6.1 $|\psi\rangle = \cos\left(\frac{\theta}{2}\right)|0\rangle + e^{i\varphi}\sin\left(\frac{\theta}{2}\right)|1\rangle$, $\vec{r} = \begin{pmatrix} r_x \\ r_y \\ r_z \end{pmatrix} = \begin{pmatrix} \cos(\varphi)\sin(\theta) \\ \sin(\varphi)\sin(\theta) \\ \cos(\theta) \end{pmatrix}$ $X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ $Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

a) $\mathbb{E}[Z] = \langle\psi|Z|\psi\rangle = \left(\cos\left(\frac{\theta}{2}\right)|0\rangle - e^{-i\varphi}\sin\left(\frac{\theta}{2}\right)|1\rangle\right)^T |\psi\rangle = \cos\left(\frac{\theta}{2}\right)^2 - \cancel{e^{i\varphi}e^{-i\varphi}}\sin\left(\frac{\theta}{2}\right)^2 =$
$\cos\left(\frac{\theta}{2}\right)^2 - \sin\left(\frac{\theta}{2}\right)^2 = \cos(\theta) = \vec{r}_z = \langle Z\rangle$

$\mathbb{E}[X] = \langle\psi|X|\psi\rangle = \left(e^{-i\varphi}\sin\left(\frac{\theta}{2}\right)|0\rangle + \cos\left(\frac{\theta}{2}\right)|1\rangle\right)^T |\psi\rangle = e^{-i\varphi}\sin\left(\frac{\theta}{2}\right)\cos\left(\frac{\theta}{2}\right) + e^{i\varphi}\sin\left(\frac{\theta}{2}\right)\cos\left(\frac{\theta}{2}\right) =$
$\sin\left(\frac{\theta}{2}\right)\cos\left(\frac{\theta}{2}\right)\left(e^{-i\varphi} + e^{i\varphi}\right) = \frac{1}{2}\sin(\theta)\left(e^{-i\varphi} + e^{i\varphi}\right) =$
$\frac{1}{2}\sin(\theta)\left[\cos(\varphi) - \cancel{i\sin(\varphi)} + \cos(\varphi) + \cancel{i\sin(\varphi)}\right] = \frac{1}{2}\sin(\theta)\left[2\cos(\varphi)\right] = \sin(\theta)\cos(\varphi) =$
$\vec{r}_x = \langle X\rangle$

b) $\langle Z\rangle^2 = \cos(\theta)^2$, $\langle Z^2\rangle = \langle\psi|Z^2|\psi\rangle = \langle\psi|I|\psi\rangle = \langle\psi|\psi\rangle = 1$
$\Delta Z = \sqrt{\langle Z^2\rangle - \langle Z\rangle^2} = \sqrt{1-\cos(\theta)^2} = \sqrt{\sin(\theta)^2 + \cancel{\cos(\theta)^2} - \cancel{\cos(\theta)^2}} = \sin(\theta)$

$\langle X\rangle^2 = \sin(\theta)^2\cos(\varphi)^2$, $\langle X^2\rangle = \langle\psi|X^2|\psi\rangle = \langle\psi|I|\psi\rangle = \langle\psi|\psi\rangle = 1$
$\Delta X = \sqrt{\langle X^2\rangle - \langle X\rangle^2} = \sqrt{1-\sin(\theta)^2\cos(\varphi)^2}$

c) $[Z,X] = ZX - XZ = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} - \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = 2\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$

$\mathbb{E}[[Z,X]] = \langle\psi|[Z,X]|\psi\rangle = 2\left(-e^{-i\varphi}\sin\left(\frac{\theta}{2}\right)|0\rangle + \cos\left(\frac{\theta}{2}\right)|1\rangle\right)^T|\psi\rangle$
$= 2\left(-e^{-i\varphi}\sin\left(\frac{\theta}{2}\right)\cos\left(\frac{\theta}{2}\right) + e^{i\varphi}\sin\left(\frac{\theta}{2}\right)\cos\left(\frac{\theta}{2}\right)\right) = 2\sin\left(\frac{\theta}{2}\right)\cos\left(\frac{\theta}{2}\right)\left(e^{i\varphi} - e^{-i\varphi}\right)$
$= \sin(\theta)\left(\cancel{\cos(\varphi)} + i\sin(\varphi) - \cancel{\cos(\varphi)} + i\sin(\varphi)\right) = 2i\sin(\theta)\sin(\varphi)$

