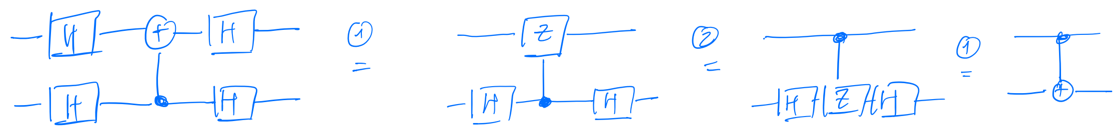
5.1 a) $|u_1\rangle = \frac{3}{5}|0\rangle + i\frac{4}{5}|1\rangle$ $\qquad |\psi\rangle = \frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle$

$|u_2\rangle = \frac{4}{5}|0\rangle - i\frac{3}{5}|1\rangle$

probability of u_1: $\langle u_1|\psi\rangle = u_1^{\dagger}\cdot\psi = \frac{3i}{5\sqrt{2}} - \frac{4i}{5\sqrt{2}} = \frac{-i}{5\sqrt{2}} \Rightarrow \frac{1}{50}$

probability of u_2: $\langle u_2|\psi\rangle = u_2^{\dagger}\cdot\psi = \frac{4i}{5\sqrt{2}} + \frac{3i}{5\sqrt{2}} = \frac{7i}{5\sqrt{2}} \Rightarrow \frac{49}{50}$

b)



① $HXH = Z$

② Z invariant to choice of control and target gate

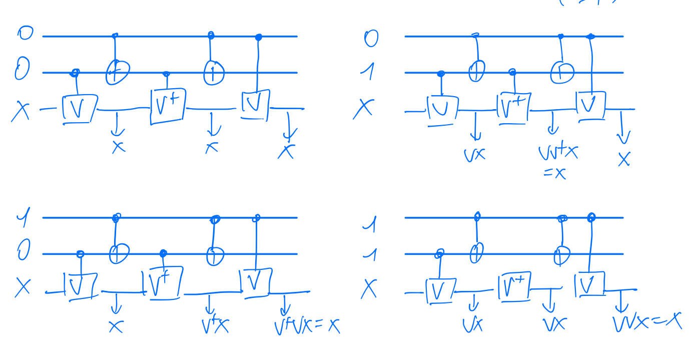
- $|++\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix} \otimes \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix} = \frac{1}{2}\left(|00\rangle+|01\rangle+|10\rangle+|11\rangle\right) \xrightarrow{CNOT} \frac{1}{2}\left(|00\rangle+|01\rangle+|11\rangle+|10\rangle\right) = |++\rangle$

- $|-+\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\end{pmatrix} \otimes \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix} = \frac{1}{2}\begin{pmatrix}1\\1\\-1\\-1\end{pmatrix} = \frac{1}{2}\left(|00\rangle+|01\rangle-|10\rangle-|11\rangle\right) \xrightarrow{CNOT}$ $\frac{1}{2}\left(|00\rangle+|01\rangle-|11\rangle-|10\rangle\right) = |-+\rangle$

- $|+-\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix} \otimes \frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\end{pmatrix} = \frac{1}{2}\begin{pmatrix}1\\-1\\1\\-1\end{pmatrix} = \frac{1}{2}\left(|00\rangle-|01\rangle+|10\rangle-|11\rangle\right) \xrightarrow{CNOT}$
$= \frac{1}{2}\left(|00\rangle-|01\rangle+|11\rangle-|10\rangle\right) = \frac{1}{2}\begin{pmatrix}1\\-1\\-1\\1\end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\end{pmatrix} \otimes \frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\end{pmatrix} =$
$|--\rangle$

- $|--\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\end{pmatrix} \otimes \frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\end{pmatrix} = \frac{1}{2}\begin{pmatrix}1\\-1\\-1\\1\end{pmatrix} = \frac{1}{2}\left(|00\rangle-|01\rangle+|11\rangle-|10\rangle\right) \xrightarrow{CNOT}$
$\frac{1}{2}\left(|00\rangle-|01\rangle+|10\rangle-|11\rangle\right) = \frac{1}{2}\begin{pmatrix}1\\-1\\1\\-1\end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix} \otimes \frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\end{pmatrix} = |+-\rangle$

