Professorship for Quantum Computing
Department of Informatics
Technical University of Munich

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Introduction to Quantum Computing

Exam:	IN2381 / Final Exam	**Date:**	Tuesday 2nd March, 2021
Examiner:	Prof. Dr. Christian Mendl	**Time:**	14:15 – 15:45

Working instructions

- This exam consists of **12 pages** with a total of **3 problems**.
  Please make sure now that you received a complete copy of the exam.
- The total amount of achievable credits in this exam is 60 credits.
- Detaching pages from the exam is prohibited.
- Allowed resources: open book
- Subproblems marked by * can be solved without results of previous subproblems.
- **Answers are only accepted if the solution approach is documented.** Give a reason for each answer unless explicitly stated otherwise in the respective subproblem.
- Do not write with red or green colors nor use pencils.

Left room from ________	to ________	/	Early submission at ________

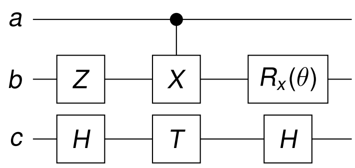
Problem 1 (20 credits)

0 1 2 3 4

a) We are given a unitary gate T defined as

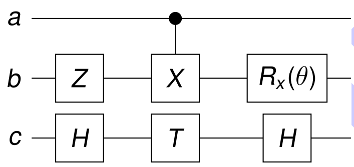
$$T = \sqrt{\frac{50}{29}} \begin{bmatrix} 0.3i & 0.7i \\ -0.7i & 0.3i \end{bmatrix}$$

Draw the circuit that performs the inverse operation of the following circuit (with $\theta \in \mathbb{R}$):

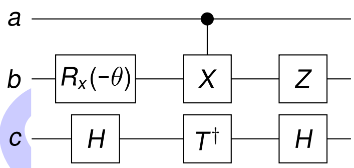


Explicitly define new gates, if any, used in the drawn circuit.

The "inverse" circuit is obtained by reversing the order of the gates and taking the adjoint (conjugate transpose) of each gate. For the exam variant with the following circuit,



the inverse circuit is thus given by:



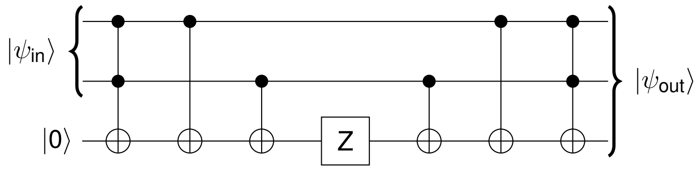
We still need to explicitly define $T^\dagger$, noting that T is not Hermitian. For example:

$$T = \sqrt{\frac{50}{29}} \begin{bmatrix} 0.3i & 0.7i \\ -0.7i & 0.3i \end{bmatrix} \rightsquigarrow$$

$$T^\dagger = \sqrt{\frac{50}{29}} \begin{bmatrix} -0.3i & 0.7i \\ -0.7i & -0.3i \end{bmatrix}$$

(Analogous for other exam variants.)

b)* Compute the output $|\psi_{\text{out}}\rangle$ of the following quantum circuit for computational basis states as input, i.e., $|\psi_{\text{in}}\rangle \in \{|00\rangle, |01\rangle, |10\rangle, |11\rangle\}$:



Based on your calculation, provide the output for the input state

$$|\psi_{\text{in}}\rangle = a\,|{+}{-}\rangle + b\,|{-}{+}\rangle$$

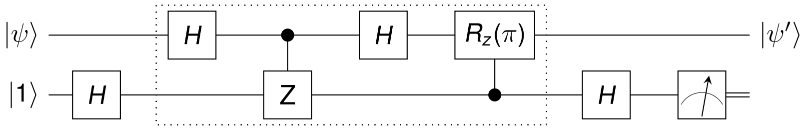
The circuit leaves $|00\rangle$ invariant, and introduces a phase factor (-1) for the other computational basis states $|01\rangle$, $|10\rangle$, $|11\rangle$.
For the exam variant with

$$\begin{aligned}
|\psi_{\text{in}}\rangle &= a\,|{+}{-}\rangle + b\,|{-}{+}\rangle \\
&= \frac{a+b}{2}\left(|00\rangle - |11\rangle\right) + \frac{a-b}{2}\left(-|01\rangle + |10\rangle\right) \quad \rightsquigarrow \\
|\psi_{\text{out}}\rangle &= \frac{a+b}{2}\left(|00\rangle + |11\rangle\right)|0\rangle + \frac{a-b}{2}\left(|01\rangle - |10\rangle\right)|0\rangle,
\end{aligned}$$

