Professorship for Quantum Computing
Department of Informatics
Technical University of Munich

Eexam
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Note:

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Introduction to Quantum Computing

Exam:	IN2381 / Retake	**Date:**	Friday 9th October, 2020
Examiner:	Prof. Dr. Christian Mendl	**Time:**	16:15 – 17:45

	P 1	P 2	P 3
I			

Working instructions

- This exam consists of **8 pages** with a total of **3 problems**.
  Please make sure now that you received a complete copy of the exam.
- The total amount of achievable credits in this exam is 60 credits.
- Detaching pages from the exam is prohibited.
- Allowed resources: one **A4 sheet** (both sides) with your own notes
- Subproblems marked by * can be solved without results of previous subproblems.
- **Answers are only accepted if the solution approach is documented.** Give a reason for each answer unless explicitly stated otherwise in the respective subproblem.
- Do not write with red or green colors nor use pencils.
- Physically turn off all electronic devices, put them into your bag and close the bag.

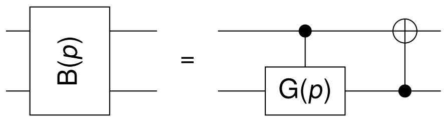
Left room from ________ to ________ / Early submission at ________

Problem 1 **(20 credits)**

In the following, we consider the so-called three-qubit W-State $|W_3\rangle$:

$$|W_3\rangle = \frac{1}{\sqrt{3}} \left(|100\rangle + |010\rangle + |001\rangle\right) .$$

We also define a two-qubit gate $B(p)$, for a real parameter $p \in [0, 1]$:



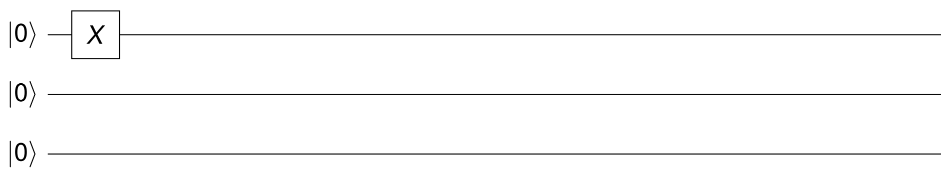
where

$$G(p) = \begin{pmatrix} \sqrt{p} & -\sqrt{1-p} \\ \sqrt{1-p} & \sqrt{p} \end{pmatrix} .$$

a) Determine the outputs of $B(p)$ for the four input basis states $|00\rangle$, $|01\rangle$, $|10\rangle$ and $|11\rangle$

b) Add $B(p)$ gates with suitably chosen values of p to the following circuit such that the output is $|W_3\rangle$, and verify your construction.

Hint: Two $B(p)$ gates are sufficient.



c)* Demonstrate that a single-qubit measurement (with respect to some arbitrary orthonormal basis) performed on the first qubit of $|W_3\rangle$ leaves the remaining two qubits entangled in general.

Hint: Describe the measurement as a projection $|\psi\rangle\langle\psi|$, where $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$ is a general single qubit state.

0 1 2 3 4 5 6

Problem 2 **(20 credits)**

A quantum operation, $\mathcal{E}$, describes in general terms how a quantum system evolves. It can be expressed using the operator-sum representation as

$$\mathcal{E}(\rho) = \sum_k E_k \rho E_k^\dagger,$$

where ρ is the density matrix of the system and the complex matrices E_k are the so-called Kraus operators.

0 1 2

a) What condition must the Kraus operators satisfy such that $\mathcal{E}$ is compatible with the laws of quantum mechanics?

