Christian B. Mendl, Pedro Hack, Irene López Gutiérrez, Keefe Huang due: 19 Dec 2022, 08:00 on Moodle

Tutorial 8 (Experimental Bell inequality)

As mentioned in the lecture, the violation of Bell inequality has been experimentally verified. Many of these experiments make use of photons as the qubits.[1] Fig. 1 shows a schematic version of the most common setup used to test the Bell inequality with photons. The source generates a pair of entangled photons, one of which is sent to Alice, one of which is sent to Bob. The coincidence monitor counts the instances with simultaneous measurements on Alice's and Bob's side in order to discard detections of environment photons not created by the source.

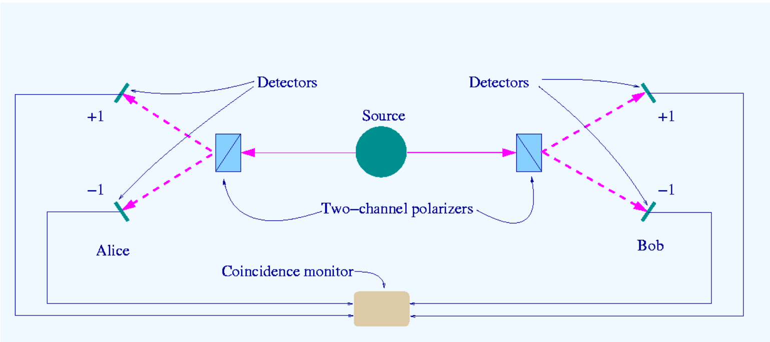

Figure 1: Photon analyzer (Source: https://commons.wikimedia.org/w/index.php?curid=641329)

The actual setup, of course, contains many more elements, like in the experiment of M. Giustina et al.:

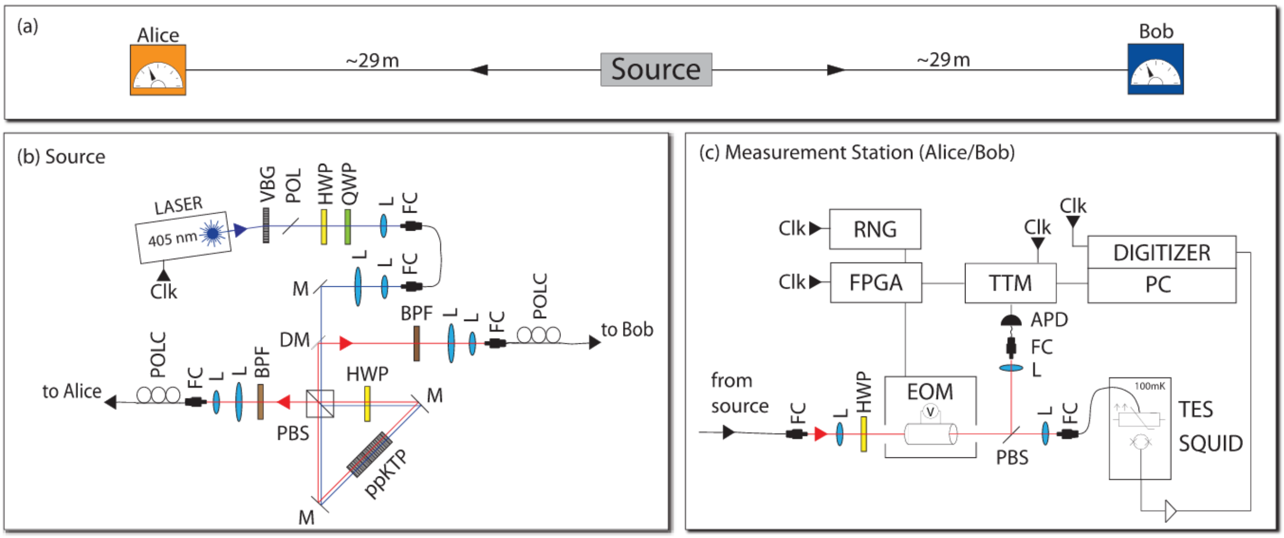

Figure 2: (a) Schematic setup. (b) Source of the entangled photons. (c) Detection station of one of the receivers.

(a) Discuss the role of the polarizers in these experiments. For simplicity, you can focus on the schematic representation from Fig. 1.

