Modern Quantum Mechanics

INSTRUCTOR SOLUTIONS MANUAL

Second Edition

J.J. Sakurai

Late, University of California, Los Angeles

Jim Napolitano

Rensselaer Polytechnic Institute, Troy

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Chapter One

1. $AC\{D,B\} = ACDB + ACBD$, $A\{C,B\}D = ACBD + ABCD$, $C\{D,A\}B = CDAB + CADB$, and $\{C,A\}DB = CADB + ACDB$. Therefore $-AC\{D,B\} + A\{C,B\}D - C\{D,A\}B + \{C,A\}DB = -ACDB + ABCD - CDAB + ACDB = ABCD - CDAB = [AB, CD]$

In preparing this solution manual, I have realized that problems 2 and 3 in are misplaced in this chapter. They belong in Chapter Three. The Pauli matrices are not even defined in Chapter One, nor is the math used in previous solution manual. – *Jim Napolitano*

2. (a) $\mathsf{Tr}(X) = a_0 \mathsf{Tr}(1) + \sum_\ell \mathsf{Tr}(\sigma_\ell) a_\ell = 2a_0$ since $\mathsf{Tr}(\sigma_\ell) = 0$. Also $\mathsf{Tr}(\sigma_k X) = a_0 \mathsf{Tr}(\sigma_k) + \sum_\ell \mathsf{Tr}(\sigma_k \sigma_\ell) a_\ell = \frac{1}{2}\sum_\ell \mathsf{Tr}(\sigma_k\sigma_\ell + \sigma_\ell\sigma_k) a_\ell = \sum_\ell \delta_{k\ell} \mathsf{Tr}(1) a_\ell = 2a_k$. So, $a_0 = \frac{1}{2}\mathsf{Tr}(X)$ and $a_k = \frac{1}{2}\mathsf{Tr}(\sigma_k X)$. (b) Just do the algebra to find $a_0 = (X_{11} + X_{22})/2$, $a_1 = (X_{12} + X_{21})/2$, $a_2 = i(-X_{21} + X_{12})/2$, and $a_3 = (X_{11} - X_{22})/2$.

3. Since $\det(\boldsymbol{\sigma} \cdot \mathbf{a}) = -a_z^2 - (a_x^2 + a_y^2) = -|\mathbf{a}|^2$, the cognoscenti realize that this problem really has to do with rotation operators. From this result, and (3.2.44), we write

$$\det\left[\exp\left(\pm\frac{i\boldsymbol{\sigma}\cdot\hat{\mathbf{n}}\phi}{2}\right)\right] = \cos\left(\frac{\phi}{2}\right) \pm i\sin\left(\frac{\phi}{2}\right)$$

and multiplying out determinants makes it clear that $\det(\boldsymbol{\sigma}\cdot\mathbf{a}') = \det(\boldsymbol{\sigma}\cdot\mathbf{a})$. Similarly, use (3.2.44) to explicitly write out the matrix $\boldsymbol{\sigma}\cdot\mathbf{a}'$ and equate the elements to those of $\boldsymbol{\sigma}\cdot\mathbf{a}$. With $\hat{\mathbf{n}}$ in the z-direction, it is clear that we have just performed a rotation (of the spin vector) through the angle ϕ.

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4. (a) $\mathsf{Tr}(XY) \equiv \sum_a \langle a|XY|a\rangle = \sum_a \sum_b \langle a|X|b\rangle\langle b|Y|a\rangle$ by inserting the identity operator. Then commute and reverse, so $\mathsf{Tr}(XY) = \sum_b \sum_a \langle b|Y|a\rangle\langle a|X|b\rangle = \sum_b \langle b|YX|b\rangle = \mathsf{Tr}(YX)$.
(b) $XY|\alpha\rangle = X[Y|\alpha\rangle]$ is dual to $\langle\alpha|(XY)^\dagger$, but $Y|\alpha\rangle \equiv |\beta\rangle$ is dual to $\langle\alpha|Y^\dagger \equiv \langle\beta|$ and $X|\beta\rangle$ is dual to $\langle\beta|X^\dagger$ so that $X[Y|\alpha\rangle]$ is dual to $\langle\alpha|Y^\dagger X^\dagger$. Therefore $(XY)^\dagger = Y^\dagger X^\dagger$.
(c) $\exp[if(A)] = \sum_a \exp[if(A)]|a\rangle\langle a| = \sum_a \exp[if(a)]|a\rangle\langle a|$
(d) $\sum_a \psi_a^*(\mathbf{x}')\psi_a(\mathbf{x}'') = \sum_a \langle\mathbf{x}'|a\rangle^*\langle\mathbf{x}''|a\rangle = \sum_a \langle\mathbf{x}''|a\rangle\langle a|\mathbf{x}'\rangle = \langle\mathbf{x}''|\mathbf{x}'\rangle = \delta(\mathbf{x}'' - \mathbf{x}')$

5. For basis kets $|a_i\rangle$, matrix elements of $X \equiv |\alpha\rangle\langle\beta|$ are $X_{ij} = \langle a_i|\alpha\rangle\langle\beta|a_j\rangle = \langle a_i|\alpha\rangle\langle a_j|\beta\rangle^*$. For spin-1/2 in the $|\pm z\rangle$ basis, $\langle +|S_z = \hbar/2\rangle = 1$, $\langle -|S_z = \hbar/2\rangle = 0$, and, using (1.4.17a), $\langle\pm|S_x = \hbar/2\rangle = 1/\sqrt{2}$. Therefore

$$|S_z = \hbar/2\rangle\langle S_x = \hbar/2| \doteq \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$$

6. $A[|i\rangle + |j\rangle] = a_i|i\rangle + a_j|j\rangle \neq [|i\rangle + |j\rangle]$ so in general it is not an eigenvector, unless $a_i = a_j$. That is, $|i\rangle + |j\rangle$ is not an eigenvector of A unless the eigenvalues are degenerate.

7. Since the product is over a complete set, the operator $\prod_{a'}(A - a')$ will always encounter a state $|a_i\rangle$ such that $a' = a_i$ in which case the result is zero. Hence for any state $|\alpha\rangle$

$$\prod_{a'}(A-a')|\alpha\rangle = \prod_{a'}(A-a')\sum_i |a_i\rangle\langle a_i|\alpha\rangle = \sum_i \prod_{a'}(a_i - a')|a_i\rangle\langle a_i|\alpha\rangle = \sum_i 0 = 0$$

If the product instead is over all $a' \neq a_j$ then the only surviving term in the sum is

$$\prod_{a'}(a_j - a')|a_i\rangle\langle a_i|\alpha\rangle$$

and dividing by the factors $(a_j - a')$ just gives the projection of $|\alpha\rangle$ on the direction $|a'\rangle$. For the operator $A \equiv S_z$ and $\{|a'\rangle\} \equiv \{|+\rangle, |-\rangle\}$, we have

$$\begin{aligned}
&\prod_{a'}(A-a') &=& \left(S_z - \frac{\hbar}{2}\right)\left(S_z + \frac{\hbar}{2}\right) & \\
\text{and}\quad &\prod_{a'\neq a''}\frac{A-a'}{a''-a'} &=& \frac{S_z + \hbar/2}{\hbar} & \text{for } a'' = +\frac{\hbar}{2} \\
&\text{or} &=& \frac{S_z - \hbar/2}{-\hbar} & \text{for } a'' = -\frac{\hbar}{2}
\end{aligned}$$

It is trivial to see that the first operator is the null operator. For the second and third, you can work these out explicitly using (1.3.35) and (1.3.36), for example

$$\frac{S_z + \hbar/2}{\hbar} = \frac{1}{\hbar}\left[S_z + \frac{\hbar}{2}1\right] = \frac{1}{2}\left[(|+\rangle\langle+|) - (|-\rangle\langle-|) + (|+\rangle\langle+|) + (|-\rangle\langle-|)\right] = |+\rangle\langle+|$$

which is just the projection operator for the state $|+\rangle$.

8. I don't see any way to do this problem other than by brute force, and neither did the previous solutions manual. So, make use of $\langle+|+\rangle = 1 = \langle-|-\rangle$ and $\langle+|-\rangle = 0 = \langle-|+\rangle$ and carry through six independent calculations of $[S_i, S_j]$ (along with $[S_i, S_j] = -[S_j, S_i]$) and the six for $\{S_i, S_j\}$ (along with $\{S_i, S_j\} = +\{S_j, S_i\}$).

9. From the figure $\hat{\mathbf{n}} = \hat{\mathbf{i}}\cos\alpha\sin\beta + \hat{\mathbf{j}}\sin\alpha\sin\beta + \hat{\mathbf{k}}\cos\beta$ so we need to find the matrix representation of the operator $\mathbf{S}\cdot\hat{\mathbf{n}} = S_x\cos\alpha\sin\beta + S_y\sin\alpha\sin\beta + S_z\cos\beta$. This means we need the matrix representations of S_x, S_y, and S_z. Get these from the prescription (1.3.19) and the operators represented as outer products in (1.4.18) and (1.3.36), along with the association (1.3.39a) to define which element is which. Thus

$$S_x \doteq \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad S_y \doteq \frac{\hbar}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad S_z \doteq \frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

We therefore need to find the (normalized) eigenvector for the matrix

$$\begin{pmatrix} \cos\beta & \cos\alpha\sin\beta - i\sin\alpha\sin\beta \\ \cos\alpha\sin\beta + i\sin\alpha\sin\beta & -\cos\beta \end{pmatrix} = \begin{pmatrix} \cos\beta & e^{-i\alpha}\sin\beta \\ e^{i\alpha}\sin\beta & -\cos\beta \end{pmatrix}$$

with eigenvalue +1. If the upper and lower elements of the eigenvector are a and b, respectively, then we have the equations $|a|^2 + |b|^2 = 1$ and

$$\begin{aligned} a\cos\beta + be^{-i\alpha}\sin\beta &= a \\ ae^{i\alpha}\sin\beta - b\cos\beta &= b \end{aligned}$$

Choose the phase so that a is real and positive. Work with the first equation. (The two equations should be equivalent, since we picked a valid eigenvalue. You should check.) Then

$$\begin{aligned} a^2(1-\cos\beta)^2 &= |b|^2\sin^2\beta = (1-a^2)\sin^2\beta \\ 4a^2\sin^4(\beta/2) &= (1-a^2)4\sin^2(\beta/2)\cos^2(\beta/2) \\ a^2[\sin^2(\beta/2)+\cos^2(\beta/2)] &= \cos^2(\beta/2) \\ a &= \cos(\beta/2) \\ \text{and so} \qquad b &= ae^{i\alpha}\frac{1-\cos\beta}{\sin\beta} = \cos(\beta/2)e^{i\alpha}\frac{2\sin^2(\beta/2)}{2\sin(\beta/2)\cos(\beta/2)} \\ &= e^{i\alpha}\sin(\beta/2) \end{aligned}$$

which agrees with the answer given in the problem.

10. Use simple matrix techniques for this problem. The matrix representation for H is

$$H \doteq \begin{bmatrix} a & a \\ a & -a \end{bmatrix}$$

Eigenvalues E satisfy $(a-E)(-a-E) - a^2 = -2a^2 + E^2 = 0$ or $E = \pm a\sqrt{2}$. Let x_1 and x_2 be the two elements of the eigenvector. For $E = +a\sqrt{2} \equiv E^{(1)}$, $(1-\sqrt{2})x_1^{(1)} + x_2^{(1)} = 0$, and for $E = -a\sqrt{2} \equiv E^{(2)}$, $(1+\sqrt{2})x_1^{(2)} + x_2^{(2)} = 0$. So the eigenstates are represented by

$$|E^{(1)}\rangle \doteq N^{(1)}\begin{bmatrix} 1 \\ \sqrt{2}-1 \end{bmatrix} \qquad \text{and} \qquad |E^{(2)}\rangle \doteq N^{(2)}\begin{bmatrix} -1 \\ \sqrt{2}+1 \end{bmatrix}$$

where $N^{(1)^2} = 1/(4-2\sqrt{2})$ and $N^{(2)^2} = 1/(4+2\sqrt{2})$.

11. It is of course possible to solve this using simple matrix techniques. For example, the characteristic equation and eigenvalues are

$$\begin{aligned} 0 &= (H_{11}-\lambda)(H_{22}-\lambda) - H_{12}^2 \\ \lambda &= \frac{H_{11}+H_{22}}{2} \pm \left[\left(\frac{H_{11}-H_{22}}{2}\right)^2 + H_{12}^2\right]^{1/2} \equiv \lambda_\pm \end{aligned}$$

You can go ahead and solve for the eigenvectors, but it is tedious and messy. However, there is a strong hint given that you can make use of spin algebra to solve this problem, another two-state system. The Hamiltonian can be rewritten as

$$H \doteq A\mathsf{1} + B\sigma_z + C\sigma_x$$

where $A \equiv (H_{11}+H_{22})/2$, $B \equiv (H_{11}-H_{22})/2$, and $C \equiv H_{12}$. The eigenvalues of the first term are both A, and the eigenvalues for the sum of the second and third terms are those of $\pm(2/\hbar)$ times a spin vector multiplied by $\sqrt{B^2+C^2}$. In other words, the eigenvalues of the full Hamiltonian are just $A \pm \sqrt{B^2+C^2}$ in full agreement with what we got with usual matrix techniques, above. From the hint (or Problem 9) the eigenvectors must be

$$|\lambda_+\rangle = \cos\frac{\beta}{2}|1\rangle + \sin\frac{\beta}{2}|2\rangle \qquad \text{and} \qquad |\lambda_-\rangle = -\sin\frac{\beta}{2}|1\rangle + \cos\frac{\beta}{2}|2\rangle$$

where $\alpha = 0$, $\tan\beta = C/B = 2H_{12}/(H_{11}-H_{22})$, and we do $\beta \to \pi - \beta$ to "flip the spin."

12. Using the result of Problem 9, the probability of measuring $+\hbar/2$ is

$$\left|\left[\frac{1}{\sqrt{2}}\langle+| + \frac{1}{\sqrt{2}}\langle-|\right]\left[\cos\frac{\gamma}{2}|+\rangle + \sin\frac{\gamma}{2}|-\rangle\right]\right|^2 = \frac{1}{2}\left[\sqrt{\frac{1+\cos\gamma}{2}} + \sqrt{\frac{1-\cos\gamma}{2}}\right]^2 = \frac{1+\sin\gamma}{2}$$

The results for $\gamma = 0$ (i.e. $|+\rangle$), $\gamma = \pi/2$ (i.e. $|S_x+\rangle$), and $\gamma = \pi$ (i.e. $|-\rangle$) are 1/2, 1, and 1/2, as expected. Now $\langle(S_x - \langle S_x\rangle)^2\rangle = \langle S_x^2\rangle - \langle S_x\rangle^2$, but $S_x^2 = \hbar^2/4$ from Problem 8 and

$$\begin{aligned}\langle S_x\rangle &= \left[\cos\frac{\gamma}{2}\langle+| + \sin\frac{\gamma}{2}\langle-|\right]\frac{\hbar}{2}\left[|+\rangle\langle-| + |-\rangle\langle+|\right]\left[\cos\frac{\gamma}{2}|+\rangle + \sin\frac{\gamma}{2}|-\rangle\right] \\ &= \frac{\hbar}{2}\left[\cos\frac{\gamma}{2}\langle-| + \sin\frac{\gamma}{2}\langle+|\right]\left[\cos\frac{\gamma}{2}|+\rangle + \sin\frac{\gamma}{2}|-\rangle\right] = \hbar\cos\frac{\gamma}{2}\sin\frac{\gamma}{2} = \frac{\hbar}{2}\sin\gamma\end{aligned}$$

so $\langle(S_x - \langle S_x\rangle)^2\rangle = \hbar^2(1-\sin^2\gamma)/4 = \hbar^2\cos^2\gamma/4 = \hbar^2/4, 0, \hbar^2 4$ for $\gamma = 0, \pi/2, \pi$.

13. All atoms are in the state $|+\rangle$ after emerging from the first apparatus. The second apparatus projects out the state $|S_n+\rangle$. That is, it acts as the projection operator

$$|S_n+\rangle\langle S_n+| = \left[\cos\frac{\beta}{2}|+\rangle + \sin\frac{\beta}{2}|-\rangle\right]\left[\cos\frac{\beta}{2}\langle+| + \sin\frac{\beta}{2}\langle-|\right]$$

and the third apparatus projects out $|-\rangle$. Therefore, the probability of measuring $-\hbar/2$ after the third apparatus is

$$P(\beta) = |\langle+|S_n+\rangle\langle S_n+|-\rangle|^2 = \cos^2\frac{\beta}{2}\sin^2\frac{\beta}{2} = \frac{1}{4}\sin^2\beta$$

The maximum transmission is for $\beta = 90°$, when 25% of the atoms make it through.

14. The characteristic equation is $-\lambda^3 - 2(-\lambda)(1/\sqrt{2})^2 = \lambda(1-\lambda^2) = 0$ so the eigenvalues are $\lambda = 0, \pm1$ and there is no degeneracy. The eigenvectors corresponding to these are

$$\frac{1}{\sqrt{2}}\begin{bmatrix}-1\\0\\1\end{bmatrix} \qquad \frac{1}{2}\begin{bmatrix}1\\\sqrt{2}\\1\end{bmatrix} \qquad \frac{1}{2}\begin{bmatrix}1\\-\sqrt{2}\\1\end{bmatrix}$$

The matrix algebra is not hard, but I did this with MATLAB using

```
M=[[0 1 0];[1 0 1];[0 1 0]]/sqrt(2)
[V,D]=eig(M)
```

These are the eigenvectors corresponding to the a spin-one system, for a measurement in the x-direction in terms of a basis defined in the z-direction. I'm not sure if there is enough information in Chapter One, though, in order to deduce this.

15. The answer is *yes*. The identity operator is $1 = \sum_{a',b'} |a',b'\rangle\langle a',b'|$ so

$$AB = AB1 = AB\sum_{a',b'} |a',b'\rangle\langle a',b'| = A\sum_{a',b'} b'|a',b'\rangle\langle a',b'| = \sum_{a',b'} b'a'|a',b'\rangle\langle a',b'| = BA$$

Completeness is powerful. It is important to note that the sum must be over both a' and b' in order to span the complete set of sets.

16. Since $AB = -BA$ and $AB|a,b\rangle = ab|a,b\rangle = BA|a,b\rangle$, we must have $ab = -ba$ where both a and b are real numbers. This can only be satisfied if $a = 0$ or $b = 0$ or both.

17. Assume there is no degeneracy and look for an inconsistency with our assumptions. If $|n\rangle$ is a nondegenerate energy eigenstate with eigenvalue E_n, then it is the *only* state with this energy. Since $[H.A_1] = 0$, we must have $HA_1|n\rangle = A_1H|n\rangle = E_nA_1|n\rangle$. That is, $A_1|n\rangle$ is an eigenstate of energy with eigenvalue E_n. Since H and A_1 commute, though, they may have simultaneous eigenstates. Therefore, $A_1|n\rangle = a_1|n\rangle$ since there is only one energy eigenstate.

Similarly, $A_2|n\rangle$ is also an eigenstate of energy with eigenvalue E_n, and $A_2|n\rangle = a_2|n\rangle$. But $A_1A_2|n\rangle = a_2A_1|n\rangle = a_2a_1|n\rangle$ and $A_2A_1|n\rangle = a_1a_2|n\rangle$, where a_1 and a_2 are real numbers. This cannot be true, in general, if $A_1A_2 \neq A_2A_1$ so our assumption of "no degeneracy" must be wrong. There is an out, though, if $a_1 = 0$ or $a_2 = 0$, since one operator acts on zero.

The example given is from a "central forces" Hamiltonian. (See Chapter Three.) The Hamiltonian commutes with the orbital angular momentum operators L_x and L_y, but $[L_x, L_y] \neq 0$. Therefore, in general, there is a degeneracy in these problems. The degeneracy is avoided, though for S-states, where the quantum numbers of L_x and L_y are both necessarily zero.

18. The positivity postulate says that $\langle\gamma|\gamma\rangle \geq 0$, and we apply this to $|\gamma\rangle \equiv |\alpha\rangle + \lambda|\beta\rangle$. The text shows how to apply this to prove the Schwarz Innequality $\langle\alpha|\alpha\rangle\langle\beta|\beta\rangle \geq |\langle\alpha|\beta\rangle|^2$, from which one derives the generalized uncertainty relation (1.4.53), namely

$$\langle(\Delta A)^2(\Delta B)^2\rangle \geq \frac{1}{4}\,|\langle[A,B]\rangle|^2$$

Note that $[\Delta A, \Delta B] = [A - \langle A\rangle, B - \langle B\rangle] = [A, B]$. Taking $\Delta A|\alpha\rangle = \lambda\Delta B|\alpha\rangle$ with $\lambda^* = -\lambda$, as suggested, so $\langle\alpha|\Delta A = -\lambda\langle\alpha|\Delta B$, for a particular state $|\alpha\rangle$. Then

$$\langle\alpha|[A,B]|\alpha\rangle = \langle\alpha|\Delta A\Delta B - \Delta B\Delta A|\alpha\rangle = -2\lambda\langle\alpha|(\Delta B)^2|\alpha\rangle$$

and the equality is clearly satisfied in (1.4.53). We are now asked to verify this relationship for a state $|\alpha\rangle$ that is a gaussian wave packet when expressed as a wave function $\langle x'|\alpha\rangle$. Use

$$\begin{aligned}
\langle x'|\Delta x|\alpha\rangle &= \langle x'|x|\alpha\rangle - \langle x\rangle\langle x'|\alpha\rangle = (x' - \langle x\rangle)\langle x'|\alpha\rangle \\
\text{and} \qquad \langle x'|\Delta p|\alpha\rangle &= \langle x'|p|\alpha\rangle - \langle p\rangle\langle x'|\alpha\rangle = \frac{\hbar}{i}\frac{d}{dx'}\langle x'|\alpha\rangle - \langle p\rangle\langle x'|\alpha\rangle \\
\text{with} \qquad \langle x'|\alpha\rangle &= (2\pi d^2)^{-1/4}\exp\left[\frac{i\langle p\rangle x'}{\hbar} - \frac{(x'-\langle x\rangle)^2}{4d^2}\right] \\
\text{to get} \qquad \frac{\hbar}{i}\frac{d}{dx'}\langle x'|\alpha\rangle &= \left[\langle p\rangle - \frac{\hbar}{i}\frac{1}{2d^2}(x' - \langle x\rangle)\right]\langle x'|\alpha\rangle \\
\text{and so} \qquad \langle x'|\Delta p|\alpha\rangle &= i\frac{\hbar}{2d^2}(x' - \langle x\rangle)\langle x'|\alpha\rangle = \lambda\langle x'|\Delta x|\alpha\rangle
\end{aligned}$$

where λ is a purely imaginary number. The conjecture is satisfied.

It is very simple to show that this condition is satisfied for the ground state of the harmonic oscillator. Refer to (2.3.24) and (2.3.25). Clearly $\langle x\rangle = 0 = \langle p\rangle$ for any eigenstate $|n\rangle$, and $x|0\rangle$ is proportional to $p|0\rangle$, with a proportionality constant that is purely imaginary.

19. Note the obvious typographical error, i.e. S_2^x should be S_x^2. Have $S_x^2 = \hbar^2/4 = S_y^2 = S_z^2$, also $[S_x, S_y] = i\hbar S_z$, all from Problem 8. Now $\langle S_x\rangle = \langle S_y\rangle = 0$ for the $|+\rangle$ state. Then $\langle(\Delta S_x)^2\rangle = \hbar^2/4 = \langle(\Delta S_y)^2\rangle$, and $\langle(\Delta S_x)^2\langle(\Delta S_y)^2 = \hbar^4/16$. Also $|\langle[S_x, S_y]\rangle|^2/4 = \hbar^2|\langle S_z\rangle|^2/4 = \hbar^4/16$ and the generalized uncertainty principle is satisfied by the equality. On the other hand, for the $|S_x+\rangle$ state, $\langle(\Delta S_x)^2\rangle = 0$ and $\langle S_z\rangle = 0$, and again the generalized uncertainty principle is satisfied with an equality.

20. Refer to Problems 8 and 9. Parameterize the state as $|\rangle = \cos\frac{\beta}{2}|+\rangle + e^{i\alpha}\sin\frac{\beta}{2}|-\rangle$, so

$$\begin{aligned}
\langle S_x\rangle &= \frac{\hbar}{2}\left[\cos\frac{\beta}{2}\langle+| + e^{-i\alpha}\sin\frac{\beta}{2}\langle-|\right][|+\rangle\langle-| + |-\rangle\langle+|]\left[\cos\frac{\beta}{2}|+\rangle + e^{i\alpha}\sin\frac{\beta}{2}|-\rangle\right] \\
&= \frac{\hbar}{2}\sin\frac{\beta}{2}\cos\frac{\beta}{2}(e^{i\alpha} + e^{-i\alpha}) = \frac{\hbar}{2}\sin\beta\cos\alpha \\
\langle(\Delta S_x)^2\rangle &= \langle S_x^2\rangle - \langle S_x\rangle^2 = \frac{\hbar^2}{4}(1 - \sin^2\beta\cos^2\alpha) \qquad \text{(see prob 12)} \\
\langle S_y\rangle &= i\frac{\hbar}{2}\left[\cos\frac{\beta}{2}\langle+| + e^{-i\alpha}\sin\frac{\beta}{2}\langle-|\right][-|+\rangle\langle-| + |-\rangle\langle+|]\left[\cos\frac{\beta}{2}|+\rangle + e^{i\alpha}\sin\frac{\beta}{2}|-\rangle\right] \\
&= i\frac{\hbar}{2}\sin\frac{\beta}{2}\cos\frac{\beta}{2}(e^{i\alpha} - e^{-i\alpha}) = -\frac{\hbar}{2}\sin\beta\sin\alpha \\
\langle(\Delta S_y)^2\rangle &= \langle S_y^2\rangle - \langle S_y\rangle^2 = \frac{\hbar^2}{4}(1 - \sin^2\beta\sin^2\alpha)
\end{aligned}$$

Therefore, the left side of the uncertainty relation is

$$\begin{aligned}
\langle(\Delta S_x)^2\rangle\langle(\Delta S_y)^2\rangle &= \frac{\hbar^4}{16}(1-\sin^2\beta\cos^2\alpha)(1-\sin^2\beta\sin^2\alpha) \\
&= \frac{\hbar^4}{16}\left(1-\sin^2\beta+\frac{1}{4}\sin^4\beta\sin^2 2\alpha\right) \\
&= \frac{\hbar^4}{16}\left(\cos^2\beta+\frac{1}{4}\sin^4\beta\sin^2 2\alpha\right) \equiv P(\alpha,\beta)
\end{aligned}$$

which is clearly maximized when $\sin 2\alpha = \pm 1$ for any value of β. In other words, the uncertainty product is a maximum when the state is pointing in a direction that is 45° with respect to the x or y axes in any quadrant, for any tilt angle β relative to the z-axis. This makes sense. The maximum tilt angle is derived from

$$\frac{\partial P}{\partial \beta} \propto -2\cos\beta\sin\beta+\sin^3\beta\cos\beta(1) = \cos\beta\sin\beta(-2+\sin^2\beta)=0$$

or $\sin\beta = \pm 1/\sqrt{2}$, that is, 45° with respect to the z-axis. It all hangs together. The maximum uncertainty product is

$$\langle(\Delta S_x)^2\rangle\langle(\Delta S_y)^2\rangle = \frac{\hbar^4}{16}\left(\frac{1}{2}+\frac{1}{4}\frac{1}{4}\right) = \frac{9}{256}\hbar^4$$

The right side of the uncertainty relation is $|\langle[S_x,S_y]\rangle|^2/4 = \hbar^2|\langle S_z\rangle|^2/4$, so we also need

$$\langle S_z\rangle = \frac{\hbar}{2}\left[\cos^2\frac{\beta}{2}-\sin^2\frac{\beta}{2}\right] = \frac{\hbar}{2}\cos\beta$$

so the value of the right hand side at maximum is

$$\frac{\hbar^2}{4}|\langle S_z\rangle|^2 = \frac{\hbar^2}{4}\frac{\hbar^2}{4}\frac{1}{2} = \frac{8}{256}\hbar^4$$

and the uncertainty principle is indeed satisfied.

21. The wave function is $\langle x|n\rangle = \sqrt{2/a}\sin(n\pi x/a)$ for $n=1,2,3,\ldots$, so we calculate

$$\begin{aligned}
\langle x|x|n\rangle &= \int_0^a \langle n|x\rangle x\langle x|n\rangle dx = \frac{a}{2} \\
\langle x|x^2|n\rangle &= \int_0^a \langle n|x\rangle x^2\langle x|n\rangle dx = \frac{a^2}{6}\left(-\frac{3}{n^2\pi^2}+2\right) \\
(\Delta x)^2 &= \frac{a^2}{6}\left(-\frac{3}{n^2\pi^2}+2-\frac{6}{4}\right) = \frac{a^2}{6}\left(-\frac{3}{n^2\pi^2}+\frac{1}{2}\right) \\
\langle x|p|n\rangle &= \int_0^a \langle n|x\rangle \frac{\hbar}{i}\frac{d}{dx}\langle x|n\rangle dx = 0 \\
\langle x|p^2|n\rangle &= -\hbar^2\int_0^a \langle n|x\rangle \frac{d^2}{dx^2}\langle x|n\rangle dx = \frac{n^2\pi^2\hbar^2}{a^2} = (\Delta p)^2
\end{aligned}$$

(I did these with MAPLE.) Since $[x,p]=i\hbar$, we compare $(\Delta x)^2(\Delta p)^2$ to $\hbar^2/4$ with

$$(\Delta x)^2(\Delta p)^2 = \frac{\hbar^2}{6}\left(-3+\frac{n^2\pi^2}{2}\right) = \frac{\hbar^2}{4}\left(\frac{n^2\pi^2}{3}-2\right)$$

which shows that the uncertainty principle is satisfied, since $n\pi^2/3 > n\pi > 3$ for all n.

