Time stable boundary treatment for compressible Navier-Stokes

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Abstract

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1 Introduction

For wave propagating problems, the computational domain is often large compared to the wavelengths, which mean that waves have to travel long distances during long times. As a result, high order accurate time marching methods, as well as high order spatially accurate schemes (at least 3rd order) are required. Such schemes, although they might be G-K-S stable [4] (convergence to the true solution as $\Delta x \to 0$), may exhibit a non-physical growth in time [1], for realistic mesh sizes. It is therefore important to device schemes which do not allow a growth in time that is not called for by the differential equation. Such schemes are called strictly (or time) stable.

One way to obtain a strictly stable and accurate scheme is to: i) Approximate the derivatives of the initial boundary value problem with accurate, non dissipative operators that satisfy a summation by parts (SBP) formula, and ii) use a specific procedure, referred to as the Simultaneous Approximation Term (SAT) method [1],[2],[11],[12], for implementation of boundary conditions.

One major advantage of using SBP operators [5],[6], [14] to discretize the equations on a multi physics domain is that in a discrete case we can mimic the boundary and interface terms from the underlying continuous problem. Combined with the SAT method we then obtain completely analogous conservation and stability properties as for the underlying Partial Differential

Equation (PDE), requiring only the continuous boundary and interface conditions (i.e. the physics). This should make this technique very attractive to physicists.

In Section 2 we introduce some definitions and discuss the SBP property for the first and second derivative difference operator, and introduce the SAT method. In Section 3 we introduce the one dimensional model for the Navier-Stokes quations and derive strongly well posed boundary conditions and it is shows how to combine the SAT method and the SBP operators to obtain strictly stable finite difference approximations. In Section 4 we consider Navier-Stokes equations on a multi-block domain and show how to obtain stable and accurate coupling of the domains. In section 5, we present computations and additional analysis for the two dimensional problem. In section 6, we draw conclusions. In appendix I we present the the SBP operators used in the computations.

2 Definitions

Consider the initial boundary value problem (IBVP),

$$\begin{array}{lll} u_t + P\,u = F(x,t) & x \in \Omega & t \geq 0 \\ Lu = g(t) & x \in \Gamma & t \geq 0 \\ u = f(x) & x \in \Omega & t = 0\,, \end{array} \tag{1}$$

where P is the differential operator, L the boundary operator, F the forcing function, f the initial data and g the boundary data.

Definition 2.1 *(1) is said to be* **strongly well posed** *if an unique solution exists and the estimate*

$$\begin{aligned} &\|u\|_\Omega^2 + \int_0^t \|u(\cdot,\tau)\|_\Gamma^2\, d\tau \\ &\quad \leq K_c e^{\eta_c t}\left(\|u(\cdot,0)\|_\Omega^2 + \int_0^t \left(\|F(\cdot,\tau)\|_\Omega^2 + \|g(\tau)\|_\Gamma^2\right)\, d\tau\right)\,, \end{aligned}$$

holds. K_c and η_c do not depend on F, f or g. $\|\cdot\|_\Omega$ and $\|\cdot\|_\Gamma$ are suitable continuous norms.

When it comes to the discrete case, a simplification is to consider the semidiscrete problem where time is left continuous. This approach is justified by Kreiss and Wu [7], where they show that a stable semidiscretisation can be discretised in time using Runge-Kutta schemes such that the fully discrete problem is stable.

The corresponding semi-discrete problem is

$$\begin{array}{lll} v_t + P\,v = F(x,t) & x \in \Omega & t \geq 0 \\ Lv = g(t) & x \in \Gamma & t \geq 0 \\ v = f & x \in \Omega & t = 0\,, \end{array} \tag{2}$$

where P and L are the corresponding discrete operators.

Definition 2.2 *(2) is said to be* **strongly stable** *if, for sufficiently small Δx, there is an unique solution that satisfies*

$$\begin{aligned} &\|v\|_\Omega^2 + \int_0^t \|v\|_\Gamma^2 \, d\tau \\ &\quad \leq K_d e^{\eta_d t} \left(\|v\|_\Omega^2 + \int_0^t \left(\|F\|_\Omega^2 + \|g\|_\Gamma^2 \right) \, d\tau \right) . \end{aligned}$$

K_d and η_d do not depend on F, f or g. $\|\cdot\|_\Omega$ and $\|\cdot\|_\Gamma$ are suitable discrete norms.

This implies that convergence to the true solution at a fixed time T is achieved as the grid size $\Delta x \to 0$.

However, a **strongly stable** scheme may exhibit a non-physical growth in time [1], for realistic mesh sizes. It is therefore important to device schemes which do not allow a growth in time that is not called for by the differential equation. Such schemes are called strictly (or time) stable.

Definition 2.3 *We call (2)* **strictly stable** *if the growth rate satisfy*

$$\eta_d \leq \eta_c + \mathcal{O}(\Delta x)$$

In fact we can show that $\eta_d \leq \eta_c + \mathcal{O}(\Delta x^p)$ holds for the (SBP+SAT) schemes we are about to describe, where p is the order of accuracy of the method.

2.1 The SBP property

Before we start describing the SBP property, some definitions are needed. Let the inner product for real valued $m \times 1$ vector functions $u, v \in L^2[l,r]$ be defined by $(u,v) = \int_l^r u^T\, v\, dx$ and the corresponding norm $\|u\|^2 = (u,u)$. The domain ($l \leq x \leq r$) is discretized using *N+1* grid points.

The numerical approximation of the $k:th$ component of u, at grid point x_j is denoted v_j^k. We define a discrete solution vector $v^T = [v_0^T, v_1^T, \cdots, v_N^T]$, where $v_0 = [v_0^1, v_0^2, \cdots, v_0^m]^T$ is the discrete approximation of u at the left boundary. We define an inner product for discrete real valued vector-functions

$u, v \in \mathbf{R}^{\mathbf{N+1\times m}}$ by $(u,v)_H = u^T\, H\, v$, where $H = H^T\ >\ 0$, and a corresponding norm $\|v\|_H^2 = v^T\, H\, v$. The vectors

$$\hat{e_0} = [1, 0, ..., 0]^T\ , \quad \hat{e_N} = [0, ..., 0, 1]^T\ , \tag{3}$$

of size $N + 1 \times 1$ will frequently be used in subsequent sections.

