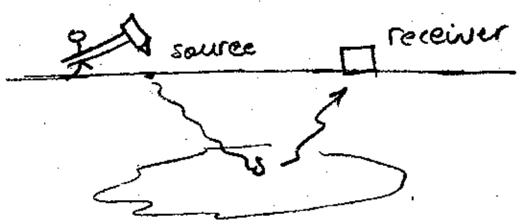
Overview of Linear Inverse Theory I.1.0

- Forward Problem
- Inverse Problem
- Fundamental challenges for inverse problem

Overview of Linear Inverse Theory.

The geophysical experiment.

Geophysicists are continually faced with a generic type of problem. Our goal is to extract quantitative information about the earth (or a physical system) without directly sampling. To carry this task out, the geophysicist sets up an experiment. (sensitive to particular physical properties that are diagnostic of the structure)



The experiment that is designed depends upon what information is sought.
- large scale (whole-earth) (find ρ, v_p, v_s)
- exploration (oil/gas, minerals) (σ, μ, η, ρ)
- environmental (utilities, salt water intrusion) μ, σ, ϵ

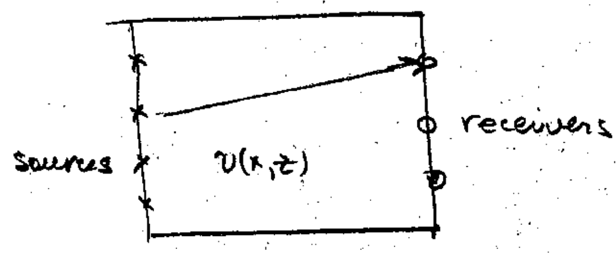
Experiment:
- energy is input to the earth
- propagation of the energy depends upon physical laws
- initial primary energy interacts with substructure to generate scattered (secondary) energy which is returned to a receiver
- receiver outputs a (set) of numbers which constitute our data

Knowledge of:
(1) the data
(2) physics about how energy propagates through the earth
(3) physical experiment (source/receiver geometry, ...

must somehow be combined to reveal information about the earth. This is the goal of inverse theory. It provides a set of mathematical tools by which to extract quantitative answers to specific questions.

Examples

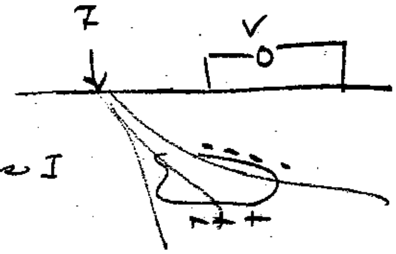
1 Cross-well tomography



measure travel time — want to know the velocity $v(x,z)$

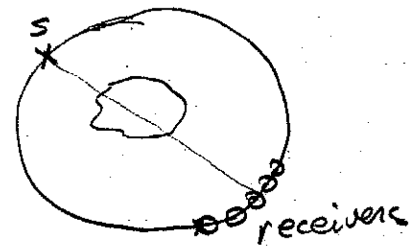
2. DC surveys

— need many sources I



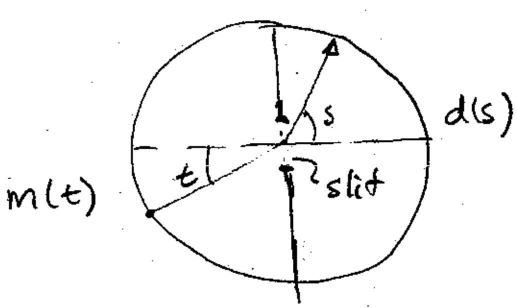
measure voltage
Want $\sigma(x,y,z)$

3. Medical Imaging — CAT —
- X-ray emitted from source
- detectors record amplitude
- find η (effectively an absorption coefficient)



(need to rotate source / receivers around to get good coverage
— Note difference with geophysics problem where we only have sources and receivers on the surface.

