Professorship for Quantum Computing
Department of Informatics
Technical University of Munich

Eexam
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Note:

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Introduction to Quantum Computing

Exam:	IN2381 / Final Exam	**Date:**	Tuesday 2nd March, 2021
Examiner:	Prof. Dr. Christian Mendl	**Time:**	14:15 – 15:45

Working instructions

- This exam consists of **12 pages** with a total of **3 problems**.
  Please make sure now that you received a complete copy of the exam.
- The total amount of achievable credits in this exam is 60 credits.
- Detaching pages from the exam is prohibited.
- Allowed resources: open book
- Subproblems marked by * can be solved without results of previous subproblems.
- **Answers are only accepted if the solution approach is documented.** Give a reason for each answer unless explicitly stated otherwise in the respective subproblem.
- Do not write with red or green colors nor use pencils.

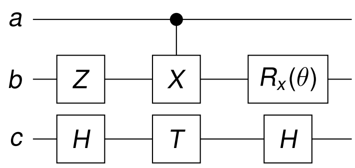
Left room from ________ to ________ / Early submission at ________

Problem 1 **(20 credits)**

a) We are given a unitary gate T defined as

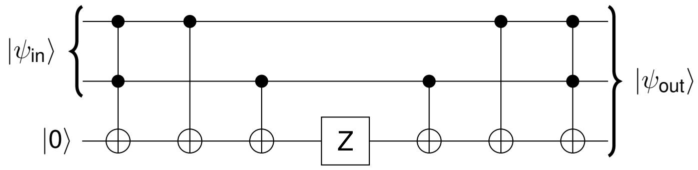
$$T = \sqrt{\frac{50}{29}} \begin{bmatrix} 0.3i & 0.7i \\ -0.7i & 0.3i \end{bmatrix}$$

Draw the circuit that performs the inverse operation of the following circuit (with $\theta \in \mathbb{R}$):



Explicitly define new gates, if any, used in the drawn circuit.

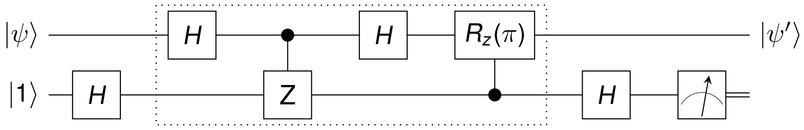
b)* Compute the output $|\psi_{\text{out}}\rangle$ of the following quantum circuit for computational basis states as input, i.e., $|\psi_{\text{in}}\rangle \in \{|00\rangle, |01\rangle, |10\rangle, |11\rangle\}$:



Based on your calculation, provide the output for the input state

$$|\psi_{\text{in}}\rangle = a\,|+-\rangle + b\,|-+\rangle$$

c)* The following quantum circuit takes as input a single qubit quantum state $|\psi\rangle$ and an ancilla qubit set to $|1\rangle$. R_z is the rotation gate along the Z-axis of the Bloch sphere. Simplify the gates inside the dotted box. (Hint: you should arrive at a single controlled gate.)



d) Calculate the output $|\psi'\rangle$ of the circuit in (c) in terms of $|\psi\rangle$ and Pauli or identity matrices, depending on whether the measurement outcome is 0 or 1.

0
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Problem 2 (20 credits)

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a) Quantum states are manipulated by quantum gates, which are described by complex, unitary matrices U, such that

$$|\psi'\rangle = U\,|\psi\rangle\,.$$

Explain why U must be unitary.

0
1
2

b)* We are running an experiment on a system described by a density matrix ρ. Determine the coefficient c such that

$$\rho = c\,|{++}\rangle\,\langle{++}| + \frac{3}{4}\,|{--}\rangle\,\langle{--}|$$

is a valid density matrix.

0
1
2

c)* Assume that we apply phase gates to each qubit of the system from part (b). Write down this operator and its action on ρ. (An explicit computation is not required here.)

d)* In real-life experiments, however, it is difficult to isolate the system from the environment, so the operation applied to ρ may not be unitary. This seems to contradict the statement in section (a). How can one reconcile both points of view? Write down a mathematical expression for such an operation.

0 1 2 3

e)* Consider a dataset containing all integers from 0 to $N-1$, where N is a power of 2. We are given a deterministic function f that processes the dataset. Alice tells us that it will output 0 for half of the entries of our dataset and 1 for the other half. Bob claims that it always outputs 1. We know one of them is correct.

In the worst case scenario, how many classical evaluations of f will we need to determine who is correct? Which quantum algorithm could we use to solve our problem? How many evaluations will we need in that case?

