Optimization in Communication Networks
Lecture 5: Discrete-time Markov Chain

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Lecture Outline

- Markov Chain
- Recurrence
- Invariant Measure
- Positive Recurrence
- Stationary Distribution
- Foster's Criteria
- Implications
- Poisson Process
- Continuous Time Markov Chain

- Many people pretend to know Markov chains!
- A very good reference book: [Bremaud, 1999]

Markov Chain: Definition and Stopping Time

- Definition. Let $X_1, \ldots, X_n, \ldots$ be a sequence of random variables taking values in some finite or countably finite space $\mathcal{E}$, such that

$$\begin{aligned} p_{ij} &= \mathbb{P}[X_{n+1} = j | X_n = i] \\ &= \mathbb{P}[X_{n+1} = j | X_n = i, X_{n-1} = i_{n-1}, \ldots, X_0 = i_0], \end{aligned}$$

  for all $i, j \in \mathcal{E}$, $n \geq 0$.

  > "For any fixed n, the future of the process is independent of $\{X_1, \ldots, X_n\}$, given X_n."

  Then, $\{X_n\}_{n \leq 0}$ is called time homogeneous markov chain (HMC). Then, the matrix $P = [p_{ij}]$ is called its transition probability matrix.

- We will denote by $\mathcal{F}_n$ the "history" $\{X_1, \ldots, X_n\}$. That is, $\mathcal{F}_n$ contains information about the past upto time n.

- Definition. A random variable T is called stopping time with respect to $\{\mathcal{F}_n\}_{n \geq 0}$, if one can answer the question "$T > n$?" by examining $\mathcal{F}_n$ for all $n \geq 0$. Formally $\{T > n\} \in \mathcal{F}_n$.

BTW, is the $\{T > n\}$ a set? Why?

Example

- Let $\mathcal{E} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$,. Initially, $X_0 = 0$, and $\forall n \leq 0$,

$$\mathbb{P}[X_{n+1} = X_n + 1 | X_n] = \mathbb{P}[X_{n+1} = X_n - 1 | X_n] = 1/2.$$

- Draw transition diagram.

- Check whether the above is Markov chain or not.

- $T = \min\{k \geq 1 | X_k = 0\}$ is a stopping time.

- Ex) If T is a stopping time, then $\{T = n\} \in \mathcal{F}_n$ (because $\{T = n\} = \{T > n - 1\} \setminus \{T > n\}$).

Strong Markov Property

- Theorem. [Strong Markov Property] Given HMC $\{X_n\}_{n\geq 0}$ with transition matrix P, and a stopping time τ. Let $X_\tau = i$ for some $i \in \mathcal{E}$. Then,

  (a) $\{X_0, \dots X_{\tau-1}\}$ and $\{X_{\tau+n}\}_{n\geq 1}$ are independent given $\{X_\tau = i\}$.
  (b) The $\{X_{\tau+n}\}_{n\geq 1}$ is HMC with the same transition matrix P.

- Proof.

  (a): We wish to establish the following: For any $k \geq 1$,

$$\mathbb{P}\Big[(X_0 = i_0, \dots, X_{\tau-1} = i_{\tau-1}); (X_{\tau+1} = j_1, \dots, X_{\tau+k} = j_k) | X_\tau = i\Big] =$$

$$\mathbb{P}\Big[X_0 = i_0, \dots, X_{\tau-1} = i_{\tau-1} | X_\tau = i\Big] \cdot \mathbb{P}\Big[X_{\tau+1} = j_1, \dots, X_{\tau+k} = j_k | X_\tau = i\Big].$$

Equivalently, we want to prove the following:

$$\mathbb{P}\Big[X_{\tau+1} = j_1, \dots, X_{\tau+k} = j_k | X_\tau = i; X_0 = i_0, \dots, X_{\tau-1} = i_{\tau-1}\Big] = \mathbb{P}\Big[X_{\tau+1} = j_1, \dots, X_{\tau+k} = j_k | X_\tau = i\Big].$$

