Chair of Scientific Computing in Computer Science
Department of Informatics
Technical University of Munich

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Introduction to Quantum Computing

Exam:	IN2381 / Final Exam	**Date:**	Thursday 2^{nd} March, 2023
Examiner:	Prof. Dr. Christian Mendl	**Time:**	17:30 – 19:00

	P 1	P 2	P 3
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Working instructions

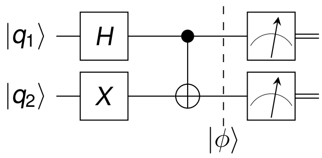
- This exam consists of **12 pages** with a total of **3 problems**.
  Please make sure now that you received a complete copy of the exam.
- The total amount of achievable credits in this exam is 60 credits.
- Detaching pages from the exam is prohibited.
- Allowed resources:
  - ein **A4 Blatt** (beidseitig) mit eigenen Notizen
  - ein **analoges Wörterbuch** Deutsch ↔ Muttersprache **ohne Anmerkungen**
- Subproblems marked by * can be solved without results of previous subproblems.
- **Answers are only accepted if the solution approach is documented.** Give a reason for each answer unless explicitly stated otherwise in the respective subproblem.
- Do not write with red or green colors nor use pencils.
- Physically turn off all electronic devices, put them into your bag and close the bag.

Left room from ________ to ________ / Early submission at ________

Problem 1 (20 credits)

0 1 2 3

a) Consider the following circuit, where the initial state of q_1 and q_2 is $|0\rangle$ in all cases:



Explicitly determine the state $|\phi\rangle$ before the measurements. List the possible measurement outcomes and their respective probabilities.

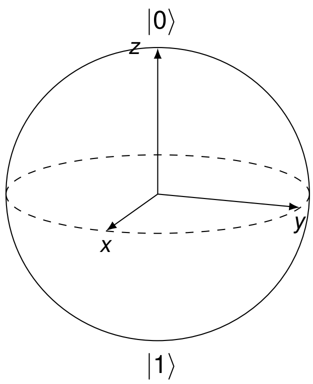
$$|\phi\rangle = \frac{1}{\sqrt{2}}\left(|01\rangle + |10\rangle\right)$$

The possible measurement outcomes are thus 01 and 10, with probabilities

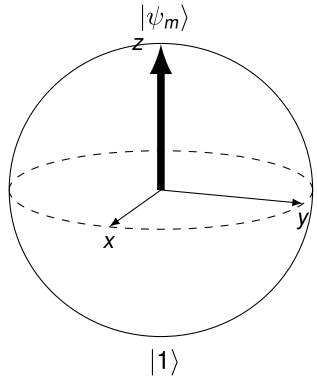
$$p(01) = 0.5,$$
$$p(10) = 0.5.$$

0 1 2 3

b) Assume that the measurements occur sequentially and the first qubit q_1 is always measured first. In the case that the first qubit collapses to $|1\rangle$ due to the measurement, what are the possible outcomes when measuring the second qubit q_2? Draw the state of q_2 after measurement in the Bloch sphere below.



The two-qubit state $|\phi\rangle$ collapses to $|10\rangle$, i.e., the second qubit is in state $|0\rangle$ with probability 1. We denote the state of q_2 after measurement as $|\psi_m\rangle$.



c)* Assume that an available quantum computer only supports R_x, R_y, R_z and S single-qubit gates. Decompose the Hadamard gate into appropriate rotation-gates up to a global phase constant. Also verify that your decomposition is correct. Hints: The decomposition can be achieved using only R_x and R_y gates, or R_y and R_z gates. You may use $\cos(\pi/4) = 1/\sqrt{2}$ and $\sin(\pi/4) = 1/\sqrt{2}$.

0 1 2 3 4 5

H can be decomposed into a $\frac{\pi}{2}$ rotation around y-axis followed by a π rotation around the x-axis:

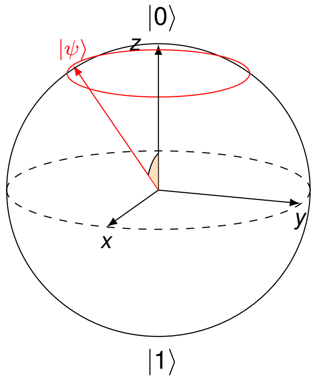
$$\begin{aligned} R_x(\pi)R_y(\pi/2) &= \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} \\ &= \begin{pmatrix} \frac{-i}{\sqrt{2}} & \frac{-i}{\sqrt{2}} \\ \frac{-i}{\sqrt{2}} & \frac{i}{\sqrt{2}} \end{pmatrix} \\ &= -iH \end{aligned}$$