22. We're looking for a "rough order of magnitude" estimate, so go crazy with the approximations. Model the ice pick as a mass m and length L, standing vertically on the point, i.e. and inverted pendulum. The angular acceleration is $\ddot{\theta}$, the moment of inertia is mL^2 and the torque is $mgL\sin\theta$ where θ is the angle from the vertical. So $mL^2\ddot{\theta} = mgL\sin\theta$ or $\ddot{\theta} = \sqrt{g/L}\sin\theta$. Since $\theta \ll 0$ as the pick starts to fall, take $\sin\theta = \theta$ so

$$\begin{aligned}
\theta(t) &= A\exp\left(\sqrt{\frac{g}{L}}t\right) + B\exp\left(-\sqrt{\frac{g}{L}}t\right) \\
x_0 \equiv \theta(0)L &= (A+B)L \\
p_0 \equiv m\dot{\theta}(0)L &= m\sqrt{\frac{g}{L}}(A-B)L = \sqrt{m^2gL}(A-B)
\end{aligned}$$

Let the uncertainty principle relate x_0 and p_0, i.e. $x_0p_0 = \sqrt{m^2gL^3}(A^2-B^2) = \hbar$. Now ignore B; the exponential decay will become irrelevant quickly. You can notice that the pick is falling when it is tilting by something like $1^\circ = \pi/180$, so solve for a time T where $\theta(T) = \pi/180$. Then

$$T = \sqrt{\frac{L}{g}}\ln\frac{\pi/180}{A} = \sqrt{\frac{L}{g}}\left(\frac{1}{4}\ln\frac{m^2gL^3}{\hbar^2} - \ln\frac{180}{\pi}\right)$$

Take $L = 10$ cm, so $\sqrt{L/g} \approx 0.1$ sec, but the action is in the logarithms. (It is worth your time to confirm that the argument of the logarithm in the first term is indeed dimensionless.) Now $\ln(180/\pi) \approx 4$ but the first term appears to be much larger. This is good, since it means that quantum mechanics is driving the result. For $m = 0.1$ kg, find $m^2gL^3/\hbar^2 = 10^{64}$, and so $T = 0.1 \text{ sec} \times (147/4 - 4) \sim 3$ sec. I'd say that's a surprising and interesting result.

23. The eigenvalues of A are obviously $\pm a$, with $-a$ twice. The characteristic equation for B is $(b-\lambda)(-\lambda)^2 - (b-\lambda)(ib)(-ib) = (b-\lambda)(\lambda^2-b^2) = 0$, so its eigenvalues are $\pm b$ with b twice. (Yes, B has degenerate eigenvalues.) It is easy enough to show that

$$AB = \begin{pmatrix} ab & 0 & 0 \\ 0 & 0 & iab \\ 0 & -iab & 0 \end{pmatrix} = BA$$

so A and B commute, and therefore must have simultaneous eigenvectors. To find these, write the eigenvector components as u_i, $i = 1, 2, 3$. Clearly, the basis states $|1\rangle$, $|2\rangle$, and $|3\rangle$ are eigenvectors of A with eigenvalues a, $-a$, and $-a$ respectively. So, do the math to find the eigenvectors for B in this basis. Presumably, some freedom will appear that allows us to linear combinations that are also eigenvectors of A. One of these is obviously $|1\rangle \equiv |a, b\rangle$, so just work with the reduced 2×2 basis of states $|2\rangle$ and $|3\rangle$. Indeed, both of these states have eigenvalues a for A, so one linear combinations should have eigenvalue $+b$ for B, and orthogonal combination with eigenvalue $-b$.

Let the eigenvector components be u_2 and u_3. Then, for eigenvalue $+b$,

$$-ibu_3 = +bu_2 \qquad \text{and} \qquad ibu_2 = +bu_3$$

both of which imply $u_3 = iu_2$. For eigenvalue $-b$,

$$-ibu_3 = -bu_2 \qquad \text{and} \qquad ibu_2 = -bu_3$$

both of which imply $u_3 = -iu_2$. Choosing u_2 to be real, then ("No, the eigenvalue alone does not completely characterize the eigenket.") we have the set of simultaneous eigenstates

Eigenvalue of		
A	B	Eigenstate
a	b	$\|1\rangle$
$-a$	b	$\frac{1}{\sqrt{2}}(\|2\rangle + i\|3\rangle)$
$-a$	$-b$	$\frac{1}{\sqrt{2}}(\|2\rangle - i\|3\rangle)$

24. *This problem also appears to belong in Chapter Three. The Pauli matrices are not defined in Chapter One, but perhaps one could simply define these matrices, here and in Problems 2 and 3.*
Operating on the spinor representation of $|+\rangle$ with $(1\sqrt{2})(1 + i\sigma_x)$ gives

$$\frac{1}{\sqrt{2}}\left[\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + i\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right]\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ i \end{pmatrix}$$

So, for an operator U such that $U \doteq (1\sqrt{2})(1+i\sigma_x)$, we observe that $U|+\rangle = |S_y; +\rangle$, defined in (1.4.17b). Similarly operating on the spinor representation of $|-\rangle$ gives

$$\frac{1}{\sqrt{2}}\left[\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + i\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right]\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} i \\ 1 \end{pmatrix} = \frac{i}{\sqrt{2}}\begin{pmatrix} 1 \\ -i \end{pmatrix}$$

that is, $U|-\rangle = i|S_y; -\rangle$. This is what we would mean by a "rotation" about the x-axis by 90°. The sense of the rotation is about the $+x$ direction vector, so this would actually be a rotation of $-\pi/2$. (See the diagram following Problem Nine.) The phase factor $i = e^{i\pi/2}$ does not affect this conclusions, and in fact leads to observable quantum mechanical effects. (This is all discussed in Chapter Three.) The matrix elements of S_z in the S_y basis are then

$$\begin{aligned}
\langle S_y; +|S_z|S_y; +\rangle &= \langle +|U^\dagger S_z U|+\rangle \\
\langle S_y; +|S_z|S_y; -\rangle &= -i\langle +|U^\dagger S_z U|-\rangle \\
\langle S_y; -|S_z|S_y; +\rangle &= i\langle -|U^\dagger S_z U|+\rangle \\
\langle S_y; -|S_z|S_y; -\rangle &= \langle -|U^\dagger S_z U|-\rangle
\end{aligned}$$

Note that $\sigma_x^\dagger = \sigma_x$ and $\sigma_x^2 = 1$, so $U^\dagger U \doteq (1\sqrt{2})(1 - i\sigma_x)(1\sqrt{2})(1 + i\sigma_x) = (1/2)(1 + \sigma_x^2) = 1$ and U is therefore unitary. (This is no accident, as will be discussed when rotation operators are presented in Chapter Three.) Furthermore $\sigma_z\sigma_x = -\sigma_x\sigma_z$, so

$$\begin{aligned}
U^\dagger S_z U &\doteq \frac{1}{\sqrt{2}}(1 - i\sigma_x)\frac{\hbar}{2}\sigma_z\frac{1}{\sqrt{2}}(1 + i\sigma_x) = \frac{\hbar}{2}\frac{1}{2}(1 - i\sigma_x)^2\sigma_z = -i\frac{\hbar}{2}\sigma_x\sigma_z \\
&= -i\frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \frac{\hbar}{2}\begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} \\
\text{so} \qquad S_z &\doteq \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \frac{\hbar}{2}\sigma_x
\end{aligned}$$

in the $|S_y; \pm\rangle$ *basis.* This can be easily checked directly with (1.4.17b), that is

$$S_z|S_y; \pm\rangle = \frac{\hbar}{2}\frac{1}{\sqrt{2}}[|+\rangle \mp i|-\rangle = \frac{\hbar}{2}|S_y; \mp\rangle$$

There seems to be a mistake in the old solution manual, finding $S_z = (\hbar/2)\sigma_y$ instead of σ_x.

25. Transforming to another representation, say the basis $|c\rangle$, we carry out the calculation

$$\langle c'|A|c''\rangle = \sum_{b'}\sum_{b''}\langle c'|b'\rangle\langle b'|A|b''\rangle\langle b''|c''\rangle$$

There is no principle which says that the $\langle c'|b'\rangle$ need to be real, so $\langle c'|A|c''\rangle$ is not necessarily real if $\langle b'|A|b''\rangle$ is real. The problem alludes to Problem 24 as an example, but not that specific question (assuming my solution is correct.) Still, it is obvious, for example, that the operator S_y is "real" in the $|S_y; \pm\rangle$ basis, but is not in the $|\pm\rangle$ basis.

For another example, also suggested in the text, if you calculate

$$\langle p'|x|p''\rangle = \int \langle p'|x|x'\rangle\langle x'|p''\rangle dx' = \int x'\langle p'|x'\rangle\langle x'|p''\rangle dx' = \frac{1}{2\pi\hbar}\int x' e^{i(p''-p')x'/\hbar}dx'$$

and then define $q \equiv p'' - p'$ and $y \equiv x'/\hbar$, then

$$\langle p'|x|p''\rangle = \frac{\hbar}{2\pi i}\frac{d}{dq}\int e^{iqy}dy = \frac{\hbar}{i}\frac{d}{dq}\delta(q)$$

so you can also see that although x is real in the $|x'\rangle$ basis, it is not so in the $|p'\rangle$ basis.

26. From (1.4.17a), $|S_x; \pm\rangle = (|+\rangle \pm |-\rangle)/\sqrt{2}$, so clearly

$$\begin{aligned}
U \doteq \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} &= \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix} \\
&= \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}[1\ 0] + \begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix}[0\ 1] \\
\Longrightarrow \quad &= |S_x : +\rangle\langle +| + |S_x : -\rangle\langle -| \doteq \sum_r |b^{(r)}\rangle\langle a^{(r)}|
\end{aligned}$$

27. The idea here is simple. Just insert a complete set of states. Firstly,

$$\langle b''|f(A)|b'\rangle = \sum_{a'} \langle b''|f(A)|a'\rangle\langle a'|b'\rangle = \sum_{a'} f(a')\langle b''|a'\rangle\langle a'|b'\rangle$$

The numbers $\langle a'|b'\rangle$ (and $\langle b''|a'\rangle$) constitute the "transformation matrix" between the two sets of basis states. Similarly for the continuum case,

$$\begin{aligned}\langle \mathbf{p}''|F(r)|\mathbf{p}'\rangle &= \int \langle \mathbf{p}''|F(r)|\mathbf{x}'\rangle\langle \mathbf{x}'|\mathbf{p}'\rangle d^3x' = \int F(r')\langle \mathbf{p}''|\mathbf{x}'\rangle\langle \mathbf{x}'|\mathbf{p}'\rangle d^3x' \\ &= \frac{1}{(2\pi\hbar)^3}\int F(r')e^{i(\mathbf{p}'-\mathbf{p}'')\cdot\mathbf{x}'/\hbar}d^3x'\end{aligned}$$

The angular parts of the integral can be done explicitly. Let $\mathbf{q} \equiv \mathbf{p}' - \mathbf{p}''$ define the "z"-direction. Then

$$\begin{aligned}\langle \mathbf{p}''|F(r)|\mathbf{p}'\rangle &= \frac{2\pi}{(2\pi\hbar)^3}\int dr' F(r')\int_0^\pi \sin\theta d\theta e^{iqr'\cos\theta/\hbar} = \frac{1}{4\pi^2\hbar^3}\int dr' F(r')\int_{-1}^{1} d\mu\; e^{iqr'\mu/\hbar} \\ &= \frac{1}{4\pi^2\hbar^3}\int dr' F(r')\frac{\hbar}{iqr'}2i\sin(qr'/\hbar) = \frac{1}{2\pi^2\hbar^2}\int dr' F(r')\frac{\sin(qr'/\hbar)}{qr'}\end{aligned}$$

28. For functions $f(q,p)$ and $g(q,p)$, where q and p are conjugate position and momentum, respectively, the Poisson bracket from classical physics is

$$[f,g]_{\text{classical}} = \frac{\partial f}{\partial q}\frac{\partial g}{\partial p} - \frac{\partial f}{\partial p}\frac{\partial g}{\partial q} \qquad \text{so} \qquad [x, F(p_x)]_{\text{classical}} = \frac{\partial F}{\partial p_x}$$

Using (1.6.47), then, we have

$$\left[x, \exp\left(\frac{ip_xa}{\hbar}\right)\right] = i\hbar\left[x, \exp\left(\frac{ip_xa}{\hbar}\right)\right]_{\text{classical}} = i\hbar\frac{\partial}{\partial p_x}\exp\left(\frac{ip_xa}{\hbar}\right) = -a\exp\left(\frac{ip_xa}{\hbar}\right)$$

To show that $\exp(ip_xa/\hbar)|x'\rangle$ is an eigenstate of position, act on it with x. So

$$x\exp\left(\frac{ip_xa}{\hbar}\right)|x'\rangle = \left[\exp\left(\frac{ip_xa}{\hbar}\right)x - a\exp\left(\frac{ip_xa}{\hbar}\right)\right]|x'\rangle = (x'-a)\exp\left(\frac{ip_xa}{\hbar}\right)|x'\rangle$$

In other words, $\exp(ip_xa/\hbar)|x'\rangle$ is an eigenstate of x with eigenvalue $x'-a$. That is $\exp(ip_xa/\hbar)|x'\rangle$ is the translation operator with $\Delta x' = -a$, but we knew that. See (1.6.36).

29. I wouldn't say this is "easily derived", but it is straightforward. Expressing $G(\mathbf{p})$ as a power series means $G(\mathbf{p}) = \sum_{nm\ell} a_{nm\ell}p_i^n p_j^m p_k^\ell$. Now

$$\begin{aligned}[x_i, p_i^n] = x_ip_ip_i^{n-1} - p_i^nx_i &= i\hbar p_i^{n-1} + p_ix_ip_i^{n-1} - p_i^nx_i \\ &= 2i\hbar p_i^{n-1} + p_i^2x_ip_i^{n-2} - p_i^nx_i \\ &\cdots \\ &= ni\hbar p_i^{n-1} \\ \text{so} \qquad [x_i, G(\mathbf{p})] &= i\hbar\frac{\partial G}{\partial p_i}\end{aligned}$$

The procedure is essentially identical to prove that $[p_i, F(\mathbf{x})] = -i\hbar\partial F/\partial x_i$. As for

$$[x^2, p^2] = x^2p^2 - p^2x^2 = x^2p^2 - xp^2x + xp^2x - p^2x^2 = x[x, p^2] + [x, p^2]x$$

make use of $[x, p^2] = i\hbar\partial(p^2)/\partial p = 2i\hbar p$ so that $[x^2, p^2] = 2i\hbar(xp+px)$. The classical Poisson bracket is $[x^2, p^2]_{\text{classical}} = (2x)(2p) - 0 = 4xp$ and so $[x^2, p^2] = i\hbar[x^2, p^2]_{\text{classical}}$ when we let the (classical quantities) x and p commute.

30. This is very similar to problem 28. Using problem 29,

$$[x_i, \mathcal{J}(\mathbf{l})] = \left[x_i, \exp\left(\frac{-i\mathbf{p}\cdot\mathbf{l}}{\hbar}\right)\right] = i\hbar\frac{\partial}{\partial p_i}\exp\left(\frac{-i\mathbf{p}\cdot\mathbf{l}}{\hbar}\right) = l_i \exp\left(\frac{-i\mathbf{p}\cdot\mathbf{l}}{\hbar}\right) = l_i\mathcal{J}(\mathbf{l})$$

We can use this result to calculate the expectation value of x_i. First note that

$$\begin{aligned}\mathcal{J}^\dagger(\mathbf{l})\,[x_i, \mathcal{J}(\mathbf{l})] &= \mathcal{J}^\dagger(\mathbf{l})x_i\mathcal{J}(\mathbf{l}) - \mathcal{J}^\dagger(\mathbf{l})\mathcal{J}(\mathbf{l})x_i = \mathcal{J}^\dagger(\mathbf{l})x_i\mathcal{J}(\mathbf{l}) - x_i \\ &= \mathcal{J}^\dagger(\mathbf{l})l_i\mathcal{J}(\mathbf{l}) = l_i\end{aligned}$$

Therefore, under translation,

$$\langle x_i\rangle = \langle\alpha|x_i|\alpha\rangle \to \langle\alpha|\mathcal{J}^\dagger(\mathbf{l})x_i\mathcal{J}(\mathbf{l})|\alpha\rangle = \langle\alpha|\mathcal{J}^\dagger(\mathbf{l})x_i\mathcal{J}(\mathbf{l})|\alpha\rangle = \langle\alpha|(x_i + l_i)|\alpha\rangle = \langle x_i\rangle + l_i$$

which is exactly what you expect from a translation operator.

31. This is a continued rehash of the last few problems. Since $[\mathbf{x}, \mathcal{J}(\mathbf{dx}')] = \mathbf{dx}'$ by (1.6.25), and since $\mathcal{J}^\dagger[\mathbf{x}, \mathcal{J}] = \mathcal{J}^\dagger\mathbf{x}\mathcal{J} - \mathbf{x}$, we have $\mathcal{J}^\dagger(\mathbf{dx}')\mathbf{x}\mathcal{J}(\mathbf{dx}') = \mathbf{x} + \mathcal{J}^\dagger(\mathbf{dx}')\mathbf{dx}' = \mathbf{x} + \mathbf{dx}'$ since we only keep the lowest order in $\mathbf{dx}'$. Therefore $\langle\mathbf{x}\rangle \to \langle\mathbf{x}\rangle + \mathbf{dx}'$. Similarly, from (1.6.45), $[\mathbf{p}, \mathcal{J}(\mathbf{dx}')] = 0$, so $\mathcal{J}^\dagger[\mathbf{p}, \mathcal{J}] = \mathcal{J}^\dagger\mathbf{p}\mathcal{J} - \mathbf{p} = 0$. That is $\mathcal{J}^\dagger\mathbf{p}\mathcal{J} = \mathbf{p}$ and $\langle\mathbf{p}\rangle \to \langle\mathbf{p}\rangle$.

32. These are all straightforward. In the following, all integrals are taken with limits from $-\infty$ to ∞. One thing to keep in mind is that odd integrands give zero for the integral, so the right change of variables can be very useful. Also recall that $\int \exp(-ax^2)dx = \sqrt{\pi/a}$, and $\int x^2\exp(-ax^2)dx = -(d/da)\int\exp(-ax^2)dx = \sqrt{\pi}/2a^{3/2}$. So, for the x-space case,

$$\begin{aligned}\langle p\rangle &= \int\langle\alpha|x'\rangle\langle x'|p|\alpha\rangle dx' = \int\langle\alpha|x'\rangle\frac{\hbar}{i}\frac{d}{dx'}\langle x'|\alpha\rangle dx' = \frac{1}{d\sqrt{\pi}}\int\hbar k\exp\left(-\frac{x'^2}{d^2}\right)dx' = \hbar k \\ \langle p^2\rangle &= -\hbar^2\int\langle\alpha|x'\rangle\frac{d^2}{dx'^2}\langle x'|\alpha\rangle dx' \\ &= -\frac{\hbar^2}{d\sqrt{\pi}}\int\exp\left(-ikx' - \frac{x'^2}{2d^2}\right)\frac{d}{dx'}\left[\left(ik - \frac{x'}{d^2}\right)\exp\left(ikx' - \frac{x'^2}{2d^2}\right)\right]dx' \\ &= -\frac{\hbar^2}{d\sqrt{\pi}}\int\left[-\frac{1}{d^2} + \left(ik - \frac{x'}{d^2}\right)^2\right]\exp\left(-\frac{x'^2}{d^2}\right)dx' \\ &= \hbar^2\left[\frac{1}{d^2} + k^2\right] - \frac{\hbar^2}{d^5\sqrt{\pi}}\int x'^2\exp\left(-\frac{x'^2}{d^2}\right)dx' = \hbar^2\left[\frac{1}{d^2} + k^2\right] - \frac{\hbar^2}{2d^2} = \frac{\hbar^2}{2d^2} + \hbar^2k^2\end{aligned}$$

Using instead the momentum space wave function (1.7.42), we have

$$\begin{aligned}
\langle p \rangle &= \int \langle \alpha | p | p' \rangle \langle p' | \alpha \rangle dp' = \int p' |\langle p' | \alpha \rangle|^2 dp' \\
&= \frac{d}{\hbar\sqrt{\pi}} \int p' \exp\left[-\frac{(p'-\hbar k)^2 d^2}{\hbar^2} \right] dp' = \frac{d}{\hbar\sqrt{\pi}} \int (q+\hbar k) \exp\left[-\frac{q^2 d^2}{\hbar^2} \right] dq = \hbar k \\
\langle p^2 \rangle &= \frac{d}{\hbar\sqrt{\pi}} \int (q+\hbar k)^2 \exp\left[-\frac{q^2 d^2}{\hbar^2} \right] dq = \frac{d}{\hbar\sqrt{\pi}} \frac{\sqrt{\pi}}{2} \frac{\hbar^3}{d^3} + (\hbar k)^2 = \frac{\hbar^2}{2d^2} + \hbar^2 k^2
\end{aligned}$$

33. I can't help but think this problem can be done by creating a "momentum translation" operator, but instead I will follow the original solution manual. This approach uses the position space representation and Fourier transform to arrive the answer. Start with

$$\begin{aligned}
\langle p' | x | p'' \rangle &= \int \langle p' | x | x' \rangle \langle x' | p'' \rangle dx' = \int x' \langle p' | x' \rangle \langle x' | p'' \rangle dx' \\
&= \frac{1}{2\pi\hbar} \int x' \exp\left[-i \frac{(p'-p'') \cdot x'}{\hbar} \right] dx' = i \frac{\partial}{\partial p'} \frac{1}{2\pi} \int \exp\left[-i \frac{(p'-p'') \cdot x'}{\hbar} \right] dx' \\
&= i\hbar \frac{\partial}{\partial p'} \delta(p'-p'')
\end{aligned}$$

Now find $\langle p' | x | \alpha \rangle$ by inserting a complete set of states $|p''\rangle$, that is

$$\langle p' | x | \alpha \rangle = \int \langle p' | x | p'' \rangle \langle p'' | \alpha \rangle dp'' = i\hbar \frac{\partial}{\partial p'} \int \delta(p'-p'') \langle p'' | \alpha \rangle dp'' = i\hbar \frac{\partial}{\partial p'} \langle p' | \alpha \rangle$$

Given this, the next expression is simple to prove, namely

$$\langle \beta | x | \alpha \rangle = \int dp' \langle \beta | p' \rangle \langle p' | x | \alpha \rangle = \int dp' \phi_\beta^*(p') i\hbar \frac{\partial}{\partial p'} \phi_\alpha(p')$$

using the standard definition $\phi_\gamma(p') \equiv \langle p' | \gamma \rangle$.

Certainly the operator $\mathcal{T}(\Xi) \equiv \exp(ix\Xi/\hbar)$ looks like a momentum translation operator. So, we should try to work out $p\mathcal{T}(\Xi)|p'\rangle = p\exp(ix\Xi/\hbar)|p'\rangle$ and see if we get $|p'+\Xi\rangle$. Take a lesson from problem 28, and make use of the result from problem 29, and we have

$$p\mathcal{T}(\Xi)|p'\rangle = \{\mathcal{T}(\Xi)p + [p, \mathcal{T}(\Xi)]\}|p'\rangle = \left\{ p'\mathcal{T}(\Xi) - i\hbar \frac{\partial}{\partial x} \mathcal{T}(\Xi) \right\} |p'\rangle = (p'+\Xi)\mathcal{T}(\Xi)|p'\rangle$$

and, indeed, $\mathcal{T}(\Xi)|p'\rangle$ is an eigenstate of p with eigenvalue $p'+\Xi$. In fact, this could have been done first, and then write down the translation operator for infinitesimal momenta, and derive the expression for $\langle p' | x | \alpha \rangle$ the same way as done in the text for infinitesimal spacial translations. (I like this way of wording the problem, and maybe it will be changed in the next edition.)

Chapter Two

1. *This solution reprinted from the solutions manual for the revised edition.*

Hamiltonian $H = \omega S_z$. The Heisenberg equations of motion are:

$$\dot{S}_x = (1/i\hbar)[S_x, H] = (\omega/i\hbar)[S_x, S_z] = -\omega S_y$$

$$\dot{S}_y = (1/i\hbar)[S_y, H] = (\omega/i\hbar)[S_y, S_z] = +\omega S_x$$

$$\dot{S}_z = 0.$$

Hence $\dot{S}_x + i\dot{S}_y = -\omega S_y + i\omega S_x = i\omega(S_x+iS_y)$ and $\dot{S}_x - i\dot{S}_y = -\omega S_y - i\omega S_x = -i\omega(S_x-iS_y)$, so $(S_x\pm iS_y)_t = (S_x\pm iS_y)_{t=0}\, e^{\pm i\omega t}$ and we have finally $S_x(t) = S_x(0)\cos\omega t - S_y(0)\sin\omega t$, $S_y(t) = S_y(0)\cos\omega t + S_x(0)\sin\omega t$, $S_z(t) = S_z(0)$.

2. *This solution reprinted from the solutions manual for the revised edition.*

The Hamiltonian is obviously not Hermitian. Physically, the particle can go from state 2 to state 1 but not from state 1 to state 2. Because H is not Hermitian the time evolution operator is not unitary. Since unitarity is important for probability conservation, we suspect that probability conservation is violated.

To illustrate this point, set $H_{11} = H_{22} = 0$ for simplicity. For the time evolution operator we get, as usual, $\mathcal{U}(t,t_o=0) = \lim_{N\to\infty} (1-itH/\hbar N)^N$ where $\mathcal{U}$ is actually not unitary. But $H^2 = H_{12}^2\, |1\rangle\langle 2|1\rangle\langle 2| = 0$, hence $H^n = 0$ for $n > 1$. This means that $\mathcal{U}(t,t_o=0) = 1 - (itH_{12}/\hbar)|1\rangle\langle 2|$ even for a <u>finite time</u> interval. Now the most general initial state is $c_1|1\rangle + c_2|2\rangle$. At a later time we have $[1 - (itH_{12}/\hbar)|1\rangle\langle 2|](c_1|1\rangle + c_2|2\rangle) = c_1|1\rangle + c_2|2\rangle - (itH_{12}/\hbar)c_2|1\rangle$. Hence the probability for being found in $|1\rangle$ is $|c_1-(itH_{12}/\hbar)c_2|^2$ and the probability for being found in $|2\rangle$ is $|c_2|^2$. But the total probability is $|c_1|^2 - 2\,\mathrm{Im}(c_1c_2^*)H_{12}t/\hbar + |c_2|^2H_{12}^2t^2/\hbar^2 + |c_2|^2 \neq |c_1|^2 + |c_2|^2$ in general, and in fact $\langle\alpha, t_o=0\,|\alpha, t_o=0\rangle \neq \langle\alpha, t_o=0; t\,|\alpha, t_o=0; t\rangle$, so probability conservation is violated!

3. *This solution reprinted from the solutions manual for the revised edition.*

At time $t = 0$, $\hat{n} = \sin\beta\ \hat{x} + \cos\beta\ \hat{z}$, and $\vec{S} = \frac{\hbar}{2}\vec{\sigma}$, and $\vec{S}\cdot\hat{n} = \frac{\hbar}{2}(\sin\beta\ \sigma_x + \cos\beta\ \sigma_z)$. The eigenvalue equation at $t = 0$ $\vec{S}\cdot\hat{n}|\psi\rangle = \frac{\hbar}{2}|\psi\rangle$ where $|\psi\rangle = \begin{pmatrix} a \\ b \end{pmatrix}$ leads to $a\cos\beta + b\sin\beta = a$, and a normalized eigenstate of form

$$\frac{(1+\cos\beta)^{\frac{1}{2}}}{2^{\frac{1}{2}}}\begin{pmatrix} 1 \\ \sin\beta/(1+\cos\beta) \end{pmatrix}. \tag{1}$$

The Hamiltonian $H = -\vec{\mu}_s\cdot\vec{B} = (g_s\mu_B/2)\sigma_z B$ is that under consideration.

(a) The time dependence of $\psi(t)$ is governed by $H|\psi\rangle = i\hbar\partial/\partial t|\psi\rangle$ or

$$-i\omega\begin{pmatrix} A(t) \\ -B(t) \end{pmatrix} = \ \partial/\partial t\begin{pmatrix} A(t) \\ B(t) \end{pmatrix} \tag{2}$$

where $\omega = g_s\mu_B B/2\hbar$. This leads to two equations $-i\omega A(t) = \partial/\partial t[A(t)]$ and $+i\omega B(t) = \partial/\partial t\ [B(t)]$, thus $A(t) = A(0)e^{-i\omega t}$ and $B(t) = B(0)e^{+i\omega t}$. Compare with (1) above, we have

$$\psi(t) = \begin{pmatrix} [(1+\cos\beta)^{\frac{1}{2}}/2^{\frac{1}{2}}]e^{-i\omega t} \\ [\sin\beta/2^{\frac{1}{2}}(1+\cos\beta)^{\frac{1}{2}}]e^{+i\omega t} \end{pmatrix}. \tag{3}$$

Next we express $|\psi(t)\rangle$ in the $|s_x;\pm\rangle$ basis as $\alpha_1|s_x;+\rangle + \alpha_2|s_x;-\rangle$ where $|s_x;\pm\rangle$ are given explicitly by (1.4.17a) and $\alpha_1 = \frac{1}{\sqrt{2}}(1,1)\begin{pmatrix} Ae^{-i\omega t} \\ Be^{+i\omega t} \end{pmatrix} = (1/\sqrt{2})Ae^{-i\omega t} + (1/\sqrt{2})Be^{i\omega t}$ and $\alpha_2 = \frac{1}{\sqrt{2}}(1,-1)\begin{pmatrix} Ae^{-i\omega t} \\ Be^{+i\omega t} \end{pmatrix} = (1/\sqrt{2})Ae^{-i\omega t} - (1/\sqrt{2})Be^{+i\omega t}$ (for short we have written $A(0) = A$ and $B(0) = B$). Hence probability of finding the electron in $s_x = \hbar/2$ state is $\alpha_1^*\alpha_1 = \frac{1}{2}[A^2+B^2+AB(e^{2i\omega t} + e^{-2i\omega t})] = \frac{1}{2}(1+\sin\beta\cos 2\omega t)$.