4 Optics ; Shaw Problem



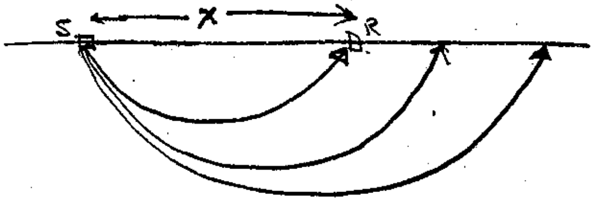
Each ray (incident angle t) scatters into a region of π

Goal: measure intensity on rhs and determine the strength of the source on lhs

(Matlab tool box, Hansen, Aster et al)

5/ Signal processing.

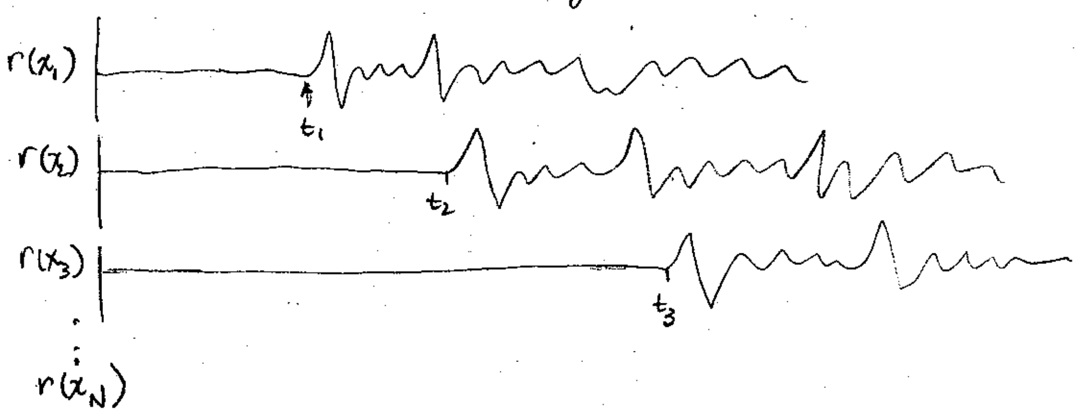
Example: Consider a seismic refraction survey over an earth characterized by a 1-D velocity function $v(z)$



(or optical problem with a high index of refraction)

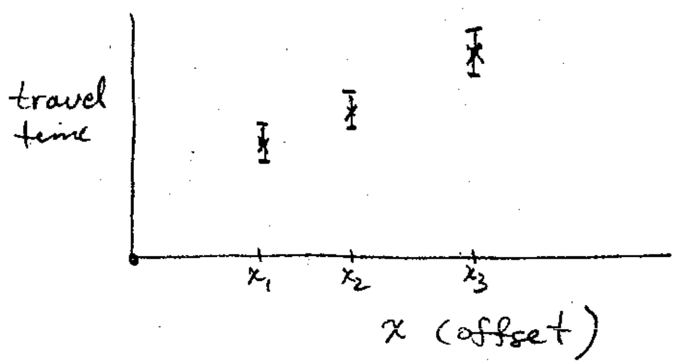
S is an impulsive source (perhaps with a source wavelet known) the seismic records at each receiver x_j is given by $r(x_j)$

For the above situation we might have



The recorded seismograms have encoded in them information about the subsurface. We could use the entire coda or we might pick out particular pieces of information. eg a first arrival and use that as a datum.

Result:



Data are: t_j $(j=1,N)$ plus some estimate about the accuracy of each datum.

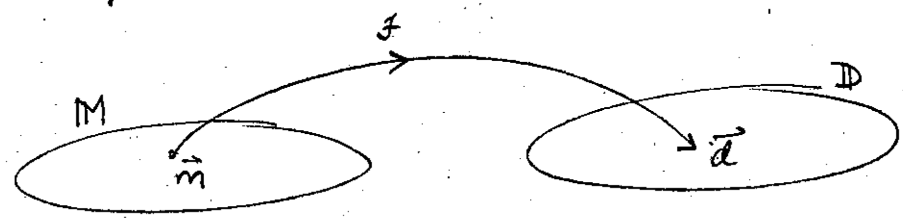
Question: Given these data, and the details about the experiment, what is $v(z)$?

