Algorithms of Scientific Computing II – Quantum Computing (IN2002, W19/20) **Exam**

Christian Mendl and Irene López Gutiérrez 11 February 2020

Last name, first name, signature: **Matrikelnummer:**

Notes:

- Allocated time: 90 minutes
- Permitted material: one A4 sheet (both sides) with your own notes
- Write your name on each sheet!
- Please use the back side if you run out of space
- The maximum number of points is the same for all problems

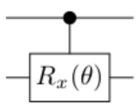
Problem	1	2	3	$\sum$
Points				

Problem 1

(a) Recall that the rotation operator R_x is defined as

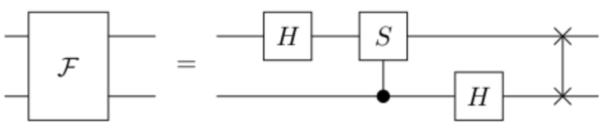
$$R_x(\theta) = \mathrm{e}^{-i\theta X/2} = \begin{pmatrix} \cos\frac{\theta}{2} & -i\sin\frac{\theta}{2} \\ -i\sin\frac{\theta}{2} & \cos\frac{\theta}{2} \end{pmatrix} \quad \text{for } \theta \in \mathbb{R}.$$

Compute the eigenvalues of $R_x(\theta)$ and of the controlled-$R_x(\theta)$ gate:



Hint: The eigenvalues of $R_x(\theta)$ are identical to the ones of $R_y(\theta)$ and $R_z(\theta)$.

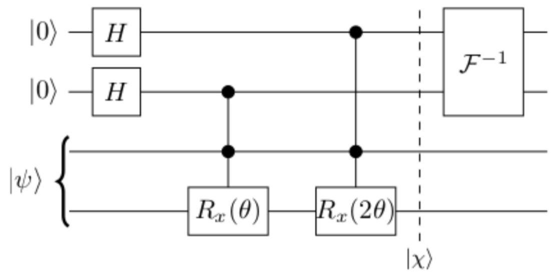
(b) The following circuit implements the quantum Fourier transform $\mathcal{F}$ for two qubits:



Here H is the Hadamard gate, $S = \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix}$ the so-called phase gate, and the last operation the swap gate. Provide a circuit which realizes the *inverse* quantum Fourier transform for two qubits.

Hint: In general, $(AB)^\dagger = B^\dagger A^\dagger$ for matrices A and B with compatible dimensions.

We now consider the following quantum circuit:



(c) Compute the intermediate state $|\chi\rangle$ of this circuit for the two cases

$$\text{(i)} \quad |\psi\rangle = \frac{1}{\sqrt{2}}|0\rangle(|0\rangle + |1\rangle) \qquad \text{and} \qquad \text{(ii)} \quad |\psi\rangle = \frac{1}{\sqrt{2}}|1\rangle(|0\rangle + |1\rangle).$$

(d) What is the overall output of the circuit for $\theta = \pi$ in the cases (i) and (ii)?

Problem 2 *Phase damping* models decoherence in realistic physical situations and is described by the quantum channel

$$\mathcal{E}(\rho) = \sum_{k=0}^{1} E_k \rho E_k^\dagger,$$

with operation elements

$$E_0 = \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{1-\lambda} \end{pmatrix}, \quad E_1 = \begin{pmatrix} 0 & 0 \\ 0 & \sqrt{\lambda} \end{pmatrix}$$

and "scattering" probability $\lambda \in [0, 1]$. We assume $0 < \lambda < 1$ in the following.

(a) A quantum channel $\mathcal{E}$ is called *unital* if $\mathcal{E}(I) = I$. Show that the phase damping channel is unital.