(b) $\langle s_x \rangle = \langle\psi(t)|s_x|\psi(t)\rangle = (A^*(t),\ B^*(t))\frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} A(t) \\ B(t) \end{pmatrix} = \frac{\hbar}{2}\ (A^*(t)B(t) + B^*(t)A(t)) = (\hbar/2)\sin\beta\cos 2\omega t$.

(c) In case (i) $\beta\to 0$, $\alpha_1^*\alpha_1 = \frac{1}{2}$ and $\langle s_x \rangle = 0$; in case (ii) $\alpha_1^*\alpha_1 = \frac{1}{2}(1+\cos 2\omega t) = \cos^2\omega t$ while $\langle s_x \rangle \to \hbar(\cos^2\omega t - \frac{1}{2})$. These answers are eminently sensible since for $\beta = 0$ $\hat{n}$ is along the z-axis, hence there is equal probability of being found in $|s_x;+\rangle$ (i.e. $\alpha_1^*\alpha_1$) and in $|s_x;-\rangle$ (i.e. $\alpha_2^*\alpha_2$) - both being $\frac{1}{2}$. Yet $\langle s_x \rangle$

= 0 as the classical analogue would also be reasonable for an electron pointed spin-wise in the z-direction. For $\beta = \pi/2$ (i.e. $\hat{n}$ along OX), at t=0 $a_1^* a_1 = 1$, and $< s_x > = \hbar/2$ are entirely reasonable in terms of initial state requirements.

4. First, restating equations from the textbook,

$$\begin{aligned} |\nu_e\rangle &= \cos\theta|\nu_1\rangle - \sin\theta|\nu_2\rangle \\ |\nu_\mu\rangle &= \sin\theta|\nu_1\rangle + \cos\theta|\nu_2\rangle \\ \text{and} \qquad E &= pc\left(1 + \frac{m^2c^2}{2p^2}\right) \end{aligned}$$

Now, let the initial state $|\nu_e\rangle$ evolve in time to become a state $|\alpha, t\rangle$ in the usual fashion

$$\begin{aligned} |\alpha, t\rangle &= e^{-iHt/\hbar}|\nu_e\rangle \\ &= \cos\theta e^{-iE_1t/\hbar}|\nu_1\rangle - \sin\theta e^{-iE_2t/\hbar}|\nu_2\rangle \\ &= e^{-ipct/\hbar}\left[e^{-im_1^2c^3t/2p\hbar}\cos\theta|\nu_1\rangle - e^{-im_2^2c^3t/2p\hbar}\sin\theta|\nu_2\rangle\right] \end{aligned}$$

The probability that this state is observed to be a $|\nu_e\rangle$ is

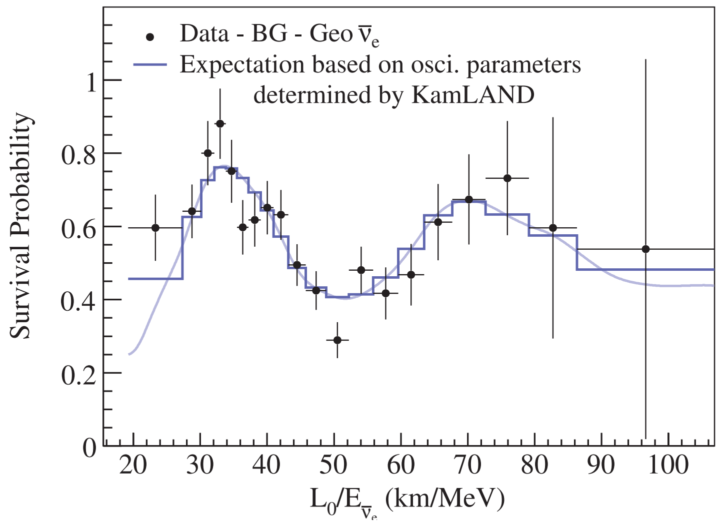
$$\begin{aligned} P(\nu_e \to \nu_e) = |\langle\nu_e|\alpha, t\rangle|^2 &= \left|e^{-im_1^2c^3t/2p\hbar}\cos^2\theta + e^{-im_2^2c^3t/2p\hbar}\sin^2\theta\right|^2 \\ &= \left|\cos^2\theta + e^{i\Delta m^2c^3t/2p\hbar}\sin^2\theta\right|^2 \\ &= \cos^4\theta + \sin^4\theta + 2\cos^2\theta\sin^2\theta\cos\left[\frac{\Delta m^2c^3t}{2p\hbar}\right] \\ &= 1 - \sin^2 2\theta\sin^2\left[\frac{\Delta m^2c^3t}{4p\hbar}\right] \end{aligned}$$

Writing the nominal neutrino energy as $E = pc$ and the flight distance $L = ct$ we have

$$P(\nu_e \to \nu_e) = 1 - \sin^2 2\theta\sin^2\left[\Delta m^2c^4\frac{L}{4E\hbar c}\right]$$

It is quite customary to ignore the factor of c^4 and agree to measure mass in units of energy, typically eV.

The neutrino oscillation probability from KamLAND is plotted here:



The minimum in the oscillation probability directly gives us $\sin^2 2\theta$, that is

$$1 - \sin^2 2\theta \approx 0.4 \qquad \text{so} \qquad \theta \approx 25^\circ$$

The wavelength gives the mass difference parameter. We have

$$40\frac{\text{km}}{\text{MeV}} = 2\pi\frac{4\hbar c}{\Delta m^2} = \frac{8\pi \times 200 \text{ MeV fm}}{\Delta m^2}$$

where we explicitly agree to measure Δm^2 in eV^2. Therefore

$$\Delta m^2 = 40\pi \times 10^{12}\text{eV}^2 \times 10^{-15}/10^3 = 1.2 \times 10^{-4} \text{ eV}^2$$

The results from a detailed analysis by the collaboration, in Physical Review Letters 100(2008)221803, are $\tan^2\theta = 0.56$ ($\theta = 37^\circ$) and $\Delta m^2 = 7.6 \times 10^{-5}$ eV^2. The full analysis not only includes the fact that the source reactors are at varying distances (although clustered at a nominal distance), but also that neutrino oscillations are over three generations.

5. *This solution reprinted from the solutions manual for the revised edition.*

First work out $x(t)$ and $p(t)$ in the Heisenberg picture. Evidently $\dot{x} = (1/i\hbar)[x, p^2/2m] = p/m$, and $\dot{p} = (1/i\hbar)[p, p^2/2m] = 0$. So $p(t) = p(0)$ and is independent of time, while $x(t) = x(0) + (p(0)/m)t$. Hence $[x(t), x(0)] = (t/m)[x(0), p(0)] = i\hbar t/m$.

6. *This solution reprinted from the solutions manual for the revised edition.*
Note: This is the proof of the so-called "dipole sum rule."

$[H, x] = [p^2/2m + V(x), x] = -i\hbar p/m$, therefore $[[H,x], x] = -\hbar^2/m$. Take the expectation value of $[[H,x], x]$ w.r.t. an energy eigenket $|a''\rangle$, we have

$$\langle a''|Hxx|a''\rangle - 2\langle a''|xHx|a''\rangle + \langle a''|xxH|a''\rangle = -\hbar^2/m. \tag{1}$$

Use next $H|a''\rangle = E_{a''}|a''\rangle$ and $\langle a''|H = E_{a''}\langle a''|$, (1) becomes

$$E_{a''}\langle a''|xx|a''\rangle - 2\langle a''|xHx|a''\rangle + E_{a''}\langle a''|xx|a''\rangle = -\hbar^2/m \tag{2a}$$

$$\text{or} \quad - E_{a''}\langle a''|xx|a''\rangle + \langle a''|xHx|a''\rangle = \hbar^2/2m \tag{2b}$$

Now using closure property, we have $\langle a''|xHx|a''\rangle = \sum_{a'} \langle a''|xH|a'\rangle\langle a'|x|a''\rangle =$ $\sum_{a'} E_{a'}\langle a''|x|a'\rangle\langle a'|x|a''\rangle = \sum_{a'} E_{a'}|\langle a''|x|a'\rangle|^2$ and $\langle a''|xx|a''\rangle = \sum_{a'}\langle a''|x|a'\rangle\langle a'|x|a''\rangle$ $= \sum_{a'}|\langle a''|x|a'\rangle|^2$. Equation (2b) becomes

$$\sum_{a'}|\langle a''|x|a'\rangle|^2 \;(E_{a'} - E_{a''}) = \hbar^2/2m. \tag{3}$$

7. *This solution reprinted from the solutions manual for the revised edition.*

Let $H = \vec{p}^2/2m + V(\vec{x})$, and we compute $[\vec{x}\cdot\vec{p}, H]$ through the following steps. $[\vec{x}\cdot\vec{p}, H] = [\vec{x}\cdot\vec{p}, \vec{p}^2/2m + V(\vec{x})] = (1/2m)[\vec{x}\cdot\vec{p}, \vec{p}^2] + [\vec{x}\cdot\vec{p}, V(\vec{x})] = (1/2m) \sum_{i,j}$ $[x_i p_i, p_j p_j] + \sum_i [x_i p_i, V(\vec{x})] = (1/2m) \sum_{i,j} (x_i[p_i, p_j p_j] + [x_i, p_j p_j]p_i) +$ $\sum_i (x_i[p_i, V(\vec{x})] + [x_i, V(\vec{x})]p_i) = (1/2m) \sum_{i,j} [x_i, p_j p_j]p_i + \sum_i x_i[p_i, V(\vec{x})]$ $= (1/2m) \sum_{i,j}([x_i, p_j]p_j p_i + p_j[x_i, p_j]p_i) + \sum_i x_i[p_i, V(\vec{x})] = (1/2m) \sum_{i,j} (i\hbar\delta_{ij}p_j p_i$ $+ p_j(i\hbar\delta_{ij})p_i) + \sum_i x_i(-i\hbar\partial V/\partial x_i) = i\hbar\,[\vec{p}^2/m - \vec{x}.\vec{\nabla}V(\vec{x})]$. Hence $\langle[\vec{x}\cdot\vec{p}, H]\rangle =$

$= i\hbar\,[\langle\vec{p}^2\rangle/m - \langle\vec{x}\cdot\vec{\nabla}V(\vec{x})\rangle] = i\hbar\, d/dt \langle\vec{x}\cdot\vec{p}\rangle$ (using Heisenberg equation of motion for $\vec{x}\cdot\vec{p}$). The condition for quantum mechanical analogue of the virial theorem is $d/dt\langle\vec{x}\cdot\vec{p}\rangle = 0$, i.e. the expectation value of $\vec{x}\cdot\vec{p}$ for a stationary state is independent of t.

8. *This solution reprinted from the solutions manual for the revised edition.*

To compute $\langle(\Delta x)^2\rangle = \langle x^2\rangle - \langle x\rangle^2$, first note that state ket is fixed in the Heisenberg picture, hence $\langle x(t)\rangle = \langle t{=}0|x(t)|t{=}0\rangle = 0$ because $\langle x(0)\rangle = \langle p(0)\rangle = 0$ and $x(t) = x(0) + (p(0)/m)t$ from problem 4 above. Next we compute

$$[x(t)]^2 = [x(0)]^2 + (t/m)[x(0)p(0)+p(0)x(0)] + (t^2/m^2)[p(0)]^2.$$

Because $\langle x(0)\rangle = \langle p(0)\rangle = 0$, hence $\Delta x = x(0) - \langle x(0)\rangle = x(0)$ while $\Delta p = p(0) - \langle p(0)\rangle = p(0)$. From problem 18(b) of Chapter 1, the minimum uncertainty wave packet satisfies $x(0)|t{=}0\rangle = \lambda p(0)|t{=}0\rangle$ where λ is a purely imaginary number. It is then evident that $\langle(x(0)p(0) + p(0)x(0))\rangle = (1/\lambda)\langle x(0)x(0)\rangle + (1/\lambda^*)\langle x^2(0)\rangle = 0$. So $\langle(\Delta x)^2\rangle_t = \langle x^2\rangle_t = \langle t{=}0|(x(0))^2|t{=}0\rangle + (t^2/m^2)\langle t{=}0|(p(0))^2|t{=}0\rangle = \langle(\Delta x)^2\rangle_{t=0} + (t^2/m^2)\langle(\Delta p)^2\rangle_{t=0} = \langle(\Delta x)^2\rangle_{t=0} + (\hbar^2 t^2/4m^2\langle(\Delta x)^2\rangle_{t=0})$. This agrees with expansion of wave packet calculated using wave mechanics.

9. *This solution reprinted from the solutions manual for the revised edition.*

(a) $H = |a'\rangle\delta\langle a''| + |a''\rangle\delta\langle a'| = \delta\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}$ as is evident since $\langle a'|H|a'\rangle = \langle a''|H|a''\rangle = 0$, while $\langle a'|H|a''\rangle = \langle a''|H|a'\rangle = \delta$. Now $H|\psi\rangle = E|\psi\rangle$ and the secular equation is $\det[H - E\underline{1}] = 0$, i.e. $E = \pm\delta$ are the energy eigenvalues. The corresponding eigenkets satisfy (with $|\psi\rangle = \begin{pmatrix}A\\ B\end{pmatrix}$)

$$\begin{pmatrix}0 & \delta\\ \delta & 0\end{pmatrix}\begin{pmatrix}A\\ B\end{pmatrix} = \pm\delta\begin{pmatrix}A\\ B\end{pmatrix}, \text{ and } |A|^2 + |B|^2 = 1 \text{ (normalization)}.$$

Obviously $\frac{1}{\sqrt{2}}\begin{pmatrix}1\\ 1\end{pmatrix}$ for $E = +\delta$ and $\frac{1}{\sqrt{2}}\begin{pmatrix}1\\ -1\end{pmatrix}$ for $E = -\delta$ are appropriate eigenket solutions.

(b) As a function of time we write $|\psi(t)\rangle = \begin{pmatrix}A(t)\\ B(t)\end{pmatrix}$, and $H|\psi(t)\rangle = i\hbar d/dt|\psi(t)\rangle$

reads $\begin{pmatrix} 0 & \delta \\ \delta & 0 \end{pmatrix}\begin{pmatrix} A(t) \\ B(t) \end{pmatrix} = i\hbar d/dt \begin{pmatrix} A(t) \\ B(t) \end{pmatrix}$ or $\delta B(t) = i\hbar dA(t)/dt$ and $\delta A(t) = i\hbar dB(t)/dt$.

Thus $A(t) = -(\hbar/\delta)^2 d^2A(t)/dt^2$ and $B(t) = -(\hbar/\delta)^2 d^2B(t)/dt^2$, and $A(t) = A_1\cos\omega t + A_2\sin\omega t$, $B(t) = B_1\cos\omega t + B_2\sin\omega t$ are the simple harmonic solutions with $\omega = \delta/\hbar$. It is evident that $|a'\rangle = |\psi(t=0)\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ hence $A_1 = 1$, $B_1 = 0$ and from normalization $B_2 = 1$, $A_2 = 0$. So $|\psi(t)\rangle = \begin{pmatrix} \cos\omega t \\ \sin\omega t \end{pmatrix}$.

(c) We need to evaluate $|\langle a''|\psi(t)\rangle|^2$ where $\langle a''| = (0,1)$. Evidently probability is $\sin^2\omega t$.

(d) The Hamiltonian $H = \delta\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = J_x$ for a spin ½ system if $\delta = \hbar/2$, hence $|\psi(t)\rangle = \begin{pmatrix} \cos\omega t \\ \sin\omega t \end{pmatrix}$ describes the evolution of a spinor in time, initially in state $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and hence an eigenstate of $J_z = \frac{\hbar}{2}\sigma_z$.

10. *This solution reprinted from the solutions manual for the revised edition.*

(a) Let the normalized energy eigenkets be written as $|E\rangle = |R\rangle\langle R|E\rangle + |L\rangle\langle L|E\rangle$. Now $H|E\rangle = E|E\rangle$ therefore $\Delta(|L\rangle\langle R| + |R\rangle\langle L|)|E\rangle = E|E\rangle$ or $\Delta(|L\rangle\langle R|E\rangle + |R\rangle\langle L|E\rangle) = E(|R\rangle\langle R|E\rangle + |L\rangle\langle L|E\rangle)$. Due to the linear independence of $|L\rangle$ and $|R\rangle$, we have $\Delta\langle R|E\rangle = E\langle L|E\rangle$ and $\Delta\langle L|E\rangle = E\langle R|E\rangle$. Now due to normalization condition $|\langle R|E\rangle|^2 + |\langle L|E\rangle|^2 = 1$, we have $\Delta^2 = E^2$ or $\Delta = \pm E$ (these define the two level system eigenvalues). Take $\Delta = +E$, and $\langle R|E\rangle = \langle L|E\rangle = 1/\sqrt{2}$, than $|+E\rangle = \frac{1}{\sqrt{2}}(|R\rangle + |L\rangle)$; for $\Delta = -E$, take $\langle R|E\rangle = -\langle L|E\rangle = 1/\sqrt{2}$ and $|-E\rangle = \frac{1}{\sqrt{2}}(|R\rangle - |L\rangle)$.

(b) Suppose at $t=0$, $|\alpha\rangle = |R\rangle\langle R|\alpha\rangle + |L\rangle\langle L|\alpha\rangle \equiv |\alpha, t=t_o=0\rangle$. The evolution of state vector $|\alpha, t_o=0; t\rangle$ is such that $e^{-iHt/\hbar}|\alpha\rangle = |\alpha, t_o=0; t\rangle$. From part (a) we have $|R\rangle = \frac{1}{\sqrt{2}}(|+E\rangle + |-E\rangle)$ and $|L\rangle = \frac{1}{\sqrt{2}}(|+E\rangle - |-E\rangle)$, therefore

$$\begin{aligned} e^{-iHt/\hbar}|\alpha\rangle &= e^{-iHt/\hbar}(\langle R|\alpha\rangle|R\rangle + \langle L|\alpha\rangle|L\rangle) \\ &= \frac{1}{\sqrt{2}}\langle R|\alpha\rangle e^{-iHt/\hbar}(|+E\rangle + |-E\rangle) + \frac{1}{\sqrt{2}}\langle L|\alpha\rangle\, e^{-iHt/\hbar}(|+E\rangle - |-E\rangle). \qquad (1) \end{aligned}$$

But $e^{-iHt/\hbar}|\pm E\rangle = e^{\mp i\Delta t/\hbar}|\pm E\rangle$, hence from (1) we have

$$|\alpha,\ t_o=0;\ t\rangle = e^{-iHt/\hbar}|\alpha\rangle = \frac{1}{\sqrt{2}}\langle R|\alpha\rangle(e^{-i\Delta t/\hbar}|+E\rangle + e^{i\Delta t/\hbar}|-E\rangle) + \frac{1}{\sqrt{2}}\langle L|\alpha\rangle(e^{-i\Delta t/\hbar}|+E\rangle - e^{i\Delta t/\hbar}|-E\rangle). \tag{2}$$

Rearrange r.h.s. of (2) back to the $\{|R\rangle, |L\rangle\}$ basis, we have

$$|\alpha,\ t_o=0;\ t\rangle = (\langle R|\alpha\rangle\cos\Delta t/\hbar - i\langle L|\alpha\rangle\sin\Delta t/\hbar)|R\rangle + (\langle L|\alpha\rangle\cos\Delta t/\hbar - i\langle R|\alpha\rangle\sin\Delta t/\hbar)|L\rangle \tag{3}$$

(c) Suppose at $t=0$, $|\alpha\rangle = |R\rangle$ with certainty, than from (3) we have $\langle L|\alpha\rangle = 0$ and $\langle R|\alpha\rangle = 1$ (normalization). We need the development of $|L\rangle$ as a function of time, this corresponds to $|\alpha,t_o=0;t\rangle = \cos\Delta t/\hbar|R\rangle - i\sin\Delta t/\hbar|L\rangle$ and $\langle L|\alpha,t_o=0;t\rangle = -i\sin\Delta t/\hbar$. The transition probability is $|\langle L|\alpha,t_o=0;t\rangle|^2 = \sin^2\Delta t/\hbar$.

(d) In the Schrödinger picture the base kets $|R\rangle$ and $|L\rangle$ remain stationary in time and the state vector obeys $i\hbar\ \partial/\partial t|\alpha,t_o=0;\ t\rangle = H|\alpha,t_o=0;\ t\rangle$. Write $|\alpha,t_o=0;\ t\rangle = \alpha_R(t)|R\rangle + \alpha_L(t)|L\rangle$ and using $H = \Delta(|L\rangle\langle R|+|R\rangle\langle L|)$, the Schrödinger equation leads to coupled equations $i\hbar d\alpha_R(t)/dt = \Delta\alpha_L(t)$ and $i\hbar d\alpha_L(t)/dt = \Delta\alpha_R(t)$ where $\alpha_R(t) = \langle R|\alpha,t_o=0;\ t\rangle$ and $\alpha_L(t) = \langle L|\alpha\ ,t_o=0;\ t\rangle$. Solutions of the coupled equations can be obtained by noting that $d^2/dt^2[\alpha_{R,L}(t)] + (\Delta^2/\hbar^2)\alpha_{R,L}(t) = 0$, hence

$$\alpha_L(t)=A\cos\Delta t/\hbar + B\sin\Delta t/\hbar,\quad \alpha_R(t)=C\cos\Delta t/\hbar + D\sin\Delta t/\hbar \tag{4}$$

At $t = 0$ $|\alpha\rangle = \langle R|\alpha\rangle|R\rangle + \langle L|\alpha\rangle|L\rangle = \alpha_R(0)|R\rangle + \alpha_L(0)|L\rangle$, hence $\alpha_R(0) = C = \langle R|\alpha\rangle$ and $\alpha_L(0) = A = \langle L|\alpha\rangle$. Next the normalization condition at t, with $t_o=0$ $\langle\alpha,t_o=0;\ t|\alpha,t_o=0;\ t\rangle = 1$ give

$$\cos^2\Delta t/\hbar + (\langle R|\alpha\rangle^* D + D^*\langle R|\alpha\rangle + \langle L|\alpha\rangle^* B + B^*\langle L|\alpha\rangle)\cos\Delta t/\hbar\ \sin\Delta t/\hbar + (|D|^2 + |B|^2)\sin^2\Delta t/\hbar = 1. \tag{5}$$

Solution of (5) is possible with $D = -i\langle L|\alpha\rangle$ and $B = -i\langle R|\alpha\rangle$, hence (4) for $\alpha_L(t)$ and $\alpha_R(t)$ gives the coefficients of $|L\rangle$ and $|R\rangle$ in (3) of (b).

(e) The lack of Hermiticity here is same as in problem 2, replacing $H = H_{12}|1\rangle\langle 2|$

by $H = \Delta|L\rangle\langle R|$. We find again $H^n = 0$ for $n>1$, and $U(t,t_o=0) = 1 - it\Delta/\hbar\ |L\rangle\langle R|$ even for a finite time interval. The initial state is $\langle R|\alpha\rangle|R\rangle + \langle L|\alpha\rangle|L\rangle$; at a later time t we have $(1 - it\Delta/\hbar\ |L\rangle\langle R|)(\langle R|\alpha\rangle|R\rangle + \langle L|\alpha\rangle|L\rangle)$, hence probability for being found in $|L\rangle$ is $|\langle L|\alpha\rangle - (it\Delta/\hbar)\langle R|\alpha\rangle|^2$ and in $|R\rangle$ is $|\langle R|\alpha\rangle|^2$, but $|\langle L|\alpha\rangle - (it\Delta/\hbar)\langle R|\alpha\rangle|^2 + |\langle R|\alpha\rangle|^2 \neq |\langle L|\alpha\rangle|^2 + |\langle R|\alpha\rangle|^2$. Thus probability conservation is violated.

11. *This solution reprinted from the solutions manual for the revised edition.*
$H = p^2/2m + \frac{1}{2} m\omega^2 x^2$ for the one dimensional simple harmonic oscillator.

(a) In the Heisenberg picture, the operators x and p obey the Heisenberg equations of motion: $dp/dt = (1/i\hbar)[p, H] = -m\omega^2 x$, $dx/dt = (1/i\hbar)[x, H] = p/m$. This implies $\ddot{x} = -\omega^2 x$ and $\ddot{p} = -\omega^2 p$ with the initial conditions $x(0) = x_o$ and $p(0) = p_o$ $\dot{x}(0) = p_o/m$ and $\dot{p}(0) = -m\omega^2 x_o$. The solutions are $x(t) = x_o\cos\omega t + (p_o/m\omega)\sin\omega t$, $p(t) = p_o\cos\omega t - m\omega x_o\sin\omega t$ which give $H = p^2(t)/2m + \frac{1}{2} m\omega^2 x^2(t) = p_o^2/2m + \frac{1}{2} m\omega^2 x_o^2$, i.e. H is time independent. Dynamical variables x and p are time-dependent in the Heisenberg picture. At $t = 0$, the Heisenberg and Schrödinger pictures coincide, thus $x_H(0) = x_S(0) = x_o$ (with $x_S(t) = x_S(0)$) and $p_H(0) = p_S(0) = p_o$ (with $p_S(t) = p_S(0)$) and we note the time-independence of dynamical variables in the Schrödinger picture.

The relationship between the Heisenberg and Schrödinger pictures is $x_H(t) = e^{iHt/\hbar}x_S e^{-iHt/\hbar}$ with $x_S = x_o$ and $p_H(t) = e^{iHt/\hbar}p_S e^{-iHt/\hbar}$ with $p_S = p_o$. Using (2.3.48) – (2.3.50), one knows $x_H(t) = x_o\cos\omega t + (p_o/m\omega)\sin\omega t$. Also

$$e^{iHt/\hbar}p_o e^{-iHt/\hbar} = p_o + (it/\hbar)[H,p_o] + (i^2t^2/2!\hbar^2)[H,[H,p_o]]$$
$$+ (i^3t^3/3!\hbar^3)[H,[H,[H,p_o]]] + \ldots = p_o - \left(\frac{t^2\omega^2}{2!}\right)p_o - tm\omega^2 x_o + \frac{t^3 m\omega^4 x_o}{3!} + \ldots\ldots$$

where we have used $[H, x_o] = -i\hbar p_o/m$, $[H, p_o] = i\hbar m\omega^2 x_o$. This implies that $p_H(t) = p_o\cos\omega t - m\omega x_o\sin\omega t$.

(b) At t=0, the general state vectors for both pictures are equal: $|\alpha\rangle_H = |\alpha\rangle_S =$

$|\alpha,t=0\rangle$, e.g. $|\alpha,t=0\rangle = \Sigma_n c_n(0)|n\rangle$. At $t\neq 0$, $|\alpha,t\rangle_H = |\alpha,t=0\rangle = \Sigma_n c_n(0)|n\rangle$ i.e. time independent, while $|\alpha,t\rangle_S = e^{-iHt/\hbar}|\alpha,t=0\rangle = \Sigma_n c_n(0)e^{-i\omega(n+\frac{1}{2})t}|n\rangle$ and is thus time dependent. (We have used $H = \hbar\omega(N+\frac{1}{2})$ which is time-independent in both pictures). We can recast $|\alpha,t\rangle_S$ as $|\alpha,t\rangle_S = \Sigma_n c_n(t)|n\rangle$ with $c_n(t) = c_n(0)e^{-i\omega(n+\frac{1}{2})t}$. Also note $i\hbar\partial/\partial t|\alpha,t\rangle_S = H|\alpha,t\rangle_S$ which is the Schrödinger equation for the Schrödinger state vector. Remarks: $c_n(t)$ can be determined in the two pictures by (a) $c_n(t) = \langle n|\alpha,t\rangle_S = c_n(0)e^{-i\omega(n+\frac{1}{2})t}$, the Schrödinger picture with base kets $|n\rangle$ time independent, and (b) $c_n(t) = \langle n,t|\alpha,t\rangle_H = \langle n|e^{-iHt/\hbar}|\alpha,t\rangle_H = c_n(0)e^{-i\omega(n+\frac{1}{2})t}$, the Heisenberg picture with base kets $|n,t\rangle = e^{iHt/\hbar}|n\rangle$ which are time-dependent.

12. *This solution reprinted from the solutions manual for the revised edition.*

For a one-dimensional SHO potential $H = p^2/2m + \frac{1}{2}m\omega^2x^2$, hence $\dot{x} = (1/i\hbar)[x,H] = p/m$, and $\dot{p} = (1/i\hbar)[p,H] = (1/i\hbar)(m\omega^2/2)[p,x^2] = (m\omega^2/2i\hbar)[-2i\hbar x] = -m\omega^2x$. Hence $\ddot{x} + \omega^2x = 0$, and solution is $x(t) = A\cos\omega t + B\sin\omega t$. At $t=0$, $x(0) = A$ while $\dot{x}(t) = -A\omega\sin\omega t+B\omega\cos\omega t$ leads to $\dot{x}(0) = B\omega$ and thus $p(0) = m\omega B$. Thus in the Heisenberg picture $x(t) = x(0)\cos\omega t + (p(0)/m\omega)\sin\omega t$.

Our state vector $|\alpha\rangle = e^{-ipa/\hbar}|0\rangle$ at $t=0$; for $t>0$ we have in the Heisenberg picture $\langle x(t)\rangle = \langle\alpha|x(t)|\alpha\rangle$. We note that

$$e^{ip(0)a/\hbar}x(0)e^{-ip(0)a/\hbar} = e^{ip(0)a/\hbar}([x(0),e^{-ip(0)a/\hbar}] + e^{-ip(0)a/\hbar}x(0))$$
$$= x(0) + a,$$

while $e^{ip(0)a/\hbar}p(0)e^{-ip(0)a/\hbar} = p(0)$. Hence

$$\langle x(t)\rangle = \langle\alpha|x(t)|\alpha\rangle = \langle 0|e^{ipa/\hbar}x(t)e^{-ipa/\hbar}|0\rangle$$
$$= \langle 0|e^{ip(0)a/\hbar}[x(0)\cos\omega t+(p(0)/m\omega)\sin\omega t]e^{-ipa/\hbar}|0\rangle.$$

Since $\langle 0|x(0)|0\rangle = \langle 0|p(0)|0\rangle = 0$, we obtain for $\langle x(t)\rangle = a\cos\omega t$.

