AGT Coursework 1 - Answers

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Question 1

- **a**: We model the scenario as a normal form game

$$G = \langle N, A, u \rangle$$

where:

  - N is the set of 11 players $\{1, \ldots, 11\}$.
  - $A = \{A_1, \ldots, A_{11}\}$ is the set of actions available to each player, such that for any player i, $A_i = \{0, 1, 2, 3, 4\}$.
  - $u = \{u_1, \ldots, u_{11}\}$ is the profile of utility function such that for any player i and an action profile $\mathbf{a}$:
  $u_i(\mathbf{a}) = \begin{cases} 1, & \text{if for any } A_j \in A, \ |\frac{2}{3} * \frac{\Sigma_{k=1}^{k=11} \mathbf{a}_k}{11} - \mathbf{a}_i| \leq |\frac{2}{3} * \frac{\Sigma_{k=1}^{k=11} \mathbf{a}_k}{11} - \mathbf{a}_j| \\ 0, & \text{otherwise} \end{cases}$
  In other words, a player is in the set of winners if their bid is closest to the average of bids multiplied by $\frac{2}{3}$.

- **b**: **Answer:** The set of action profiles in which all players bid the same number.
Argument: First notice that if all players bid the same number, by definition of the utility function they receive the maximal payoff, so no agent is willing to change their bid.

Now consider a strategy profile in which agents are not bidding uniformly. Let us firstly consider a case in which agents pick at least 3 distinct values. Take such an action profile $\mathbf{a}$ and denote $\frac{2}{3}$ of the average of $\mathbf{a}$ as μ. Now consider numbers a, b represented in $\mathbf{a}$ such that μ lies between them. Then consider a bid c different than a and b. Notice that a player bidding c would become a winner by picking a, if a is closer to μ than b, or b otherwise. So, we cannot have a PNE if agents pick at least 3 distinct values. Further, suppose that agents pick exactly two distinct numbers, a and b. Notice that if $\mu \neq \frac{(a+b)}{2}$, we cannot have a PNE. Indeed, suppose that $\mu \neq \frac{(a+b)}{2}$ and assume w.l.o.g that a is closer to μ than b. Then, players choosing b would benefit from moving their bid to a. Let us then show, that in the considered game it is always the case that $\mu \neq \frac{(a+b)}{2}$. Suppose that $\mu = \frac{(a+b)}{2}$ and denote as x the number of players bidding a. Notice that x is a natural number and $x < 11$. Then, we have that $\frac{2}{3} * (\frac{ax+b*(11-x)}{11}) = \frac{a+b}{2}$. We can transform this equation to $x = \frac{33(a+b)-44b}{4(a-b)}$. Now notice that in order for x to be a positive integer, we need to have that 4 divides $a+b$ and $a > b$. So, we need to have that (a,b) is $(4,0)$ or $(3,1)$. But then by calculation we get that in both cases x is not a natural number smaller than 11 which contradicts the assumptions.

- **c**: **Answer**: There is no such threshold.
  Argument: Consider any real number r and the action profile $\mathbf{a} = (1,\ldots,1)$. Then notice that all submitted bids are equidistant from the average bid multiplied by r, so by definition of the utility function, all agents receive the maximal payoff. Hence, $\mathbf{a}$ is a PNE.

Question 2

- **a)** We model the scenario as a normal form game

$$G = \langle N, A, u \rangle$$

, where:

  - $N = \{1, 2\}$ is the set of agents (two vendors)
  - $A = \{A_1, A_2\}$ is the set of actions available to agents, such that $A_1 = A_2 = \{0, \ldots, k\}$ for some $k \in \mathbb{N}$
  - $\mathbf{u}$ is the profile of is the utility functions (u_1, u_2) such that for a player a_i (with the other player denoted as a_j) and an action profile $\mathbf{a}$:

$$u_i(\mathbf{a}) = \begin{cases} \frac{a_i+a_j+1}{2}, & \text{if } a_i < a_j \\ k+1-\frac{a_i+a_j+1}{2}, & \text{if } a_i > a_j \\ \frac{k+1}{2}, & \text{otherwise} \end{cases}$$

- **b) Answer:** The set of PNE is $P = \{(4,4), (4,5), (5,4), (5,5)\}$.
Argument: Let us first notice that for all action profiles in P, both vendors receive the payoff 5. Then let us consider the outcome of IESDS in this game. Notice first that for both players action 0 is strictly dominated by action 1: $u_i((0,0)) = 5$, while $u_i((1,0)) = 9$. Also, if $0 < a_j$ the utility of a_i increases by moving their bid to 1. Thus, action 0 is eliminated in IESDS for both players. Similarly we can show that action 9 is strictly dominated by action 8 for both players. Notice that we can proceed with eliminating extreme actions of the reduced game by the same argument, until the game is reduced to central positions 4 and 5. Thus, possible action profiles have been restricted to P.

