3.1

a) $\{\sigma_1,\sigma_2\} = \sigma_1\sigma_2+\sigma_2\sigma_1 = \begin{pmatrix}0&1\\1&0\end{pmatrix}\begin{pmatrix}0&-i\\i&0\end{pmatrix} + \begin{pmatrix}0&-i\\i&0\end{pmatrix}\begin{pmatrix}0&1\\1&0\end{pmatrix} = \begin{pmatrix}i&0\\0&-i\end{pmatrix} + \begin{pmatrix}-i&0\\0&i\end{pmatrix} = \begin{pmatrix}0&0\\0&0\end{pmatrix}$

$\{\sigma_2,\sigma_3\} = \sigma_2\sigma_3+\sigma_3\sigma_2 = \begin{pmatrix}0&-i\\i&0\end{pmatrix}\begin{pmatrix}1&0\\0&-1\end{pmatrix} + \begin{pmatrix}1&0\\0&-1\end{pmatrix}\begin{pmatrix}0&-i\\i&0\end{pmatrix} = \begin{pmatrix}0&i\\i&0\end{pmatrix} + \begin{pmatrix}0&-i\\-i&0\end{pmatrix} = \begin{pmatrix}0&0\\0&0\end{pmatrix}$

$\{\sigma_3,\sigma_1\} = \sigma_3\sigma_1+\sigma_1\sigma_3 = \begin{pmatrix}1&0\\0&-1\end{pmatrix}\begin{pmatrix}0&1\\1&0\end{pmatrix} + \begin{pmatrix}0&1\\1&0\end{pmatrix}\begin{pmatrix}1&0\\0&-1\end{pmatrix} = \begin{pmatrix}0&1\\-1&0\end{pmatrix} + \begin{pmatrix}0&-1\\1&0\end{pmatrix} = \begin{pmatrix}0&0\\0&0\end{pmatrix}$

b) $[\sigma_1,\sigma_2] = \sigma_1\sigma_2-\sigma_2\sigma_1 = \begin{pmatrix}i&0\\0&-i\end{pmatrix} - \begin{pmatrix}-i&0\\0&i\end{pmatrix} = 2i\begin{pmatrix}1&0\\0&-1\end{pmatrix} = 2i\sigma_3$

$[\sigma_2,\sigma_3] = \sigma_2\sigma_3-\sigma_3\sigma_2 = \begin{pmatrix}0&i\\i&0\end{pmatrix} - \begin{pmatrix}0&-i\\-i&0\end{pmatrix} = 2i\begin{pmatrix}0&1\\1&0\end{pmatrix} = 2i\sigma_1$

$[\sigma_3,\sigma_1] = \sigma_3\sigma_1-\sigma_1\sigma_3 = \begin{pmatrix}0&1\\-1&0\end{pmatrix} - \begin{pmatrix}0&-1\\1&0\end{pmatrix} = 2\begin{pmatrix}0&1\\-1&0\end{pmatrix} = 2i\begin{pmatrix}0&-i\\i&0\end{pmatrix} = 2i\sigma_2$

c) $e^{U^\dagger AU} = \sum_{k=0}^{\infty}\frac{(U^\dagger AU)^k}{k!} \overset{①}{=} \sum_{k=0}^{\infty}\frac{U^\dagger A^k U}{k!} = U^\dagger\left(\sum_{k=0}^{\infty}\frac{A^k}{k!}\right)U = U^\dagger e^A U$

① Proof by induction: $(U^\dagger AU)^k = U^\dagger A^k U$

Base case $(k=0)$:

$(U^\dagger AU)^0 = I$

$U^\dagger A^0 U = U^\dagger I U = U^\dagger U = I$

Inductive step (proof $k+1$ when know the case for k):

$(U^\dagger AU)^{k+1} = (U^\dagger AU)^k(U^\dagger AU) = U^\dagger A^k \overset{\text{Unitary}}{\cancel{UU^\dagger}} AU = U^\dagger A^{k+1}U$

d) $HXH = \frac{1}{\sqrt2}\begin{pmatrix}1&1\\1&-1\end{pmatrix}\begin{pmatrix}0&1\\1&0\end{pmatrix}\frac{1}{\sqrt2}\begin{pmatrix}1&1\\1&-1\end{pmatrix} = \frac12\begin{pmatrix}1&1\\-1&1\end{pmatrix}\begin{pmatrix}1&1\\1&-1\end{pmatrix} = \frac12\begin{pmatrix}2&0\\0&-2\end{pmatrix} = \begin{pmatrix}1&0\\0&-1\end{pmatrix} = Z$

$HZH = \frac{1}{\sqrt2}\begin{pmatrix}1&1\\1&-1\end{pmatrix}\begin{pmatrix}1&0\\0&-1\end{pmatrix}\frac{1}{\sqrt2}\begin{pmatrix}1&1\\1&-1\end{pmatrix} = \frac12\begin{pmatrix}1&-1\\1&1\end{pmatrix}\begin{pmatrix}1&1\\1&-1\end{pmatrix} = \frac12\begin{pmatrix}0&2\\2&0\end{pmatrix} = \begin{pmatrix}0&1\\1&0\end{pmatrix} = X$

e) $HR_x(\theta)H = He^{-i\frac{\theta}{2}X}H \overset{①}{=} H^\dagger e^{-i\frac{\theta}{2}X}H \overset{②}{=} e^{H^\dagger(-\frac{i\theta}{2}X)H} = e^{-\frac{i\theta}{2}H^\dagger XH} \overset{③}{=} e^{-\frac{i\theta}{2}Z} = R_z(\theta)$