13. *This solution reprinted from the solutions manual for the revised edition.*

(a) The wave function in problem 11 takes form $\langle x'|\alpha\rangle = \langle x'|e^{-ipa/\hbar}|0\rangle$. Since $e^{ipa/\hbar}|x'\rangle = |x'-a\rangle$ (hence $\langle x'|e^{-ipa/\hbar} = \langle x'-a|$), we have $\langle x'|\alpha\rangle = \langle x'-a|0\rangle$. Hence $\langle x'|\alpha\rangle = \pi^{-\frac{1}{4}} x_0^{-\frac{1}{2}} \exp[-(x'-a)^2/2x_0^2]$.

(b) The ground state wave function is

$$\langle x'|0\rangle = \pi^{-\frac{1}{4}} x_0^{-\frac{1}{2}} \exp[-\frac{x'^2}{2x_0^2}] .$$

The probability of finding $|\alpha\rangle$ in the ground state is

$$P = \int\langle\alpha|x'\rangle\langle x'|0\rangle dx' = (1/\pi^{\frac{1}{2}}x_0)\int_{-\infty}^{\infty} \exp[-\{(x'-a)^2+x'^2\}/2x_0^2]\ dx' = e^{-a^2/2x_0^2}.$$

P is time independent and hence does not change for $t>0$.

14. *This solution reprinted from the solutions manual for the revised edition.*

(a) From the given information, we can write

$$x = \sqrt{\hbar/2m\omega}(a+a^\dagger),\ p = i\sqrt{\hbar m\omega/2}(a^\dagger-a) \tag{1}$$

$x|n\rangle = \sqrt{\hbar/2m\omega}(\sqrt{n}|n-1\rangle+\sqrt{n+1}|n+1\rangle)$ and $p|n\rangle = i\sqrt{\hbar m\omega/2}(\sqrt{n+1}|n+1\rangle-\sqrt{n}|n-1\rangle)$. Remember also that $a^\dagger a = N$ where N is number operator and $N|n\rangle = n|n\rangle$ while $\langle m|n\rangle = \delta_{mn}$. Therefore $\langle m|x|n\rangle = \frac{1}{2}\sqrt{2\hbar/m\omega}(\sqrt{n}\delta_{m,n-1} + \sqrt{n+1}\delta_{m,n+1})$, likewise $\langle m|p|n\rangle = (m\omega i/2)\sqrt{2\hbar/m\omega}(\sqrt{n+1}\delta_{m,n+1}-\sqrt{n}\delta_{m,n-1})$. Computation of $\langle m|\{x,p\}|n\rangle = \langle m|xp|n\rangle + \langle m|px|n\rangle$ is obtained by using (1) and $\langle m|x = \sqrt{\hbar/2m\omega}(\sqrt{m}\langle m-1|+\sqrt{m+1}\langle m+1|)$ as well as $\langle m|p = -i\sqrt{\hbar m\omega/2}(\sqrt{m+1}\langle m+1|-\sqrt{m}\langle m-1|)$ [sign change comes from complex conjugation when passing to dual space]. The calculation is then straightforward leading to $\langle m|\{x,p\}|n\rangle = -i\hbar(\sqrt{n(m+1)}\delta_{m+1,n-1}+\sqrt{m(n+1)}\delta_{m-1,n+1})$. For $\langle m|x^2|n\rangle = \langle m|xx|n\rangle$, try evaluate the scalar product $\langle m|x$ and $x|n\rangle$, the answer is $\langle m|x^2|n\rangle = (\hbar/2m\omega)\{\sqrt{n(n-1)}\delta_{m,n-2}+(2n+1)\delta_{m,n}+\sqrt{(n+1)(n+2)}\delta_{m,n+2}\}$. Likewise $\langle m|p^2|n\rangle = \langle m|pp|n\rangle$ and we evaluate the scalar product $\langle m|p$ and $p|n\rangle$, the answer is $\langle m|p^2|n\rangle = -\frac{\hbar m\omega}{2}\{\sqrt{n(n-1)}\delta_{m,n-2}-(2n+1)\delta_{m,n}+\sqrt{(n+1)(n+2)}\delta_{m,n+2}\}$.

(b) Virial theorem states $\langle p^2/m\rangle = \langle\vec{x}\cdot\vec{\nabla}V\rangle$, hence in one dimension we have $\langle p^2/m\rangle = \langle xdV/dx\rangle$. For the SHO, $H = p^2/2m + V(x) = p^2/2m + \frac{1}{2}m\omega^2x^2$, therefore $xdV/dx = m\omega^2x^2$. Now $\langle p^2/m\rangle = \frac{1}{m}\langle n|p^2|n\rangle = \frac{\hbar\omega}{2}(2n+1) = \hbar\omega(n+\frac{1}{2})$, while $\langle xdV/dx\rangle = m\omega^2\langle x^2\rangle = \frac{m\omega^2\hbar}{2m\omega}(2n+1) = \hbar\omega(n+\frac{1}{2})$. Therefore the virial theorem is verified.

15. *This solution reprinted from the solutions manual for the revised edition.*

(a) $\langle x'|p'\rangle = (2\pi\hbar)^{-1/2}e^{ip'x'/\hbar}$ or $\langle p'|x'\rangle = (2\pi\hbar)^{-1/2}e^{-ip'x'/\hbar}$, hence $\langle p'|x|\alpha\rangle$ $= \int dx'\langle p'|x|x'\rangle\langle x'|\alpha\rangle = \int dx'\ x'\langle p'|x'\rangle\langle x'|\alpha\rangle$. Note that explicitly we have $i\hbar\partial/\partial p'[(2\pi\hbar)^{-1/2}e^{-ip'x'/\hbar}] = x'\langle p'|x'\rangle$. Hence $\langle p'|x|\alpha\rangle = \int dx' i\hbar\partial/\partial p'(\langle p'|x'\rangle\langle x'|\alpha\rangle)$ $= i\hbar\partial/\partial p'\ \langle p'|\alpha\rangle$ where we assume that differentiation and integration can be interchanged.

(b) For $H = p^2/2m + \frac{1}{2}m\omega^2x^2$, the state vector $|\ \rangle_S$ satisfies in Schrödinger picture

$$(p^2/2m + \tfrac{1}{2}m\omega^2x^2)|\ \rangle_S = i\hbar\partial/\partial t|\ \rangle_S\ . \qquad (1)$$

In the momentum representation, we have

$$\langle p'|(p^2/2m + \tfrac{1}{2}m\omega^2x^2)|\ \rangle_S = i\hbar\partial/\partial t\ \langle p'|\ \rangle_S \qquad (2)$$

and thus

$$\frac{p'^2}{2m}\langle p'|\ \rangle_S + \tfrac{1}{2}m\omega^2(-\hbar^2\partial^2/\partial p'^2)\langle p'|\ \rangle_S = i\hbar\partial/\partial t\ \langle p'|\ \rangle_S \qquad (3)$$

where in (3) we have used identity $\langle p'|xx|\ \rangle_S = i\hbar\partial/\partial p'\ \langle p'|x|\ \rangle_S = -\hbar^2\frac{\partial^2}{\partial p'^2}\langle p'|\ \rangle_S$.

For the SHO problem there is a complete symmetry between x and p. So the energy eigenfunctions in momentum space must be of the form $e^{-p^2/2p_o^2}\ H_n(p/p_o)$ up to normalization ($p_o \equiv \sqrt{\hbar m\omega}$) in analogy with $e^{-x^2/2x_o^2}\ H_n(x/x_o)$ ($x_o = \sqrt{\hbar/m\omega}$) in position space.

16. *This solution reprinted from the solutions manual for the revised edition.*

From (2.3.45a), we have $x(t) = x(0)\cos\omega t + (p(0)/m\omega)\sin\omega t$, and $x(t)x(0) = [x(0)]^2\cos\omega t + (p(0)x(0)/m\omega)\sin\omega t$. Simple harmonic oscillator (SHO) ground state is from (2.3.30) $\langle x'|0\rangle = (1/\pi^{1/4}x_o^{1/2})\exp[-\frac{1}{2}(x'/x_o)^2]$, $x_o = \sqrt{\hbar/m\omega}$. Then

$$\begin{aligned} C(t) &= \langle 0|x(t)x(0)|0\rangle \\ &= \iint\langle 0|x'\rangle\langle x'|[(x(0))^2\cos\omega t + (p(0)x(0)/m\omega)\sin\omega t]|x''\rangle\langle x''|0\rangle\ dx'dx'' \\ &= \int\langle 0|x'\rangle\langle x'|0\rangle x'^2\cos\omega t\ dx' + (\sin\omega t/m\omega)\langle 0|p(0)x(0)|0\rangle. \end{aligned}$$

The term $\langle 0|p(0)x(0)|0\rangle$ vanishes (c.f. problem 13 with m=n=0 or by explicit evaluation in $|x'\rangle$ representation). Hence C(t) is given by

$$C(t) = \cos\omega t\int_{-\infty}^{\infty}(x'^2/\pi^{1/2}x_o)\exp[-(x'/x_o)^2]\ dx' = (\hbar/2m\omega)\cos\omega t.$$

17. *This solution reprinted from the solutions manual for the revised edition.*

(a) Let linear combination be $|\alpha\rangle = a|0\rangle + b|1\rangle$. Then $\langle x\rangle = (a^*\langle 0| + b^*\langle 1|)x(a|0\rangle + b|1\rangle)$ or $\langle x\rangle = a^*a\langle 0|x|0\rangle + a^*b\langle 0|x|1\rangle + b^*a\langle 1|x|0\rangle + b^*b\langle 1|x|1\rangle$. From problem 13 we have $\langle m|x|n\rangle = \frac{1}{2}\sqrt{2\hbar/m\omega}(\sqrt{n}\,\delta_{m,n-1} + \sqrt{n+1}\,\delta_{m,n+1})$, hence $\langle x\rangle = \frac{a^*b}{2}\sqrt{2\hbar/m\omega} + \frac{b^*a}{2}\sqrt{2\hbar/m\omega}$ or $\langle x\rangle = \frac{1}{2}\sqrt{2\hbar/m\omega}(a^*b + b^*a)$. Without loss of generality choose a,b to be real and normalized such that $a^2+b^2 = 1$, then $\langle x\rangle = \sqrt{2\hbar/m\omega}\,a\sqrt{1-a^2}$. Maximum of $\langle x\rangle$ then requires $d\langle x\rangle/da = 0$ or $a = +1/\sqrt{2}$ and likewise $b = +1/\sqrt{2}$. Hence $\langle x\rangle_{max} = \frac{1}{2}\sqrt{2\hbar/m\omega}$ and up to a common phase $|\alpha\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$.

(b) The state vector in Schrödinger representation evolves for $t>0$ as $|\alpha,t_o;t\rangle = U(t,t_o)|\alpha,t_o\rangle$ where $U(t,t_o) = e^{-iH(t-t_o)/\hbar}$ and $H = p^2/2m + \frac{1}{2}m\omega^2x^2$ is independent of time. Taking $t_o=0$, we have $|\alpha,t\rangle = e^{-iHt/\hbar}(|0\rangle+|1\rangle)/\sqrt{2}$, but since $\{|n\rangle\}$ are energy eigenkets with energy eigenvalues $E_n = \hbar\omega(n+\frac{1}{2})$, we write $|\alpha,t\rangle = (e^{-i\omega t/2}|0\rangle + e^{-3i\omega t/2}|1\rangle)/\sqrt{2}$ as the state vector for $t>0$ in the Schrödinger picture.

(i) In the Schrödinger picture

$$\begin{aligned}\langle\alpha,t|x|\alpha,t\rangle &= \tfrac{1}{2}(e^{i\omega t/2}\langle 0| + e^{3i\omega t/2}\langle 1|)x(e^{-i\omega t/2}|0\rangle + e^{-3i\omega t/2}|1\rangle)\\ &= \tfrac{1}{2}(\langle 0|x|0\rangle + e^{-i\omega t}\langle 0|x|1\rangle + e^{i\omega t}\langle 1|x|0\rangle + \langle 1|x|1\rangle)\\ &= \tfrac{1}{2}(e^{-i\omega t}\,\tfrac{1}{2}\sqrt{2\hbar/m\omega} + e^{i\omega t}\,\tfrac{1}{2}\sqrt{2\hbar/m\omega}) = \tfrac{1}{2}\sqrt{2\hbar/m\omega}\cos\omega t.\end{aligned}$$

(ii) In the Heisenberg picture $|\alpha\rangle = (|0\rangle+|1\rangle)/\sqrt{2}$, $x(t) = x(0)\cos\omega t + \frac{p(0)}{m\omega}\sin\omega t$, hence $\langle x\rangle = \langle\alpha|x(t)|\alpha\rangle = \frac{1}{2}\{\cos\omega t\langle 0|x(0)|0\rangle + \frac{\sin\omega t}{m\omega}\langle 0|p(0)|0\rangle + \cos\omega t\langle 0|x(0)|1\rangle + \frac{\sin\omega t}{m\omega}\langle 0|p(0)|1\rangle + \cos\omega t\langle 1|x(0)|0\rangle + \frac{\sin\omega t}{m\omega}\langle 1|p(0)|0\rangle + \cos\omega t\langle 1|x(0)|1\rangle + \frac{\sin\omega t}{m\omega}\langle 1|p(0)|1\rangle\}$. The evaluation of $\langle n|x(0)|m\rangle$ and $\langle n|p(0)|m\rangle$ have been given in problem 13, and give for $\langle\alpha|x(t)|\alpha\rangle = \langle x(t)\rangle_H = \frac{1}{2}\sqrt{2\hbar/m\omega}\cos\omega t$ as in (i).

(c) We evaluate $\langle(\Delta x)^2\rangle_t$ in the Schrödinger picture for definiteness. Here

$$\langle(\Delta x)^2\rangle = \langle x^2\rangle - \langle x\rangle^2,$$

with $\langle x\rangle = \frac{\cos\omega t}{2}\sqrt{2\hbar/m\omega}$ from (b). Again $\langle\alpha,t|x^2|\alpha,t\rangle = \frac{1}{2}(\langle 0|x^2|0\rangle + e^{-i\omega t}\langle 0|x^2|1\rangle + e^{i\omega t}\langle 1|x^2|0\rangle + \langle 1|x^2|1\rangle)$. Use again the expression for $\langle m|x^2|n\rangle$ from problem 13, i.e. $\langle 0|x^2|0\rangle = \hbar/2m\omega$, $\langle 1|x^2|1\rangle = 3\hbar/2m\omega$, $\langle 0|x^2|1\rangle = \langle 1|x^2|0\rangle = 0$. Therefore $\langle x^2\rangle = \langle\alpha,t|x^2|\alpha,t\rangle = \frac{1}{2}(2\hbar/m\omega) = \hbar/m\omega$, and

$$\langle(\Delta x)^2\rangle = \frac{\hbar}{m\omega}(1-\cos^2\omega t/2).$$

18. *This solution reprinted from the solutions manual for the revised edition.*

If we work in the Schrödinger picture, than

$$\langle 0|e^{ikx}|0\rangle = \int_{-\infty}^{\infty} \psi_0^*(x)e^{ikx}\psi_0(x)dx$$

where $\psi_0(x) = (m\omega/\pi\hbar)^{\frac{1}{4}}\exp[-m\omega x^2/2\hbar]$. First we note from problem 13 that $\langle 0|x^2|0\rangle = \hbar/2m\omega$, therefore $\exp[-k^2\langle 0|x^2|0\rangle/2] \equiv \exp[-k^2\hbar/4m\omega]$. Now explicitly $\langle 0|e^{ikx}|0\rangle = (m\omega/\pi\hbar)^{\frac{1}{2}}\int_{-\infty}^{\infty} e^{ikx-m\omega x^2/\hbar}dx$, this can be evaluated by noting that $\int_{-\infty}^{\infty} e^{-(ax^2+bx+c)}dx = \sqrt{\pi/a}\; e^{(b^2-4ac)/4a}$. Hence $\langle 0|e^{ikx}|0\rangle = (m\omega/\pi\hbar)^{\frac{1}{2}}(\hbar\pi/m\omega)^{\frac{1}{2}}\; e^{-k^2\hbar/4m\omega} = \exp[-k^2\langle 0|x^2|0\rangle/2]$.

19. *This solution reprinted from the solutions manual for the revised edition.*

(a) Take $a|\lambda\rangle = \exp[-|\lambda|^2/2]\; a\exp[\lambda a^\dagger]\;|0\rangle = \exp[-|\lambda|^2/2]\; a\sum_{n=0}^{\infty}(\lambda^n/n!)(a^\dagger)^n|0\rangle$; but we know that $(a^\dagger)^k|n\rangle = \sqrt{(n+1)(n+2)\ldots\ldots(n+k)}\,|n+k\rangle$ hence $(a^\dagger)^k|0\rangle = \sqrt{k!}\,|k\rangle$ and $a(a^\dagger)^k|0\rangle = \sqrt{k!}\; a|k\rangle = \sqrt{k}\sqrt{k!}\,|k-1\rangle$. Thus $a|\lambda\rangle = e^{-|\lambda|^2/2}\sum_{n=1}^{\infty}\lambda^n\frac{\sqrt{n}\sqrt{n!}}{n!}|n-1\rangle =$

$= e^{-|\lambda|^2/2}\sum_{n=0}^{\infty}\lambda^{n+1}(\sqrt{n+1}/\sqrt{(n+1)!})|n\rangle$. But $(n+1)!/(n+1) = n!$, hence

$$a|\lambda\rangle = e^{-|\lambda|^2/2}\sum_{n=0}^{\infty}(\lambda^{n+1}/\sqrt{n!})|n\rangle = \lambda e^{-|\lambda|^2/2}\sum_{n=0}^{\infty}(\lambda^n/\sqrt{n!})|n\rangle. \qquad (1)$$

The r.h.s. of (1) is $\lambda e^{-|\lambda|^2/2}e^{\lambda a^\dagger}|0\rangle$ by noting that $e^{\lambda a^\dagger}|0\rangle = \sum_{n=0}^{\infty}(\lambda a^\dagger)^n/n!\;|0\rangle = \sum_{n=0}^{\infty}\lambda^n|n\rangle/\sqrt{n!}$. Hence with $|\lambda\rangle = e^{-|\lambda|^2/2}e^{\lambda a^\dagger}|0\rangle$, we have indeed $a|\lambda\rangle = \lambda|\lambda\rangle$ with λ in general a complex number. For normalization we find

$$\begin{aligned}\langle\lambda|\lambda\rangle &= e^{-|\lambda|^2}\langle 0|e^{\lambda^* a}e^{\lambda a^\dagger}|0\rangle = e^{-|\lambda|^2}\langle 0|e^{\lambda^* a}\sum_{n=0}^{\infty}\lambda^n|n\rangle/\sqrt{n!} \\ &= e^{-|\lambda|^2}\langle 0|\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}(\lambda^n/\sqrt{n!})(\lambda^* a)^m/m!\;|n\rangle\,,\end{aligned} \qquad (2)$$

but $a^m|n\rangle = \sqrt{n(n-1)\ldots\ldots(n-m+1)}\,|n-m\rangle$, hence (2) contributes by orthonormality of states only when $n-m = 0$, i.e.

$$\langle\lambda|\lambda\rangle = e^{-|\lambda|^2}\langle 0| \sum_{n=0}^{\infty} \frac{\lambda^n(\lambda^*)^n}{\sqrt{n!}(n!)} \sqrt{n!}|0\rangle = e^{-|\lambda|^2}e^{+|\lambda|^2} = 1.$$

Therefore $|\lambda\rangle$ is a normalized coherent state.

(b) $\langle(\Delta x)^2\rangle = \langle x^2\rangle - \langle x\rangle^2$, $x = \sqrt{\hbar/2m\omega}(a+a^\dagger)$, where $a|\lambda\rangle = \lambda|\lambda\rangle$ and $\langle\lambda|a^\dagger = \langle\lambda|\lambda^*$. So $\langle x\rangle = \langle\lambda|x|\lambda\rangle = \sqrt{\hbar/2m\omega}(\langle\lambda|(a+a^\dagger)|\lambda\rangle) = \sqrt{\hbar/2m\omega}(\lambda+\lambda^*)$, and $\langle x\rangle^2 = (\hbar/2m\omega)(\lambda^2+\lambda^{*2}+2\lambda\lambda^*) = (\hbar/2m\omega)(\lambda+\lambda^*)^2$. Now $x^2 = xx = (\hbar/2m\omega)[a^{\dagger 2}+a^2+aa^\dagger+a^\dagger a] = (\hbar/2m\omega)[a^{\dagger 2}+a^2+2a^\dagger a+1]$, hence $\langle x^2\rangle = (\hbar/2m\omega)[\lambda^{*2}+\lambda^2+2\lambda^*\lambda+1] = (\hbar/2m\omega)[(\lambda^*+\lambda)^2+1]$. Likewise $\langle p\rangle^2 = -(\hbar m\omega/2)[\lambda^*-\lambda]^2$ and $\langle p^2\rangle = (\hbar m\omega/2)[1-(\lambda^*-\lambda)^2]$, using $p = i\sqrt{\hbar m\omega/2}(a^\dagger-a)$. Hence $\langle(\Delta p)^2\rangle = \langle p^2\rangle - \langle p\rangle^2 = \hbar m\omega/2$ and $\langle(\Delta x)^2\rangle = \hbar/2m\omega$ and $\langle(\Delta x)^2\rangle\langle(\Delta p)^2\rangle = \hbar^2/4$.

(c) Write $|\lambda\rangle = e^{-|\lambda|^2/2}\sum_{n=0}^{\infty}(\lambda^n/\sqrt{n!})|n\rangle = \sum_{n=0}^{\infty} f(n)|n\rangle$, hence $f(n) = e^{-|\lambda|^2/2}(\frac{\lambda^n}{\sqrt{n!}})$. Therefore $|f(n)|^2 = e^{-|\lambda|^2}|\lambda|^{2n}/n!$ and is a Poisson distribution

$$P(\lambda',n) = e^{-\lambda'}\lambda'^n/n!, \text{ where } \lambda' = |\lambda|^2.$$

Now $\Gamma(n+1) = n!$, hence $|f(n)|^2 = e^{-|\lambda|^2}\,|\lambda|^{2n}/\Gamma(n+1)$. The maximum value is obtained by noting that $\ln|f(n)|^2 = -|\lambda|^2 + n\ln[|\lambda|^2] - \ln\Gamma(n+1)$, and $\frac{\partial}{\partial n}\ln|f(n)|^2 = \ln|\lambda|^2 - \frac{\partial}{\partial n}\ln\Gamma(n+1) = 0$. The latter equation defines n_{max} where for large n, $\frac{\partial}{\partial n}\ln\Gamma(n+1) \sim \ln n$. Hence $n_{max} = |\lambda|^2$.

(d) The translation operator $e^{-ip\ell/\hbar}$ where p is momentum operator and ℓ just the displacement distance, can be rewritten as

$$e^{-ip\ell/\hbar} = e^{\ell\sqrt{m\omega/2\hbar}(a^\dagger-a)} = e^{\ell\sqrt{m\omega/2\hbar}a^\dagger}e^{-\ell\sqrt{m\omega/2\hbar}a}\,e^{-\frac{1}{2}(-\ell^2)(m\omega/2\hbar)[a^\dagger,a]}$$

$$= e^{-\frac{1}{2}(-\ell^2)(m\omega/2\hbar)[a^\dagger,a]}e^{\ell\sqrt{m\omega/2\hbar}a^\dagger}e^{-\ell\sqrt{m\omega/2\hbar}a} = e^{\frac{-\ell^2 m\omega}{4\hbar}}\,e^{\ell\sqrt{m\omega/2\hbar}a^\dagger}\,e^{-\ell\sqrt{m\omega/2\hbar}a}.$$

Note $e^{-\ell\sqrt{m\omega/2\hbar}a}|0\rangle = |0\rangle$ because $a|0\rangle = 0$. Hence

$$e^{-ip\ell/\hbar}|0\rangle = e^{-|\lambda|^2/2}\,e^{\lambda a^\dagger}|0\rangle, \text{ where } \lambda = \ell\sqrt{m\omega/2\hbar}$$

[We have used here the identity $e^{A+B} = e^A e^B e^{-\frac{1}{2}[A,B]}$, true for any pair of operators A and B that commute with [A,B], c.f. R. J. Glauber, Phys. Rev. 84, 399 (1951).]

20. *This solution reprinted from the solutions manual for the revised edition.*

We know that $[a_\pm, a_\pm^\dagger] = 1$ and $[a_+, a_-] = [a_+^\dagger, a_-] = 0$ (since oscillators are independent), then $[J_z, J_+] = \frac{\hbar^2}{2}(a_+^\dagger a_+ a_+^\dagger a_- - a_-^\dagger a_- a_+^\dagger a_- - a_+^\dagger a_- a_+^\dagger a_+ + a_+^\dagger a_- a_-^\dagger a_-) =$
$\frac{\hbar^2}{2}(a_+^\dagger a_+ a_+^\dagger a_- - a_-^\dagger a_- a_+^\dagger a_- - a_+^\dagger a_-(a_+ a_+^\dagger - 1) + a_- a_-^\dagger a_+^\dagger a_-) = \frac{\hbar^2}{2}(a_+^\dagger a_+ - a_-^\dagger a_- - a_+^\dagger a_+ + 1$
$+ a_- a_-^\dagger) a_+^\dagger a_- = \frac{\hbar^2}{2}([a_-, a_-^\dagger] + 1) a_+^\dagger a_- = \frac{\hbar^2}{2}(2) a_+^\dagger a_- = \hbar J_+$. Similarly $[J_z, J_-] = -\hbar J_-$.
and $\vec{J}^2 = J_x^2 + J_y^2 + J_z^2 = J_+ J_- - \hbar J_z + J_z^2$ is such that $[\vec{J}^2, J_z] = J_+ J_- J_z - \hbar J_z^2 + J_z^3$
$- J_z J_+ J_- + \hbar J_z^2 - J_z^3 = [J_+ J_-, J_z] = [J_+, J_z] J_- + J_+ [J_-, J_z] = -\hbar J_+ J_- + \hbar J_+ J_- = 0$.
Explicitly $J_+ J_- - \hbar J_z + J_z^2 = \hbar^2 a_+^\dagger a_- a_-^\dagger a_+ - \frac{\hbar^2}{2}(a_+^\dagger a_+ - a_-^\dagger a_-) + \frac{\hbar^2}{4}(a_+^\dagger a_+ a_+^\dagger a_+ -$
$a_+^\dagger a_+ a_-^\dagger a_- - a_-^\dagger a_- a_+^\dagger a_+ + a_-^\dagger a_- a_-^\dagger a_-) = \hbar^2 a_+^\dagger a_+(1 + a_-^\dagger a_-) - \frac{\hbar^2}{2}(a_+^\dagger a_+ - a_-^\dagger a_-) + \frac{\hbar^2}{4}(a_+^\dagger a_+ a_+^\dagger a_+$
$-2 a_+^\dagger a_+ a_-^\dagger a_- + a_-^\dagger a_- a_-^\dagger a_-) = \frac{\hbar^2}{2} a_+^\dagger a_+(1 + a_-^\dagger a_- + \frac{1}{2} a_+^\dagger a_+) + \frac{\hbar^2}{2} a_-^\dagger a_-(1 + \frac{1}{2} a_-^\dagger a_-) =$
$\frac{\hbar^2}{4} a_+^\dagger a_+(a_+^\dagger a_+ + a_-^\dagger a_-) + \frac{\hbar^2}{2}(a_+^\dagger a_+ + a_-^\dagger a_-)(1 + \frac{1}{2} a_-^\dagger a_-) = \frac{\hbar^2}{2} N(a_+^\dagger a_+/2 + 1 + a_-^\dagger a_-/2) =$
$\frac{\hbar^2}{2} N(N/2 + 1)$.

21. Starting with (2.5.17a), namely $g(x,t) = \exp(-t^2 + 2tx)$, carry out the suggested integral

$$\begin{aligned} \int_{-\infty}^{\infty} g(x,t) g(x,s) e^{-x^2} dx &= = \int_{-\infty}^{\infty} e^{2st-(t+s)^2+2x(t+s)-x^2} dx \\ &= e^{2st} \int_{-\infty}^{\infty} e^{-[x-(t+s)]^2} dx = \pi^{1/2} e^{2st} \\ \text{i.e.} \qquad \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \left[\int_{-\infty}^{\infty} H_n(x) H_m(x) e^{-x^2} dx \right] \frac{1}{(n!)^2} t^n s^m &= \pi^{1/2} \sum_{n=0}^{\infty} \frac{2^n}{n!} t^n s^n \end{aligned}$$

The sum on the right only includes terms where t and s have the same power, so the normalization integral on the left must be zero if $n \neq m$. When $n = m$ this gives

$$\begin{aligned} \left[\int_{-\infty}^{\infty} H_n(x) H_n(x) e^{-x^2} dx \right] \frac{1}{(n!)^2} &= \pi^{1/2} \frac{2^n}{n!} \\ \text{or} \qquad \int_{-\infty}^{\infty} H_n^2(x) e^{-x^2} dx &= \pi^{1/2} 2^n n! \end{aligned}$$

which is (2.5.29). In order to normalize the wave function (2.5.28), we compute

$$\int_{-\infty}^{\infty} u_n^*(x) u_n(x) dx = |c_n|^2 \int_{-\infty}^{\infty} H_n^2 \left(x \sqrt{\frac{m\omega}{\hbar}} \right) e^{-m\omega x^2/\hbar} dx = |c_n|^2 \sqrt{\frac{\hbar}{m\omega}} \pi^{1/2} 2^n n! = 1$$

so that $c_n = (m\omega/\pi\hbar)^{1/4} (2^n n!)^{-1/2}$, taking c_n to be real. Compare to (B.4.3).

22. *This solution reprinted from the solutions manual for the revised edition.*
In the region $x>0$, ψ obeys the same differential equation as the two-sided harmonic oscillator; however, the only acceptable solutions are those that vanish at the origin. Therefore, the eigenvalues are those of the ordinary harmonic oscillator belonging to wave functions of odd parity. Now the parity of the SHO wave functions alternates with increasing n, starting with an even-parity ground state. Hence,

$$E = (4n+3)\hbar\omega/2 = (4n+3)\hbar\sqrt{k/m}/2 \quad \text{with } n=0,1,2,\ldots\ldots$$

(a) Ground state energy $= 3\hbar\sqrt{k/m}/2$ for $n=0$.

