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Tutorial 13 (Experimentally resolving the quantum measurement process[1])
Recall from the lecture that a projective measurement is described by a Hermitian operator M. Writing its spectral decomposition as $M = \sum_m \lambda_m P_m$, where P_m is the projector onto the eigenspace of eigenvalue λ_m, the P_m's take the role of the measurement operators. If the measurement outcome is not recorded, then the overall process is represented by the quantum channel

$$\mathcal{E}_{\text{proj}}(\rho) = \sum_m P_m \rho P_m.$$

In the history of quantum mechanics, the interpretation as mathematical projection onto subspaces traces back to an article by G. Lüders[2]. He concluded that the quantum superposition within an eigenspace of dimension 2 or larger "survives" the measurement process, and that two commuting observables M, $\tilde{M}$, $[M, \tilde{M}] = 0$, are "compatible" with each other, i.e., measuring M does not affect the outcome statistics of $\tilde{M}$. In this tutorial, we discuss an experimental realization [1] of such a "Lüders process" retaining the superposition. (The term "Lüders process" and "ideal measurement" refer to a projective measurement here.)

The principal quantum system is formed by three electronic states $|0\rangle$, $|1\rangle$ and $|2\rangle$ of a $^{88}\text{Sr}^+$ ion, as indicated in Fig. 1(a). Such a "qutrit" is a generalization of qubits to statevectors from $\mathbb{C}^3$. The ion has an additional short-lived excited state $|e\rangle$. In the experiment, a laser with variable power drives

$$|0\rangle \to g_0 |0\rangle + g_1 |e\rangle .$$

$|e\rangle$ quickly decays to $|0\rangle$, emitting a photon in the process: $|e\rangle |n = 0\rangle \to |0\rangle |n = 1\rangle$, where $|n\rangle$ is the quantum state of the photon environment. The (indirect) measurement process consists of the detection of the emitted photon, which indicates the occupancy of $|0\rangle$, but leaves a superposition between $|1\rangle$ and $|2\rangle$ intact. The coefficients g_0 and g_1 satisfy $|g_0|^2 + |g_1|^2 = 1$ and are used to demonstrate a transition from "no measurement" ($g_0 = 1$) to an ideal measurement ($g_0 = 0$). (In the experiment, fluorescence detection is actually only employed at the end for state tomography, but not during the measurement, i.e., one ignores the outcome.)

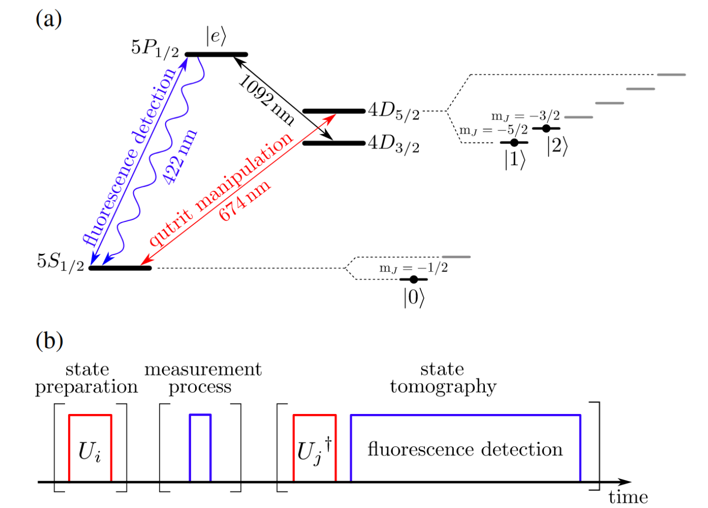

Figure 1: (a) Electronic states of the ion $^{88}\text{Sr}^+$. (b) Experimental sequence to characterize the process.

(a) The system is initialized to

$$|\psi\rangle = (\alpha_0 |0\rangle + \alpha_1 |1\rangle + \alpha_2 |2\rangle) |n = 0\rangle .$$

What is the state of the system, $|\psi'\rangle$, after the excitation by the laser and $|e\rangle$ has decayed back into $|0\rangle$?

The Kraus operators which model this whole operation (i.e., the drive into $|e\rangle$ and the subsequent decay) are

$$E_0 = g_1 |0\rangle \langle 0| \quad \text{and} \quad E_1 = g_0 |0\rangle \langle 0| + |1\rangle \langle 1| + |2\rangle \langle 2| .$$

(b) Compute the reduced density matrix of the ion at the beginning of the experiment, $\rho_{\text{ion}} = \text{tr}_{\text{env}}[|\psi\rangle \langle\psi|]$, and apply the quantum operation to ρ_{ion}.

