Linear Stability of Plane Poiseuille Flow in a Continuous Time Domain

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Abstract— **In this paper, we present a linear stability analysis of a plane Pouiselle flow formulated in a continuous time domain. Contrary to a conventional approach, which utilizes a normal mode decomposition, where solution is decomposed into a combination of the traveling wave modes, and each mode is analyzed separately, we keep the temporal variable continuous, so that no separability assumption on the solution is necessary. Stability analysis is performed by first casting the corresponding linearized partial differential equation into a partial integral equation (PIE) framework, and subsequently employing a linear partial inequality (LPI) stability test, which searches for a corresponding Lyapunov function parameterized through polynomial expansions to prove or disprove stability. Stability results of the continuous-time formulation for the plane Pouiselle flow are compared with a traditional eigenvalue-based neutral stability curve, and a good agreement is obtained.**

I. INTRODUCTION

Properties of a laminar-to-turbulent transition in fluid flows have been a subject of intense research for many decades due to its utmost importance, both from a canonical perspective, and in practical applications. When stability of fluid flows around an underlying laminar profile is concerned, considering small perturbations to this laminar profile and analyzing asymptotic stability of such perturbations in the linearized equations is a reasonable avenue to proceed, which gives rise to a well-known linear stability theory (LST) [1], [2]. A traditional approach of the LST is to decompose a perturbation solution into a sum of the contributing modes, in the form of traveling waves, and analyze stability of each of the traveling waves separately.

This normal-mode approach has two important drawbacks: 1) It assumes a separability of the solution into a (finite) sum of traveling wave modes; 2) It ignores the mode interaction during stability analysis, which might be important. Indeed, it appears that the linearized Navier-Stokes (LNS) operator is not normal, which leads to non-orthogonality of the corresponding eigenmodes, prompting their interactions in the temporal dynamics of the perturbation field [3], [4].

Further analysis of the LNS operator revealed that the most amplified disturbances (in a finite time horizon) do not resemble that of the normal modes, but take upon a shape determined by an interaction of these modes [5], [6]. Despite importance of these interactions, as demonstrated in the transient response and input-output analysis of the LNS system [4], [7], the tools to account for the mode interaction in the asymptotic (infinite-time horizon) analysis of the dynamics, thus far have been lacking.

In the current paper, we introduce a novel approach to analyze stability of the linearized Navier-Stokes equations, that leverages a recently developed partial-integral equation (PIE) framework [8] that allows to apply a Lyapunov-based LPI stability test to a continuous form of equations (both in space, here represented by a wall-normal direction, and in time), so that neither a mode decomposition nor a spatial discretization is required to analyze stability. We show how to reformulate the corresponding two-dimensional LNS equation, Fourier-transformed in the streamwie direction, in a PIE format, extend the available stability proof [8] to a new (modified) form of the PIE equation resulting from the 4th-order continuous LNS operator, and demonstrate that the corresponding LPI stability condition yields results comparable with the conventional eigenvalue stability analysis. An advantage of the developed stability analysis formulation in a continuous-time domain is that it provides a pathway towards design of infinite-dimensional controllers [9], to be considered in the future work.

II. PROBLEM FORMULATION

We consider a two-dimensional (2D) fluid flow between two parallel plates governed by the incompressible Navier-Stokes equations

$$\begin{gathered}\frac{\partial \mathbf{u}}{\partial t}+\mathbf{u}\cdot\nabla\mathbf{u}=-\nabla p+\frac{1}{Re}\nabla^2\mathbf{u},\\ \nabla\cdot\mathbf{u}=0,\end{gathered} \tag{1}$$

where $\mathbf{u}=(u,v)$ is the velocity in x (streamwise) and y (wall-normal) directions, respectively, p is the pressure, and $Re=U_c\,\delta/\nu$ is the Reynolds number based on the characteristic velocity U_c and the channel half-width δ. With this definition, the velocities in (1) are normalized with U_c, and the spatial variables are normalized with δ. Boundary conditions are the no-slip at the plate walls $\mathbf{u}|_{y=\pm 1}=0$. We decompose instantaneous variables into a sum of the corresponding laminar solution and the perturbations as $\mathbf{u}=\mathbf{U}+\mathbf{u}'$, $p=P+p'$, with the parallel mean flow approximation $\mathbf{U}=\{U(y),0\}$, and linearize Eqs. (1) around the laminar solution to yield

$$\begin{gathered}\frac{\partial \mathbf{u}'}{\partial t}+\mathbf{U}\cdot\nabla\mathbf{u}'+\mathbf{u}'\cdot\nabla\mathbf{U}=-\nabla p'+\frac{1}{Re}\nabla^2\mathbf{u}',\\ \nabla\cdot\mathbf{u}'=0.\end{gathered} \tag{2}$$

