2.1 a) $|\psi\rangle = \frac{i}{2}|0\rangle - \frac{\sqrt{3}}{2}|1\rangle$ Find φ and θ

- $|\psi\rangle = e^{i\gamma}\left(\cos\frac{\theta}{2}|0\rangle + e^{i\varphi}\sin\frac{\theta}{2}|1\rangle\right) = \cos\frac{\theta}{2}|0\rangle + e^{i\varphi}\sin\frac{\theta}{2}|1\rangle$

let $\alpha = \frac{i}{2}$, $\beta = -\frac{\sqrt{3}}{2}$. $\alpha^* = -\frac{i}{2}$ $|\alpha| = \frac{1}{2}$

multiplication by a phase, not relevant for the block angles

We have $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle \Rightarrow \frac{\alpha^*}{|\alpha|}|\psi\rangle = \frac{|\alpha|^2}{|\alpha|}|0\rangle + \frac{\beta\alpha^*}{|\alpha|}|1\rangle = |\alpha||0\rangle + \frac{\beta\alpha^*}{|\alpha|}|1\rangle$

With $\cos\frac{\theta}{2} = |\alpha| \Rightarrow \theta = 2\arccos(|\alpha|) = 2\arccos\left(\frac{1}{2}\right) = \boxed{2\pi/3 = \theta}$

Then $\frac{\beta\alpha^*}{|\alpha|} = e^{i\varphi}\sin\frac{\theta}{2} = e^{i\varphi}\sin\frac{\pi}{3} = e^{i\varphi}|\beta| \Leftrightarrow \frac{\beta\alpha^*}{|\alpha||\beta|} = e^{i\varphi} \Leftrightarrow \varphi = \arg\left(\frac{\beta\alpha^*}{|\alpha||\beta|}\right)$

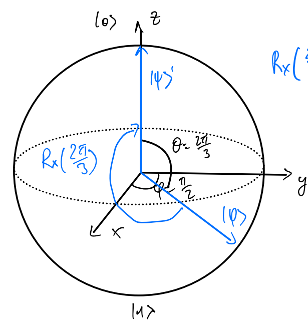
$= \arg\left(\frac{-\frac{\sqrt{3}}{2}\left(-\frac{i}{2}\right)}{\frac{1}{2}\frac{\sqrt{3}}{2}}\right) = \arg(i) = \lim_{x\to 0}\tan^{-1}\left(\frac{1}{x}\right) = \boxed{\frac{\pi}{2} = \varphi}$. // Alternatively: $e^{i\varphi} = i \Leftrightarrow \cos(\varphi) + i\sin(\varphi) = i$
$\Leftrightarrow \sin(\varphi) = 1 \Leftrightarrow \varphi = \frac{\pi}{2}$

$\boxed{\vec{r} = (0, \sqrt{3}/2, -1/2)}$

b) $R_x\left(\frac{2\pi}{3}\right) = \cos\left(\frac{\pi}{3}\right)I - i\sin\left(\frac{\pi}{3}\right)\sigma_1 = \begin{bmatrix}\cos(\frac{\pi}{3}) & -i\sin(\frac{\pi}{3})\\ -i\sin(\frac{\pi}{3}) & \cos(\frac{\pi}{3})\end{bmatrix} = \begin{bmatrix}\frac{1}{2} & -\frac{\sqrt{3}}{2}i\\ -\frac{\sqrt{3}}{2}i & \frac{1}{2}\end{bmatrix}$

$R_x\left(\frac{2\pi}{3}\right)|\psi\rangle = \begin{bmatrix}\frac{1}{2} & -\frac{\sqrt{3}}{2}i\\ -\frac{\sqrt{3}}{2}i & \frac{1}{2}\end{bmatrix}\left(\frac{i}{2}|0\rangle - \frac{\sqrt{3}}{2}|1\rangle\right) = \begin{bmatrix}\frac{1}{2} & -\frac{\sqrt{3}}{2}i\\ -\frac{\sqrt{3}}{2}i & \frac{1}{2}\end{bmatrix}\begin{bmatrix}\frac{i}{2}\\ -\frac{\sqrt{3}}{2}\end{bmatrix} =$

$\begin{bmatrix}\frac{1}{4}i + \frac{3}{4}i\\ -\frac{\sqrt{3}}{4}i^2 - \frac{\sqrt{3}}{4}\end{bmatrix} = \begin{bmatrix}i\\ 0\end{bmatrix} = i|0\rangle$



