Algorithms - 2 Data Structures

Data Structure

DS is a way to store and organize data in order to facilitate access and modifications

Basic Abstract Data Types (ADTs)

A mathematical model of the data methods used to modify and access the data, we don't care the implementation

Including: list, stack, queue, set,dictionary

stack

> PUSH(S,x), POP(S)

queue

> ENQUEUE(Q,x), DEQUEUE(Q)
> Q.head, Q.tail, Q.length

> **Priority queue**
>
>> data with a key
>> INSERT(P,x), EXTRACT-MAX(P), MAX(P), INCREASE-KEY(P,x,k)
>>
>> Applications: OS(manage jobs), Graph Algorithms(Dijkstra), arrange events, compression(Huffman encoding), heapsort...
>>
>> How should we implement the priority queue:
>> binary heap,..

Common data structures

---

1. Heap

> **Binary Heap**

>> **structure**
>>
>>> array structure, visualized as nearly complete tree(only the left of last layer may be not full)
>>>
>>> root = A[1], PARENT(i) = $\frac{i}{2}$, LEFT(i)= $2i$, RIGHT(i)=$2i$+1
>>> attributes: A.length
>>>
>>> **Theorem: A nearly complete binary tree of n nodes has height** $\theta(logn)$
>>> prove:
>>>
>>>> a complete binary tree with height h has $\sum_{i=0}^{h} 2^i = \frac{1-2^{h+1}}{1-2} = 2^{h+1} - 1$ nodes
>>>> so for a nearly complete tree $1 + \sum_{i=0}^{h-1} 2^i = 2^h \leq n \leq 2^{h+1} - 1$
>>>
>>> Some other properties about the tree:
>>> https://cs.stackexchange.com/questions/841/proving-a-binary-heap-has-lceil-n-2-rceil-leaves
>>
>> **Attribute**
>>
>>> min binary heap PARENT(A[i]) $\leq$ A[i]
>>
>> **Operations** - e.g. Min-heap

```
HEAP-MINIMUM(A)
    return A[1]

HEAP-EXTRACT-MIN(A)
    if A.heap-size < 1
        error 'heap underflow'
    min = A[1]
    A[1] = A[A.heap-size]
    A.heap-size = A.heap-size - 1
    MIN-HEAPIFY(A, 1)
    return min

MIN-HEAP-INSERT(A, key)
    A.heap-size = A.heap-size + 1
    A[A.heap-size] = inf
    HEAP-DECREASER-KEY(A, A.heap-size, key)
```

```
HEAP-DECREASE-KEY(A,i,key)
    if key > A[i]
        error 'new key is larger'
    A[i] = key
    while i > 1 and A[PARENT(i)] > A[i]
        exchange A[i] with A[PARENT(i)]
        i = PARENT(i)

BUID-MIN-HEAP(A)
    A.heap-size = A.length
    for i = floor(A.length/2) downto 1
        MAX-HEAPIFY(A,i)

MIN_HEAPIFY(A,i)
    l = LEFT(i)
    r = RIGHT(i)
    if l <= A.heap-size and A[l] < A[i]
        smallest = l
    else smallest = i
    if r <= A.heap-size and A[r] < A[smallest]
        smallest = r
    if smallest != i:
        exchange A[i] and A[smallest]
        MIN-HEAPIFY(A, smallest)

HEAPSORT(A) #O(nlogn)
    BUILD-MAX-HEAP(A)
    for i = A.length downto 2
        exchange A[l] with A[1]
        A.heap-size == A.heap-size - 1
        MAX-HEAPIFY(A, 1)
```

Time complexity of build a heap: O(n)

