Professorship for Quantum Computing
Department of Informatics
Technical University of Munich

Ecorrection
Place student sticker here

Note:

- During the attendance check a sticker containing a unique code will be put on this exam.
- This code contains a unique number that associates this exam with your registration number.
- This number is printed both next to the code and to the signature field in the attendance check list.

Introduction to Quantum Computing

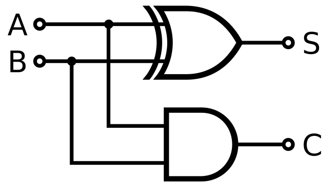
Exam:	IN2381 / Final Exam	**Date:**	Tuesday 20th July, 2021
Examiner:	Christian Mendl	**Time:**	14:15 – 15:45

Working instructions

- This exam consists of **10 pages** with a total of **3 problems**.
  Please make sure now that you received a complete copy of the exam.
- The total amount of achievable credits in this exam is 60 credits.
- Detaching pages from the exam is prohibited.
- Allowed resources: open book
- Subproblems marked by * can be solved without results of previous subproblems.
- **Answers are only accepted if the solution approach is documented.** Give a reason for each answer unless explicitly stated otherwise in the respective subproblem.
- Do not write with red or green colors nor use pencils.

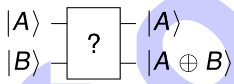
Problem 1 (20 credits)

In this problem, you will build a quantum version of a half adder – the basic building block of addition on a classical computer. The most important part of such a circuit is the half-adder:

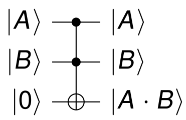


where A and B are classical bits, $S = A \oplus B$ is the sum modulo two and $C = A \cdot B$ is ordinary multiplication of A and B called the carry. The carry is the part of the summation that adds to the next digit (it is only 1 if both $A = 1$ and $B = 1$).

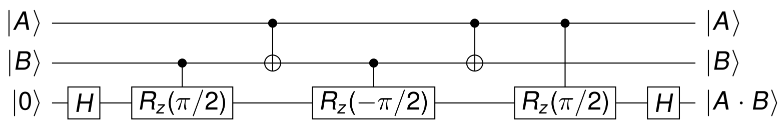
a) Assume you start in the arbitrary two-qubit state $|AB\rangle$. Provide a quantum gate / series of quantum gates that performs the operation:



b) The $A \cdot B$ operation can be performed by a Toffoli gate:



Verify that the circuit below performs that operation up to a global phase constant.



Hint: Follow the state of each qubit through the circuit for all 4 possible input basis states $|00\rangle$, $|01\rangle$, $|10\rangle$ and $|11\rangle$ separately.

c) Build a quantum half-adder using the Toffoli gate and the result from a): i.e. find the circuit that performs the operation:

$$\begin{array}{rcl} |A\rangle & \multirow{3}{*}{\boxed{?}} & |A\rangle \\ |B\rangle & & |A \oplus B\rangle \\ |0\rangle & & |A \cdot B\rangle \end{array}$$

You do not need to write out the Toffoli decomposition explicitly.

0 1 2 3 4

d) Given $|A \oplus B\rangle$ and $|A \cdot B\rangle$, is it possible to determine $|A\rangle$ and $|B\rangle$ in all cases? Provide reasoning for your answer.

Problem 2 **(20 credits)**

Consider an ensemble of quantum states $\{p_i, |\psi_i\rangle\}$, where the quantum system is in state $|\psi_i\rangle$ with probability p_i. Recall from the lecture that the density operator ρ of such an ensemble is defined as:

$$\rho = \sum_i p_i |\psi_i\rangle \langle\psi_i|$$

a) Given the ensemble $\left\{\left(\frac{1}{2}, \frac{|0\rangle+|1\rangle}{\sqrt{2}}\right), \left(\frac{1}{2}, \frac{|0\rangle+i|1\rangle}{\sqrt{2}}\right)\right\}$, compute ρ and write it in the form:

$$\rho = \frac{1}{2}I + \alpha_x X + \alpha_y Y + \alpha_z Z.$$

What is the connection of $\vec{\alpha} = (\alpha_x, \alpha_y, \alpha_z)$ with the Bloch sphere representation?

0 1 2 3 4 5 6

b) Now consider the ensemble

$$\left\{\left(\frac{1}{2}, |0\rangle\right), \left(\frac{1}{2}, |1\rangle\right)\right\},$$

and compute its density matrix ρ. Draw a Bloch sphere, clearly labeling $|0\rangle$ and $|1\rangle$, and indicate the position of this ensemble within the sphere.