2.1.1 The first derivative

Consider the hyperbolic scalar equation, $u_t + u_x = 0$ (excluding the boundary condition). Notice first that $(u, u_t) + (u_t, u) = d/dt\|u\|^2$. Integration by parts leads to,

$$\frac{d}{dt}\|u\|^2 = -(u, u_x) - (u_x, u) = -u^2|_l^r\ , \tag{4}$$

where we introduce the notation $u^2|_l^r \equiv u^2(r,t) - u^2(l,t)$. To simplify the notation for the continuos problem we will denote $u(k,t)$ by u_k. A discrete approximation can be written $v_t + D\,v = 0$. We introduce the following definition

Definition 2.4 *A difference operator $D = H^{-1}Q$ approximating $\partial/\partial\, x$ is said to be a first derivative SBP operator if i) $H = H^T > 0$ and ii) $Q + Q^T = B = diag\,(-1, 0 \ldots, 0, 1)$.*

By multiplying the semi discrete approximation by $v^T H$, adding the transpose and utilizing definition 2.4 we obtain

$$v^T H\, v_t + v_t^T H^T\, v_t = \frac{d}{dt}\|v\|_H^2 = -v^T(Q + Q^T)v = v_0^2 - v_N^2\ . \tag{5}$$

Equation (5) is a discrete analog to the integration by parts (IBP) formula (4) in the continuous case.

2.1.2 The second derivative

For parabolic problems, we need an SBP operator also for the second derivative. Consider the heat equation $u_t = u_{xx}$. Multiplying by u and integration by parts leads to

$$\frac{d}{dt}\|u\|^2 = (u, u_{xx}) + (u_{xx}, u) = 2uu_x|_a^b - 2\|u_x\|^2\ . \tag{6}$$

A discrete approximation is given by $v_t + D_2 v = 0$. We introduce the following definition

Definition 2.5 *A difference operator $D_2 = H^{-1}(-M + BS)$ approximating $\partial^2/\partial x^2$ is said to be a second derivative SBP operator if $x^T(M + M^T)x \geq 0$, if S includes an approximation of the first derivative operator at the boundary and $B = diag\,(-1, 0 \ldots, 0, 1)$.*

By multiplying the semi discrete approximation by $v^T H$, adding the transpose and utilizing definition 2.5 we obtain

$$\frac{d}{dt}\|v\|_H^2 = 2v_N(Sv)_N - 2v_0(Sv)_0 - v^T(M + M^T)v \; . \qquad (7)$$

To obtain an energy estimate it suffice that $x^T(M + M^T)\,x \geq 0$, assuming that the boundary terms are correctly implemented.

To obtain stability estimates for a mixed hyperbolic-parabolic problem it is necessary that the norm H used in the construction of the first and second derivative SBP operators are the same. High order accurate second derivative SBP operators were developed in [9]. For completion the 2nd 4th and 6th order operators based on diagonal norms, for both the first and second derivative approximations are listed in Appendix I. The diagonal norm second derivatives have the additional useful property (as will be shown in Section 3.1) that $M = M^T$. We introduce yet another definition

Definition 2.6 *A difference operator $H^{-1}(-A+BS)$ approximating $\partial^2/\partial x^2$ is said to be a symmetric second derivative SBP operator if it is an SBP operator and if $M = M^T$.*

To summarize, an SBP operator mimic the behavior of the corresponding continuous operator with regard to the inner product mentioned above.

2.2 The SAT method

By using an SBP operator, a strict stable approximation for a Cauchy problem is obtained. Nevertheless, the SBP property alone does not guarantee strict stability for an initial boundary value problem, a specific boundary treatment is also required. To impose the boundary condition explicitly, i.e. to combine the difference operator and the boundary operator into a modified operator, usually destroys the SBP property. In general, this makes it impossible to obtain an energy estimate. This boundary procedure, often used in practical calculations, is referred to as the injection method and can result in an unwanted exponential growth of the solution, see for example [15], [8].

The basic idea behind the SAT method is to impose the boundary conditions weakly, as a penalty term, such that the SBP property is preserved and such that we get an energy estimate.

As an example of the simple, yet powerful SAT boundary procedure, we consider the hyperbolic scalar equation,

$$u_t + u_x = 0, \quad l \leq x \leq r, \quad t \geq 0, \quad u_l = g_l \ . \tag{8}$$

Integration by parts leads to,

$$\frac{d}{dt}\|u\|^2 = g_l^2 - u_r^2 \ . \tag{9}$$

A discrete approximation of (8) using an SBP operator to discretize the domain combined with the SAT method for the boundary conditions is given by

$$Hv_t + Qv = \tau \hat{e}_0 \left\{ v_0 - g_l \right\} \ , \tag{10}$$

where $\hat{e}_0$ is defined in (3).

The energy method on (10) leads to

$$\frac{d}{dt}\|v\|_H^2 = \frac{\tau^2}{2\,\tau - 1} g_l^2 - v_N^2 - (2\tau - 1)\left(v_0 - \frac{\tau}{2\,\tau - 1} g_l \right)^2 \ .$$

Clearly an energy estimate exist for $\tau > 1/2$. By choosing $\tau = 1$, the energy method leads to

$$\frac{d}{dt}\|v\|_H^2 = g_l^2 - v_N^2 - (v_0 - g_l)^2 \ . \tag{11}$$

Equation (11) is a discrete analog of the integration by parts formula (9) in the continuous case, where the extra term $(v_0 - g_l)^2$ introduces a small additional damping. Note that no artificial dissipation is included.

3 Boundary conditions

Consider the system

$$u_t + A\,u_x = (B\,u_x)_x + F \quad l \leq x \leq r, \quad t \geq 0 \ , \tag{12}$$

where F is a forcing function and A is a symmetric matrix . We consider (12) to be a one dimensional model for the compressible Navier-Stokes equations. The energy method applied to (12) leads to

$$\frac{d}{dt}\|u\|^2 = BT_l + BT_r + DI + FO \ , \tag{13}$$

where $BT_{l,r}$ denotes the boundary terms at the right and left boundary respectively, DI stands for the physical dissipation and $FO = \eta\|u\|^2 + \frac{1}{\eta}\|F\|^2$

(for some $\eta > 0$) corresponds to the forcing terms. To simplify notation, with no loss of generality, we will consider A and B to be constant coefficient matrices and apply zero forcing. The main focus in this paper is to obtain a stable boundary treatment. The nonlinear case $A = A(u)$ may require some damping (artificial dissipation or filtering) mechanism to avoid instabilities due to unresolved features.

3.1 The inviscid case

We start by consider the inviscid case i.e., (12) with $B = 0$. The number of boundary conditions at the inflow and outflow boundaries are given by the number of positive and negative eigenvalues to A respectively. We will consider characteristic boundary conditions. Since A is symmetric

$$A = X \Lambda X^T , \tag{14}$$

where the columns of X are the eigenvectors to A and Λ the eigenvalue matrix. By introducing the charcteristic variables $c = X^T u$, (12) transforms to

$$c_t + \Lambda c_x = 0 \quad l \leq x \leq r, \quad t \geq 0 , \tag{15}$$

where

$$c = \begin{bmatrix} \rho \bar{c} u - p \\ \sqrt{\frac{2}{\gamma - 1}} (\rho \bar{c}^2 - \rho) \\ \rho \bar{c} u + p \end{bmatrix} \; ; \; \Lambda = \begin{bmatrix} u - \bar{c} & & \\ & u & \\ & & u - \bar{c} \end{bmatrix} .$$

Here p is pressure ρ, the density and $\bar{c}$ the speed of sound.

