Chair of Scientific Computing in Computer Science
Department of Informatics
Technical University of Munich

Esolution
Place student sticker here

Note:

- During the attendance check a sticker containing a unique code will be put on this exam.
- This code contains a unique number that associates this exam with your registration number.
- This number is printed both next to the code and to the signature field in the attendance check list.

Introduction to Quantum Computing

Exam:	IN2381 / Final Exam	**Date:**	Friday 25th February, 2022
Examiner:	Prof. Dr. Christian Mendl	**Time:**	14:15 – 15:45

	P 1	P 2	P 3
I			

Working instructions

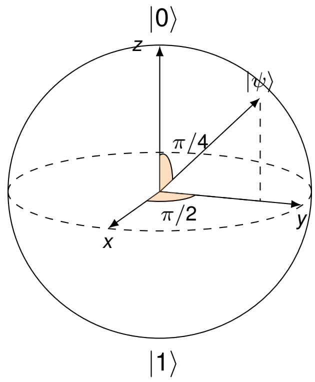
- This exam consists of **12 pages** with a total of **3 problems**.
  Please make sure now that you received a complete copy of the exam.
- The total amount of achievable credits in this exam is 60 credits.
- Detaching pages from the exam is prohibited.
- Allowed resources:
  - one **A4 sheet** (both sides) with your own notes
  - one **analog dictionary** English ↔ native language
- Subproblems marked by * can be solved without results of previous subproblems.
- **Answers are only accepted if the solution approach is documented.** Give a reason for each answer unless explicitly stated otherwise in the respective subproblem.
- Do not write with red or green colors nor use pencils.
- Physically turn off all electronic devices, put them into your bag and close the bag.

Left room from ________ to ________ / Early submission at ________

Problem 1 Bloch Sphere (20 credits)

0 1 2 3 4 5

a) We consider the quantum state $|\psi\rangle$ specified by its Bloch vector as:



Determine the coefficients α and β in the standard basis representation $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$. What is the probability to measure 0 and 1, respectively?

We insert $\theta = \frac{\pi}{4}$ and $\varphi = \frac{\pi}{2}$ into

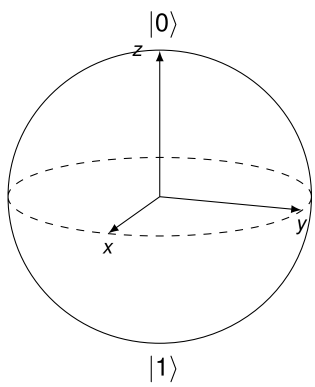
$$|\psi\rangle = \cos(\theta/2)|0\rangle + e^{i\varphi}\sin(\theta/2)|1\rangle = \underbrace{\cos(\pi/8)}_{\alpha}|0\rangle + \underbrace{i\sin(\pi/8)}_{\beta}|1\rangle\,.$$

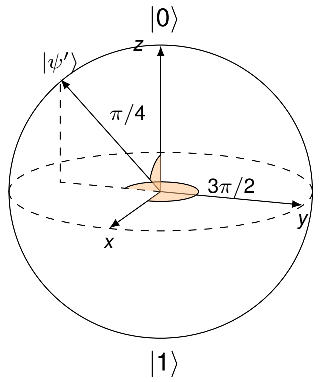
The measurement probabilities are

$$p(0) = \cos(\pi/8)^2,$$
$$p(1) = |i\sin(\pi/8)|^2 = \sin(\pi/8)^2.$$

0 1 2 3 4

b) We now apply the Pauli-Z gate to the system. **Mark** the resulting state (including angles) on the Bloch sphere below, and again provide the probabilities of measuring 0 or 1.





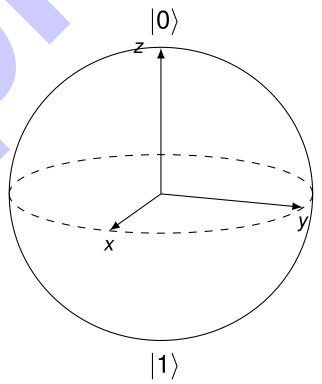
$p(0) = \cos(\pi/8)^2,$

$p(1) = |e^{3i\pi/2}\sin(\pi/8)|^2 = \sin(\pi/8)^2.$

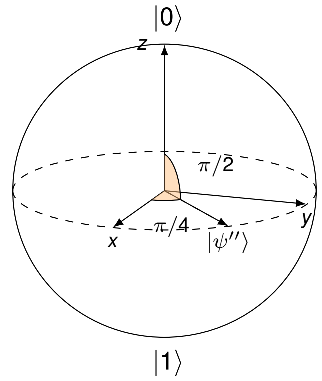
Alternatively, it is also fine to argue that the probabilities are the same because only the phase angle changed, or first computing the basis coefficients of the new quantum state and then the probabilities.

c) Now, apply the Hadamard gate to the state from (b). Again **mark** the resulting state (including angles) on the Bloch sphere below.