> Amortized analysis: determine worst-case of a sequence of data structure operations
>
> | layer | height | depth | nodes | time = relate node*c |
> |---|---|---|---|---|
> | 1 | 3 | 0 | $1 = 2^0$ | $c \cdot 3 \cdot 2^0$ |
> | 2 | 2 | 1 | $2 = 2^1$ | $c \cdot 2 \cdot 2^1$ |
> | 3 | 1 | 2 | $4 = 2^2$ | $c \cdot 1 \cdot 2^0$ |
> | 4 | 0 | 3 | <=8 $= 2^3$ | |
>
> h is the height of the tree
> $T(n) \leq c \sum_{i=1}^{h} 2^{h-i} \cdot i \leq c \cdot s^h \sum_{i=1}^{h} s^{h-i} \cdot i \leq c \cdot 2^{logn} \sum_{i=1}^{h} 2^{-i} \cdot i \cdot \leq cn \sum_{i=1}^{\text{inf}} 2^{-i} \cdot i$
> $= cn \sum_{i=1}^{inf} (\frac{1}{2})^i \cdot i = \frac{cn}{2} \sum_{i=1}^{inf} i \cdot (\frac{1}{2})^{i-1} = \frac{cn}{2} \frac{1}{(1-1/2)^2} = O(n)$
>
> notice that:
> $\frac{\partial \sum_{i=0}^{inf} x^i}{\partial x} = \sum_{i=0}^{inf} i \cdot x^{i-1}$
> $\frac{\partial \frac{1}{1-x}}{\partial x} = \frac{1}{(1-x)^2}$
> $\sum_{i=0}^{inf} x^i = \frac{1}{1-x}$

2. Hash

---

dynamic Set operations:

> key(x) = k(satellite data)
> SEARCH(S,k)
> MINIMUN(S)/MAXIMUM(S)
> SUCCESSOR(S,x)
> PREDECESSOR(S,x)
>
> **dictionary operation**
>
> > SERACH(S,k) - returns a pointer x, where x.key = k or nil if not found
> > INSERT(S,x)
> > DELETE(S,x)
>
> **dictionary data structure**
>
> > universe of keys U
> > a data structure that stores a subset $S \subseteq U$
> > operations:
> > SEARCH(S,k): given $k \in u$
> > INSERT(S,x): given x, a pointer to $k \notin S$
> > DELETE(S,x): given x, a pointer to $k \in S$

Hashing and Hash Tables

> a hash table is an array T of size m
> a hash function creates and index in the array from an $k \in U, h: U \rightarrow \{0, \ldots, m-1\}$ (hashes to slot h(k) in T)
>
> **Collision solution - chaining:**
>
> > Operations:
> >
> > > CHAINED-HASH-INSERT(S,x): insert x at the head of lst T[x.key]
> > > CHAINED-HASH-DELETE(S,x): delete x from the list T[x.key]
> >
> > Assumption:
> >
> > > Simple uniform hashing: any key is equally likely to hash to any of the m slots
> > > for all $i \neq j, pr[h(k_i) = h(k_j)] = \frac{1}{m}$
> >
> > Analysis for unsuccessful search
> >
> > > Let S be the items already in the table where $x \notin S, |S| = n$
> > > $C_{x,y} = \begin{cases} 1 & h(x) = h(y) \\ o & otherwise \end{cases}$
> > > $E[C_{x,y}] = 1 \cdot P[h(x) = h(y)] + 0 \cdot P[h(x) \neq h*(y)] = \frac{1}{m}$
> > > $C_x = \sum_{y \in S.y \neq x} C_{x,y}$ = # of items that x has a collision with
> > > $E[C_x] = \sum_{y \neq x} E[C_{x,y}] = \frac{n}{m}$ = # items in the table divided by the size of the table
> > > So, if $n = O(m)$ then $\alpha = n/m = O(m)/m = O(1)$, for insertion, deletion, find

Universal Hashing:

