Solution 3: Production and Supply

Problem 1 (*Production Function*)

(a) Positive and decreasing marginal products of both inputs

(i) Marginal Product of Labor:

$$\frac{\partial F(L,K)}{\partial L} = aL^{a-1}K^b > 0, \quad \frac{\partial^2 F(L,K)}{\partial L^2} = (a-1)aL^{a-2}K^b < 0$$

(ii) Marginal Product of Capital:

$$\frac{\partial F(L,K)}{\partial K} = bL^aK^{b-1} > 0, \quad \frac{\partial^2 F(L,K)}{\partial K^2} = (b-1)bL^aK^{b-2} < 0$$

(b) Returns to Scale: Provided that $L > 0$ and $K > 0$, a multiplication of both inputs by a constant λ implies a multiplication of output by λ^{a+b}.

$$F(\lambda L, \lambda K) = (\lambda L)^a \cdot (\lambda K)^b = \lambda^{a+b}L^aK^b = \lambda^{a+b}F(L,K)$$

Thus, the production function exhibits

(i) constant returns to scale if $a + b = 1$,

(ii) increasing returns to scale if $a + b > 1$,

(iii) decreasing returns to scale if $a + b < 1$.

(c) Strictly Convex Isoquants: The MRTS is the slope of the isoquant (in absolute value) for any input bundle.

$$\text{MRTS} = \frac{\frac{\partial F(L,K)}{\partial L}}{\frac{\partial F(L,K)}{\partial K}} = \frac{aL^{a-1}K^b}{bL^aK^{b-1}} = \frac{a}{b}\frac{K}{L}$$

The MRTS decreases when L is substituted for K at constant output.

$$\frac{d\text{MRTS}}{dL} = \frac{a}{b}\frac{\frac{dK}{dL}L - K}{L^2} < 0$$

Thus, the isoquant is strictly convex.

Problem 2 (*Cost Minimization*)

(a) Optimal input bundle

(i) The optimal input bundle must be located on the isoquant representing the given output level.

$$q = F(L, K) = (L \cdot K)^{\frac{1}{2}}$$

(ii) For any interior solution, the isocost line through the optimal input bundle must be tangent to the isoquant, i.e. the MRTS must equal the input price ratio.

$$\underbrace{\frac{\frac{\partial F}{\partial L}}{\frac{\partial F}{\partial K}}}_{\text{MRTS}} = \frac{w}{r} \quad \Rightarrow \quad \frac{\frac{1}{2}L^{-\frac{1}{2}}K^{\frac{1}{2}}}{\frac{1}{2}L^{\frac{1}{2}}K^{-\frac{1}{2}}} = \frac{w}{r} \quad \Leftrightarrow \quad K = \frac{w}{r}L \quad \Leftrightarrow \quad L = \frac{r}{w}K$$

Substituting (ii) into (i) yields

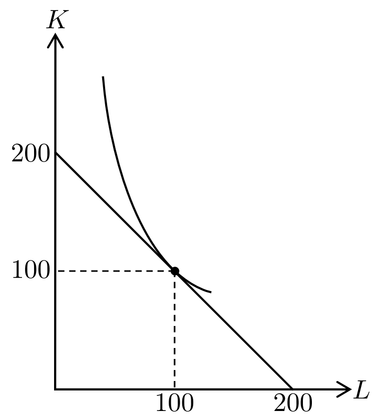
$$L = q\left(\frac{r}{w}\right)^{\frac{1}{2}}, \quad K = q\left(\frac{w}{r}\right)^{\frac{1}{2}}.$$

Given the output $q = 100$ and input prices $w = 2.5$ and $r = 2.5$, the optimal input bundle is

$$L = 100, \quad K = 100,$$

and the corresponding input costs are

$$c = wL + rK = 2.5 \cdot 100 + 2.5 \cdot 100 = 500.$$



Optimal Input Employment

(b) Now, the same output $q = 100$ shall be produced given a higher wage rate for labor $w' = 10$ and an unchanged rental rate for capital $r = 2.5$.