and for

$$\begin{aligned}
|\psi_{\text{in}}\rangle &= a\,|{+}{+}\rangle + b\,|{-}{-}\rangle \\
&= \frac{a+b}{2}\left(|00\rangle + |11\rangle\right) + \frac{a-b}{2}\left(|01\rangle + |10\rangle\right) \quad \rightsquigarrow \\
|\psi_{\text{out}}\rangle &= \frac{a+b}{2}\left(|00\rangle - |11\rangle\right)|0\rangle - \frac{a-b}{2}\left(|01\rangle + |10\rangle\right)|0\rangle
\end{aligned}$$

c)* The following quantum circuit takes as input a single qubit quantum state $|\psi\rangle$ and an ancilla qubit set to $|1\rangle$. R_z is the rotation gate along the Z-axis of the Bloch sphere. Simplify the gates inside the dotted box. (Hint: you should arrive at a single controlled gate.)



For this solution, the top wire (initial state $|\psi\rangle$) is wire 1 and the bottom wire (initial state $|1\rangle$) is wire 2. Notation for single qubit control gates follows U_{ct}, where c is the control wire and t is the target wire.

First note that $CZ_{12} = CZ_{21}$, since for a controlled-Z gate the role of control and target can be interchanged.
Thus we recognize that $(H \otimes I) \cdot CZ_{12} \cdot (H \otimes I)$ is equivalent to CX_{21} (controlled-X with $|\psi\rangle$ as target):

$$(H \otimes I) \cdot CZ_{12} \cdot (H \otimes I) = (H \otimes I) \cdot CZ_{21} \cdot (H \otimes I) = CX_{21}.$$

Here we have used that $HZH = X$ and $H^2 = I$.
We then determine that

$$\begin{aligned} R_z(\pi)X &= \begin{pmatrix} e^{-i\frac{\pi}{2}} & 0 \\ 0 & e^{i\frac{\pi}{2}} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \\ &= \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = Y. \end{aligned}$$

Since both X and $R_z(\pi)$ are controlled by the second qubit, we can substitute their product by a controlled-Y gate.
In summary, the gates in the dotted box are equivalent to performing a CY_{21} gate.

d) Calculate the output $|\psi'\rangle$ of the circuit in (c) in terms of $|\psi\rangle$ and Pauli or identity matrices, depending on whether the measurement outcome is 0 or 1.

0 1 2 3 4 5

(Note that the overall circuit is inspired by Tutorial 3.)

The quantum state directly before the measurement is:

$$\begin{aligned}
|\chi\rangle &= (I \otimes H) \cdot CY_{21} \cdot (|\psi\rangle \otimes H|1\rangle) \\
&= (I \otimes H) \cdot CY_{21} \cdot \frac{1}{\sqrt{2}} \big(|\psi\rangle \otimes |0\rangle - |\psi\rangle \otimes |1\rangle\big) \\
&= \frac{1}{\sqrt{2}} \big(|\psi\rangle \otimes H|0\rangle - Y|\psi\rangle \otimes H|1\rangle\big) \\
&= \frac{1}{2}|\psi\rangle \otimes \big(|0\rangle + |1\rangle\big) - \frac{1}{2}Y|\psi\rangle \otimes \big(|0\rangle - |1\rangle\big) \\
&= \frac{I - Y}{2}|\psi\rangle \otimes |0\rangle + \frac{I + Y}{2}|\psi\rangle \otimes |1\rangle .
\end{aligned}$$

The two possible final states $|\psi'\rangle$ are thus:

$$|\psi'\rangle = \begin{cases} \frac{I-Y}{2}|\psi\rangle & \text{for measurement result 0} \\ \frac{I+Y}{2}|\psi\rangle & \text{for measurement result 1} \end{cases}$$

Problem 2 (20 credits)

0 1 2

a) Quantum states are manipulated by quantum gates, which are described by complex, unitary matrices U, such that

$$|\psi'\rangle = U|\psi\rangle .$$

Explain why U must be unitary.

Two possible solutions:

- It has to be unitary to preserve normalization.
- Physical gates are discrete steps of the time evolution operator, which is unitary.