5.2 a)



i) $VV^{\dagger} = V^{\dagger}V = I$

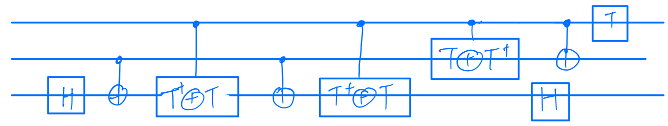
ii) $VV = U$

b) $X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ spectral composition of X (from previous exercise):

$$X = H \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} H$$

$$V^2 = X \Leftrightarrow V = \sqrt{X} = H \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix} H = \begin{pmatrix} 1/\sqrt{2} & \frac{1}{\sqrt{2}} i \\ 1/\sqrt{2} & -\frac{1}{\sqrt{2}} i \end{pmatrix} H = \begin{pmatrix} \frac{1+i}{2} & \frac{1-i}{2} \\ \frac{1-i}{2} & \frac{1+i}{2} \end{pmatrix}$$

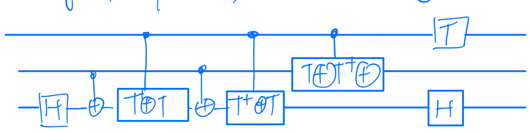
c) i) $TT^\dagger = T^\dagger T = I$. T is unitary, therefore we can simplify some gates:



$$T \oplus T^\dagger = \begin{pmatrix} 1 & 0 \\ 0 & e^{-i\pi/4} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & e^{i\pi/4} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ e^{-i\pi/4} & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & e^{i\pi/4} \end{pmatrix} = \begin{pmatrix} 0 & e^{i\pi/4} \\ e^{-i\pi/4} & 0 \end{pmatrix}$$

$$T^\dagger \oplus T = \begin{pmatrix} 1 & 0 \\ 0 & e^{i\pi/4} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & e^{-i\pi/4} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ e^{i\pi/4} & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & e^{-i\pi/4} \end{pmatrix} = \begin{pmatrix} 0 & e^{-i\pi/4} \\ e^{i\pi/4} & 0 \end{pmatrix}$$

ii) As the gates $T \oplus T^\dagger$ and $\oplus$ from the second qubit are all controlled by the first qubit, we can merge them



$$T \oplus T^\dagger \oplus = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & e^{i\pi/4} \\ e^{-i\pi/4} & 0 \end{pmatrix} = \begin{pmatrix} e^{-i\pi/4} & 0 \\ 0 & e^{i\pi/4} \end{pmatrix}$$

iii) We know that $(\oplus)^2$, $(H)^2$ and $(T^\dagger \oplus T)^2$ are equal to I, therefore we can say that the $\oplus T \oplus T^\dagger \oplus T^\dagger \oplus T$ from the last qubit is only relevant when both the first and second qubit are equal to 1:

$$H \oplus^b (T^\dagger \oplus T)^a \oplus^b (T^\dagger \oplus T)^a H$$

$a = b = 0$

$\Downarrow$

$HIIIIH$
$= HH = I$

$a = 0,\ b = 1$

$\Downarrow$

$H \oplus I \oplus I H$
$= H \oplus \oplus H$
$= HIH = HH = I$

$a = 1,\ b = 0$

$\Downarrow$

$HI(T^\dagger \oplus T) I (T^\dagger \oplus T) H$

$= H (T^\dagger \oplus T)^2 H = HIH = HH = I$

$a = 1,\ b = 1$

$\Downarrow$

$H \oplus T^\dagger \oplus T \oplus T^\dagger \oplus T H$

$= H \begin{pmatrix} 0 & e^{-\frac{i\pi}{4}} \\ e^{\frac{i\pi}{4}} & 0 \end{pmatrix} \oplus \begin{pmatrix} 0 & e^{-\frac{i\pi}{4}} \\ e^{\frac{i\pi}{4}} & 0 \end{pmatrix} \oplus H$

$= H \begin{pmatrix} e^{-\frac{i\pi}{4}} & 0 \\ 0 & e^{\frac{i\pi}{4}} \end{pmatrix} \begin{pmatrix} e^{-\frac{i\pi}{4}} & 0 \\ 0 & e^{\frac{i\pi}{4}} \end{pmatrix} H$

$= H \begin{pmatrix} e^{-\frac{i\pi}{2}} & 0 \\ 0 & e^{\frac{i\pi}{2}} \end{pmatrix} H = H \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix} H$

$= -iH \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} H = -iHZH \equiv$

$= HZH = X$

And for the T and $T \oplus T^\dagger \oplus$ gates acting on the first and second qubit it doesn't matter if it's activated or not as they doesn't change the probabilities of the qubit states. Therefore, we can conclude that this circuit acts as a CCNOT, applying Pauli-X when $a = 1$, $b = 1$