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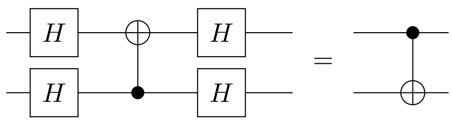
Exercise 5.1 (Basis transformation and measurement)

(a) Compute the probabilities when measuring $|\psi\rangle = \frac{i}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle$ with respect to the orthonormal basis $\{|u_1\rangle, |u_2\rangle\}$ given by $|u_1\rangle = \frac{3}{5}|0\rangle + i\frac{4}{5}|1\rangle$ and $|u_2\rangle = \frac{4}{5}|0\rangle - i\frac{3}{5}|1\rangle$.

Hint: You can obtain the coefficients of $|\psi\rangle$ with respect to these basis states by computing the inner products $\langle u_j|\psi\rangle$ for $j = 1, 2$.

(b) The role of the control and target qubit of a CNOT gate can be reversed by switching to a different basis! First show that



with H the Hadamard gate: $H = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$. Use this identity to derive the following relations:

$$
\begin{aligned}
|+\rangle|+\rangle &\overset{\text{CNOT}}{\mapsto} |+\rangle|+\rangle \\
|-\rangle|+\rangle &\overset{\text{CNOT}}{\mapsto} |-\rangle|+\rangle \\
|+\rangle|-\rangle &\overset{\text{CNOT}}{\mapsto} |-\rangle|-\rangle \\
|-\rangle|-\rangle &\overset{\text{CNOT}}{\mapsto} |+\rangle|-\rangle
\end{aligned}
$$

with $|\pm\rangle$ defined as $|\pm\rangle = \frac{1}{\sqrt{2}}(|0\rangle \pm |1\rangle)$. In other words, with respect to the $|\pm\rangle$ basis, the second qubit assumes the role of the control and the first qubit the role of the target.

Hint: Use that $H|+\rangle = |0\rangle$ and $H|-\rangle = |1\rangle$, and conversely $H|0\rangle = |+\rangle$ and $H|1\rangle = |-\rangle$.

Solution

(a) We see that $\{|u_1\rangle, |u_2\rangle\}$ forms an orthonormal basis as $|u_1\rangle$ and $|u_2\rangle$ are normalized and $\langle u_1|u_2\rangle = 0$.

We then compute the inner products of $|\psi\rangle$ with $|u_1\rangle$ and $|u_2\rangle$:

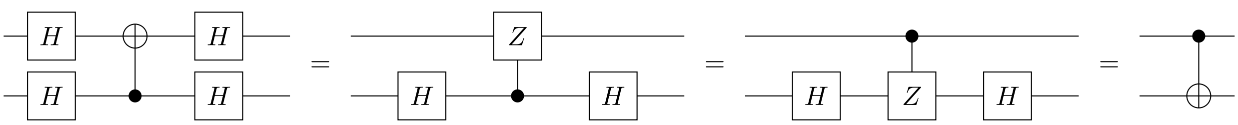
$$
\begin{aligned}
\alpha_1 &= \langle u_1|\psi\rangle = -\frac{i}{5\sqrt{2}}, \\
\alpha_2 &= \langle u_2|\psi\rangle = \frac{7i}{5\sqrt{2}}.
\end{aligned}
$$

Thus in terms of the new basis, $|\psi\rangle = \alpha_1|u_1\rangle + \alpha_2|u_2\rangle$.

The measurement probabilities are then the squared absolute values of the coefficients α_j:

$$
\begin{aligned}
p(u_1) &= |\alpha_1|^2 = \frac{1}{50}, \\
p(u_2) &= |\alpha_2|^2 = \frac{49}{50}.
\end{aligned}
$$

(b) Conjugating the Pauli-X gate with the Hadamard gate results in the Pauli-Z gate and other way around, i.e., $HXH = Z$ and $HZH = X$, see Exercise 3.1(d). Moreover, applying the Hadamard gate twice gives the identity, $H^2 = I$. This justifies the first and last step of the following identities:



The second equal sign uses the fact that the controlled-Z operation is a diagonal matrix, with only a flipped sign on $|11\rangle$. Hence, this gate is invariant when interchanging the roles of the control and target qubits.

The relations then follow immediately by noting that the Hadamard gate switches between the $\{|0\rangle, |1\rangle\}$ and $\{|+\rangle, |-\rangle\}$ bases.