0 1 2 3 4 5 6 7 8

b)* An example of a quantum operation is the depolarizing channel, which models quantum noise. Its Kraus operators (for a real parameter $p \in [0, \frac{3}{4}]$) are

$$E_0 = \sqrt{1-p}I, \quad E_1 = \sqrt{p/3}X, \quad E_2 = \sqrt{p/3}Y \quad \text{and} \quad E_3 = \sqrt{p/3}Z.$$

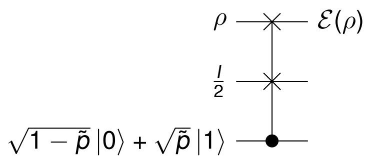
Compute $\mathcal{E}(\rho)$ using the Bloch sphere representation of the density matrix, $\rho = (I + \vec{r} \cdot \vec{\sigma})/2$ with $\vec{r} \in \mathbb{R}^3$, $\|\vec{r}\| \leq 1$, to show that the depolarizing channel acts as a contraction of the Bloch vector $\vec{r}$. You may use without proof that

$$XY = -YX = iZ$$
$$YZ = -ZY = iX$$
$$ZX = -XZ = iY.$$

c) The depolarizing channel can also be written as

$$\mathcal{E}(\rho) = \tilde{p}\frac{I}{2} + (1 - \tilde{p})\rho$$

with $\tilde{p} = \frac{4}{3}p \in [0, 1]$. A corresponding circuit implementation is



where the middle and bottom wires serve as environment. Describe how this circuit realizes the depolarizing channel.

d)* The "purity" of a quantum state with density matrix ρ is defined as $\mathrm{Tr}(\rho^2)$. Compute the minimum purity of a single qubit quantum state.

Hint: You might recall an expression for $\mathrm{Tr}(\rho^2)$ in terms of the Bloch vector of ρ. Alternatively, why can we assume that ρ is a diagonal matrix for evaluating $\mathrm{Tr}(\rho^2)$?

Problem 3 (20 credits)

We consider a quantum system of n qubits, and use the notation X_j, Y_j, Z_j to denote that one of the Pauli matrices acts on the jth qubit; e.g., $X_1 Z_3 \equiv X \otimes I \otimes Z$ for $n = 3$.
Conjugation by U refers to the transformation $UgU^\dagger$ of a quantum gate g by a unitary operation U. The following table summarizes several conjugation transformations:

U	X	X	X	H	H	H	S	S	S
g	X	Y	Z	X	Y	Z	X	Y	Z
$UgU^\dagger$	X	$-Y$	$-Z$	Z	$-Y$	X	Y	$-X$	Z

Here $S = \left(\begin{smallmatrix} 1 & 0 \\ 0 & i \end{smallmatrix}\right)$ is the phase gate.

0 1 2 3 4 5

a) State the check matrix representation of $g_1, g_2 \in G_4$ given by

$$g_1 = Y \otimes I \otimes Z \otimes X,$$
$$g_2 = X \otimes Z \otimes Y \otimes X.$$

Based on this representation, show that g_1 commutes with g_2.

0 1 2 3 4 5

b)* Specify an element g of the Pauli group G_4 such that

$$R = \langle X_1 Y_3, Y_1 X_2 Z_3 Y_4, g \rangle$$

stabilizes a non-trivial vector space and the three generators of R are independent. Also state the properties which g must satisfy (a proof of them is not required).

c)* We consider the subgroup

$$T = \langle X_1 Z_2 Y_3, Y_1 Z_2 X_3 \rangle$$

of the Pauli group G_3. Show that the vector space V_T stabilized by T is invariant under multiplication by $(S \otimes X \otimes S)$ (with S the phase gate), in other words, $\psi \in V_T$ if and only if $(S \otimes X \otimes S)\psi \in V_T$.

0 1 2 3 4 5

d)* We consider the three qubit bit flip code $C = \text{span}\{|0_L\rangle, |1_L\rangle\} = \text{span}\{|000\rangle, |111\rangle\}$, affected by amplitude damping noise on the first qubit. Recall that the operator-sum representation of the amplitude damping quantum channel is given by

$$\mathcal{E}_{\text{AD}}(\rho) = E_0 \rho E_0^\dagger + E_1 \rho E_1^\dagger \quad \text{with} \quad E_0 = \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{1-\gamma} \end{pmatrix}, \quad E_1 = \begin{pmatrix} 0 & \sqrt{\gamma} \\ 0 & 0 \end{pmatrix},$$

and a real parameter $\gamma \in (0, 1)$. Show that this noise process acting on C is not error-correctable.

0 1 2 3 4 5

Additional space for solutions–clearly mark the (sub)problem your answers are related to and strike out invalid solutions.