Algorithms - 4 Divide and conquer

Divide: into a number of subproblems that are smaller instances of the same problem
Conquer: solve subproblems recursively
Base case: if the subproblems are small enough, solve by brute force
Combine: merge subproblem solutions to give a solution to the original problem

Recurrence Relations

> transform recurrence relations in to closed form - asymptotic solution
>
> **substitution**- need to have a good guess
>
> > guess: the form of solution
> > check: mathematical induction to find the constants and show it works)
> >
> > > name the constant in the asymptotic notation
> >
> > e.g.
> > Given : T(n) = 2T(n/2) + cnlogn
> > **Guess:** $T(n) \le dnlog^2 n$
> > **Inductive hypothesis(IH):**
> > IH: $T(k) \le dk(logk)^2$ k<n and for positive constant d, c
> > $T(n) = 2T(n/2) + cnlogn \le 2d \times \frac{n}{2}(log\frac{n}{2})^2 + cnlogn$
> > $= dn(logn - log2)^2 + cnlogn$
> > $= dn(logn)^2 - 2dnlogn + dn + cnlogn$
> > $= dn(logn)^2 - (2d - c)nlogn + dn$
> > Since our base case is $n \ge 2 = n_0 \implies dn \le dnlogn$
> > $T(n) \le dn(logn)^2 - (2d - c)nlogn + dnlogn$
> > $= dn(logn)^2 - (d - c)nlogn$
> > Only if we choose d that $d - c \ge 0 \implies d \ge c$
> > we can make sure that: $T(n) \le dn(logn)^2$
> > There for $T(n) \le dnlog^2 n$
> >
> > if we need to prove θ, we need to prove O, Ω separatly
>
> **recursion**
>
> > recursion tree：
> > e.g $T(n) = \begin{cases} 2T(\dfrac{n}{2}) + O(n) & else \\ O(1) & if\ n = 1 \end{cases}$

depth	#node	node size	time complexity
0	1	n	$c \cdot n$
1	2	n/2	$c \cdot n$
...	...		...
i	2^i	$n/2^i$	$c \cdot n$
...	...		...
lgn	n	n/n	$c_2 \cdot n$

So total time complexity is $c_2 n + \sum_{i=1}^{logn} c \cdot n = nlogn(n)$

master method

$T(n) = aT(n/b) + f(n)$
a = # recursive calls >= 1 b > 1
f(n) >= 0 (for divide/combine)

case	f(n)	T(n)	situation
1	$O(n^{log_b a-\epsilon})$ constant $\epsilon > 0$	$\theta(n^{log_b a})$	cost dominated by the leaves
2	$\theta(n^{log_b a} log^k n)$ constant $k \geq 0$	$\theta(n^{log_b a} log^{k+1} n)$	cost is same at each level $log_b n$ levels
3	$\Omega(n^{log_b a+\epsilon})$ constant $\epsilon > 0$ and f(n) satisfies the regularity condition: $af(n/b) \leq cf(n)$ for some constant $c < 1$ and sufficiently large n	$\theta(f(n))$	cot dominated by the root

simplified:
$T(n) = aT(n/b) + \theta(n^d)$
$a \geq 1, b > 1, d \geq 0, assumen = b^k$

case	f(n)	T(n)	situation
1	$log_b a > d \quad or \quad a > b^d$	$\theta(n^{log_b a})$	cost dominated by the leaves
2	$log_b a = d \quad or \quad a = b^d$	$\theta(n^d logn)$	cost is same at each level $log_b n$ levels
3	$log_b a < d \quad or \quad a < b^d$	$\theta(n^d)$	cot dominated by the root

simplified ++: $\theta(n^d) \to n^d$
prove:

$$T(n) = \begin{cases} aT(\frac{n}{b}) + n^d & else \\ O(1) & if\ n = 1 \end{cases}$$

