Chair of Scientific Computing in Computer Science
Department of Informatics
Technical University of Munich

Esolution
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Algorithms of Scientific Computing II (Quantum Computing)

Exam:	IN2002 / Retake	**Date:**	Thursday 9th July, 2020
Examiner:	Christian B. Mendl	**Time:**	10:45 – 12:15

	P 1	P 2	P 3
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Working instructions

- This exam consists of **10 pages** with a total of **3 problems**.
  Please make sure now that you received a complete copy of the exam.
- The total amount of achievable credits in this exam is 60 credits.
- Detaching pages from the exam is prohibited.
- Allowed resources: one **A4 sheet** (both sides) with your own notes
- Subproblems marked by * can be solved without results of previous subproblems.
- **Answers are only accepted if the solution approach is documented.** Give a reason for each answer unless explicitly stated otherwise in the respective subproblem.
- Do not write with red or green colors nor use pencils.
- Physically turn off all electronic devices, put them into your bag and close the bag.

Left room from __________ to __________ / Early submission at __________

Problem 1 **(20 credits)**

A *tripartite state* consists of three entangled qubits. Two well-known tripartite states are the Greenberger-Horne-Zeilinger (GHZ) state and the W-state. The classical GHZ state is defined as

$$|\text{GHZ}\rangle = \frac{|000\rangle + |111\rangle}{\sqrt{2}}.$$

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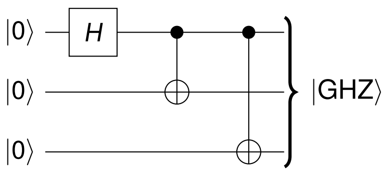
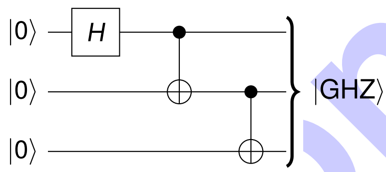
a) Specify a quantum circuit which generates $|\text{GHZ}\rangle$ for input $|000\rangle$.



or (alternatively)



Suppose that three parties, Alice, Bob and Charlie, share a $|\text{GHZ}\rangle$ state (each taking one entangled qubit, denoted $|q_a\rangle$, $|q_b\rangle$, $|q_c\rangle$, respectively). Alice controls another qubit $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$ (with $\alpha, \beta \in \mathbb{C}$, $|\alpha|^2 + |\beta|^2 = 1$) that she wishes to pass to Charlie. However, Alice, Bob and Charlie are limited to exchanging only classical information.

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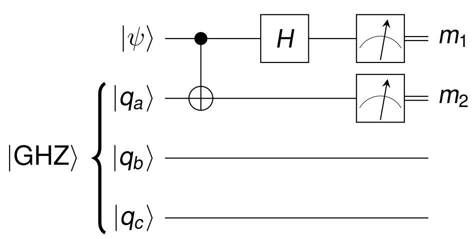
b)* Name an algorithm that could be modified to allow Charlie to recreate the state $|\psi\rangle$, given these constraints.

The quantum teleportation circuit.

We now consider the following circuit:



c)* Alice performs a standard measurement of both her qubits, as shown in the circuit diagram. The qubits owned by Bob and Charlie, $|q_b, q_c\rangle$, then collapse in dependence of Alice's measurement result. Suppose that $m_2 = 1$, determine $|q_b, q_c\rangle$ for the two cases $m_1 = 0$ and $m_1 = 1$.