Note that

$$u^T A u = u^T X^T \Lambda X \, u = c^T \Lambda \, c = c^T \Lambda_+ \, c + c^T \Lambda_- \, c \equiv c_+^T \Lambda_+ \, c_+ + c_-^T \Lambda_- \, c_- \; , \tag{16}$$

where $c_{+,-}$ are the characteristic variables corresponding to the positive (right going) and negative (left going) eigenvalues where

$$2 \, \Lambda_+ = \Lambda + |\Lambda| \; ; \; 2 \, \Lambda_- = \Lambda - |\Lambda| \; .$$

Hence, the boundary conditions at the left and right boundary respectively are given by

$$c_+ = g_l \; , \; c_- = g_r \; . \tag{17}$$

For simple subsonic flows

$$c_- = \begin{bmatrix} \rho \bar{c} u - p \\ 0 \\ 0 \end{bmatrix} \; ; \; \Lambda_+ = \begin{bmatrix} 0 & & \\ & u & \\ & & u - \bar{c} \end{bmatrix} .$$

The energy method on (15) and (17) leads to

$$\frac{d}{dt}\|c\|^2 = -c^T\Lambda c|_l^r = -(c^T\Lambda_+ c)_r - g_r^T\Lambda_- g_r + (c^T\Lambda_- c)_l + g_l^T\Lambda_+ g_l \ , \tag{18}$$

i.e., a strongly well posed problem.

The semi discrete problem using the SAT method is given by

$$c_t + (D\otimes\Lambda)c = -(H^{-1}\otimes\Sigma_L)e_0\otimes\Lambda_+(c_0 - g_l) + (H^{-1}\otimes\Sigma_R)e_N\otimes\Lambda_-(c_N - g_r) \ . \tag{19}$$

Here we have introduced the Kronecker product

$$C\otimes D = \begin{bmatrix} c_{0,0}\,D & \cdots & c_{0,q-1}\,D \\ \vdots & & \vdots \\ c_{p-1,0}\,D & \cdots & c_{p-1,q-1}\,D \end{bmatrix}$$

where C is a $p\times q$ matrix and D is a $m\times n$ matrix. Two useful rules for the Kronecker product are $(A\otimes B)(C\otimes D) = (AC)\otimes(BD)$, and $(A\otimes B)^T = A^T\otimes B^T$.

The energy method (multiply (19) from the left by $c^T H$ and add the transpose) leads to

$$\begin{aligned}\frac{d}{dt}\|c\|_H^2 = {} & (I-2\Sigma_L)\left(c_0 + \frac{\Sigma_L g_l}{I-2\Sigma_L}\right)^T \Lambda_+ \left(c_0 + \frac{\Sigma_L g_l}{I-2\Sigma_L}\right) \\ & - (I-2\Sigma_R)\left(c_N + \frac{\Sigma_R g_r}{I-2\Sigma_R}\right)^T \Lambda_- \left(c_N + \frac{\Sigma_R g_r}{I-2\Sigma_R}\right) \\ & - (\Sigma_L g_l)^T \frac{\Lambda_+}{I-2\Sigma_L}(\Sigma_L g_l) + (\Sigma_R g_r)^T \frac{\Lambda_-}{I-2\Sigma_R}(\Sigma_R g_r) \\ & - c_0^T\Lambda_+ c_0 + c_N^T\Lambda_- c_N \ .\end{aligned} \tag{20}$$

An energy estimate analog to the continuous case (18) is obtained if

$$\Sigma_L > \frac{I}{2} \quad , \quad \Sigma_r > \frac{I}{2} \ . \tag{21}$$

With zero boundary data we can use $\Sigma_{L,R} = \frac{I}{2}$. The choice $\Sigma_{L,R} = I$ leads to upwinding at the boundaries i.e.,

$$\begin{aligned}\frac{d}{dt}\|c\|_H^2 = {} & -(c_0 - g_l)^T\Lambda_+(c_0 - g_l) + (c_N - g_r)^T\Lambda_-(c_N - g_r) \\ & - c_0^T\Lambda_+ c_0 + c_N^T\Lambda_- c_N + g_l^T\Lambda_+ g_l - g_r^T\Lambda_- g_r \ .\end{aligned} \tag{22}$$

To solve (12) numerically it is not necessary to transform the system of equations to characteristic form. The transformation is done to simplify the stability analysis i.e, to find the penalty matrices $\Sigma_{L,R}$. Transforming (19) back to the original variables we obtain

$$v_t + (D \otimes A)v = P_l^I + P_r^I \; . \tag{23}$$

where the modified penalties are given by

$$\begin{array}{l} P_l^I = -(H^{-1} \otimes \Sigma_L) e_0 \otimes A_+ (v_0 - \tilde{g}_l) \; , \\ P_r^I = +(H^{-1} \otimes \Sigma_L) e_N \otimes A_- (v_N - \tilde{g}_r) \; , \end{array} , \tag{24}$$

and

$$A_{+,-} = X \Lambda_{+,-} X^T \; , \; u_{0,N} = X c_{0,N} \; , \; \tilde{g}_{l,r} = X g_{l,r} \; .$$

Remark Due to the simple form (21) of the penalty matrices $\Sigma_{L,R}$ and the fact that $X^T X = I$, the penalty matrices do not change when we transform back to the original variables (23). It is alright to specify all of the variables (v) in both penalties (left and right boundary) since we are scaling the penalties by $A_{+,-}$. Hence, on the right (or left) boundary we are in fact only penalizing the left (or right) going characteristics variables.

3.2 The viscid case

When characteristic boundary conditions are used for incomplete parabolic systems (like the Navier-Stokes equations), there are some theoretical problems, see for example [10], [9]. Here we will present an alternative way of imposing boundary conditions for the Navier-Stokes equations that will lead to strongly well posed as well as strictly stable numerical schemes for the linearised problem..

There are basically two different types of well posed boundary conditions for the Navier-Stokes equations: i) Robin type, or ii) Dirichlet type. The first type are suitable at subsonic outflow or when boundary data are not well known. The second type is suitable for solid walls or otherwise when boundary data exists.