WLOG: $n = b^k$

depth	#node	node size	time complexity(non recursive cost)
0	a^0	$\frac{n}{b^0}$	$n^d = (\frac{n}{b^0})^d$

depth	#node	node size	time complexity(non recursive cost)
1	a^1	$\frac{n}{b^1}$	$n^d = (\frac{n}{b^1})^d$
...	...		...
i	a^i	$\frac{n}{b^i}$	$n^d = (\frac{n}{b^i})^d$
...	...		...
lgn	$a^k = a^{log_b n}$	$\frac{n}{b^k} = 1$	1

work for each level: $a^i(\frac{n}{b^i})^d = n^d(\frac{a}{b^d})^i$
total work : $n^d \sum_{i=0}^{log_b n}(\frac{a}{b^d})^i$
notice: a -> represent the increase of subproblem, b^d -> represents the decrease in cost of the subproblem

case3: $(\frac{a}{b^d})^i < 1 \iff log_b a < d$

> $a = (\frac{a}{b^d})^0 \leq \sum_{i=0}^{log_b n}(\frac{a}{b^d})^i \leq \sum_{i=0}^{\inf}(\frac{a}{b^d})^i$
>
> Thus $\sum_{i=0}^{log_b n}(\frac{a}{b^d})^i = \theta(1), T(n) = \theta(n^d)$

case2: $(\frac{a}{b^d})^i < 1 \iff log_b a < d$

> T(n) = $\sum_{i=0}^{log_b n}(\frac{a}{b^d})^i = n^d \sum_{i=0}^{log_b n} 1^i = n^d(log_b n + 1)$
> Thus $T(n) = \theta(n^d logn)$

case1: $(\frac{a}{b^d})^i > 1 \iff log_b a > d$

> $T(n) = n^d \sum_{i=0}^{log_b n}(\frac{a}{b^d})^i \in \theta(n^d \frac{a^{log_b n}}{b^{d log_b n}}) = \theta(n^d \frac{a^{log_b n}}{n^d}) = \theta(a^{log_b n}) = \theta(a^{\frac{log_a n}{log_a b}} = a^{log_a n log_b a} = n^{log_b a})$
> Notice in first step :
> sum of the geometric series, $a \neq b^d$, and $a/b^d - 1$ is a constant:
> $\sum_{i=0}^{log_b n}(a/b^d)^i = \frac{(a/b^d)^{log_b n+1}-1}{a/b^d-1} \in \theta((a/b^d)^{log_b n}))$

Problem - Matrix Multiplication

> Sol1: normally: $O(n^3)$
> Sol2: Recursively compute the matrix multiplication each matrix -> 4 squares - $O(n^3)$
>
> > $$T(n) = \begin{cases} 8T(\frac{n}{2}) + O(n^2) & else \\ O(1) & if\ n = 1 \end{cases}$$
> > Still $O(n^3)$
>
> Sol3: Strassen's Matrix
>
> > A11 A12 B11 B 12 = C11 C12
> > A21 A22 * B21 B22 = C21 C22
> >
> > we can just calculate 7 sub multiplication
> > P1 = A11(B12 - B22)
> > P2 = (A11 + A12)B22
> > P3 = (A21 + A22)B11
> > P4 = A22(B21 - B11)
> > P5 = (A11 + A22)(B11 + B22)
> > P6 = (A12 - A22)(B21 + B22)
> > P7 = (A11 - A21)(B11 + B12)

> > C11 = P5 + P4 - P2 + p6
> > C12 = P3+ P4
> > C21 = P3 + P4
> > C22 = P5 + P1 - P3 - P7
> >
> > $$T(n) = \begin{cases} 7T(\frac{n}{2}) + O(n^2) & else \\ O(1) & if\, n = 1 \end{cases}$$
> > $n^{2.80} \leq O(n^{log_2 7}) \leq n^{2.81}$
>
> Recurrence
> substitution method: guess and check method