Taking a curl of the momentum equation and using the continuity equation allows one to eliminate the pressure from

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the system (2) and arrive at a single linear PDE to fully describe the 2D LNS operator

$$\left[\left(\frac{\partial}{\partial t}+U\,\frac{\partial}{\partial x}\right)\nabla^2-\frac{\partial^2 U}{\partial y^2}\frac{\partial}{\partial x}-\frac{1}{Re}\nabla^4\right]\psi=0, \tag{3}$$

where the stream function $\psi(x,y,t)$ has been introduced, such that $u'=\partial\psi/\partial y, v'=-\partial\psi/\partial x$. Owing to the problem periodicity in streamwise direction, we perform a Fourier transform of (3) in x yielding, for a streawise wave-number k,

$$\left[\left(\frac{\partial}{\partial t}+i\,k\,U\right)\hat{\Delta}^2-i\,k\frac{\partial^2 U}{\partial y^2}-\frac{1}{Re}\hat{\Delta}^4)\right]\hat{\psi}=0, \tag{4}$$

where $\hat{\psi}(y,t)$ is the corresponding Fourier coefficient of the stream function for the wave-number k, i is an imaginary unit, and the one-dimensional differential operator $\hat{\Delta}^2=\partial^2/\partial y^2-k^2$ with $\hat{\Delta}^4=(\hat{\Delta}^2)^2$. Boundary conditions on $\hat{\psi}$ can be derived from the boundary conditions on $\mathbf{u}$ as $\hat{\psi}|_{\pm1}=\hat{\psi}_y|_{\pm1}=0$.

III. REPRESENTATION AS A PARTIAL-INTEGRAL EQUATION

We now consider a representation of Eq. (4) in a partial-integral equation (PIE) form. PIE representation allows one to transform the boundary conditions into the equation dynamics, which, in turn, makes the formulation amenable to a stability analysis in a continuous framework [8]. For that, we first rewrite Eq. (4) in its state-space form

$$\hat{\Delta}^2\dot{\hat{\psi}}=ik\frac{\partial^2 U}{\partial y^2}\hat{\psi}-ik\,U\hat{\Delta}^2\hat{\psi}+\frac{1}{Re}\hat{\Delta}^4\hat{\psi}, \tag{5}$$

where we use the notation $\dot{\hat{\psi}}=\partial\hat{\psi}/\partial t$ for compactness.

Since Eq. (5) is in a complex form, while the PIE framework, including the corresponding open-source software PIETOOLS for manipulating PIEs [10], was previously developed for real-valued functions, we let $\hat{\psi}=\hat{\psi}_R+i\,\hat{\psi}_I$, and decompose Eq. (5) into a coupled system of equations for the real and imaginary components as

$$\begin{aligned}
\begin{bmatrix}-\frac{1}{k^2} & 0\\ 0 & -\frac{1}{k^2}\end{bmatrix}\begin{bmatrix}\dot{\hat{\psi}}_{R\,yy}\\ \dot{\hat{\psi}}_{I\,yy}\end{bmatrix}+\begin{bmatrix}\dot{\hat{\psi}}_R\\ \dot{\hat{\psi}}_I\end{bmatrix}=\\
\begin{bmatrix}-\frac{1}{k^2Re} & 0\\ 0 & -\frac{1}{k^2Re}\end{bmatrix}\begin{bmatrix}\hat{\psi}_{R\,yyyy}\\ \hat{\psi}_{I\,yyyy}\end{bmatrix}+\begin{bmatrix}\frac{2}{Re} & -\frac{U}{k}\\ \frac{U}{k} & \frac{2}{Re}\end{bmatrix}\begin{bmatrix}\hat{\psi}_{R\,yy}\\ \hat{\psi}_{I\,yy}\end{bmatrix}+\\
\begin{bmatrix}-\frac{k^2}{Re} & \frac{1}{k}U_{yy}+k\,U\\ -\frac{1}{k}U_{yy}-k\,U & -\frac{k^2}{Re}\end{bmatrix}\begin{bmatrix}\hat{\psi}_R\\ \hat{\psi}_I\end{bmatrix},
\end{aligned} \tag{6}$$

where $[\,]_{yy}$ and $[\,]_{yyyy}$ denote 2^{nd} and 4^{th} partial derivatives with respect to y. We define the set of boundary constraints as

$$B(\hat{\psi}_R,\hat{\psi}_I):\hat{\psi}_R|_{\pm1}=\hat{\psi}_{R\,y}|_{\pm1}=\hat{\psi}_I|_{\pm1}=\hat{\psi}_{I\,y}|_{\pm1}=0 \tag{7}$$