(b) Recall that an arbitrary density operator ρ for a mixed state qubit can be represented as

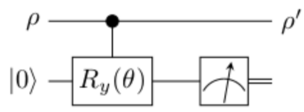
$$\rho = \frac{I + \vec{r} \cdot \vec{\sigma}}{2}, \tag{1}$$

with $\vec{r} \in \mathbb{R}^3$ the *Bloch vector* of ρ and $\vec{\sigma}$ the vector of Pauli matrices. Compute the Bloch vector $\vec{r}'$ of the output state $\rho' = \mathcal{E}(\rho)$ of the phase damping channel in dependence of $\vec{r}$.

(c) In which case(s) does $\mathcal{E}(\rho)$, with $\mathcal{E}$ the phase damping channel, describe a pure quantum system?

(d) Compute the density matrix after n repeated applications of the phase damping operation, $\mathcal{E}(\ldots\mathcal{E}(\mathcal{E}(\rho)))$, and take the limit $n \to \infty$. You may work with a symbolic 2×2 matrix representation of ρ, or the Bloch representation (1) and part (b).

(e) Show that the following circuit describes the phase damping operation, and relate the angle θ to the parameter λ:



Hint: The operation elements corresponding to the circuit have entries $(E_k)_{\ell,m} = \langle \ell, k|U|m, 0\rangle$ with $k, \ell, m \in \{0, 1\}$ and U the controlled-$R_y(\theta)$ operation.

Problem 3 We consider a quantum system of n qubits, and use the notation X_j, Y_j, Z_j to denote that one of the Pauli matrices acts on the jth qubit; e.g., $X_1 Z_3 \equiv X \otimes I \otimes Z$ for $n = 3$.

Conjugation by U refers to the transformation $UgU^\dagger$ of a quantum gate g by a unitary operation U. The following table summarizes several conjugation transformations:

U	H	H	H	S	S	S	CNOT	CNOT	CNOT	CNOT
g	X	Y	Z	X	Y	Z	X_1	X_2	Z_1	Z_2
$UgU^\dagger$	Z	$-Y$	X	Y	$-X$	Z	X_1X_2	X_2	Z_1	Z_1Z_2

Here $S = \left(\begin{smallmatrix} 1 & 0 \\ 0 & i \end{smallmatrix}\right)$ is the phase gate, and the CNOT gate has the first qubit as control and the second qubit as target.

(a) Verify that indeed $SXS^\dagger = Y$, $SZS^\dagger = Z$, and $\text{CNOT} \cdot Z_2 \cdot \text{CNOT}^\dagger = Z_1 Z_2$.

(b) Compute $\text{CNOT} \cdot Y_2 \cdot \text{CNOT}^\dagger$.

Hint: $Y = iXZ$, and $Ug\tilde{g}U^\dagger = UgU^\dagger U\tilde{g}U^\dagger$.

(c) We consider the subgroup $R = \{I, X_1 Z_2\}$ of the Pauli group G_2. Compute the two-qubit subspace V_R stabilized by R, such that $g|\psi\rangle = |\psi\rangle$ for all $g \in R$ and $|\psi\rangle \in V_R$.

(d) The subgroup $T = \langle X_1 Y_2, Y_2 Z_3 \rangle$ of the Pauli group G_3 stabilizes the subspace $V_T = \mathrm{span}\{|\chi_0\rangle, |\chi_1\rangle\}$ with

$$|\chi_0\rangle = |000\rangle + i|010\rangle + |100\rangle + i|110\rangle, \qquad |\chi_1\rangle = |001\rangle - i|011\rangle - |101\rangle + i|111\rangle.$$

(A proof of this statement is not required here.) Find a subgroup T' of the Pauli group G_3 which stabilizes $V_{T'} = \mathrm{span}\{(S \otimes S \otimes S)|\chi_0\rangle, (S \otimes S \otimes S)|\chi_1\rangle\}$.

(e) Let U be a unitary, *diagonal* 2×2 matrix which maps the set $\{\pm X, \pm Y\}$ to itself by conjugation. Show that U can be written as matrix products of phase gates (including $S^0 = I$).