What is the appropriate methodology.

Irrespective of details, a minimum criterion for acceptance of a velocity structure is that it reproduce (adequately) the observations. In order to make progress here we need to be able to simulate the observations that would be observed for an arbitrary $v(z)$. We need to be able to solve the forward problem

Forward Problem

The forward problem consists of computing the output given the input, the earth, geometry of the experiment and the physics of the problem. Effectively it is mapping from "model" space to "data" space.



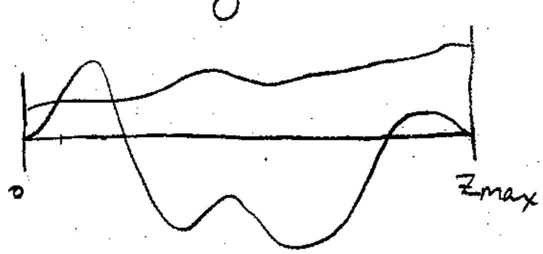
model space   data space

Model space: is a vector space whose elements consist of all possible functions or vectors which are permitted to serve as possible candidates for the earth model. Usually model space will be a Hilbert space.

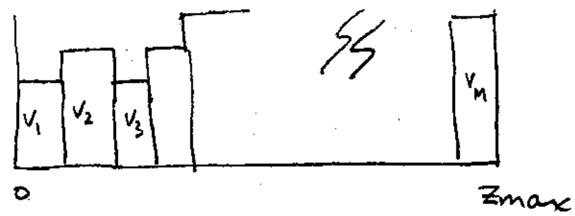
eg: Let z_{max} be a maximum depth in the earth in which the seismic information can penetrate.

The "model" in the seismic problem is a velocity function $v(z)$. Since there is a minimum and maximum depth (zero and z_{max}), we can define our model space to be $C^0[0, z_{max}]$. This is the set of all continuous functions defined on the range $[0, z_{max}]$

(Could have $z_{max} = \infty$)



Remark: Sometimes our models will be functions. Other times they can be represented as vectors.



$$\text{model} = \vec{v} = (v_1, v_2, \cdots v_M)^T$$

So model space is a vector space $\mathbb{R}^M$ (ordered M-tuples of numbers)

So our model space will be different for different problems but we will always specify its properties at the outset. In most of this course we shall restrict ourselves to Hilbert spaces. This requires that we equip our model space with

(i) inner product (measure angles)
(ii) norm (measure length)

Data space.

Geophysical observations are real numbers. For some problems they can be considered to be complex numbers but that presents no real difficulty. For this course we will generally assume that the observations are real numbers.

If we have N data, $\{d_1, d_2, \cdots d_N\}$ then we can define an element of data space to be a vector of length N

$$\vec{d} = (d_1, d_2, \cdots d_N)^T \qquad \in \mathbb{R}^N$$

Data space is therefore a usual vector space composed of elements which are N-tuples of numbers.

$\mathcal{F}$: is a mapping (which we refer to as the forward mapping) which shows how the data are related to the model. Symbolically we write

$$\mathcal{F} : \mathbb{M} \rightarrow \mathbb{D}$$

as a notation for the forward mapping.

To be more explicit about the relationship between data and model spaces we look at each datum separately. Each datum d_j can be written as

$$d_j = \mathcal{F}_j[\bar{m}] \qquad (1)$$

$\mathcal{F}_j$ is a <u>functional</u>. It is a rule which unambiguously assigns a single real number to each element $\bar{m}$ from model space.

Being able to evaluate d_j in (1) for any permissible $\bar{m}$ is said to being able to solve the forward problem.

Remark: Note that in solving the forward problem one must know
(1) physical laws pertaining to the experiment
(2) geometry
(3) details about sources and receivers.