Problem - Integer Multiplication

> original: x, y - n digit $O(n^2)$
> using merge and sort:
>
> > x = $2^{n/2} x_L + x_R$
> > y = $2^{n/2} y_L + y_R$
> > $x \cdot y = (2^{n/2} x_L + x_R)(2^{n/2} y_L + y_R) = x_L y_L 2^n + (X_R y_L + x_L y_R) 2^{n/2} + y_R x_R$
> > $T(n) = 4T(n/2) + O(n) -> O(n^2)$
> >
> > change:
> > $P_1 = x_L y_L, P_2 = x_R y_R, P_3 = (x_L + x_R)(y_L + y_R), x \cdot y = P_1 2^n + (P_3 - P_1 - P_2) 2^{n/2} + P_2$
> > $T(n) = 3T(n/2) + O(n) = O(n^{log_2 3})$

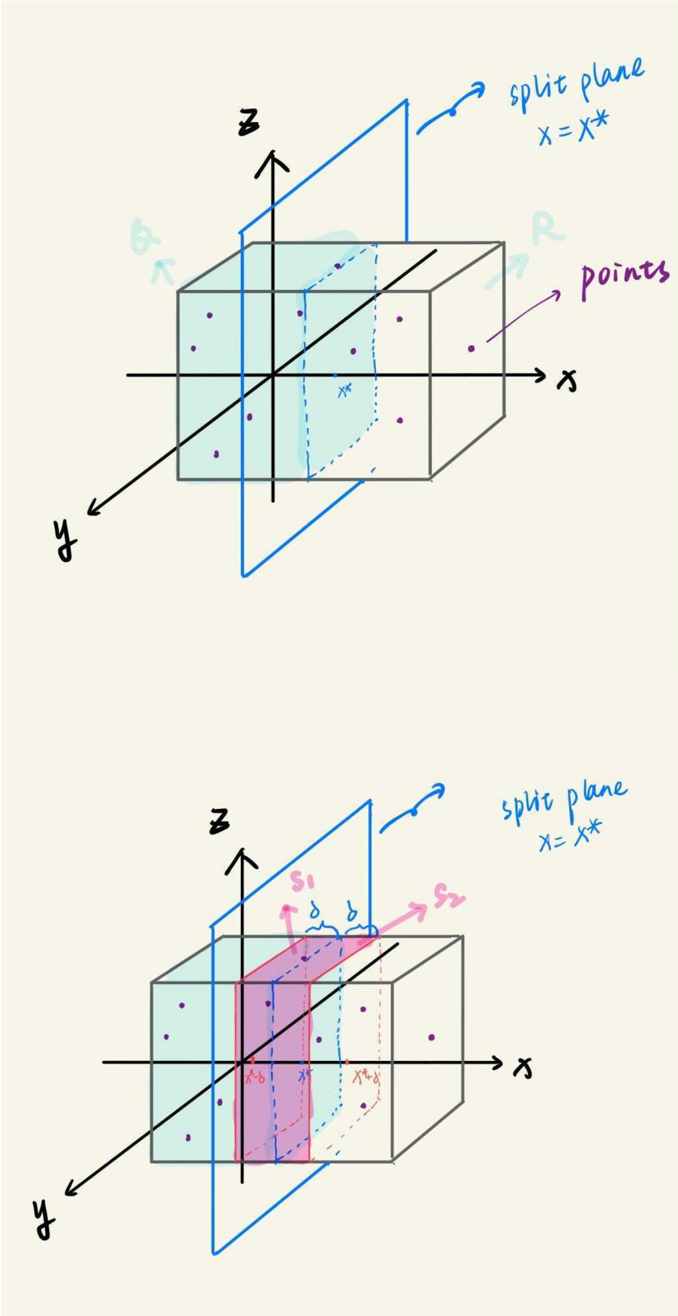
Problem - Closest Pair of Points

> Given a set of points, find the closest pair of points
>
> brute search: O(n^2)
> divide and conquer:
>
> ```
> CLOSET_PAIR_REC(px) # px are points sorted by x-value
>     if |px| <= 3
>         solve and return
>     CONSTRUCT Qx and Rx
>     (q1, q2) = CLOSEST_PAIR_REC(Qx)
>     (r1, r2) = CLOSEST_PAIR_REC(Rx)
>     theta = min{d(q1, q2), d(r1, r2)}
>     x_mean = max x-coordianate in Qx
>     CONSTRUCT Y = (s1, s2, ...sm) where si =(xi, yi) and |xi-xmean| <= theta # sorted by y-value
>     for each si in Y
>         for j = 1 to 7 # not tight ? may be smaller?
>             if d(si, si+j) < theta
>                 (s1', s2') = (si, si+j)
>                 theta = d(s1', s2')
> ```
>
> $T(n) = 2T(b/2) + O(nlongn) => O(nlog^2 n)$
> To improve: ? just input y, not x, y
>
> > O(nlogn)
>
> https://sites.math.rutgers.edu/~ajl213/CLRS/Ch33.pdf

3D

The question is quite similar, but instead of x, y, we need consider x, y, z now:

1. divide: split points to two sets according to their x coordinates with a plane x = x*, lets call two separated set Q and R. And here need to maintain four ordered array Qx, Qz, Rx, Rz, for Qx, Rx we can use index to get in O(1), but for Qz Rz we need O(n) of time (similar to $Y_L, Y_R -> Y$ in question 6) .