We will prove that above by showing that $LHS = RHS = p_{ij}$, where $k = 1$. Then, the similar things can be proved for other k by using induction on k. Let

$$(A) \triangleq \mathbb{P}\Big[X_{\tau+1} = j_1 | X_\tau = i; (X_0 = i_0, \ldots, X_{\tau-1} = i_{\tau-1})\Big] = \frac{\mathbb{P}\Big[X_{\tau+1} = j_1; X_\tau = i; X_0^{\tau-1} = i_0^{\tau-1}\Big]}{\mathbb{P}\Big[X_\tau = i; X_0^{\tau-1} = i_0^{\tau-1}\Big]},$$

where we use the notation $X_0^{\tau-1} = (X_0, \ldots, X_{\tau-1})$, and $i_0^{\tau-1} = (i_0, \ldots, i_{\tau-1})$.

Then, the numerator of (A) reads

$$\sum_{n \geq 0} \mathbb{P}\Big[\tau = n, X_{n+1} = j_1; X_n = i; X_0^{n-1} = i_0^{n-1}\Big] =$$

$$\sum_{n \geq 0} \mathbb{P}\Big[X_{n+1} = j_1 | X_n = i; X_0^{n-1} = i_0^{n-1}, \tau = n\Big] \cdot \mathbb{P}\Big[\tau = n; X_n = i; X_0^{n-1} = i_0^{n-1}\Big]$$

Now, note that $\{\tau = n\} \in \mathcal{F}_n$. Thus, by (weak) Markovian property of X_n, we get:

$$\mathbb{P}\Big[X_{n+1} = j_1 | X_n = i; X_0^{n-1} = i_0^{n-1}, \tau = n\Big] = \mathbb{P}\Big[X_{n+1} = j | X_n = i\Big] = p_{ij}.$$

Then, easy to prove:

$$\text{Num. of (A)} = p_{ij} \cdot \text{Denum. of (A)},$$

i.e., (A) $= p_{ij}$. Thus, $LHS = p_{ij}$. Similarly, we can prove that $RHS = p_{ij}$.
(b): We wish to establish that

$$\mathbb{P}\Big[X_{\tau+1}^{\tau+k} = i_1^k | X_\tau = i_0\Big] = \prod_{l=0}^{k-1} p_{i_l i_{l+1}}.$$

The proof for $k = 1$ follows using the exact same argument as above. Thus, the result follows by induction on k. $\square$

Definitions

- Definition. Given HMC with transition matrix P, P^n is the n-step transition matrix, i.e., $P^n = [p_{ij}(n)]$, where $p_{ij}(n) =$ probability of visiting j in the n-step starting from i.

- Definition. Node i communicates with j if there exist $n_1, n_2 \geq 0$, s.t. $p_{ij}(n_1) > 0$ and $p_{ji}(n_2) > 0$, denoted by $i \leftrightarrow j$.

- Definition. Communication defines "equivalence class" of HMC: (i) if $i \leftrightarrow j$, and $j \leftrightarrow k$, then $i \leftrightarrow k$, and (ii) $i \leftrightarrow i$ (since $p_{ii}(0) = 1$).

- Definition. A Markov chain is called irreducible if there is only one communication class. Any state can be reachable starting from any other state.

- An example of a Markov chain that is not irreducible?

- Henceforth, we only consider a irreducible Markov chain.

Aperiodic HMC

- The period of $d(i)$ of state $i \in \mathcal{E}$ is defined by

$$d(i) = gcd\{n : p_{ii}(n) > 0\}.$$

  We call i periodic if $d(i) > 1$ and aperiodic if $d(i) = 1$.

- An irreducible HMC is called aperiodic if all of its period is aperiodic.

- Period is a class property, i.e., if i and j communicate, then they have the same period.