Examples of some functionals are

$$I_j[m] = \int_a^b g_j(x)\, m(x)\, dx \qquad m \in C^0[a,b]$$

$$N_1[\bar{x}] = |x_1| + |x_2| + |x_3| + \cdots |x_N| \qquad \bar{x} \in \mathbb{R}^N$$

$$D_2[f] = \left.\frac{d^2 f}{dx^2}\right|_{x=0} \qquad f \in C^2[a,b]$$

<u>Linear Functionals.</u>

There are two types of functionals that are encountered in geophysics. (1) linear
(2) nonlinear.

A functional F is linear iff it obeys the following relationship. If α and β are constants and if f and g denote two arbitrary functions in the domain of F, then F is linear iff

$$F[\alpha f + \beta g] = \alpha F[f] + \beta F[g]$$

if the functional does not satisfy this relationship then it is non linear.

Remarks: In geophysics most of the functionals encountered are nonlinear. Since forward problem is nonlinear this requires that the inverse problem is also nonlinear. Nonlinear inversal problems are more difficult and their solution is saved for a graduate course in inverse theory. Nevertheless, many nonlinear inverse problems can be solved by linearizing the equations and using techniques of linear inverse theory in an iterative manner. This fact, plus the observation that there are a number of linear geophysical problems lead us to concentrate on linear functionals.

In fact, most of our linear functionals will have the form of

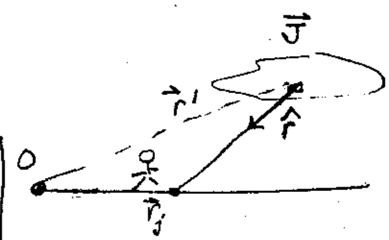
$$I_j[m] = \int_a^b g_j(x)\, m(x)\, dx$$

eg: Convolution:

$$\boxed{y(t_j) = \int_{-\infty}^{\infty} m(x)\, w(t_j - x)\, dx}$$

Biot - Savart law

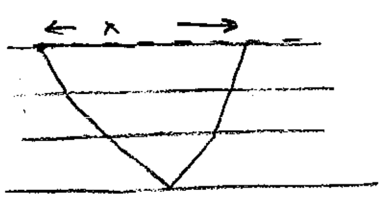
$$\boxed{\vec{B}(r_j) = \frac{\mu_0}{4\pi} \int_V \frac{\vec{J}(\vec{r}\,') \times \hat{r}}{|\vec{r}\,' - \vec{r}_j|^2}\, dV}$$



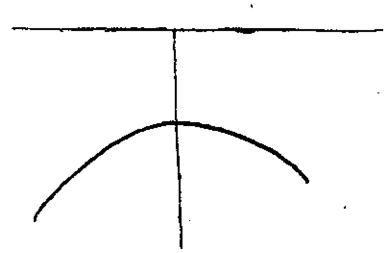
RMS - interval velocity

$$\boxed{V(t_j)^2 = \frac{1}{t_j} \int_0^{t_j} v^2(t)\, dt}$$

Aside: $v(z) \rightarrow v(t)$ $\qquad t = \int_0^z \frac{du}{v(u)}$



$$t^2(x) \approx t_0^2 + \frac{x^2}{(v_{rms})^2}$$



In fact any datum that is governed by an ordinary differential equation with constant coefficients can be written in this manner

$$\mathcal{L}u(\bar{x}) = m(\bar{x}) \qquad \mathcal{L}: \text{ linear differential operator}$$

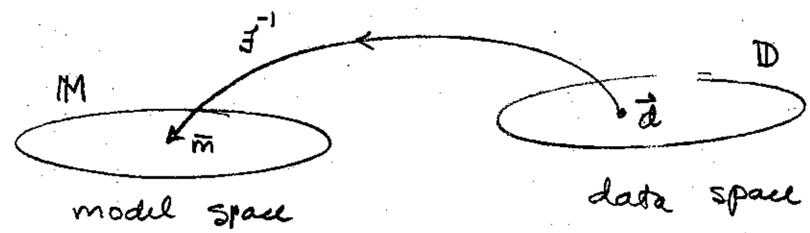
$$u(\bar{x}) = \int g(x;x_0)\, m(x_0)\, dx_0$$

where $g(x_0;x)$ is the Green's function for the problem $\mathcal{L}g(x;x_0) = \delta(x-x_0)$

So the list of linear functionals encountered is really quite large. This shows that if we can solve the inverse problem for linear mappings that we will have done a great deal.
(*) In practice one solves the adjoint equation.

