Chair of Scientific Computing in Computer Science
Department of Informatics
Technical University of Munich

Esolution
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Note:

- During the attendance check a sticker containing a unique code will be put on this exam.
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Introduction to Quantum Computing

Exam:	IN2381 / Final Exam	**Date:**	Thursday 2nd March, 2023
Examiner:	Prof. Dr. Christian Mendl	**Time:**	17:30 – 19:00

	P 1	P 2	P 3
I			

Working instructions

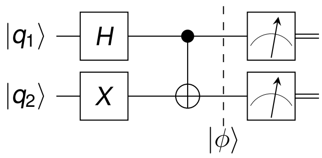
- This exam consists of **12 pages** with a total of **3 problems**.
  Please make sure now that you received a complete copy of the exam.
- The total amount of achievable credits in this exam is 60 credits.
- Detaching pages from the exam is prohibited.
- Allowed resources:
  - ein **A4 Blatt** (beidseitig) mit eigenen Notizen
  - ein **analoges Wörterbuch** Deutsch ↔ Muttersprache **ohne Anmerkungen**
- Subproblems marked by * can be solved without results of previous subproblems.
- **Answers are only accepted if the solution approach is documented.** Give a reason for each answer unless explicitly stated otherwise in the respective subproblem.
- Do not write with red or green colors nor use pencils.
- Physically turn off all electronic devices, put them into your bag and close the bag.

Left room from ________ to ________	/	Early submission at ________

Problem 1 (20 credits)

0 1 2 3

a) Consider the following circuit, where the initial state of q_1 and q_2 is $|0\rangle$ in all cases:



Explicitly determine the state $|\phi\rangle$ before the measurements. List the possible measurement outcomes and their respective probabilities.

0 1 2 3

b) Assume that the measurements occur sequentially and the first qubit q_1 is always measured first. In the case that the first qubit collapses to $|1\rangle$ due to the measurement, what are the possible outcomes when measuring the second qubit q_2? Draw the state of q_2 after measurement in the Bloch sphere below.

c)* Assume that an available quantum computer only supports R_x, R_y, R_z and S single-qubit gates. Decompose the Hadamard gate into appropriate rotation-gates up to a global phase constant. Also verify that your decomposition is correct. Hints: The decomposition can be achieved using only R_x and R_y gates, or R_y and R_z gates. You may use $\cos(\pi/4) = 1/\sqrt{2}$ and $\sin(\pi/4) = 1/\sqrt{2}$.

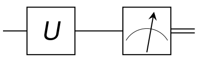
0
1
2
3
4
5

d)* Consider an experimental setup which prepares a (fixed but unknown) qubit state $|\psi\rangle$. We run the experiment for 1000 repetitions, and each time perform a standard basis measurement on the qubit. In 900 instances, we obtain the result 0, and in the remaining 100 instances the result 1.
Assuming that the measurement outcomes precisely reflect the underlying state, draw the possible locations of $|\psi\rangle$ on the Bloch sphere, and provide a mathematical expression for its polar angle θ.
Hint: If a continuum of states is possible, you can illustrate this by a curve or marking the area.

e)* Your task is to measure a qubit with respect to the orthonormal basis states

$$|a\rangle = \frac{1}{5}\begin{pmatrix} 3 \\ 4i \end{pmatrix} \quad \text{and} \quad |b\rangle = \frac{1}{5}\begin{pmatrix} 4 \\ -3i \end{pmatrix}.$$

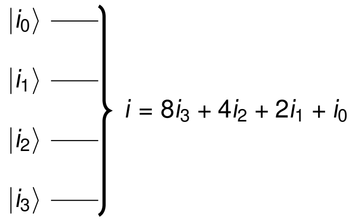
Since you only have a standard basis measurement setup available, you apply a gate U before the standard measurement:



Write down the matrix representation of the gate U to solve the task.