A solution to Eq. (6) satisfying the boundary constraints (7), $\hat{\psi}(y,t)=[\hat{\psi}_R,\,\hat{\psi}_I]^T\in H_4[-1,1]\cap B(\hat{\psi}_R,\hat{\psi}_I)$, where $H_4[-1,1]$ is the Sobolev space of functions with square-integrable derivatives up to 4^{th} order, will be denoted as a PDE state.

We seek a representation of the above system (6), with the boundary conditions (7), as a PIE. To formulate an equation as a PIE, we first need to define a fundamental (PIE) state $z(y,t)$, typically expressed as a vector of the highest spatial derivatives of the states entering the PDE, $z(y,t)=[z_R,z_I]^T=[\hat{\psi}_{R\,yyyy},\,\hat{\psi}_{I\,yyyy}]^T$ in the current case. Note that, by definition, the fundamental state $z(y,t)\in L_2[-1,1]$, where $L_2[-1,1]$ is a space of square-integrable functions on the domain $y\in[-1,1]$. The next step is to define a map between the PDE state $\hat{\psi}(y,t)=[\hat{\psi}_R,\,\hat{\psi}_I]^T$ (satisfying the boundary conditions) and the fundamental state $z(y,t)$ as

$$\hat{\psi}(y,t)=\mathcal{T}z(y,t). \tag{8}$$

This can be accomplished by multiple application of a fundamental theorem of calculus while taking into account the corresponding boundary conditions [8]. It can be shown that the operator $\mathcal{T}$ resulting from such a map can be written in a form of a partial-integral (PI) operator as $\mathcal{T}=\mathcal{T}_{\{R_0,R_1,R_2\}}$ defined as follows.

Definition 1. *A partial integral operator $\mathcal{P}=\mathcal{P}_{\{R_0,R_1,R_2\}}$ is defined as a three-component operator acting on a fundamental state as*

$$\begin{aligned}\mathcal{P}z(y,t)=\mathcal{P}_{\{R_0,R_1,R_2\}}z(y,t)=R_0(y)z(y,t)+\\ \int_{-1}^{y}R_1(y,s)z(s,t)\,ds+\int_{y}^{1}R_2(y,s)z(s,t)\,ds,\end{aligned} \tag{9}$$

where $\{R_0(y),R_1(y,s),R_2(y,s)\}$ are the matrices with the entries that are polynomials in the variables y and s [8].

Considering the left-hand side of Eq. (6), we also need to define an auxiliary map between the second-derivative state $\hat{\phi}_{yy}(y,t)=[\hat{\phi}_{R\,yy},\hat{\phi}_{I\,yy}]^T$ and the fundamental state as $\hat{\phi}_{yy}(y,t)=\mathcal{T}_2\,z(y,t)$. In a general case, the maps $\mathcal{T},\mathcal{T}_2$ are domain and boundary-conditions specific [8], [10]. For the current case of $y\in[-1,1]$ with homogeneous Dirichlet and Neumann boundary conditions on $\hat{\psi}_R$ and $\hat{\psi}_I$, Eq. (7), we find that

$$\begin{aligned}\mathcal{T}_{\{R_1\}}=\big[&-\frac{1}{24}y^3s^3+\frac{1}{8}(y^3s-y^2s^2+ys^3)+\frac{1}{12}(y^3-s^3)\\&+\frac{1}{4}(ys^2-y^2s)-\frac{1}{8}(y^2-y\,s+s^2)+\frac{1}{24}\big]\,I,\\
\mathcal{T}_{\{R_2\}}=\big[&-\frac{1}{24}y^3s^3-\frac{1}{8}(y^3s-y^2s^2+ys^3)+\frac{1}{12}(y^3-s^3)\\&+\frac{1}{4}(ys^2-y^2s)-\frac{1}{8}(y^2-ys+s^2)+\frac{1}{24}\big]\,I,\end{aligned} \tag{10}$$