- Thus, it suffices to check one state's aperiodicity for a irreducible Markov chain, if you want to check the aperiodicity of the entire Markov chain.

  Proof. As $i \leftrightarrow j$, there exists integers N, M, such that $p_{ij}(M) > 0$ and $p_{ji}(N) > 0$. For any $k \geq 1$,

$$p_{ii}(M + nk + N) \geq p_{ij}(M)(p_{jj}(k))^n p_{ji}(N).$$

why?

Thus, for any $k \geq 1$, such that $p_{jj}(k) > 0$, we have $p_{ii}(M + nk + N) > 0$ for all $n \geq 1$. Thus, d_i divides $M + nk +_N$ for all $n \geq 1$, and in particular, d_i divides k. Thus, d_i divides all k, such that $p_{jj}(k) > 0$, in particular, d_i divides d_j. By symmetry, d_j divides d_i. Thus, $d_i = d_j$.

Example. Two states 1 and 2. $p_{12} = 1$ and $p_{21} = 1$.

Recurrence

- Definition. Let $T_i = \min\{k \geq 1 | X_k = i\}$. Then, mentioned earlier, T_i is a stopping time. State i is called recurrent if $\mathbb{P}_i[T_i] \triangleq \mathbb{P}[T_i < \infty | X_0 = i] = 1$, otherwise called transient.

  Staring from a state i, I will return to the state i within a finite time with probability 1.

- Let $f_{ii}^{(n)} = \mathbb{P}[T_i = n \mid X_o = i]$, which is the probability that the first return time from i to i is n. Then, from the definition

  Recurrent if $\sum_{n=1}^{\infty} f_{ii}^{(n)} = 1$, and transient if $\sum_{n=1}^{\infty} f_{ii}^{(n)} < 1$.

- Lemma. Let $N_i = \sum_{n \geq 1} \mathbf{1}_{\{X_n = i\}}$ be the number of times state i is visited. Then,

$$\mathbb{P}_i[T_i < \infty] = 1, \quad \text{iff} \quad \mathbb{E}_i[N_i] = \infty.$$

Recurrent state i iff I visit state i infinite times!

Proof. Let $f_{ii} = \sum_{n=1}^{\infty} f_{ii}^{(n)} = \mathbb{P}_i[T_i < \infty]$. Let $0 = \tau_0, \tau_1, \ldots,$ be times of visit of state i. Now, suppose $f_{ii} < 1$. For $r \geq 1$, using strong Markov property,

$$\mathbb{P}_i[N_i = r] = \mathbb{P}_i[\tau_1 < \infty, \tau_2 - \tau_1 < \infty, \ldots, \tau_{r+1} - \tau_r = \infty]$$
$$= \Big(\prod_{j=1}^{r} \mathbb{P}_i[\tau_j - \tau_{j-1} < \infty]\Big)\mathbb{P}_i[\tau_{r+1} - \tau_r = \infty] = f_{ii}^r(1 - f_{ii}).$$

Thus, $\mathbb{E}_i[N_i] = \sum_r r f_{ii}^r (1 - f_{ii}) = 1/(1 - f_{ii})$. □

- Note that $\mathbb{E}_i[N_i] = \sum_{n=0}^{\infty} p_{ii}(n)$.

- Lemma. For an irreducible HMC, if some $i \in \mathcal{E}$ is recurrent then any other $j \in \mathcal{E}$ is recurrent.

  Recurrence is a property of the equivalent communication class

- Proof. As $i \leftrightarrow j$, there exists integers N, M, such that $p_{ij}(M) > 0$ and $p_{ji}(N) > 0$. We have that:

$$p_{ii}(M + n + N) \geq \alpha \times p_{jj}(n),$$

where $\alpha = p_{ij}(M)p_{ji}(N)$. Similarly, we get:

$$p_{jj}(N + n + M) \geq \alpha \times p_{ii}(n).$$

The above means that $\sum_{n=0}^{\infty} p_{ii}(n)$ and $\sum_{n=0}^{\infty} p_{jj}(n)$ either both converge or both diverge. $\square$

Invariant Measure

- Definition. Let $x = (x_i)_{i \in \mathcal{E}}$ be s.t. $x_i \in (0, \infty)$, for all $i \in \mathcal{E}$. and $x^T = x^T P$: that is

$$x_i = \sum_{j \in \mathcal{E}} x_j P_{ji}.$$

Then, x is called an invariant measure.

