Optimization in Communication Networks
Lecture 5-2: Continuous Time Markov Chain, Poisson Process, and Embedded Markov Chain

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Poisson Process

- Definition. [Poisson Process] It is a random point process on $\mathcal{R}_+$ (also called a counting process), defined by monotonically non-decreasing sequence of r.v.s. $\{T_n\}_{n\geq 0}$ that satisfy the following conditions:

  (a) $T_0 = 0$,

  (b) $T_n - T_{n-1} \overset{\mathrm{D}}{=} \exp(\lambda)$: λ: parameter of process

  (c) $(T_n - T_{n-1})$ are i.i.d.

- Let $N((a,b]) = \sum_{n\geq 0} \mathbf{1}_{(a,b]}(T_n)$. Then, $N(t) = N((0,t])$ is the number of "points" of process upto time t; which captures the essence of the process.

- Property

  (i) (Independent Increments) For all $0 = t_0 \leq t_1 \leq \ldots \leq t_k$; $N((t_i, t_{i+1}])$, $i \geq 0$ are independent.

  (ii) (Stationary Increments) $N((a, b])$ is Poisson r.v. with mean $\lambda(b-a)$, i.e.,

$$\mathbb{P}\Big[N(a,b] = k\Big] = \exp(-\lambda(b-a))\frac{(\lambda(b-a))^k}{k!}$$

  – (i) and (ii) are often used as the definition of Poisson process.
  – How to prove (i) and (ii)?

Poisson Process: Splitting and Merging

1. Let P_1 and P_2 be independent Poisson process of parameters λ_1 and λ_2. Then, the union of P_1 and P_2 is also Poisson process of parameter $\lambda_1 + \lambda_2$.

2. Let P be a Poisson process of parameter λ. Let's split P by marking each point of P by 1 with prob. p and 2 with prob $1 - p$ independently. Then, points marked by 1 (resp. 2) form a Poisson process of parameter λp (resp. $\lambda(1-p)$).

Bernoulli Process: Discrete-time version of Poisson Process

- Bernoulli process: A sequence, Y_1, $Y_2, \ldots,$ of IID binary random variables, where the event $\{Y_i = 1\}$ represents an arriving customer at time i, and $\{Y_i = 1\}$ otherwise. Then, it is easy to show that the inter-arrival time has a geometric distribution.

- Inter-arrival time: Exponential in Poisson process

- Inter-arrival time: Geometric in Bernoulli process

(Homogeneous) Continuous Time HMC

- Let $\mathcal{E}$ be finite or countable state space. Let $X(t), t \geq 0$ be a process living in $\mathcal{E}$. It satisfies the following conditions:

  (a)

  $$\mathbb{P}\Big[X(t+s) = j | X(s) = i, X(s_1), \ldots, X(s_l)\Big] = \mathbb{P}\Big[X(t+s) = j | X(s) = i\Big],$$

  for any $0 \leq s_l \leq s_1 \leq s$,

  (b) $\mathbb{P}[X(t+s) = j | X(s) = i] = \mathbb{P}[X(t+s') = j | X(s') = i] = p_{ij}(t)$.

  Let $P(t) = [p_{ij}(t)]$ be called the transition semi-group of continuous time HMC

  Question. We have $p_{ij}(t)$ that depends on time t. So, this continuous MC is non-homogeneous MC? No! Just t-step matrix, not time-dependent. In other words

  $$\mathbb{P}[X(t+s) = j \mid X(s) = i]$$

  is independent of s.

- Let T_i be the amount of time that the process stays in state i before making a transition. Then, it is easy to see that the following memoryless property:

$$\mathbb{P}[T_i > s + t \mid T_i > s] = \mathbb{P}[T_i > t].$$

- Thus, a continous-time Markov chain is a stochastic process that moves from state to state in accordance with a (discrete-time) Markov chain, but is such that the amount of time it spends in each state, before proceeding to the next state, is exponentially distributed.

- Transition Rate Matrix (also called *infinitesimal generator of the semi-group* $P(t)$), $Q = [q_{ij}]$, defined by:

$$
\begin{aligned}
q_i &\triangleq \lim_{h\to 0} \frac{1 - p_{ii}(h)}{h}, \\
q_{ij} &\triangleq \lim_{h\to 0} \frac{p_{ij}(h)}{h}, \\
q_{ii} &\triangleq -q_i.
\end{aligned}
$$

- Thus, it is often a continous time markov chain is given by the transition rate matrix Q.

- What is the row-sum of Q?

- In other words, for small h,

$$
\begin{aligned}
p_{ij}(h) &= q_{ij}h + o(h) \approx q_{ij}h \\
p_{ii}(h) &= 1 + q_{ii}h + o(h) \approx 1 - q_i h.
\end{aligned}
$$

- Recall: a continous-time Markov chain is a stochastic process that moves from state to state in accordance with a (discrete-time) Markov chain, where in each state it stays for an exponentially distributed time.

- Question, Given Q, and a state i, how T_i is distributed?

- Theorem. T_i is exponentially distributed with parameter $-q_{ii} = q_i$.