0 1 2

b)* We are running an experiment on a system described by a density matrix ρ. Determine the coefficient c such that

$$\rho = c\,|{++}\rangle\langle{++}| + \frac{3}{4}\,|{--}\rangle\langle{--}|$$

is a valid density matrix.

The states $\{|+\rangle, |-\rangle\}$ result from an orthonormal base change starting from the standard basis $\{|0\rangle, |1\rangle\}$; this change of basis leaves the matrix trace invariant. Thus the question is equivalent to whether

$$\tilde{\rho} = c\,|00\rangle\langle 00| + d\,|11\rangle\langle 11|$$

is a valid density matrix, where d is the coefficient in front of $|{--}\rangle\langle{--}|$ (randomized for different variants of the exam).
The trace must be equal to one. Since

$$\mathrm{tr}[\rho] = \mathrm{tr}[\tilde{\rho}] = c + d,$$

it must hold that $c = 1 - d$.

0 1 2

c)* Assume that we apply phase gates to each qubit of the system from part (b). Write down this operator and its action on ρ. (An explicit computation is not required here.)

There are two qubits in this system, so

$$U = S \otimes S.$$

The output is given by

$$\rho' = U\rho U^{\dagger}.$$

d)* In real-life experiments, however, it is difficult to isolate the system from the environment, so the operation applied to ρ may not be unitary. This seems to contradict the statement in section (a). How can one reconcile both points of view? Write down a mathematical expression for such an operation.

There is a unitary operation acting on both ρ and the environment. The general operation on ρ results from the unobservable environment, mathematically from "tracing out" the environment. The sought mathematical expression is the definition of quantum operations:

$$\mathcal{E}(\rho) = \sum_k E_k \rho E_k^\dagger$$

with complex matrices (Kraus operators) E_k obeying $\sum_k E_k^\dagger E_k \leq I$.

e)* Consider a dataset containing all integers from 0 to $N-1$, where N is a power of 2. We are given a deterministic function f that processes the dataset. Alice tells us that it will output 0 for half of the entries of our dataset and 1 for the other half. Bob claims that it always outputs 1. We know one of them is correct.

In the worst case scenario, how many classical evaluations of f will we need to determine who is correct? Which quantum algorithm could we use to solve our problem? How many evaluations will we need in that case?

We need $N/2+1$ classical evaluations to determine with certainty whether f is constant or balanced. With a quantum computer we could use the Deutsch-Jozsa algorithm; in this case we only need one evaluation of f.

f)* The quantum algorithm from (e) takes advantage of the quantum superposition principle to apply f simultaneously to all entries of the dataset. We want to preprocess qubits which are all currently in the state $|-\rangle$ to obtain the equal superposition state. Write down how many qubits we need in terms of N, and a preprocessing operation to arrive at the equal superposition state.

We need $n = \log_2(N)$ qubits. As preprocessing step, we apply $Z^{\otimes n} = Z \otimes \cdots \otimes Z$ to the qubits, since

$$Z^{\otimes n} |-\rangle^{\otimes n} = (Z\,|-\rangle)^{\otimes n} = |+\rangle^{\otimes n} = \frac{1}{\sqrt{2^n}} \sum_{x\in\{0,1\}^n} |x\rangle\,.$$

0 1 2 3 4 5 6

g) Consider the Hermitian gate U_f defined on computational basis states as

$$U_f |x\rangle |z\rangle = |x\rangle |z \oplus f(x)\rangle .$$

We want to apply U_f to the equal superposition state, denoted $|\psi\rangle$ here, and an ancilla qubit which is in state $|-\rangle$. However, before we apply U_f, a noise process affects the ancilla qubit, namely a phase flip with probability p, defined by the two matrices (Kraus operators)

$$E_1 = \sqrt{1-p}I \quad \text{and} \quad E_2 = \sqrt{p}Z.$$

Compute the output density matrix ρ' of the ancilla qubit after undergoing this phase flip operation. Then, provide an expression for U_f applied to $|\psi\rangle \langle\psi| \otimes \rho'$. Finally, assess how the outcome of the algorithm from (e) is affected by the noise.