Alternative solution:

$$H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = \underbrace{\frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}}_{R_y(\pi/2)} \underbrace{\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}}_{iR_z(\pi)} = iR_y(\tfrac{\pi}{2})R_z(\pi).$$

0 1 2 3 4 5

d)* Consider an experimental setup which prepares a (fixed but unknown) qubit state $|\psi\rangle$. We run the experiment for 1000 repetitions, and each time perform a standard basis measurement on the qubit. In 900 instances, we obtain the result 0, and in the remaining 100 instances the result 1.
Assuming that the measurement outcomes precisely reflect the underlying state, draw the possible locations of $|\psi\rangle$ on the Bloch sphere, and provide a mathematical expression for its polar angle θ.
Hint: If a continuum of states is possible, you can illustrate this by a curve or marking the area.



From the definition of the angles of the Bloch sphere, we have

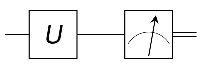
$$|\cos(\theta/2)|^2 = 0.9,$$
$$\theta = 2\arccos\left(\sqrt{0.9}\right)$$

0 1 2 3 4

e)* Your task is to measure a qubit with respect to the orthonormal basis states

$$|a\rangle = \frac{1}{5}\begin{pmatrix} 3 \\ 4i \end{pmatrix} \quad \text{and} \quad |b\rangle = \frac{1}{5}\begin{pmatrix} 4 \\ -3i \end{pmatrix}.$$

Since you only have a standard basis measurement setup available, you apply a gate U before the standard measurement:



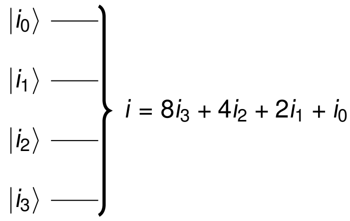
Write down the matrix representation of the gate U to solve the task.

U needs to transform the basis states $|a\rangle$ and $|b\rangle$ to the standard basis. This is achieved by

$$U = \begin{pmatrix} \langle a| \\ \langle b| \end{pmatrix} = \begin{pmatrix} \frac{3}{5} & -i\frac{4}{5} \\ \frac{4}{5} & i\frac{3}{5} \end{pmatrix}.$$

Problem 2 (20 credits)

Assume that we have a quantum computer available which supports 5 qubits. We use a 4-qubit binary encoding of integers as computational basis states, i.e., for the state $|i_3 i_2 i_1 i_0\rangle$ (with $i_0, i_1, i_2, i_3 \in \{0, 1\}$):



a) Assuming that we intend to find a specific set of integers that can be expressed in this encoding. What is the search space of this problem? 0 1

The search space is $2^4 = 16$.

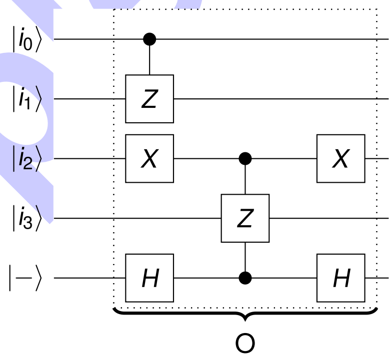
b) Recalling Grover's algorithm, provide the definition of the oracle per the lecture and its effective action if the ancilla qubit is in the $|-\rangle$ state. 0 1 2

The oracle performs a bit flip in the ancilla qubit if the input state is a solution. Given that the ancilla qubit is in the $|-\rangle$ state, its action is

$$|x\rangle\,|-\rangle \longrightarrow -\,|x\rangle\,|-\rangle$$

if $|x\rangle$ is a solution to the oracle. If $|x\rangle$ is not a solution, the oracle acts as the identity.

c) In the following circuit, an oracle O used for Grover's search implements the effective action described (b): 0 1 2 3 4 5 6 7 8



How many integer(s) does this oracle mark as a solution? Which integers are they? (Describing them in binary form is sufficient) Provide a clear justification or explicit derivation for your answer.

Hint: For this subproblem, you should only work in gate representation. There is no need to expand to matrix form in your working.

We note that this circuit applies a -1 phase if the input is a solution identified by the oracle. We then identify sections of the circuit that introduce a -1 phase as follows:

1. The controlled-Z gate introduces a -1 phase precisely if all qubits are in the $|1\rangle$ state (and leaves other computational basis states invariant). Hence, a -1 phase is introduced if i_0 and i_1 are both 1.

