Notes on Kostrikin's Introduction to Alegbra

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1.1 Vector Space

Definition 1.1 (Vector)**.** A vector is a nth-tuple denoted by $(x_1, x_2, \cdots, x_n)$ which is element of $\mathbb{R}^n$.

Definition 1.2 (Vector Space)**.** A triple (V, +, ·) satisfies the following Axioms is called a **vector space** over field K:

1. X + Y = Y + X for all X, Y $\in$ V
2. (X + Y) + Z = X + (Y + Z)
3. $\exists$ 0 $\in$ V s.t. $\forall$ X $\in$ V we have X + 0 = X
4. $\exists$ a negative element of X denoted by -X s.t. X + (-X) = 0
5. 1 $\in$ K s.t. 1X=X for all vector X
6. $\forall$ α, β $\in$ K, ($\alpha\beta$)X = α(βX)
7. (α + β) X = α X + β X
8. α(X + Y) = α X + α Y

And we denote the culomn vector: $\begin{pmatrix} x_1 \\ x_2 \\ \cdots \\ x_n \end{pmatrix} = [x\_1, x\_2, \cdots,$ x_n]

1.1.1 Linear Span

Definition 1.3 (Linear combinition)**.** We say vector X is a linear conbination of vectors $X_1, \cdots, X_n$ if there is some numbers $k_1, \cdots, k_n \in K$ s.t.

$$X = \sum_{i=1}^{n} k_i X_i$$

And use this definition, it's easy to say that if V is the set of all combinitions of vectors $X_1, \cdots, X_n$ we have $\forall X, Y \in V \implies \alpha_1 X + \alpha_2 Y \in V$.

It's obvious that vector 0 is always in V.

The V is called the linear span of vectors $X_1, , X_n$, and denoted by $\langle X_1, \dots, X_n \rangle$.

Then we can define the *linear span of any subset $S \in \mathbb{R}^n$, $\langle S \rangle$, the linear conbination of any finite vectors in S. It's obviously that if V is a linear span, then $\langle V \rangle = V$.

Example 1.1. Let:

$$U_m = (\lambda_1, \cdots, \lambda_m, 0, \dots, 0)$$

Obviously, it's a linear span of vectors

$$e_1, \dots, e_m$$

1.1.2 Linear dependent and linear independent

Definition 1.4 (Linear dependent and linear independent)**.** If there is some numbers k_1, …, k_n which is not all equals to 0 s.t.

$$k_1 X_1 + \cdots + k_1 X_n = 0$$

, the vectors $X_1, \dots, X_n$ is called **linear dependent**, and if $k_1 X_1 + \cdots + k_1 X_n = 0 \implies k_1 = \cdots = k_n = 0$, the vectors is called **linear independent**

线性独立是指一组向量不能表示为另一组向量的线性组合。线性相关是指一组向量可以表示为另一组向量的线性组合。

Theorem 1.1. *The following declarations are valid:*

1. *If a part group of vectors $X_1, \cdots, X_n$ is linear dependent, then the vectors are linear dependent*
2. *Any part of the vectors $X_1, \cdots, X_n$ are linear independent*
3. *At least one of the vectors $X_1, \cdots, X_n$ is the linear combinition of other vectors if $X_1, \cdots, X_n$ are linear dependent*
4. *If one of vectors $X_1, \cdots, X_n$ is the linear combinition of others, the vectors are linear dependent*
5. *If vectors $X_1, \cdots, X_n$ are linear independent, $X_1, \cdots, X_n, X$ are linear dependent, X is the linear combinition of vectors {X_1, · · ·, X_n}.*
6. *If vectors $X_1, \cdots, X_n$ are linear independent, and vector X_{n+1} is not linear combinition of them, then vectors $X_1, \cdots, X_n, X_{n+1}$ are linear independent*

Then we prove the theorem.

证明. The proof is easy, just need to grasp the concept of linear dependent and linear independent, the only thing is move some term to other side of sign "=".

1. We take $X_1, \cdots, X_s$ where $s < n$,

$$\alpha_1 X_1 + \cdots + \alpha_s X_s = 0$$

and take $\alpha_{s+1} = \cdots = \alpha_n = 0$

2. If there is vectors $X_1, \cdots, X_s$ where s < n are linear dependent, then the vectors $X_1, \cdots, X_n$ are also linear dependent, which leads a contradiction to the condition.
3. 线性相关 $\implies k \in 1, \cdots, n$ s.t. $\alpha_k \neq 0$, 移项使得 X_k 变成其余向量的线性组合。

$\square$