$$\begin{aligned}\mathcal{T}_{2\{R_1\}}=&\big[-\frac{1}{4}ys^3+\frac{3}{4}ys-\frac{1}{4}s^2+\frac{1}{2}y-\frac{1}{2}s-\frac{1}{4}\big]\,I,\\
\mathcal{T}_{2\{R_2\}}=&\big[-\frac{1}{4}ys^3+\frac{3}{4}ys-\frac{1}{4}s^2-\frac{1}{2}y+\frac{1}{2}s-\frac{1}{4}\big]\,I,\end{aligned} \tag{11}$$

$\mathcal{T}_{\{R_0\}}=\mathcal{T}_{2\,\{R_0\}}=0$, with I being a 2×2 identity matrix. Substituting the corresponding mappings with the operators

$\mathcal{T}$, $\mathcal{T}_2$ defined by Eqs. (10), (11) into Eq. (6), we obtain its equivalent PIE representation as

$$\mathcal{M}\,\dot{z} = \mathcal{A}\,z, \tag{12}$$

where the solution vector $z(y,t) = [z_R, z_I]^T \in L_2[-1,1]$, and thus is free of boundary conditions. The operators $\mathcal{M}$, $\mathcal{A}$ in Eq. (12) are given by

$$\mathcal{M} = -\frac{1}{k^2}\mathcal{T}_2 + \mathcal{T}, \tag{13}$$

$$\begin{aligned}\mathcal{A} = &\left[-\frac{1}{k^2 Re} + \frac{2}{Re}\mathcal{T}_2 - \frac{k^2}{Re}\mathcal{T}\right]\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix} + \\ &\left[\frac{U}{k}\mathcal{T}_2 - \frac{1}{k}U_{yy}\mathcal{T} - \,kU\mathcal{T}\right]\begin{bmatrix}0 & -1\\ 1 & 0\end{bmatrix}.\end{aligned} \tag{14}$$

It can be seen that $\mathcal{M}$ is a PI operator by construction, while $\mathcal{A}$ is also a PI operator, as long as the mean velocity profile $U(y)$ is polynomial in y. In the current analysis, $U(y)$ is given by $U(y) = 1 - y^2$, corresponding to a Pouiselle flow between two parallel plates, normalized with the centerline velocity U_c.

IV. STABILITY ANALYSIS USING LINEAR PARTIAL INEQUALITIES

We define an exponential stability of the PDE system (6) representing a 2D LNS approximation for a plane shear flow in the following

Definition 2. *The PDE system (6) with boundary conditions (7) is exponentially stable in L_2 if there exist constants K, $\gamma > 0$ such that for any $\hat{\psi}(y,0) \in B(\hat{\psi}_R, \hat{\psi}_I)$, a solution $\hat{\psi}(y,t)$ of the PDE with (7) satisfies*

$$\|\hat{\psi}(y,t)\|_{L_2} \le K\|\hat{\psi}(y,0)\|_{L_2} e^{-\gamma\, t}. \tag{15}$$

Similarly, we define an exponential stability of the PIE system as

Definition 3. *The PIE system (12) is exponentially stable in L_2 if there exist constants K, $\gamma > 0$ such that for any $z(y,0)$, a solution $z(y,t)$ of the PIE satisfies*

$$\|\mathcal{M}z(y,t)\|_{L_2} \le K\|\mathcal{M}z(y,0)\|_{L_2} e^{-\gamma\, t}. \tag{16}$$

Exponential stability of PIE is tested by defining and verifying a feasibility of a linear partial inequality (LPI) encapsulated in the following theorem.

Theorem 1. *Suppose there exists $\beta, \delta > 0$, and a self-adjoint coercive PI operator $\mathcal{P}_{\{R_0,R_1,R_2\}}$ such that $\mathcal{P} = \mathcal{P}^*$, $\langle z, \mathcal{P}z\rangle_{L_2} \ge \beta\|z\|^2_{L_2}$, and*

$$\mathcal{A}^*\mathcal{P}\mathcal{M} + \mathcal{M}^*\mathcal{P}\mathcal{A} < -\delta\mathcal{M}^*\mathcal{M}, \tag{17}$$

where $\mathcal{M}$, $\mathcal{A}$ are as defined in Eqs. (13), (14). Then any solution of the PIE system (12) satisfies

$$\|\mathcal{M}z(y,t)\|_{L_2} \le \left(\frac{\xi_M}{\beta}\right)^{1/2}\|\mathcal{M}z(y,0)\|_{L_2} e^{-\delta/(2\xi_M)t}, \tag{18}$$

where $\xi_M = \|\mathcal{M}\|_{\mathcal{L}(L_2)}$.

