z(\beta)R_y(\gamma)R_z(\delta).$$

Find the Z-Y decomposition of the Hadamard gate $H = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$.
Hint: There exists a solution with $\beta = 0$.

Solution

(a)

$$|\psi\rangle = \frac{i}{2}|0\rangle - \frac{\sqrt{3}}{2}|1\rangle = i\left(\frac{1}{2}|0\rangle + i\frac{\sqrt{3}}{2}|1\rangle\right) \overset{!}{=} \mathrm{e}^{i\gamma}\left(\cos\frac{\theta}{2}|0\rangle + \mathrm{e}^{i\varphi}\sin\frac{\theta}{2}|1\rangle\right)$$

for $\theta = \frac{2\pi}{3}$, $\varphi = \frac{\pi}{2}$ and $\gamma = \frac{\pi}{2}$ (since $\mathrm{e}^{i\pi/2} = i$). Inserted into the Bloch vector results in

$$\vec{r} = (\cos(\varphi)\sin(\theta), \sin(\varphi)\sin(\theta), \cos(\theta)) = \left(0, \tfrac{\sqrt{3}}{2}, -\tfrac{1}{2}\right).$$

We observe that the Bloch vector lies in the y-z-plane.

(b) We first evaluate the rotation operator:

$$R_x(\tfrac{2\pi}{3}) = \cos(\tfrac{\pi}{3})I - i\sin(\tfrac{\pi}{3})X = \begin{pmatrix} \frac{1}{2} & -i\frac{\sqrt{3}}{2} \\ -i\frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix}.$$

Applying $R_x(\frac{2\pi}{3})$ to $|\psi\rangle$ gives

$$R_x(\tfrac{2\pi}{3})|\psi\rangle = \begin{pmatrix} \frac{1}{2} & -i\frac{\sqrt{3}}{2} \\ -i\frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix}\begin{pmatrix} \frac{i}{2} \\ -\frac{\sqrt{3}}{2} \end{pmatrix} = \begin{pmatrix} i \\ 0 \end{pmatrix} = i|0\rangle.$$

On the Bloch sphere, R_x is a rotation about the x-axis; here $|\psi\rangle$ is rotated within the y-z-plane to the north pole. (The prefactor i in $i|0\rangle$ does not affect the Bloch vector representation.)

(c)

$$H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = \underbrace{\frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}}_{R_y(\frac{\pi}{2})} \underbrace{\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}}_{iR_z(\pi)} = \mathrm{e}^{i\pi/2} R_y(\tfrac{\pi}{2}) R_z(\pi),$$

thus the parameters of the Z-Y decomposition are $\alpha = \frac{\pi}{2}$, $\beta = 0$, $\gamma = \frac{\pi}{2}$ and $\delta = \pi$.