0 1 2 3 4 5 6 7

The overall quantum state before applying the CNOT and Hadamard gate is:

$$|\chi\rangle = |\psi\rangle \otimes |\mathrm{GHZ}\rangle = (\alpha |0\rangle + \beta |1\rangle) \otimes \frac{1}{\sqrt{2}}(|000\rangle + |111\rangle).$$

Then, Alice applies the two gates

$$(\mathrm{CNOT} \otimes I^{\otimes 2}) |\chi\rangle = \alpha |0\rangle \otimes \frac{1}{\sqrt{2}}(|000\rangle + |111\rangle) + \beta |1\rangle \otimes \frac{1}{\sqrt{2}}(|100\rangle + |011\rangle),$$

and

$$\begin{aligned}(H \otimes I^{\otimes 3})(\mathrm{CNOT} \otimes I^{\otimes 2}) |\chi\rangle &= \frac{\alpha}{2}(|0\rangle + |1\rangle) \otimes (|000\rangle + |111\rangle) + \frac{\beta}{2}(|0\rangle - |1\rangle) \otimes (|100\rangle + |011\rangle) \\ &= \frac{\alpha}{2} |00\rangle_{\psi,q_a} \otimes |00\rangle_{q_b,q_c} + \frac{\beta}{2} |00\rangle_{\psi,q_a} \otimes |11\rangle_{q_b,q_c} \\ &+ \frac{\alpha}{2} |01\rangle_{\psi,q_a} \otimes |11\rangle_{q_b,q_c} + \frac{\beta}{2} |01\rangle_{\psi,q_a} \otimes |00\rangle_{q_b,q_c} \\ &+ \frac{\alpha}{2} |10\rangle_{\psi,q_a} \otimes |00\rangle_{q_b,q_c} - \frac{\beta}{2} |10\rangle_{\psi,q_a} \otimes |11\rangle_{q_b,q_c} \\ &+ \frac{\alpha}{2} |11\rangle_{\psi,q_a} \otimes |11\rangle_{q_b,q_c} - \frac{\beta}{2} |11\rangle_{\psi,q_a} \otimes |00\rangle_{q_b,q_c}.\end{aligned}$$

Accordingly, the state $|q_b, q_c\rangle$ after measurement (and normalization) is then:

$$\begin{aligned}|q_b, q_c\rangle_{01} &= \alpha |11\rangle + \beta |00\rangle, \quad \text{or} \\ |q_b, q_c\rangle_{11} &= \alpha |11\rangle - \beta |00\rangle.\end{aligned}$$

d)* In the setting of part (c), how can Charlie recover Alice's original qubit $|\psi\rangle$? What additional information from Bob is required? Describe the overall process.

Hint: Bob should perform a measurement in a basis that indicates what Charlie should do ...

If unable to solve part (c), start working with the two possible states:

$$|q_b, q_c\rangle_{01} = \alpha\,|01\rangle + \beta\,|10\rangle\,,$$
$$|q_b, q_c\rangle_{11} = \alpha\,|01\rangle - \beta\,|10\rangle\,,$$

with the notation $|q_b, q_c\rangle_{m_1 m_2}$ (using the measurement results m_1 and m_2 as indices).

Bob should perform a measurement on his qubit using the X basis $|+\rangle$ and $|-\rangle$:

$$|+\rangle = \frac{|0\rangle + |1\rangle}{\sqrt{2}}, \quad |-\rangle = \frac{|0\rangle - |1\rangle}{\sqrt{2}}.$$

The required action by Charlie to recover the original state $|\psi\rangle$ depends on the measurement by Bob, who has to send this information to Charlie.

$$\langle + |q_b, q_c\rangle_{01} = \frac{\beta}{\sqrt{2}}\langle 0\,|0\rangle\,|0\rangle + \frac{\alpha}{\sqrt{2}}\langle 1\,|1\rangle\,|1\rangle \to \beta\,|0\rangle + \alpha\,|1\rangle$$
$\to$ Charlie must apply an X gate

$$\langle - |q_b, q_c\rangle_{01} = \frac{\beta}{\sqrt{2}}\langle 0\,|0\rangle\,|0\rangle - \frac{\alpha}{\sqrt{2}}\langle 1\,|1\rangle\,|1\rangle \to \beta\,|0\rangle - \alpha\,|1\rangle$$
$\to$ Charlie must apply ZX / XZ gates

$$\langle + |q_b, q_c\rangle_{11} = -\frac{\beta}{\sqrt{2}}\langle 0\,|0\rangle\,|0\rangle + \frac{\alpha}{\sqrt{2}}\langle 1\,|1\rangle\,|1\rangle \to -\beta\,|0\rangle + \alpha\,|1\rangle$$
$\to$ Charlie must apply an ZX / XZ gates

$$\langle - |q_b, q_c\rangle_{11} = -\frac{\beta}{\sqrt{2}}\langle 0\,|0\rangle\,|0\rangle - \frac{\alpha}{\sqrt{2}}\langle 1\,|1\rangle\,|1\rangle \to -\beta\,|0\rangle - \alpha\,|1\rangle$$
$\to$ Charlie must apply an X gate

