Professorship for Quantum Computing
Department of Informatics
Technical University of Munich

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Introduction to Quantum Computing

Exam:	IN2381 / Retake	**Date:**	Friday 9th October, 2020
Examiner:	Prof. Dr. Christian Mendl	**Time:**	16:15 – 17:45

	P 1	P 2	P 3
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Working instructions

- This exam consists of **8 pages** with a total of **3 problems**.
  Please make sure now that you received a complete copy of the exam.
- The total amount of achievable credits in this exam is 60 credits.
- Detaching pages from the exam is prohibited.
- Allowed resources: one **A4 sheet** (both sides) with your own notes
- Subproblems marked by * can be solved without results of previous subproblems.
- **Answers are only accepted if the solution approach is documented.** Give a reason for each answer unless explicitly stated otherwise in the respective subproblem.
- Do not write with red or green colors nor use pencils.
- Physically turn off all electronic devices, put them into your bag and close the bag.

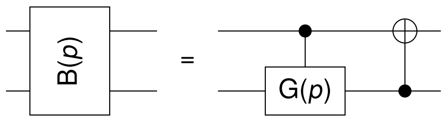
Left room from __________ to __________ / Early submission at __________

Problem 1 (20 credits)

In the following, we consider the so-called three-qubit W-State $|W_3\rangle$:

$$|W_3\rangle = \frac{1}{\sqrt{3}}\left(|100\rangle + |010\rangle + |001\rangle\right).$$

We also define a two-qubit gate $B(p)$, for a real parameter $p \in [0, 1]$:



where

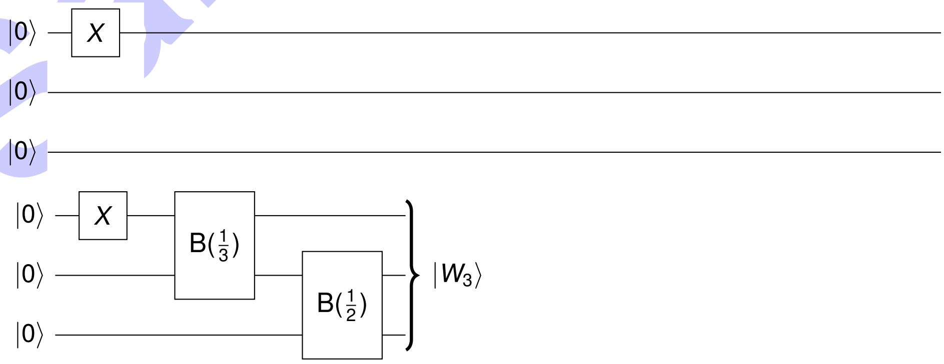
$$G(p) = \begin{pmatrix} \sqrt{p} & -\sqrt{1-p} \\ \sqrt{1-p} & \sqrt{p} \end{pmatrix}.$$

a) Determine the outputs of $B(p)$ for the four input basis states $|00\rangle$, $|01\rangle$, $|10\rangle$ and $|11\rangle$

$$|00\rangle \xrightarrow{\text{no change}} |00\rangle$$
$$|01\rangle \xrightarrow{G(p)} |01\rangle \xrightarrow{\text{CNOT}} |11\rangle$$
$$|10\rangle \xrightarrow{G(p)} \sqrt{p}\,|10\rangle + \sqrt{1-p}\,|11\rangle \xrightarrow{\text{CNOT}} \sqrt{p}\,|10\rangle + \sqrt{1-p}\,|01\rangle$$
$$|11\rangle \xrightarrow{G(p)} -\sqrt{1-p}\,|10\rangle + \sqrt{p}\,|11\rangle \xrightarrow{\text{CNOT}} \sqrt{1-p}\,|10\rangle + \sqrt{p}\,|01\rangle$$

b) Add $B(p)$ gates with suitably chosen values of p to the following circuit such that the output is $|W_3\rangle$, and verify your construction.