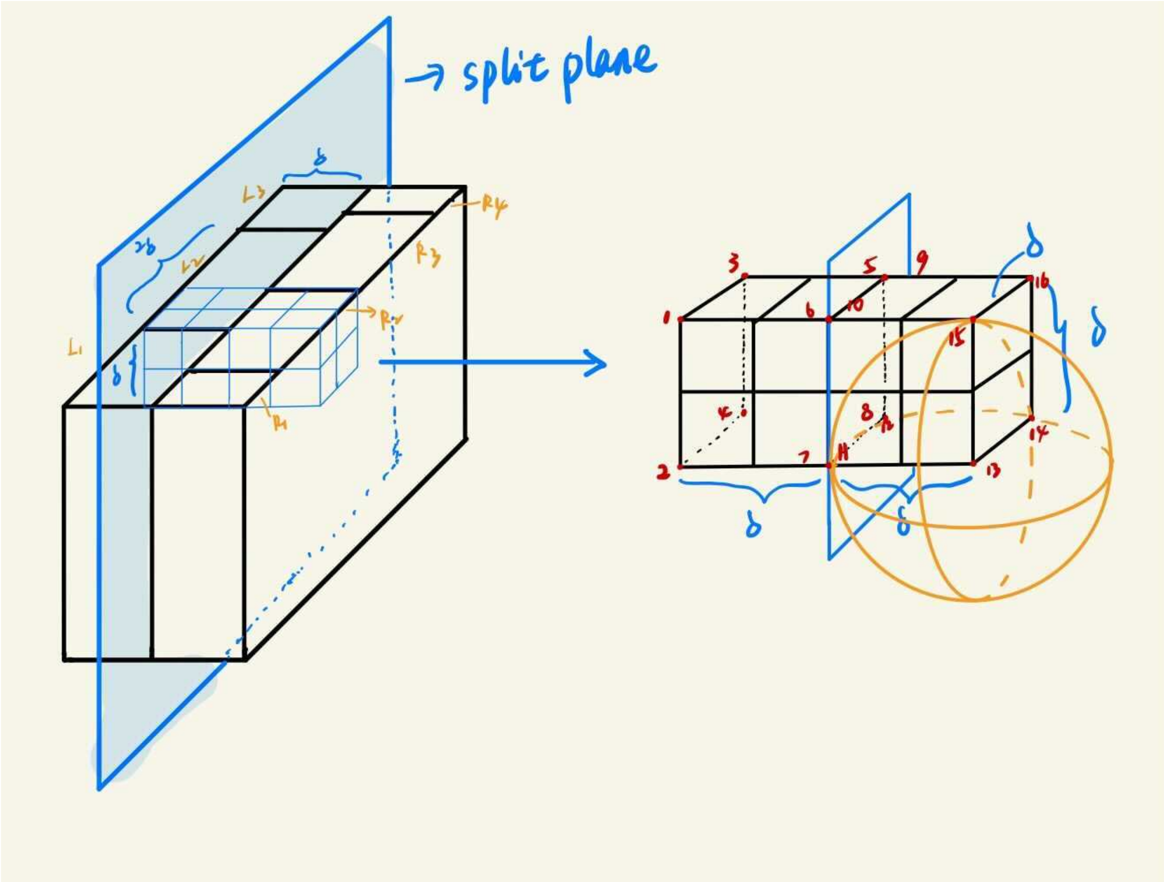
   

2. Merge: find the smallest distance δ_1 in R with points q_1, q_2 and δ_2 in Q with points r_1, r_2, $\delta = min(\delta_1, \delta_2)$ In order to check the pair across two sets, we need to:

   analysis two areas - S1 with x coordinates in [x* - δ, x*] , S2 with x coordinates in [x*, x*+ δ]
   Now even though we have consider the range of x coordinates, there still have two dimension (y, z):

   One way to solve this is to check all the points that the distance at y dimension is less than δ. Then for each point, how many near point we need to check?

   > Here is a trick for simpler coding:
   > We can just go through all point in S1(not all the points both in S1 and S2), but we must check two directions - for points x',y',z' in S2, if y' - y<= δ, y-y' <= δ
   >
   > according to pigeonhole principle we can only check 15 points in one direction, and we need to check 30 neighbor points for each points in S1.




```
FIND-SMALLEST-DISTANCE(P) # p are points
    sort P according to x -> Px
    sort P according to z -> Pz
    return Closest-Pair(Px, Pz)

CLOSEST-PAIR(Px, Pz)
    if |Px| <= 3:
        solve and return
    CONSTRUCT Qx, Rx, Qz, Rz
    (q1, q2) = CLOSEST-PAIR(Qx, Qz)
    (r1, r2) = CLOSEST_PAIR_REC(Rx, Rz)
    theta = min{d(q1, q2, q3), d(r1, r2, r3)}
    x_mean = max x-coordianate in Qx
    CONSTRUCT S1 where points (px, py, pz) satisfies 0 <= x_mean-px <= theta
    CONSTRUCT S2 where points (px, py, pz) satisfies 0 <= px - x_mean <= theta