3.2.1 Case 1

We start by considering the Robin type boundary conditions i.e,

$$\alpha_l A_+ u + \beta_l B\, u_x = g_l, \; x = l \quad ; \quad \alpha_r A_- u + \beta_r B\, u_x = g_r, \; x = r \; . \tag{25}$$

To simplify the notation (without loss of generality) we assume zero boundary data. The energy method on (12) imposing the Robin type boundary conditions (25) leads to

$$\begin{aligned}\frac{d}{dt}\|u\|^2 =& u_l^T A_+ \left(I + \frac{2\alpha_l}{\beta_l} I\right) u_l + u_l^T A_- u_l \\ & - u_r^T A_- \left(I + \frac{2\alpha_r}{\beta_r} I\right) u_r - u_r^T A_+ u_r \\ & - \int_l^r u_x^T (B + B^T) u_x \, dx \ .\end{aligned} \tag{26}$$

An energy estimate exists for

$$I + \frac{2\alpha_{l,\,r}}{\beta_{l,\,r}} \leq 0 \ . \tag{27}$$

Remark By chosing $\alpha_{l,r}$ and $\alpha_{l,r}$ we can introduce energy dissipation at the boundaries that can introduce a stabilizing effect for problems where the boundary data is not well known. See for example [Jannes avhandling]

The semi discrete problem using the SAT method is given by

$$\begin{aligned}v_t + (D \otimes A)v = (D_2 \otimes B)v - (H^{-1} \otimes \Omega_L) e_0 \otimes (\alpha_l A_+ v_0 + \beta_l B (Su)_0 - g_l) \\ + (H^{-1} \otimes \Omega_R) e_N \otimes (\alpha_r A_- v_N + \beta_r B (Su)_N - g_r)\end{aligned} \tag{28}$$

The energy method leads to

$$\begin{aligned}\frac{d}{dt}\|v\|_H^2 =& v_0^T A_+ (I - 2\Omega_l \alpha_l) v_0 + v_0^T A_- v_0 \\ & - v_N^T A_- (I - 2\Omega_r \alpha_r) v_N - v_N^T A_+ v_N \\ & - v_0^T B (I + \Omega_l \beta_l) (Sv)_0 + v_N^T B (I + \Omega_r \beta_r) (Sv)_N \\ & - (Sv)_0^T \left(I + \Omega_l^T \beta_l\right) B \, v_0 + (Sv)_N^T \left(I + \Omega_r^T \beta_r\right) B \, v_N \\ & - v^T \left(M \otimes B + M^T \otimes B^T\right) \, v \ .\end{aligned} \tag{29}$$

If D_2 is a symmetric second derivative SBP operator (see Definition 2.6) and

$$\Omega_{l,\,r} = -\frac{I}{\beta_{l,\,r}} \tag{30}$$

hold, we obtain

$$\begin{aligned}\frac{d}{dt}\|v\|_H^2 =& v_0^T A_+ \left(I + \frac{2\alpha_l}{\beta_l} I\right) v_0 + v_0^T A_- v_0 \\ & - v_N^T A_- \left(I + \frac{2\alpha_r}{\beta_r} I\right) v_N - v_N^T A_+ v_N \\ & - v^T \left(B + B^T \otimes M\right) \, v \ ,\end{aligned} \tag{31}$$

i.e., an energy estimate completely analog to the continuous case (26) .

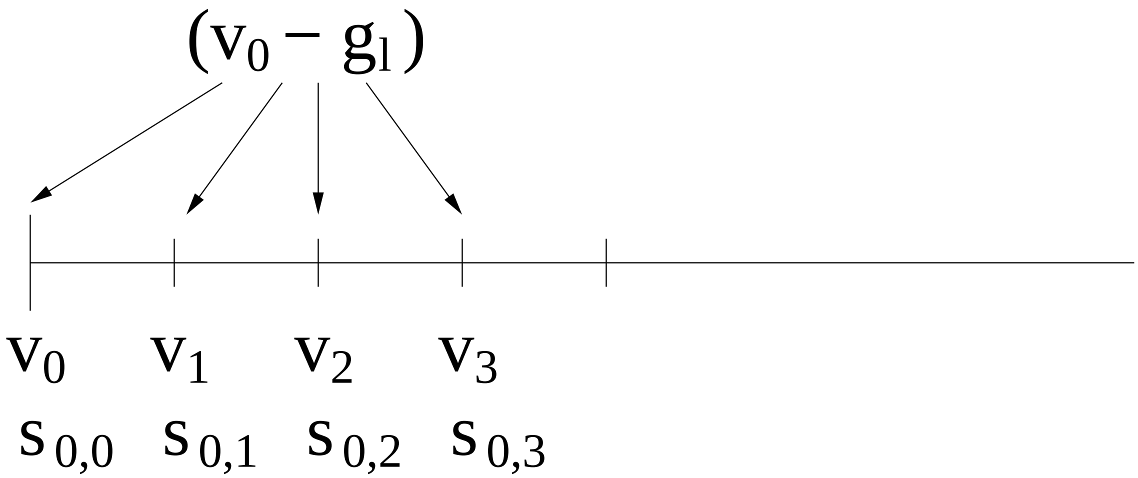


Figure 1: An illustration how the viscid penalty is imposed weakly at the four points closest to the boundary. Here $s_{0,3} = \frac{1}{3}$ is the 4th parameter in S.

3.2.2 Case 2

Now we turn to the second case, specifying only the variables. The SAT method for this type of boundary condition leads to

$$v_t + (D \otimes A)v = (D_2 \otimes B)v + P_l^I + P_r^I + P_l^v + P_r^v \tag{32}$$

where $P_{l,r}^I$ are the inviscid penalties (24) and $P_{l,r}^V$ the corresponding viscous penalties given by

$$\begin{aligned} P_l^V &= -(H^{-1} \otimes \Gamma_l)(S^T \otimes B_l)e_0 \otimes (v_0 - \tilde{g}_l) \\ P_r^V &= +(H^{-1} \otimes \Gamma_r)(S^T \otimes B_r)e_N \otimes (v_N - \tilde{g}_r) \end{aligned} \quad . \tag{33}$$

Remark Since we are scaling the viscid penalties by S^T the Dirichlet boundary conditions will be imposed weakly at all points used in the reconstruction of the boundary derivatives i.e., $(Sv)_{0,N}$. The left boundary derivative approximation for the $4th$ order accurate operator (see Appendix I) requires 4 points in the reconstruction. This means that the first four points, see figure 1, will be penalized, with the weights given by the coefficients in S.

