Algorithms of Scientific Computing II – Quantum Computing (IN2002, W19/20) **Exam solution**

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20 points per problem, so 60 points in total.

Problem 1

(a) The eigenvalues of $R_z(\theta)$ are $\mathrm{e}^{\pm i\theta/2}$ since

$$R_z(\theta) = \mathrm{e}^{-i\theta Z/2} = \begin{pmatrix} \mathrm{e}^{-i\theta/2} & 0 \\ 0 & \mathrm{e}^{i\theta/2} \end{pmatrix}.$$

Thus, according to the hint, these are the eigenvalues of $R_x(\theta)$, too.

In general, the matrix representation of a controlled-U gate reads

$$\begin{pmatrix} 1 & & \\ & 1 & \\ & & U \end{pmatrix}$$

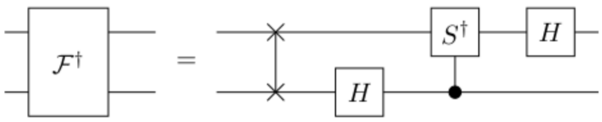
where the empty fields are all 0 and U occupies the lower right 2×2 block. The eigenvalues of the controlled-$R_x(\theta)$ gate are thus $\{1, \mathrm{e}^{i\theta/2}, \mathrm{e}^{-i\theta/2}\}$.

[2 points for eigenvalues of $R_x(\theta)$ and 2 points for eigenvalues of controlled-$R_x(\theta)$]

(b) $\mathcal{F}$ is a unitary quantum operation, i.e., $\mathcal{F}^{-1} = \mathcal{F}^\dagger$. Following the hint, we obtain $\mathcal{F}^\dagger$ by reversing the order of the gates and taking the adjoint (conjugate transpose) of each gate. The Hadamard and swap gates are their own adjoints, while

$$S^\dagger = \begin{pmatrix} 1 & 0 \\ 0 & -i \end{pmatrix};$$

the adjoint of the controlled-S gate is likewise the controlled-$S^\dagger$ gate (clear from matrix representation). In summary, one arrives at

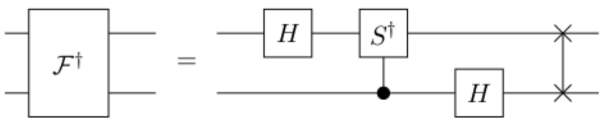


[6 points]

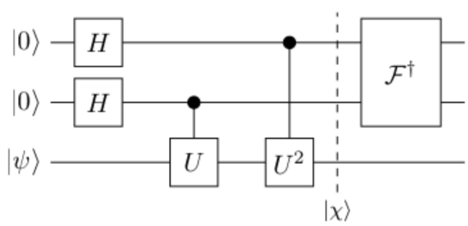
Alternative solution: The inverse Fourier transform differs from the forward Fourier transform only by negative signs in the exponential terms, i.e., $\mathrm{e}^{-2\pi ijk/N}$ instead of $\mathrm{e}^{2\pi ijk/N}$. Following the construction of the quantum Fourier circuit, this amounts to replacing

$$R_k = \begin{pmatrix} 1 & 0 \\ 0 & \mathrm{e}^{2\pi i/2^k} \end{pmatrix}$$

by $R_k^\dagger$. For two qubits, the only rotation gate in the Fourier circuit is $S = R_2$. Thus the following circuit realizes the inverse Fourier transform:



(c) Let U be the controlled-$R_x(\theta)$ gate. Then one identifies the circuit by the *phase estimation* circuit for $t = 2$:



Namely, U^2 is equal to the controlled-$R_x(2\theta)$ gate since $R_x(\theta)^2 = \mathrm{e}^{-i\theta X} = R_x(2\theta)$.