Using the quantum operation formula from the solution of (d):

$$\begin{aligned} \rho' &= E_1 |-\rangle \langle-| E_1 + E_2 |-\rangle \langle-| E_2 \\ &= (1-p) |-\rangle \langle-| + p |+\rangle \langle+| . \end{aligned}$$

Therefore, together with $U_f^\dagger = U_f$,

$$\begin{aligned} U_f(|\psi\rangle \langle\psi| \otimes \rho')U_f^\dagger &= U_f((1-p) |\psi-\rangle \langle\psi-| + p |\psi+\rangle \langle\psi+|)U_f \\ &= (1-p)U_f |\psi-\rangle \langle\psi-| U_f + pU_f |\psi+\rangle \langle\psi+| U_f. \end{aligned}$$

With probability $1-p$ the output after applying U_f will be the pure state

$$U_f(|\psi-\rangle) = \sum_{x\in\{0,1\}^n} \frac{(-1)^{f(x)} |x\rangle}{\sqrt{N}} |-\rangle$$

which corresponds to the Deutsch-Jozsa algorithm without noise. With probability p the output after U_f will be

$$U_f |\psi+\rangle = |\psi+\rangle .$$

Nothing happens here because $|0 \oplus f(x)\rangle + |1 \oplus f(x)\rangle = |0\rangle + |1\rangle$ no matter what $f(x)$ is.

The Deutsch-Jozsa algorithm then applies Hadamard gates to the first n qubits and measures them. Because of this second term resulting from noise, we will measure all 0s with probability p. So there is a chance of finding that the function is constant even though it is balanced.

Problem 3 (20 credits)

We consider a quantum system of n qubits, and use the notation X_j, Y_j, Z_j to denote that one of the Pauli matrices acts on the jth qubit; e.g., $X_1 Z_3 \equiv X \otimes I \otimes Z$ for $n = 3$.
Conjugation by U refers to the transformation $UgU^\dagger$ of a quantum gate g by a unitary operation U. The following table summarizes several conjugation transformations:

U	Z	Z	Z	H	H	H	S	S	S
g	X	Y	Z	X	Y	Z	X	Y	Z
$UgU^\dagger$	$-X$	$-Y$	Z	Z	$-Y$	X	Y	$-X$	Z

Here $S = \left(\begin{smallmatrix} 1 & 0 \\ 0 & i \end{smallmatrix}\right)$ is the phase gate.

a) We encode a logical qubit by two physical qubits as

$$|0_L\rangle = |01\rangle\,, \quad |1_L\rangle = |10\rangle\,.$$

The physical qubits are affected by bit flip errors $|0\rangle \leftrightarrow |1\rangle$. Describe and briefly explain a measurement for *error detection*, i.e., diagnosing whether a single bit flip has occurred. Is it also possible to recover from such an error?

A single bit flip error on the first qubit sends

$$|01\rangle \mapsto |11\rangle\,, \quad |10\rangle \mapsto |00\rangle\,,$$

and a single bit flip error on the second qubit

$$|01\rangle \mapsto |00\rangle\,, \quad |10\rangle \mapsto |11\rangle\,.$$

In all cases the resulting basis states have an *even* number of $|1\rangle$ states.

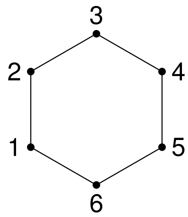
For error detection, we can thus use the following projection operators:

$$\begin{aligned} P_0 &= |01\rangle\langle 01| + |10\rangle\langle 10| && \text{no error occurred,} \\ P_1 &= |00\rangle\langle 00| + |11\rangle\langle 11| && \text{bit flip occurred} \end{aligned}$$

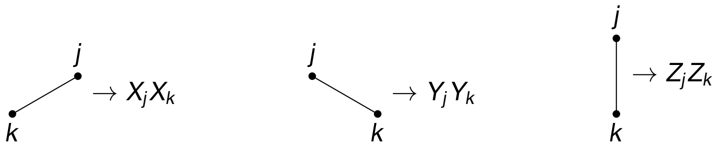
An alternative representation is a measurement of the observable $Z_1 Z_2$: since $Z_1 Z_2 |ab\rangle = (-1)^{a+b} |ab\rangle$ for all $a, b \in \{0, 1\}$, the eigenvalue 1 will indicate that no error has occurred ($a = b$), whereas the eigenvalue -1 indicates that a bit flip happened.

Recovery is not possible: for example the state $|00\rangle$ can originate both from $|0_L\rangle$ and $|1_L\rangle$ with equal probability.

b)* Six qubits are assigned to the vertices of a hexagon:



We define $W = X_1 Y_2 Z_3 X_4 Y_5 Z_6$, and the following operators depending on the *orientation* of an edge:



It turns out that for each of the six edges of the hexagon, the corresponding edge operator commutes with W. Prove this statement for edge $3-4$ and $4-5$.