- Lemma. [existence] Given an irreducible recurrent HMC, there is at least one invariant measure. Specifically consider some $o \in \mathcal{E}$. Define,

$$x_i^o = \mathbb{E}_o \Big[\sum_{n \geq 1} \mathbf{1}_{\{X_n = i\}} \mathbf{1}_{\{n \leq T_o\}} \Big],$$

with $T_o = \min\{k \geq 1 : X_k = o\}$. Then, such an $x^o = (x_i^o)$ is an invariant measure.

irreducibility and recurrence $\rightarrow$ existence of invariant measure

- What is x_i^o?

  Starting from o, the expected number of "meeting" i until the first return to o.

- Property of x^o:

$$\begin{aligned}\sum_{i\in\mathcal{E}} x_i^o &= \sum_{i\in\mathcal{E}} \mathbb{E}_o\Big[\sum_{n\geq 1} \mathbf{1}_{\{X_n=i\}}\mathbf{1}_{\{n\leq T_o\}}\Big] \\ &= \mathbb{E}_o\Big[\sum_{n\geq 1} \mathbf{1}_{\{n\leq T_o\}}\big(\sum_{i\in\mathcal{E}} \mathbf{1}_{\{X_n=i\}}\big)\Big] \\ &= \mathbb{E}_o\Big[\mathbf{1}_{\{n\leq T_o\}}\Big] = E_o[T_o] \qquad (1)\end{aligned}$$

  Ah-ha! $\sum_{i\in\mathcal{E}} x_i^o$ is nothing but an expected minimum time of starting from o, the chain gets backs to o.

  Proof. First, note that

$$x_o^o = \mathbb{E}_o\Big[\sum_{n\geq 1} \mathbf{1}_{\{X_n=o\}}\mathbf{1}_{\{n\leq T_o\}}\Big] = 1$$

Why? Because $X_n = o$ only when $n = T_o$ for any $n \leq T_o$. Define:

$$\phi_i(n) = \mathbb{P}_o\Big[X_1 \neq o, \ldots, X_{n-1} \neq o, X_n = i\Big], \text{ for any } i \in \mathcal{E}.$$

$\phi_i(n)$ is the probability that I am at i at the n-step, but not visiting o before n.

Then, $x_i^o = \sum_{n \geq 1} \phi_i(n)$. Note that $\phi_i(1) = p_{oi}$. Using MC's property, for $n \geq 2$,

$$\phi_i(n) = \sum_{j \neq o} \phi_j(n-1) p_{ji}.$$

Summing over n gives:

$$\begin{aligned} x_i^o &= \sum_{j \neq o} \Big(\sum_{n \geq 2} \phi_j(n-1) p_{ji}\Big) + p_{oi} \\ &= \sum_{j \neq o} \Big(\sum_{n \geq 1} \phi_j(n) p_{ji}\Big) + p_{oi} \\ &= \sum_{j \neq o} x_j^o p_{ji} + x_o^o p_{oj} \end{aligned}$$

$$= \sum_{j \in \mathcal{E}} x_j^o p_{ji}.$$

Thus, x^o is an invariant measure as long as we show that $x_i^o \in (0, \infty)$ for all $i \in \mathcal{E}$. Left as an exercise.

- Lemma. [uniqueness] For an irreducible HMC, let $x = (x_i)$, $y = (y_i)$ be two invariant measures. If HMC is recurrent then there exists $c > 0$, s.t. $x_i = cy_i$ for all $i \in \mathcal{E}$.

  irreducibility and recurrence $\rightarrow$ uniqueness of invariant measure upto a multiplicative constant.

