Professorship for Quantum Computing
Department of Informatics
Technical University of Munich

Ecorrection
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Note:

- During the attendance check a sticker containing a unique code will be put on this exam.
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Introduction to Quantum Computing

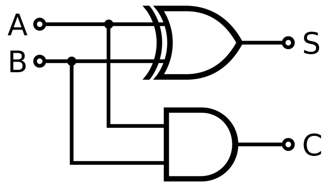
Exam:	IN2381 / Final Exam	**Date:**	Tuesday 20th July, 2021
Examiner:	Christian Mendl	**Time:**	14:15 – 15:45

Working instructions

- This exam consists of **10 pages** with a total of **3 problems**.
  Please make sure now that you received a complete copy of the exam.
- The total amount of achievable credits in this exam is 60 credits.
- Detaching pages from the exam is prohibited.
- Allowed resources: open book
- Subproblems marked by * can be solved without results of previous subproblems.
- **Answers are only accepted if the solution approach is documented.** Give a reason for each answer unless explicitly stated otherwise in the respective subproblem.
- Do not write with red or green colors nor use pencils.

Problem 1 (20 credits)

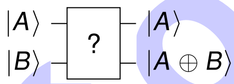
In this problem, you will build a quantum version of a half adder – the basic building block of addition on a classical computer. The most important part of such a circuit is the half-adder:

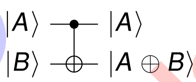


where A and B are classical bits, $S = A \oplus B$ is the sum modulo two and $C = A \cdot B$ is ordinary multiplication of A and B called the carry. The carry is the part of the summation that adds to the next digit (it is only 1 if both $A = 1$ and $B = 1$).

0 1 2 3

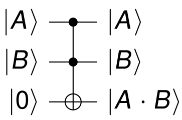
a) Assume you start in the arbitrary two-qubit state $|AB\rangle$. Provide a quantum gate / series of quantum gates that performs the operation:



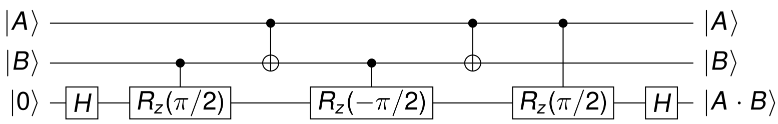


0 1 2 3 4 5 6 7 8 9

b) The $A \cdot B$ operation can be performed by a Toffoli gate:



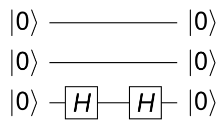
Verify that the circuit below performs that operation up to a global phase constant.



Hint: Follow the state of each qubit through the circuit for all 4 possible input basis states $|00\rangle$, $|01\rangle$, $|10\rangle$ and $|11\rangle$ separately.

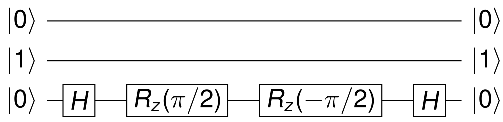
We account for the activated controlled gates:

Input $|00\rangle$:



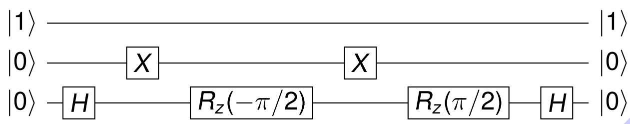
1 credit

Input $|01\rangle$:



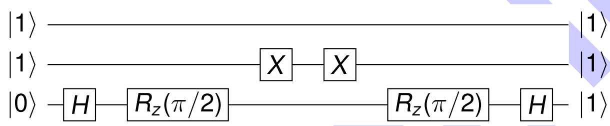
2 credits

Input $|10\rangle$:



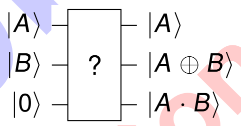
2 credits

Input $|11\rangle$:

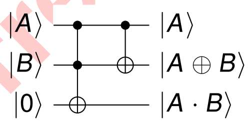


Here we have used that $HR_z(\pi/2)R_z(\pi/2)H = HR_z(\pi)H = -iHZH = -iX$, so there is a global phase constant of $-i$. 4 credits

c) Build a quantum half-adder using the Toffoli gate and the result from a): i.e. find the circuit that performs the operation:



You do not need to write out the Toffoli decomposition explicitly.