Hint: Two $B(p)$ gates are sufficient.



$$X_1 |000\rangle \to |100\rangle$$
$$B_{1,2}(\tfrac{1}{3}) \left(|100\rangle\right) \to \tfrac{1}{\sqrt{3}} |100\rangle + \tfrac{2}{\sqrt{3}} |010\rangle$$
$$B_{2,3}(\tfrac{1}{2}) \left(\tfrac{1}{\sqrt{3}} |100\rangle + \sqrt{\tfrac{2}{3}} |010\rangle \right) \to \tfrac{1}{\sqrt{3}} |100\rangle + \sqrt{\tfrac{2}{3}\tfrac{1}{2}} |010\rangle + \sqrt{\tfrac{2}{3}\tfrac{1}{2}} |001\rangle = |W_3\rangle$$

c)* Demonstrate that a single-qubit measurement (with respect to some arbitrary orthonormal basis) performed on the first qubit of $|W_3\rangle$ leaves the remaining two qubits entangled in general.
Hint: Describe the measurement as a projection $|\psi\rangle \langle\psi|$, where $|\psi\rangle = \alpha |0\rangle + \beta |1\rangle$ is a general single qubit state.

0 1 2 3 4 5 6

Based on the POVM formalism, the single qubit measurement can be represented by a projection $|\psi\rangle \langle\psi|$. Since $|W_3\rangle$ is invariant under a permutation of qubits, we can without loss of generality assume that the measurement is performed on the first qubit. The quantum state after the measurement (up to normalization) is then

$$|W_3'\rangle = |\psi\rangle_1 \langle\psi|_1 W_3\rangle \quad \text{with} \quad \langle\psi|_1 W_3\rangle = \frac{\alpha^*}{\sqrt{3}} |01\rangle + \frac{\alpha^*}{\sqrt{3}} |10\rangle + \frac{\beta^*}{\sqrt{3}} |00\rangle =: |\tilde{W}_2\rangle\,.$$

We now demonstrate that $|\tilde{W}_2\rangle$ is entangled since it cannot be represented as the tensor product of two single qubit states. Let $|\phi_0\rangle = a|0\rangle + b|1\rangle$ and $|\phi_1\rangle = c|0\rangle + d|1\rangle$, $a, b, c, d \in \mathbb{C}$ denote two arbitrary states. The hypothetical representation

$$|\tilde{W}_2\rangle \propto |\phi_0\rangle \otimes |\phi_1\rangle$$

requires that the following equations are satisfied (for some scaling factor $\tau > 0$):

$$ac = \frac{\beta^*\tau}{\sqrt{3}}$$
$$ad = \frac{\alpha^*\tau}{\sqrt{3}}$$
$$bc = \frac{\alpha^*\tau}{\sqrt{3}}$$
$$bd = 0$$

These equations do not (in general) have a solution. Hence, $|\tilde{W}_2\rangle$ is an entangled state.

Problem 2 (20 credits)

A quantum operation, $\mathcal{E}$, describes in general terms how a quantum system evolves. It can be expressed using the operator-sum representation as

$$\mathcal{E}(\rho) = \sum_k E_k \rho E_k^\dagger,$$

where ρ is the density matrix of the system and the complex matrices E_k are the so-called Kraus operators.

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a) What condition must the Kraus operators satisfy such that $\mathcal{E}$ is compatible with the laws of quantum mechanics?

$$\sum_k E_k^\dagger E_k \leq I$$

0 1 2 3 4 5 6 7 8

b)* An example of a quantum operation is the depolarizing channel, which models quantum noise. Its Kraus operators (for a real parameter $p \in [0, \frac{3}{4}]$) are

$$E_0 = \sqrt{1-p}I, \quad E_1 = \sqrt{p/3}X, \quad E_2 = \sqrt{p/3}Y \quad \text{and} \quad E_3 = \sqrt{p/3}Z.$$