  Proof. Omitted.

- Remark. There exists an HMC that are irreducible and possess an invariant measure, yet not recurrent. Consider an asymmetric random walk, where $x_i = 1$, $\forall i \in \mathcal{E}$ is an invariant measure.

Positive Recurrence

- **Definition.** State i of an HMC is positive recurrent if $\mathbb{E}_i[T_i] < \infty$. Clearly, a state is recurrent if it is positive recurrent. But, not otherwise.

  HMC is positive recurrent if all states are positive recurrent.

- Note that $\mathbb{E}_i[T_i] = \sum_{n=1}^{\infty} n f_{ii}^{(n)}$.

- Can you tell just "recurrence" from "positive-recurrence"?

- **Lemma.** [Alternate definition] State $o \in \mathcal{E}$ is positive recurrent, iff

$$\sum_{i \in \mathcal{E}} x_i^o < \infty.$$

Proof. See (1).

- Lemma. [equivalence of positive recurrence] Given an irreducible HMC, if some $o \in \mathcal{E}$ is positive recurrent then all $i \in \mathcal{E}$ are positive recurrent.

  Positive-recurrence is also a property of the equivalent communication class
  Proof. Omitted.

Finite State HMC

- Lemma. Given an irreducible HMC, it is positive recurrent if $\mathcal{E}$ is finite.

  State finiteness just with irreducibility automatically implies positive recurrence

  The intuition is that if the number of states is finite, then I can get back to my state within a "short" time.

  Proof. (Sketch)

  1. First prove that it is recurrent

  2. Then, for the irreducible MC, we know that x^o is an invariant measure, i.e., $x_i^o \in (0, \infty)$.

  3. Since $\mathcal{E}$ is finite, we should have $\sum_{i \in \mathcal{E}} x_i^o < \infty$.

Stationary Distribution: Existence and Uniqueness

- Definition. Let $\{\pi(i)\}_{i \in \mathcal{E}}$ be an invariant measure of HMC P such that $\sum_{i \in \mathcal{E}} \pi(i) = 1$. Then, $\pi = [\pi(i)]$ is called the stationary distribution of HMC.

  Stationary distribution $=$ Invariant measure $+$ distribution, i.e., $\sum_i \pi(i) = 1$.

- Lemma. For an irreducible positive recurrent HMC, there exists the unique stationary distribution.

  Proof. (Sketch)

  1. irreduciblility and (positive) recurrence $\rightarrow x^o$ is an invariant measure with $\sum_{i \in \mathcal{E}} x_i^o < \infty$.

  2. define $\pi(i)$ be the scaled x_i^o by $\sum_{i \in \mathcal{E}} x_i^o$. Then, uniqueness of invariant measure upto a multiplicative constant proves the lemma.

Stationary Distribution: Convergence

- Definition. Given distributions μ and ν on $\mathcal{E}$, define a distance between μ and ν as

$$d_{TV}(\mu, \nu) = \sup_{A \in \mathcal{E}} [\mu(A) - \nu(A)].$$

  "TV" means "Total Variation" used to measure the distance between two distributions.

- Lemma. Given an irreducible, aperiodic, and positive recurrent HMC on countable state-space $\mathcal{E}$, starting from any distributions μ and ν on $\mathcal{E}$

$$\lim_{n \to \infty} d_{TV}(\mu^T P^n, \nu^T P^n) = 0,$$

  i.e., $\lim_{n \to \infty} |\mu^T P^n - \pi| = 0$.

  Start the MC with any initial state that we randomly choose. Then, by running the MC for a long, long time, we go to a stationary regime that is unique.

Proof. Omitted.