2 credits for Toffoli, 1 for CNOT; -2 if the order is wrong.

0 1 2 3 4

d) Given $|A \oplus B\rangle$ and $|A \cdot B\rangle$, is it possible to determine $|A\rangle$ and $|B\rangle$ in all cases? Provide reasoning for your answer.

It is not possible because both $|01\rangle$ and $|10\rangle$ result in $|A \oplus B\rangle = |1\rangle$ and $|A \cdot B\rangle = |0\rangle$. This operation is only reversible with an extra qubit.

4 credits for complete correct answer. 2 credits if this is shown in only one case.

Problem 2 **(20 credits)**

Consider an ensemble of quantum states $\{p_i, |\psi_i\rangle\}$, where the quantum system is in state $|\psi_i\rangle$ with probability p_i. Recall from the lecture that the density operator ρ of such an ensemble is defined as:

$$\rho = \sum_i p_i |\psi_i\rangle \langle\psi_i|$$

a) Given the ensemble $\left\{\left(\frac{1}{2}, \frac{|0\rangle+|1\rangle}{\sqrt{2}}\right), \left(\frac{1}{2}, \frac{|0\rangle+i|1\rangle}{\sqrt{2}}\right)\right\}$, compute ρ and write it in the form:

$$\rho = \frac{1}{2}I + \alpha_x X + \alpha_y Y + \alpha_z Z.$$

What is the connection of $\vec{\alpha} = (\alpha_x, \alpha_y, \alpha_z)$ with the Bloch sphere representation?

Simply by plugging in, expanding and rearranging:

$$\begin{aligned}
\rho &= \frac{1}{2}\frac{(|0\rangle + |1\rangle)}{\sqrt{2}}\frac{(\langle 0| + \langle 1|)}{\sqrt{2}} + \frac{1}{2}\frac{(|0\rangle + i|1\rangle)}{\sqrt{2}}\frac{(\langle 0| - i\langle 1|)}{\sqrt{2}} \\
&= \frac{1}{4}\left(|0\rangle\langle 0| + |0\rangle\langle 1| + |1\rangle\langle 0| + |1\rangle\langle 1|\right) + \frac{1}{4}\left(|0\rangle\langle 0| - i|0\rangle\langle 1| + i|1\rangle\langle 0| + |1\rangle\langle 1|\right) \\
&= \frac{1}{2}\left(|0\rangle\langle 0| + |1\rangle\langle 1|\right) + \frac{1}{4}\left(|0\rangle\langle 1| + |1\rangle\langle 0|\right) + \frac{1}{4}\left(-i|0\rangle\langle 1| + i|1\rangle\langle 0|\right) \\
&= \frac{1}{2}I + \frac{1}{4}X + \frac{1}{4}Y
\end{aligned}$$

which means that $(\alpha_x, \alpha_y, \alpha_z) = \left(\frac{1}{4}, \frac{1}{4}, 0\right)$. Recall that a density operator can be represented as

$$\rho = \frac{I + \vec{r}\cdot\vec{\sigma}}{2},$$

where $\vec{r}$ is the Bloch vector. So $\vec{\alpha} = \frac{\vec{r}}{2}$.
4 credits for correct ρ, 1 credit for ρ in right form, 1 credit for realising this is the Bloch vector over 2

b) Now consider the ensemble

$$\left\{\left(\frac{1}{2}, |0\rangle\right), \left(\frac{1}{2}, |1\rangle\right)\right\},$$

and compute its density matrix ρ. Draw a Bloch sphere, clearly labeling $|0\rangle$ and $|1\rangle$, and indicate the position of this ensemble within the sphere.

$$\rho = \frac{1}{2}\left(|0\rangle\langle 0| + |1\rangle\langle 1|\right) = \frac{1}{2}I$$

