20190312, Ch1 intro to QM

Wednesday, March 6, 2019 1:48 PM

recap

→ §5 representation theory

§6 measurement

Homework:

1. Sakurai 1.28 (b), (c) (without (a))
2. Sakurai 1.33
3. with $|\alpha\rangle$ a Gaussian wavepacket,

$$\langle x|\alpha\rangle = \frac{1}{\pi^{1/4}\sqrt{d}} \exp\left[ikx - \frac{x^2}{2d^2}\right]$$

proof $\langle p|\alpha\rangle = \sqrt{\frac{d}{\hbar\sqrt{\pi}}} \exp\left(-\frac{(p-\hbar k)^2 d^2}{2\hbar^2}\right)$

hint: $\int_{-\infty}^{+\infty} dx \exp\left(-\frac{(x-\xi)^2}{d^2}\right) = d\sqrt{\pi}$, ξ complex

4. read chapter 2 of Sakurai (no need to hand in notes for this one)

In-class quiz: 03/26, 60 min, Homework level difficulty chapter 1 One double sided A4 cheat sheet allowed

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§7 infinite dimension Hilbert space

two dimensional $|0\rangle, |1\rangle$

$|z+\rangle, |z-\rangle$

position operator

$$\hat{X}|x\rangle = x|x\rangle$$

eigen value eigenstate

$\{|x\rangle\}$ forms a basis, typically $-\infty < x < +\infty$

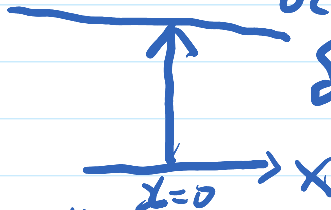
2. basis $\{|x\rangle\}$ $\{|x\rangle\}$

$\{|x\rangle\}$ forms a basis, typically $-\infty < x < +\infty$

basis $\{|\alpha_i\rangle\}$ $\qquad$ $\{|x\rangle\}$

$$\langle \alpha_i | \alpha_j \rangle = \delta_{ij} = \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases}$$

$$\langle x' | x'' \rangle = \delta(x' - x'')$$



$$\delta(x) = \begin{cases} \infty, & x=0 \\ 0, & \text{else} \end{cases}$$

$$\underline{\mathbb{1}} = \sum_i |\alpha_i\rangle\langle\alpha_i| \qquad \mathbb{1} = \int_{-\infty}^{+\infty} dx\, |x\rangle\langle x|$$

$$\int_{-\infty}^{+\infty} dx\, \delta(x) = 1$$

$$|\psi\rangle = \sum_i C_i |\alpha_i\rangle \qquad \underline{|\psi\rangle} = \mathbb{1} \cdot |\psi\rangle = \int_{-\infty}^{+\infty} dx\, |x\rangle \underline{\langle x| \cdot |\psi\rangle}$$

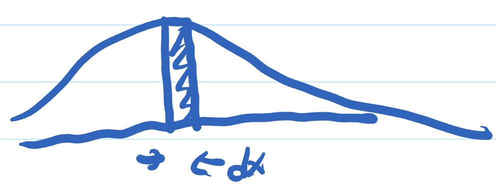
wave function

$$\psi(x) \equiv \langle x | \psi \rangle$$

$$|\psi\rangle = \int_{-\infty}^{+\infty} \underline{dx}\, \underline{|x\rangle}\, \underline{\psi(x)}$$

probability amplitudy of

C_i $\qquad$ $\psi(x) \cdot dx$



probability of finding a particle

$$P_i = |\langle \alpha_i | \psi \rangle|^2 = |C_i|^2 \qquad P(x) = |\langle x | \psi \rangle|^2 \cdot dx = |\psi(x)|^2 \cdot dx$$

finding $[x_0, x_1]$

$$P = \int_{x_0}^{x_1} P(x) = \int_{x_0}^{x_1} |\psi(x)|^2 dx$$

$$|\alpha_i\rangle = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{pmatrix} \leftarrow i^{th}$$

representation

$\langle \varphi | \psi \rangle = 0$

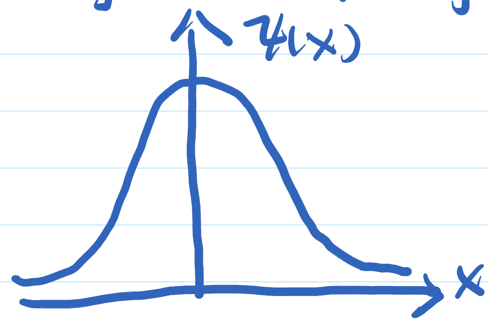
$$\langle \varphi | \psi \rangle = \langle \varphi | \mathbb{1} | \psi \rangle$$
$$= \sum_i \langle \varphi | \alpha_i \rangle \langle \alpha_i | \psi \rangle$$
$$= (d_0^* \ d_1^* \ \cdots) \cdot \begin{pmatrix} c_0 \\ c_1 \\ \vdots \\ \vdots \end{pmatrix}$$