(b) From (2.3.31), $\langle x'|1\rangle = \psi_1 = \frac{1}{\sqrt{2}\, x_0}(x'-x_0^2 \frac{d}{dx'})(1/\pi^{1/4}\sqrt{x_0})\exp[-\frac{1}{2}(x'/x_0)^2]$

(where $x_0 \equiv \sqrt{\hbar/m\omega}$). Hence $\psi_1 = (2/\sqrt{2}\, x_0^{3/2}\pi^{1/4})x'\exp[-\frac{1}{2}(x'/x_0)^2]$ and $\langle x^2\rangle =$

$(2/x_0^3\pi^{1/2})\int_0^\infty x'^4\exp[-(x'/x_0)^2]dx' = (2\Gamma(5/2)/x_0^3\pi^{1/2}2)x_0^5 = \frac{3}{4}x_0^2 = \frac{3\hbar}{4m\omega}$.

23. *This solution reprinted from the solutions manual for the revised edition.*
The solution to the particle trapped between the rigid wall (one dimension) is

$$\psi_n(x) = A_n \sin(n\pi x/L),\ n = 1,2,3,\ldots\ldots \tag{1}$$

Now, $P(x,t)dx = |\psi(x,t)|^2 dx$ is the probability that the particle described by the wave function $\psi(x,t)$ may be found between x and $x+dx$, therefore in order for the particle to be exactly at $x = L/2$ for $t=0$, $\psi(x,0) = a\delta(x-L/2)$ where $a = 1$ via normalization.

The eigenvalues corresponding to $\psi_n(x)$ are $E_n = n^2\pi^2\hbar^2/2mL^2$, $n = 1,2,3,\ldots.$ and by the expansion postulate

$$\psi(x,t) = \sum_n c_n e^{-iE_n t/\hbar}\psi_n(x), \tag{2}$$

the transition amplitude c_n is then given by

$$c_n = \int_0^L \psi_n^*(x)\psi(x,0)dx = \int_0^L A_n \sin(n\pi x/L)\delta(x-L/2)dx = A_n\sin(\tfrac{n\pi}{2}). \tag{3}$$

Hence $c_n = (-1)^{\frac{n-1}{2}}A_n$ (for n odd) and $c_n = 0$ (for n even). Therefore (relative) probabilities are $P = |c_n|^2 = |A_n|^2\delta_{n,odd}$, and (2) reads (using (3))

$$\psi(x,t) = {}_n\Sigma_{odd}\, A_n^2 \exp[-i(n^2\pi^2/L^2)\hbar t/2m]\,\sin(n\pi x/L)\,(-1)^{\frac{n-1}{2}} \tag{4}$$

where in fact for normalized $\psi_n(x)$ in (1), $A_n = \sqrt{2/L}$ (independent of n).

24. *This solution reprinted from the solutions manual for the revised edition.*

Our Schrödinger equation is $(-\hbar^2/2m)d^2\psi/dx^2 - v_o\delta(x)\psi = -E\psi$ (i.e. $E>0$ hence $-E<0$ is bound state energy). Integrate above equation from $-\epsilon$ to $+\epsilon$, and let $\epsilon \to 0$, we have

$$\frac{-\hbar^2}{2m}\left(d\psi/dx\big|_{x=\epsilon} - d\psi/dx\big|_{x=-\epsilon}\right) - v_o\psi(0) = 0. \qquad (1)$$

For $x\neq 0$,

$$\frac{-\hbar^2}{2m}d^2\psi/dx^2 = -E\psi \qquad (2)$$

with bound solutions $\psi(x>0) = A\exp[-(2mE/\hbar^2)^{1/2}x]$ and $\psi(x<0) = A\exp[+(2mE/\hbar^2)^{1/2}x]$. Substitute these solutions into (1), we have $(-\hbar^2/2m)[-(2mE/\hbar^2)^{1/2} - (2mE/\hbar^2)^{1/2}] - v_o = 0$ or $E = v_o^2 m/2\hbar^2$. This is an unique solution, no excited bound states are expected.

25. *This solution reprinted from the solutions manual for the revised edition.*

Using the result of problem 22, where $2mE/\hbar^2 = \lambda^2 m^2/\hbar^4$ in our notation, we have $\psi(x,t=0) = A\exp[-m\lambda|x|/\hbar^2]$. The normalization is then $2A^2\int_0^\infty \exp[-2m\lambda x/\hbar^2]dx = 1$ or $2A^2[\hbar^2/2m\lambda] = 1$ and hence $A = (m\lambda/\hbar^2)^{1/2}$. From (2.5.7) and (2.5.16), we have

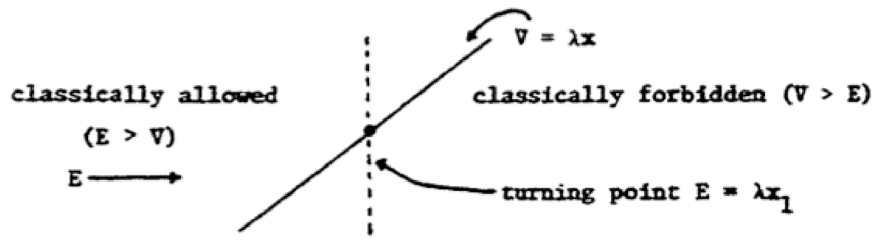
$$\psi(x,t>0) = \int_{-\infty}^{\infty} dx'\psi(x',0)K(x,x';t)$$

$$= (m\lambda/\hbar^2)^{1/2}(m/2\pi i\hbar t)^{1/2}\int_{-\infty}^{\infty} \exp[-m\lambda|x'|/\hbar^2]\exp[i(x-x')^2 m/2\hbar t]dx'$$

where we have used $\psi(x',0) = (m\lambda/\hbar^2)^{1/2}\exp[-m\lambda|x'|/\hbar^2]$.

26. *This solution reprinted from the solutions manual for the revised edition.*

(a)

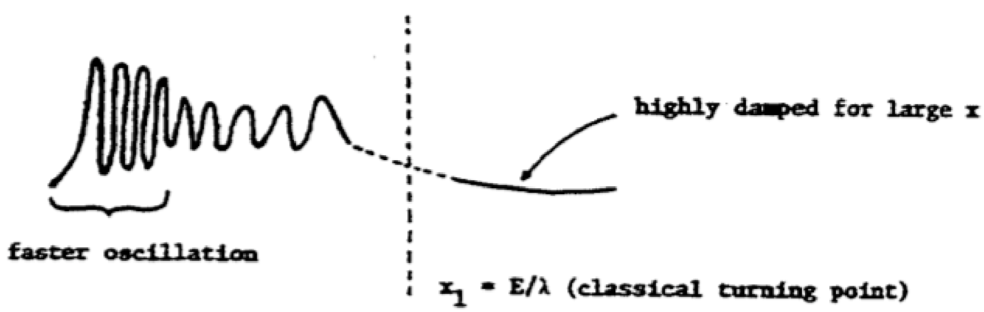


The energy spectrum is continuous. Aside from normalization, the wave functions are:-

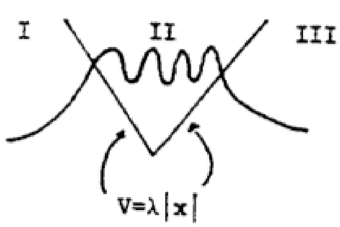
Classically allowed region: $\dfrac{1}{(E-\lambda x)^{\frac{1}{4}}}\, e^{\pm(i/\hbar)\int^x \sqrt{2m(E-\lambda x')}\,dx'}$

classically forbidden region: $\dfrac{1}{(\lambda x-E)^{\frac{1}{4}}}\, e^{-(1/\hbar)\int^x \sqrt{2m(\lambda x'-E)}\,dx'}$

These expressions are not valid near $x=x_1 = E/\lambda$ (classical turning point). The sketch of energy eigenfunction specified by E looks as follow



(b) The most important change is that the energy spectrum is now discrete, and the wave functions are:



Region I: $\dfrac{1}{(\lambda x-E)^{\frac{1}{4}}}\, e^{+(1/\hbar)\int^x\sqrt{2m(\lambda x'-E)}\,dx'}$

Region II: $\dfrac{1}{(E-\lambda x)^{\frac{1}{4}}} \left\{\begin{matrix}\sin\\ \cos\end{matrix}\right\} \left(\dfrac{1}{\hbar}\int^x\sqrt{2m(E-\lambda x')}\,dx'\right)$

27. *Note: This was Problem 36 in Chapter Five in the Revised Edition. It was moved to this chapter because "density of states" is explicitly worked out now in this chapter. It seems, though, that I should have reworded the problem a bit.* Refer back to the discussion in Section 2.5. The wave function is

$$u_E(\mathbf{x}) = \frac{1}{L}e^{i\mathbf{k}\cdot\mathbf{x}} \qquad \text{where} \qquad k_x = \frac{2\pi}{L}n_x \qquad \text{and} \qquad k_y = \frac{2\pi}{L}n_y$$

and n_x and n_y are integers, with $\mathbf{p} = \hbar\mathbf{k}$. The energy is

$$\begin{aligned} E &= \frac{\mathbf{p}^2}{2m} = \frac{\hbar^2}{2m}(k_x^2 + k_y^2) = \frac{2\pi^2\hbar^2}{mL^2}(n_x^2 + n_y^2) = \frac{2\pi^2\hbar^2}{mL^2}\mathbf{n}^2 \\ \text{so} \qquad dE &= \frac{4\pi^2\hbar^2}{mL^2}ndn \end{aligned}$$

The number of states with $|\mathbf{n}|$ between n and $n + dn$, and ϕ and $\phi + d\phi$, is

$$dN = ndnd\phi = m\left(\frac{L}{2\pi\hbar}\right)^2 dEd\phi$$

so the density of states is simply $m(L/2\pi\hbar)^2$. Remarkably, this result independent of energy.

28. *This solution reprinted from the solutions manual for the revised edition.*

The electron is confined to the interior of a hollow cylindrical shell, where using cylindrical coordinates (ρ, θ, z) the boundary conditions are:-

$$\psi(\rho_a, \theta, z) = \psi(\rho_b, \theta, z) = \psi(\rho, \theta, 0) = \psi(\rho, \theta, L) = 0$$

(a) Inside the cylindrical shell, the Schrödinger equation in cylindrical coordinates reads

$$-\frac{\hbar^2}{2m_e}\left[\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial\psi}{\partial\rho}\right) + \frac{1}{\rho^2}\frac{\partial^2\psi}{\partial\theta^2} + \frac{\partial^2\psi}{\partial z^2}\right] = E\psi = -|E|\psi \text{ (bound states).}$$

Using the method of separation of variables $\psi = R(\rho)Q(\theta)Z(z)$ and

$$\psi(\rho, \theta, z) = (AJ_m(\kappa\rho) + BN_m(\kappa\rho))(Ce^{im\theta} + De^{-im\theta})(Ee^{\alpha z} + Fe^{-\alpha z})$$

are the energy eigenfunctions where m is an integer (to preserve single-valued ψ), $\kappa \equiv \sqrt{\alpha^2 - 2m_e|E|/\hbar^2}$, and with $x = \kappa\rho$, $R(x) = AJ_m(x) + BN_m(x)$ satisfies Bessel equation

$$\frac{d^2R}{dx^2} + \frac{1}{x}\frac{dR}{dx} + (1 - m^2/x^2)R = 0$$

Impose next the boundary conditions; $\psi(\rho, \theta, 0) = 0$ implies $E = -F$, hence $Z(z) = E(e^{\alpha z} - e^{-\alpha z}) = 2E\sinh\alpha z$. Now ψ will not vanish at $z = L$ unless α is complex, so write $\alpha = ik$ and $Z(z) = 2Ei\sin kz$, thence $Z(L) = 0$ if and only if $kL = \ell\pi$ (ℓ is non zero integer). Since $\alpha^2 < 0$, κ is also imaginary. Vanishing of solution at $\rho = \rho_a$ and $\rho = \rho_b$ leads to

$$AJ_m(\kappa\rho_a) + BN_m(\kappa\rho_a) = 0, \quad AJ_m(\kappa\rho_b) + BN_m(\kappa\rho_b) = 0$$

and eliminating A/B we have $J_m(\kappa\rho_b)N_m(\kappa\rho_a) - N_m(\kappa\rho_b)J_m(\kappa\rho_a) = 0$. Now $\alpha^2 = -k^2 = -\ell^2\pi^2/L^2 = \kappa^2 + 2m|E|/\hbar^2$, therefore $E = (\hbar^2/2m_e)[\kappa^2 + \ell^2\pi^2/L^2]$. If we write $\kappa = k_{mn}$, the n^{th} root of the transcendental equation $J_m(k_{mn}\rho_b)N_m(k_{mn}\rho_a) - N_m(k_{mn}\rho_b)J_m(k_{mn}\rho_a) = 0$, than the energy can be written as

$$E_{\ell mn} = (\hbar^2/2m_e)[k_{mn}^2 + (\ell\pi/L)^2] \quad \left\{ \begin{array}{l} \ell = 1,2,3,\ldots. \\ m = 0,1,2\ldots.. \end{array} \right.$$

(b) In the field free region between $\rho_a<\rho<\rho_b$ of cylindrical shell, we can have case (a) above with $\vec{A} = \phi = 0$ and $(-i\hbar\vec{\nabla})^2/2m\ \psi = E\psi$, or the gauge-invariant form (with $\phi = 0$) $\frac{1}{2m}(\frac{\hbar}{i}\vec{\nabla} - \frac{e}{c}\vec{A})^2\psi' = E\psi'$, $\psi' = e^{+ief/\hbar c}\psi$ and $\vec{A} = \vec{\nabla}f$ (with $\vec{\nabla}\times\vec{A} = \vec{B} = 0$). So to find solution with field coupling terms ($\vec{A}\neq 0$), we find the solution ψ with $\vec{A} = 0$ and then multiply by phase factor $e^{+ief/\hbar c}$, where $f(\vec{r},t) = \int^r d\vec{r}'\cdot\vec{A}(\vec{r}',t)$. Let us choose a gauge in which $A_z = A_\rho = 0$, $A_\theta = (G/\rho)\hat{\theta}$ with G a constant. Then $d\vec{r}' = \rho' d\theta'\hat{\theta}$ and $f(\vec{r},t) = \int_0^\theta \rho' d\theta' G/\rho' = G\theta$, and $\psi' = e^{ie\theta G/\hbar c}\psi$. Now G can be determined using Stoke's theorem that $\int(\vec{\nabla}\times\vec{A})\cdot d\vec{S} = \int_C \vec{A}\cdot d\vec{\ell}$ where C is a closed contour inside cylindrical shell. We have $B\rho_a^2\pi = 2\pi G$, and hence

$$\psi' = e^{ie\theta B\rho_a^2/2\hbar c}\psi = e^{i\beta\theta}\psi \tag{1}$$

It is evident that the solution ψ' (by symmetry) is of form $\psi' = R(\rho)e^{i\beta\theta}Q(\theta)Z(z)$. Except for $\tilde{Q}(\theta) = e^{i\beta\theta}Q(\theta)$, the forms of $R(\rho)$ and $Z(z)$ are the same as in part (a) of problem but with a different separation constant for $R(\rho)$. Now $Q''(\theta) + m^2Q(\theta) = 0$, hence $\tilde{Q}''(\theta) - 2i\beta\tilde{Q}' + (m^2-\beta^2)\tilde{Q} = 0$. The separated equation for R and $\tilde{Q}$ reads

$$\frac{\rho}{R}\frac{d}{d\rho}(\rho dR/d\rho) + \frac{1}{\tilde{Q}}\frac{d^2\tilde{Q}}{d\theta^2} - \frac{2i\beta}{\tilde{Q}}\frac{d\tilde{Q}}{d\theta} + \rho^2(k^2+2mE/\hbar^2) = 0.$$

Again as in part (a) we have $\kappa^2 = 2m_eE/\hbar^2 - k^2 = 2m_eE/\hbar^2 - \ell^2\pi^2/L^2$ or writing $\kappa = k_{\gamma n}$ with $\gamma^2 = m^2-\beta^2$, we have

$$E = \frac{\hbar^2}{2m_e}[k_{\gamma n}^2 + (\ell\pi/L)^2]$$

where $k_{\gamma n}$ is the n^{th} root of transcendental equation

$$0 = J_\gamma(k_{\gamma n}\rho_b)N_\gamma(k_{\gamma n}\rho_a) - N_\gamma(k_{\gamma n}\rho_b)J_\gamma(k_{\gamma n}\rho_a)$$

($N_\gamma \to J_{-\gamma}$ for γ not an integer). Note because γ depends on β (hence B), the energy eigenvalues are influenced by $\vec{B}$ even though the electron never "touches" the magnetic field.

(c) The ground state of B=0 case is

$$E_{101} = \frac{\hbar^2}{2m_e}[k_{01}^2 + \pi^2/L^2]$$

with $J_0(k_{01}\rho_b)N_0(k_{01}\rho_a) = N_0(k_{01}\rho_b)J_0(k_{01}\rho_a)$, while for $B \neq 0$

$$E_{ground} = \frac{\hbar^2}{2m_e}[k_{\gamma n}^2 + \pi^2/L^2]$$

where γ is not necessarily an integer. However if we require the ground state energy to be unchanged in the presence of B, then

$$\gamma^2 = m^2 - \beta^2 = 0, \quad m \text{ integer, and}$$

$$\pm m = eB\rho_a^2/2\hbar c \to \pi\rho_a^2 B = 2\pi N\hbar c/e,$$

where $N = \pm m = 0, \pm 1, \pm 2, \pm 3, \ldots\ldots$

29. *This solution reprinted from the solutions manual for the revised edition.*

$\psi \propto \exp[iS(x,t)/\hbar]$ and $H = i\hbar\partial\psi/\partial t$, where $H = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + V(x)$. Thus $-(\hbar^2/2m)[\frac{\partial}{\partial x}(\frac{i\partial S}{\hbar\partial x}\psi)] + V(x)\psi = i\hbar[\frac{i\partial S}{\hbar\partial t}\psi]$ which simplifies to

$$-\frac{\hbar^2}{2m}[\frac{i\partial^2 S}{\hbar\partial x^2}\psi + (\frac{i\partial S}{\hbar\partial x})(\frac{i\partial S}{\hbar\partial x})\psi] + V(x)\psi = i\hbar[\frac{i\partial S}{\hbar\partial t}\psi]. \quad (1)$$

If $\lim \hbar \to 0$ in some sense, (1) reduces to $\frac{1}{2m}(\partial S/\partial x)^2 + V(x) = -\partial S/\partial t$ and this is the Hamilton-Jacobi equation. For $V(x) = 0$ we have $\frac{1}{2m}(\partial S/\partial x)^2 = -\partial S/\partial t$ and seek a solution of separable form $S(x,t) = X(x) + T(t)$. Then $\frac{1}{2m}(\partial X/\partial x)^2 = -\partial T/\partial t = \alpha$(a constant), so $T(t) = -\alpha t$ + const and $X(x) = \sqrt{2\alpha m}\, x$ + const. Hence $\psi(x,t) \propto \exp[i(\sqrt{2\alpha m}\, x - \alpha t)/\hbar]$, a plane wave wave function. Our procedure works because S is linearly dependent on x (i.e. $\partial^2 S/\partial x^2 = 0$).

30. *This solution reprinted from the solutions manual for the revised edition.*

From (2.4.16), the flux $\vec{j}(\vec{x},t) = (-i\hbar/2m)[\psi^*\vec{\nabla}\psi - (\vec{\nabla}\psi^*)\psi]$, and the wave function for a hydrogen atom is $\psi = R_{n\ell}(r)Y_{\ell m_\ell}(\theta,\phi)$ with $Y_{\ell m_\ell}(\theta,\phi) = C_{\ell m_\ell} P_\ell^{m_\ell}(\cos\theta)e^{im_\ell\phi}$.

In spherical coordinates:

$$\vec{\nabla} = \hat{e}_r\frac{\partial}{\partial r} + \hat{e}_\theta\frac{1}{r}\frac{\partial}{\partial\theta} + \hat{e}_\phi\frac{1}{r\sin\theta}\frac{\partial}{\partial\phi} \;,$$

hence $\vec{j} = (m_\ell/mr\sin\theta)|\psi|^2\hat{e}_\phi$, and thus $\vec{j}$ vanishes for $m_\ell = 0$. For $m_\ell \neq 0$, $j>0$ if $m_\ell>0$ and $\vec{j}<0$ if $m_\ell<0$, where $\vec{j}>0$ means that $\vec{j}$ has the same direction as $\hat{e}_\phi$ (i.e. in the direction of increasing ϕ) while $\vec{j}<0$ means that $\vec{j}$ has the opposite direction to $\hat{e}_\phi$(i.e. in the direction of decreasing ϕ).

31. *This solution reprinted from the solutions manual for the revised edition.*

From (2.5.15) we have $K(x'',t;\, x',t_o) = \frac{1}{2\pi\hbar}\int_{-\infty}^{\infty} dp'\exp[\frac{ip'(x''-x')}{\hbar} - \frac{ip'^2(t-t_o)}{2m\hbar}]$.

The exponent can be written after completion of the square as the following

$$\frac{-i(t-t_o)}{2m\hbar}\{p'^2 - \frac{p'(x''-x')2m}{(t-t_o)}\} = \frac{-i(t-t_o)}{2m\hbar}[p' - \frac{m(x''-x')}{(t-t_o)}]^2 + \frac{im(x''-x')^2}{2\hbar(t-t_o)}.$$

Then with $\xi' = p'-m(x''-x')/(t-t_o)$, $d\xi' = dp'$, we have

$$K(x'',t;x',t_o) = \frac{1}{2\pi\hbar}\exp[\frac{im(x''-x')^2}{2\hbar(t-t_o)}]\int_{-\infty}^{\infty} d\xi\exp[-i(t-t_o)\xi^2/2m\hbar]$$

$$= \frac{1}{2\pi\hbar}\exp[\frac{im(x''-x')^2}{2\hbar(t-t_o)}][\frac{2\pi m\hbar}{i(t-t_o)}]^{\frac{1}{2}} = \{\frac{m}{2\pi\hbar i(t-t_o)}\}^{\frac{1}{2}}\exp[\frac{im(x''-x')^2}{2\hbar(t-t_o)}]$$

hence we have established (2.5.16). The three dimensional generalization is evidently

$$K(\vec{x}'',t;\vec{x}',t_o) = \{\frac{m}{2\pi\hbar i(t-t_o)}\}^{\frac{3}{2}}\exp[im(\vec{x}''-\vec{x}')^2/2\hbar(t-t_o)]$$

32. *This solution reprinted from the solutions manual for the revised edition.*

$Z = \int d^3x'\ K(\vec{x}',t;\ \vec{x}',0)\big|_{\beta = it/\hbar} = \sum_{a'} \exp[-\beta E_{a'}]$ from (2.5.22). The probability $P(E_{a'}) = \exp[-\beta E_{a'}]/Z$, hence the ground state energy (c.f. (1.4.6))

$$U = \sum_{a'} E_{a'} P(E_{a'}) = \sum_{a'} E_{a'} \exp[-\beta E_{a'}]/Z = -\frac{1}{Z}\frac{\partial Z}{\partial \beta}\ .$$

For a particle in a one dimensional box (with periodic boundary condition), $da' = \frac{L}{2\pi}dk = \frac{L}{2\pi\hbar}dp$, hence $Z = \sum_{a'} \exp[-\beta E_{a'}] \to (L/2\pi\hbar)\int_{-\infty}^{\infty} \exp[-\frac{p^2\beta}{2m}]dp = (\frac{2L}{2\pi\hbar})\int_0^{\infty} e^{-p^2\beta/2m}dp$, $= (L/\pi\hbar)\int_0^{\infty} e^{-p^2\beta/2m}dp$. Let $u^2 = p^2\beta/2m$, $p = \sqrt{2m/\beta}\ u$, $dp = \sqrt{2m/\beta}\ du$, then $Z = (L/\pi\hbar)\sqrt{2m/\beta}\int_0^{\infty}\exp[-u^2]du = (L/\pi\hbar)\sqrt{2m/\beta}\ \sqrt{\pi}/2 = (L/\hbar)\sqrt{m/2\pi\beta}$. The ground state energy for a particle in a one dimensional "box" is

$$-\frac{1}{Z}\frac{dZ}{d\beta} = -\frac{1}{(L/\hbar)(m/2\pi\beta)^{1/2}}(L/\hbar)\sqrt{m/2\pi}\ (-\tfrac{1}{2})\beta^{-3/2} = 1/2\beta.$$

(Note in thermodynamics $\beta = 1/kT$, hence ground state energy $= kT/2$, an entirely reasonable result).

33. *This solution reprinted from the solutions manual for the revised edition.*

Analogous to (2.5.26) for $K(\vec{x}'',t;\vec{x}',t_0)$, we expect

$$K(\vec{p}'',t;\vec{p}',t_0) = \sum_{a'} \langle\vec{p}''|a'\rangle\langle a'|\vec{p}'\rangle \exp[-iE_{a'}(t-t_0)/\hbar]$$

$$= \sum_{a'} \langle\vec{p}''|\exp[-iHt/\hbar]|a'\rangle\langle a'|\exp[+iHt_0/\hbar]|\vec{p}'\rangle = \langle\vec{p}'',t|\vec{p}',t_0\rangle.$$

For a free particle, $H = p^2/2m$, hence

$$\langle\vec{p}'',t|\vec{p}',t_0\rangle = \sum_{a'} \langle\vec{p}''|\exp[-\frac{ip^2t}{2m\hbar}]|a'\rangle\langle a'|\exp[ip^2t_0/2m\hbar]|\vec{p}'\rangle.$$

34. *This solution reprinted from the solutions manual for the revised edition.*

(a) The classical Lagrangian for a SHO is $L = \frac{1}{2}m\dot{x}^2 - \frac{1}{2}m\omega^2x^2$. The classical action is $S(t,t_o) = \int_{t_o}^{t} dt\,(\frac{1}{2}m\dot{x}^2 - \frac{1}{2}m\omega^2x^2)$. For a finite time interval $\Delta t = t_n - t_{n-1}$ and $\Delta x = x_n - x_{n-1}$, we have $S(n,n-1) = \Delta t\cdot\frac{m}{2}\{(x_n-x_{n-1})^2/\Delta t^2 - \omega^2\left(\frac{x_n+x_{n-1}}{2}\right)^2\}$
$= \frac{m}{2}\{\frac{(x_n-x_{n-1})^2}{\Delta t} - \omega^2(x_n - \frac{1}{2}\Delta x)^2\Delta t\}$. Hence

$$S(n,n-1) \cong \frac{m}{2}\{\frac{(x_n-x_{n-1})^2}{\Delta t} - \omega^2x_n^2\Delta t\} \tag{1}$$

where terms of order $\Delta x\Delta t$ and $(\Delta x)^2\Delta t$ have been neglected.

(b) The transition amplitude obtained from (1) is

$$\langle x_nt_n|x_{n-1}t_{n-1}\rangle = \sqrt{m/2\pi i\hbar\Delta t}\,\exp[iS(n,n-1)/\hbar]$$
$$= \sqrt{m/2\pi i\hbar\Delta t}\,\exp\{\frac{im}{2\hbar}[\frac{(x_n-x_{n-1})^2}{\Delta t} - \omega^2x_n^2\Delta t]\}\ . \tag{2}$$

From (2.5.18) and (2.5.26)

$$\langle x_nt_n|x_{n-1}t_{n-1}\rangle = K(x_n,t_n;x_{n-1},t_{n-1})$$
$$= \sqrt{m\omega/2\pi i\hbar\sin(\omega\Delta t)}\exp[\{im\omega/2\hbar\sin(\omega\Delta t)\}\{(x_n^2+x_{n-1}^2)\cos(\omega\Delta t) - 2x_nx_{n-1}\}]. \tag{3}$$

Up to order $(\Delta t)^2$, we have $\sin(\omega\Delta t) \cong \omega\Delta t$, $(x_n^2+x_{n-1}^2)\cos(\omega\Delta t) - 2x_nx_{n-1} \cong$

$(x_n-x_{n-1})^2-[(x_n^2+x_{n-1}^2)/2]\omega^2\Delta t^2 \cong (x_n-x_{n-1})^2 - \omega^2x_n^2\Delta t^2$, where we have neglected also a term of order $\Delta x\Delta t^2$ because $(x_n^2+x_{n-1}^2)/2 = x_n^2 - [(x_n+x_{n-1})/2]\Delta x$ implies $(x_n^2+x_{n-1}^2)\Delta t^2/2 \cong x_n^2\Delta t^2$. Thus (3) becomes (up to order $(\Delta t)^2$)

$$\langle x_nt_n|x_{n-1}t_{n-1}\rangle \cong \sqrt{m/2\pi i\hbar\Delta t}\,\exp\{\frac{im}{2\hbar}[(x_n-x_{n-1})^2/\Delta t - \omega^2x_n^2\Delta t]$$

in agreement with (2).