① H is hermitian
② showed in part c (H is self-inverse)
③ showed in part d

3.2

a) • $X = \begin{pmatrix}0&1\\1&0\end{pmatrix}$ $\quad |\lambda I - X| = \begin{vmatrix}\lambda&-1\\-1&\lambda\end{vmatrix} = \lambda^2-1 \overset{!}{=} 0 \Rightarrow \lambda = \pm1$

$\lambda_1 = 1 \Rightarrow Xv_1 = \lambda_1 v_1 = v_1$: $\begin{pmatrix}0&1\\1&0\end{pmatrix}\begin{pmatrix}a\\b\end{pmatrix} = \begin{pmatrix}a\\b\end{pmatrix} \Leftrightarrow \begin{pmatrix}b\\a\end{pmatrix} = \begin{pmatrix}a\\b\end{pmatrix}$

with the normalization constraint we have $1 = \sqrt{2a^2} \Rightarrow a = b = \frac{1}{\sqrt2} \Rightarrow v_1 = \begin{pmatrix}\frac{1}{\sqrt2}\\\frac{1}{\sqrt2}\end{pmatrix}$

$\lambda_2 = -1 \Rightarrow Xv_2 = \lambda_2 v_2 = -v_2$: $\begin{pmatrix}0&1\\1&0\end{pmatrix}\begin{pmatrix}a\\b\end{pmatrix} = -\begin{pmatrix}a\\b\end{pmatrix} \Leftrightarrow \begin{pmatrix}b\\a\end{pmatrix} = -\begin{pmatrix}a\\b\end{pmatrix} \rightarrow v_2 = \begin{pmatrix}\frac{1}{\sqrt2}\\-\frac{1}{\sqrt2}\end{pmatrix}$

• $Y = \begin{pmatrix}0&-i\\i&0\end{pmatrix}$ $\quad |\lambda I - Y| = \begin{vmatrix}\lambda&i\\-i&\lambda\end{vmatrix} = \lambda^2+i^2 = \lambda^2-1 \overset{!}{=} 0 \Rightarrow \lambda = \pm1$

$\lambda_1 = 1 \Rightarrow Yv_1 = v_1$: $\begin{pmatrix}0&-i\\i&0\end{pmatrix}\begin{pmatrix}a\\b\end{pmatrix} = \begin{pmatrix}a\\b\end{pmatrix} \Leftrightarrow \begin{pmatrix}-bi\\ai\end{pmatrix} = \begin{pmatrix}a\\b\end{pmatrix} \Leftrightarrow v_1 = \begin{pmatrix}\frac{1}{\sqrt2}\\\frac{1}{\sqrt2}i\end{pmatrix}$

$$\lambda_2 = -1 \Rightarrow Y v_2 = -v_2 : \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = -\begin{pmatrix} a \\ b \end{pmatrix} \Leftrightarrow \begin{pmatrix} -bi \\ ai \end{pmatrix} = -\begin{pmatrix} a \\ b \end{pmatrix} \Leftrightarrow v_2 = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} i \end{pmatrix}$$

- $Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ $\quad |\lambda I - Z| = \begin{vmatrix} \lambda-1 & 0 \\ 0 & \lambda+1 \end{vmatrix} = \lambda^2 - 1 \overset{!}{=} 0 \Rightarrow \lambda = \pm 1$

$$\lambda_1 = 1 \Rightarrow Z v_1 = v_1 : \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix} \Leftrightarrow \begin{pmatrix} a \\ -b \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix} \Rightarrow v_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$\lambda_2 = -1 \Rightarrow Z v_1 = -v_1 : \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = -\begin{pmatrix} a \\ b \end{pmatrix} \Leftrightarrow \begin{pmatrix} a \\ -b \end{pmatrix} = \begin{pmatrix} -a \\ -b \end{pmatrix} \Rightarrow v_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

b) $H|0\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix} \Rightarrow$ both probs are $\left(\frac{1}{\sqrt{2}}\right)^2 = \frac{1}{2}$.

Similary we can see that with the eigenvectors of X:

$|0\rangle = (|+\rangle + |-\rangle)\frac{1}{\sqrt{2}}$ $\quad |+\rangle = v_1$, $|-\rangle = v_2$ eigenvectors of pauli-x

This matches the second figure, after exiting the X magnetic field, there are $\frac{1}{2}$ prob for $|+\rangle$ and $|-\rangle$

c) Since it corresponds to a standard Z-axis measurement and the output state of the previous stage consists of $|+\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$, then the output probability of the third stage will be $1/2$ for both $|\uparrow\rangle$ and $|\downarrow\rangle$.

If we replace it with x-axis measurement, then the output from last stage remains unchanged, as it's just a repeated consequent measurement.

In conclusion we can say that measuring the spin over an axis somehow "erases" the information over the previous axes.