Hint: You can use the geometric interpretation of the Hadamard operation, or compute the resulting quantum state algebraically first.



On the Bloch sphere, the Hadamard gate is a rotation about the y-axis by $\pi/2$, followed by a rotation about the x-axis by π. This results in:



Alternatively, can we evaluate this operation algebraically first, and then read off the Bloch angles:

$$
\begin{aligned}
|\psi''\rangle = H\,|\psi'\rangle &= \frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}\begin{pmatrix}\cos(\pi/8)\\ -i\sin(\pi/8)\end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix}\cos(\pi/8) - i\sin(\pi/8)\\ \cos(\pi/8) + i\sin(\pi/8)\end{pmatrix}\\
&= \frac{1}{\sqrt{2}}\begin{pmatrix}e^{-i\pi/8}\\ e^{i\pi/8}\end{pmatrix} = \frac{e^{-i\pi/8}}{\sqrt{2}}\begin{pmatrix}1\\ e^{i\pi/4}\end{pmatrix} = e^{-i\pi/8}\left(\cos(\pi/4)\,|0\rangle + e^{i\pi/4}\sin(\pi/4)\,|1\rangle\right).
\end{aligned}
$$

0 1 2 3 4 5

d)* We now consider a single-qubit quantum system described by the following density matrix:

$$\rho = \frac{1}{2}\begin{pmatrix}1 & \frac{1}{2}\\ \frac{1}{2} & 1\end{pmatrix}.$$

Compute the updated density matrix after applying $R_y(\pi/2)$ to the system.

We can read off the Bloch vector of ρ based on $\rho = \frac{1}{2}(I + \vec{r}\cdot\vec{\sigma})$, namely $\vec{r} = (\frac{1}{2}, 0, 0)$. Since $R_y(\pi/2)$ corresponds to a geometric rotation of 90 degrees around the y-axis, the updated Bloch vector is $\vec{r}' = (0, 0, -\frac{1}{2})$, and the new density matrix

$$\rho' = \frac{1}{2}(I + \vec{r}'\cdot\vec{\sigma}) = \frac{1}{2}\left(I - \frac{1}{2}Z\right) = \frac{1}{2}\begin{pmatrix}\frac{1}{2} & 0\\ 0 & \frac{3}{2}\end{pmatrix}.$$

Alternative solution: We compute ρ' algebraically.
As first step, the matrix representation of the rotation gate is:

$$R_y(\pi/2) = \begin{pmatrix}\cos(\pi/4) & -\sin(\pi/4)\\ \sin(\pi/4) & \cos(\pi/4)\end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & -1\\ 1 & 1\end{pmatrix}.$$

We now apply the gate:

$$
\begin{aligned}
\rho' = R_y(\pi/2)\rho R_y(\pi/2)^\dagger &= \frac{1}{4}\begin{pmatrix}1 & -1\\ 1 & 1\end{pmatrix}\begin{pmatrix}1 & \frac{1}{2}\\ \frac{1}{2} & 1\end{pmatrix}\begin{pmatrix}1 & 1\\ -1 & 1\end{pmatrix}\\
&= \frac{1}{4}\begin{pmatrix}1 & -1\\ 1 & 1\end{pmatrix}\begin{pmatrix}\frac{1}{2} & \frac{3}{2}\\ -\frac{1}{2} & \frac{3}{2}\end{pmatrix}\\
&= \frac{1}{4}\begin{pmatrix}1 & 0\\ 0 & 3\end{pmatrix}.
\end{aligned}
$$

Problem 2 Grover's Search Algorithm (20 credits)

We consider using Grover's search as a form of database search. We have determined that 6 qubits are sufficient to fully represent the search space and have encoded each item in the database into the computational basis. The search criterion is $f(\mathbf{x}) = 1$, given an input state $|\mathbf{x}\rangle = |x_1x_2x_3x_4x_5x_6\rangle$ which represents the $\left(x_1 2^5 + x_2 2^4 + x_3 2^3 + x_4 2^2 + x_5 2^1 + x_6 2^0\right)$-th item in the database, and

$$f(\mathbf{x}) = x_3 \oplus x_5 \oplus 1.$$

a) Find the input states that satisfy $f(\mathbf{x}) = 1$ and determine the number of viable solutions. 0 1 2

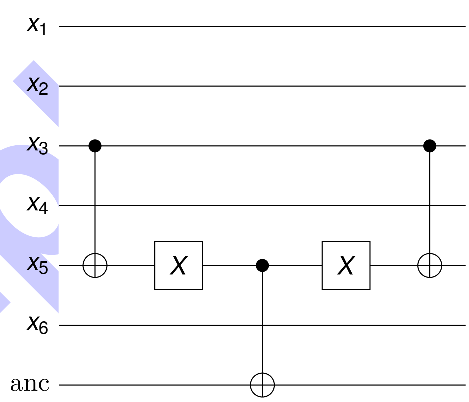
$f(\mathbf{x})$ checks if the 3-rd and 5-th qubit are in the same state. An example would be $|001010\rangle$.