(c) Verify that your result for (b) matches

$$\rho'_{\text{ion}} = \text{tr}_{\text{env}}[|\psi'\rangle \langle\psi'|].$$

(d) Which measurement process corresponds to the case when $g_1 = 1$?

The experiment uses *process tomography* for characterization. A detailed explanation is beyond the scope of this tutorial; as brief summary: an additional laser (shown in red in Fig. 1) performs the initial state preparation, formally by applying a unitary matrix to $|0\rangle$. In the experiment, nine specific initial states $|\psi_i\rangle = U_i |0\rangle$ are used in different runs, corresponding to the unitaries $\{U_i\}$. Before the final fluorescence detection, the red laser realizes the action of one of the adjoint unitaries $U_j^\dagger$.

[1] F. Pokorny et al.: *Tracking the dynamics of an ideal quantum measurement*. Phys. Rev. Lett. 124, 080401 (2020)
[2] G. Lüders: *Über die Zustandsänderung durch den Meßprozeß*. Ann. Phys. 443, 322–328 (1950)

(e) Show that, in general, for a projective measurement with operators P_m, applying a unitary $U^\dagger$ beforehand changes the outcome probabilities as if using the operators $UP_mU^\dagger$.

The experiment represents the process in terms of the so-called Choi matrix, as shown in Fig. 2. As $g_0 \to 0$, the process becomes an ideal (projective) measurement.

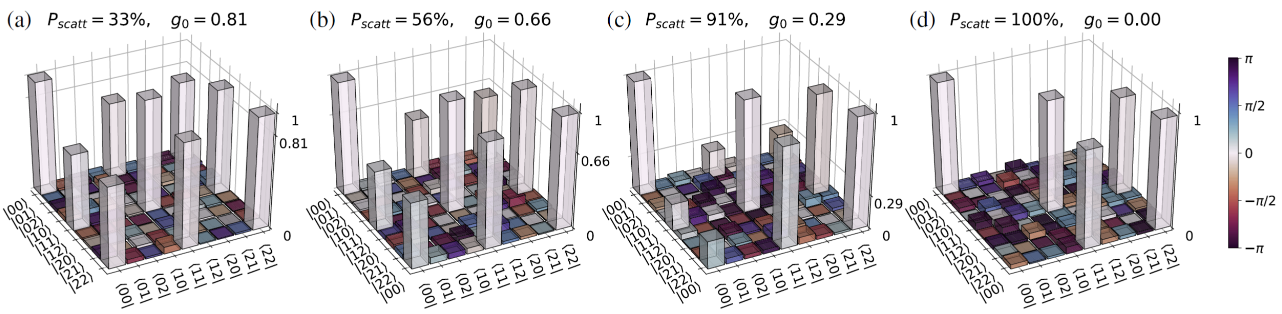

Figure 2: Choi matrices reconstructed from experimental data for different values of g_0, from [1].

(f) Which feature of the Choi matrix indicates that the superposition between $|1\rangle$ and $|2\rangle$ is preserved?

Solution

(a) Right after the excitation has happened, the state of the system is

$$(\alpha_0 g_0 |0\rangle + \alpha_0 g_1 |e\rangle + \alpha_1 |1\rangle + \alpha_2 |2\rangle) |n = 0\rangle .$$

After the decay, this becomes

$$|\psi'\rangle = (\alpha_0 g_0 |0\rangle + \alpha_1 |1\rangle + \alpha_2 |2\rangle) |n = 0\rangle + \alpha_0 g_1 |0\rangle |n = 1\rangle .$$

(b) Initially the ion and the environment are unentangled, so we can directly read off the density matrix

$$\rho_{\text{ion}} = (\alpha_0 |0\rangle + \alpha_1 |1\rangle + \alpha_2 |2\rangle)(\alpha_0^* \langle 0| + \alpha_1^* \langle 1| + \alpha_2^* \langle 2|) = \begin{pmatrix} |\alpha_0|^2 & \alpha_0\alpha_1^* & \alpha_0\alpha_2^* \\ \alpha_1\alpha_0^* & |\alpha_1|^2 & \alpha_1\alpha_2^* \\ \alpha_2\alpha_0^* & \alpha_2\alpha_1^* & |\alpha_2|^2 \end{pmatrix} .$$