This means that the Bloch vector of this ensemble is $(0, 0, 0)$, i.e., it is located at the origin of the Bloch sphere.
1 credit for correct ρ, 2 credits for correct location of ensemble, 1 credit for correct drawing of a Bloch sphere

0 1 2 3 4

c)* The unitary operation

$$U = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \sqrt{1-\lambda} & -i\sqrt{\lambda} & 0 \\ 0 & -i\sqrt{\lambda} & \sqrt{1-\lambda} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

where $0 \le \lambda \le 1$, acts on a system of two qubits. The first qubit is initially in an arbitrary state ρ and the second one is initialized at $|0\rangle$. Trace out the second qubit to obtain the two operators E_0 and E_1 which represent the action of U on the first one.

From Exercise 11.2 we recall that

$$\mathcal{E}(\rho) = \sum_k E_k \rho E_k^\dagger.$$

$$(E_k)_{\ell,m} = \langle \ell, k|U|m, 0\rangle$$

2 credits for these or related formulas which show recognition of this being a quantum operation.
Therefore,

$$E_0 = \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{1-\lambda} \end{pmatrix} \quad \text{and} \quad E_1 = \begin{pmatrix} 0 & -i\sqrt{\lambda} \\ 0 & 0 \end{pmatrix}$$

Full credits if these two matrices are correct, even if no explanation.

0 1 2 3 4 5 6

d) Compute the effect of the operators you found on the general density matrix $\rho = \frac{1}{2}I + \alpha_x X + \alpha_y Y + \alpha_z Z$. Interpret their action on the Bloch sphere.

One possible solution is by writing E_0 and E_1 in terms of the Pauli matrices:

$$E_0 = \frac{1}{2}(I+Z) + \frac{\sqrt{1-\lambda}}{2}(I-Z) = \left(\frac{1+\sqrt{1-\lambda}}{2}\right) I + \left(\frac{1-\sqrt{1-\lambda}}{2}\right) Z = aI + bZ$$

$$E_1 = \frac{\sqrt{\lambda}}{2}(-iX+Y) = \frac{-i\sqrt{\lambda}}{2}X + \frac{\sqrt{\lambda}}{2}Y = cX + dY$$

For a general ρ, $\mathcal{E}(\rho) = \frac{1}{2}\mathcal{E}(I) + \alpha_x\mathcal{E}(X) + \alpha_y\mathcal{E}(Y) + \alpha_z\mathcal{E}(Z)$. We compute each term:

$$\begin{aligned}\mathcal{E}(I) &= (aI+bZ)I(aI+bZ) + (cX+dY)I(c^*X+dY) \\ &= a^2I + 2abZ + b^2I + |c|^2I + icdZ - idc^*Z + d^2I = I + \lambda Z\end{aligned}$$

$$\begin{aligned}\mathcal{E}(X) &= (aI+bZ)X(aI+bZ) + (cX+dY)X(c^*X+dY) \\ &= a^2X - b^2X + |c|^2X - d^2X = \sqrt{1-\lambda}\end{aligned}$$

$$\begin{aligned}\mathcal{E}(Y) &= (aI+bZ)Y(aI+bZ) + (cX+dY)Y(c^*X+dY) \\ &= a^2Y - b^2Y - |c|^2Y + d^2X = \sqrt{1-\lambda}\end{aligned}$$

$$\begin{aligned}\mathcal{E}(Z) &= (aI+bZ)Z(aI+bZ) + (cX+dY)Z(c^*X+dY) \\ &= a^2Z + 2abI + b^2Z - |c|^2Z - icdI + idc^*I - d^2Z = (1-\lambda)Z\end{aligned}$$

4 credits for these or similar
Therefore,

$$\mathcal{E}(\rho) = \frac{1}{2}I + \sqrt{1-\lambda}\alpha_x X + \sqrt{1-\lambda}\alpha_y Y + \left(\frac{\lambda}{2} + (1-\lambda)\alpha_z\right)Z,$$

and $\vec{\alpha}' = \left(\sqrt{1-\lambda}\alpha_x, \sqrt{1-\lambda}\alpha_y, (\frac{\lambda}{2} + (1-\lambda)\alpha_z)\right)$. 1 credit for this or similar
This is an amplitude damping channel. (This identification is not required.) It shrinks the Bloch sphere (0.5 credits) towards the north pole. (0.5 credits)