Proof: Suppose $z(y,t)$ solves the PIE system (12) for some $z(y,0)$. Consider the candidate Lyuapunov function defined as

$$V(z) = \langle\mathcal{M}z, \mathcal{P}\mathcal{M}\,z\rangle_{L_2} \ge \beta\|\mathcal{M}\,z\|^2_{L_2}. \tag{19}$$

The derivative of V along the solution trajectory $z(y,t)$ is

$$\begin{aligned}\dot{V}(z) = \langle\mathcal{M}\dot{z}, \mathcal{P}\mathcal{M}z\rangle_{L_2} + \langle\mathcal{M}z, \mathcal{P}\mathcal{M}\,\dot{z}\rangle_{L_2} = \\ \langle\mathcal{A}z, \mathcal{P}\mathcal{M}\,z\rangle_{L_2} + \langle\mathcal{M}z, \mathcal{P}\mathcal{A}\,z\rangle_{L_2} = \\ \langle z, (\mathcal{A}^*\mathcal{P}\mathcal{M} + \mathcal{M}^*\mathcal{P}\mathcal{A})\,z\rangle_{L_2} \le -\delta\|\mathcal{M}z\|^2_{L_2}.\end{aligned} \tag{20}$$

Applying Gronwall-Bellman lemma to (19), (20), and using $\xi_M = \|\mathcal{M}\|_{\mathcal{L}(L_2)}$, proves the theorem.

We now prove the main result of the current paper, namely the equivalence of the PIE stability condition tested by Theorem 1, and the stability of the original 2D LNS PDE system.

Theorem 2. *Suppose there exists $\beta, \delta > 0$, and a self-adjoint coercive PI operator $\mathcal{P}_{\{R_0,R_1,R_2\}}$ such that $\mathcal{P} = \mathcal{P}^*$, $\langle z, \mathcal{P}z\rangle_{L_2} \ge \beta\|z\|^2_{L_2}$, and*

$$\mathcal{A}^*\mathcal{P}\mathcal{M} + \mathcal{M}^*\mathcal{P}\mathcal{A} < -\delta\mathcal{M}^*\mathcal{M}, \tag{21}$$

where $\mathcal{M}$, $\mathcal{A}$ are as defined in Eqs. (13), (14). Then there exists a constant $C > 0$, such that any solution of the PDE system (6) with boundary conditions (7) satisfies

$$\|\hat{\psi}(y,t)\|_{L_2} \le C\|\hat{\psi}(y,0)\|_{L_2} e^{-\delta/(2\xi)t}. \tag{22}$$

Proof: Denote $\xi_M = \|\mathcal{M}\|_{\mathcal{L}(L_2)}$, $\xi_T = \|\mathcal{T}\|_{\mathcal{L}(L_2)}$, $\xi_{T_2} = \|\mathcal{T}_2\|_{\mathcal{L}(L_2)}$. From Theorem (1), we have that

$$\|\mathcal{M}z(y,t)\|_{L_2} \le \left(\frac{\xi_M}{\beta}\right)^{1/2}\|\mathcal{M}z(y,0)\|_{L_2} e^{-\delta/(2\xi)t}. \tag{23}$$

Considering that $\hat{\psi}(y,t) = \mathcal{T}z(y,t)$, and $\mathcal{T} = \mathcal{M} + \mathcal{T}_2/k^2$, we have

$$\begin{aligned}\|\hat{\psi}(y,t)\|_{L_2} &\le \|\mathcal{M}z(y,t)\|_{L_2} + \frac{1}{k^2}\|\mathcal{T}_2\,z(y,t)\|_{L_2} \\ &\le \|\mathcal{M}z(y,t)\|_{L_2} + \frac{\xi_{T_2}}{k^2}\,\|z(y,t)\|_{L_2}.\end{aligned} \tag{24}$$