0 1 2 3 4 5

0 1 2 3 4

c)* The unitary operation

$$U = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \sqrt{1-\lambda} & -i\sqrt{\lambda} & 0 \\ 0 & -i\sqrt{\lambda} & \sqrt{1-\lambda} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

where $0 \leq \lambda \leq 1$, acts on a system of two qubits. The first qubit is initially in an arbitrary state ρ and the second one is initialized at $|0\rangle$. Trace out the second qubit to obtain the two operators E_0 and E_1 which represent the action of U on the first one.

0 1 2 3 4 5 6

d) Compute the effect of the operators you found on the general density matrix $\rho = \frac{1}{2}I + \alpha_x X + \alpha_y Y + \alpha_z Z$. Interpret their action on the Bloch sphere.

Problem 3 **(20 credits)**

We consider a quantum system of n qubits, and use the notation X_j, Y_j, Z_j to denote that one of the Pauli matrices acts on the jth qubit; e.g., $X_1 Z_3 \equiv X \otimes I \otimes Z$ for $n = 3$.
Conjugation by U refers to the transformation $UgU^\dagger$ of a quantum gate g by a unitary operation U. The following table summarizes several conjugation transformations:

U	Z	Z	Z	H	H	H	S	S	S
g	X	Y	Z	X	Y	Z	X	Y	Z
$UgU^\dagger$	$-X$	$-Y$	Z	Z	$-Y$	X	Y	$-X$	Z

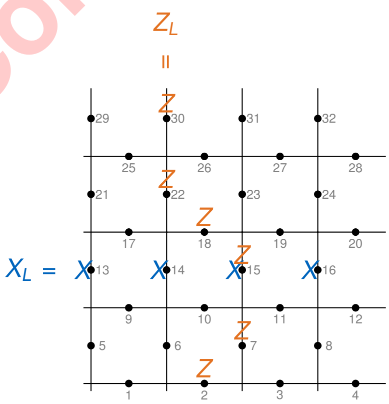
Here $S = \left(\begin{smallmatrix} 1 & 0 \\ 0 & i \end{smallmatrix}\right)$ is the phase gate.

a) State the check matrix representation of $g_1, g_2 \in G_4$ given by

$$g_1 = Z \otimes Y \otimes X \otimes X,$$
$$g_2 = X \otimes Z \otimes I \otimes Y.$$

Based on this representation, show that g_1 *anti*-commutes with g_2.

b)* Given a square lattice, a qubit is associated with each *edge* of the lattice (dots in the figure below). We define logical Pauli operators X_L and Z_L as tensor products of strings of X and Z operators: $X_L = X_{13}X_{14}X_{15}X_{16}$ and $Z_L = Z_2 Z_7 Z_{15} Z_{18} Z_{22} Z_{30}$, as visualized in the figure.



Show that X_L and Z_L anti-commute, i.e., $X_L Z_L = -Z_L X_L$. How can one define a logical Y_L operator such that X_L, Y_L, Z_L satisfy the anti-commutation relations of the Pauli-matrices?

0
1
2
3
4
5

c)* The subgroup $R = \langle X_1 Y_2 Z_3, Y_1 Y_2 Y_3 \rangle$ of the Pauli group G_3 stabilizes the subspace $V_R = \mathrm{span}\{|\chi_0\rangle, |\chi_1\rangle\}$ with

$$|\chi_0\rangle = \frac{1}{2}\big(|000\rangle + |001\rangle + i\,|110\rangle - i\,|111\rangle \big), \qquad |\chi_1\rangle = \frac{1}{2}\big(|010\rangle + |011\rangle - i\,|100\rangle + i\,|101\rangle \big).$$

(A proof of this statement is not required here.) Determine the result (eigenvalue) when measuring the operator $Y_1 Y_2 Y_3$ with respect to the quantum state $(S \otimes H \otimes (SH))\,|\chi_0\rangle$, where S is the phase gate.

d)* We consider the two qubit code $C = \text{span}\{|0_L\rangle, |1_L\rangle\}$ with $|0_L\rangle = |00\rangle$ and $|1_L\rangle = |01\rangle$. It is affected by a simultaneous bit flip noise process described by the operation elements $E_0 = \frac{1}{\sqrt{2}} I_4$ and $E_1 = \frac{1}{\sqrt{2}} X \otimes X$, where I_n the $n \times n$ identity matrix. Show that this noise process is error-correctable.

0
1
2
3
4
5

Additional space for solutions–clearly mark the (sub)problem your answers are related to and strike out invalid solutions.