> **choosing a hash function**
>
> > using universal Hashing - randomness
> > Solve:
> > 1.Unifoma random
> > 2.fundamental weakness
> > for any hash function, we can find a set of keys that are hashed to the same spot
>
> **Universal Class of Hash Functions:**
>
> > $H = h_1, h_2, \ldots, h_w$ is a universal hash function family if for all $k, k' \in U, Pr_{h\in H}[h(k) = h(k') \leq \frac{1}{m}]$, m is the table size
>
> **How to construct a universal Hash functions family:**
>
> > $H_{pm} = \{h_{a,b}(x) = ((ax+b)\ mod\ p)\ \}\ 1 \leq a \leq p-1, 0 \leq b \leq p-1,\ and\ p \geq all\ keys$
> >
> > Proof:
> > Let $k, l \in U$ are two keys, WLOG $k > l, k \neq l.$
> >
> > 1. **If $h_{ab}(k) = h_{ab}(l)$ it is because they collided after mod p or mod m   - Property of prime numbers**
> >   p is a prime larger than any key, and p > m
> > 2. **If $h_{ab}(k) = h_{ab}(l)$ it is because they collided after mod m - Modulo arithmetic**
> >   let r = (ak+b) mod p, s = (sl + b) mod p
> >   since a < p,   k,l <p => a doesn't divide p, (k-l) doesn't divide p
> >   the collision cannot be mod p since o/w 0 $\neq s - r \equiv a(k-l)\ mod\ p$
> > 3. **There is a 1-1 correspondence between the (r,s) pairs and the (a,b) pairs**
> >
> >   a = ((r-s)((k-l)^-1 mod p)) mod p
> >   b = (r-ak) mod p
> >   There are p(p-1) possible pairs (r,s) that $r \neq s$, thus there is a 1-1 correspondence between pairs (a,b) and pairs (r,s). Thus if a, b is chosen randomly the pair (r,s) is equally likely to be any pair of distinct values modulo p
> > 4. **If we randomly choose an (r,s) pair, the probability that they collided mode m is <= 1/m.** Thus if I randomly choose an (a. b) pair the probability that $h_{a,b}(k) = h_{a,b}(l)$ is at most $\frac{1}{m}$.
> >   # of choice for s such that $r \neq r$ and $s \equiv r$ is at most $\lceil \frac{p}{m} \rceil - 1$
> >   (just list all the possible s that collide with r: r'= r mod m
> >   r', r'+m, r'+2m, r'+3m, r'+4m, ..., r' + $(\lfloor \frac{p}{m} \rfloor - 1)m)$ are all less than p, r' + $(\lceil \frac{p}{m} \rceil - 1)m)$ may be less than p
>
> **Theorem**
>
> > If for all $k, k' \in U, Pr_{h\in H}[h(k) = h(k')] \leq \frac{1}{m}$, then for any $S \subseteq U$, for any $x \in U - S$ the expected number of collisions between x and the other elements in S is at most n/m, when |S|=n
> >
> > Proof:
> >
> > Corollary 1: If   for all $k, k' \in U, Pr_{h\in H}[h(k) = h(k')] \leq \frac{1}{m}$, then for any $S \subseteq U$, for any $x \in U - S$ thee expected number of comparisons for a successful search is O(1+n/m) and for an unsuccessful search is O(n/m) where |S| = n.
> >
> > Corollary 2: Using universal hashing and collision resolution by chaining in an initially empty table with m slots, it takes expected time $\theta(n)$ to handle any sequence of n INSERT, SEARCH, and DELETE operations containing O(m) INSERT operations where m is the table size

Perfect Hashing

> **Static**
>
> > We can do better when keys are static:
> > Given a fixed set of n keys we can construct a static hash table of size $m = \theta(n)$ such that search takes $\theta(1)$ time in the worst case
> >
> > > **Theorem:**
> > > Hashing n keys into $m = n^2$ slots using $h \in_R H$ then E(# collisions) < 1/2
> > >
> > > **Proof:**
> > > probability that two keys collide is $1/m = 1/n^2$, and # pairs of key = $\binom{n}{2}$
> > > E(# collisions) $\leq \binom{n}{2} \cdot \frac{1}{n^2} = \frac{n(n-1)}{2} \cdot \frac{1}{n^2} < \frac{1}{2}$
> > >
> > > **Corollary:**
> > > Probability of no collisions > 1/2
> > > Proof: X be a R.V. holding the number of collisions, according to Markov inequality:
> > > $Pr[X \geq 1] \leq \frac{E[X]}{1} < \frac{1}{2}$
> > > (FYI: Markov's inequality can refer to Math.md)
> >
> > **To reduce the space: two levels**
> >
> > > **Theorem:**
> > > Let m=n be the size of the hash table at level 1
> > > Let n_j^2 be the size of the hash table at index j
> > > Let h be randomly chosen from a universal set H
> > > Then $Pr[\sum_{j=0}^{m-1}(n_j)^2 \geq 4n] < 1/2$
> > >
> > > **Proof:**
> > > the number of pairs that collide at the first layer:
> > > $\sum_{x \in S}\sum_{y \in S} C_{x,y} = \sum_{j=0}^{m-1}\sum_{x \in S_j}\sum_{y \in S_j} C_{x,y} = \sum_{j=0}^{m-1}(n_j)^2$
> > > $E[\sum_{j=0}^{m-1} n_i^2] = E[\sum_{x \in S}\sum_{y \in S} C_{xy}] = n + \sum_{x \in S}\sum_{y \in S-x} E[C_{xy}] \leq n + n(n-1)/m < 2n$
> > > Markov's inequality: $Pr[\sum_{i=1}^{m-1}(n_i)^2 \geq 4n] < 2n/4n = 1/2$
>
> **dynamic**
>
> TBC