The energy method on (32) leads to

$$\begin{aligned} \frac{d}{dt}\|v\|_H^2 = {} & v_0^T A_+ \left(I - 2\Sigma_l\right) v_0 + v_0^T A_- v_0 \\ & - v_N^T A_- \left(I - 2\Sigma_r\right) v_N - v_N^T A_+ v_N \\ & - 2v_0^T B \left(I + \Gamma_l\right) (Sv)_0 + 2v_N^T B \left(I + \Gamma_r\right) (Sv)_N \\ & - v^T M \otimes \left(B + B^T\right) v \ . \end{aligned} \tag{34}$$

An energy estimate analog to the continuous case is obtained if the inviscid stability conditions (21) hold and the viscid stability conditions are given by

$$\Gamma_{l,r} = -I \ . \tag{35}$$

By chosing the inviscid penalties to $\Sigma_{l,r} = \frac{I}{2}$ we obtain

$$\begin{aligned} \frac{d}{dt}\|v\|_H^2 = v_0^T A_- v_0 - v_N^T A_+ v_N \ . \\ - v^T M \otimes \left(B + B^T\right) v \ . \end{aligned} \tag{36}$$

4 Interface conditions

Consider the following system of equations with an interface at $x = i$

$$\begin{aligned} u_t + Au_x &= (B_l u_x)_x + F_l \quad l \leq x \leq i, \quad t \geq 0 \ , \\ p_t + Ap_x &= (B_r p_x)_x + F_r \quad i \leq x \leq r, \quad t \geq 0 \ . \end{aligned} \tag{37}$$

The energy method applied to (37) leads to

$$\frac{d}{dt}(\|u\|_l^2 + \|p\|_r^2) = BT_l + BT_r + DI + FO + IT_i \ , \tag{38}$$

where $BT_{l,r}$ denotes the (outher) boundary terms, FO the forcing terms, DI correspond to the physical dissipation and $IT_i = u^T(Au - 2B_l u_x) - p^T(Ap - 2B_r p_x)$ the interface coupling term. The interface conditions at the interface $x = i$ are given by

$$u = p \ , \ B_l u_x = B_r p_x \ . \tag{39}$$

The main focus here is on obtaining a stable and accurate interface coupling for the corresponding semi discrete problem.

Remark We can have different forcing $F_l \neq F_r$ and different physical viscosity $B_l \neq B_r$ in the two regions, the following analysis still holds.

The semi discrete finite difference approximation to (37) and (39) can be written

$$\begin{aligned} v_t + (D_l \otimes A)v &= ((D_2)_l \otimes B_l)v + P_l^I + P_l^V + P_{li}^I + P_{li}^V \\ w_t + (D_r \otimes A)p &= ((D_2)_r \otimes B_r)w + P_r^I + P_r^V + P_{ri}^I + P_{ri}^V \end{aligned} \tag{40}$$

where the inviscid interface penalties are given by

$$\begin{aligned} P_{li}^I &= (H_l^{-1} \otimes \Pi_l) e_N \otimes A_-(v_N - w_0) \\ P_{ri}^I &= -(H_r^{-1} \otimes \Pi_r) e_0 \otimes A_+(w_0 - v_N) \ , \end{aligned} \tag{41}$$

and corresponding viscid interface penalties

$$
\begin{aligned}
P_{li}^{V} &= + (H_l^{-1} \otimes \Psi_l)e_N \otimes (B_l(S_l v)_N - B_r(S_r w)_0) \\
&\quad + (H_l^{-1} \otimes \Phi_l)(S_l^T \otimes B_l)e_N \otimes (v_N - w_0) \\
P_{ri}^{V} &= - (H_r^{-1} \otimes \Psi_r)e_0 \otimes (B_r(S_r w)_0 - B_l(S_l v)_N) \\
&\quad - (H_r^{-1} \otimes \Phi_r)(S_r^T \otimes B_r)e_0 \otimes (w_0 - v_N)
\end{aligned} . \tag{42}
$$

Notice that we can have different SBP discretizations in the left (l) and right (r) domain respectively, the analysis still holds. The energy method applied to (40) leads to

$$
\frac{d}{dt}(\|v\|_{H_l}^2 + \|w\|_{H_r}^2) = IT^V + IT^I + BT + DI \ . \tag{43}
$$

BT corresponds to the outher boundary terms, DI the physical dissipation and $IT^I = y_l^T M_l \, y_l + y_r^T M_r \, y_r$ the inviscid part of interface coupling, where

$$
M_r = - \begin{bmatrix} I & -\Pi_r \\ -\Pi_r & 2\,\Pi_r - I \end{bmatrix} \ , \ M_l = - \begin{bmatrix} 2\,\Pi_l - I & -\Pi_l \\ -\Pi_l & I \end{bmatrix} \ ,
$$

and

$$
y_r = \begin{bmatrix} |\Lambda_+|^{1/2} X v \\ |\Lambda_+|^{1/2} X w \end{bmatrix} \ , y_l = \begin{bmatrix} |\Lambda_-|^{1/2} X v \\ |\Lambda_-|^{1/2} X w \end{bmatrix} \ .
$$

The viscid part of the interface coupling is given by

$$
\begin{aligned}
IT^V &= 2 v_N^T B_l (Sv)_N (I + \Psi_l + \Phi_l) + 2 w_N^T B_l (Sw)_0 (-I + \Psi_r + \Phi_r) \\
&\quad - 2 v_N^T B_r (Sw)_0 (\Psi_l + \Phi_r) - 2 w_0^T B_l (Sv)_N (\Psi_r + \Phi_l)
\end{aligned} .
$$

A stable coupling requires that i) $IT^I \leq 0$, and ii) $IT^V = 0$. The first condition holds if

$$
\Pi_l = I \ , \ \Pi_r = I \ . \tag{44}
$$

The second condition holds if

$$
\Psi_r = \Psi, \ \Psi_l = \Psi - 1, \ \Phi_l = -\Psi, \ \Phi_r = I - \Psi \ . \tag{45}
$$

Notice that Ψ is a free parameter, but its value will affect the eigenvalues i.e., the stiffness of the problem. To minimize stiffness we chose $\Psi = \frac{I}{2}$. Together the interface conditions (44) and (45) leads to a stable interface coupling given by

$$
IT^V + IT^I = +(v_N - w_0)^T A_- (v_N - w_0) - (v_N - w_0)^T A_+ (v_N - w_0) \, .
$$

5 Computations and further analysis

In this section we compare the numerical results of two dimensional vortex calculations in free space, in order to test the accuracy and stability of the boundary and interface treatment. We consider both the viscid and inviscid case. We compare two different finite difference approximations. The spatial derivatives in both methods are approximated with fourth-order accurate central-differences in the interior with second order boundary closures. The first discretization (see [17]) have a non-SBP closure while the other (see Appendix I) is SBP. We use implicit time-advancement with approximate factorization. The equations are written in primitive variables,

$$u_t + A_1 u_x + A_2 u_y = B_{11} u_{xx} + +B_{22} u_{yy} + B_{12} u_{xy} + B_{21} u_{yx} + F \ ,$$

where

$$u^T = [\rho,\, u,\, v,\, T] \ .$$

Details of the numerical implementation of the compressible code can be found in [17].