Implications

- Positive recurrence implies the existence of stationary distribution
  - Suppose $\mathcal{E} = \{0, 1, 2, \ldots\}$
  - $\pi = (\pi(i))$ be a stationary distribution, that is, $\sum_{i \in \mathcal{E}} \pi(i) = 1$. Hence,

$$P_\pi([n, \infty)) = \sum_{i \geq n} \pi(i) \xrightarrow{n \to \infty} 0.$$

  - That is, with respect to π , the value of MC is finite with probability 1.

- Aperiodicity established that positive recurrent irreducible HMC converges to stationary distribution.

- Thus, in "equilibrium" an aperiodic, irreducible positive recurrent HMC is finite with probability 1.

- Ergodic MC: positive recurrent and aperiodic.

- In many papers, ergodicity $\rightarrow$ HMC is finite w.p. 1 (In that case, we implicitly assume "irreducibility").

- Stationary Distribution Criteria: If we can compute the stationary distribution, then we know that it is positive-recurrent.

  Cannot do it for many applications

  Are there other methods for testing positive-recurrence?

  Yes. The next slides ...

Test For Positive Recurrence: Foster's Criteria

- Lemma. [Foster's criteria] Given an irreducible HMC on countable state space $\mathcal{E}$, let there exist non-negative valued function $V : \mathcal{E} \mapsto \mathcal{R}_+$ such that

  (a) $\sum_{j \in \mathcal{E}} p_{ij} V(j) < \infty$ for all $i \in \mathcal{E}$,
  (b) $\sum_{j \in \mathcal{E}} p_{ij} V(j) < V(i) - \epsilon$, for all $i \notin \mathcal{F}$, where $\epsilon > 0$, and $\mathcal{F}$ a finite subset of $\mathcal{E}$.

  Then, HMC is positive-recurrent.

- Intuition?

- Proof. Very long proof by proving the following:

  1. Under hypothesis of Lemma, for any $i \in \mathcal{F}$, $\mathbb{E}_i[T(\mathcal{F})] < \infty$, where $T(\mathcal{F}) = \min\{k \geq 1 \mid X_k \in \mathcal{F}\}$

  2. For an irreducible HMC, if there is a finite set $\mathcal{F}$ s.t. for any $i \in \mathcal{F}$, $\mathbb{E}_i[T(\mathcal{F})] < \infty$, then HMC is positive recurrent.

Poisson Process

- Definition. [Poisson Process] It is a random point process on $\mathcal{R}_+$ (also called a counting process), defined by monotonically non-decreasing sequence of r.v.s. $\{T_n\}_{n\geq 0}$ that satisfy the following conditions:

  (a) $T_0 = 0$,
  (b) $T_n - T_{n-1} \overset{\mathrm{D}}{=} \exp(\lambda)$: λ: parameter of process
  (c) $(T_n - T_{n-1})$ are i.i.d.

- Let $N((a,b]) = \sum_{n\geq 0} \mathbf{1}_{(a,b]}(T_n)$. Then, $N(t) = N((0,t])$ is the number of "points" of process upto time t; which captures the essence of the process.

- Property

  (i) For all $0 = t_0 \leq t_1 \leq \ldots \leq t_k$; $N((t_i, t_{i+1}])$, $i \geq 0$ are independent.
  (ii) $N((a,b])$ is Poisson r.v. with mean $\lambda(b-a)$, i.e.,

$$\mathbb{P}\Big[N(a,b] = k\Big] = \exp(-\lambda(b-a))\frac{(\lambda(b-a))^k}{k!}$$

Splitting and Merging

- How to approximate Poisson process with discrete time process?

- Exercise

1. Let P_1 and P_2 be independent Poisson process of parameters λ_1 and λ_2. Then, the union of P_1 and P_2 is also Poisson process of parameter $\lambda_1 + \lambda_2$.
2. Let P be a Poisson process of parameter λ. Let's split P by marking each point of P by 1 with prob. p and 2 with prob $1 - p$ independently. Then, points marked by 1 (resp. 2) form a Poisson process of parameter λp (resp. $\lambda(1 - p)$).

