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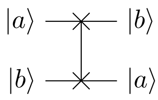
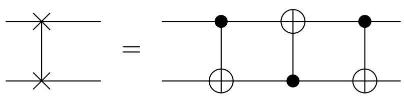
Exercise 4.1 (Basic quantum circuits)

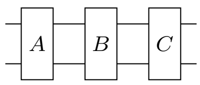
(a) Find the matrix representation (with respect to the computational basis states $|00\rangle$, $|01\rangle$, $|10\rangle$, $|11\rangle$) of the *swap-gate* $|a,b\rangle \mapsto |b,a\rangle$, which is written in circuit form as



Also show that the swap operation is equivalent to the following sequence of three CNOT gates:

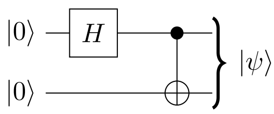


Hint: You can either work directly with basis states, e.g. $|a,b\rangle \overset{\text{CNOT}}{\mapsto} |a, a\oplus b\rangle$, or use matrix representations. In the latter case, note that a sequence of gates like



(with A, B, C unitary 4×4 matrices) corresponds to the matrix product CBA since the circuit is read from left to right, but the input vector in the matrix representation is multiplied from the right.

(b) Compute the output $|\psi\rangle$ of the following "entanglement circuit" applied to the input $|00\rangle$:



with $H = \frac{1}{\sqrt{2}}\left(\begin{smallmatrix} 1 & 1 \\ 1 & -1 \end{smallmatrix}\right)$ denoting the Hadamard gate.

(c) Build the CNOT gate from the controlled-Z gate and two Hadamard gates, and verify your construction.

Solution

(a) The swap gate interchanges $|01\rangle \leftrightarrow |10\rangle$, thus

$$U_{\text{swap}} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Concerning the sequence of three CNOT-gates:

$$|a,b\rangle \overset{\text{CNOT}}{\mapsto} |a, a\oplus b\rangle \overset{\text{flipped CNOT}}{\mapsto} |a \oplus (a\oplus b), a\oplus b\rangle = |b, a\oplus b\rangle \overset{\text{CNOT}}{\mapsto} |b, (a\oplus b)\oplus b\rangle = |b,a\rangle\,.$$

Here we have used that $a \oplus a = 0$ for $a \in \{0,1\}$.

Alternative solution using matrix representations:

$$U_{\text{CNOT}} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad U_{\text{flipped CNOT}} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix},$$

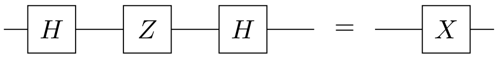
which gives as matrix representation of the three CNOT-gates:

$$U_{\mathsf{CNOT}} \cdot U_{\text{flipped CNOT}} \cdot U_{\mathsf{CNOT}} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = U_{\text{swap}}.$$

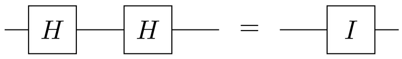
(b)

$$|00\rangle \overset{H\otimes I}{\mapsto} \frac{|0\rangle + |1\rangle}{\sqrt{2}} |0\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |10\rangle) \overset{\mathsf{CNOT}}{\mapsto} \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) = |\psi\rangle$$

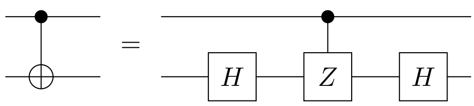
(c) Based on the matrix representation of the Hadamard gate and Pauli matrices, one directly verifies that $HZH = X$ and $H^2 = I$ (identity operation). Expressed in circuit form:



and



These properties lead to the following identity:



The circuit on the right is the requested construction.