Compute $\mathcal{E}(\rho)$ using the Bloch sphere representation of the density matrix, $\rho = (I+\vec{r}\cdot\vec{\sigma})/2$ with $\vec{r} \in \mathbb{R}^3$, $\|\vec{r}\| \leq 1$, to show that the depolarizing channel acts as a contraction of the Bloch vector $\vec{r}$. You may use without proof that

$$\begin{aligned} XY &= -YX = iZ \\ YZ &= -ZY = iX \\ ZX &= -XZ = iY. \end{aligned}$$

We first compute the action of each Kraus operator:

$$\begin{aligned} E_0 \rho E_0^\dagger &= (1-p)\rho, \\ E_1 \rho E_1^\dagger &= \frac{p}{6}(I + r_1 X - r_2 Y - r_3 Z), \\ E_2 \rho E_2^\dagger &= \frac{p}{6}(I - r_1 X + r_2 Y - r_3 Z), \\ E_3 \rho E_3^\dagger &= \frac{p}{6}(I - r_1 X - r_2 Y + r_3 Z). \end{aligned}$$

Then we calculate the sum to obtain the output of the channel:

$$\begin{aligned} \mathcal{E}(\rho) &= (1-p)\rho + \frac{p}{6}\left(3I - \vec{r}\cdot\vec{\sigma}\right) \\ &= \frac{1}{2}(I+\vec{r}\cdot\vec{\sigma}) - \frac{p}{2}(I+\vec{r}\cdot\vec{\sigma}) + \frac{p}{2}\left(I - \frac{1}{3}\vec{r}\cdot\vec{\sigma}\right) \\ &= \frac{1}{2}(I+\vec{r}\cdot\vec{\sigma}) - \frac{1}{2}\frac{4p}{3}(\vec{r}\cdot\vec{\sigma}) \\ &= \frac{1}{2}\left(I + \left(1 - \frac{4p}{3}\right)\vec{r}\cdot\vec{\sigma}\right). \end{aligned}$$

Thus the Bloch vector parametrizing $\mathcal{E}(\rho)$ is

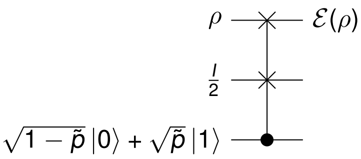
$$\vec{r}' = \left(1 - \frac{4p}{3}\right)\vec{r},$$

which is a contracted (scaled) version of $\vec{r}$.

c) The depolarizing channel can also be written as

$$\mathcal{E}(\rho) = \tilde{p}\frac{I}{2} + (1 - \tilde{p})\rho$$

with $\tilde{p} = \frac{4}{3}p \in [0, 1]$. A corresponding circuit implementation is



where the middle and bottom wires serve as environment. Describe how this circuit realizes the depolarizing channel.

As intuitive explanation, the circuit swaps the identity matrix for the input density matrix ρ with probability $\tilde{p}$, which matches the above formula for the channel. The probabilistic swapping stems from the control by the bottom wire.

d)* The "purity" of a quantum state with density matrix ρ is defined as $\mathrm{Tr}(\rho^2)$. Compute the minimum purity of a single qubit quantum state.

Hint: You might recall an expression for $\mathrm{Tr}(\rho^2)$ in terms of the Bloch vector of ρ. Alternatively, why can we assume that ρ is a diagonal matrix for evaluating $\mathrm{Tr}(\rho^2)$?

The mentioned expression (see exercise 7.2) is

$$\mathrm{Tr}\left(\rho^2\right) = \frac{1}{2}\left(1 + \|\vec{r}\|^2\right),$$

where $\vec{r} \in \mathbb{R}^3$, $\|\vec{r}\| \leq 1$ is the Bloch vector of ρ. In particular, the minimum is $\frac{1}{2}$, obtained for $\vec{r} = 0$. (Note that in this case ρ is proportional to the identity matrix.)