Problem 3 (20 credits)

We consider a quantum system of n qubits, and use the notation X_j, Y_j, Z_j to denote that one of the Pauli matrices acts on the jth qubit; e.g., $X_1 Z_3 \equiv X \otimes I \otimes Z$ for $n = 3$.
Conjugation by U refers to the transformation $UgU^\dagger$ of a quantum gate g by a unitary operation U. The following table summarizes several conjugation transformations:

U	Z	Z	Z	H	H	H	S	S	S
g	X	Y	Z	X	Y	Z	X	Y	Z
$UgU^\dagger$	$-X$	$-Y$	Z	Z	$-Y$	X	Y	$-X$	Z

Here $S = \left(\begin{smallmatrix} 1 & 0 \\ 0 & i \end{smallmatrix}\right)$ is the phase gate.

a) State the check matrix representation of $g_1, g_2 \in G_4$ given by

$$g_1 = Z \otimes Y \otimes X \otimes X,$$
$$g_2 = X \otimes Z \otimes I \otimes Y.$$

Based on this representation, show that g_1 *anti*-commutes with g_2.

The check matrix of (g_1, g_2) is

$$\begin{pmatrix} r(g_1) \\ r(g_2) \end{pmatrix} = \begin{pmatrix} 0 & 1 & 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 1 & 0 & 1 \end{pmatrix}.$$

(3 credits)
Now we recall that elements of the Pauli group either commute or anti-commute. (0.5 credits) According to exercise 12.2 (e), g_1 commutes with g_2 precisely if $r(g_1)\Lambda r(g_2)^T = 0 \mod 2$, with

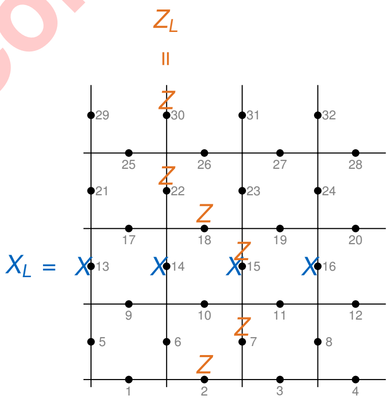
$$\Lambda = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}.$$

(0.5 credits)
Λ effectively interchanges the first with the second half of a check row. We obtain

$$r(g_1)\Lambda r(g_2)^T = \begin{pmatrix} 0 & 1 & 1 & 1 & 1 & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 & 1 & 1 & 0 & 0 & 1 \end{pmatrix}^T = 3 = 1 \mod 2,$$

i.e., they do not commute, and hence must anti-commute. (1 credit)

b)* Given a square lattice, a qubit is associated with each *edge* of the lattice (dots in the figure below). We define logical Pauli operators X_L and Z_L as tensor products of strings of X and Z operators: $X_L = X_{13}X_{14}X_{15}X_{16}$ and $Z_L = Z_2 Z_7 Z_{15} Z_{18} Z_{22} Z_{30}$, as visualized in the figure.



Show that X_L and Z_L anti-commute, i.e., $X_L Z_L = -Z_L X_L$. How can one define a logical Y_L operator such that X_L, Y_L, Z_L satisfy the anti-commutation relations of the Pauli-matrices?

One observes that X_L and Z_L share only a single qubit which they both act on, namely qubit 15 . Since X_j always commutes with Z_k for $j \neq k$, we obtain

$$\begin{aligned} X_L Z_L &= Z_2 Z_7 X_{13} X_{14} X_{15} Z_{15} X_{16} Z_{18} Z_{22} Z_{30} \\ &= Z_2 Z_7 X_{13} X_{14} (-Z_{15} X_{15}) X_{16} Z_{18} Z_{22} Z_{30} = -Z_L X_L. \end{aligned}$$

3 credits

The Pauli-Y matrix can be expressed in terms of X and Z via $Y = iXZ$. We can use this relation to analogously define $Y_L = iX_L Z_L$. Then Y_L anti-commutes with X_L since X_L commutes with itself and anti-commutes with Z_L, as we have just shown. Likewise, Y_L anti-commutes with Z_L.

