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Tutorial 6 (The no-cloning theorem[1])
The *no-cloning theorem* states that, surprisingly, one cannot make a copy of an unknown quantum state. In more detail, we consider a system of two qubits (source and target): the first qubit is the source state $|\psi\rangle$, and the second qubit starts out in some standard state $|s\rangle$, for example $|s\rangle = |0\rangle$. Thus the initial state is $|\psi\rangle \otimes |s\rangle \equiv |\psi\rangle\,|s\rangle$. One would like to copy $|\psi\rangle$ into $|s\rangle$, that is, find some unitary transformation $U \in \mathbb{C}^{4\times 4}$ such that

$$|\psi\rangle \otimes |s\rangle \mapsto U(|\psi\rangle \otimes |s\rangle) = |\psi\rangle \otimes |\psi\rangle\,.$$

Show that such a copying procedure is impossible: the equation cannot hold for arbitrary source qubits $|\psi\rangle$.

Solution Suppose the copying procedure works for two particular states $|\psi\rangle$ and $|\varphi\rangle$. Then

$$\begin{aligned} U(|\psi\rangle \otimes |s\rangle) &= |\psi\rangle \otimes |\psi\rangle\,, \\ U(|\varphi\rangle \otimes |s\rangle) &= |\varphi\rangle \otimes |\varphi\rangle\,. \end{aligned}$$

Taking inner products gives for the left sides

$$\Big\langle U(|\psi\rangle \otimes |s\rangle)\Big|U(|\varphi\rangle \otimes |s\rangle)\Big\rangle = (\langle\psi| \otimes \langle s|)U^\dagger U(|\varphi\rangle \otimes |s\rangle) = (\langle\psi| \otimes \langle s|)(|\varphi\rangle \otimes |s\rangle) = \langle\psi|\varphi\rangle\langle s|s\rangle = \langle\psi|\varphi\rangle$$

and for the right sides

$$\Big\langle\, |\psi\rangle \otimes |\psi\rangle \,\Big|\, |\varphi\rangle \otimes |\varphi\rangle \,\Big\rangle = \langle\psi|\varphi\rangle\langle\psi|\varphi\rangle = (\langle\psi|\varphi\rangle)^2,$$

leading to $\langle\psi|\varphi\rangle = (\langle\psi|\varphi\rangle)^2$. But this can only hold for $\langle\psi|\varphi\rangle = 1$ or $\langle\psi|\varphi\rangle = 0$, so either $|\psi\rangle = |\varphi\rangle$ or $|\psi\rangle$ and $|\varphi\rangle$ are orthogonal. In other words, general quantum cloning is impossible; for example, a potential quantum cloner cannot copy the qubit states $|\psi\rangle = |0\rangle$ and $|\varphi\rangle = (|0\rangle + |1\rangle)/\sqrt{2}$.

[1] M. A. Nielsen, I. L. Chuang: *Quantum Computation and Quantum Information*. Cambridge University Press (2010), page 532