Inverse Problem.

The inverse mapping (operator) undoes the original mapping



$$\mathcal{F}^{-1}: \mathbb{D} \rightarrow \mathbb{M}$$

In the inverse problem we begin with an element in data space and try to find the model element which generated the data.

eg: in the RMS velocity problem, the goal would be to recover v knowing $V_j = V(t_j)$ $(j=1,N)$.

$$V_j^2 = \frac{1}{t_j}\int_0^{t_j} v^2(t)\, dt \qquad j=1,N \qquad \left\{ \text{note: this is linear in } v^2(t) \right.$$

Forward problem: Given $v(t)$ find V_j $(j=1,N)$

Inverse problem: Given V_j $(j=1,N)$ find v

Remark: The RMS problem "typifies" many of the problems we encounter. A great portion of this course will be concerned with solving the inverse problem for linear integral functionals of the form

$$d_j = \int_a^b g_j(x)\, m(x)\, dx \qquad j = 1, N$$

d_j : j^{th} datum
m : model
g_j : kernel function for the j^{th} datum.

We'll sometimes refer to this as a Fredholm equation of the first kind.

<u>Questions pertaining to the solution of the inverse problem.</u>

(1) Does a solution exist?
(2) How does one construct a solution?
(3) Is it unique?
(4) If the solution is not unique, are there properties that are uniquely determined?

Remark: The answers to these questions depend upon the data. The different possibilities include

(i) infinite amount of accurate data
(ii) finite amount of accurate data
(iii) finite amount of inaccurate data.

Just to set the intuition for some of the important things to come, lets look at the RMS example for each of the three data types above.

<u>Perfect data</u>. In the limit that sampling is continuous, the RMS equation is

$$V^2(t) = \frac{1}{t}\int_0^t v^2(u)\, du$$

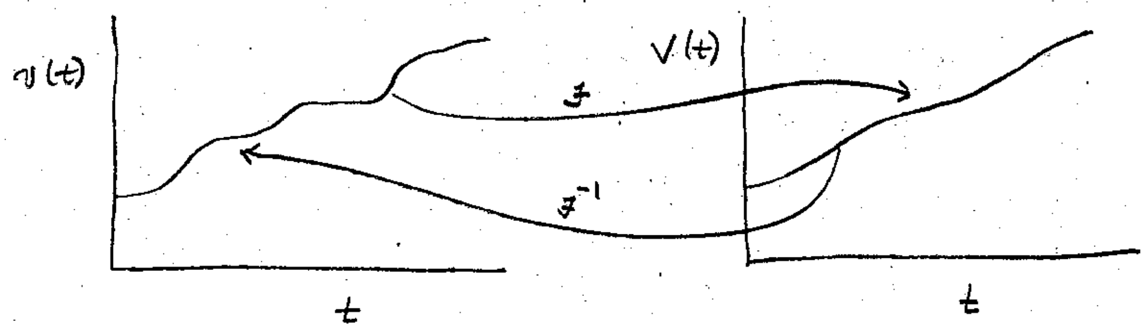
This has an analytic inverse. Multiply by t and differentiate w.r.t. time.

$$t V^2(t) = \int_0^t v^2(u)\, du$$

$$V^2(t) + 2t\,V(t)\,V'(t) = v^2(t)$$

so
$$v^2(t) = V^2(t)\left(1 + \frac{2t\,V'(t)}{V(t)}\right)$$

so
$$\boxed{v(t) = V(t)\left(1 + \frac{2t\,V'(t)}{V(t)}\right)^{1/2}}$$

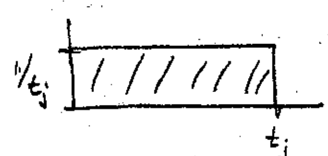


Comments.