(i) $|\psi\rangle$ is an eigenvector of U with eigenvalue 1 (in other words, U acts as identity in this case), since the control qubit in $|\psi\rangle$ is set to $|0\rangle$. In terms of the phase estimation algorithm, the phase $\varphi = 0$. The intermediate state is thus

$$|\chi\rangle = \frac{|0\rangle + |1\rangle}{\sqrt{2}} \frac{|0\rangle + |1\rangle}{\sqrt{2}} |\psi\rangle = \frac{1}{2}\big(|00\rangle + |01\rangle + |10\rangle + |11\rangle\big)|\psi\rangle.$$

[2 points]

(ii) Since $\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$ is an eigenvector of $R_x(\theta)$ with eigenvalue $\mathrm{e}^{-i\theta/2}$, $|\psi\rangle$ is an eigenvector of U with the same eigenvalue (control qubit is enabled in $|\psi\rangle$). Accordingly, the intermediate state equals

$$\begin{aligned}|\chi\rangle = \frac{|0\rangle + \mathrm{e}^{-i\theta}|1\rangle}{\sqrt{2}} \frac{|0\rangle + \mathrm{e}^{-i\theta/2}|1\rangle}{\sqrt{2}} |\psi\rangle &= \frac{1}{2}\left(|00\rangle + \mathrm{e}^{-i\theta/2}|01\rangle + \mathrm{e}^{-i\theta}|10\rangle + \mathrm{e}^{-i3\theta/2}|11\rangle\right)|\psi\rangle \\ &= \left(\frac{1}{2}\sum_{k=0}^{3} \mathrm{e}^{-ik\theta/2}|k\rangle\right)|\psi\rangle.\end{aligned}$$

[4 points]

(d) (i) Applying the inverse Fourier transform to the first two qubits leads to the output

$$(\mathcal{F}^\dagger \otimes I)|\chi\rangle = |00\rangle|\psi\rangle,$$

in agreement with $\varphi = 0$ (i.e., the first two output qubits are the digital representation of the phase).
[2 points]

(ii) For $\theta = \pi$, we note that $\mathrm{e}^{-i\theta/2} = -i = \mathrm{e}^{2\pi i\varphi}$ with $\varphi = \frac{3}{4}$, which has the exact binary representation $\varphi = 0.\varphi_1\varphi_0 = 0.11$. According to the phase estimation algorithm, the output of the first two qubits is φ encoded as quantum state: $|\varphi_1\varphi_0\rangle = |11\rangle$, and thus the overall output is $|11\rangle|\psi\rangle$.
Alternatively, first inserting $\theta = \pi$ into $|\chi\rangle$ yields $|\chi\rangle = \frac{1}{2}(|00\rangle - i|01\rangle - |10\rangle + i|11\rangle)|\psi\rangle$. Applying the inverse Fourier transform to the first two qubits leads to the output

$$(\mathcal{F}^\dagger \otimes I)|\chi\rangle = |11\rangle|\psi\rangle.$$

[2 points]

Problem 2

(a) Since $\lambda \in (0,1)$, $\sqrt{1-\lambda}$ and $\sqrt{\lambda}$ are both real numbers. Inserting the identity matrix leads directly to

$$\mathcal{E}(I) = E_0 E_0^\dagger + E_1 E_1^\dagger = \begin{pmatrix} 1 & 0 \\ 0 & 1-\lambda \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & \lambda \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I.$$

[2 points]

(b) We first note that

$$\begin{aligned} E_0 X E_0^\dagger &= \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{1-\lambda} \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & \sqrt{1-\lambda} \end{pmatrix} = \sqrt{1-\lambda}\, X, \\ E_1 X E_1^\dagger &= \begin{pmatrix} 0 & 0 \\ 0 & \sqrt{\lambda} \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & \sqrt{\lambda} \end{pmatrix} = 0 \quad \text{(zero matrix)}, \end{aligned}$$

and similarly $E_0 Y E_0^\dagger = \sqrt{1-\lambda}\, Y$ and $E_1 Y E_1^\dagger = 0$. Analogous to (a), one computes that $\mathcal{E}(Z) = Z$. Now inserting the Bloch representation of ρ with $\vec{r} = (r_1, r_2, r_3)$ leads to

$$\mathcal{E}(\rho) = \frac{1}{2}\left(\mathcal{E}(I) + \sum_{\alpha=1}^{3} r_\alpha\, \mathcal{E}(\sigma_\alpha)\right) = \frac{1}{2}\left(I + \sqrt{1-\lambda}\, r_1\, X + \sqrt{1-\lambda}\, r_2\, Y + r_3\, Z\right) = \frac{I + \vec{r}' \cdot \vec{\sigma}}{2}$$

with $\vec{r}' = (\sqrt{1-\lambda}\, r_1, \sqrt{1-\lambda}\, r_2, r_3)$.
[5 points]