2 credits; alternative solutions possible

0 1 2 3 4 5

c)* The subgroup $R = \langle X_1 Y_2 Z_3, Y_1 Y_2 Y_3 \rangle$ of the Pauli group G_3 stabilizes the subspace $V_R = \mathrm{span}\{|\chi_0\rangle, |\chi_1\rangle\}$ with

$$|\chi_0\rangle = \frac{1}{2}\big(|000\rangle + |001\rangle + i|110\rangle - i|111\rangle\big), \qquad |\chi_1\rangle = \frac{1}{2}\big(|010\rangle + |011\rangle - i|100\rangle + i|101\rangle\big).$$

(A proof of this statement is not required here.) Determine the result (eigenvalue) when measuring the operator $Y_1 Y_2 Y_3$ with respect to the quantum state $(S \otimes H \otimes (SH))|\chi_0\rangle$, where S is the phase gate.

In general, an eigenvalue equation $M|\psi\rangle = \lambda|\psi\rangle$ is equivalent to $UMU^\dagger U|\psi\rangle = \lambda U|\psi\rangle$ for any unitary matrix U. Setting $U = S \otimes H \otimes (SH)$ here, we search for an operator M such that $UMU^\dagger = Y_1 Y_2 Y_3$, with formal solution

$$M = U^\dagger (Y_1 Y_2 Y_3) U = (S^\dagger Y S) \otimes (HYH) \otimes (HS^\dagger Y S H).$$

Here we have already used that $H^\dagger = H$. From the above conjugation table, we see that $SXS^\dagger = Y$, i.e., $S^\dagger Y S = X$, as well as $HYH = -Y$ and $HS^\dagger Y S H = HXH = Z$. In summary,

$$M = X \otimes (-Y) \otimes Z = -X_1 Y_2 Z_3.$$

Since R stabilizes $|\chi_0\rangle$ and $X_1 Y_2 Z_3 \in R$, the state $|\chi_0\rangle$ is an eigenstate of M with eigenvalue (-1). In particular, the measurement result will be (-1) with probability 1.

d)* We consider the two qubit code $C = \text{span}\{|0_L\rangle, |1_L\rangle\}$ with $|0_L\rangle = |00\rangle$ and $|1_L\rangle = |01\rangle$. It is affected by a simultaneous bit flip noise process described by the operation elements $E_0 = \frac{1}{\sqrt{2}} I_4$ and $E_1 = \frac{1}{\sqrt{2}} X \otimes X$, where I_n the $n \times n$ identity matrix. Show that this noise process is error-correctable.

0 1 2 3 4 5

We make use of the quantum error-correction conditions (see lecture). The projector onto C in the present case is $P = |00\rangle\langle 00| + |01\rangle\langle 01|$. The code is error-correctable precisely if

$$PE_k^\dagger E_\ell P = \alpha_{k\ell} P$$

for all $k, \ell \in \{0, 1\}$ and some Hermitian matrix $(\alpha_{k\ell})$ of complex numbers. We first note that $E_k^\dagger E_k = \frac{1}{2} I_4$ for $k \in \{0, 1\}$ since $X^2 = I_2$. Thus the condition is satisfied via $\alpha_{kk} = \frac{1}{2}$ in the case $k = \ell$.
We now explicitly compute

$$PE_0^\dagger E_1 P = PE_1^\dagger E_0 P = \frac{1}{2} P(X \otimes X)P = (|00\rangle\langle 00| + |01\rangle\langle 01|)(|11\rangle\langle 00| + |10\rangle\langle 01|) = 0,$$

i.e., the condition is satisfied via $\alpha_{01} = \alpha_{10} = 0$ for $k \neq \ell$. In summary, the quantum error-correction conditions hold true for all combinations of $k, \ell \in \{0, 1\}$.

Additional space for solutions–clearly mark the (sub)problem your answers are related to and strike out invalid solutions.