d) • $\Delta Z\Delta X = \sin(\theta)\sqrt{1-\sin^2(\theta)\cos^2(\varphi)}$

• $\frac{|\mathbb{E}[Z,X]|}{2} = \frac{|2i\sin(\theta)\sin(\varphi)|}{2} = \frac{\sqrt{-4i^2\sin(\theta)^2\sin(\varphi)^2}}{2} = \frac{2\sin(\theta)\sin(\varphi)}{2} = \sin(\theta)\sin(\varphi)$

$\cancel{\sin(\theta)}\sqrt{1-\sin^2(\theta)\cos^2(\varphi)} \geq \cancel{\sin(\theta)}\sin(\varphi) \iff \sqrt{1-\sin^2(\theta)\cos^2(\varphi)} \geq \sin(\varphi)$
$\iff 1 - \sin^2(\theta)\cos^2(\varphi) \geq \sin^2(\varphi) \iff -\sin^2(\theta)\cos^2(\varphi) \geq \cancel{\sin^2(\varphi)} - \cancel{\sin^2(\varphi)} - \cos^2(\varphi) \iff$
$\sin^2(\theta)\cos^2(\varphi) \leq \cos^2(\varphi) \iff \sin^2(\theta) \leq 1 \iff \sin^2(\theta) \leq \sin^2(\theta) + \cos^2(\theta)$ ✓

6.2 a) We can compare the result of applying U to $|\psi\rangle$ directly and going through the circuit step-by step.

$|\psi\rangle = x_0|000\rangle + x_1|001\rangle + \dots + x_7|111\rangle$
$U|\psi\rangle = (ax_0 + cx_7)|000\rangle + x_1|001\rangle + \dots + x_6|110\rangle + (bx_0 + dx_7)|111\rangle$

Interpreting the circuit step-by-step from left to right:

- $|\psi\rangle \to |\psi_1\rangle = x_0|001\rangle + x_1|000\rangle + x_2|010\rangle + \dots + x_7|111\rangle$ flip $|00x\rangle \to |00\bar{x}\rangle$
- $|\psi_1\rangle \to |\psi_2\rangle = x_0|011\rangle + x_1|000\rangle + x_2|010\rangle + x_3|001\rangle + \dots + x_7|111\rangle$ flip $|0x1\rangle \to |0\bar{x}1\rangle$
- $|\psi_2\rangle \to |\psi_3\rangle = x_0(\tilde{U}|0\rangle|11\rangle) + \dots + x_7(\tilde{U}|1\rangle|11\rangle) =$
  $ax_0|011\rangle + bx_0|111\rangle + \dots + cx_7|011\rangle + dx_7|111\rangle = (ax_0 + cx_7)|011\rangle + \dots + (bx_0 + dx_7)|111\rangle$ $|x11\rangle \to |\tilde{U}x11\rangle$
- $|\psi_3\rangle \to |\psi_4\rangle = (ax_0 + cx_7)|001\rangle + x_1|000\rangle + x_2|010\rangle + x_3|011\rangle + \dots$ flip back $|0x1\rangle \to |0\bar{x}1\rangle$
- $|\psi_4\rangle \to |\psi_5\rangle = (ax_0 + cx_7)|000\rangle + x_1|001\rangle + x_2|010\rangle + x_3|011\rangle + x_4|100\rangle + x_5|101\rangle + x_6|110\rangle + (x_0b + x_7d)|111\rangle$
  flip back $|00x\rangle \to |00\bar{x}\rangle$

b) We can rewrite the controlled flips with Toffoli and X gates (from prev exercise). And as we have seen, these gates can be decomposed by single qubit gates and CNOT.

c) From the matrix we can see that the U targets $|010\rangle$ and $|111\rangle$. Taking inspiration from previous circuit design we can add a controlled $\tilde{U}$ in the first qubit that is triggered only when the other 2 qubits are $|1\rangle$. Then, we only need two other layers to make the $|010\rangle \to |011\rangle$ and later flip back.