$\{ |\alpha_i\rangle \}$

$$P = \int_{x_0}^{x_1} \rho(x) = \int_{x_0}^{x_1} |\psi(x)|^2 dx$$

no analog w/ matrix

$\{ |x'\rangle \} \longleftrightarrow \{ |x\rangle \}$



wavepacket

$$\langle \varphi | \psi \rangle = \langle \varphi | \mathbb{1} | \psi \rangle$$
$$= \int_{-\infty}^{+\infty} dx \, \langle \varphi | x \rangle \langle x | \psi \rangle$$
$$= \underline{\int_{-\infty}^{+\infty} dx \, \varphi^*(x) \cdot \psi(x)}$$

$\{ \psi(x) \}$

$$\langle \psi | \psi \rangle = 1 = \underline{\int_{-\infty}^{+\infty} dx \, \psi^*(x) \, \psi(x)}$$
$$= \int_{-\infty}^{+\infty} |\psi(x)|^2 dx$$

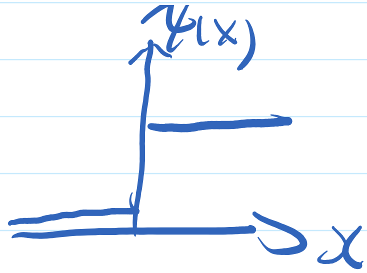
requirements of a "physical" wavefunction $\psi(x)$

① single value

$\uparrow \psi(x)$

① single value

② continuous



③ $x \to \pm\infty$, $\psi(x) \to 0$

④ norm square integratable

ensures $\int_{x_0}^{x_1} |\psi(x)|^2 dx$ is physical

§8 momentum operator

- translation operator

位移

$\hat{\mathcal{T}}(dx')$

$$\underline{\hat{\mathcal{T}}(dx')|x'\rangle = |x' + dx'\rangle}$$

$$\hat{\mathcal{T}}(0) = \mathbb{1}$$

$$A \cdot B|\psi\rangle = A(B|\psi\rangle)$$

$$\left\{\begin{array}{l} \hat{\mathcal{T}}(-dx')\cdot\hat{\mathcal{T}}(dx') = \mathbb{1} \\ UU^\dagger = \mathbb{1} \\ \hat{\mathcal{T}}(dx)^\dagger\,\hat{\mathcal{T}}(dx) = \mathbb{1} \end{array}\right. \Rightarrow \underline{\hat{\mathcal{T}}(dx')^\dagger = \hat{\mathcal{T}}(-dx')}$$

$$\hat{\mathcal{T}}(dx') \doteq \mathbb{1} + \hat{\xi}\, dx' + O(dx'^2) + \cdots$$

$$\hat{\mathcal{T}}(dx')^\dagger = \mathbb{1} + \hat{\xi}^\dagger dx' \;\; \boxed{1 + \quad 1}$$

$$\hat{\mathcal{T}}(dx')^{\dagger} = \hat{\mathbb{1}} + \hat{\xi}^{\dagger} dx'$$
$$\hat{\mathcal{T}}(-dx') = \mathbb{1} - \hat{\xi} dx'$$
$$\boxed{\hat{\xi}^{\dagger} = -\hat{\xi}}$$

let $\hat{\xi} = -i\hat{k}$

$i\hat{k}^{\dagger} = i\hat{k} \Rightarrow \hat{k}^{\dagger} = \hat{k}$ Hermitian

$$\boxed{\hat{\mathcal{T}}(dx') = \underset{\text{unitless}}{\mathbb{1}} - \underline{i}\,\hat{k}\, dx'}$$

$\hat{k}$ Hermitian op.