35. *This solution reprinted from the solutions manual for the revised edition.*

The Schwinger action principle states that the following condition determines the transformation function $\langle x_2t_2|x_1t_1\rangle$ in terms of a given quantum mechanical Lagrangian L

$$\delta\langle x_2t_2|x_1t_1\rangle = (i/\hbar)\langle x_2t_2|\delta\int_{t_1}^{t_2} Ldt|x_1t_1\rangle.$$

To obtain $\langle x_2t_2|x_1t_1\rangle$, let $\delta\langle x_2t_2|x_1t_1\rangle = (i/\hbar)\langle x_2t_2|\delta W_{21}|x_1t_1\rangle$ where W_{21} is action in going from initial state x_1t_1 to final state x_2t_2. Also, let $\delta W_{21} = \delta\omega_{21}$ where $\delta\omega_{21}$ is the well-ordered form (c.f. Finkelstein (1973), p.164) of δW_{21}. Then $\delta\langle x_2t_2|x_1t_1\rangle = \frac{i}{\hbar}\langle x_2t_2|\ \delta\omega_{21}|x_1t_1\rangle = \frac{i}{\hbar}\ \delta\omega'_{21}\langle x_2t_2|x_1t_1\rangle$ and thus $\delta\ln\langle x_2t_2|x_1t_1\rangle = \frac{i}{\hbar}\delta\omega'_{21}$ or

$$\langle x_2t_2|x_1t_1\rangle = \exp[\tfrac{i}{\hbar}\omega'_{21}]. \tag{1}$$

The corresponding Feynman expression for $\langle x_2t_2|x_1t_1\rangle$ [c.f. Finkelstein (1973), p.144] is

$$\langle x_2t_2|x_1t_1\rangle = \frac{1}{N}\sum_{\text{paths}} \exp[(i/\hbar)S_{21}]. \tag{2}$$

The classical limit of (2) is such that as $\hbar/S \to$ small, the probability amplitude $\langle x_2t_2|x_1t_1\rangle$ will be important only for those varied paths which lie in a narrow tube between x_1t_1 and x_2t_2 enclosing the classical path. On the other hand, to describe the classical limit for (1) (which has a well-ordered exponent instead of a sum over paths), is to consider first the operator Hamilton-Jacobi equation (c.f. Finkelstein (1973), p.166)

$$H(\frac{\partial\omega}{\partial x}.....x....) + \partial\omega/\partial t = 0. \tag{3}$$

Since ω'_{21} satisfies (3), which arises from a variation of the final state (and is similar to the Schrödinger picture), it is seen that the correspondence limit of ω'_{21} is S, i.e. the probability amplitude (1) approaches the consideration of all possible paths as in the Feynman path integral case (2). Thus in the classical limit, (1) and (2) become equal provided they both are modulated by the factor 1/N (N = total number of individual steps in going from $x_1t_1 \to x_2t_2$).

36. *This solution reprinted from the solutions manual for the revised edition.*

Take the plane wave $\psi(\vec{r},t) = e^{i(\vec{k}\cdot\vec{r}-\omega t)} = e^{i(\vec{p}\cdot\vec{r}/\hbar - \omega t)} = e^{i\phi(\vec{r},t)}$, where $E_k = p^2/2m = \hbar^2k^2/2m$. Also $\vec{r} = \vec{v}t$, hence $\phi(\vec{r},t) = \vec{p}\cdot\vec{r}/\hbar - \omega r/v$. Let us examine again Fig. 2.5 of text, like before the gravity-induced phase change associated with AB and also with CD are present, but the effects cancel as we compare the two alternative paths. Hence we are concerned with the phase changes $\Delta\phi_{BD}$ and $\Delta\phi_{AC}$, and their difference. Because we are concerned with a time-independent potential the sum of the kinetic energy and potential energy is constant, i.e. $\vec{p}^2/2m + mgz = E$, but the difference in height between level BD and level AC implies a slight difference in $\vec{p}$, or $\lambda\!\!\!^{-}$. As a result there is an accumulation of phase differences due to $\lambda\!\!\!^{-}$ difference. Along AC, $\Delta\phi_{AC} = p_{AC}\ell_1/\hbar - \omega\ell_1/v_{AC}$ while along BD $\Delta\phi_{BD} = p_{BD}\ell_1/\hbar - \omega\ell_1/v_{BD}$, where $p_{AC} = mv_{AC}$, $p_{BD} = mv_{BD}$, and [from $p^2/2m + mgz = \text{const}$] we have

$$v_{BD} = (2/m)^{\frac{1}{2}}[mv_{AC}^2/2 - mg\ell_2\sin\delta]^{\frac{1}{2}}.$$

The accumulation of phase difference is $\Delta\phi = |\Delta\phi_{BD} - \Delta\phi_{AC}| = |\frac{m}{\hbar}\ell_1(v_{BD}-v_{AC})| = (m^2g\ell_1\ell_2\lambda\!\!\!^{-}\sin\delta)/\hbar^2$ where $p_{AC} = mv_{AC} = \hbar/\lambda\!\!\!^{-}$.

37. *This solution reprinted from the solutions manual for the revised edition.*

(a) To verify (2.6.25), i.e. $[\Pi_i,\Pi_j] = (i\hbar e/c)\epsilon_{ijk}B_k$, we note that $\Pi_i = p_i - eA_i/c$ and $\Pi_j = p_j - eA_j/c$ while $p_{i,j} = \frac{\hbar}{i}\frac{\partial}{\partial x_{i,j}}$ and $\vec{B} = \vec{\nabla}\times\vec{A}$. Explicit calculation of $[\Pi_i,\Pi_j] = [p_i - eA_i/c,\ p_j - eA_j/c] = [p_i,p_j] + [p_i,-eA_j/c] + [-eA_i/c,p_j] + [-eA_i/c, -eA_j/c] = -e[p_i,A_j/c] - e[A_i/c,p_j]$. From problem 29 of Chapter 1 we have $[p_i, F(\vec{x})] = -i\hbar\partial F/\partial x_i$, hence setting $F \equiv A(\vec{x})$ we have $[\Pi_i,\Pi_j] = (i\hbar e/c)\epsilon_{ijk}B_k$.

To verify (2.6.27), with $H = (\vec{p}-e\vec{A}/c)^2 + e\phi$, let us note that from (2.6.22) we have $d\vec{x}/dt = (\vec{p}-e\vec{A}/c)/m$, hence $d^2\vec{x}/dt^2 = \frac{1}{m}(d\vec{p}/dt - \frac{e}{c}d\vec{A}/dt)$. Now $d\vec{p}/dt = \frac{1}{i\hbar}[\vec{p}, H]$, hence explicitly $d\vec{p}/dt = -e\vec{\nabla}\phi + \frac{e}{c}\vec{\nabla}(\frac{d\vec{x}}{dt}\cdot\vec{A})$ and $d\vec{A}/dt = \partial\vec{A}/\partial t + \frac{1}{i\hbar}[\vec{A},H] = \vec{\nabla}\cdot\vec{A}\ d\vec{x}/dt + \partial\vec{A}/\partial t$. Thus $\frac{d}{dt}(\vec{p}-e\vec{A}/c) = -e\vec{\nabla}\phi + \frac{e}{c}\vec{\nabla}(\frac{d\vec{x}}{dt}\cdot\vec{A}) - \frac{e}{c}\vec{\nabla}\cdot\vec{A}\ d\vec{x}/dt - \frac{e}{c}\frac{\partial\vec{A}}{\partial t}$ or $\frac{d}{dt}(\vec{p}-e\vec{A}/c) = -e(\vec{\nabla}\phi + \frac{1}{c}\frac{\partial\vec{A}}{\partial t}) + \frac{e}{c}[d\vec{x}/dt \times (\vec{\nabla}\times\vec{A})]$. By symmetrization this can be written as $d\vec{\Pi}/dt = e[\vec{E} + \frac{1}{2c}(d\vec{x}/dt \times \vec{B} - \vec{B}\times d\vec{x}/dt\ )]$ and hence (2.6.27).

(b) To verify $\partial\rho/\partial t + \vec{\nabla}\cdot\vec{J} = 0$ (the continuity equation) with $\vec{J} = \frac{\hbar}{m}\,\mathrm{Im}(\psi^*\vec{\nabla}\psi) - \frac{e}{mc}\vec{A}|\psi|^2$ which can be written as $\vec{J} = \frac{\hbar}{2im}[\psi^*\vec{\nabla}\psi - \psi\vec{\nabla}\psi^*] - \frac{e}{mc}\vec{A}|\psi|^2$. Let us work in Coulomb gauge $\vec{\nabla}\cdot\vec{A} = 0$ (because of gauge invariance this is no loss of generality), we find

$$\vec{\nabla}\cdot\vec{J} = \frac{\hbar}{2mi}[\psi^*(\vec{\nabla}^2\psi) - \psi(\vec{\nabla}^2\psi^*)] - \frac{e}{mc}(\psi^*\vec{A}\cdot\vec{\nabla}\psi + \vec{\nabla}\psi^*\cdot\vec{A}\psi). \tag{1}$$

This can be simplified further by using the time-dependent Schrödinger equation

$$i\hbar\partial\psi/\partial t = \{\frac{-\hbar^2}{2m}\vec{\nabla}^2\psi + \frac{ie\hbar}{mc}(\vec{A}\cdot\vec{\nabla})\psi + \frac{e^2\vec{A}^2}{2m^2c^2}\psi + \phi\psi\} \tag{2}$$

$$-i\hbar\psi^*/\partial t = \{-\frac{\hbar^2}{2m}\vec{\nabla}^2\psi^* - \frac{ie\hbar}{mc}(\vec{A}\cdot\vec{\nabla})\psi^* + \frac{e^2\vec{A}^2}{2m^2c^2}\psi^* + \phi\psi^*\} \tag{3}$$

From (2) and (3) we may eliminate $\psi^*\vec{\nabla}^2\psi - \psi(\vec{\nabla}^2\psi^*)$ in (1), the result is

$$\vec{\nabla}\cdot\vec{J} = -(\psi^*\partial\psi/\partial t + \psi\partial\psi^*/\partial t) = -\frac{\partial}{\partial t}(\psi^*\psi) = -\partial\rho/\partial t.$$

38. *This solution reprinted from the solutions manual for the revised edition.*

Take $H_o = p^2/2m + \phi(r)$, then $H = (\vec{p}-e\vec{A}/c)^2/2m + \phi(r)$. Now $(\vec{p}-e\vec{A}/c)^2 = \vec{p}^2 - [\frac{e}{c}\vec{A}\cdot\vec{p} + \frac{e}{c}\vec{p}\cdot\vec{A}] + \frac{e^2}{c^2}\vec{A}^2$, and we note that $\vec{A}\cdot\vec{p}$ can be written as

$$\vec{A}\cdot\vec{p} = \tfrac{1}{2}(\vec{B}\times\vec{r})\cdot\vec{p} = \frac{\vec{B}}{2}\cdot(\vec{r}\times\vec{p}) = \tfrac{1}{2}\,\vec{B}\cdot\vec{L}$$

while $\vec{A}^2 = \tfrac{1}{4}(\vec{B}\times\vec{r})^2 = \tfrac{1}{4}\,B^2(x^2+y^2)$ when we choose B to be an uniform magnetic field along $\hat{z}$ - direction. Thus in Coulomb gauge $\vec{\nabla}\cdot\vec{A} = 0$, we have

$$H = H_o - \frac{eBL_z}{2mc} + \frac{e^2B^2}{8mc^2}(x^2+y^2).$$

Hence we are led to the correct expression for the interaction of the orbital magnetic moment $(e/2mc)\vec{L}$ with the magnetic field $\vec{B}$. There is also the quadratic Zeeman effect contribution proportional to $B^2(x^2+y^2)$ in H which contributes to the "diamagnetic susceptibility" χ appearing as an energy shift $= -\tfrac{1}{2}\,\chi B^2$.

39. *This solution reprinted from the solutions manual for the revised edition.*

(a) $[p_x-eA_x/c,\ p_y-eA_y/c] = -\frac{e}{c}[p_x,\ A_y] + \frac{e}{c}[p_y,\ A_x] = \frac{ie\hbar}{c}(\partial A_y/\partial x - \partial A_x/\partial y) = ie\hbar B/c$. Hence $[\Pi_x,\ \Pi_y] = ie\hbar B/c$.

(b) From the relation $[\Pi_x,\Pi_y] = ie\hbar B/c$, it is suggestive that we define $X = -c\Pi_y/eB$, then $[X,\Pi_x] = i\hbar$ (just like $[x,p] = i\hbar$). The Hamiltonian then reads

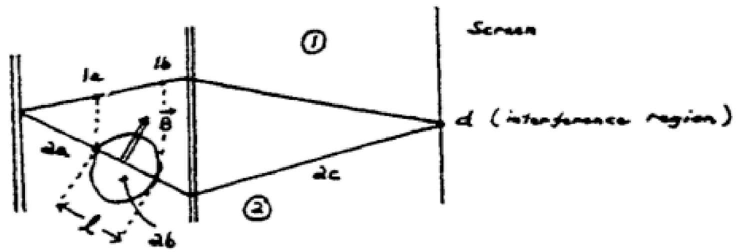
$$H = \Pi_x^2/2m + \Pi_y^2/2m + p_z^2/2m = \Pi_x^2/2m + e^2B^2X^2/2mc^2 + p_z^2/2m \tag{1}$$

where p_z is same as Π_z because $A_z = 0$. Compare Eq.(1) with the one-dimensional simple harmonic oscillator

$$H_{SHO} = p^2/2m + m\omega^2x^2/2 \tag{2}$$

for which we know $E_n = \hbar\omega(n+\tfrac{1}{2})$. So evidently the substitution $\omega \to |e\vec{B}|/mc$, we immediately get the energy eigenvalues of (1). (Note: Π_x and X satisfy the same commutation relations as p and x for the harmonic oscillator problem.) We must still add translational kinetic energy in the z-direction. The eigenvalues of p_z, $\hbar k$, are continuous. So the final answer is $E_{k,n} = \hbar^2k^2/2m + \hbar\frac{|eB|}{mc}(n+\tfrac{1}{2})$, where $n = 0,1,2,\ldots\ldots.$

40. *This solution reprinted from the solutions manual for the revised edition.*



Consider the paths ① and ②, and the two wave functions ψ_1 and ψ_2 where $\vec{B} = 0$. Then $\psi_2 = e^{i\delta}\psi_1$ since by symmetry $|\psi_2|^2 = |\psi_1|^2$ for $\vec{B} = 0$. If $\vec{B}$ is turned on in a region (drawn above) of length ℓ, the neutrons will cross the above length in a time T given by

$$v = \ell/T \text{ and } p \equiv m\ell/T \equiv \hbar/\lambda\!\!\!^{-}.$$

Therefore $T = m\ell\lambda\!\!\!^{-}/\hbar$, and is the time in which the external B-field is acting on the particle. Now let us focus our attention on path ②; the Hamiltonian is

$$H = \begin{cases} H_o = p^2/2m \text{ for 2a, 2c regions} \\ H' = p^2/2m + g_n\mu\vec{\sigma}\cdot\vec{B} \text{ for 2b region} \end{cases}$$

where $\mu = -e\hbar/2mc$.

Now ψ_{2b} is related to ψ_{2a} via the time evolution operator viz: $\psi_{2b} = e^{-iHT/\hbar}\psi_{2a}$. Furthermore ψ_{2d} (wave function at screen via path ②) is given by

$$\psi_{2d} = \exp[-iH_ot/\hbar]\exp[-iH'T/\hbar]\psi_{2a},$$

where t is the time of transit along ② from 2b to 2d. Noting that $p^2/2m = \hbar^2/2m\lambda\!\!\!^{-2}$, we find $\exp[-iH'T/\hbar] = \exp[-(iT/\hbar)\{\hbar^2/2m\lambda\!\!\!^{-2} + g_n\mu\vec{\sigma}\cdot\vec{B}\}]$. Choose next $\vec{B} = B\hat{e}_z$ (and remind that $T = m\ell\lambda\!\!\!^{-}/\hbar$), we find

$$e^{-iH'T/\hbar} = e^{-i\phi}, \text{ where } \phi = \ell/2\lambda\!\!\!^{-} + g_n\mu\sigma_zBm\ell\lambda\!\!\!^{-}/\hbar^2.$$

and

$$\psi_{2d} = e^{-iH_ot/\hbar}e^{-i\phi}\psi_{2a}. \qquad (1)$$

A change in B, produces the following change in ϕ

$$\Delta\phi = g_n \mu \sigma_z m \ell \lambdabar \Delta B/\hbar^2 = \frac{g_n \mu m \ell \lambdabar}{\hbar^2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \Delta B$$

For path 1 we see

$$\psi_{1d} = e^{-iH_o t/\hbar} \, e^{-iH_o T/\hbar} \psi_{1a} \tag{2}$$

where ψ_{1d} is wave function at screen via path 1 and t is the time of transit from 1b to 1d. From Eqs. (1) and (2) we see that maxima occur for $\Delta\phi = 2\pi$(i.e. no "effect" on phase in region 2a to 2b), therefore

$$2\pi/\Delta B = g_n \mu m \ell \lambdabar/\hbar^2 \tag{3}$$

and with $|\mu| = |e|\hbar/2mc$, we have $|\Delta B| = 4\pi\hbar c/|e| g_n \lambdabar \ell$.

Chapter Three

1. *Note: The original solution manual does not answer this problem correctly.* The eigenvalues λ satisfy $\lambda^2 - i(-i) = \lambda^2 - 1 = 0$, i.e. $\lambda = \pm 1$, as they must be, since $S_y \doteq (\hbar/2)\sigma_y$ has eigenvalues $\pm\hbar/2$. The eigenvectors are well known by now, namely

$$|S_y;+\rangle \doteq \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ i \end{bmatrix} \qquad \text{and} \qquad |S_y;-\rangle \doteq \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -i \end{bmatrix} \qquad \text{so, for} \qquad |\psi\rangle \doteq \begin{bmatrix} \alpha \\ \beta \end{bmatrix}$$

where $|\alpha|^2 + |\beta|^2 = 1$, the probability of finding $S_y = +\hbar/2$ is $|\langle S_y;+|\psi\rangle|^2 = |(\alpha - i\beta)/\sqrt{2}|^2 = (1 + \mathrm{Im}(\alpha^*\beta))/2$. Clearly this gives the right answer for $|\psi\rangle = |S_y;\pm\rangle$. It might have been more interesting, though, to ask for the expectation value of S_y, namely

$$\langle\psi|S_y|\psi\rangle = \frac{\hbar}{2}\begin{bmatrix} \alpha^* & \beta^* \end{bmatrix}\begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}\begin{bmatrix} \alpha \\ \beta \end{bmatrix} = i\frac{\hbar}{2}(\beta^*\alpha - \alpha^*\beta) = \frac{\hbar}{2}\mathrm{Im}(\alpha^*\beta)$$

2. Since $\mathbf{S} \doteq (\hbar/2)\boldsymbol{\sigma}$, the matrix representation of the Hamiltonian is

$$H \doteq \mu\begin{pmatrix} -B_z & -B_x + iB_y \\ -B_x - iB_y & B_z \end{pmatrix}$$

Therefore, the characteristic equation for the eigenvalues λ is

$$(-\mu B_z - \lambda)(\mu B_z - \lambda) - \mu^2(-B_x - iB_y)(-B_x + iB_y) = -\mu^2(B_z^2 + B_x^2 + B_y^2) + \lambda^2 = 0$$

so the eigenvalues are $\lambda = \pm\mu B$ where $B^2 = B_x^2 + B_y^2 + B_z^2$. Of course.

3. *This solution reprinted from the solutions manual for the revised edition.*

(a) Write U as $U = (a_o + i\vec{\sigma}.\vec{a})(a_o - i\vec{\sigma}.\vec{a})^{-1} = A(A^\dagger)^{-1}$, than $UU^\dagger = A(A^\dagger)^{-1}A^{-1}A^\dagger$ $= A(AA^\dagger)^{-1}A^\dagger = A\dfrac{1}{a_o^2+a_1^2+a_2^2+a_3^2}A^\dagger = 1$. Likewise $U^\dagger U = 1$, therefore U is unitary.

Now since $A = \begin{pmatrix} a_o+ia_3 & ia_1+a_2 \\ ia_1-a_2 & a_o-ia_3 \end{pmatrix}$ and $A^\dagger = \begin{pmatrix} a_o-ia_3 & -ia_1-a_2 \\ -ia_1+a_2 & a_o+ia_3 \end{pmatrix}$, we have

$\det A = \det A^\dagger = a_o^2+a_1^2+a_2^2+a_3^2$ while from $\det(A^\dagger(A^\dagger)^{-1}) = \det A^\dagger \det(A^{\dagger -1}) = 1$,

it is evident that $\det(A^{\dagger -1}) = 1/\det(A^\dagger) = 1/(a_o^2+a_1^2+a_2^2+a_3^2)$. Thus $\det U =$ $\det[A(A^\dagger)^{-1}] = \det A \det(A^\dagger)^{-1} = 1$, therefore U is unimodular.

(b) Since $AA^\dagger = (a_o^2+a_1^2+a_2^2+a_3^2)\underline{1} = \alpha\underline{1}$ say, we find

$$U = A(A^\dagger)^{-1} = A^2/\alpha = \frac{1}{\alpha}\begin{pmatrix} a_o^2 - |\vec{a}|^2+2ia_oa_3 & 2a_oa_2+2ia_oa_1 \\ -2a_oa_2 + 2ia_oa_1 & a_o^2 - |\vec{a}|^2-2ia_oa_3 \end{pmatrix}.$$

Compare with (3.3.7) and (3.3.10), we find angle and axis of rotation appropriate for U as $\cos\frac{\phi}{2} = (a_o^2 - \vec{a}^2)/\alpha$, $\sin\frac{\phi}{2} = 2a_o|\vec{a}|/\alpha$, $n_x = -a_1/|\vec{a}|$, $n_y = -a_2/|\vec{a}|$, and $n_z = -a_3/|\vec{a}|$.

4. *This solution reprinted from the solutions manual for the revised edition.*

The coupled representation has: $|11\rangle = |++\rangle$, $|10\rangle = \frac{1}{2^{1/2}}(|+-\rangle + |-+\rangle)$, $|1{-1}\rangle = |--\rangle$, and $|00\rangle = \frac{1}{2^{1/2}}(|+-\rangle - |-+\rangle)$ while $\vec{S}_1 . \vec{S}_2 = (\vec{S}^2 - \vec{S}_1^2 - \vec{S}_2^2)/2$. We are interested in the spin function of the system given by $\chi_+^{(e^-)}\chi_-^{(e^+)}$ hence in the $+-$ contribution arising from $|10\rangle$ and $|00\rangle$. So we are interested in the piece of Hamiltonian

$$(H) = \begin{pmatrix} \langle 10|H|10\rangle & \langle 10|H|00\rangle \\ \langle 00|H|10\rangle & \langle 00|H|00\rangle \end{pmatrix} = \begin{pmatrix} A\hbar^2/4 & eB\hbar/mc \\ eB\hbar/mc & -3A\hbar^2/4 \end{pmatrix}.$$

The eigenvalue equation is $(H)\psi = E\psi$, where E satisfies $\det[(H) - E\underline{1}] = 0$. We have $E_\pm = -\frac{1}{4}(A\hbar^2) \pm \frac{1}{2}[(A\hbar^2)^2 + 4(eB\hbar/mc)^2]^{\frac{1}{2}} = -\frac{1}{4}A\hbar^2(1 \mp 2/\cos\theta)$, where $\tan\theta = 2eB/mcA\hbar$.

For $\psi = \begin{pmatrix} x \\ y \end{pmatrix}$, the eigenvalue equation leads to normalized eigenvectors

$$\psi_+ = \begin{pmatrix} \cos\frac{\theta}{2} \\ \sin\frac{\theta}{2} \end{pmatrix} \text{ for } E_+ \text{ and } \psi_- = \begin{pmatrix} \sin\frac{\theta}{2} \\ -\cos\frac{\theta}{2} \end{pmatrix} \text{ for } E_-$$

(a) In the case $A \to 0$, $eB/mc \neq 0$ we have $\theta = \pi/2$, hence $\psi_+ = \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix}$ for $E_+ = +eB\hbar/mc$ and $\psi_- = \begin{pmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{pmatrix}$ for $E_- = -eB\hbar/mc$. However the spin function of system is $\chi_+^{(e^-)}\chi_-^{(e^+)}$ and $|+-\rangle = \frac{1}{2^{1/2}}|10\rangle + \frac{1}{2^{1/2}}|00\rangle$ which corresponds to ψ_+ with $E_+ = +eB\hbar/mc$ as the respective eigenvector and eigenvalue.

(b) In the case $eB/mc \to 0$, $A \neq 0$, we have $\theta \to 0$. Hence $\psi_+ = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ for $E_+ = +A\hbar^2/4$ and $\psi_- = \begin{pmatrix} 0 \\ -1 \end{pmatrix}$ for $E_- = -3A\hbar^2/4$. Our spin function $|+-\rangle = \frac{1}{2^{1/2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ is not therefore an eigenvector corresponding to a definite energy eigenvalue. The expectation value can be computed by noting that $\langle +-|H|+-\rangle = \frac{1}{2}[\langle 10|H|10\rangle + \langle 00|H|10\rangle + \langle 10|H|00\rangle + \langle 00|H|00\rangle] = \frac{1}{2}[A\hbar^2/4 - 3A\hbar^2/4] = -\frac{1}{4}A\hbar^2$.

5. *This solution reprinted from the solutions manual for the revised edition.*

Choose a representation in which $\vec{S}^2$, and S_z are diagonal, so $\vec{S}^2|s,m\rangle = s(s+1)\hbar^2|s,m\rangle$ and $S_z|s,m\rangle = m\hbar|s,m\rangle$. Using ladder operations $S_+ = S_x+iS_y$, $S_- = S_x-iS_y$ where $S_\pm|s,m\rangle = [s(s+1)-m(m\pm1)]^{1/2}\hbar|s,m\pm1\rangle$, we have for $s = 1$ (spin 1 particle)

$$S_x = \frac{\hbar}{2^{1/2}}\begin{bmatrix}0&1&0\\1&0&1\\0&1&0\end{bmatrix}, \quad S_y = \frac{\hbar}{2^{1/2}}\begin{bmatrix}0&-i&0\\i&0&-i\\0&i&0\end{bmatrix}, \quad S_z = \hbar\begin{bmatrix}1&0&0\\0&0&0\\0&0&-1\end{bmatrix}, \quad S^2 = 2\hbar^2\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}.$$

(a) $S_z(S_z+\hbar I)(S_z-\hbar I) = S_z(S_z^2-\hbar^2 I) \equiv 0$. (b) $S_x(S_x+\hbar I)(S_x-\hbar I) = S_x(S_x^2-\hbar^2 I) = (\hbar^3/2\sqrt{2})\times[0]$ where $[0]$ is the null matrix. This result is physically reasonable, since same quantity is considered with quantization axis S_x instead of S_z.

6. *This solution reprinted from the solutions manual for the revised edition.*

The Heisenberg equation of motion is $d\vec{K}/dt = \frac{i}{\hbar}[H, \vec{K}]$. Substitute $\vec{K}$ and H into this equation, we have $2d\vec{K}/dt = \frac{i}{\hbar}[K_1^2/I_1+K_2^2/I_2+K_3^2/I_3,\ K_1\vec{e}_1+K_2\vec{e}_2+K_3\vec{e}_3]$. Take the first component for definiteness, we have $2dK_1/dt = \frac{i}{\hbar}[K_2^2/I_2+K_3^2/I_3,\ K_1]$. Now $[K_2^2/I_2,\ K_1] = \frac{1}{I_2}\{K_2,[K_2,K_1]\}$, and since $[K_1,K_2] = -i\hbar K_3$ (true for a <u>rotating system of axis</u>), we have $[K_2^2/I_2,K_1] = i\hbar/I_2\ \{K_2,K_3\}$ and similarly $[K_3^2/I_3,\ K_1] = i\hbar/I_3\ \{K_1,K_3\}$. So $dK_1/dt = \frac{I_2-I_3}{2I_2I_3}\{K_2,K_3\}$, and similarly $dK_2/dt = \frac{I_3-I_1}{2I_3I_1}\{K_3,K_1\}$, $dK_3/dt = \frac{I_1-I_2}{2I_1I_2}\{K_1,K_2\}$.

The correspondence limit gives $K_iK_j = K_jK_i$ and $K_i = I_i\omega_i$, hence $dK_i/dt = I_i\dot{\omega}_i$. Then the Heisenberg equation of motion for $\vec{K}$, reduces to $I_i\dot{\omega}_i = (I_j-I_k)\omega_j\omega_k$ (i,j,k cyclic permutation of 1,2,3) – that is the Euler's equation of motion.

7. *This solution reprinted from the solutions manual for the revised edition.*

If U represents the rotation with Euler angles α,β,γ, then U must satisfy for infinitesimal rotation angle ε (c.f. (3.1.7)) $U_x(\varepsilon)U_y(\varepsilon) - U_y(\varepsilon)U_x(\varepsilon) = U_z(\varepsilon^2) - 1$. Obviously $U_x(\varepsilon) = e^{iG_1\varepsilon}$, $U_y(\varepsilon) = e^{iG_2\varepsilon}$, and $U_z(\varepsilon) = e^{iG_3\varepsilon}$, and represent infinitesimal rotations around x,y,z axes respectively. In terms of Euler angle rotation $U_x(\varepsilon) = e^{-iG_3\pi/2}e^{iG_2\varepsilon}e^{iG_3\pi/2}$, etc. where we have used (3.3.19). Expand $e^{iG_1\varepsilon}$, $e^{iG_2\varepsilon}$, and $e^{iG_3\varepsilon^2}$ in terms of Taylor series in $U_x(\varepsilon)U_y(\varepsilon)-U_y(\varepsilon)U_x(\varepsilon) = U_z(\varepsilon^2) - 1$, and compare coefficients of ε^2 on both sides. We have $[G_1,G_2] = iG_3$, and similarly $[G_2,G_3] = iG_1$ and $[G_3,G_1] = iG_2$, i.e. $[G_i,G_j] = i\varepsilon_{ijk}G_k$. Compare with commutation relations for $\vec{J}$, viz:- $[J_i,J_j] = i\hbar\varepsilon_{ijk}J_k$, we find $G_i = J_i/\hbar$.

8. *This solution reprinted from the solutions manual for the revised edition.*

A_ℓ are unrotated operators while $U^{-1}A_kU$ are operators under rotation. So $U^{-1}A_kU = \sum_\ell R_{k\ell}A_\ell$ is the connecting equation between unrotated operators and operators obtained after rotation. The operators after rotation are just combinations of unrotated operators. From $U^{-1}A_kU = A'_k = \sum_\ell R_{k\ell}A_\ell$, we obtain for matrix elements $\langle m|A'_k|n\rangle = \sum_\ell R_{k\ell}\langle m|A_\ell|n\rangle$. But this is the same as vector transformation $V'_k = \sum_\ell R_{k\ell}V_\ell$, hence $\langle m|A_k|n\rangle$ transforms like a vector.