Furthermore, we can write

$$\|\mathcal{M}z(y,t)\|_{L_2} \le \xi_M\|z(y,t)\|_{L_2} = C_M\|z(y,t)\|_{L_2}, \tag{25}$$

where $C_M > 0$ is some constant such that $C_M \le \xi_M$, from where we have that

$$\|z(y,t)\|_{L_2} = \frac{\|\mathcal{M}z(y,t)\|_{L_2}}{C_M}. \tag{26}$$

Continuing with Eq. (24), we have

$$\begin{aligned}\|\hat{\psi}(y,t)\|_{L_2} \le \|\mathcal{M}z(y,t)\|_{L_2}\left(1 + \frac{1}{k^2}\frac{\xi_{T_2}}{C_M}\right) \\ \le \left(\frac{\xi_M}{\beta}\right)^{1/2}\left(1 + \frac{1}{k^2}\frac{\xi_{T_2}}{C_M}\right)\|\mathcal{M}z(y,0)\|_{L_2} e^{-\delta/(2\xi)t}.\end{aligned} \tag{27}$$

Considering an estimate similar to Eq. (26) for the $\mathcal{T}$ operator, we have that

$$\|z(y,t)\|_{L_2} = \frac{\|\mathcal{T}z(y,t)\|_{L_2}}{C_T}, \tag{28}$$

with $0 < C_T \le \xi_T$. Applying Eq. (28) to initial conditions and substituting into Eq. (27), we have

$$\|\hat{\psi}(y,t)\|_{L_2} \le \left(\frac{\xi_M}{\beta}\right)^{1/2}\left(1+\frac{1}{k^2}\frac{\xi_{T_2}}{C_M}\right)\left(\frac{\xi_M}{C_T}\right)\|\hat{\psi}(y,0)\|_{L_2} e^{-\delta/(2\xi)t}, \tag{29}$$

which proves the theorem.

V. RESULTS

A. VERIFICATION OF THE PIE SYSTEM

We verify our representation of the 2D LNS equation as a PIE through the Method of Manufactured Solutions (MMS). With MMS, we construct an analytical solution to the PDE (6), with the boundary conditions (7), by first specifying the form of the solution as

$$\hat{\psi}_R(y,t) = f(y)e^{\alpha t},\ \hat{\psi}_I(y,t) = g(y)e^{\alpha t}, \tag{30}$$

and substituting this form of the solution into Eq. (6) to yield

$$\begin{bmatrix} -\frac{1}{k^2} & 0 \\ 0 & -\frac{1}{k^2} \end{bmatrix}\begin{bmatrix} \dot{\hat{\psi}}_{R\,yy} \\ \dot{\hat{\psi}}_{I\,yy} \end{bmatrix} + \begin{bmatrix} \dot{\hat{\psi}}_R \\ \dot{\hat{\psi}}_I \end{bmatrix} = \begin{bmatrix} -\frac{1}{k^2 Re} & 0 \\ 0 & -\frac{1}{k^2 Re} \end{bmatrix}\begin{bmatrix} \hat{\psi}_{R\,yyyy} \\ \hat{\psi}_{I\,yyyy} \end{bmatrix} + \begin{bmatrix} \frac{2}{Re} & -\frac{U}{k} \\ \frac{U}{k} & \frac{2}{Re} \end{bmatrix}\begin{bmatrix} \hat{\psi}_{R\,yy} \\ \hat{\psi}_{I\,yy} \end{bmatrix} + \begin{bmatrix} -\frac{k^2}{Re} & \frac{1}{k}U_{yy} + k\,U \\ -\frac{1}{k}U_{yy} - k\,U & -\frac{k^2}{Re} \end{bmatrix}\begin{bmatrix} \hat{\psi}_R \\ \hat{\psi}_I \end{bmatrix} + \begin{bmatrix} Q_R(y,t) \\ Q_I(y,t) \end{bmatrix}, \tag{31}$$

with the forcing terms $Q_R(y,t)$, $Q_I(y,t)$ expressed as

$$\begin{aligned} Q_R(y,t) =& [\frac{1}{k^2 Re}f_{yyyy} - (\frac{\alpha}{k^2}+\frac{2}{Re})f_{yy} + (\alpha + \frac{k^2}{Re})f \\ &- (\frac{U_{yy}}{k} + kU)g + \frac{U}{k}g_{yy}]e^{\alpha t}, \end{aligned} \tag{32}$$

$$\begin{aligned} Q_I(y,t) =& [\frac{1}{k^2 Re}g_{yyyy} - (\frac{\alpha}{k^2}+\frac{2}{Re})g_{yy} + (\alpha + \frac{k^2}{Re})g \\ &+ (\frac{U_{yy}}{k} + kU)f - \frac{U}{k}f_{yy}]e^{\alpha t}. \end{aligned} \tag{33}$$

