Chair of Scientific Computing in Computer Science
Department of Informatics
Technical University of Munich

Eexam
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Note:

- During the attendance check a sticker containing a unique code will be put on this exam.
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Algorithms of Scientific Computing II (Quantum Computing)

Exam:	IN2002 / Retake	**Date:**	Thursday 9th July, 2020
Examiner:	Christian B. Mendl	**Time:**	10:45 – 12:15

	P 1	P 2	P 3
I			

Working instructions

- This exam consists of **10 pages** with a total of **3 problems**.
  Please make sure now that you received a complete copy of the exam.
- The total amount of achievable credits in this exam is 60 credits.
- Detaching pages from the exam is prohibited.
- Allowed resources: one **A4 sheet** (both sides) with your own notes
- Subproblems marked by * can be solved without results of previous subproblems.
- **Answers are only accepted if the solution approach is documented.** Give a reason for each answer unless explicitly stated otherwise in the respective subproblem.
- Do not write with red or green colors nor use pencils.
- Physically turn off all electronic devices, put them into your bag and close the bag.

Left room from __________ to __________ / Early submission at __________

Problem 1 **(20 credits)**

A *tripartite state* consists of three entangled qubits. Two well-known tripartite states are the Greenberger-Horne-Zeilinger (GHZ) state and the W-state. The classical GHZ state is defined as

$$|\mathrm{GHZ}\rangle = \frac{|000\rangle + |111\rangle}{\sqrt{2}}.$$

0 1 2 3 4

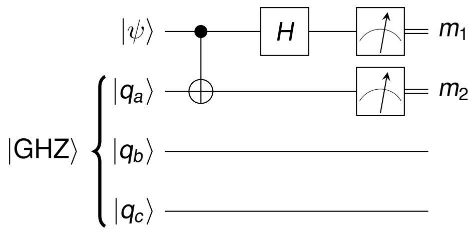
a) Specify a quantum circuit which generates $|\mathrm{GHZ}\rangle$ for input $|000\rangle$.

Suppose that three parties, Alice, Bob and Charlie, share a $|\mathrm{GHZ}\rangle$ state (each taking one entangled qubit, denoted $|q_a\rangle$, $|q_b\rangle$, $|q_c\rangle$, respectively). Alice controls another qubit $|\psi\rangle = \alpha\,|0\rangle + \beta\,|1\rangle$ (with $\alpha, \beta \in \mathbb{C}$, $|\alpha|^2 + |\beta|^2 = 1$) that she wishes to pass to Charlie. However, Alice, Bob and Charlie are limited to exchanging only classical information.

0 1

b)* Name an algorithm that could be modified to allow Charlie to recreate the state $|\psi\rangle$, given these constraints.

We now consider the following circuit:



c)* Alice performs a standard measurement of both her qubits, as shown in the circuit diagram. The qubits owned by Bob and Charlie, $|q_b, q_c\rangle$, then collapse in dependence of Alice's measurement result. Suppose that $m_2 = 1$, determine $|q_b, q_c\rangle$ for the two cases $m_1 = 0$ and $m_1 = 1$.

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0 1 2 3 4 5 6 7 8

d)* In the setting of part (c), how can Charlie recover Alice's original qubit $|\psi\rangle$? What additional information from Bob is required? Describe the overall process.

Hint: Bob should perform a measurement in a basis that indicates what Charlie should do . . .

If unable to solve part (c), start working with the two possible states:

$$|q_b, q_c\rangle_{01} = \alpha\,|01\rangle + \beta\,|10\rangle\,,$$
$$|q_b, q_c\rangle_{11} = \alpha\,|01\rangle - \beta\,|10\rangle\,,$$

with the notation $|q_b, q_c\rangle_{m_1 m_2}$ (using the measurement results m_1 and m_2 as indices).

Problem 2 **(20 credits)**

a) Determine whether the single qubit density matrix

$$\rho = \begin{pmatrix} \frac{9}{10} & -\frac{i}{5} \\ \frac{i}{5} & \frac{1}{10} \end{pmatrix}$$

describes a pure quantum system.

0 1 2 3 4 5 6

b)* Pure quantum states evolve under a unitary matrix U as

$$|\psi\rangle = U\,|\psi_0\rangle\,,$$

where $|\psi_0\rangle$ is the initial state. Write down the density matrix ρ_0 corresponding to $|\psi_0\rangle$, and an equivalent expression for the evolution of the density matrix governed by U.