There are $2 \times 2^4 = 32$ possible solutions in this search space.

b)* State the definition of a quantum oracle used in the lecture. Design a suitable oracle O based on this definition for the search criterion, by drawing a quantum circuit implementing the oracle using **only** CNOTs (with a single control qubit), Pauli-X and Pauli-Z gates. 0 1 2 3 4 5 6

A quantum oracle introduces a bit flip in an additional "oracle qubit" if the input state satisfies the search criterion:

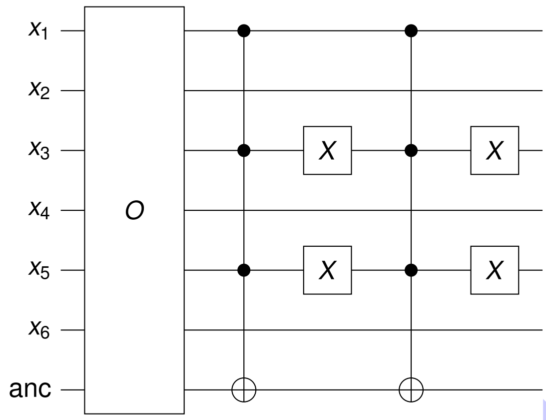
$$O\,|x\rangle\,|\text{anc}\rangle = |x\rangle\,|\text{anc} \oplus f(x)\rangle$$



Alternative solutions may exist

0 1 2 3 4 5

c)* The circuit below shows a modified oracle, where O is the oracle as specified in part (b).



Explain the action of this circuit. Identify the modified search criterion corresponding to the new oracle. What is the size of the new solution space?

The modified oracle no longer flips the ancilla qubit $|\text{anc}\rangle$ whenever $x_1 = 1$.

Thus, the modified search function is now

$$\tilde{f}(\mathbf{x}) = \begin{cases} 0 & x_1 = 1 \\ x_3 \oplus x_5 \oplus 1 & \text{otherwise} \end{cases}$$

The new solution space has dimension $2 \times 2^3 = 16$.

0 1 2 3 4

d)* Determine the rotation angle per application of the Grover operator for a search space dimension $N = 64$ and $M = 2$ solutions.

Specify the optimal number of rotations. (Symbolic expressions are sufficient.) What is the resulting probability of obtaining a solution after applying these Grover rotations?

Recalling from lecture, we let $\sin\left(\frac{\theta}{2}\right) = \sqrt{\frac{M}{N}}$, where M is the number of solutions and N is the size of the search space.

Each application of the Grover's operator performs a rotation by $\theta = 2\arcsin\left(\frac{1}{4\sqrt{2}}\right)$ radians.

Recalling that the initial state is given by

$$|\psi\rangle = \cos\left(\frac{\theta}{2}\right)|\alpha\rangle + \sin\left(\frac{\theta}{2}\right)|\beta\rangle$$

with

$$|\alpha\rangle = \frac{1}{\sqrt{N-M}} \sum_{\substack{x=0 \\ f(x)=0}}^{N-1} |x\rangle$$

$$|\beta\rangle = \frac{1}{\sqrt{M}} \sum_{\substack{x=0 \\ f(x)=1}}^{N-1} |x\rangle$$

We want to solve for a number of rotations k that maximizes the sin term. Thus, the ideal number of applications is:

$$\left(\frac{\pi}{2} - \frac{\theta}{2}\right) \div \theta \approx k,\ k \in \mathbb{Z}^+,$$

where the ideal number of applications is k.

The probability of measuring a viable solution is $\sin^2\left(\left(k + \frac{1}{2}\right)\theta\right)$.

e)* Consider the scenario that we wish for multiple search criteria to be satisfied simultaneously. Given a quantum oracle for each criterion, how could one combine these into a large oracle that ensures that all criteria are satisfied?

Hint: You can use multiple ancilla qubits.

☐ 0
☐ 1
☐ 2
☐ 3

One could use an ancilla for each individual oracle. Then one introduces another "global" ancilla qubit which is flipped precisely if all the individual ancillas are in the $|1\rangle$ state (for example via a multiple-controlled CNOT gate).
(Alternative solution possible.)

Problem 3 Quantum Operations (20 credits)

Consider a single-qubit system A with the following density matrix:

$$\rho_A = \frac{1}{4}\begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix}$$

0 1 2

a)* Is this a pure or a mixed state? Clearly state your reasoning.