The overall quantum operation is

$$\mathcal{E}(\rho_{\text{ion}}) = \sum_k E_k \rho_{\text{ion}} E_k^\dagger .$$

Here

$$E_0 \rho_{\text{ion}} E_0^\dagger = \begin{pmatrix} g_1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} |\alpha_0|^2 & \alpha_0\alpha_1^* & \alpha_0\alpha_2^* \\ \alpha_1\alpha_0^* & |\alpha_1|^2 & \alpha_1\alpha_2^* \\ \alpha_2\alpha_0^* & \alpha_2\alpha_1^* & |\alpha_2|^2 \end{pmatrix} \begin{pmatrix} g_1^* & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} |g_1|^2|\alpha_0|^2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

and

$$\begin{aligned} E_1 \rho_{\text{ion}} E_1^\dagger &= \begin{pmatrix} g_0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} |\alpha_0|^2 & \alpha_0\alpha_1^* & \alpha_0\alpha_2^* \\ \alpha_1\alpha_0^* & |\alpha_1|^2 & \alpha_1\alpha_2^* \\ \alpha_2\alpha_0^* & \alpha_2\alpha_1^* & |\alpha_2|^2 \end{pmatrix} \begin{pmatrix} g_0^* & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} |g_0|^2|\alpha_0|^2 & g_0\alpha_0\alpha_1^* & g_0\alpha_0\alpha_2^* \\ g_0^*\alpha_1\alpha_0^* & |\alpha_1|^2 & \alpha_1\alpha_2^* \\ g_0^*\alpha_2\alpha_0^* & \alpha_2\alpha_1^* & |\alpha_2|^2 \end{pmatrix} . \end{aligned}$$

Adding both of these matrices, and using the fact that $|g_0|^2 + |g_1|^2 = 1$, we obtain

$$\mathcal{E}(\rho_{\text{ion}}) = \begin{pmatrix} |\alpha_0|^2 & g_0\alpha_0\alpha_1^* & g_0\alpha_0\alpha_2^* \\ g_0^*\alpha_1\alpha_0^* & |\alpha_1|^2 & \alpha_1\alpha_2^* \\ g_0^*\alpha_2\alpha_0^* & \alpha_2\alpha_1^* & |\alpha_2|^2 \end{pmatrix} .$$

(c) We trace out the environment by

$$\rho'_{\text{ion}} = \sum_{i=0}^{1} \langle n=i|\psi'\rangle \langle\psi'|n=i\rangle \,.$$

Note that

$$\langle n=0|\psi'\rangle = (\alpha_0 g_0 |0\rangle + \alpha_1 |1\rangle + \alpha_2 |2\rangle)$$

and

$$\langle n=1|\psi'\rangle = \alpha_0 g_1 |0\rangle \,.$$

This leads us to

$$\rho'_{\text{ion}} = \begin{pmatrix} |\alpha_0|^2 & g_0\alpha_0\alpha_1^* & g_0\alpha_0\alpha_2^* \\ g_0^*\alpha_1\alpha_0^* & |\alpha_1|^2 & \alpha_1\alpha_2^* \\ g_0^*\alpha_2\alpha_0^* & \alpha_2\alpha_1^* & |\alpha_2|^2 \end{pmatrix}$$

as in the previous section.

(d) When $g_1 = 1$, $g_0 = 0$ and therefore the Kraus operators are $E_0 = |0\rangle\langle 0|$ and $E_1 = I - |0\rangle\langle 0| = |1\rangle\langle 1| + |2\rangle\langle 2|$, i.e., this process is a projective measurement into the space spanned by $|0\rangle$ and its orthogonal complement.

(e) Denoting the quantum state before the measurement by $|\psi\rangle$, outcome m occurs with probability

$$p(m) = \langle\psi| P_m |\psi\rangle \,.$$

(Note that $P_m^2 = P_m$ by definition of a projection operator.) Applying $U^\dagger$ beforehand changes the probability to

$$\tilde{p}(m) = \langle\psi| U P_m U^\dagger |\psi\rangle = \langle\psi| \tilde{P}_m |\psi\rangle$$

with $\tilde{P}_m = U P_m U^\dagger$.

(f) These are the four columns corresponding to $(|11\rangle + |22\rangle)(\langle 11| + \langle 22|)$.