If we hypothesize that the experiment can be described by hidden variables, denoted λ, then the probability that Alice measures a and Bob measures b is given by:

$$P(a,b) = \sum_{x,y,\lambda} P(a|x,\lambda)P(b|y,\lambda)P(x,y)P(\lambda),$$

where x is the choice of the measurement basis used by Alice, and y is the one used by Bob. This expression is not able to reproduce all results from experiments according to Bell's inequality, so the consensus is that hidden variable theories are incorrect. However, there are *two* additional implicit assumptions leading to this expression:

[1] see, .e.g.,
M. Giustina et al.: *Significant-loophole-free test of Bell's theorem with entangled photons.* Phys. Rev. Lett. 115, 250401 (2015)
The BIG Bell Test Collaboration: *Challenging local realism with human choices.* Nature 557, 212-216 (2018)
D. Rauch et al.: *Cosmic Bell test using random measurement settings from high-redshift quasars.* Phys. Rev. Lett. 121, 080403 (2018)

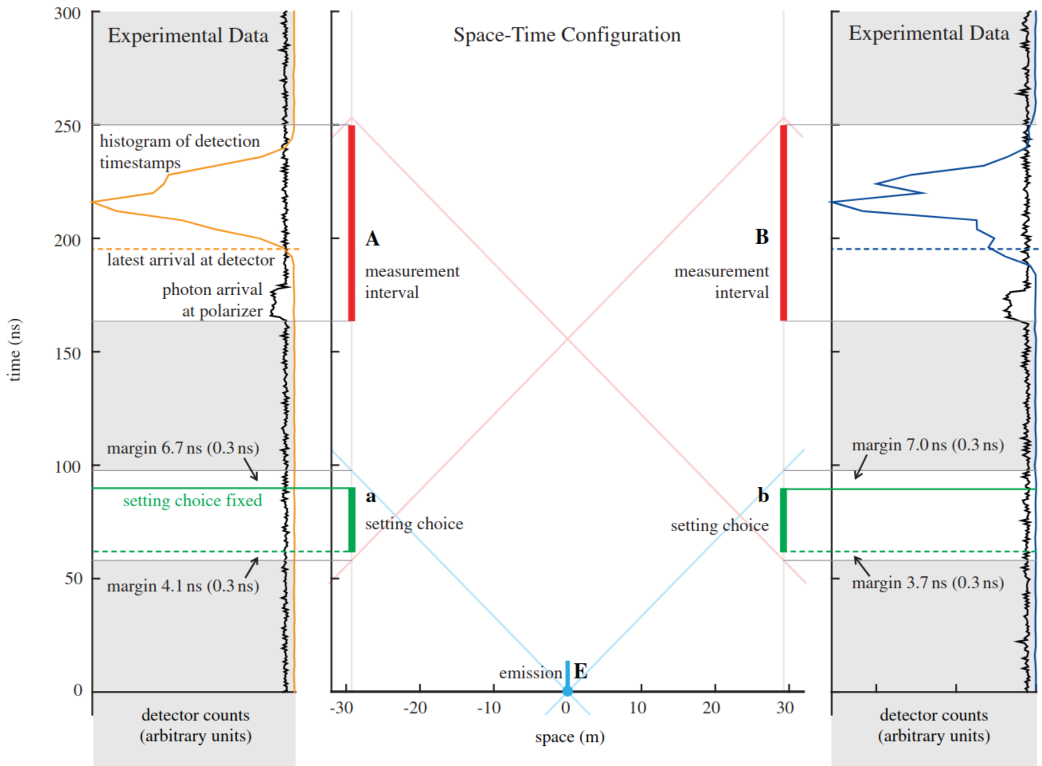

Figure 3: Space and time configuration of the experiment by M. Giustina et al. [Phys. Rev. Lett. 115, 250401 (2015)]

(b) One of these assumptions is locality. In a lab experiment the photon detection is not instantaneous. Why could this create a loophole, and what can be done to close this loophole? As an example, Fig. 3 shows the space-time configuration and specific times each experimental step takes.

(c) Identify the remaining assumption and discuss how one can make sure that it holds during the experiment.