Several different methods for the coupling procedure are tested. Two types of penalization are considered: penalization of variables (non-characteristic penalization) and penalization of characteristics. Conventional method of Riemann invariants (without penalization) is also implemented for comparison. Also, two types of differential operators are compared: SBP and non-SBP. Five numerical test cases, referred to as cases A, B, C, D, E (see Table 1) are analyzed, representing different combinations of the parameters defining the coupling method.

	penalty variable	penalty charact.	Riemann
SBP	A	B	
Non-SBP	C	D	E

Table 1: Methods for specifying interface conditions

5.1 Inviscid vortex

The inviscid vortex model is presented in [3]. It satisfies the two-dimensional Euler equations, under the assumption of isentropy. In [13] it is shown that the solution is steady in the frame of reference moving with the freestream. The scaled vortex has the velocity field

$$v_\Theta = \frac{\epsilon r}{2\pi} exp\left(\frac{1-r^2}{2}\right) \ , \tag{46}$$

where ϵ is the nondimensional circulation, (r, Θ) are the polar coordinates. The analytic solution in a fixed frame of reference (x, y, t) becomes

$$\begin{aligned} u &= 1 - \tfrac{\epsilon y}{2\pi} exp\left(\tfrac{f(x,y,t)}{2}\right) \,, & v &= \tfrac{\epsilon(x-(t-T))}{2\pi} exp\left(\tfrac{f(x,y,t)}{2}\right) \\ \varrho &= \left(1 - \tfrac{\epsilon^2(\gamma-1)M^2}{8\pi^2} exp\left(f(x,y,t)\right)\right)^{\frac{1}{\gamma-1}} \,, & p &= \tfrac{\varrho^\gamma}{\gamma M^2} \end{aligned} \tag{47}$$

where $f(x, y, t) = 1 - (((x - x_0) - t)^2 + y^2)$, M is the Mach number, $\gamma = c_p/c_v$ and x_0 is the initial position of the vortex (in the x-direction). The vortex is introduced into the computational domain by using the analytic solution as boundary data and initial data.

The convergence rate is calculated as,

$$q = \log_{10}\left(\frac{\|u - v^{h1}\|_h}{\|u - v^{h2}\|_h}\right) / \log_{10}\left(\frac{h_1}{h_2}\right) \,, \tag{48}$$

where u is the analytic solution and v^{h1} the corresponding numerical solution with grid size h_1. $\|u - v^{h1}\|_h$ is the $l_2 - error$. The results are presented in Table 2 [Ioulia, this is just an example, taken from a previous convergence study].

N	$2:nd$	q	$3:rd$	q	$5:th$	q
50	8.32e-6		1.52e-6		6.75e-7	
100	2.58e-6	1.68	2.24e-7	2.77	1.06e-8	6.00
150	1.24e-6	1.80	7.30e-8	2.76	1.12e-9	5.53
200	7.20e-7	1.90	3.24e-8	2.83	2.49e-10	5.24

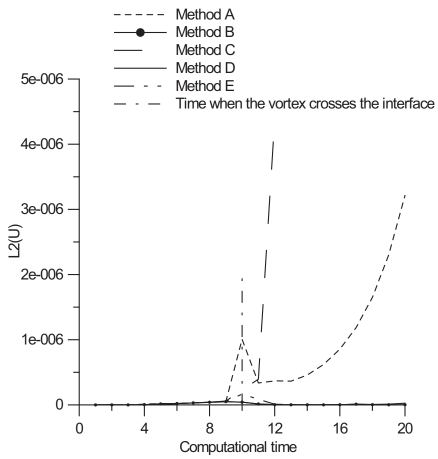
Table 2: l_2-error and convergence rate q, tested for a vortex in free space.

5.2 Viscid vortex

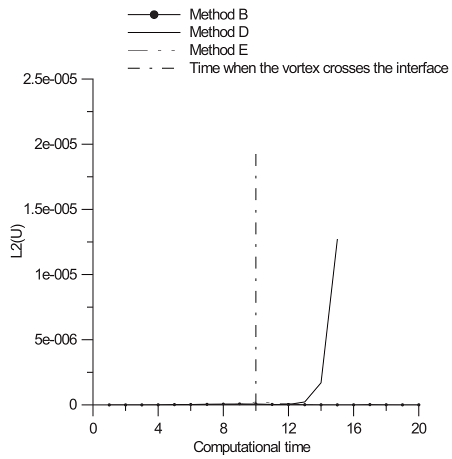
Here we want to test also the viscid boundary treatment by convecting aTaylor vortex [16] across the interface coupling of the two domains. The Taylor vortex is an analytic solution to Navier Stokes equations and has the following form:

$$v_\phi = \frac{\mathcal{M}r}{16\pi\nu^2t^2}\exp(\frac{-r^2}{4\nu t});\ v_r = v_z = 0,$$

where v_ϕ is tangential velocity, v_r is radial velocity, v_z is spanwise velocity and r is the distance from the center of the vortex. This circular vortex is being convected with the free-stream from one computational domain to another. The size of the domains is approximately 18x15 in terms of the radius



(a) $Re = 330$



(b) $Re = \infty$

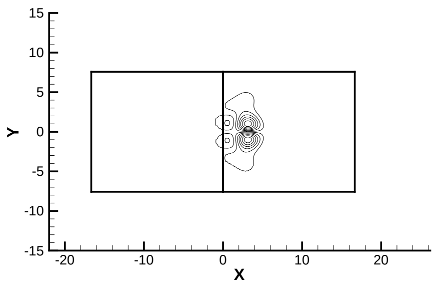
Figure 2: Behavior of L2 error of streamwise velocity U versus time

of the vortex with 144x96 computational grid points in each domain. The comparison of L2 error of streamwise velocity component U versus the computational time can be seen in Figure 2(a) for rather law Reynolds number $Re = 330$ with respect to the radius of the vortex.

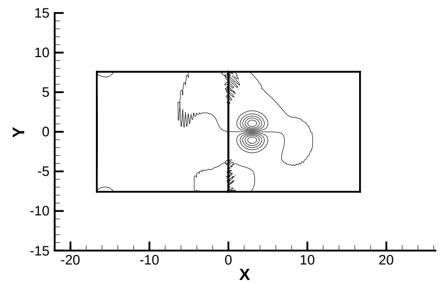
It is clearly seen from the Figure 2(a) that methods A and C, corresponding to the non-characteristic penalization, are unstable for the given problem even for such a low Reynolds number. It is consistent with the well-known fact that only characteristic-type boundary conditions properly describe the propagation of information in and out of the compressible medium.

To test methods B, D and E further, viscosity is reduced to zero, being a very rigorous test for stability. No artificial dissipation is added in the code. L2 errors of U for methods B, D and E are shown in Figure 2(b). It is seen that method D, which is the combination of characteristics penalties with non-SBP boundary operators, also proves to be unstable. This is expected, since for non-SBP type operators the cancelation of boundary terms during penalization does not occur and the technique is not guaranteed to be stable.