1. We first note that, for a controlled-Z gate, the role of the control and target qubits can be interchanged. We can then use $HZH = X$ and $H^2 = I$ to replace the Hadamard gates and controlled-Z (acting on the ancillary qubit) by a controlled-X gate with target on the ancilla. Must recall from (b) or explicitly state that applying a bitflip to the $|-\rangle$ introduces a -1 phase.
2. We additionally see that applying an X gate before and after a control means that the is 'negated'. This means the control is active if the input state is $|0\rangle$. Hence, a -1 phase is applied when i_2 and i_3 are 0 and 1 respectively.
3. Finally, we also see that applying a -1 phase twice is equivalent to applying the identity.

By these steps, we see that a -1 phase is applied when the input is: $|11xx\rangle$ or $|xx01\rangle$, where $x \in \{0, 1\}$, except for $|1101\rangle$ when a -1 phase is applied twice.
We note that there are then 6 solutions.
(Alternative solutions, for example via a truth table, possible as well.)

0 1 2 3 4

d) In general, the circuit segment G in part (b) must be run multiple times before performing a measurement at the end to obtain a solution. Given the oracle in (b), how often must G be applied?

We recall that applying the Grover operator G k times to the initial equal superposition state $|\psi\rangle$ results in

$$G^k |\psi\rangle = \cos\left((\tfrac{1}{2} + k)\theta\right) |\alpha\rangle + \sin\left((\tfrac{1}{2} + k)\theta\right) |\beta\rangle$$

with $|\beta\rangle$ a superposition of the sought solutions. The angle θ is defined via $\sin(\frac{\theta}{2}) = \sqrt{M/N}$, with $N = 2^4 = 16$ here, and $M = 6$ solutions according to (c). Thus, $\theta = 2sin^{-1}(\sqrt{3/8})$.
We need to apply the gate $k \approx \frac{\pi}{4sin^{-1}\left(\sqrt{3/8}\right)} - \frac{1}{2}$ times.

Carry over error from (a) & (b) to be considered if usage of equations is correct.

0 1 2 3 4 5

e) We define a gate $P = e^{-i\pi|\psi\rangle\langle\psi|}$, where $|\psi\rangle$ is the equal superposition state.
Show that P implements a reflection about the equal superposition state.

We see that $|\psi\rangle\langle\psi|$ is not Hermitian. Thus, the 'trick' with the general rotation gate does not help us here. Instead, we can use the Taylor expansion to express the matrix exponential, further recalling that projection operators are idempotent and $\sum_{k=0}^{\infty} \frac{x^k}{k!} = e^x$:

$$\begin{aligned}
e^{-i\pi|\psi\rangle\langle\psi|} &= \sum_{k=0}^{\infty} \frac{(-i\pi)^k}{k!} |\psi\rangle\langle\psi|^k \\
&= I + |\psi\rangle\langle\psi| \sum_{k=1}^{\infty} \frac{(-i\pi)^k}{k!} \\
&= I + |\psi\rangle\langle\psi| \left(e^{-i\pi} - 1\right) \\
&= I - 2|\psi\rangle\langle\psi|
\end{aligned}$$

This exactly implements a reflection over the equal superposition state.

Problem 3 (20 credits)

a) If $H \in \mathbb{C}^{n\times n}$ is a Hermitian matrix, show that $U = e^{iH}$ is unitary.

The question was covered in Tutorial 3 (a): first note that, for all $A \in \mathbb{C}^{n\times n}$,

$$(e^A)^\dagger = \sum_{k=0}^{\infty} \frac{1}{k!}(A^k)^\dagger = \sum_{k=0}^{\infty} \frac{1}{k!}(A^\dagger)^k = e^{(A^\dagger)}.$$

Together with the property that H is Hermitian and thus $(-iH)^\dagger = iH$, one obtains

$$U^\dagger U = e^{iH}e^{-iH} = e^{i(H-H)} = e^0 = I.$$

b)* Let $\mathcal{E}$ be a quantum channel with Kraus operators $\{E_k\}_{k=1}^n$. Provide a Kraus decomposition of the channel $\mathcal{F}$ that first applies a unitary gate V and then the channel $\mathcal{E}$.