The following is the reference solution for edges $1-2$ and $2-3$; the argumentation for the other edges works analogously.

We have to show that, for edge $1-2$,

$$[(Z_1 Z_2), W] = 0 \quad \text{i.e.,} \quad (Z_1 Z_2) W = W (Z_1 Z_2)$$

and analogously for edge $2-3$:

$$[(X_2 X_3), W] = 0 \quad \text{i.e.,} \quad (X_2 X_3) W = W (X_2 X_3).$$

Now we use that any two differing Pauli matrices *anti-commute*, e.g., $XZ = -ZX$ (see Exercise 11.2). Thus, for edge $1-2$:

$$\begin{aligned}(Z_1 Z_2) W &= (ZX) \otimes (ZY) \otimes Z \otimes X \otimes Y \otimes Z = (-XZ) \otimes (-YZ) \otimes Z \otimes X \otimes Y \otimes Z \\ &= (XZ) \otimes (YZ) \otimes Z \otimes X \otimes Y \otimes Z = W (Z_1 Z_2),\end{aligned}$$

and for edge $2-3$:

$$\begin{aligned}(X_2 X_3) W &= X \otimes (XY) \otimes (XZ) \otimes X \otimes Y \otimes Z = X \otimes (-YX) \otimes (-ZX) \otimes X \otimes Y \otimes Z \\ &= X \otimes (YX) \otimes (ZX) \otimes X \otimes Y \otimes Z = W (X_2 X_3).\end{aligned}$$

c)* The subgroup $T = \langle X_1 Y_2 Z_3, X_1 X_2 Y_3, -Z_1 Y_2 Y_3 \rangle$ of the Pauli group G_3 stabilizes the one-dimensional subspace $V_T = \text{span}\{|\psi\rangle\}$ with

$$|\psi\rangle = \frac{1}{2\sqrt{2}} \big(|000\rangle - |001\rangle + |010\rangle + |011\rangle - i\,|100\rangle + i\,|101\rangle + i\,|110\rangle + i\,|111\rangle \big).$$

(A proof of this statement is not required here.) Find a subgroup T' of the Pauli group G_3 which stabilizes $V_{T'} = \text{span}\{(S \otimes H \otimes Z)\,|\psi\rangle\}$.

We observe that $g\,|\psi\rangle = |\psi\rangle$ is equivalent to $UgU^\dagger U\,|\psi\rangle = U\,|\psi\rangle$ for any unitary matrix U, setting $U = S \otimes H \otimes Z$ here. Thus we only need to conjugate the generators of T by U to obtain T' (which will likewise consist of conjugated group elements); together with the above table, this leads to:

$$\begin{aligned}
T' &= \left\langle U(X \otimes Y \otimes Z)U^\dagger, U(X \otimes X \otimes Y)U^\dagger, U(-Z \otimes Y \otimes Y)U^\dagger \right\rangle \\
&= \left\langle (SXS^\dagger) \otimes (HYH^\dagger) \otimes (ZZZ^\dagger), (SXS^\dagger) \otimes (HXH^\dagger) \otimes (ZYZ^\dagger), -(SZS^\dagger) \otimes (HYH^\dagger) \otimes (ZYZ^\dagger) \right\rangle \\
&= \langle -(Y \otimes Y \otimes Z), -(Y \otimes Z \otimes Y), -(Z \otimes Y \otimes Y) \rangle.
\end{aligned}$$

d)* Specify an element g of the Pauli group G_4 such that

$$R = \langle Y_2 Z_3, Y_1 Z_2 X_3 Z_4, g \rangle$$

stabilizes a non-trivial vector space and the three generators of R are independent. Also state the properties which g must satisfy (a proof of them is not required).

0 1 2 3 4 5

We need to choose $g \in G_4$, $g \neq I$, such that it

- commutes with the other two generators, and
- cannot be written as product of them.

For example $g = Y_1 Z_4$, $g = Z_1 Y_4$ or $g = X_1 X_4$ has this properties, since it clearly commutes with $Y_2 Z_3$ (acting on different qubits) and also with $Y_1 Z_2 X_3 Z_4$. The product of the other two generators acts non-trivially on all four qubits, and is thus different from g. (Other choices of g possible as well.)

(Solution analogous for exam variants with permutations of the Pauli matrices.)

Additional space for solutions–clearly mark the (sub)problem your answers are related to and strike out invalid solutions.