To illustrate the nature of instability for penalty method for non-characteristic penalization and for non-SBP operators we plot the contours of streamwise velocity when vortex just crossed the interface for methods A (SBP, but non-characteristic penalty) and D (characteristic penalty, but not SBP) for $Re = \infty$ in Figure 3. One can see that the nature of instability is completely different for these two methods. For method A, which is SBP, but non-characteristic penalization, the vortex itself is distorted, which is due to the violation of characteristic information propagation. However, for the method



(a) Method A



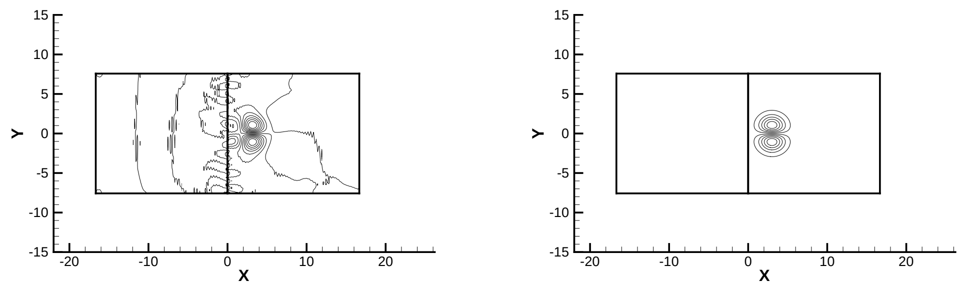
(b) Method D

Figure 3: Contours of streamwise velocity when the vortex just crossed the interface

D, which is characteristic penalty, but not SBP, the vortex itself seems to be perfectly transformed, but the numerical errors are being accumulated at the interface because of the lack of cancelation.

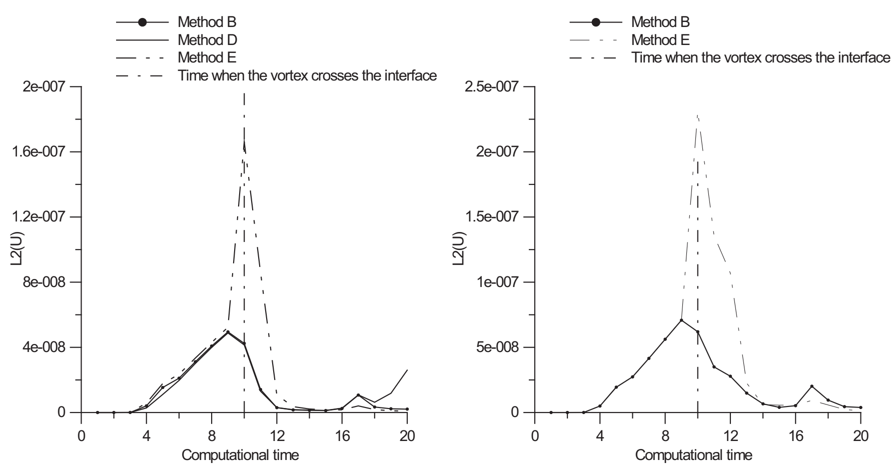
The same contour plots for methods B and C are plotted in Figure 4. As expected, for method C, which is non-SBP and non-characteristic penalization, we get both numerical errors at the interface and distortion of the vortex. And for method B, which is SBP and characteristics penalty, we get the perfect shape of the vortex with no numerical noise accumulated at the interface.

Only methods B, corresponding to characteristic penalization with SBP operators, and E, corresponding to conventional and robust method of information exchange through Riemann invariants, can be used for stable coupling of two compressible codes. To compare the accuracy of two stable methods B and E we plot L2 errors of U for both Reynolds numbers $Re = 330$ and $Re = \infty$ in Figure 5. It is clearly observed that characteristics penalization technique with SBP gives an error, which is approximately 4 times smaller than the Riemann invariants method for both Reynolds numbers. Errors of the other quantities (vertical velocity, density and pressure) show the similar behavior. This leads to the conclusion that the penalty method with SBP-type differential operators and characteristic penalization is the best choice for coupling compressible to compressible codes.



(a) Method C (b) Method B

Figure 4: Contours of streamwise velocity when the vortex just crossed the interface



(a) $Re = 330$ (b) $Re = \infty$

Figure 5: Behavior of L2 error of streamwise velocity U versus time for methods B and E

6 Conclusions

APPENDIX

I SBP operators

What I have denoted D is denoted by D_1.

I.1 First order accuracy at the boundary

The discrete norm H and the discrete second order accurate SBP operator $H^{-1}Q$ approximating $\frac{d}{dx}$ are given by

$$H = h\begin{bmatrix} \frac{1}{2} & & & & \\ & 1 & & & \\ & & \ddots & & \\ & & & 1 & \\ & & & & \frac{1}{2} \end{bmatrix} \quad H^{-1}Q = \frac{1}{h}\begin{bmatrix} -1 & 1 & & & \\ -\frac{1}{2} & 0 & \frac{1}{2} & & \\ & \ddots & \ddots & \ddots & \\ & & -\frac{1}{2} & 0 & \frac{1}{2} \\ & & & -1 & 1 \end{bmatrix}.$$

The discrete second order accurate SBP operator $D_2 = H^{-1}(-A + BS)$ approximating $\frac{d^2}{dx^2}$ and the boundary derivative operator BS are given by,

$$D_2 = \frac{1}{h^2}\begin{bmatrix} 1 & -2 & 1 & & \\ 1 & -2 & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & 1 & -2 & 1 \\ & & 1 & -2 & 1 \end{bmatrix} \quad S = \frac{1}{h}\begin{bmatrix} -\frac{3}{2} & 2 & -\frac{1}{2} & & \\ & 1 & & & \\ & & \ddots & & \\ & & & 1 & \\ & & \frac{1}{2} & -2 & \frac{3}{2} \end{bmatrix}.$$

I.2 Second order accuracy at the boundary

The discrete norm H is defined:

$$H = h\begin{bmatrix} \frac{17}{48} & & & & & \\ & \frac{59}{48} & & & & \\ & & \frac{43}{48} & & & \\ & & & \frac{49}{48} & & \\ & & & & 1 & \\ & & & & & \ddots \end{bmatrix}.$$

The discrete difference SBP operator approximating $\frac{d}{dx}$ we denote $D_1 = H^{-1}Q$.

$$D_1 = \frac{1}{h}\begin{bmatrix} -\frac{24}{17} & \frac{59}{34} & -\frac{4}{17} & -\frac{3}{34} & 0 & 0 & 0 & \\ -\frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 & \\ \frac{4}{43} & -\frac{59}{86} & 0 & \frac{59}{86} & -\frac{4}{43} & 0 & 0 & \\ \frac{3}{98} & 0 & -\frac{59}{98} & 0 & \frac{32}{49} & -\frac{4}{49} & 0 & \\ 0 & 0 & \frac{1}{12} & -\frac{2}{3} & 0 & \frac{2}{3} & -\frac{1}{12} & \\ & & & \ddots & \ddots & \ddots & \ddots & \ddots \end{bmatrix}.$$