The functions $f(y)$, $g(y)$ in (30) are chosen as the polynomials satisfying the boundary conditions (7) as

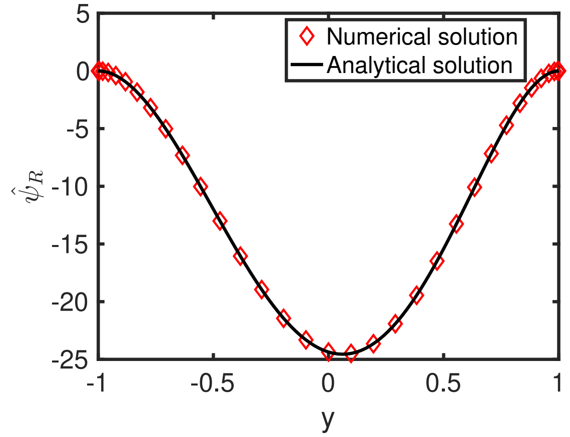
$$\begin{aligned} f(y) &= (-\frac{1}{2}y^5 - 2y^4 + y^3 + 4y^2 - \frac{1}{2}y - 2), \\ g(y) &= (-4y^5 - \frac{3}{2}y^4 + 8y^3 + 3y^2 - 4y - \frac{3}{2}). \end{aligned} \tag{34}$$

The PDE equation (31) with the boundary conditions (7) is transformed into a PIE as

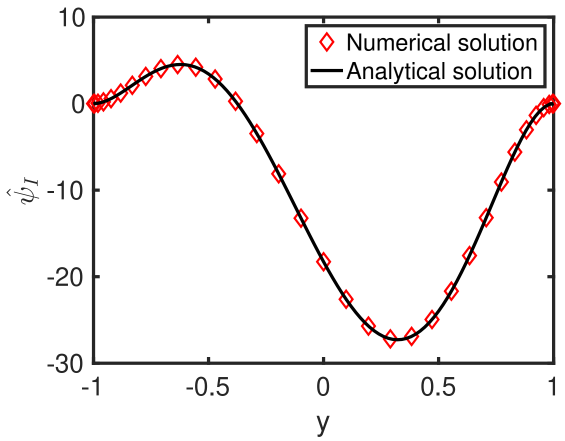
$$\mathcal{M}\,\dot{z} = \mathcal{A}\,z + Q(y,t), \tag{35}$$

where $Q(y,t) = [Q_R(y,t), Q_I(y,t)]^T$, with $\mathcal{M}$, $\mathcal{A}$, $Q_R(y,t)$ and $Q_I(y,t)$ given by Eqs. (13), (14), (32) and (33), respectively. The analytical solution to the PIE equation constructed with MMS is given by

$$z_R(y,t) = f_{yyyyy}e^{\alpha t},\ z_I(y,t) = g_{yyyyy}e^{\alpha t}. \tag{36}$$



(a) Real part, $\hat{\psi}_R(y, t = 5)$



(b) Imaginary part, $\hat{\psi}_I(y, t = 5)$

Fig. 1: Verification of the PIE system for 2D LNS with PIESIM [11].

PIE system (35) with the initial conditions $z_R(y,0) = f_{yyyy}, z_I(y,0) = g_{yyyy}$ was numerically solved by a recently developed computational methodology for solving partial-integral equation systems implemented in the open-source numerical solver PIESIM [11], which is a part of the PIE analysis software PIETOOLS [10]. In PIESIM, the PIE state variables $z(y,t)$, together with the forcing functions $Q(y,t)$, are decomposed into a series of Chebyshev polynomials $z(y,t) = \sum_{i=1}^N a_k(t)T_k(y)$, $Q(y,t) = \sum_{i=1}^N q_k(t)T_k(y)$ with $a_k(t), q_k(t)$ being the vector-valued Chebyshev coefficients, and $T_k(y)$ are the Chebyshev polynomials of the first kind [12]. The actions of the PI operators $\mathcal{M}$, $\mathcal{A}$ on the Chebyshev polynomials $T_k(y)$ is evaluated analytically using recursive relations for multiplication and integration of Chebyshev polynomials [12], [13]. This allows one to obtain a system of the ODE equations for the Chebyshev coefficients, which can be integrated in time analytically or using traditional time-stepping techniques [11]. Once the PIE solution $z(y,t)$ is obtained, the PDE solution $\hat{\psi}(y,t)$ is reconstructed via a PIE-to-PDE map $\hat{\psi}(y,t) = \mathcal{T}z(y,t)$, discretized in Chebyshev space using the same techniques as the ones employed for the PIE system.