  It now suffices to show that all profiles in P are a PNE in the reduced game. This follows immediately, as they provide uniform payoff for both vendors.

- **c) Answer:** The outcome of IESDS is $P = \{(\frac{k}{2}, \frac{k}{2})\}$ when k is even.
Argument: Notice that IESDS is conducted symmetrically to the procedure described in the answer to the question 2 b). In each iteration of IESDS the extreme actions are removed and thus the procedure results in the game reduced to the single action profile $(\frac{k}{2}, \frac{k}{2})$.

Question 3

- **a) Answer**: $a > 1$.
  Argument: Suppose that $a > 1$. Then the payoff of the row player is strictly greatest in the action profile (T, L). So, (T, L) cannot be Pareto dominated by any action profile and hence is Pareto optimal. Suppose now that $a \leq 1$. Then, consider the action profile (B, L). There, the outcome of the column player is strictly greater than in (T, L) and, as $a \leq 1$, the outcome of the row player is at least as good as in (T, L). So, (T, L) is Pareto dominated by (B, L). Thus, (T, L) is Pareto optimal only when $a > 1$.

- **b) Answer**: $a \leq 1$.
  Argument: Suppose that $a \leq 1$. Then, no agent can improve by changing their action unilaterally from the action profile (B, L). Indeed, the column player would lower her utility from 1 to 0, while the row player would not increase her utility, as $a \leq 1$. Suppose now that $a > 1$. Then, the row player would benefit from switching to (T, L), as in (B, L) she would have received the payoff 1, and a in (T, L).

- **c) Answer:** The set of Nash equilibria for this game is $\{\langle(\frac{1}{2}, \frac{1}{2}), (\frac{1}{2}, \frac{1}{2})\rangle\}$.
  Argument: Let us first notice that there are no PNE in this game. Indeed, the column player would benefit from switching her action in profiles (T, L) and (B, R), while the row player would benefit from deviating in profiles (B, L) and (T, R). Let us now calculate other mixed Nash equilibria. Let p be the probability that the row player chooses T, and q that the column player chooses L. Row player needs to be indifferent, so we have that: $q * 2 + (1 - q) * 0 = 1 * q + 1 - q$. So, $q = \frac{1}{2}$. Further, column player needs to be indifferent, so $0 * p + 1 - p = p + 0 * (1 - p)$, so $p = \frac{1}{2}$.

- **d) Answer**: This is not a potential game.
  Argument: Suppose that it is. Then, take a potential function P for this game. Then, consider $P(T, L)$. Notice that as the row player would receive the same payoff in (T, L) and in (B, L), $P(T, L) = P(B, L)$ by definition of a potential function. Then, the column player would have lost one point by deviating to (B, R) from (B, L), so, $P(B, L) = P(B, R) - 1 = P(T, L) - 1$. Then, as the row player would lose 1 point from switching to (T, R) from (B, R), $P(T, R) = P(B, R) - 1 = P(T, L) - 2$. But then, as the column player would lose 1 point from deviating to (T, L) from (T, R),$P(T, L) = P(T, R) - 1 = P(T, L) - 3$. Contradiction.

Question 4

- **a)** Let us denote as x the fraction of agents choosing to travel via the bottom route. Then, the equilibrium flow occurs when $x = 1$. Indeed, if $x = 1$, the cost of an agent traveling via the bottom route equals 3, which is equal to the cost of traveling via the top route. So, agents don't have an incentive to switch to the top route. However, if $x < 1$, the cost of traveling via the bottom route is strictly lower than the cost of traveling via the top route, so agents have an incentive to switch from top to bottom.

- **b)** Let us calculate the optimal flow. That is, we need to calculate x minimizing the global cost of the flow. Hence, we need to find $x \in [0, 1]$ minimizing the value of $(1 - x) * 3 + x * (x^2 + x + 1)$, which is equivalent to minimizing $3 - 3x + x^3 + x^2 + x = x^3 + x^2 - 2x + 3$. This amounts to $\frac{\sqrt{7}-1}{3}$.

- **c)** Let $\sigma = \frac{\sqrt{7}-1}{3}$. Let us calculate the price of anarchy (PoA) for this game. By definition, PoA is the cost of the equilibrium flow divided by the cost of the optimal flow. Notice that the cost of the equilibrium flow is $1 * (1^3 + 1^2 + 1) = 3$. Let us now calculate the cost of the optimal flow. Notice that this amounts to $\sigma^3 + \sigma^2 - 2\sigma + 3$, which can be estimated as 2.37. So, PoA$\approx \frac{3}{2.37} = 1.27$.