Consistent Hashing

TBC

Bloom-filer

> is a space-efficient probabilistic data structure for test membership
>
> property:
>
> > False positive - increase with n
> > never generate false negative result
> > can't delete element
> > can always adding element with the prize that FP increaser
>
> Operations:
>
> > insert(x), lookup(x)

> Probability of FP:
>
> > m: size of bit array,
> > k: number of hash functions
> > n: number of expected elements to be inserted in the filter
> > $P = (1-(1-\frac{1}{m})^{kn})^k$

3. Tree

> we may need more operations than dictionary
> - maintain order information(for fast retrieval) , support insert/ delete/ search
>
> each node: .key, .satellite_data, .p, .left, .right
>
> ## Binary Search Tree
>
> > the left tree are all smaller, and the right trees are all greater
>
> ## Self Balancing Tree
>
> > Balanced: height is O(logn)
> > Insertion:
> > local transformation preserve global property: 1. tree is ordered, 2. perfectly balanced
> > SEARCH(T,k) O(logn)
> > INSERT(T,k) O(logn)
> > DELETE(T,k) O(logn)
> >
> > #### 2-3 Tree
> >
> > > at most 3 children, all external nodes have same depth(perfectly balanced)
> > > 2-node: 1 key 2 links
> > > 3-node: 2 keys 3 links
> > > $\lfloor log_3 n \rfloor \le h \le \lfloor log_2 n \rfloor$
> > >
> > > Search
> > > Insert
> > >
> > > > always insert into an existing node
> > > > always maintain depth condition
> > > > split node if a 3-node:
> > > >
> > > > > create a new node, v', push middle key up
> >
> > #### 2-4 Tree
> >
> > > at most 4 children, all external nodes have same depth(perfectly balanced)
> > > 2-node: 1 key 2 links
> > > 3-node: 2 keys 3 links
> > > 4-node: 3 keys 4 links
> > >
> > > $\lfloor log_3 n \rfloor \le h \le \lfloor log_2 n \rfloor$
> > >
> > > always insert into an existing node
> > > always maintain depth condition
> > > split node if a 4-node:
> > >
> > > > create a new node, v', push middle key up

> > Note: different insert order may cause different structure
> > https://stackoverflow.com/questions/38258701/when-would-a-2-3-4-tree-not-have-the-same-structure
>
> relax the idea that the tree has to be perfectly balanced - RB Tree

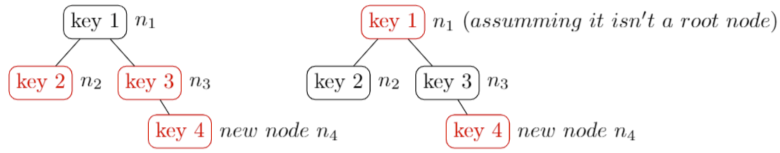
Red-Black Tree

> is a binary search tree that obeys **5 properties**:
>
> > 1.Every node is colored red or black
> > 2.The root is black
> > 3.Every leaf(T.nil) is black and doesn't contain any data
> > 4.Both children of a red node are black
> > 5.For every node in the tree, all paths from that node down to a leaf(nil) have the same number of black nodes along the path
>
> Creating a Red-Black BST from a 2-3-4 tree
>
> > 4-node: one black -> two red
> > 3-node: black -> red have two ways
> > 2-node: black
>
> Creating a 2-3-4 tree from Red-Black BST
>
> > merge every red node into its black parent
>
> **Height of a node:** # edges in a longest path to a leaf
> **Black height of a node x:** bh(x) # black nodes encountered on a path to a leaf, not including the node itself.
>
> **Claim: A node with height h has black-height at least h/2**
>
> > Proof: same as $h \leq 2 * bh$
>
> **Claim: The subtree rooted at a node, x, contains at least $2^{bh(x)} - 1$ internal nodes (non nil)**
>
> > Basis: if x has height 0, it is a leaf(nil node) and bh(x) = 0, the subtree rooted at x has 0 internal nodes $2^0 - 1 = 0$
> >
> > Inductive hypothesis: if x has height h and b= bh(x), then its children have height at most h-1. If the child is red it has black-height b=bh(x), otherwise it has black-heigh b-1 = bh(x) -1. By the inductive hypothesis each child has at least $2^{bh(x)-1} - 1$ internal nodes. Thus the subtree rooted at x contains at least $2(2^{bh(x)-1} - 1) + 1 = 2^{bh(x)} - 1$ internal nodes.
>
> **Lemma: A red-black tree with n internal nodes has height at most 2lg(n+1)**
>
> > Proof: Let h be the height, and b= bh(x) be the black height of the root node = x. By the previous two claims $n \geq 2^b - 1 \geq 2^{h/2} - 1$. Thus $n + 1 \geq 2^b \geq 2^{h/2} \implies lg(n+1) \geq h/2 \implies h \leq 2lg(n+1)$
>
> **Corollary: On a red-black tree with n nodes, we can implement dynamic-set operations SEARCH, MINIMUM, MAXIMUM, PREDECCESOR.... In (logn)**
>
> **Modify the Tree:**
>
> > Insert it as a red node, and then handle violation
> > There are three possible violations:
> > Case1: sibling of parent is red