(i) At the new input price ratio, the input bundle $L = 100$ and $K = 100$ no longer minimizes costs. It causes input costs of

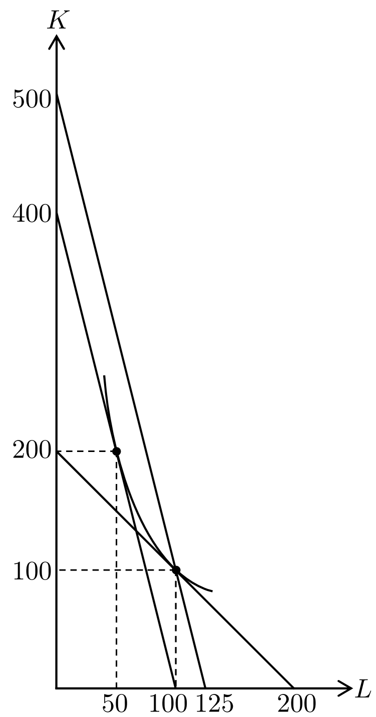
$$c = w'L + rK = 10 \cdot 100 + 2.5 \cdot 100 = 1{,}250.$$

(ii) The new cost minimizing input bundle is

$$L' = 50, \quad K' = 200,$$

and the corresponding input costs are

$$c' = w'L' + rK' = 10 \cdot 50 + 2.5 \cdot 200 = 1{,}000.$$



(Optimal) Input Employment

Problems 3-5 (*Cost Minimization*)

Problem 3

Provided that $L > 0$ and $K > 0$, a multiplication of both inputs by 4 implies a multiplication of output by 2.

$$F(4L, 4K) = (4L \cdot 4K)^{\frac{1}{4}} = 2(L \cdot K)^{\frac{1}{4}} = 2F(L, K)$$

$\Rightarrow$ **(B)** is correct.

Optimal input bundle

(i) The optimal input bundle must be located on the isoquant representing the given output level.

$$q = F(L, K) = (L \cdot K)^{\frac{1}{4}}$$

(ii) For any interior solution, the isocost line through the optimal input bundle must be tangent to the isoquant, i.e. the MRTS must equal the input price ratio.

$$\underbrace{\frac{\frac{\partial F}{\partial L}}{\frac{\partial F}{\partial K}}}_{\text{MRTS}} = \frac{w}{r} \quad \Rightarrow \quad \frac{\frac{1}{4}L^{-\frac{3}{4}}K^{\frac{1}{4}}}{\frac{1}{4}L^{\frac{1}{4}}K^{-\frac{3}{4}}} = \frac{w}{r} \quad \Leftrightarrow \quad K = \frac{w}{r}L \quad \Leftrightarrow \quad L = \frac{r}{w}K$$

Substituting (ii) into (i) yields

$$L = q^2\left(\frac{r}{w}\right)^{\frac{1}{2}}, \quad K = q^2\left(\frac{w}{r}\right)^{\frac{1}{2}}.$$

Thus, variable costs (minimum input costs as a function of output) are

$$c(q) = w \cdot q^2\left(\frac{r}{w}\right)^{\frac{1}{2}} + r \cdot q^2\left(\frac{w}{r}\right)^{\frac{1}{2}} = 2(wr)^{\frac{1}{2}}q^2.$$

Problem 4

If the wage rate for labor is given by $w = 16$, and the rental rate for capital is given by $r = 4$, variable costs are $c(q) = 16q^2$.

$\Rightarrow$ **(D)** is correct.

Problem 5

If the wage rate for labor is given by $w = 18$, and the rental rate for capital is given by $r = \frac{1}{2}$, variable costs are $c(q) = 6q^2$.

$\Rightarrow$ **(C)** is correct.

Problems 6-8 (*Profit Maximization*)

Marginal Costs:	$MC(q) = 4q$
Average Total Costs:	$AC(q) = \frac{200}{q} + 2q, \quad \forall q > 0$
Average Variable Costs:	$ac(q) = 2q$

Problem 6

For $q = 20$, marginal costs are higher than average total costs.

$$MC(20) = 80 > 50 = AC(20)$$

$\Rightarrow$ **(B)** is correct.

Problem 7

If the firm produces $q > 0$, profit maximization requires $p = MC(q)$. For $p = 20$, this yields

$$20 = 4q \quad \Leftrightarrow \quad q = 5.$$

In the short run, the firm produces $q = 5$, because the price is higher than average variable costs at $q = 5$.

$$p = 20 > 10 = ac(5)$$

In the long run, the firm produces $q = 0$, because the price is lower than average total costs at $q = 5$.

$$p = 20 < 50 = AC(5)$$

Thus, if $p = 20$, the firm's supply is 5 in the short run and 0 in the long run.

$\Rightarrow$ **(B)** is correct.

Problem 8

The threshold price, above which the firm's supply is $q > 0$ in the long run, corresponds to the quantity for which marginal costs equal average total costs, i.e. where average total costs reach their minimum.

$$MC(q) = 4q = \frac{200}{q} + 2q = AC(q) \quad \Rightarrow \quad q = 10$$

Thus, the threshold price is

$$p = MC(10) = 40.$$

$\Rightarrow$ **(D)** is correct.