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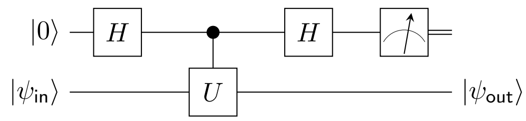
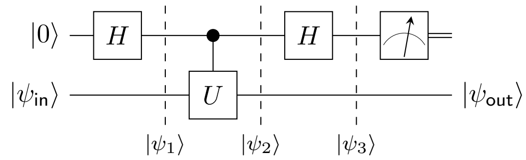
Tutorial 5 (Measuring an operator[1])
Suppose U is a single qubit operator with eigenvalues ± 1, so that U is both Hermitian and unitary, i.e., it can be regarded both as an observable and a quantum gate. Suppose we wish to measure the observable U. That is, we desire to obtain a measurement result indicating one of the two eigenvalues, and leaving a post-measurement state which is the corresponding eigenvector. Show that this is implemented by the following quantum circuit:



This tutorial requires the concept of an orthogonal projection (see also the linear algebra cheatsheet): a square matrix $P \in \mathbb{C}^{n \times n}$ is called an *orthogonal projection matrix* if P is Hermitian ($P^\dagger = P$) and $P^2 = P$, i.e., applying P a second time does not change the result any more. Note that a geometric projection is a special case of this abstract definition.

Solution We compute the intermediate two-qubit states $|\psi_1\rangle$, $|\psi_2\rangle$, $|\psi_3\rangle$ shown below, which result from applying the circuit gates from left to right:



$$|\psi_1\rangle = (H\,|0\rangle) \otimes |\psi_{\text{in}}\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \otimes |\psi_{\text{in}}\rangle\,, \tag{1}$$

$$|\psi_2\rangle = (\text{controlled-}U)\,|\psi_1\rangle = \frac{1}{\sqrt{2}}\,|0\rangle \otimes |\psi_{\text{in}}\rangle + \frac{1}{\sqrt{2}}\,|1\rangle \otimes (U\,|\psi_{\text{in}}\rangle), \tag{2}$$

$$\begin{aligned}
|\psi_3\rangle &= \frac{1}{\sqrt{2}}(H\,|0\rangle) \otimes |\psi_{\text{in}}\rangle + \frac{1}{\sqrt{2}}(H\,|1\rangle) \otimes (U\,|\psi_{\text{in}}\rangle) \\
&= \frac{1}{2}(|0\rangle + |1\rangle) \otimes |\psi_{\text{in}}\rangle + \frac{1}{2}(|0\rangle - |1\rangle) \otimes (U\,|\psi_{\text{in}}\rangle) \\
&= |0\rangle \otimes \frac{I+U}{2}\,|\psi_{\text{in}}\rangle + |1\rangle \otimes \frac{I-U}{2}\,|\psi_{\text{in}}\rangle \\
&= |0\rangle \otimes (P_+\,|\psi_{\text{in}}\rangle) + |1\rangle \otimes (P_-\,|\psi_{\text{in}}\rangle)
\end{aligned} \tag{3}$$

where we have defined $P_\pm = \frac{1}{2}(I \pm U)$. The $P_\pm$ are orthogonal projectors: they are Hermitian since U is Hermitian by assumption, and

$$P_\pm^2 = \frac{1}{4}(I \pm U)^2 = \frac{1}{4}\left(I \pm 2U + U^2\right) = \frac{1}{2}(I \pm U) = P_\pm.$$

In the last step we have used that $U^2 = U^\dagger U = I$. Moreover, the $P_\pm$ project onto orthogonal subspaces since

$$P_+ P_- = \frac{1}{4}(I+U)(I-U) = \frac{1}{4}\left(I - U^2\right) = 0.$$

Since $U = 1 \cdot P_+ + (-1) \cdot P_-$, we have found the spectral decomposition of U, i.e., the $P_\pm$ project onto the eigenspaces of U corresponding to the eigenvalues ± 1.

Now we show that the circuit can indeed be interpreted as measurement of $|\psi_{\text{in}}\rangle$ with measurement operators $P_\pm$: first, they satisfy the completeness relation since $P_+ + P_- = I$. Moreover, according to the last line of Eq. (3), $|\psi_3\rangle$ is a sum of two orthogonal states, and the probability that the measurement (in the circuit diagram) of the first qubit gives 0 or 1 is equal to the squared norm of the first and second state, respectively:

$$p(0) = \| |0\rangle \otimes (P_+\,|\psi_{\text{in}}\rangle) \|^2 = \|P_+\,|\psi_{\text{in}}\rangle\|^2 = \langle\psi_{\text{in}}| P_+^\dagger P_+ \,|\psi_{\text{in}}\rangle = \langle\psi_{\text{in}}| P_+ \,|\psi_{\text{in}}\rangle$$

[1] M. A. Nielsen, I. L. Chuang: *Quantum Computation and Quantum Information*. Cambridge University Press (2010), Exercise 4.34

and correspondingly $p(1) = \langle\psi_{\mathsf{in}}|P_- |\psi_{\mathsf{in}}\rangle$. Directly after the measurement, the second qubit will be in the state

$$|\psi_{\mathsf{out}}\rangle = \frac{P_+ |\psi_{\mathsf{in}}\rangle}{\|P_+ |\psi_{\mathsf{in}}\rangle\|} \quad \text{if measured } 0,$$
$$|\psi_{\mathsf{out}}\rangle = \frac{P_- |\psi_{\mathsf{in}}\rangle}{\|P_- |\psi_{\mathsf{in}}\rangle\|} \quad \text{if measured } 1$$

which agrees with the definition of a quantum measurement with operators $P_\pm$.