The solution if starting from part (d):

$$\langle + |q_b, q_c\rangle_{01} = \frac{\alpha}{\sqrt{2}}\langle 0\,|0\rangle\,|1\rangle + \frac{\beta}{\sqrt{2}}\langle 1\,|1\rangle\,|0\rangle \to \beta\,|0\rangle + \alpha\,|1\rangle$$
$\to$ Charlie must apply an X gate

$$\langle - |q_b, q_c\rangle_{01} = \frac{\alpha}{\sqrt{2}}\langle 0\,|0\rangle\,|1\rangle - \frac{\beta}{\sqrt{2}}\langle 1\,|1\rangle\,|0\rangle \to -\beta\,|0\rangle + \alpha\,|1\rangle$$
$\to$ Charlie must apply ZX / XZ gates

$$\langle + |q_b, q_c\rangle_{11} = \frac{\alpha}{\sqrt{2}}\langle 0\,|0\rangle\,|1\rangle - \frac{\beta}{\sqrt{2}}\langle 1\,|1\rangle\,|0\rangle \to \alpha\,|1\rangle - \beta\,|0\rangle$$
$\to$ Charlie must apply an ZX / XZ gates

$$\langle - |q_b, q_c\rangle_{11} = \frac{\alpha}{\sqrt{2}}\langle 0\,|0\rangle\,|1\rangle + \frac{\beta}{\sqrt{2}}\langle 1\,|1\rangle\,|0\rangle \to \beta\,|0\rangle + \alpha\,|1\rangle$$
$\to$ Charlie must apply an X gate

Problem 2 (20 credits)

a) Determine whether the single qubit density matrix

$$\rho = \begin{pmatrix} \frac{9}{10} & -\frac{i}{5} \\ \frac{i}{5} & \frac{1}{10} \end{pmatrix}$$

describes a pure quantum system.

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We can represent any single qubit density matrix in the form

$$\rho = \frac{I + \vec{r} \cdot \vec{\sigma}}{2}$$

for some $\vec{r} \in \mathbb{R}^3$. ρ describes a pure state precisely if $\|\vec{r}\| = 1$. For the present example, $\vec{r} = (0, \frac{2}{5}, \frac{4}{5})$, and $\|\vec{r}\| = \sqrt{\frac{4}{25} + \frac{16}{25}} = \frac{2}{\sqrt{5}} < 1$, thus ρ does not describe a pure state.

b)* Pure quantum states evolve under a unitary matrix U as

$$|\psi\rangle = U |\psi_0\rangle\,,$$

where $|\psi_0\rangle$ is the initial state. Write down the density matrix ρ_0 corresponding to $|\psi_0\rangle$, and an equivalent expression for the evolution of the density matrix governed by U.

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The corresponding density matrix is $\rho_0 = |\psi_0\rangle \langle\psi_0|$. It transforms as

$$\rho = U |\psi_0\rangle \langle\psi_0| U^\dagger = U \rho_0 U^\dagger.$$

c) Consider a quantum system consisting of two subsystems A and B, initially in the state described by the density matrix $\rho_0 = \rho_0^A \otimes \rho_0^B$, where ρ_0^A and ρ_0^B are density matrices on subsystems A and B, respectively. Specify the density matrix of subsystem A after an evolution governed by a unitary operator U acting on the combined system.