$\mathrm{tr}\left[\rho_A^2\right] = \frac{1}{16}(1+9) < 1$, therefore it is a mixed state.

0 1 2 3 4

b)* Recall that there exists a process called purification, by which we can extend the system A into a larger quantum system AR, such that $\rho_A = \mathrm{tr}_R[|\psi\rangle\langle\psi|]$. Find such a state $|\psi\rangle$ on the extended system, using as the environment an additional qubit.

For any pure state $|\psi\rangle$ we can write its Schmidt decomposition

$$|\psi\rangle = \sum_i \lambda_i |i_A\rangle |i_B\rangle ,$$

where λ_i^2 are the eigenvalues of the reduced density matrices on both subsystems. Therefore we need to compute the eigenvalues p_i and eigenvectors $|\phi_i\rangle$ of ρ_A, and set

$$|\psi\rangle = \sum_{i=0}^{1} \sqrt{p_i}\, |\phi_i\rangle |i\rangle .$$

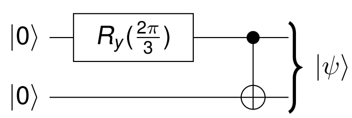
ρ is already a diagonal matrix, and we see that $(p_0, |\phi_0\rangle) = (\frac{1}{4}, |0\rangle)$ and $(p_1, |\phi_1\rangle) = (\frac{3}{4}, |1\rangle)$. Thus

$$|\psi\rangle = \frac{1}{2}|0\rangle|0\rangle + \frac{\sqrt{3}}{2}|1\rangle|1\rangle$$

is a purification for ρ_A.

0 1 2 3 4

c)* Assume that originally both subsystems A and R were in state $|0\rangle$. Draw a circuit which outputs the state $|\psi\rangle$ you found in part (b). Hint: You only need a rotation gate and one two-qubit gate. The relation $\cos(\frac{\pi}{3}) = \frac{1}{2}$ might be helpful. If you didn't solve part (b), you may use: $|\psi\rangle = \frac{1}{\sqrt{5}}|00\rangle + \frac{2}{\sqrt{5}}|11\rangle$.



For the state given in the hint, the rotation angle is: $2\arccos(\frac{1}{\sqrt{5}})$

d)* We consider a principal quantum system in state $|\phi\rangle$ and an environment qubit initialized at $|0\rangle$. Now a unitary operator U acts on the combined system. Provide a formula for computing the resulting density matrix describing the principal system.

0 1 2 3 4

This is precisely the scenario of a quantum operation as discussed in the lecture:

$$\mathcal{E}(\rho) = \mathrm{tr}_{\mathrm{env}}[U(\rho \otimes |0\rangle\langle 0|)U^\dagger],$$

here applied to $|\phi\rangle\langle\phi|$, i.e., the resulting density matrix is $\mathcal{E}(|\phi\rangle\langle\phi|)$.
Alternatively:

$$\mathcal{E}(\rho) = \sum_{k=0}^{1} E_k \rho E_k^\dagger \quad \text{with} \quad (E_k)_{\ell m} = \langle \ell, k|U|m, 0\rangle$$

e) If we ignore the environment qubit, we observe that subsystem A underwent the following transformation:

$$|0\rangle\langle 0| \mapsto \rho_A,$$

with ρ_A specified above. Starting from the circuit you have constructed in part (c), find the operation $\mathcal{E}$ acting on A. Check that it indeed gives ρ_A as output.

0 1 2 3 4 5 6

The unitary from the circuit is

$$U = \frac{1}{2}\begin{pmatrix} 1 & 0 & -\sqrt{3} & 0 \\ 0 & 1 & 0 & -\sqrt{3} \\ 0 & \sqrt{3} & 0 & 1 \\ \sqrt{3} & 0 & 1 & 0 \end{pmatrix}.$$

Tracing out the environment, we obtain the Kraus operators

$$E_0 = \frac{1}{2}\begin{pmatrix} 1 & -\sqrt{3} \\ 0 & 0 \end{pmatrix}$$

and

$$E_1 = \frac{1}{2}\begin{pmatrix} 0 & 0 \\ \sqrt{3} & 1 \end{pmatrix}.$$

Applying the operation to the state $|0\rangle\langle 0|$ yields

$$\begin{aligned} \mathcal{E}(|0\rangle\langle 0|) &= \sum_{k=0}^{1} E_k |0\rangle\langle 0| E_k^\dagger \\ &= \frac{1}{4}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \frac{1}{4}\begin{pmatrix} 0 & 0 \\ 0 & 3 \end{pmatrix} = \rho_A, \end{aligned}$$

as required.

Additional space for solutions–clearly mark the (sub)problem your answers are related to and strike out invalid solutions.