Alternative solution: A unitary transformation $\rho \to U\rho U^\dagger$ does not change the purity, since

$$\mathrm{Tr}\left((U\rho U^\dagger)^2\right) = \mathrm{Tr}\left(U\rho U^\dagger U\rho U^\dagger\right) = \mathrm{Tr}\left(U^\dagger U\rho^2\right) = \mathrm{Tr}\left(\rho^2\right).$$

(Here we have used the cyclic invariance of the trace.) Thus we can diagonalize ρ without affecting the purity, in other words, without loss of generality assume that ρ is a diagonal matrix.
Since $\mathrm{Tr}(\rho) = 1$, we may represent ρ as

$$\rho = \frac{1}{2}\begin{pmatrix} 1+\lambda & 0 \\ 0 & 1-\lambda \end{pmatrix}, \quad \lambda \in [-1, 1].$$

Thus

$$\mathrm{Tr}\left(\rho^2\right) = \frac{1}{4}\left((1+\lambda)^2 + (1-\lambda)^2\right) = \frac{1}{4}\left(2 + 2\lambda^2\right),$$

which obtains the minimum value $\frac{1}{2}$ for $\lambda = 0$.

Problem 3 **(20 credits)**

We consider a quantum system of n qubits, and use the notation X_j, Y_j, Z_j to denote that one of the Pauli matrices acts on the jth qubit; e.g., $X_1Z_3 \equiv X \otimes I \otimes Z$ for $n = 3$.
Conjugation by U refers to the transformation $UgU^\dagger$ of a quantum gate g by a unitary operation U. The following table summarizes several conjugation transformations:

U	X	X	X	H	H	H	S	S	S
g	X	Y	Z	X	Y	Z	X	Y	Z
$UgU^\dagger$	X	$-Y$	$-Z$	Z	$-Y$	X	Y	$-X$	Z

Here $S = \left(\begin{smallmatrix} 1 & 0 \\ 0 & i \end{smallmatrix}\right)$ is the phase gate.

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a) State the check matrix representation of $g_1, g_2 \in G_4$ given by

$$g_1 = Y \otimes I \otimes Z \otimes X,$$
$$g_2 = X \otimes Z \otimes Y \otimes X.$$

Based on this representation, show that g_1 commutes with g_2.

The check matrix of (g_1, g_2) is

$$\begin{pmatrix} r(g_1) \\ r(g_2) \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 1 & 0 & 1 & 1 & 0 \end{pmatrix}.$$

According to exercise 13.2 (e), g_1 commutes with g_2 precisely if $r(g_1)\Lambda r(g_2)^T = 0 \mod 2$, with

$$\Lambda = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}.$$

Λ effectively interchanges the first with the second half of a check row. We obtain

$$r(g_1)\Lambda r(g_2)^T = \begin{pmatrix} 1 & 0 & 0 & 1 & 1 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & 1 & 0 & 1 & 0 & 1 & 1 \end{pmatrix}^T = 2 = 0 \mod 2.$$

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b)* Specify an element g of the Pauli group G_4 such that

$$R = \langle X_1Y_3, Y_1X_2Z_3Y_4, g \rangle$$

stabilizes a non-trivial vector space and the three generators of R are independent. Also state the properties which g must satisfy (a proof of them is not required).

We need to choose $g \in G_4$, $g \neq I$, such that it commutes with the other two generators and cannot be written as product of them. For example $g = X_2Y_4$, $g = Y_2X_4$ or $g = Z_2Z_4$ has this properties, since it clearly commutes with X_1Y_3 (acting on different qubits) and also with $Y_1X_2Z_3Y_4$. The product of the other two generators acts non-trivially on all four qubits, and is thus different from g. (Other choices of g possible as well.)

c)* We consider the subgroup

$$T = \langle X_1 Z_2 Y_3, Y_1 Z_2 X_3 \rangle$$

of the Pauli group G_3. Show that the vector space V_T stabilized by T is invariant under multiplication by $(S \otimes X \otimes S)$ (with S the phase gate), in other words, $\psi \in V_T$ if and only if $(S \otimes X \otimes S)\psi \in V_T$.