Continuous Time HMC

- Let $\mathcal{E}$ be finite or countable state space. Let $X(t), t \geq 0$ be a process living in $\mathcal{E}$. It satisfies the following conditions:

  (a)

  $$\mathbb{P}\Big[X(t+s) = j | X(s) = i, X(s_1), \ldots, X(s_l)\Big] = \mathbb{P}\Big[X(t+s) = j | X(s) = i\Big],$$

  for any $0 \leq s_l \leq s_1 \leq s$,

  (b) $\mathbb{P}[X(t+s) = j | X(s) = i] = \mathbb{P}[X(t+s') = j | X(s') = i] = p_{ij}(t)$.

  Let $P(t) = [p_{ij}(t)]$ be called the transition semi-group of continuous time HMC

  Question. We have $p_{ij}(t)$ that depends on time t. So, this continuous MC is non-homogeneous MC? No! Just t-step matrix, not time-dependent.

- Transition Rate Matrix (also called *infinitesimal generator of the semi-group* $P(t)$), $Q = [q_{ij}]$, defined by:

$$\begin{aligned} q_i &\triangleq \lim_{h \to 0} \frac{1 - p_{ii}(h)}{h}, \\ q_{ij} &\triangleq \lim_{h \to 0} \frac{p_{ij}(h)}{h}, \\ q_{ii} &\triangleq -q_i \end{aligned}$$

- In other words,

$$\begin{aligned} p_{ij}(h) &= q_{ij}h + o(h) \\ p_{ii}(h) &= 1 + q_{ii}h + o(h) \end{aligned}$$

Embedded Markov Chain

- We are interested in a special type of continuous time HMC.

  Given a Poisson process with λ, let $\{T_n\}$ be its jump times. Let $\{\hat{X}_n\}_{n\geq 0}$ be a discrete time HMC, inpdendent of Poisson process. Let us define a continuous time random process $X(t)$ as follows:

$$X(t) \triangleq \hat{X}_{N(t)}$$

  Then $X(t)$ is a continuous time HMC. Why? Can you visualize this continuous chain?

  Check. (a) and (b) hold for this definition?

- We call $\{\hat{X}_n\}_{n\geq 0}$ embedded HMC of $X(t)$.

- Used for analysis of systems modeled by continuous MC through discrete MC. See the next slide.

Remark: How to study continuous MC through discrete MC?

A. The definition of $X(t)$ implies that for $\lambda > 0$, w.p. 1, $N(t) \to \infty$ as $t \to \infty$, and $T_n \to \infty$ as $n \to \infty$.

   Thus, property of irreducibility, recurrence, and positive recurrence remain identical for $\hat{X}_n$ and $X(t)$. That is, we can carry over the **technology** of discrete time HMC for such continuous time HMCs.

B. Let π be time-stationary distribution of $X(t)$. Then, it must be the time-stationary distribution of $\hat{X}_n$. This is primarily due to property of Poisson process:

$$
\begin{aligned}
\mathbb{P}\Big[X(t) = j | N(t, t+\delta) = 1\Big] &= \frac{\mathbb{P}[X(t) = j; N(t, t+\delta) = 1]}{\mathbb{P}[N(t, t+\delta) = 1]} \\
&= \frac{\mathbb{P}[X(t) = j] \cdot \mathbb{P}[N(t, t+\delta) = 1]}{\mathbb{P}[N(t, t+\delta) = 1]}
\end{aligned}
$$

Why is the last equality true?

- The above implies that sampling according to time is the same as sampling according to the Poisson process. Thus, if π is stationary distribution for $X(t)$ then so is for $\hat{X}_n(t)$ and vice-versa.

References

[Bremaud, 1999] Bremaud, P. (1999). *Markov Chaing: Gibbs Fields, Monte Carlo Simulation, and Queues*. Springer.