case 1: double red problem

red-black tree:

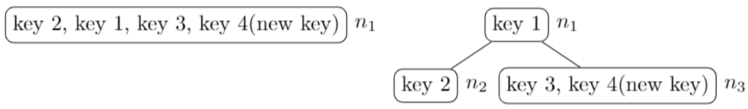
situation: both parent n_2 and sibling of parent n_1 are red nodes



fix: change both parent n_3 and parent's sibling n_2 into black, and change parent's parent n_3 into red, finally set root to be black

2-3-4 tree:

situation: the node to be inserted has already three keys



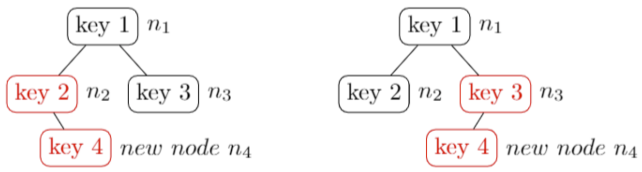
fix: since the origin node has 3 keys, it split into two children, choose the middle one key_1 to be parent.

Case2:

case 2: one black and one red problem

red-black tree:

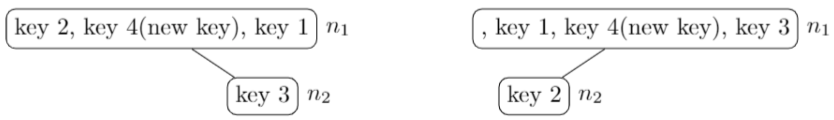
situation: uncle node is black,the new node and new nodes' parent are in the different side



fix: change rotate new node and new node's parent, to make sure they are in the same side, then it would be case 2.

2-3-4 tree:

situation: the node to be inserted into has 2 keys

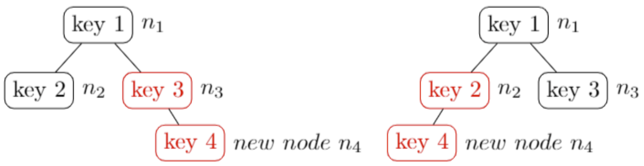


fix: the node to be inserted only two keys, so we can directly insert new key into corresponding node.

Case3:

red-black tree:

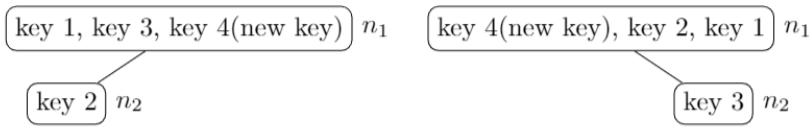
situation: sibling of parent is black, the new node and new nodes' parent are in the same side



fix: rotate the new node' parent and new node's parent's parent, make new node and new node's parent step into higher level, let new node's parent's parent to be the another children of new node's parent, and than change new node's parent into black, and change new node's parent's parent into red.