$R_x\left(\frac{2\pi}{3}\right)|\psi\rangle = |\psi\rangle'$

π

c) $H = e^{i\alpha}R_z(\beta)R_y(\gamma)R_z(\delta) = e^{i\alpha}\begin{bmatrix}e^{-i\left(\frac{\beta+\delta}{2}\right)}\cos\left(\frac{\gamma}{2}\right) & -e^{-i\left(\frac{\beta-\delta}{2}\right)}\sin\left(\frac{\gamma}{2}\right)\\ e^{i\left(\frac{\beta-\delta}{2}\right)}\sin\left(\frac{\gamma}{2}\right) & e^{i\left(\frac{\beta+\delta}{2}\right)}\cos\left(\frac{\gamma}{2}\right)\end{bmatrix}$

$\beta = 0$
$= e^{i\alpha}\begin{bmatrix}e^{-i\frac{\delta}{2}}\cos\left(\frac{\gamma}{2}\right) & -e^{i\frac{\delta}{2}}\sin\left(\frac{\gamma}{2}\right)\\ e^{-i\frac{\delta}{2}}\sin\left(\frac{\gamma}{2}\right) & e^{i\frac{\delta}{2}}\cos\left(\frac{\gamma}{2}\right)\end{bmatrix}$

$H = \begin{pmatrix}\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}}\end{pmatrix}$ $\arccos\left(\frac{1}{\sqrt{2}}\right) = \arcsin\left(\frac{1}{\sqrt{2}}\right) = \frac{\pi}{4} \Rightarrow \frac{\pi}{4} = \frac{\gamma}{2} \Rightarrow \gamma = \frac{\pi}{2}$

$= e^{i\alpha}\begin{bmatrix}e^{-i\frac{\delta}{2}}\frac{1}{\sqrt{2}} & -e^{i\frac{\delta}{2}}\frac{1}{\sqrt{2}}\\ e^{-i\frac{\delta}{2}}\frac{1}{\sqrt{2}} & e^{i\frac{\delta}{2}}\frac{1}{\sqrt{2}}\end{bmatrix} \Leftrightarrow \begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix} = e^{i\alpha}\begin{bmatrix}e^{-i\frac{\delta}{2}} & -e^{i\frac{\delta}{2}}\\ e^{-i\frac{\delta}{2}} & e^{i\frac{\delta}{2}}\end{bmatrix}$

$= \begin{bmatrix}\cos(-\frac{\delta}{2}+\alpha) + i\sin(-\frac{\delta}{2}+\alpha) & -(\cos(\frac{\delta}{2}+\alpha) + i\sin(\frac{\delta}{2}+\alpha))\\ \cos(-\frac{\delta}{2}+\alpha) + i\sin(-\frac{\delta}{2}+\alpha) & \cos(\frac{\delta}{2}+\alpha) + i\sin(\frac{\delta}{2}+\alpha)\end{bmatrix}$

- $\sin(\alpha - \frac{\delta}{2}) = \sin(\alpha + \frac{\delta}{2}) = 0 \longrightarrow \alpha + \frac{\delta}{2} = \alpha - \frac{\delta}{2} = \pi n \quad \forall n \in \mathbb{N}$

- $\begin{cases} \cos(\alpha - \frac{\delta}{2}) = 1 \\ \cos(\alpha + \frac{\delta}{2}) = -1 \\ \alpha - \frac{\delta}{2} = \pi n \quad \forall n \in \mathbb{N} \end{cases} \longrightarrow \begin{matrix} \alpha = \pi/2 + 2\pi n \quad \forall n \in \mathbb{N} \\ \delta = \pi \end{matrix}$

$$H = e^{i\frac{\pi}{2}} R_z(0) R_y(\tfrac{\pi}{2}) R_z(\pi)$$

2.2 a) X) 1. $X|0\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix} = |1\rangle$ or $X|1\rangle = |0\rangle$

2. $X \cdot \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$

Y) 1. $Y|0\rangle = \begin{pmatrix} 0 \\ i \end{pmatrix} = i|0\rangle$ or $Y|1\rangle = -i|0\rangle$

2. $Y \cdot \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) = \frac{1}{\sqrt{2}} i \begin{pmatrix} -1 \\ 1 \end{pmatrix} = \frac{1}{\sqrt{2}} i (|1\rangle - |0\rangle)$

Z) 1. $Z|0\rangle = |0\rangle$ or $Z|1\rangle = -|1\rangle$

2. $Z \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$

H) 1. $H|0\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$ or $H|1\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$

2. $H \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) = \frac{1}{\sqrt{2}^2}(|0\rangle + |1\rangle + |0\rangle - |1\rangle) = \frac{1}{2}(2|0\rangle) = |0\rangle$

b) The 3/4 probability of successful guess can be achieved with gate H:

- If we receive $|1\rangle$: we always guess the case 1. This will give a success rate of 100% as after the gate H, case 2 can only produce $|0\rangle$.

- If we see $|0\rangle$: we always guess it comes from the case 2. Doing so we can ensure at least 50% of success. Given that case 1 and 2 are performed with same prob.