The discrete 4th order accurate SBP operator $D_2 = H^{-1}(-A + BS)$ approximating $\frac{d^2}{dx^2}$ is given by,

$$D_2 = \frac{1}{h^2}\begin{bmatrix} 2 & -5 & 4 & -1 & 0 & 0 & 0 & \\ 1 & -2 & 1 & 0 & 0 & 0 & 0 & \\ -\frac{4}{43} & \frac{59}{43} & -\frac{110}{43} & \frac{59}{43} & -\frac{4}{43} & 0 & 0 & \\ -\frac{1}{49} & 0 & \frac{59}{49} & -\frac{118}{49} & \frac{64}{49} & -\frac{4}{49} & 0 & \\ 0 & 0 & -\frac{1}{12} & \frac{4}{3} & -\frac{5}{2} & \frac{4}{3} & \frac{1}{12} & \\ & & & \ddots & \ddots & \ddots & \ddots & \ddots \end{bmatrix},$$

and the 3rd order accurate boundary derivative operator BS is given by,

$$S = \frac{1}{h}\begin{bmatrix} -\frac{11}{6} & 3 & -\frac{3}{2} & \frac{1}{3} & & & \\ & 1 & & & & & \\ & & & \ddots & & & \\ & & & & & 1 & \\ & & & -\frac{1}{3} & \frac{3}{2} & -3 & \frac{11}{6} \end{bmatrix}.$$

I.3 Third order accuracy at the boundary

The discrete norm H is defined:

$$H = h\begin{bmatrix} \frac{13649}{43200} & & & & & & & \\ & \frac{12013}{8640} & & & & & & \\ & & \frac{2711}{4320} & & & & & \\ & & & \frac{5359}{4320} & & & & \\ & & & & \frac{7877}{8640} & & & \\ & & & & & \frac{43801}{43200} & & \\ & & & & & & 1 & \\ & & & & & & & \ddots \end{bmatrix}.$$

The discrete difference operator approximating $\frac{d}{dx}$ we denote (here) $D_1 = H^{-1}Q$, obeying our wanted SBP property:

$$
D1 = \frac{1}{h}\begin{bmatrix}
-\frac{21600}{13649} & \frac{104009}{54596} & \frac{30443}{81894} & -\frac{33311}{27298} & \frac{16863}{27298} & -\frac{15025}{163788} & 0 & 0 & 0 & 0 \\
-\frac{104009}{240260} & 0 & -\frac{311}{72078} & \frac{20229}{24026} & -\frac{24337}{48052} & \frac{36661}{360390} & 0 & 0 & 0 & 0 \\
-\frac{30443}{162660} & \frac{311}{32532} & 0 & -\frac{11155}{16266} & \frac{41287}{32532} & -\frac{21999}{54220} & 0 & 0 & 0 & 0 \\
\frac{33311}{107180} & -\frac{20229}{21436} & \frac{485}{1398} & 0 & \frac{4147}{21436} & \frac{25427}{321540} & \frac{72}{5359} & 0 & 0 & 0 \\
-\frac{16863}{78770} & \frac{24337}{31508} & -\frac{41287}{47262} & -\frac{4147}{15754} & 0 & \frac{342523}{472620} & -\frac{1296}{7877} & \frac{144}{7877} & 0 & 0 \\
\frac{15025}{525612} & -\frac{36661}{262806} & \frac{21999}{87602} & -\frac{25427}{262806} & -\frac{342523}{525612} & 0 & \frac{32400}{43801} & -\frac{6480}{43801} & \frac{720}{43801} & 0 \\
0 & 0 & 0 & -\frac{1}{60} & \frac{3}{20} & -\frac{3}{4} & 0 & \frac{3}{4} & -\frac{3}{20} & \frac{1}{60} \\
 & & & & & & & & & \\
 & & & & & \ddots & \ddots & \ddots & \ddots &
\end{bmatrix}
$$

The discrete difference operator approximating $\frac{d^2}{dx^2}$ obeying our wanted SBP property is written $D_2 = H^{-1}\left(-A + BS\right)$, where $A = A^T \leq 0$. The interior stencil is the standard sixth order accurate central scheme $h^2\left(D_2\, v\right)_j = \frac{1}{90}\, v_{j-3} - \frac{3}{20}\, v_{j-2} + \frac{3}{2}\, v_{j-1} - \frac{49}{18}\, v_j + \frac{3}{2}\, v_{j+1} - \frac{3}{120}\, v_{j+2} + \frac{1}{90}\, v_{j+3}$. At the boundary the operator becomes:

$$
D2 = \frac{1}{h^2}\begin{bmatrix}
\frac{114170}{40947} & -\frac{438107}{54596} & \frac{336409}{40947} & -\frac{276997}{81894} & \frac{3747}{13649} & \frac{21035}{163788} & 0 & 0 & 0 & 0 \\
\frac{6173}{5860} & -\frac{2066}{879} & \frac{3283}{1758} & -\frac{303}{293} & \frac{2111}{3516} & -\frac{601}{4395} & 0 & 0 & 0 & 0 \\
-\frac{52391}{81330} & \frac{134603}{32532} & -\frac{21982}{2711} & \frac{112915}{16266} & -\frac{46969}{16266} & \frac{30409}{54220} & 0 & 0 & 0 & 0 \\
\frac{68603}{321540} & -\frac{12423}{10718} & \frac{112915}{32154} & -\frac{75934}{16077} & \frac{53369}{21436} & -\frac{54899}{160770} & \frac{48}{5359} & 0 & 0 & 0 \\
-\frac{7053}{39385} & \frac{86551}{94524} & -\frac{46969}{23631} & \frac{53369}{15754} & -\frac{87904}{23631} & \frac{820271}{472620} & -\frac{1296}{7877} & \frac{96}{7877} & 0 & 0 \\
\frac{21035}{525612} & -\frac{24641}{131403} & \frac{30409}{87602} & -\frac{54899}{131403} & \frac{820271}{525612} & -\frac{117600}{43801} & \frac{64800}{43801} & -\frac{6480}{43801} & \frac{480}{43801} & 0 \\
0 & 0 & 0 & \frac{1}{90} & -\frac{3}{20} & 3/2 & -\frac{49}{18} & 3/2 & -\frac{3}{20} & \frac{1}{90} \\
 & & & & & & & & & \\
 & & & & & \ddots & \ddots & \ddots & \ddots &
\end{bmatrix}
$$

and the boundary derivative operator of 4th order accuracy is:

$$
S = \frac{1}{h}\begin{bmatrix}
-\frac{25}{12} & 4 & -3 & \frac{4}{3} & -\frac{1}{4} & & \\
 & & 1 & & & & \\
 & & & \ddots & & & \\
 & & & & 1 & & \\
 & & \frac{1}{4} & -\frac{4}{3} & 3 & -4 & \frac{25}{12}
\end{bmatrix}.
$$

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