variables

(x_1, y_1), (x_2, y_2) and (x_3, y_3) where $x_1 < x_2 < x_3$

first derivative

Taylor expansions around x_2:

$$y_1 = y_2 + f^{'}(x_2)(x_1 - x_2) + \frac{f^{''}(x_2)}{2}(x_1 - x_2)^2 + \mathcal{O}((x_1 - x_2)^3)$$

$$y_3 = y_2 + f^{'}(x_2)(x_3 - x_2) + \frac{f^{''}(x_2)}{2}(x_3 - x_2)^2 + \mathcal{O}((x_3 - x_2)^3)$$

Neglect higher order terms and eliminate terms with $f^{''}(x_2)$:

$$(y_1 - y_2)(x_3 - x_2)^2 - (y_3 - y_2)(x_1 - x_2)^2 = f^{'}(x_2)[(x_1 - x_2)(x_3 - x_2)^2 - (x_3 - x_2)(x_1 - x_2)^2]$$

So that:

$$f^{'}(x_2) = \frac{(y_1 - y_2)(x_3 - x_2)^2 - (y_3 - y_2)(x_1 - x_2)^2}{(x_1 - x_2)(x_3 - x_2)(x_3 - x_1)}$$

We can see that when x_1, x_2, x_3 are evenly spaced ($x_2 - x_1 = x_3 - x_2 = h$)

$$f^{'}(x_2) = \frac{y_3 - y_1}{2h}$$

reduced to central differencing formula.

second derivative

Taylor expansions around x_2:

$$y_1 = y_2 + f^{'}(x_2)(x_1 - x_2) + \frac{f^{''}(x_2)}{2}(x_1 - x_2)^2 + \frac{f^{'''}(x_2)}{6}(x_1 - x_2)^3 + \mathcal{O}((x_1 - x_2)^4)$$

$$y_3 = y_2 + f^{'}(x_2)(x_3 - x_2) + \frac{f^{''}(x_2)}{2}(x_3 - x_2)^2 + \frac{f^{'''}(x_2)}{6}(x_3 - x_2)^3 + \mathcal{O}((x_3 - x_2)^4)$$

Neglect higher order terms and eliminate terms with $f^{'''}(x_2)$:

$$(y_1 - y_2)(x_3 - x_2)^3 - (y_3 - y_2)(x_1 - x_2)^3 = f^{'}(x_2)[(x_1 - x_2)(x_3 - x_2)^3 - (x_3 - x_2)(x_1 - x_2)^3]$$

$$+\frac{f^{''}(x_2)}{2}[(x_1-x_2)^2(x_3-x_2)^3-(x_3-x_2)^2(x_1-x_2)^3]$$

So we have:

$$f^{''}(x_2)=2\frac{-(y_1-y_2)(x_3-x_2)+(y_3-y_2)(x_1-x_2)}{(x_1-x_2)(x_3-x_2)(x_3-x_1)}$$

We can see that when x_1, x_2, x_3 are evenly spaced ($x_2-x_1=x_3-x_2=h$)

$$f^{';}(x_2)=\frac{-h(y_1-y_2)-h(y_3-y_2)}{-2h^3}=\frac{y_1-2y_2+y_3}{2h^2}$$

reduced to central differencing formula.