0 1 2

c) Consider a quantum system consisting of two subsystems A and B, initially in the state described by the density matrix $\rho_0 = \rho_0^A \otimes \rho_0^B$, where ρ_0^A and ρ_0^B are density matrices on subsystems A and B, respectively. Specify the density matrix of subsystem A after an evolution governed by a unitary operator U acting on the combined system.

0 1 2

0 1 2 3 4

d) Given the setting of part (c), let $(|e_k\rangle)_k$ be an orthonormal basis of the state space of subsystem B, and assume $\rho_0^B = |e_0\rangle\langle e_0|$. Show that one can represent $\rho^A = \sum_k E_k \rho_0^A E_k^\dagger$, and provide an explicit expression for E_k.

0 1 2 3 4 5 6

e)* Consider the quantum bit flip operation

$$\rho \mapsto \mathcal{E}(\rho) = \sum_{k=0}^{1} E_k \rho E_k^\dagger \quad \text{with} \quad E_0 = \sqrt{1-p}\,I, \quad E_1 = \sqrt{p}X$$

for a real parameter $p \in [0, 1]$. Design a circuit that performs this operation.

Hint: A possible circuit consists of a single qubit wire for the principal system described by ρ, and another qubit wire for the environment initialized to $|0\rangle$. The rotation operator

$$R_y(\theta) = \begin{pmatrix} \cos\frac{\theta}{2} & -\sin\frac{\theta}{2} \\ \sin\frac{\theta}{2} & \cos\frac{\theta}{2} \end{pmatrix}$$

might be helpful for encoding the probability p.

Problem 3 **(20 credits)**

We consider a quantum system of n qubits, and use the notation X_j, Y_j, Z_j to denote that one of the Pauli matrices acts on the jth qubit; e.g., $X_1 Z_3 \equiv X \otimes I \otimes Z$ for $n = 3$.

Conjugation by U refers to the transformation $UgU^\dagger$ of a quantum gate g by a unitary operation U. The following table summarizes several conjugation transformations:

U	H	H	H	S	S	S
g	X	Y	Z	X	Y	Z
$UgU^\dagger$	Z	$-Y$	X	Y	$-X$	Z

Here $S = \left(\begin{smallmatrix} 1 & 0 \\ 0 & i \end{smallmatrix}\right)$ is the phase gate.

a) The quantum state

$$|\psi\rangle = \alpha\,|000\rangle + \beta\,|111\rangle$$

with $\alpha, \beta \in \mathbb{C}$ is affected by a single qubit bit flip error, resulting in a state $|\psi'\rangle$. A subsequent measurement of the observables $Z_1 Z_2$ and $Z_1 Z_3$ leads to the measurement outcomes (eigenvalues) -1 and 1, respectively. Specify $|\psi'\rangle$, and a quantum gate to recover the original $|\psi\rangle$ when applied to $|\psi'\rangle$.

0 1 2 3 4 5 6

b)* Recall that the *commutator* of two operators A and B is defined as $[A, B] = AB - BA$, and that they commute if $[A, B] = 0$. Show that the operators $X_1 Y_2 Z_3$ and $Z_1 Y_2 X_3$ commute.

0 1 2 3 4

0 1 2 3 4 5 6

c)* The subgroup $R = \langle X_1 Y_2 Z_3, Z_1 Y_2 X_3 \rangle$ of the Pauli group G_3 stabilizes the subspace $V_R = \mathrm{span}\{|\chi_0\rangle, |\chi_1\rangle\}$ with

$$|\chi_0\rangle = \frac{1}{2}\big(i\,|001\rangle - |010\rangle + i\,|100\rangle + |111\rangle\big), \qquad |\chi_1\rangle = \frac{1}{2}\big(|000\rangle + i\,|011\rangle - |101\rangle + i\,|110\rangle\big).$$

(A proof of this statement is not required here.) Determine the result (eigenvalue) when measuring the operator $X_1 Y_2 Z_3$ with respect to the quantum state $(H \otimes H \otimes H)\,|\chi_0\rangle$.

0 1 2 3 4

d)* We consider the subgroup $T = \langle X_1 X_2, X_1 Z_2 \rangle$ of the Pauli group G_2. Compute the subspace V_T stabilized by T, such that $g\,|\psi\rangle = |\psi\rangle$ for all $g \in T$ and $|\psi\rangle \in V_T$.

Additional space for solutions–clearly mark the (sub)problem your answers are related to and strike out invalid solutions.