(1) If $V(t)$ is known exactly then $v(t)$ can be recovered exactly. The solution exists and is unique.

(2) The forward problem smooths

$$V^2(t) = \frac{1}{t}\int_0^t v^2(u)\, du$$



RMS velocity squared is an average of interval velocity squared. (This smoothing is characteristic of almost all geophysical forward problems)

(3) The inverse problem (must) roughen.

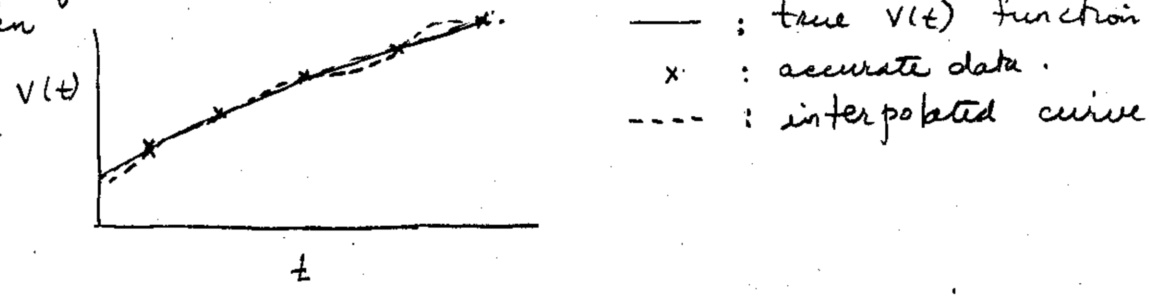
$$v(t) = V(t)\left(1 + \frac{2t}{V(t)}\,\frac{dV(t)}{dt}\right)^{1/2}$$

The effects of gradients in $V(t)$ are magnified by the factor t. $t\,V'(t)$ transforms small changes in $V(t)$ at late times into large changes in $v(t)$.

Inverse problem using a finite number of accurate data.

The major point to be emphasized here is that the solution to the inverse problem is nonunique. We understand this from the following logic.

Given



One way to invert the data:

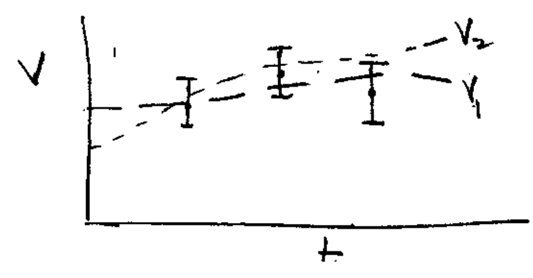
- Begin with the accurate data points and interpolate them so as to define V(t) everywhere
- Use the analytic formula.

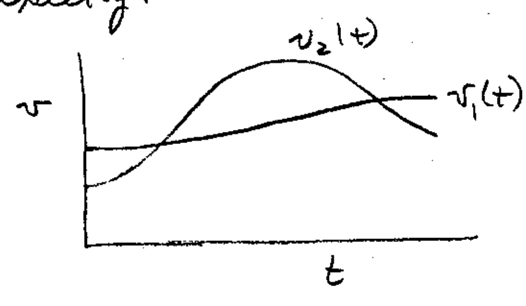
But: There are clearly infinitely many ways to interpolate between a finite set of points. Each interpolated function yields a different V(t) and hence a different answer to v(t). This illustrates in an intuitive manner that the solution to the inverse problem is nonunique when only a finite # of accurate points are provided.

Remark: Nonuniqueness is not just a term used by mathematicians. It has very practical geophysical consequences.

Inverse Problem using a finite number of inaccurate data

The same arguments as above argue that the nonuniqueness worsens. To make headway with this we have to specify how well the data are to be reproduced. Since the data are inaccurate we do not want to reproduce them exactly.