Exercise 8.1 (Bell inequality violation by quantum mechanics)

(a) Let $\{|a\rangle, |b\rangle\}$ be an orthonormal basis of $\mathbb{C}^2$, and $\alpha, \beta, \gamma, \delta \in \mathbb{C}$ chosen such that

$$|0\rangle = \alpha |a\rangle + \beta |b\rangle,$$
$$|1\rangle = \gamma |a\rangle + \delta |b\rangle.$$

Verify the relation

$$\frac{|01\rangle - |10\rangle}{\sqrt{2}} = (\alpha\delta - \beta\gamma)\frac{|ab\rangle - |ba\rangle}{\sqrt{2}}.$$

(b) In the quantum experiment violating the Bell inequality, Charlie prepares the “spin singlet” quantum state

$$|\psi\rangle = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle)$$

and sends the first qubit to Alice and the second to Bob. Alice then measures the observable Q or R on her qubit and Bob the observable S or T on his qubit, defined as

$$Q = Z, \qquad S = \frac{-Z - X}{\sqrt{2}},$$
$$R = X, \qquad T = \frac{Z - X}{\sqrt{2}}.$$

Verify the following average values (which violate the Bell inequality):

$$\langle\psi| Q \otimes S |\psi\rangle = \frac{1}{\sqrt{2}}, \quad \langle\psi| R \otimes S |\psi\rangle = \frac{1}{\sqrt{2}}, \quad \langle\psi| R \otimes T |\psi\rangle = \frac{1}{\sqrt{2}}, \quad \langle\psi| Q \otimes T |\psi\rangle = -\frac{1}{\sqrt{2}}.$$

Exercise 8.2 (CHSH game)
The CHSH game demonstrates the power of quantum correlations and gives rise to a Bell inequality through a simple game in a real-world setting.

Alice and Bob play a game whose inputs and objective are described in the following. Alice and Bob each receive a completely random bit denoted by x resp. y (independently distributed, with equal probability for 0 or 1). After receiving their input they are not allowed to communicate until the end of the game. Alice and Bob each have to produce an output $a \in \{0,1\}$ resp. $b \in \{0,1\}$. They win the game whenever their inputs and outputs fulfill the condition

$$x \cdot y = a + b \mod 2.$$

As a resource Alice and Bob receive an EPR pair

$$|\beta_{00}\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$$

shared between them.

(a) Draw a sketch of the setting described above.

(b) Before starting the game, Alice and Bob are allowed to agree on a strategy determining which outputs to generate for a given input. Note that Alice only has access to her input x and Bob only to his input y and they are not allowed to communicate once they received their input. In case Alice and Bob do not make use of their shared entanglement, the maximum winning probability they can achieve with a deterministic classical strategy is 75%. Give a deterministic classical strategy which Alice and Bob should follow in order to achieve the best possible classical winning probability.

We now give a strategy for Alice and Bob making use of their quantum resources. Define the unitary operator

$$U_\theta := \begin{pmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{pmatrix} \quad \text{for } \theta \in \mathbb{R}$$

and parameters

$$\theta_0 = 0, \qquad \theta_1 = \frac{\pi}{4},$$
$$\tau_0 = \frac{\pi}{8}, \qquad \tau_1 = -\frac{\pi}{8}.$$

Based on their inputs x, y Alice and Bob will apply operators U_{θ_x} and U_{τ_y} on their local system of the shared EPR pair. Afterwards, they measure their local register in the computational basis with outcomes a, b. The probability of receiving outcomes a, b dependent on inputs x, y is given by

$$P(a, b \mid x, y) = \langle (U_{\theta_x} \otimes U_{\tau_y})\beta_{00} \mid P_a \otimes P_b \mid (U_{\theta_x} \otimes U_{\tau_y})\beta_{00} \rangle,$$

where $P_a = |a\rangle\langle a|$ and $P_b = |b\rangle\langle b|$ are projections onto the computational basis states for $a, b \in \{0,1\}$. Alice and Bob use the outcome of their measurement a, b as output for the game.

(c) Show that Alice and Bob can win with probability roughly 85% when following the above quantum strategy.
Hint: The relation $(V \otimes W)|\beta_{00}\rangle = (V \cdot W^T \otimes I)|\beta_{00}\rangle$ for any $V, W \in \mathbb{C}^{2\times 2}$ might be helpful.