9. *This solution reprinted from the solutions manual for the revised edition.*

We are given that $\mathcal{D}^{(1/2)}(\alpha,\beta,\gamma)$ is such that (c.f. (3.3.21))

$$\mathcal{D}^{(1/2)}(\alpha,\beta,\gamma) = \begin{pmatrix} e^{-i(\alpha+\gamma)/2}\cos\frac{\beta}{2} & -e^{-i(\alpha-\gamma)/2}\sin\frac{\beta}{2} \\ e^{i(\alpha-\gamma)/2}\sin\frac{\beta}{2} & e^{i(\alpha+\gamma)/2}\cos\frac{\beta}{2} \end{pmatrix}, \tag{1}$$

but this is equivalent to (c.f. (3.2.45))

$$\mathcal{D}^{(1/2)}(\hat{n};\theta) = e^{-\frac{i}{2}(\vec{\sigma}\cdot\hat{n})\theta} = \begin{pmatrix} \cos\frac{\theta}{2} - in_z\sin\frac{\theta}{2} & (-in_x-n_y)\sin\frac{\theta}{2} \\ (-in_x+n_y)\sin\frac{\theta}{2} & \cos\frac{\theta}{2} + in_z\sin\frac{\theta}{2} \end{pmatrix} \tag{2}$$

corresponding to rotation about some axis $\hat{n}$ through an angle θ. Since $\mathcal{D}^{(1/2)}(\hat{n};\theta)$ is equivalent to $\mathcal{D}^{(1/2)}(\alpha,\beta,\gamma)$, we have $\mathrm{Tr}\,\mathcal{D}^{(1/2)}(\hat{n};\theta) = \mathrm{Tr}\,\mathcal{D}^{(1/2)}(\alpha,\beta,\gamma)$, thence

$$2\cos\frac{\theta}{2} = 2\cos\frac{\beta}{2}\cos\frac{(\alpha+\gamma)}{2} \quad \text{or } \theta = 2\cos^{-1}\left[\cos\frac{\beta}{2}\cos\frac{(\alpha+\gamma)}{2}\right]$$

10. *This solution reprinted from the solutions manual for the revised edition.*

(a) A general state in spin ½ system can be written as (suitably normalized)

$$|\alpha\rangle = \cos\frac{\beta}{2}e^{i\alpha/2}|+\rangle + \sin\frac{\beta}{2}e^{-i\alpha/2}|-\rangle.$$

Then $\langle S_x\rangle = \langle\alpha|S_x|\alpha\rangle = \frac{\hbar}{2}\langle\alpha|(|+\rangle\langle-| + |-\rangle\langle+|)|\alpha\rangle = \frac{\hbar}{2}[\cos\frac{\beta}{2}e^{-i\alpha/2}\langle-| + \sin\frac{\beta}{2}e^{i\alpha/2}\langle+|]|\alpha\rangle$

$= \frac{\hbar}{2}[\cos\frac{\beta}{2}e^{-i\alpha/2}\sin\frac{\beta}{2}e^{-i\alpha/2} + \sin\frac{\beta}{2}e^{+i\alpha/2}\cos\frac{\beta}{2}e^{+i\alpha/2}] = \frac{\hbar}{2}\sin\beta\cos\alpha$. Similarly $\langle S_z\rangle = \frac{\hbar}{2}\cos\beta$ and $\langle S_y\rangle = -\frac{\hbar}{2}\sin\beta\sin\alpha$. If we know $\langle S_x\rangle$, $\langle S_z\rangle$ we can obtain β and $\cos\alpha$. However to know the sign of $\sin\alpha$ and hence specify α we need to know sign ($\langle S_y\rangle$) but not the magnitude of $\langle S_y\rangle$.

(b) Let $\rho = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ be the density matrix in the S_z basis. The ensemble average of an operator O is $[O] = \mathrm{Tr}[\rho O]$. We have

$$[S_x] = \frac{\hbar}{2}\,\mathrm{Tr}[\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}] = \frac{\hbar}{2}(b+c) \tag{1}$$

$$[S_y] = \frac{\hbar}{2}\,\mathrm{Tr}[\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}] = \frac{i\hbar}{2}(b-c) \tag{2}$$

$$[S_z] = \frac{\hbar}{2}\,\mathrm{Tr}[\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}] = \frac{\hbar}{2}(a-d) \tag{3}$$

and the normalization condition is:

$$\mathrm{Tr}\,\rho = 1 \text{ or } (a+d) = 1. \tag{4}$$

Solving Eqs. (1)–(4), we obtain for elements of the density matrix

$$a = \tfrac{1}{2}[1+2[S_z]/\hbar],\; b = \frac{1}{\hbar}[[S_x]-i[S_y]],\; c = \frac{1}{\hbar}[[S_x]+i[S_y]],\; d = \tfrac{1}{2}[1-2[S_z]/\hbar].$$

11. *This solution reprinted from the solutions manual for the revised edition.*

(a) Take (3.4.8) at time t, the density operator $\rho(t)$ reads

$$\rho(t) = \sum_i w_i|\alpha_i,t\rangle\langle\alpha_i,t|.$$

In the Schrödinger picture $|\alpha_i,t\rangle = U(t,t_o)|\alpha_i,t_o\rangle$, then

$$\rho(t) = \sum_i w_i U(t,t_o)|\alpha_i,t_o\rangle\langle\alpha_i,t_o|U^\dagger(t,t_o) = U(t,t_o)(\sum_i w_i|\alpha_i,t_o\rangle\langle\alpha_i,t_o|)\times$$
$$U^\dagger(t,t_o) = U(t,t_o)\rho(t_o)U^\dagger(t,t_o)$$

(b) $\rho^2(t) = U(t,t_o)\rho(t_o)U^\dagger(t,t_o)U(t,t_o)\rho(t_o)U^\dagger(t,t_o) = U(t,t_o)\rho^2(t_o)U^\dagger(t,t_o)$.
At $t=0$ we have a pure ensemble (hence idempotent (3.4.13)) i.e. $\rho^2(t_o) = \rho(t_o)$.
But $\rho^2(t) = U(t,t_o)\rho(t_o)U^\dagger(t,t_o) = \rho(t)$ and is also idempotent hence we have a pure ensemble at time t also.

12. *This solution reprinted from the solutions manual for the revised edition.*

From (3.4.9) we see that the density matrix of an ensemble of spin 1 systems has form

$$\rho = \begin{pmatrix} a & b & c \\ b^* & d & e \\ c^* & e^* & f \end{pmatrix}$$

where a,d,f are real, and b,c,e complex, i.e. 9 independent variables. However since $\mathrm{Tr}\,\rho = 1$ (3.4.11), we have $a+d+f = 1$, and only 8 independent parameters are needed to characterize the density matrix. If we know $[S_x]$, $[S_y]$, $[S_z]$, we need five more independent quantities. They are: $[S_xS_y]$, $[S_yS_z]$, $[S_zS_x]$, $[S_x^2]$, $[S_y^2]$. Note $[S_xS_y]$, $[S_yS_z]$, and $[S_zS_x]$ may not be real, however the extra conditions (over 3) are not independent of $[S_x]$, $[S_y]$, $[S_z]$. Physically $[S_{x,y,z}]$ are related to measurement of dipole moments of the particles and to completely characterize a spin 1 system we need the five components of the quadrupole tensor.

13. *This solution reprinted from the solutions manual for the revised edition.*

Rotated state is given by

$$U_R|j,m=j\rangle = (1 - iJ_y\varepsilon/\hbar - (J_y^2)\varepsilon^2/2\hbar^2\ldots\ldots)|j,m=j\rangle.$$

Probability amplitude for being found in the original state is

$$\langle j,m=j|U_R|j,m=j\rangle.$$

We must evaluate the expectation values of J_y, J_y^2, where from $J_y = (J_+-J_-)/2i$ we have $J_y^2 = -\frac{1}{4}[J_+^2+J_-^2-J_+J_--J_-J_+]$. Evidently $\langle J_y\rangle_{j,m=j} = 0$ and from (3.5.39) and (3.5.40) $\langle(J_y)^2\rangle_{j,m=j} = \frac{1}{4}\langle j,m=j|J_+J_-|j,m=j\rangle = 2j\hbar^2/4$. So

$$\langle j,m=j|U_R|j,m=j\rangle = 1 - (2j\hbar^2/2\hbar^2 4)\varepsilon^2 + \ldots\ldots\ldots\ldots$$

Hence probability to order $\varepsilon^2 = |\langle j,m=j|U_R|j,m=j\rangle|^2 = 1 - \frac{1}{2}j\varepsilon^2$.

Alternative solution:

Calculate the probability amplitude for being found in states other than j=m. To order ε(in the amplitude) only m=j-1 state gets populated. $U_R|j,m=j\rangle = |j,m=j\rangle - (i/\hbar)J_y\varepsilon|j,m=j\rangle = |j,m=j\rangle - \frac{\varepsilon}{2}\frac{1}{2}\sqrt{j}|j,m=j-1\rangle$. The probability for being found in the original state is reduced by $\varepsilon^2 j/2$. So the answer (for our problem) is $1 - \varepsilon^2 j/2$.

14. *This solution reprinted from the solutions manual for the revised edition.*

Looking at the matrix elements we have

$$\begin{aligned}
[G_i,G_j]_{\ell n} &= [G_iG_j-G_jG_i]_{\ell n} = (G_i)_{\ell m}(G_j)_{mn} - (G_j)_{\ell m}(G_i)_{mn} \\
&= -\hbar^2[\varepsilon_{i\ell m}\varepsilon_{jmn} - \varepsilon_{j\ell m}\varepsilon_{imn}] = -\hbar^2[\varepsilon_{mi\ell}\varepsilon_{mnj} - \varepsilon_{mj\ell}\varepsilon_{mni}] \\
&= -\hbar^2[(\delta_{in}\delta_{\ell j} - \delta_{ij}\delta_{\ell n}) - (\delta_{jn}\delta_{\ell i} - \delta_{ji}\delta_{\ell n})] \\
&= \hbar^2(\delta_{i\ell}\delta_{jn} - \delta_{in}\delta_{j\ell}) = \hbar^2\varepsilon_{kij}\varepsilon_{k\ell n} = i\hbar\varepsilon_{ijk}(-i\hbar\varepsilon_{k\ell n}) \\
&= i\hbar\varepsilon_{ijk}(G_k)_{\ell n}.
\end{aligned}$$

Therefore $[G_i,G_j] = i\hbar\varepsilon_{ijk}G_k$. Let us find the unitary matrix which transforms G_i to J_i with J_3 diagonal, than $J_i = U^\dagger G_i U$ where U is made up of the eigenvectors of G_3. The explicit form of G_3 (from $(G_i)_{jk} = -i\hbar\varepsilon_{ijk}$ where j and k are the row and column indices) is

$$G_3 = i\hbar\begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

and the eigenvalues and eigenvectors are obtained from equation $(G_3-\lambda I)\vec{r}_\lambda = 0$ where λ is a root of $|G_3-\lambda I| = 0$. The eigenvalues and orthonormal eigenvectors can be readily seen to be

$$\lambda=0,\ \vec{r}_o = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix};\quad \lambda=+\hbar,\ \vec{r}_+ = \frac{1}{2^{1/2}}\begin{pmatrix} 1 \\ i \\ 0 \end{pmatrix};\quad \lambda=-\hbar,\ \vec{r}_- = \frac{1}{2^{1/2}}\begin{pmatrix} -1 \\ i \\ 0 \end{pmatrix}.$$

Hence

$$U = \frac{1}{2^{1/2}}\begin{pmatrix} 1 & i & 0 \\ 0 & 0 & \sqrt{2} \\ -1 & i & 0 \end{pmatrix},$$

and this unitary matrix transforms the Cartesian space representation of the angular momentum operator, i.e. $\vec{G}$, to its spherical basis representation, $\vec{J}$ (j=1). Since the G's and J's satisfy the same Lie algebra (and they both form a group), they are just different representations of the rotation group (irreducible). Therefore, the J's and G's are related via a rotation in the group space. This finite rotation can be obtained from compounding the infinitesimal rotation $\vec{V} \to \vec{V} + \hat{n}\delta\phi \times\vec{V}$ (or $\vec{G} \to \vec{G} + \hat{n}\delta\phi\times\vec{G}$).

15. *This solution reprinted from the solutions manual for the revised edition.*

(a) $J_+J_- = (J_x+iJ_y)(J_x-iJ_y) = J_x^2 + iJ_yJ_x - iJ_xJ_y + J_y^2 = J_x^2 + J_y^2 - i(J_xJ_y - J_yJ_x)$ $= \mathbf{J}^2 - J_z^2 - iJ_z(i\hbar) = \mathbf{J}^2 - J_z^2 + \hbar J_z$. So $\mathbf{J}^2 = J_z^2 + J_+J_- - \hbar J_z$.

(b) We have on the one hand $\langle jm|J_+J_-|jm\rangle = |c_-|^2$, while using $J_+J_- = \mathbf{J}^2 - J_z^2 + \hbar J_z$ we have on the other hand $\langle jm|J_+J_-|jm\rangle = (j(j+1) - m^2 + m)\hbar^2$. So $|c_-|^2 = [j(j+1) - m^2 + m]\hbar^2 = (j+m)(j-m+1)\hbar^2$, and by convention we choose $c_- = \sqrt{(j+m)(j-m+1)}\,\hbar$. Thus $J_-|jm\rangle = c_-|j,m-1\rangle$ (or $J_-\psi_{jm} = c_-\psi_{j,m-1}$).

16. These proofs are straightforward. Just work with separate components. For example

$$\begin{aligned} [L_z, \mathbf{p}^2] = [xp_y - yp_x, p_x^2 + p_y^2 + p_z^2] &= [xp_y, p_x^2] - [yp_x, p_y^2] \\ &= \left(i\hbar\frac{\partial}{\partial p_x}p_x^2\right)p_y - \left(i\hbar\frac{\partial}{\partial p_y}p_y^2\right)p_x \\ &= 2i\hbar[p_x, p_y] = 0 \end{aligned}$$

where I have made use of problem 1.29, although it is easy enough just to write it out. It works similarly for L_x and L_y. The commutator with $\mathbf{x}^2$ is done the same way, that is

$$\begin{aligned} [L_z, \mathbf{x}^2] = [xp_y - yp_x, x^2 + y^2 + z^2] &= [xp_y, y^2] - [yp_x, x^2] \\ &= x\left(-i\hbar\frac{\partial}{\partial y}y^2\right) - y\left(-i\hbar\frac{\partial}{\partial x}x^2\right) \\ &= -2i\hbar[x, y] = 0 \end{aligned}$$

17. *This solution reprinted from the solutions manual for the revised edition.*

Rewrite the wave function in spherical coordinates, i.e. $\psi(\vec{x}) = rf(r)(\sin\theta\cos\phi + \sin\theta\sin\phi + 3\cos\theta)$.

(a) Since $Y_{11} \propto \sin\theta e^{i\phi}$, $Y_{1-1} \propto \sin\theta e^{-i\phi}$, $Y_{10} \propto \cos\theta$, while $e^{\pm i\phi} = \cos\phi \pm i\sin\phi$, it is evident that $\psi(\vec{x})$ is an eigenfunction of $\vec{L}^2$ with $\ell = 1$.

(b) Let us write

$$\sin\theta\cos\phi + \sin\theta\sin\phi + 3\cos\theta = \sin\theta\frac{(e^{i\phi}+e^{-i\phi})}{2} + \sin\theta\frac{(e^{i\phi}-e^{-i\phi})}{2i} + 3\cos\theta$$

$$= (4\pi/3)^{\frac{1}{2}}\{(1-i)Y_{11}/\sqrt{2} - (1+i)Y_{1-1}/\sqrt{2} + 3Y_{10}\} \; . \tag{1}$$

The probability for the particle to be found in the $m_\ell = 0$ state is $9/(9+1+1) = 9/11 = P_o$. Similarly the probabilities for particle to be found in the state $m_\ell = 1$ is $P_1 = 1/11$, and in state $m_\ell = -1$ is $P_{-1} = 1/11$.

(c) The procedure for finding the potential $V(r)$ is first to substitute the wave function into Schrödinger equation, and then use the fact that the wave function is the eigenfunction of $\vec{L}^2$. Now our $\psi(\vec{x}) = R(r)F(\theta,\phi)$, while the Schrödinger equation is $(-\hbar^2/2m)\nabla^2\psi + V(r)\psi = E\psi$. In spherical coordinates

$$\nabla^2\psi = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\psi}{\partial r}\right) + \frac{1}{r^2}\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\psi}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2\psi}{\partial\phi^2}\right]$$

$$= \left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial R}{\partial r}\right) - \frac{\ell(\ell+1)R(r)}{r^2}\right]F(\theta,\phi) \tag{2}$$

where in (2) we have used (3.6.28) and the fact that $F(\theta,\phi)$ is a linear combination of spherical harmonics (c.f. (1)). Hence for $\ell = 1$, the Schrödinger equation leads to $\frac{-\hbar^2}{2m}\left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial R}{\partial r}\right) - \frac{2}{r^2}R\right] + V(r)R(r) = ER(r)$, and therefore

$$V(r) = E - \hbar^2/mr^2 + \frac{\hbar^2}{2m}\frac{1}{rR}\frac{d^2}{dr^2}(rR) \tag{3}$$

18. *This solution reprinted from the solutions manual for the revised edition.*

From $L_\pm = L_x \pm iL_y$, we have $L_x = \frac{1}{2}(L_+ + L_-)$ and $L_y = \frac{-i}{2}(L_+ - L_-)$, and from (3.5.39) and (3.5.40) $L_\pm|\ell,m\rangle = c_\pm(\ell,m)|\ell,m\pm1\rangle = \hbar[\ell(\ell+1)-m(m\pm1)]^{\frac{1}{2}}|\ell,m\pm1\rangle$. Hence $\langle L_x\rangle = \langle \ell m|\frac{1}{2}(L_+ + L_-)|\ell m\rangle = 0$ since $\langle \ell m|\ell m'\rangle = \delta_{mm'}$. Similarly $\langle L_y\rangle = \langle \ell m|L_y|\ell m\rangle = 0$. Now $\langle L_x^2\rangle = \langle \ell m|\frac{1}{4}(L_+L_+ + L_+L_- + L_-L_+ + L_-L_-)|\ell m\rangle$. But $L_+L_-|\ell m\rangle = c_-(\ell,m) \times c_+(\ell,m-1)|\ell m\rangle$ and $L_-L_+|\ell m\rangle = c_+(\ell,m)c_-(\ell,m+1)|\ell m\rangle$ while $\langle \ell m|L_+L_+|\ell m\rangle = \langle \ell m|L_-L_-|\ell m\rangle = 0$ since states of different m values are orthogonal. Hence $\langle L_x^2\rangle = \frac{1}{4}\langle \ell m|L_+L_- + L_-L_+|\ell m\rangle = \frac{1}{4}\{c_-(\ell,m)c_+(\ell,m-1) + c_+(\ell,m)c_-(\ell,m+1)\} = \frac{1}{4}\{c_-^2(\ell,m) + c_+^2(\ell,m)\} = \frac{\hbar^2}{4}\{\ell(\ell+1)-m(m-1)+\ell(\ell+1)-m(m+1)\} = \frac{\hbar^2}{2}\{\ell(\ell+1)-m^2\}$. Similarly $\langle L_y^2\rangle = \langle \ell m|-\frac{1}{4}(L_+L_+ - L_+L_- - L_-L_+ + L_-L_-)|\ell m\rangle = \frac{1}{4}\langle \ell m|(L_+L_- + L_-L_+)|\ell m\rangle = \langle L_x^2\rangle$.

<u>Semiclassical interpretation</u>: We know that $\vec{L}^2|\ell m\rangle = \hbar^2\ell(\ell+1)|\ell m\rangle$, $L_z^2|\ell m\rangle = \hbar^2 m^2\,|\ell m\rangle$. Thus $\langle \vec{L}^2\rangle = \ell(\ell+1)\hbar^2$ and $\langle L_z^2\rangle = m^2\hbar^2$. In the classical correspondence $\vec{L}^2 = L_x^2 + L_y^2 + L_z^2$ expresses itself in terms of the corresponding expectation values, and indeed $\langle L_x^2\rangle + \langle L_y^2\rangle + \langle L_z^2\rangle = \frac{1}{2}\hbar^2(\ell(\ell+1)-m^2) + \frac{1}{2}\hbar^2(\ell(\ell+1)-m^2) + m^2\hbar^2 = \ell(\ell+1)\hbar^2 = \langle \vec{L}^2\rangle$.

19. *This solution reprinted from the solutions manual for the revised edition.*

Since (c.f. (3.6.13)) $L_\pm = -i\hbar e^{\pm i\phi}[\pm i\frac{\partial}{\partial\theta} - \cot\theta\frac{\partial}{\partial\phi}]$, and we may deduce as usual that $Y_{\frac{1}{2},\frac{1}{2}}(\theta,\phi) = e^{i\phi/2}\sqrt{\sin\theta}$ from $L_+Y_{\frac{1}{2},\frac{1}{2}}(\theta,\phi) = 0$.

(a) Apply L_- to $Y_{\frac{1}{2},\frac{1}{2}}$ gives

$$\begin{aligned} Y_{\frac{1}{2},-\frac{1}{2}}(\theta,\phi) &= e^{-i\phi}[-i\frac{\partial}{\partial\theta}(e^{i\phi/2}\sqrt{\sin\theta}) - \cot\theta(1/2)e^{i\phi/2}\sqrt{\sin\theta}] \\ &= e^{-i\phi/2}[\sin\theta]^{-\frac{1}{2}}\cos\theta\ . \end{aligned}$$

(b) From $0 = (-i\frac{\partial}{\partial\theta} - \cot\theta\frac{\partial}{\partial\phi})Y_{\frac{1}{2},-\frac{1}{2}}(\theta,\phi)$, we solve for $Y_{\frac{1}{2},-\frac{1}{2}}(\theta,\phi)$ in form $Y_{\frac{1}{2},-\frac{1}{2}} \sim e^{-i\phi/2}f(\theta)$ and obtain solution for $f(\theta)$ from defining differential equation. The answer is $Y_{\frac{1}{2},-\frac{1}{2}} = e^{-i\phi/2}(\sin\theta)^{-\frac{1}{2}}$.

Comparing (a) and (b) lead to contradictory results. So this is another argument against half integer ℓ for orbital angular momentum.

20. *This solution reprinted from the solutions manual for the revised edition.*

From (3.6.46) and (3.6.48), we have

$$\mathcal{D}(R)|\ell,m\rangle = \sum_{m'} |\ell,m'\rangle\langle\ell,m'|\mathcal{D}(R)|\ell,m\rangle = \sum_{m'} |\ell,m'\rangle\mathcal{D}^{(\ell)}_{m'm}(R)$$

where $m = 0$ initially. So the probability for finding $|\ell,m'\rangle$ is given by (c.f. (3.6.51))

$$|\mathcal{D}^{(\ell)}_{m'0}(\alpha=0,\beta,\gamma=0)|^2 = |(4\pi/2\ell+1)^{\frac{1}{2}}Y^{m'*}_{\ell}(\theta=\beta,\phi=0)|^2.$$

From table for $Y^m_{\ell=2}$ (c.f. Appendix A), the probabilities are

$$m'=0:\ \tfrac{1}{4}(3\cos^2\beta-1)^2;\quad m'=\pm1:\ \tfrac{3}{2}\cos^2\beta\sin^2\beta;\quad m'=\pm2:\ \tfrac{3}{8}\sin^4\beta.$$

It is easy to check that the total probability (summed over m') is unity as expected.

21. We need to determine the quantities $\langle \mathbf{n}|qlm\rangle$. We only care about states degenerate in energy, which is simple to see by inserting H and operating both left and right. If energies are different, the inner product has to be zero. (Same old proof of orthogonality of states for hermitian operators.) For degenerate energies, we get the equation

$$n_x + n_y + n_z = 2q + l \equiv N$$

Below, we won't distinguish between the operator N and the value N. Also, to avoid confusion, we will always write inner products with the spherical state on the right and the cartesian state on the left.

First, work out the form of the angular momentum operator in terms of creation and annihilation operators.

$$\begin{aligned} L_i &= \epsilon_{ijk}x_jp_k \\ &= i\epsilon_{ijk}\frac{\hbar}{2}(a_j + a_j^\dagger)(-a_k + a_k^\dagger) \\ &= i\hbar\epsilon_{ijk}a_ja_k^\dagger \end{aligned} \tag{1}$$

Summation of repeated indices is implied. Write this out explicitly for L_z, so

$$\begin{aligned} L_z &= xp_y - yp_x \\ &= i\frac{\hbar}{2}\left[(a_x + a_x^\dagger)(-a_y + a_y^\dagger) - (a_y + a_y^\dagger)(-a_x + a_x^\dagger)\right] \\ &= i\hbar\left[a_xa_y^\dagger - a_x^\dagger a_y\right] \end{aligned}$$

Sandwich the left and right sides of this equation with the spherical and cartesian states, that is

$$\begin{aligned}
\langle n_x n_y n_z | L_z | qlm \rangle &= m\hbar \langle n_x n_y n_z | qlm \rangle \\
\text{and} \qquad \langle n_x n_y n_z | L_z | qlm \rangle &= i\hbar \langle n_x n_y n_z | \left[a_x a_y^\dagger - a_x^\dagger a_y \right] | qlm \rangle
\end{aligned}$$

which leads us to the equation

$$\begin{aligned}
m \langle n_x n_y n_z | qlm \rangle &= i\sqrt{(n_x+1)n_y} \langle n_x+1, n_y-1, n_z | qlm \rangle \\
&- i\sqrt{n_x(n_y+1)} \langle n_x-1, n_y+1, n_z | qlm \rangle \qquad (2)
\end{aligned}$$

This is enough to decompose the first excited state, with $N = 1$, that has threefold degeneracy. We have

$$\begin{aligned}
m\langle 100|01m\rangle &= -i\langle 010|01m\rangle \\
m\langle 010|01m\rangle &= +i\langle 100|01m\rangle \\
m\langle 001|01m\rangle &= 0
\end{aligned}$$

Therefore, since

$$|qlm\rangle = \sum_{n_x n_y n_z} |n_x n_y n_z\rangle \langle n_x n_y n_z | qlm \rangle$$

we can write

$$\begin{aligned}
|011\rangle &= \frac{1}{\sqrt{2}}|100\rangle + \frac{i}{\sqrt{2}}|010\rangle \\
|010\rangle &= |001\rangle \\
|01,-1\rangle &= \frac{1}{\sqrt{2}}|100\rangle - \frac{i}{\sqrt{2}}|010\rangle
\end{aligned}$$

We can check that these are correct by considering the angular dependence in coordinate space, and remembering that for Hermite polynomials $H_1(w) = 2w$. Thus, these three equations say, in turn,

$$\begin{aligned}
Y_1^1(\theta,\phi) &\propto x + iy = re^{i\phi}\sin\theta \\
Y_1^0(\theta,\phi) &\propto z = r\cos\theta \\
Y_1^{-1}(\theta,\phi) &\propto x - iy = re^{-i\phi}\sin\theta
\end{aligned}$$

which are indeed correct.

For the second excited state, we need to find coefficients to find the five states $|02m\rangle$ plus the one state $|200\rangle$ in terms of the six states $|n_x n_y n_z\rangle = |200\rangle$, $|020\rangle$, $|002\rangle$, $|110\rangle$, $|101\rangle$, and

$|011\rangle$. Consider first the state $|qlm\rangle = |200\rangle$. Equation (2) give us

$$
\begin{aligned}
0 &= \langle 110|200\rangle && \text{(3a)}\\
0 &= \langle 011|200\rangle && \text{(3b)}\\
0 &= \langle 101|200\rangle && \text{(3c)}\\
0 &= \langle 200|200\rangle - \langle 020|200\rangle && \text{(3d)}
\end{aligned}
$$

but no information on the inner product $\langle 002|200\rangle$.

For more information we have to look for an equation using $\mathbf{L}^2$. We will need to flip operator order, so use

$$\left[a_i, a_j^\dagger\right] = a_i a_j^\dagger - a_j^\dagger a_i = \delta_{ij}$$

We will also need the "double epsilon" formula; see the derivation step in the second line of (3.6.17). Put all this together with (1) to find

$$
\begin{aligned}
\mathbf{L}^2 = L_i^2 &= (-\hbar^2)\epsilon_{ijk} a_j a_k^\dagger \epsilon_{ilm} a_l a_m^\dagger \\
&= (-\hbar^2)\left[\left(\delta_{jl}\delta_{km} - \delta_{jm}\delta_{kl}\right) a_j a_k^\dagger a_l a_m^\dagger\right] \\
&= (-\hbar^2)\left[a_j a_k^\dagger a_j a_k^\dagger - a_j a_k^\dagger a_k a_j^\dagger\right] \\
&= (-\hbar^2)\left[\left(a_k^\dagger a_j + \delta_{jk}\right)\left(a_k^\dagger a_j + \delta_{jk}\right) - a_j a_k^\dagger\left(a_j^\dagger a_k + \delta_{jk}\right)\right] \\
&= (-\hbar^2)\left[a_k^\dagger a_j a_k^\dagger a_j + 2N + 3 - a_j a_k^\dagger a_j^\dagger a_k - a_j a_j^\dagger\right] \\
&= (-\hbar^2)\left[a_k^\dagger\left(a_k^\dagger a_j + \delta_{jk}\right) a_j + 2N + 3 - a_j a_j^\dagger a_k^\dagger a_k - a_j a_j^\dagger\right] \\
&= (-\hbar^2)\left[a_k^\dagger a_k^\dagger a_j a_j + 3N + 3 - a_j a_j^\dagger (N+1)\right] \\
&= (-\hbar^2)\left[a_k^\dagger a_k^\dagger a_j a_j + 3N + 3 - (N+3)(N+1)\right] \\
\text{or}\qquad \mathbf{L}^2 &= (\hbar^2)\left[N(N+1) - a_k^\dagger a_k^\dagger a_j a_j\right] && \text{(4)}
\end{aligned}
$$

Don't forget the implied summation of repeated indices.