Verification of the 2D LNS equation solution in a PIE form using the second-order backward differentiation scheme for time advancement with the time step $\Delta t = 10^{-3}$, $N = 32$, $U(y) = (1 - y^2)$ and $(k, Re, \alpha) = (1, 180, 0.5)$ is presented in Fig. 1 at $t = 5$.

B. STABILITY ANALYSIS USING LPIS

We perform stability analysis of the two-dimensional linearized Navier-Stokes equations system in its continuous spatio-temporal formulation by testing feasibility of the LPI condition stated in Theorem 1. The feasibility test is accomplished via an open-source MATLAB-based software PIETOOLS developed for analysis and manipulation of the PIE equations [10]. In PIETOOLS, the feasibility problem is formulated as a convex optimization problem, which enforces a positivity of a PI operator parameterized by polynomial functions [8], [10]. Once formulated, a convex optimization problem is solved via a semi-definite programming solver SeDuMi of the package YALMIP [14].

Figure 2 documents the results of the stability test of the 2D LNS equation for a plane Pouiselle flow in a continuous formulation compared to a conventional Orszag's neutral curve obtained via a normal mode decomposition and eigenvalue analysis [2]. It is seen that the developed LPI stability test shows a good agreement with the eigenvalue-based stability estimates. Further analysis comparing the differences between the two methods, especially on or in a close proximity to the neutral curve, will be useful.

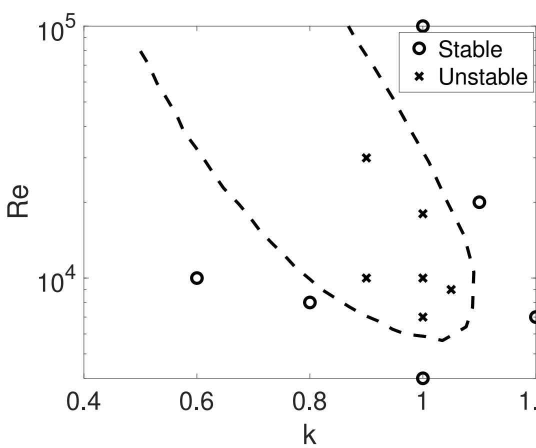


Fig. 2: Stability of 2D LNS equation via LPIs compared to an eigenvalue stability for a plane Pouiselle flow. Dashed line - neutral curve from Orszag [2], symbols - evaluation from the continuous PDE-PIE form.

VI. CONCLUSIONS

The current paper presents a methodology for a linear stability analysis of the fluid flow between two parallel plates based on the continuous form of the governing equations. Navier-Stokes equations in a two-dimensional formulation are first linearized around the mean parallel velocity profile, and then Fourier-transformed in the streamwise direction to arrive at, for each streamwise wave-number, a linear fourth-order PDE equation in time and a wall-normal coordinate. As opposed to a conventional normal mode decomposition, stability of the 2D LNS PDE equation is performed in a continuous setting, employing the results from the optimal control theory. For that, the PDE together with the boundary conditions is first transformed into a partial integral equation (PIE), with boundary conditions now being implicitly embedded into the form of the partial integral operators, thus making the PIE solution reside in an L_2 space, free of boundary conditions. The PIE form makes it possible to apply a stability test in an infinite-dimensional setting by testing a feasibility of a linear partial inequality. We have proved that the PIE stability implies the stability of the underlying PDE system, and vice versa. The presented stability analysis is compared with the conventional eigenvalue-based stability results for a plane Pouiselle flow, and a good agreement is obtained. Further work will include a synthesis of the infinite-dimensional stabilizing controllers for the unstable regimes of the flow perturbations, which is enabled by reformulation of the 2D LNS PDE problem as a partial integral equation [9]. In fact, a stabilizing controller synthesis can be posed as an LPI feasibility problem [9], and the developed PDE to PIE transformation for the parallel shear flow equations will serve as a departure point to pursue these efforts in the near future.

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