(c) The Bloch vector $\vec{r}$ of any density matrix ρ satisfies $\|\vec{r}\| \le 1$, with equality if and only if ρ describes a pure state (see exercise 7.2). According to (b), the quantum channel $\mathcal{E}$ scales the first and second entry of the input Bloch vector $\vec{r}$ by the factor $\sqrt{1-\lambda} < 1$. Thus the Bloch vector $\vec{r}'$ of $\mathcal{E}(\rho)$ has unit length, $\|\vec{r}'\| = 1$, precisely if $\vec{r}' = \vec{r} = (0, 0, \pm 1)$ (north or south pole of the Bloch sphere). The corresponding density matrices for these cases are $\rho = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)$ and $\rho = \left(\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right)$, describing a quantum system in state $|0\rangle$ and $|1\rangle$, respectively.

[4 points]

(d) According to (b), the Bloch vector after n repeated applications of the channel is equal to

$$\left(\sqrt{1-\lambda}^{\,n} r_1, \sqrt{1-\lambda}^{\,n} r_2, r_3\right) \overset{n\to\infty}{\to} (0, 0, r_3)$$

since $\sqrt{1-\lambda} < 1$ by assumption. The matrix representation of the limiting density matrix is thus

$$\rho^{(\infty)} = \frac{1}{2}\begin{pmatrix} 1 + r_3 & 0 \\ 0 & 1 - r_3 \end{pmatrix},$$

that is, the phase damping channel "damps" the off-diagonal entries to zero.

[4 points]

Note: physically, $\rho^{(\infty)}$ represents a (classical) ensemble of the basis states $|0\rangle$ and $|1\rangle$, without any quantum superposition of these states.

(e) Recall that $R_y(\theta)$ is the rotation operator

$$R_y(\theta) = \mathrm{e}^{-i\theta Y/2} = \cos(\theta/2) I - i \sin(\theta/2) Y = \begin{pmatrix} \cos\frac{\theta}{2} & -\sin\frac{\theta}{2} \\ \sin\frac{\theta}{2} & \cos\frac{\theta}{2} \end{pmatrix}.$$

Thus the controlled-$R_y(\theta)$ gate has the following matrix representation with respect to the standard computational basis $(|00\rangle, |01\rangle, |10\rangle, |11\rangle)$:

$$U = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cos\frac{\theta}{2} & -\sin\frac{\theta}{2} \\ 0 & 0 & \sin\frac{\theta}{2} & \cos\frac{\theta}{2} \end{pmatrix}.$$

According to the hint, E_0 and E_1 are submatrices of this matrix:

$$E_0 = \begin{pmatrix} \langle 00|U|00\rangle & \langle 00|U|10\rangle \\ \langle 10|U|00\rangle & \langle 10|U|10\rangle \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & \cos\frac{\theta}{2} \end{pmatrix} \overset{!}{=} \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{1-\lambda} \end{pmatrix},$$

$$E_1 = \begin{pmatrix} \langle 01|U|00\rangle & \langle 01|U|10\rangle \\ \langle 11|U|00\rangle & \langle 11|U|10\rangle \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & \sin\frac{\theta}{2} \end{pmatrix} \overset{!}{=} \begin{pmatrix} 0 & 0 \\ 0 & \sqrt{\lambda} \end{pmatrix}.$$

Therefore θ is the (unique) angle satisfying $\cos(\theta/2) = \sqrt{1-\lambda}$ and $\sin(\theta/2) = \sqrt{\lambda}$; such a solution always exists since $\sqrt{1-\lambda}^2 + \sqrt{\lambda}^2 = 1$, and lies in the interval $[0, \pi]$ since $\sqrt{1-\lambda} > 0$ and $\sqrt{\lambda} > 0$.