- What is the probability that the chain jumps from state i to state j? It's $-\frac{q_{ij}}{q_{ii}}$. The proof sketch is:

$$\mathbb{P}[\text{jumps to } j \mid \text{it jumps}] \approx \frac{p_{ij}(h)}{1 - p_{ii}(h)} \approx -\frac{q_{ij}}{q_{ii}}.$$

Embedded Markov Chain: 1

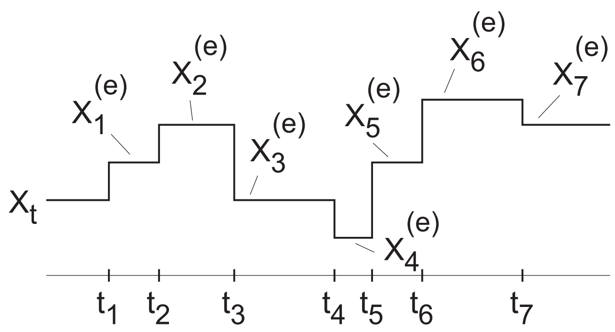
Embedded Markov chain

With every continuous time Markov process X_t we can associate a discrete time Markov chain, so called embedded Markov chain or jump chain $X_n^{(e)}$.

- Focus is on the transitions of X_t (when they occur), i.e. on the sequence of (different) states visited by X_t.
- Let the state transitions of X_t occur at instants $t_0, t_1, \ldots$
- Define $X_n^{(e)}$ to be the value of X_t immediately after the transition at time t_n (at the instant t_n^+) or the value of X_t in (t_n, t_{n+1}).

$$\boxed{X_n^{(e)} = X_{t_n^+}}$$

Since X_t is a Markov process, the embedded chain $X_n^{(e)}$ constitutes a Markov chain.



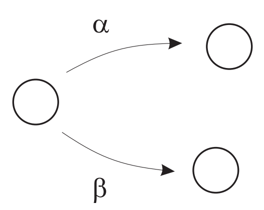
Embedded Markov Chain: 2

Embedded Markov chain (continued)

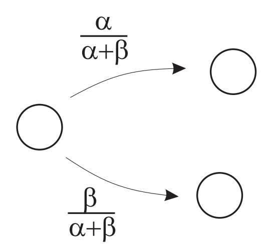
The states of a Markov process can be classified by the classification provided by the embedded Markov chain (transient, absorbing, recurrent,...).

The transition probabilities of the embedded chain

$$\begin{aligned} p_{i,j} &= \lim_{\Delta t \to 0} \mathrm{P}\{X_{t+\Delta t} = j \,|\, X_{t+\Delta t} \neq i, X_t = i\} \\ &= \lim_{\Delta t \to 0} \frac{\mathrm{P}\{X_{t+\Delta t} = j, X_{t+\Delta t} \neq i \,|\, X_t = i\}}{\mathrm{P}\{X_{t+\Delta t} \neq i \,|\, X_t = i\}} \\ &= \begin{cases} \dfrac{q_{i,j}}{\sum_j q_{i,j}} & i \neq j \quad \text{cf. } \mathrm{P}\{\min(X_1, \dots, X_n) = X_i\} = \frac{\lambda_i}{\lambda_1 + \dots + \lambda_n}, \text{ when } X_i \sim \mathrm{Exp}(\lambda_i) \\ 0 & i = j \end{cases} \end{aligned}$$



Markov process, transition rates $q_{i,j}$
equilibrium probabilities π_i



Embedded Markov chain, transition probabilities $p_{i,j}$
equilibrium probabilities $\pi_i^{(e)}$

Remark: How to study continuous MC through discrete MC?

A. The definition of $X(t)$ implies that for $\lambda > 0$, w.p. 1, $N(t) \to \infty$ as $t \to \infty$, and $T_n \to \infty$ as $n \to \infty$.

   Thus, property of irreducibility, recurrence, and positive recurrence remain identical for both chains. That is, we can carry over the **technology** of discrete time HMC for such continuous time HMCs.

   In other words, if you want to prove the positive recurrence of a CTMC, it is enough to show it for its EMC.

B. Especially, we consider the case where we sample a CTMC based on the given Poisson process, i.e., I look at the state whenever a new arrival comes according a Poisson process. Then, we have:

   Let π be time-stationary distribution of $X(t)$. Then, it must be the time-stationary distribution of $\hat{X}_n$. This is primarily due to property of Poisson process:

$$\mathbb{P}\Big[X(t) = j | N(t, t+\delta) = 1\Big] \quad = \quad \frac{\mathbb{P}[X(t) = j; N(t, t+\delta) = 1]}{\mathbb{P}[N(t, t+\delta) = 1]}$$

$$= \frac{\mathbb{P}[X(t) = j] \cdot \mathbb{P}[N(t, t+\delta) = 1]}{\mathbb{P}[N(t, t+\delta) = 1]}$$

Why is the last equality true?

- The above implies that sampling according to time is the same as sampling according to the Poisson process. Thus, if π is stationary distribution for $X(t)$ then so is for $\hat{X}_n(t)$ and vice-versa.

References