Christian B. Mendl, Pedro Hack, Keefe Huang, Irene López Gutiérrez

Exercise 1.2 (Linear algebra basics)

(a) Compute (with "pen and paper") the matrix-vector product

$$\begin{pmatrix} 2 & -i & 5 \\ 3 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 4 \\ i \\ -3 \end{pmatrix},$$

and the matrix-matrix product

$$\begin{pmatrix} -2 & 7 \\ 3 & 1+2i \end{pmatrix} \cdot \begin{pmatrix} 5 & -4 \\ 6i & 0 \end{pmatrix}.$$

(b) Find a 2×2 matrix which is not normal.
Hint: you can restrict your search to real-valued matrices.

(c) Show that the following matrix is normal, and compute its characteristic polynomial, eigenvalues and an eigenvector corresponding to one of the eigenvalues:

$$A = \begin{pmatrix} 0 & \frac{3}{5} & \frac{4}{5} \\ -\frac{3}{5} & 0 & 0 \\ -\frac{4}{5} & 0 & 0 \end{pmatrix}.$$

(d) Show that the following matrix is unitary (with $\theta \in \mathbb{R}$ a real parameter):

$$\begin{pmatrix} \cos(\theta) & i\sin(\theta) \\ i\sin(\theta) & \cos(\theta) \end{pmatrix}.$$

(e) Let $U \in \mathbb{C}^{n \times n}$ be a unitary matrix. Show that

$$|\det(U)| = 1,$$

where $|\cdot|$ denotes the absolute value.
Hint: consider $\det(U^\dagger U)$.

Solution

(a)

$$\begin{pmatrix} 2 & -i & 5 \\ 3 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 4 \\ i \\ -3 \end{pmatrix} = \begin{pmatrix} 2 \cdot 4 - i \cdot i + 5 \cdot (-3) \\ 3 \cdot 4 + 0 \cdot i + 1 \cdot (-3) \end{pmatrix} = \begin{pmatrix} -6 \\ 9 \end{pmatrix}$$

$$\begin{pmatrix} -2 & 7 \\ 3 & 1+2i \end{pmatrix} \cdot \begin{pmatrix} 5 & -4 \\ 6i & 0 \end{pmatrix} = \begin{pmatrix} -10+42i & 8 \\ 3+6i & -12 \end{pmatrix}$$

(b) A matrix A is normal when $AA^\dagger = A^\dagger A$. An example of a matrix that is not normal is

$$\begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix}$$

(c) First we show that it is normal:

$$AA^\dagger = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{9}{25} & \frac{12}{25} \\ 0 & \frac{12}{25} & \frac{16}{25} \end{pmatrix} = A^\dagger A.$$

Its characteristic polynomial is $-\lambda^3 - \lambda = 0$. And therefore its eigenvalues are $\lambda_1 = 0$, $\lambda_2 = i$ and $\lambda_3 = -i$. Its eigenvectors are

$$v_1 = \begin{pmatrix} 0 \\ -\frac{4}{3} \\ 1 \end{pmatrix}, \quad v_2 = \begin{pmatrix} -\frac{5i}{4} \\ \frac{3}{4} \\ 1 \end{pmatrix}, \quad v_3 = \begin{pmatrix} \frac{5i}{4} \\ \frac{3}{4} \\ 1 \end{pmatrix}$$

(d) A matrix U is unitary if $U^\dagger U = I$. In this case

$$\begin{pmatrix} \cos(\theta) & -i\sin(\theta) \\ -i\sin(\theta) & \cos(\theta) \end{pmatrix} \cdot \begin{pmatrix} \cos(\theta) & i\sin(\theta) \\ i\sin(\theta) & \cos(\theta) \end{pmatrix} = \begin{pmatrix} \cos^2(\theta)+\sin^2(\theta) & 0 \\ 0 & \cos^2(\theta)+\sin^2(\theta) \end{pmatrix} = I$$

(e) Here we will use two properties of the determinant: $\det(AB) = \det(A)\det(B)$ and $\det(A^\dagger) = \det(A)^*$. Therefore,

$$\det(U^\dagger U) = |\det(U)|^2.$$

We also know that

$$\det(U^\dagger U) = \det(I) = 1.$$

Combining these two equations we arrive at

$$|\det(U)| = 1$$