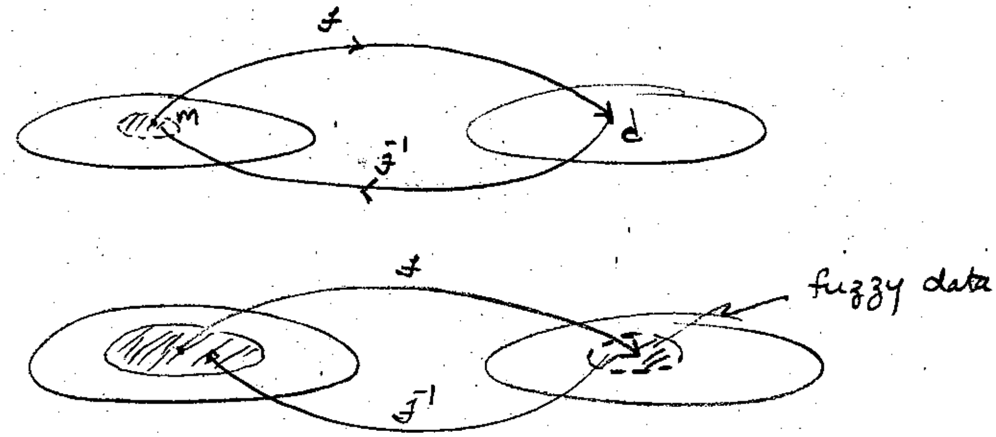


There are simply more ways to draw a possible curve $V(t)$ through the finite number of data

Effectively:

accurate data

inaccurate data



Ill - conditioned

In addition to the nonuniqueness problem, it is also known that the inverse problem is ill-conditioned. This means that numerical algorithms to compute the inverse are unstable. Effectively unregularized algorithms have the property that small changes in the data can give rise to large changes in the recovered model.

eg: Let d_t denote the true data
δd some contamination
$\mathcal{F}^{-1}$ be the inverse mapping

$$\mathcal{F}^{-1}[d_t] = m_c \quad \text{(constructed model)}$$

$$\mathcal{F}[d^o] = \mathcal{F}^{-1}[d_t + \delta d] = m_c + \delta m$$

Question: Suppose $\|\delta d\|$ is small. Does that also mean that $\|\delta m\|$ is small. In general the answer is no! This arises because of the general properties that the forward problem smooths and hence the inverse problem roughens.

Return to the RMS problem. Suppose $v(t) = v_0$ constant. Then $V(t) = v_0$. Allow a perturbation to the data so the observed data are

$$V(t) = v_0 + a \sin \omega_0 t \qquad \begin{cases} a : \text{constant} \\ \omega_0 : \text{arbitrary} \end{cases}$$

$v'(t) = \omega_0 a \cos\omega_0 t$ $\qquad\qquad$ $|a| \ll 1$

$$v(t) = v_0\left(1 + \frac{t a \omega_0 \cos\omega_0 t}{v_0 + a\sin\omega_0 t}\right)^{1/2} \sim v_0\left(1 + \frac{a t \omega_0 \cos\omega_0 t}{v_0}\right)^{1/2}$$

For any value of 'a' there is a time t such that $\left|\frac{a t \omega_0}{v_0}\right| > \delta$ where δ is an arbitrary value. The deviation from v_0 (true solution) therefore increases without bound! So $\|\delta V(t)\|$ small does not require that $\|\delta v\|$ is small.

To summarize: When solving a geophysical inverse problem we are faced with two fundamental problems:

Ill-posed

(i) The solution is nonunique. It may suffer from inherent nonuniqueness (eg. gravity data) but in addition, the nonuniqueness is enhanced because we must always deal with a finite number of inaccurate data. If there is one model which fits the data there will generally be infinitely many.

(ii) The inverse problem is ill-conditioned. It is unstable. Some type of regularization must be incorporated at the outset.

Remark: The nonuniqueness is fundamental. We will attempt to tackle this by

(i) model construction. Formulate our algorithms to generate a particular type of model. Minimize an appropriate norm of the model

(ii) appraisal: compute unique averages of the model (eg. using the methodology of Backus & Gilbert)

(iii) inference: Find bounds on predetermined linear functionals of the model.