Problem 2 **(20 credits)**

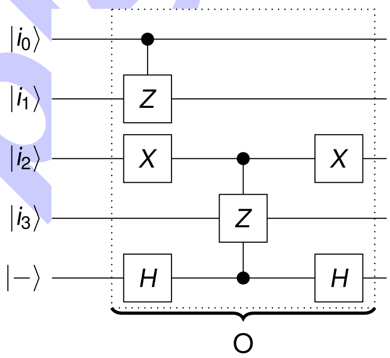
Assume that we have a quantum computer available which supports 5 qubits. We use a 4-qubit binary encoding of integers as computational basis states, i.e., for the state $|i_3 i_2 i_1 i_0\rangle$ (with $i_0, i_1, i_2, i_3 \in \{0, 1\}$):



a) Assuming that we intend to find a specific set of integers that can be expressed in this encoding. What is the search space of this problem? 0 1

b) Recalling Grover's algorithm, provide the definition of the oracle per the lecture and its effective action if the ancilla qubit is in the $|-\rangle$ state. 0 1 2

c) In the following circuit, an oracle O used for Grover's search implements the effective action described (b): 0 1 2 3 4 5 6 7 8



How many integer(s) does this oracle mark as a solution? Which integers are they? (Describing them in binary form is sufficient) Provide a clear justification or explicit derivation for your answer.

Hint: For this subproblem, you should only work in gate representation. There is no need to expand to matrix form in your working.

0
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4

d) In general, the circuit segment G in part (b) must be run multiple times before performing a measurement at the end to obtain a solution. Given the oracle in (b), how often must G be applied?

0
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2
3
4
5

e) We define a gate $P = e^{-i\pi|\psi\rangle\langle\psi|}$, where $|\psi\rangle$ is the equal superposition state. Show that P implements a reflection about the equal superposition state.

Problem 3 **(20 credits)**

a) If $H \in \mathbb{C}^{n\times n}$ is a Hermitian matrix, show that $U = e^{iH}$ is unitary.

b)* Let $\mathcal{E}$ be a quantum channel with Kraus operators $\{E_k\}_{k=1}^{n}$. Provide a Kraus decomposition of the channel $\mathcal{F}$ that first applies a unitary gate V and then the channel $\mathcal{E}$.

c)* Consider a single-qubit quantum system described by a density matrix ρ, which yields the following expectation values of the Pauli matrices:

$$\langle X\rangle = \mathrm{tr}[\rho X] = 0, \quad \langle Y\rangle = \mathrm{tr}[\rho Y] = \frac{3}{5}, \quad \langle Z\rangle = \mathrm{tr}[\rho Z] = -\frac{1}{4}.$$

Compute the matrix representation of ρ. Hint: start from the Bloch vector of ρ.

0 1 2 3 4

d)* Let ρ_A be the density operator of some quantum system A. Based on the spectral decomposition of ρ_A, construct a pure state $|\psi\rangle$ on an extended quantum system AR such that $\rho_A = \mathrm{tr}_R(|\psi\rangle\langle\psi|)$.

0 1 2 3 4 5 6

e)* Consider the quantum phase flip operation

$$\rho \mapsto \mathcal{E}(\rho) = \sum_{k=0}^{1} E_k \rho E_k^\dagger \quad \text{with} \quad E_0 = \sqrt{1-p}\,I, \quad E_1 = \sqrt{p}\,Z$$

for a real parameter $p \in [0, 1]$. Design a circuit that performs this operation.

Hint: A possible circuit consists of a single qubit wire for the principal system described by ρ, and another qubit wire for the environment initialized to $|0\rangle$. The rotation operator

$$R_y(\theta) = \begin{pmatrix} \cos\frac{\theta}{2} & -\sin\frac{\theta}{2} \\ \sin\frac{\theta}{2} & \cos\frac{\theta}{2} \end{pmatrix}$$

might be helpful for encoding the probability p.

Additional space for solutions–clearly mark the (sub)problem your answers are related to and strike out invalid solutions.