0 1 2

The system evolves as described in part (b): $\rho = U \rho_0 U^\dagger$. To find the state of A, one has to trace out B:

$$\rho^A = \mathrm{tr}_B \left[U \left(\rho_0^A \otimes \rho_0^B \right) U^\dagger \right].$$

d) Given the setting of part (c), let $(|e_k\rangle)_k$ be an orthonormal basis of the state space of subsystem B, and assume $\rho_0^B = |e_0\rangle\langle e_0|$. Show that one can represent $\rho^A = \sum_k E_k \rho_0^A E_k^\dagger$, and provide an explicit expression for E_k.

Starting from the formula in (c),

$$\rho^A = \mathrm{tr}_B\left[U\left(\rho_0^A \otimes \rho_0^B\right)U^\dagger\right] = \sum_k \langle e_k|\, U\left(\rho_0^A \otimes |e_0\rangle\langle e_0|\right)U^\dagger\, |e_k\rangle\,.$$

We define the matrices E_k with entries $(E_k)_{\ell,m} = \langle \ell, e_k|\, U\, |m, e_0\rangle$. Then ρ^A can be represented as

$$\rho^A = \sum_k E_k \rho_0^A E_k^\dagger.$$

e)* Consider the quantum bit flip operation

$$\rho \mapsto \mathcal{E}(\rho) = \sum_{k=0}^{1} E_k \rho E_k^\dagger \quad \text{with} \quad E_0 = \sqrt{1-p}\,I, \quad E_1 = \sqrt{p}X$$

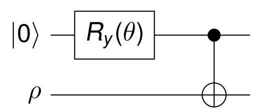
for a real parameter $p \in [0,1]$. Design a circuit that performs this operation.

Hint: A possible circuit consists of a single qubit wire for the principal system described by ρ, and another qubit wire for the environment initialized to $|0\rangle$. The rotation operator

$$R_y(\theta) = \begin{pmatrix} \cos\frac{\theta}{2} & -\sin\frac{\theta}{2} \\ \sin\frac{\theta}{2} & \cos\frac{\theta}{2} \end{pmatrix}$$

might be helpful for encoding the probability p.

The following circuit realizes the bit flip quantum operation:



The rotation angle has to be chosen such that the probability of applying X is p. So, $\sin\frac{\theta}{2} = \sqrt{p}$, and $\theta = 2\arcsin(\sqrt{p})$.

Problem 3 (20 credits)

We consider a quantum system of n qubits, and use the notation X_j, Y_j, Z_j to denote that one of the Pauli matrices acts on the jth qubit; e.g., $X_1 Z_3 \equiv X \otimes I \otimes Z$ for $n = 3$.
Conjugation by U refers to the transformation $UgU^\dagger$ of a quantum gate g by a unitary operation U. The following table summarizes several conjugation transformations:

U	H	H	H	S	S	S
g	X	Y	Z	X	Y	Z
$UgU^\dagger$	Z	$-Y$	X	Y	$-X$	Z

Here $S = \left(\begin{smallmatrix} 1 & 0 \\ 0 & i \end{smallmatrix}\right)$ is the phase gate.

a) The quantum state

$$|\psi\rangle = \alpha\,|000\rangle + \beta\,|111\rangle$$

with $\alpha, \beta \in \mathbb{C}$ is affected by a single qubit bit flip error, resulting in a state $|\psi'\rangle$. A subsequent measurement of the observables $Z_1 Z_2$ and $Z_1 Z_3$ leads to the measurement outcomes (eigenvalues) -1 and 1, respectively. Specify $|\psi'\rangle$, and a quantum gate to recover the original $|\psi\rangle$ when applied to $|\psi'\rangle$.

A single bit flip error is equivalent to applying the gate X_j for $j \in \{1, 2, 3\}$, for example

$$X_1\,|\psi\rangle = \alpha\,|100\rangle + \beta\,|011\rangle\,.$$

The measurement result (-1) for $Z_1 Z_2$ indicates that either the first or second qubit was flipped, and the result 1 for $Z_1 Z_3$ that the first and third qubit remained intact; thus the second qubit must have been flipped:

$$|\psi'\rangle = \alpha\,|010\rangle + \beta\,|101\rangle\,.$$

We recover $|\psi\rangle$ by applying X_2.

b)* Recall that the *commutator* of two operators A and B is defined as $[A, B] = AB - BA$, and that they commute if $[A, B] = 0$. Show that the operators $X_1 Y_2 Z_3$ and $Z_1 Y_2 X_3$ commute.