By definition, $\mathcal{F}$ acts on density matrices as

$$\mathcal{F}(\rho) = \mathcal{E}\left(V\rho V^\dagger\right).$$

Thus $\mathcal{F}$ has the Kraus decomposition

$$\mathcal{F}(\rho) = \sum_{k=1}^{n} F_k \rho F_k^\dagger,$$

where $F_k = E_k V$ for $k = 1, \dots, n$.

c)* Consider a single-qubit quantum system described by a density matrix ρ, which yields the following expectation values of the Pauli matrices:

$$\langle X\rangle = \mathrm{tr}[\rho X] = 0, \quad \langle Y\rangle = \mathrm{tr}[\rho Y] = \frac{3}{5}, \quad \langle Z\rangle = \mathrm{tr}[\rho Z] = -\frac{1}{4}.$$

Compute the matrix representation of ρ. Hint: start from the Bloch vector of ρ.

Following the hint, we use the Bloch sphere representation of a density matrix:

$$\rho = \frac{1}{2}\left(I + \vec{r}\cdot\vec{\sigma}\right),$$

with $\vec{r} \in \mathbb{R}^3$ the Bloch vector and $\vec{\sigma} = (X, Y, Z)$ the Pauli vector.
The expectation values are precisely the entries of $\vec{r}$; namely,

$$\mathrm{tr}[\rho X] = \frac{1}{2}\left(\mathrm{tr}[X] + r_1\,\mathrm{tr}[XX] + r_2\,\mathrm{tr}[YX] + r_3\,\mathrm{tr}[ZX]\right) = \frac{1}{2}\left(0 + r_1\,\mathrm{tr}[I] + r_2\,\mathrm{tr}[-iZ] + r_3\,\mathrm{tr}[iY]\right) = r_1,$$

where we have used that the Pauli matrices are traceless. By an analogous calculation, $\mathrm{tr}[\rho Y] = r_2$ and $\mathrm{tr}[\rho Z] = r_3$.
Inserting the expectation values gives $\vec{r} = (0, \frac{3}{5}, -\frac{1}{4})$ and thus

$$\rho = \frac{1}{2}\left(I + \vec{r}\cdot\vec{\sigma}\right) = \frac{1}{2}\left(I + \tfrac{3}{5}Y - \tfrac{1}{4}Z\right) = \begin{pmatrix} \frac{3}{8} & -\frac{3i}{10} \\ \frac{3i}{10} & \frac{5}{8} \end{pmatrix}.$$

0 1 2 3 4

d)* Let ρ_A be the density operator of some quantum system A. Based on the spectral decomposition of ρ_A, construct a pure state $|\psi\rangle$ on an extended quantum system AR such that $\rho_A = \mathrm{tr}_R(|\psi\rangle\langle\psi|)$.

This corresponds to Tutorial 12 (c): by the spectral decomposition of ρ_A, there exist orthonormal eigenvectors $|\phi_j\rangle_{j=1,\dots,k}$ and corresponding non-negative eigenvalues p_j such that $\rho_A = \sum_{j=1}^k p_j |\phi_j\rangle\langle\phi_j|$, where k denotes the dimension of A.
Introduce a system R with the same dimension k and orthonormal basis states $|\chi_j\rangle_{j=1,\dots,k}$, and define the following state on the combined system:

$$|\psi\rangle = \sum_{j=1}^{k} \sqrt{p_j}\,|\phi_j\rangle\,|\chi_j\rangle\,.$$

We obtain that

$$\mathrm{tr}_R[|\psi\rangle\langle\psi|] = \sum_{j=1}^{k} p_j\,|\phi_j\rangle\langle\phi_j| = \rho_A,$$

as required.

0 1 2 3 4 5 6

e)* Consider the quantum phase flip operation

$$\rho \mapsto \mathcal{E}(\rho) = \sum_{k=0}^{1} E_k \rho E_k^\dagger \quad \text{with} \quad E_0 = \sqrt{1-p}\,I, \quad E_1 = \sqrt{p}\,Z$$

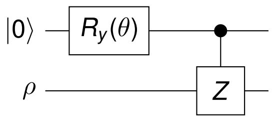
for a real parameter $p \in [0,1]$. Design a circuit that performs this operation.

Hint: A possible circuit consists of a single qubit wire for the principal system described by ρ, and another qubit wire for the environment initialized to $|0\rangle$. The rotation operator

$$R_y(\theta) = \begin{pmatrix} \cos\frac{\theta}{2} & -\sin\frac{\theta}{2} \\ \sin\frac{\theta}{2} & \cos\frac{\theta}{2} \end{pmatrix}$$

might be helpful for encoding the probability p.

The following circuit realizes the phase flip quantum operation:



The rotation angle has to be chosen such that the probability of applying Z is p. So, $\sin(\frac{\theta}{2}) = \sqrt{p}$, and $\theta = 2\arcsin(\sqrt{p})$.

Additional space for solutions–clearly mark the (sub)problem your answers are related to and strike out invalid solutions.