$[k] \sim \frac{1}{[x]}$ wavenumber

photons $k = \frac{2\pi}{\lambda}$

$\hat{k} \propto \hat{p}$

$$\hat{\mathcal{T}}(dx')|\psi\rangle = \hat{\mathcal{T}}(dx')\mathbb{1}|\psi\rangle = \int_{-\infty}^{+\infty} dx'' \, \hat{\mathcal{T}}(dx')|x''\rangle\langle x''|\psi\rangle$$
$$= \int_{-\infty}^{+\infty} dx'' \, |x'' + dx'\rangle \psi(x'')$$

$x''' = x'' + dx'$

$$= \int_{-\infty}^{+\infty} d(x''' - dx') \, |x'''\rangle \psi(x''' - dx')$$
$$= \int_{-\infty}^{+\infty} dx''' \, |x'''\rangle \psi(x''' - dx')$$

$$\psi(x''' - dx') \simeq \psi(x''') - dx' \frac{\partial}{\partial x'''}\psi(x'') + O(dx'^2)$$

$$= \int_{-\infty}^{+\infty} dx''' \, |x'''\rangle \psi(x''') - dx' \int_{-\infty}^{+\infty} dx''' |x'''\rangle \frac{\partial}{\partial x''} \psi(x'')$$
$$= |\psi\rangle - dx' \int_{-\infty}^{+\infty} dx''' \, |x'''\rangle \frac{\partial}{\partial x''} \psi(x''')$$

$$\boxed{\hat{k}|\psi\rangle = -i\int_{-\infty}^{+\infty} dx \left(\frac{\partial}{\partial x'}\psi(x')\right)\Big|_{x'=x} |x\rangle}$$

$$\langle x''|\hat{k}|\psi\rangle = -i\int_{-\infty}^{+\infty} dx \left(\qquad \right) \underbrace{\langle x''|x\rangle}_{\delta(x''-x)}$$

$$\int_{-\infty}^{+\infty} dx\, f(x)\,\delta(x'-x) = f(x')$$

$$\langle x|\hat{k}|\psi\rangle = -i\frac{\partial}{\partial x}\psi(x)$$

$$[\hat{x},\hat{k}] = \hat{x}\cdot\hat{k} - \hat{k}\cdot x$$

$$\langle x|[\hat{x},\hat{k}]|\psi\rangle = \langle x|\hat{x}\hat{k} - \hat{k}\hat{x}|\psi\rangle$$

$$= x\langle x|\hat{k}|\psi\rangle - \langle x|\hat{k}\hat{x}|\psi\rangle$$

$$|\phi\rangle \equiv \hat{x}|\psi\rangle$$

$$\phi(x) = \langle x|\hat{x}|\psi\rangle = x\psi(x)$$

right part $\langle x|\hat{k}\hat{x}|\psi\rangle = -i\frac{\partial}{\partial x}(x\psi(x))$

$$= (-i)\left(\frac{\partial x}{\partial x}\psi(x) + x\frac{\partial}{\partial x}\psi(x)\right)$$

$$= -i\psi(x) + x\frac{\partial}{\partial x}\psi(x)(-i)$$

$$\langle x|[\hat{x},\hat{k}]|\psi\rangle = \cancel{x(-i)\frac{\partial}{\partial x}\psi(x)} - \left(-i\psi(x) \cancel{- ix\frac{\partial}{\partial x}\psi(x)}\right)$$

$$= i\psi(x) = i\langle x|\psi\rangle$$

$= i\,\psi(x) = i\langle x|\psi\rangle$

$$\boxed{[\hat{x}, \hat{k}] = i\hat{\mathbb{1}}}$$

$k \longleftrightarrow$ wavenumber

$\underline{\{x, p\}}$ Poisson bracket

define momentum operator

$$\hat{p} = \eta\hat{k} = \hbar\hat{k}, \quad \hbar = \frac{h}{2\pi}$$

$$\boxed{[\hat{x}, \hat{p}] = i\hbar\,\mathbb{1}}$$

$[x][p] \sim [\hbar] \sim [J\cdot s]$

$e^{ixp/\hbar}$

$$\boxed{\begin{cases} p \longleftrightarrow -i\hbar\frac{\partial}{\partial x} \\ k \longleftrightarrow -i\frac{\partial}{\partial x} \end{cases}}$$

① $\left(-i\hbar\frac{\partial}{\partial x}\right)^{\dagger} = -i\hbar\frac{\partial}{\partial x} = +p$

$\boxed{p^{\dagger} = +p}$

② $p \longleftrightarrow -i\hbar\frac{\partial}{\partial x}$ ← true only with the x repres.