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In general, multiplying the elements of V_T by a unitary matrix U results in the subspace $V_{T'}$ stabilized by T', with

$$T' := \left\{ U g U^\dagger \,|\, g \in T \right\}.$$

Namely, $\psi \in V_T$ precisely if $g\psi = \psi$ for all $g \in T$, which is equivalent to $(UgU^\dagger)U\psi = U\psi$ for all $g \in T$, i.e., $U\psi \in T'$. In the present case $U = (S \otimes X \otimes S)$.
Now we compute $UgU^\dagger$ for the generators of T, using the above conjugation table:

$$\begin{aligned}&(S \otimes X \otimes S)(X_1 Z_2 Y_3)(S \otimes X \otimes S)^\dagger \equiv (S \otimes X \otimes S)(X \otimes Z \otimes Y)(S \otimes X \otimes S)^\dagger \\ &= (SXS^\dagger) \otimes (XZX) \otimes (SYS^\dagger) = Y \otimes (-Z) \otimes (-X) = Y \otimes Z \otimes X \equiv Y_1 Z_2 X_3,\end{aligned}$$

and similarly

$$(S \otimes X \otimes S)(Y_1 Z_2 X_3)(S \otimes X \otimes S)^\dagger = X_1 Z_2 Y_3.$$

Since the two generators are mapped to each other, one concludes that $T' = T$ and in particular $V_{T'} = V_T$.

Alternative solution: We first compute V_T explicitly: $V_T = \text{span}\{|\chi_0\rangle, |\chi_1\rangle\}$ with

$$\begin{aligned}|\chi_0\rangle &= (|000\rangle + i\,|101\rangle)/\sqrt{2}, \\ |\chi_1\rangle &= (|010\rangle - i\,|111\rangle)/\sqrt{2}.\end{aligned}$$

Then observe that $(S \otimes X \otimes S)\,|\chi_0\rangle = |\chi_1\rangle$ and $(S \otimes X \otimes S)\,|\chi_1\rangle = |\chi_0\rangle$.

d)* We consider the three qubit bit flip code $C = \text{span}\{|0_L\rangle, |1_L\rangle\} = \text{span}\{|000\rangle, |111\rangle\}$, affected by amplitude damping noise on the first qubit. Recall that the operator-sum representation of the amplitude damping quantum channel is given by

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$$\mathcal{E}_{\text{AD}}(\rho) = E_0 \rho E_0^\dagger + E_1 \rho E_1^\dagger \quad \text{with} \quad E_0 = \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{1-\gamma} \end{pmatrix}, \quad E_1 = \begin{pmatrix} 0 & \sqrt{\gamma} \\ 0 & 0 \end{pmatrix},$$

and a real parameter $\gamma \in (0, 1)$. Show that this noise process acting on C is not error-correctable.

We make use of the quantum error-correction conditions (see lecture). The projector onto C in the present case is $P = |000\rangle\langle 000| + |111\rangle\langle 111|$. The code is error-correctable precisely if

$$P E_k^\dagger E_\ell P = \alpha_{k\ell} P$$

for all k, ℓ and some Hermitian matrix $(\alpha_{k\ell})$ of complex numbers, where the operation elements $\{E_k\}$ act on the first qubit in the present setting. We compute

$$E_1 P = (E_1\,|000\rangle)\,\langle 000| + (E_1\,|111\rangle)\,\langle 111| = \sqrt{\gamma}\,|011\rangle\,\langle 111|$$

and thus

$$P E_1^\dagger E_1 P = (E_1 P)^\dagger (E_1 P) = \gamma\,|111\rangle\,\langle 111|.$$

This cannot be written in the form $\alpha_{11} P$ for any $\alpha_{11} \in \mathbb{C}$.

Additional space for solutions–clearly mark the (sub)problem your answers are related to and strike out invalid solutions.