    res_p1 = res_p2 = None
    for each p1 in S1
        for each p2 in ( all possible_points in s2) # no more than 30 points
            if d(p1, p2) < theta
                (res_p1, res_p2) = (p1, p2)
                theta = d(p1, p2)
    if res_p1 == None
        return smaller pair points between (q1, q2) (r1, r2)
    else
        return (res_p1, res_p2)
```

> https://blog.csdn.net/Carl_Rabbit/article/details/106841239 - only Chinese version, the graph is good
> https://sites.cs.ucsb.edu/~suri/cs235/ClosestPair.pdf
> http://wxr.logdown.com/articles/2014/03/14/recent-points-in-space
> https://people.csail.mit.edu/indyk/6.838-old/handouts/lec17.pdf
> http://euro.ecom.cmu.edu/people/faculty/mshamos/1976ShamosBentley.pdf

I'th Order Statistic

> given an array, we want to find i^{th} number

Deterministic Linear Selection

```
DETERMINISTICSELECT(A, n, i)
    # find the ith smallest of n items in A
    divide the elements of the input array A into groups of 5 # theta(n)
    find the medium of each group of 5 items and put them into another array B # theta(n)
    x = DETERMINISTICSELECT(B, n/4 , n/10) # T(n/5)
    partition A-{x} into two sets A1, A2 such that # theta(n)
    A1 = {k|k<x}
    A2 = {k|k>x}
 # ?
 if i = |A1| + 1 return x
    else
    if i < |A1| +1 return DETERMINISTICSELECT(A1,|A1|,1)
        else return DETERMINISITICSELECT(A2, |A2|, i-|A1|-1)
```

time analysis:

everytime we can split at least 3n/10 - 6 (don't need to check any more)
$T(n) \leq O(n) + T(n/5) + T(7n/10 + 6)$
$T(n) = O(n)$