$\langle x|p|\psi\rangle = -i\hbar\frac{\partial}{\partial x}\psi(x)$ ← ~~alway true~~

- momentum op. eigenvalue & eigenstate.

$\hat{x}|x\rangle = x|x\rangle$

because $\hat{p}$ is Hermitian

$\hat{p}|p\rangle = p|p\rangle$

p: e.value, $|p\rangle$: e.state

how to express $|p\rangle$, in position representation what is $\langle x | p \rangle = \phi(x)$

$$\underline{\langle x | \hat{p} | p \rangle} = \langle x | p | p \rangle = \underline{p \langle x | p \rangle}$$

$$= -i\hbar \frac{\partial}{\partial x} (\langle x | p \rangle)$$

$$p\,\phi(x) = -i\hbar\,\phi'(x) \Rightarrow \phi(x) = \underline{C \cdot e^{ipx/\hbar}}$$

$$k = p/\hbar \qquad = C\, e^{ikx}$$

$$\underline{\langle p' | p \rangle = \delta(p' - p)}$$

$$= \langle p' | \mathbb{1} | p \rangle = \int_{-\infty}^{+\infty} dx\, \langle p' | x \rangle \langle x | p \rangle = |C|^2 \int_{-\infty}^{+\infty} e^{-i p' \frac{x}{\hbar}} e^{i \frac{px}{\hbar}} dx$$

$$= \int_{-\infty}^{+\infty} dx\, e^{\frac{i(p-p')x}{\hbar}} |C|^2 = 2\pi\hbar\, \delta(p - p') |C|^2$$

$$1 = 2\pi\hbar |C|^2, \text{ choose } C = \frac{1}{\sqrt{2\pi\hbar}}$$

$$\boxed{\langle x | p \rangle = \frac{1}{\sqrt{2\pi\hbar}} e^{ixp/\hbar}}$$

- momentum wavefunction

$$\langle p | \underline{\underline{\psi}} \rangle \equiv \underline{\psi(p)} \qquad \langle x | p \rangle = \frac{1}{\sqrt{\hbar 2\pi}} e^{\frac{ixp}{\hbar}} = (\langle p | x \rangle)^{+}$$

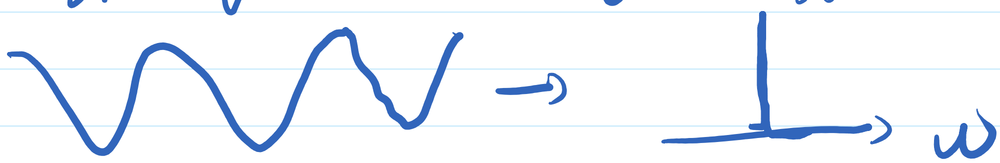
$$\langle p | \mathbb{1} | \psi \rangle = \int_{-\infty}^{+\infty} dx\, \langle p | x \rangle \langle x | \underline{\psi} \rangle$$

$$\boxed{\psi(p) = \int_{-\infty}^{+\infty} dx\, \frac{1}{\sqrt{2\pi\hbar}} e^{-\frac{ixp}{\hbar}} \psi(x)}$$

Fourier transformation

more about Fourier trans.

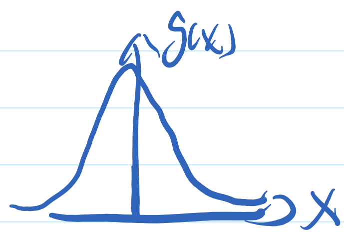
$$\sin(\omega_0 t) \xrightarrow{FT} \delta(\omega - \omega_0)$$