2-3-4 tree:

situation: the node to be inserted into has 2 keys



fix: since the origin node has only two keys, just insert new key into the node is fine.

```
#      x                          y
#  t1      y       =>        x       t3
#        t2   t3          t1   t2
LEFT-ROTATE(T,x) # no color change
    y = x.right
    # move the t2 from y to x.right
    x.right = y.left
    if y.left != T.nil:
        y.left.p = x
    y.p = x.p
    # attack y to x's orginal parent
    if x.p == T.nil
        T.root = y
    elif x == x.p.left
        x.p.left = y
    else
        x.p.right = y
    # attack x to y
    y.left = x
    x.p = y

RB-INSERT(T,z)
    y = T.nil
    x = T.root
    # find where to insert
    while x!= T.nil
        y = x
        if z.key < x.key
            x = x.left
        else: x = x.right
    #create node and attach it
    z.p = y
```

```
    if y == T.nil: # tree T was empty
        T.root = z
    else if z.key < y.key
        y.left = z
    else y.right = z
    # solve color violation
    z.left = T.nil
    z.right = T.nil
    z.color = RED
    RB-INSERT-FIXUP(T,z)

RB-INSERT-FIXUP(T,z)
    while z.p.color == RED
        if z.p == z.p.p.right: # parent is a left child
            y = z.p.p.left # y is uncle
            if y.color = RED # case 1: just change the color
                z.p.color = BLACK
                y.color = BLACK
                z.p.p.color = RED
                z = z.p.p
            else # case 2 or case 3
                if z == z.p.left # z is a left child case 2
                    z = z.p
                    RIGHT-ROTATE(T,z)
                # handle case 3
                z.p.color = BLACK
                z.p.p.color = RED
                LEFT-ROTATE(T,z.p.p)
        else (same as then clause with right and left exchanged)
  T.root = Black
```

Proof we maintain the Red-Black tree properties:

> **Loop invariant:**
> At the start of each iteration of the while loop, node z is red.
> There is at most one red-black tree violation either:
> a. Z is a red root
> b. z and z.p are both red
>
> **Initialization:**
> before we add the new red node z, so if there is violation after insert z, it must between z and z.parent
>
> Termination: loop terminates when z.p is black. Thus, there is no red-red violation. And finally we set root to be black. = > so there will no violation.
>
> **Maintenance:**
> If we enter the loop, then z and its parent are both red, and no other violation occurs in the red-black tree.
>
> WLOG we only consider the 3 cases, where z.p is the right child of z.p.p let y = z.p.p.left // y is z's uncle. z.p.p is black, since z and z.p are red and there is only one red-red violation
>
> **Case1:** y is red and z.p.p is black, by making y and z.p black and z.p.p red. Thus black height property is maintained, but we might have created a red-red violation between z.p.p and its parent. Assign z = z.p.p
>
> **Case2:** y is black and z is a left child. Set z = z.p right rotate around z. Now z is now a right child.Both z and z.p are red. Only one case 3 red-red violation.
>
> **Case3:** y is black z is a right child, make z.p black and z.p.p red left rotate on z.p.p

Augmented Data structure

Selection problem

> how to find the kth largest item in an array

Determine what to add
operation

> Dynamic order statistics
> OS-SELECT(T,i)
> OS-RANK(T,x)
>
> x.size = x.left.size + x.right.size + 1
>
> ```
> x.size = x.left.size + x.right.size + 1
>
> OS-SELECT(x,i)
>     r = x.left.size +1
>     if i == r
>         return x
>     elsif i<r
>         return OS-SELECT(x.left, i)
>     else
>         return OS-SLECT(x.right, i-r)
>
> OS-RANK(T,x)
>     r = x.left.size + 1
>     y = x
>     while y!= T.root
>         if y == y.p.right
>             r = r + y.p.left.size + 1
> ```
>
> ```
> INTERVAL-SEARCH(T,i)
>     x = T.root
>     while x != T.nil and i does not overlap x.int:
>         if x.left != T.nil and x.left.max >= i.low
>             x = x.left
>         else x = x.right
>     return x
> ```

B trees

> a-b Trees
> Height of B-tree
> Theorem
>
> ```
> B-Tree-Search(x,k)
>     i = 1
>     while i<= x.n and k > x.key_i
> ```

```
        i = i + 1
    if i <= x.n and k ==x.key_i
        return (x,i)
    elif x.leaf
        return NIL
    else Dist-Read(x,ci)

B-Tree-Create(T)
    x = Allocate-Node()
    x.leaf = True
    x.n = 0
    Disk-Write(x)
    T.root(x)

B-Tree-Split-Child(x,i)
```

application

red-black trees: https://www.quora.com/What-are-some-real-world-applications-of-Red-Black-trees-today
https://www.cnblogs.com/yufeng218/p/12465694.html mysql 引擎的实现
https://zhuanlan.zhihu.com/p/87124501 b+ 面试题
https://www.codenong.com/300935/ key 唯一吗？要

https://stackoverflow.com/questions/36084032/how-to-deal-with-duplicates-in-red-black-trees