We use that X and Z *anti-commute*, i.e., $XZ = -ZX$, to derive that

$$(X_1 Y_2 Z_3)(Z_1 Y_2 X_3) = (XZ)\otimes(YY)\otimes(ZX) = (-ZX)\otimes(YY)\otimes(-XZ) = (ZX)\otimes(YY)\otimes(XZ) = (Z_1 Y_2 X_3)(X_1 Y_2 Z_3).$$

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c)* The subgroup $R = \langle X_1 Y_2 Z_3, Z_1 Y_2 X_3 \rangle$ of the Pauli group G_3 stabilizes the subspace $V_R = \mathrm{span}\{|\chi_0\rangle, |\chi_1\rangle\}$ with

$$|\chi_0\rangle = \frac{1}{2}\big(i\,|001\rangle - |010\rangle + i\,|100\rangle + |111\rangle\big), \qquad |\chi_1\rangle = \frac{1}{2}\big(|000\rangle + i\,|011\rangle - |101\rangle + i\,|110\rangle\big).$$

(A proof of this statement is not required here.) Determine the result (eigenvalue) when measuring the operator $X_1 Y_2 Z_3$ with respect to the quantum state $(H \otimes H \otimes H)\,|\chi_0\rangle$.

We first recall that the Hadamard gate is Hermitian, $H^\dagger = H$, and self-inverse, $H^2 = I$. These properties are inherited by $H \otimes H \otimes H$. Following the above conjugation table, we compute

$$(H \otimes H \otimes H)(X_1 Y_2 Z_3)(H \otimes H \otimes H) = (HXH) \otimes (HYH) \otimes (HZH) = Z \otimes (-Y) \otimes X = -Z_1 Y_2 X_3,$$

and after multiplying by $(H \otimes H \otimes H)$ on both sides: $H^{\otimes 3}(Z_1 Y_2 X_3)H^{\otimes 3} = -X_1 Y_2 Z_3$. Thus $H^{\otimes 3}\,|\chi_0\rangle$ is an eigenstate of $X_1 Y_2 Z_3$ with eigenvalue (-1):

$$(X_1 Y_2 Z_3)(H^{\otimes 3}\,|\chi_0\rangle) = -H^{\otimes 3}(Z_1 Y_2 X_3)H^{\otimes 3}(H^{\otimes 3}\,|\chi_0\rangle) = -H^{\otimes 3}(Z_1 Y_2 X_3)\,|\chi_0\rangle = -H^{\otimes 3}\,|\chi_0\rangle\,.$$

For the last equal sign we have used that R stabilizes $|\chi_0\rangle$. In particular, the measurement result will be (-1) with probability 1.

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d)* We consider the subgroup $T = \langle X_1 X_2, X_1 Z_2 \rangle$ of the Pauli group G_2. Compute the subspace V_T stabilized by T, such that $g\,|\psi\rangle = |\psi\rangle$ for all $g \in T$ and $|\psi\rangle \in V_T$.

Since the two generators $X_1 X_2$ and $X_1 Z_2$ of T *anti-commute*, i.e., $(X_1 X_2)(X_1 Z_2) = -(X_1 Z_2)(X_1 X_2)$, the subspace V_T is the trivial subspace: $V_T = \{0\}$.
Recall that in general, for anti-commuting operators g and g', a state $|\psi\rangle$ stabilized by both of them satisfies

$$|\psi\rangle = gg'\,|\psi\rangle = -g'g\,|\psi\rangle = -\,|\psi\rangle$$

and thus $|\psi\rangle = 0$.

Additional space for solutions–clearly mark the (sub)problem your answers are related to and strike out invalid solutions.