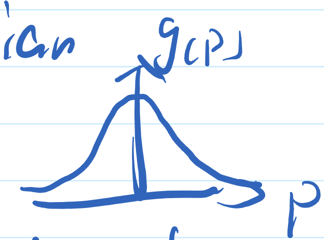
fact: a FT of Gaussian function is still a Gaussian function

$$g(x) = C \exp\left(-\frac{x^2}{2\sigma^2}\right)$$

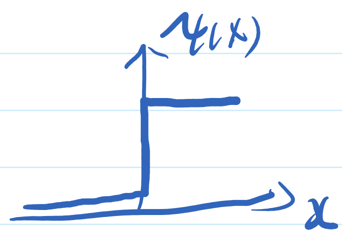
$|g\rangle$



. $g(p)$ is also a Gaussian



example: a discontinuous wavefunction



$$\hat{p} \longleftrightarrow -i\hbar \frac{\partial}{\partial x}, \quad \langle x|\hat{p}|\psi\rangle = -i\hbar \frac{\partial}{\partial x} \psi(x)$$

$$\hat{x} \longleftrightarrow i\hbar \frac{\partial}{\partial p} \quad \langle p|\hat{x}|\psi\rangle = i\hbar \frac{\partial}{\partial p} \psi(p)$$

$$\mathbb{I} = \int_{-\infty}^{+\infty} dx\, |x\rangle\langle x| = \int_{-\infty}^{+\infty} dp\, |p\rangle\langle p|.$$

∘ uncertainty principle

$\underline{[\hat{x}, \hat{p}] = i\hbar \mathbb{I}}$ has a funny consequence

$[X, p] = i\hbar \mathbb{1}$ has a funny consequence

$\boxed{\Delta x \cdot \Delta p \geq \frac{\hbar}{2}}$ "$\geq$" larger than or equal to

$$\Delta x \equiv \sqrt{\langle (x - \langle x \rangle)^2 \rangle}$$

$$\langle x \rangle = \langle \psi | X | \psi \rangle$$

$$\langle (\hat{X} - \langle \hat{X} \rangle)^2 \rangle = \langle \hat{X}^2 - 2\langle \hat{X} \rangle \hat{X} + \langle \hat{X} \rangle^2 \rangle$$

$$= \langle \hat{X}^2 \rangle - \underbrace{2\langle \hat{X} \rangle \langle \hat{X} \rangle}_{2\langle \hat{X} \rangle^2} + \langle \hat{X} \rangle^2$$

$$= \langle \hat{X}^2 \rangle - \langle \hat{X} \rangle^2$$

$\left\{ \begin{array}{l} \text{certain } |\psi\rangle \\ \text{and a more general uncertainty} \end{array} \right.$

$\hat{A}, \hat{B}$

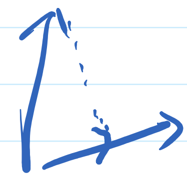
$$\boxed{\Delta A^2 \Delta B^2 \geq \frac{1}{4} |\langle [\hat{A}, \hat{B}] \rangle|^2}$$

$(\Delta x)^2 (\Delta p)^2 \geq \frac{1}{4} \hbar^2 \to \Delta x \, \Delta p \geq \frac{\hbar}{2}$ ✓

let $|\alpha\rangle = \Delta \hat{A} |\psi\rangle$, $|\beta\rangle = \Delta B |\psi\rangle$

$$\langle \alpha | \alpha \rangle \langle \beta | \beta \rangle \geq |\langle \alpha | \beta \rangle|^2$$

Schwartz inequality



proof: define $|\alpha'\rangle = \frac{|\alpha\rangle}{\sqrt{\langle \alpha | \alpha \rangle}}$

$$\langle \alpha' | \alpha' \rangle = 1$$

$$1 \geq |\langle \alpha' | \beta' \rangle|^2$$

$$1 \geq |\langle\alpha'|\beta'\rangle|^2$$

$\{|\alpha'\rangle\}$ basis $\quad |\beta'\rangle = \sum_i c_i |\alpha'_i\rangle, \ \sum_i |c_i|^2 = 1$

$$= \int_{-\infty}^{+\infty} dx\, |x\rangle\langle x|\beta'\rangle$$

$$|\langle\alpha|\beta\rangle|^2 = \frac{1}{4}\langle \underbrace{[\Delta\hat{A}, \Delta\hat{B}]}_{\text{imaginary}}\rangle^2 + \frac{1}{4}\langle \underline{\{\Delta\hat{A}, \Delta\hat{B}\}}\rangle^2$$

$\{A, B\} = AB + BA$

if Hermitian

$$\langle(\Delta A)^2\rangle\langle(\Delta B)^2\rangle \geq \frac{1}{4}\langle[\Delta\hat{A}, \Delta\hat{B}]\rangle^2 \text{ real}$$

$$\Delta x \cdot \Delta p \geq \frac{\hbar}{2}, \quad [\hat{x}, \hat{p}] = i\hbar \mathbb{1}$$

lower bound

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