0 1 2

f)* The quantum algorithm from (e) takes advantage of the quantum superposition principle to apply f simultaneously to all entries of the dataset. We want to preprocess qubits which are all currently in the state $|-\rangle$ to obtain the equal superposition state. Write down how many qubits we need in terms of N, and a preprocessing operation to arrive at the equal superposition state.

0 1 2 3

0 1 2 3 4 5 6

g) Consider the Hermitian gate U_f defined on computational basis states as

$$U_f \left|x\right\rangle \left|z\right\rangle = \left|x\right\rangle \left|z \oplus f(x)\right\rangle .$$

We want to apply U_f to the equal superposition state, denoted $\left|\psi\right\rangle$ here, and an ancilla qubit which is in state $\left|-\right\rangle$. However, before we apply U_f, a noise process affects the ancilla qubit, namely a phase flip with probability p, defined by the two matrices (Kraus operators)

$$E_1 = \sqrt{1-p}I \quad \text{and} \quad E_2 = \sqrt{p}Z.$$

Compute the output density matrix ρ' of the ancilla qubit after undergoing this phase flip operation. Then, provide an expression for U_f applied to $\left|\psi\right\rangle \left\langle\psi\right| \otimes \rho'$. Finally, assess how the outcome of the algorithm from (e) is affected by the noise.

Problem 3 **(20 credits)**

We consider a quantum system of n qubits, and use the notation X_j, Y_j, Z_j to denote that one of the Pauli matrices acts on the jth qubit; e.g., $X_1Z_3 \equiv X \otimes I \otimes Z$ for $n = 3$.
Conjugation by U refers to the transformation $UgU^\dagger$ of a quantum gate g by a unitary operation U. The following table summarizes several conjugation transformations:

U	Z	Z	Z	H	H	H	S	S	S
g	X	Y	Z	X	Y	Z	X	Y	Z
$UgU^\dagger$	$-X$	$-Y$	Z	Z	$-Y$	X	Y	$-X$	Z

Here $S = \left(\begin{smallmatrix} 1 & 0 \\ 0 & i \end{smallmatrix}\right)$ is the phase gate.

a) We encode a logical qubit by two physical qubits as

$$|0_L\rangle = |01\rangle\,, \quad |1_L\rangle = |10\rangle\,.$$

The physical qubits are affected by bit flip errors $|0\rangle \leftrightarrow |1\rangle$. Describe and briefly explain a measurement for *error detection*, i.e., diagnosing whether a single bit flip has occurred. Is it also possible to recover from such an error?

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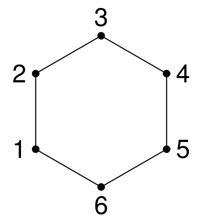
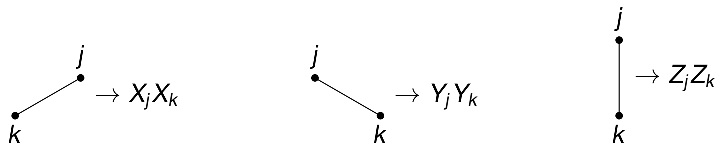
b)* Six qubits are assigned to the vertices of a hexagon:



We define $W = X_1 Y_2 Z_3 X_4 Y_5 Z_6$, and the following operators depending on the *orientation* of an edge:



It turns out that for each of the six edges of the hexagon, the corresponding edge operator commutes with W. Prove this statement for edge $3-4$ and $4-5$.

c)* The subgroup $T = \langle X_1 Y_2 Z_3, X_1 X_2 Y_3, -Z_1 Y_2 Y_3 \rangle$ of the Pauli group G_3 stabilizes the one-dimensional subspace $V_T = \mathrm{span}\{|\psi\rangle\}$ with

$$|\psi\rangle = \frac{1}{2\sqrt{2}}\big(|000\rangle - |001\rangle + |010\rangle + |011\rangle - i\,|100\rangle + i\,|101\rangle + i\,|110\rangle + i\,|111\rangle\big).$$

(A proof of this statement is not required here.) Find a subgroup T' of the Pauli group G_3 which stabilizes $V_{T'} = \mathrm{span}\{(S \otimes H \otimes Z)\,|\psi\rangle\}$.

d)* Specify an element g of the Pauli group G_4 such that

$$R = \langle Y_2 Z_3, Y_1 Z_2 X_3 Z_4, g \rangle$$

stabilizes a non-trivial vector space and the three generators of R are independent. Also state the properties which g must satisfy (a proof of them is not required).

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Additional space for solutions–clearly mark the (sub)problem your answers are related to and strike out invalid solutions.