Solution 7: Economic Growth

Problem 1 (*Golden-Rule Steady State*)

(a) The production function

$$Y = F(L, K) = L^{\frac{1}{2}} K^{\frac{1}{2}}$$

exhibits constant returns to scale; multiplication with $\frac{1}{L}$ yields

$$\frac{Y}{L} = F\left(1, \frac{K}{L}\right) = \left(\frac{K}{L}\right)^{\frac{1}{2}}.$$

Accordingly, output per worker as a function of capital per worker is

$$y = f(k) = k^{\frac{1}{2}}.$$

(b) In a steady-state, savings per worker equal break-even investment.

$$sf(k^*) = (\delta + n)k^*$$

Using the production function and the given parameter values yields

$$s\,(k^*)^{\frac{1}{2}} = \frac{k^*}{20} \quad \Leftrightarrow \quad (k^*)^{\frac{1}{2}} = 20s = y^*.$$

Hence, steady-state consumption per worker as a function of the saving rate is

$$c^*(s) = (1 - s)y^* = 20s - 20s^2.$$

(c) The golden-rule saving rate maximizes steady-state consumption per worker.

$$\max_s \quad c^*(s) = 20s - 20s^2$$

Necessary condition:

$$\frac{dc^*(s)}{ds} = 20 - 40s = 0 \quad \Leftrightarrow \quad s_{gold} = \frac{1}{2}$$

Problems 2-6 (*Golden-Rule Steady State*)

The production function

$$Y = F(L, K) = L^{\frac{1}{3}} K^{\frac{2}{3}}$$

exhibits constant returns to scale; multiplication with $\frac{1}{L}$ yields

$$\frac{Y}{L} = F\left(1, \frac{K}{L}\right) = \left(\frac{K}{L}\right)^{\frac{2}{3}}.$$

Accordingly, output per worker as a function of capital per worker is

$$y = f(k) = k^{\frac{2}{3}}.$$

In a steady-state, savings per worker equal break-even investment.

$$sf(k^*) = (\delta + n)k^*$$

Using the production function and the given parameter values yields

$$s\left(k^*\right)^{\frac{2}{3}} = \frac{k^*}{3},$$

which simplifies to

$$\left(k^*\right)^{\frac{2}{3}} = 9s^2. \tag{1}$$

Accordingly, the steady state output per worker is

$$y^* = 9s^2. \tag{2}$$

The golden-rule capital stock per worker must satisfy

$$f'(k^*_{gold}) = n + \delta.$$

Using the production function and the given parameter values yields

$$\frac{2}{3}\left(k^*_{gold}\right)^{-\frac{1}{3}} = \frac{1}{3},$$

which simplifies to

$$k^*_{gold} = 8. \tag{3}$$

Problem 2

Substituting the steady-state output $y^* = 1$ into equation (2) yields the corresponding saving rate.

$$1 = 9s^2 \quad \Leftrightarrow \quad s = \frac{1}{3}$$

$\Rightarrow$ **(C)** is correct.

Problem 3

Substituting the saving rate $s = \frac{1}{3}$ into equation (1) yields the corresponding steady-state capital stock per worker.

$$(k^*)^{\frac{2}{3}} = 1 \quad \Leftrightarrow \quad k^* = 1$$

Thus, at $k = 1$, the economy is in a steady state, implying that over time, all per-worker quantities are constant, and all aggregate quantities increase at a constant rate, namely the rate of population growth $n = \frac{1}{6}$.

$\Rightarrow$ **(B)** is correct.

Problem 4

Substituting equation (3) into equation (1) yields the golden-rule saving rate.

$$4 = 9s^2 \quad \Leftrightarrow \quad s_{gold} = \frac{2}{3}$$

$\Rightarrow$ **(D)** is correct.

Problem 5

Substituting $s_{gold} = \frac{2}{3}$ into equation (2) yields output per worker in the golden-rule steady state.

$$y^*_{gold} = 4$$

Thus, consumption per worker in the golden-rule steady state is

$$c^*_{gold} = (1 - s)y^* = \frac{4}{3}.$$

$\Rightarrow$ **(C)** is correct.

Problem 6

Any saving rate satisfying $s \in \left[0, \frac{1}{3}\right)$ is strictly below the golden-rule saving rate $s_{gold} = \frac{2}{3}$ and thus implies a dynamically efficient steady state.

$\Rightarrow$ **(A)** is correct.