[4 points for computing E_0 and E_1, 1 point for relating θ to λ]

Problem 3

(a) By explicit calculation,

$$SXS^\dagger = \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -i \end{pmatrix} = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = Y,$$

$$SZS^\dagger = \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -i \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = Z.$$

We now use matrix representations with respect to the computational basis states $|00\rangle, |01\rangle, |10\rangle, |11\rangle$:

$$\text{CNOT} \cdot Z_2 \cdot \text{CNOT}^\dagger = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = Z_1 Z_2.$$

[4 points]

(b) Following the hint, one computes

$$\mathsf{CNOT} \cdot Y_2 \cdot \mathsf{CNOT}^\dagger = i(\mathsf{CNOT} \cdot X_2 \cdot \mathsf{CNOT}^\dagger) \cdot (\mathsf{CNOT} \cdot Z_2 \cdot \mathsf{CNOT}^\dagger) = iX_2(Z_1 Z_2) = Z_1(iX_2 Z_2) = Z_1 Y_2.$$

For the second equal sign we have used the conjugation table.

[4 points]

Alternatively, one can also work with 4×4 matrix representations of the involved operators.

(c) V_R is precisely the eigenspace of $X_1 Z_2$ corresponding to eigenvalue 1. We use that $|\pm\rangle = \frac{1}{\sqrt{2}}(|0\rangle \pm |1\rangle)$ are the two eigenvectors of X with eigenvalues ± 1, respectively, and that $|0\rangle, |1\rangle$ are eigenstates of Z. Thus $\{|+\rangle|0\rangle, |+\rangle|1\rangle, |-\rangle|0\rangle, |-\rangle|1\rangle\}$ forms a basis of eigenvectors of $X_1 Z_2$, with corresponding eigenvalues $\{1, -1, -1, (-1)^2\}$. One concludes that $V_R = \mathrm{span}\{|+\rangle|0\rangle, |-\rangle|1\rangle\}$.

[4 points]

(d) We observe that $g|\psi\rangle = |\psi\rangle$ is equivalent to $UgU^\dagger U|\psi\rangle = U|\psi\rangle$ for any unitary matrix U, setting $U = S \otimes S \otimes S$ here. Thus we only need to conjugate the generators of T by U to obtain T' (which will likewise consist of conjugated group elements):

$$T' = \left\langle U(X_1 Y_2)U^\dagger, U(Y_2 Z_3)U^\dagger \right\rangle = \left\langle (SX_1 S^\dagger)(SY_2 S^\dagger), (SY_2 S^\dagger)(SZ_3 S^\dagger) \right\rangle = \langle -Y_1 X_2, -X_2 Z_3 \rangle.$$

[4 points]

(e) Since U is diagonal and unitary, we can represent U (up to an irrelevant global phase factor) as

$$U = \begin{pmatrix} 1 & 0 \\ 0 & \mathrm{e}^{i\varphi} \end{pmatrix}$$

for some angle $\varphi \in \mathbb{R}$. One computes

$$UXU^\dagger = \begin{pmatrix} 1 & 0 \\ 0 & \mathrm{e}^{i\varphi} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & \mathrm{e}^{-i\varphi} \end{pmatrix} = \begin{pmatrix} 0 & \mathrm{e}^{-i\varphi} \\ \mathrm{e}^{i\varphi} & 0 \end{pmatrix} \stackrel{!}{\in} \{\pm X, \pm Y\}.$$

Thus necessarily $\mathrm{e}^{i\varphi} \in \{\pm 1, \pm i\}$. If $\mathrm{e}^{i\varphi} = 1$ then $U = I = S^0$, if $\mathrm{e}^{i\varphi} = i$ then $U = S$, if $\mathrm{e}^{i\varphi} = -1$ then $U = S^2$ and similarly if $\mathrm{e}^{i\varphi} = -i$ then $U = S^3$.

[4 points]