Now (4) gives us information on the inner product $\langle 002|200\rangle$. We know that

$$
\begin{aligned}
\langle 002|a_k^\dagger a_k^\dagger a_j a_j &= \langle 002|\left(a_x^\dagger a_x^\dagger + a_y^\dagger a_y^\dagger + a_z^\dagger a_z^\dagger\right)\left(a_x a_x + a_y a_y + a_z a_z\right) \\
&= \sqrt{2}\langle 000|\left(a_x a_x + a_y a_y + a_z a_z\right) \\
&= 2\left(\langle 200| + \langle 020| + \langle 002|\right) && \text{(5)}
\end{aligned}
$$

and so (4) gives us

$$0 = 6\langle 002|200\rangle - 2\left(\langle 200|200\rangle + \langle 020|200\rangle + \langle 002|200\rangle\right)$$

which combined with (3) tells us that

$$|200\rangle = \frac{1}{\sqrt{3}}|200\rangle + \frac{1}{\sqrt{3}}|020\rangle + \frac{1}{\sqrt{3}}|002\rangle$$

Since $H_2(w) = 4w^2 - 2$, converting this to coordinate space combines all the angular dependence into $x^2 + y^2 + z^2 = r^2$, that is, no dependence on θ or ϕ. Since the spherical harmonic here is $Y_0^0(\theta, \phi)$, this is once again correct.

For second excited states $|qlm\rangle = |02m\rangle$ we have

$$\mathbf{L}^2 = 6\hbar^2 = N(N+1)\hbar^2$$

so that application of (4) tells us that

$$0 = \langle n_x n_y n_z| \left(a_x^\dagger a_x^\dagger + a_y^\dagger a_y^\dagger + a_z^\dagger a_z^\dagger\right) (a_x a_x + a_y a_y + a_z a_z) |02m\rangle$$

This equation yields no information for the states $\langle n_x n_y n_z| = \langle 110|$, $\langle 101|$, and $\langle 011|$. However for states $\langle n_x n_y n_z| = \langle 200|$, $\langle 020|$, and $\langle 002|$, (5) gives the same result each time. So

$$0 = \langle 200|02m\rangle + \langle 020|02m\rangle + \langle 002|02m\rangle \tag{6}$$

We need to go back to (2) for enough information to solve for these inner products. We have

$$\begin{aligned}
m\langle 110|02m\rangle &= i\sqrt{2}\left(\langle 200|02m\rangle - \langle 020|02m\rangle\right) && \text{(7a)}\\
m\langle 101|02m\rangle &= -i\langle 011|02m\rangle && \text{(7b)}\\
m\langle 011|02m\rangle &= +i\langle 101|02m\rangle && \text{(7c)}\\
m\langle 200|02m\rangle &= -i\sqrt{2}\langle 110|02m\rangle && \text{(7d)}\\
m\langle 020|02m\rangle &= +i\sqrt{2}\langle 110|02m\rangle && \text{(7e)}\\
m\langle 002|02m\rangle &= 0 && \text{(7f)}
\end{aligned}$$

Now we can consider the state $|qlm\rangle = |020\rangle$. We have

$$\begin{aligned}
0 &= \langle 101|020\rangle = \langle 011|020\rangle = \langle 110|020\rangle\\
0 &= \langle 200|020\rangle - \langle 020|020\rangle\\
\text{and} \qquad 0 &= \langle 200|020\rangle + \langle 020|020\rangle + \langle 002|020\rangle
\end{aligned}$$

which means that

$$|020\rangle = \frac{1}{\sqrt{6}}|200\rangle + \frac{1}{\sqrt{6}}|020\rangle - \frac{2}{\sqrt{6}}|002\rangle$$

and the behavior in coordinate space has angular dependence

$$(4x^2 - 2) + (4y^2 - 2) - 2(4z^2 - 2) = 4(\sin^2\theta - 2\cos^2\theta) = 4(1 - 3\cos^2\theta)$$

which is proportional to $Y_2^0(\theta, \phi)$, once again, as it should be.

Next, consider the state $|qlm\rangle = |022\rangle$. From (7) we have $\langle 020|022\rangle = -\langle 200|022\rangle$ and $\langle 110|022\rangle = i\sqrt{2}\langle 200|022\rangle$. We also find $\langle 002|022\rangle = \langle 101|022\rangle = \langle 011|022\rangle = 0$. Therefore

$$|022\rangle = \frac{1}{\sqrt{10}}|200\rangle - \frac{1}{\sqrt{10}}|020\rangle + i\sqrt{\frac{2}{10}}|110\rangle$$

which gives a (relatively normalized) angular dependence in coordinate space

$$\begin{aligned}
& \frac{1}{2\sqrt{2}}(4x^2-2) - \frac{1}{2\sqrt{2}}(4y^2-2) + i\sqrt{2}\frac{1}{\sqrt{2}}2x\frac{1}{\sqrt{2}}2y \\
= \;& \sqrt{2}\left[(x^2-y^2)+2ixy\right] \\
= \;& \sqrt{2}r^2\left[(\cos^2\phi - \sin^2\phi)\sin^2\theta + 2i\cos\phi\sin\phi\sin^2\theta\right] \\
= \;& \sqrt{2}r^2\left[\cos 2\phi + i\sin 2\phi\right]\sin^2\theta = \sqrt{2}r^2e^{2i\phi}\sin^2\theta \propto Y_2^2(\theta,\phi)
\end{aligned}$$

Finally, consider the state $|qlm\rangle = |021\rangle$. From (6) and (7) we find that all inner products are zero except for $\langle 011|021\rangle = i\langle 101|021\rangle$. Therefore

$$|021\rangle = \frac{1}{2\sqrt{2}}|101\rangle + \frac{i}{2\sqrt{2}}|011\rangle$$

which has an angular dependence in coordinate space

$$xz + iyz = r^2(\cos\phi + i\sin\phi)\sin\theta\cos\theta = r^2e^{i\phi}\sin\theta \propto Y_2^1(\theta,\phi)$$

that is, once again, the correct answer.

22. *Note: See the correction discussed at the end of this solution.* For convenience, here is reproduced the generating function of the Laguerre polynomials:

$$g(x,t) \equiv \frac{e^{-xt/(1-t)}}{1-t} \equiv \sum_{n=0}^{\infty} L_n(x)\frac{t^n}{n!}$$

Therefore

$$g(0,t) = \frac{1}{1-t} = \sum_{n=0}^{\infty} t^n = \sum_{n=0}^{\infty} L_n(0)\frac{t^n}{n!}$$

which shows that $L_n(0) = n!$. Also

$$g(x,t) = \left[1 - \frac{xt}{1-t} + \cdots\right] \times \left[1 + t + t^2 + \cdots\right]$$

which shows that the coefficient of t^0 is just unity for all x, so $L_0(x) = 1$. Now differentiate with respect to x and proceed

$$\begin{aligned}
\frac{\partial g}{\partial x} = -\frac{t}{1-t}g(x,t) &= \sum_{n=0}^{\infty} L_n'(x)\frac{t^n}{n!} \\
-\sum_{n=0}^{\infty} L_n(x)\frac{t^{n+1}}{n!} &= \sum_{n=0}^{\infty} L_n'(x)\frac{t^n}{n!} - \sum_{n=0}^{\infty} L_n'(x)\frac{t^{n+1}}{n!} \\
-\sum_{m=1}^{\infty} L_{m-1}(x)m\frac{t^m}{m!} &= \sum_{n=0}^{\infty} L_n'(x)\frac{t^n}{n!} - \sum_{m=1}^{\infty} L_{m-1}'(x)m\frac{t^m}{m!}
\end{aligned}$$

where we set $m = n-1$ in two of the summations. Now, the $n = 0$ term in the first summation on the right is just $L_0'(x) = 0$ since $L_0(x) = 1$ for all x. Therefore all summations can be taken from $n, m = 1$. So, combining this expression and equating term by term gives

$$L_n'(x) = nL_{n-1}'(x) - nL_{n-1}(x)$$

We now have what we need in order to calculate the $L_n(x)$ for $n \geq 1$. Proceeding

$$\begin{aligned}
L_1'(x) = L_0'(x) - L_0(x) = -1 &\qquad \text{so} \qquad L_1(x) = 1 - x \\
L_2'(x) = 2L_1'(x) - 2L_1(x) = -4 + 2x &\qquad \text{so} \qquad L_2(x) = 2 - 4x + x^2 \\
L_3'(x) = 3L_2'(x) - 3L_2(x) = -18 + 18x - 3x^2 &\qquad \text{so} \qquad L_3(x) = 6 - 18x + 9x^2 - x^3
\end{aligned}$$

Note that mathematicians frequently define the $L_n(x)$ such that they are smaller by a factor of $n!$. Now differentiate with respect to t to get $\partial g/\partial t$ both ways, i.e.

$$\begin{aligned}
\left[\frac{1}{1-t} - \frac{xt}{(1-t)^2} - \frac{x}{1-t}\right] g(x,t) &= \sum_{n=0}^{\infty} L_n(x)\frac{nt^{n-1}}{n!} \\
\frac{1-x-t}{(1-t)^2}\sum_{n=0}^{\infty} L_n(x)\frac{t^n}{n!} &= \sum_{n=0}^{\infty} L_n(x)n\frac{t^{n-1}}{n!} = \sum_{n=0}^{\infty} L_{n+1}(x)\frac{t^n}{n!} \\
\sum_{n=0}^{\infty} L_n(x)\frac{1}{n!}\left[(1-x)t^n - t^{n+1}\right] &= \sum_{n=0}^{\infty} L_{n+1}(x)\frac{1}{n!}\left[t^n - 2t^{n+1} + t^{n+2}\right] \\
\sum_{n=0}^{\infty} \left[(1-x)L_n(x) - nL_{n-1}(x)\right]\frac{t^n}{n!} &= \sum_{n=0}^{\infty} \left[L_{n+1}(x) - 2nL_n(x) + n(n-1)L_{n-1}(x)\right]\frac{t^n}{n!}
\end{aligned}$$

where we note that the first two terms in the summation for the second and third terms on the right are explicitly zero. So, equating term by term, the recursion relation becomes

$$L_{n+1}(x) = (2n+1-x)L_n(x) - n^2L_{n-1}(x)$$

Now combine the recursion relations to get a differential equation for $L_n(x)$. First, use the two recursion relations to get different expressions for $L_{n+1}'(x)$, namely

$$\begin{aligned}
L_{n+1}' &= (n+1)(L_n' - L_n) \\
\text{and} \qquad L_{n+1}' &= (2n+1-x)L_n' - L_n - n^2L_{n-1}'
\end{aligned}$$

Equate these two expressions and cancel common terms to find

$$-nL_n = (n-x)L_n' - n^2 L_{n-1}' = n(L_n' - nL_{n-1}') - xL_n' = -n^2 L_{n-1} - xL_n'$$

where we used the first recursion relation once again. Now we have a recursion relation that just needs the function one order below, not two. Simplify, differentiate, and subtract to get

$$\begin{aligned} xL_n' &= nL_n - n^2 L_{n-1} \\ xL_n'' + L_n' &= nL_n' - n^2 L_{n-1}' \\ xL_n'' + (1-x)L_n' &= nL_n' - nL_n - n^2(L_{n-1}' - L_{n-1}) = nL_n' - nL_n - nL_n' = -nL_n \end{aligned}$$

where the first recursion relation is once again used, in the second-to-last step. This is what we are going for, namely a differential equation for $L_n(x)$:

$$xL_n''(x) + (1-x)L_n'(x) + nL_n(x) = 0$$

Contrary to what is said in the problem statement, this is *not* Kummer's equation. Instead, one needs to work with the "associated" Laguerre polynomials, which can be defined as

$$L_n^k(x) = (-1)^k \frac{d^k}{dx^k}\left[L_{n+k}(x)\right]$$

and which satisfy instead the equation

$$xL_n^{k\prime\prime}(x) + (k+1-x)L_n^{k\prime}(x) + nL_n^k(x) = 0$$

This is easy enough to see, starting from the differential equation for $L_{n+k}(x)$, namely

$$xL_{n+k}'' + (1-x)L_{n+k}' + (n+k)L_{n+k} = 0$$

and taking the derivative of the left side k times. The first term will give $xd^k L_{n+k}''/dx^k$ plus k occurrences of $d^k L_{n+k}'/dx^k$, which adds a k inside the parentheses in the second term. The $-x$ inside theses parentheses will remain for half of the k derivatives, and also contribute k terms $d^k L_{n+k}/dx^k$ with alternating signs. These alternating signs exactly cancel the factor of k in the third term, when the $(-1)^k$ is included in the definition of the associated Laguerre polynomial, so the associated Laguerre equation is satisfied. (Work it out for a couple of cases like $k=1$ and $k=2$ and watch this in action.) This is Kummer's Equation (3.7.46) with $c = 2(l+1) = k+1$, i.e. $k = 2l+1$, and $a = l+1-\rho_0/2 = -n$, or $n = \rho_0/2 - l - 1 = N$ as defined in (3.7.50).

23. *This solution reprinted from the solutions manual for the revised edition.*

Here $K_+ \equiv a_+^\dagger a_-^\dagger$ and $K_- = a_+ a_-$. Hence in the Schwinger scheme

$$K_+|n_+,n_-\rangle = \sqrt{(n_++1)(n_-+1)}\,|n_++1,n_-+1\rangle,\quad K_-|n_+,n_-\rangle = \sqrt{n_+n_-}\,|n_+-1,n_--1\rangle. \quad (1)$$

Let $j = (n_++n_-)/2$ and $m = (n_+-n_-)/2$, and $|n_+,n_-\rangle \to |j,m\rangle$. Than (1) can be rewritten as

$$K_+|j,m\rangle = \sqrt{(j+m+1)(j-m+1)}\,|j+1,m\rangle,\quad K_-|j,m\rangle = \sqrt{(j+m)(j-m)}\;|j-1,m\rangle \quad (2)$$

i.e. K_+, K_- are the raising and lowering operators for $j = (n_++n_-)/2$ where n_++n_- corresponds to the total number of spin $\frac{1}{2}$ "particles". The nonvanishing matrix elements of $K_\pm$ are from (2)

$$\begin{aligned} \langle j',m'|K_+|j,m\rangle &= \sqrt{(j+m+1)(j-m+1)}\,\delta_{j',j+1}\delta_{m',m}\,, \\ \langle j',m'|K_-|j,m\rangle &= \sqrt{(j+m)(j-m)}\,\delta_{j',j-1}\delta_{m',m}\,. \end{aligned} \quad (3)$$

24. *This solution reprinted from the solutions manual for the revised edition.*

We are to add angular momenta $j_1 = 1$ and $j_2 = 1$ to form $j = 2,1,0$ states. Express all nine $\{j,m\}$ eigenkets in terms of $|j_1j_2,m_1m_2\rangle$. The simplest states are $j_1=1, m_1=\pm1$; $j_2=1, m_2=\pm1$, i.e. $|j=2,m=2\rangle = |{++}\rangle$ and likewise $|j=2,m=-2\rangle = |{--}\rangle$. Using the ladder operator method we have $J_- = J_{1-} \oplus J_{2-}$ and (setting $\hbar = 1$ for convenience) from (3.5.40) $J_-|j,m\rangle = \sqrt{(j+m)(j-m+1)}\,|j,m-1\rangle$. So $J_-|j=2,m=2\rangle = \sqrt{4}\,|j=2,m=1\rangle = (J_{1-} \oplus J_{2-})|j_1=1,j_2=1;\ m_1=1,m_2=1\rangle = \sqrt{2}\,|0+\rangle + \sqrt{2}\,|{+0}\rangle$, i.e.

$|j=2,m=1\rangle = \frac{1}{2^{1/2}}(|0+\rangle + |+0\rangle)$. Now $J_-|j=2,m=1\rangle = \sqrt{6}|j=2,m=0\rangle = (J_{1-} \oplus J_{2-}) \times [\frac{1}{2^{1/2}}(|0+\rangle + |+0\rangle)] = |-+\rangle + 2|00\rangle + |+-\rangle$. Hence $|j=2,m=0\rangle = \frac{1}{6^{1/2}}(|-+\rangle + 2|00\rangle + |+-\rangle)$. Also $J_-|j=2,m=0\rangle = \sqrt{6}|j=2,m=-1\rangle = \frac{1}{6^{1/2}}(\sqrt{2}|-0\rangle + 2\sqrt{2}|0-\rangle + 2\sqrt{2}|-0\rangle + \sqrt{2}|0-\rangle)$, therefore $|j=2,m=-1\rangle = \frac{1}{2^{1/2}}(|-0\rangle + |0-\rangle)$.

For the j=1 states, let us recognize that $|11\rangle = a|0+\rangle + b|+0\rangle$ with normalization $|a|^2 + |b|^2 = 1$. Since $\langle 21|11\rangle = 0$ by orthogonality, we have $a+b = 0$. Choosing our phase convention to be real, we can write $|11\rangle = \frac{1}{2^{1/2}}(|+0\rangle - |0+\rangle)$. Applying next $J_- = J_{1-} \oplus J_{2-}$ to the two sides respectively, we have $|10\rangle = \frac{1}{2^{1/2}}(|+-\rangle - |-+\rangle)$ and similarly $|1-1\rangle = \frac{1}{2^{1/2}}(|0-\rangle - |-0\rangle)$.

Finally we may write $|j=0,m=0\rangle = \alpha|+-\rangle + \beta|00\rangle + \gamma|-+\rangle$, determine α,β,γ by normalization $|\alpha|^2 + |\beta|^2 + |\gamma|^2 = 1$ and orthogonality to $|j=1,m=0\rangle$ and $|j=2,m=0\rangle$. Choosing α,β,γ to be real we have $|j=0,m=0\rangle = \frac{1}{3^{1/2}}(|+-\rangle - |00\rangle + |-+\rangle)$.

25. *This solution reprinted from the solutions manual for the revised edition.*

(a) Recall (3.5.50) and (3.5.51) that $d^{(j)}_{mm'}(\beta) = \langle jm|\mathcal{D}(\alpha=0,\beta,\gamma=0)|jm'\rangle = \langle jm|\mathcal{D}(R)|jm'\rangle$ where $\mathcal{D}^\dagger(R)J_z\mathcal{D}(R) = \sum_{q'} \mathcal{D}^{(1)*}_{oq'}(R)T^{(1)}_{q'}$ (from (3.10.22a)) and recognizing that J_z is a first rank tensor with q=0, i.e. $T^{(1)}_0$, we have

$$\frac{1}{\hbar}\langle jm'|\mathcal{D}^\dagger(R)J_z\mathcal{D}(R)|jm'\rangle = \frac{1}{\hbar}\sum_{m=-j}^{j}\langle jm'|\mathcal{D}^\dagger(R)J_z|jm\rangle\langle jm|\mathcal{D}(R)|jm'\rangle$$
$$= \sum_{m=-j}^{j}|\langle jm|\mathcal{D}(R)|jm'\rangle|^2 m. \qquad (1)$$

Similarly since only $q' = 0$ contributes, we have

$$\frac{1}{\hbar}\langle jm'|\sum_{q'}\mathcal{D}^{(1)*}_{oq'}T^{(1)}_{q'}|jm'\rangle = \frac{1}{\hbar}\langle jm'|\mathcal{D}^{(1)*}_{oo}(R)J_z|jm'\rangle$$
$$= (4\pi/2\ell+1)^{1/2}\, Y^o_o(\theta=\beta,\phi=0)m' = m'\cos\beta. \qquad (2)$$

So finally from (1) and (2), we have

$$\sum_{m=-j}^{j}|d^{(j)}_{mm'}(\beta)|^2 m = m'\cos\beta. \qquad (3)$$

Check for $j=\frac{1}{2}$, we have from (3.5.52) $d^{(1/2)} = \begin{pmatrix}\cos\frac{\beta}{2} & -\sin\frac{\beta}{2}\\ \sin\frac{\beta}{2} & \cos\frac{\beta}{2}\end{pmatrix}$. For $m' = \frac{1}{2}$ case, l.h.s. of (3) $= \frac{1}{2}\cos^2\frac{\beta}{2} + (-\frac{1}{2})\sin^2\frac{\beta}{2} = \frac{1}{2}\cos\beta$ = r.h.s. of (3); for $m' = -\frac{1}{2}$ case,

l.h.s. of (3) $= \frac{1}{2}(-\sin\frac{\beta}{2})^2 + (-\frac{1}{2})\cos^2\frac{\beta}{2} = -\frac{1}{2}\cos\beta$ = r.h.s. of (3).

(b) From (3.5.51), with $\hbar=1$, we note $d^{(j)}_{m'm}(\beta) = \langle jm'|e^{-i\beta J_y}|jm\rangle$. Now

$$\sum_{m=-j}^{j} m^2 |d^{(j)}_{m'm}(\beta)|^2 = \sum_{m=-j}^{j} m^2 \langle jm'|e^{-i\beta J_y}|jm\rangle\langle jm|e^{i\beta J_y}|jm'\rangle$$

$$\sum_{m=-j}^{j} \langle jm'|e^{-i\beta J_y}J_z^2|jm\rangle\langle jm|e^{i\beta J_y}|jm'\rangle = \langle jm'|e^{-i\beta J_y}J_z^2 e^{i\beta J_y}|jm'\rangle$$

$$= \langle jm'|\mathcal{D}(R)J_z^2\mathcal{D}^\dagger(R)|jm'\rangle \tag{4}$$

If we examine the rotational properties of J_z^2 using the spherical (irreducible) tensor language, we find $J_z^2 = \frac{1}{3}(\vec{J}^2 + Y_o^{(2)})$ where $\vec{J}^2$ is a scalar under rotation and $Y_o^{(2)}$ is a spherical tensor of rank 2. Hence $\mathcal{D}(R)J_z^2\mathcal{D}^\dagger(R) = \frac{1}{3}\vec{J}^2 + \frac{1}{3}\mathcal{D}(R)Y_o^{(2)}\mathcal{D}^\dagger(R)$ with $\mathcal{D}(R)Y_o^{(2)}\mathcal{D}^\dagger(R) = \sum_{k'=-2}^{2}\mathcal{D}^{(2)}_{k'o}Y^{(2)}_{k'}$. Therefore (4) can be recast as

$$\sum_{m=-j}^{j} m^2|d^{(j)}_{m'm}(\beta)|^2 = \frac{1}{3}j(j+1) + \frac{1}{3}\sum_{k'=-2}^{2}\langle jm'|\mathcal{D}^{(2)}_{k'o}Y^{(2)}_{k'}|jm'\rangle. \tag{5}$$

In the last term on r.h.s. of (5), only k'=0 contributes and $\mathcal{D}^{(2)}_{oo} = \frac{(3\cos^2\theta-1)}{2}$ (from (3.6.53), (3.5.50), and (3.5.51)). Hence

$$\sum_{m=-j}^{j} m^2|d^{(j)}_{m'm}(\beta)|^2 = \frac{1}{3}j(j+1) + \frac{1}{3}\langle jm'|\mathcal{D}^{(2)}_{oo}(3J_z^2-\vec{J}^2)|jm'\rangle$$

$$= \frac{j(j+1)}{2}\sin^2\beta + \frac{m'^2}{2}(3\cos^2\beta - 1)$$

26. *This solution reprinted from the solutions manual for the revised edition.*

(a) We have $J_y = \frac{1}{2i}(J_+ - J_-)$, then using (3.5.41) we derive easily

$$\langle jm'|J_y|jm\rangle = \frac{\hbar}{2i}[\sqrt{j(j+1)-m(m+1)}\langle jm'|j,m+1\rangle - \sqrt{j(j+1)-m(m-1)}\langle jm'|j,m-1\rangle]$$

and therefore for m and m' = +1,0,−1 and j=1 one finds the matrix form for $\langle j=1,m'|J_y|j=1,m\rangle$ as depicted in (3.5.54).

(b) Unlike the $j=\frac{1}{2}$ case, for j=1 only $[J_y^{(j=1)}]^2$ is <u>independent</u> of $\underline{1}$ and $J_y^{(j=1)}$, and in fact we have $(J_y/\hbar)^{2m+1} = (J_y/\hbar)$ and $(J_y/\hbar)^{2n} = (J_y/\hbar)^2$ where m and n are positive integers. By expansion of the exponential $e^{-iJ_y\beta/\hbar}$ in power series

$$e^{-iJ_y\beta/\hbar} = \sum_{n=o}^{\infty}\frac{(-iJ_y\beta/\hbar)^{2n}}{(2n)!} + \sum_{m=o}^{\infty}\frac{(-iJ_y\beta/\hbar)^{2m+1}}{(2m+1)!}$$

$$= \underline{1} + (J_y/\hbar)^2 \sum_{n=1}^{\infty} \frac{(+\beta)^{2n}(-1)^n}{(2n)!} - i(J_y/\hbar) \sum_{m=0}^{\infty} \frac{(+\beta)^{2m+1}(-1)^m}{(2m+1)!}$$

$$= \underline{1} - (J_y/\hbar)^2(1-\cos\beta) - i(J_y/\hbar)\sin\beta \ .$$

(c) Insert the 3×3 matrix form for J_y from (a), i.e. (3.5.54), into the exponential of part (b) above, we find

$$d^{(j=1)}(\beta) = e^{-iJ_y\beta/\hbar} = \begin{pmatrix} \frac{1+\cos\beta}{2} & -\sin\beta/\sqrt{2} & \frac{1-\cos\beta}{2} \\ \sin\beta/\sqrt{2} & \cos\beta & -\sin\beta/\sqrt{2} \\ \frac{1-\cos\beta}{2} & \sin\beta/\sqrt{2} & \frac{1+\cos\beta}{2} \end{pmatrix}$$

which is (3.5.57).

27. *This solution reprinted from the solutions manual for the revised edition.*

$$\sum_{j,j'} \langle \alpha_2\beta_2\gamma_2|jmn\rangle\langle jmn|J_3^2|j'm'n'\rangle\langle j'm'n'|\alpha_1\beta_1\gamma_1\rangle = \langle \alpha_2\beta_2\gamma_2|J_3^2|\alpha_1\beta_1\gamma_1\rangle \quad (1)$$

where we note that $\langle jmn|J_3^2|j'm'n'\rangle = n^2\delta_{nn'}\delta_{jj'}\delta_{mm'}$. The l.h.s. of (1) is therefore $\sum_{j,m,n} n^2 \mathcal{D}^j_{mn}(\alpha_2\beta_2\gamma_2)\mathcal{D}^{j*}_{mn}(\alpha_1\beta_1\gamma_1)$.

(Solution courtesy of Professor Thomas Fulton)

28. *This solution reprinted from the solutions manual for the revised edition.*

We will represent states as in (3.7.15). For $S_{tot.} = 0$: $\psi = \frac{1}{2^{\frac{1}{2}}}[|+-\rangle - |-+\rangle]$.

(a) Since B makes no measurement there are equal probabilities for measuring s_{1z} to be $\hbar/2$ and $-\hbar/2$. The same is true for s_{1x} because there is no preferred spatial direction.

(b) Now B measures $s_{2z} = \hbar/2$. (i) Since $s_{1z} + s_{2z} = 0$, A must obtain $-\hbar/2$. Now s_{2z} has picked the second piece of ψ which is $\sim -|-+\rangle$, therefore $s_{1z}\{-|-+\rangle\} = +\frac{\hbar}{2}|-+\rangle$. (ii) Since we know that $s_{1z}\psi = (-\hbar/2)\psi$ we cannot predict s_{1x} because $[s_{1x},s_{1z}] \neq 0$ and $|\hat{z}-\rangle = \frac{1}{2^{\frac{1}{2}}}[|\hat{x}+\rangle - |\hat{x}-\rangle]$ as in (3.9.3) yield equal probabilities for $s_{1x} = \hbar/2$ and $-\hbar/2$.

29. *This solution reprinted from the solutions manual for the revised edition.*

$$\sum_{q'} d^{(1)}_{qq'} V^{(1)}_{q'} = \tfrac{1}{2}\begin{pmatrix} 1+\cos\beta & -\sqrt{2}\sin\beta & 1-\cos\beta \\ \sqrt{2}\sin\beta & 2\cos\beta & -\sqrt{2}\sin\beta \\ 1-\cos\beta & \sqrt{2}\sin\beta & 1+\cos\beta \end{pmatrix}\begin{pmatrix} V^{(1)}_{+} \\ V^{(1)}_{o} \\ V^{(1)}_{-} \end{pmatrix} = \begin{pmatrix} V^{(1)'}_{+} \\ V^{(1)'}_{o} \\ V^{(1)'}_{-} \end{pmatrix}. \qquad (1)$$

Rewrite r.h.s. in terms of (V_x, V_y, V_z), we have

$$\sum_{q'} d^{(1)}_{qq'} V^{(1)}_{q'} = \begin{pmatrix} -\cos\beta\, V_x/\sqrt{2} - iV_y/\sqrt{2} - \sin\beta\, V_z/\sqrt{2} \\ -\sin\beta\, V_x + \cos\beta\, V_z \\ \cos\beta\, V_x/\sqrt{2} - iV_y/\sqrt{2} + \sin\beta\, V_z/\sqrt{2} \end{pmatrix}. \qquad (2)$$

But a rotation through angle β about y-axis leads to $V_x \to V'_x = V_x\cos\beta + V_z\sin\beta$, $V'_y = V_y$, $V_z \to V'_z = V_z\cos\beta - V_x\sin\beta$. Therefore $V^{(1)'}_{+} = -(V'_x + iV'_y)/\sqrt{2} = \frac{-1}{2^{\frac{1}{2}}}\cos\beta\, V_x - iV_y/\sqrt{2} - \sin\beta\, V_z/\sqrt{2}$, $V^{(1)'}_{o} = V'_z = -\sin\beta\, V_x + V_z\cos\beta$, and $V^{(1)'}_{-} = (V'_x - iV'_y)/\sqrt{2} = \cos\beta\, V_x/\sqrt{2} - iV_y/\sqrt{2} + \sin\beta\, V_z/\sqrt{2}$. Thus the r.h.s. of (2) indeed gives r.h.s. of (1) which are just the expectations from the transformation properties of $V_{x,y,z}$ under rotations about the y-axis.

30. *This solution reprinted from the solutions manual for the revised edition.*

(a) Let us take (3.10.27) where $X_{q_1}^{(k_1)}$ and $Z_{q_2}^{(k_2)}$ are irreducible spherical tensors of rank k_1 and k_2 respectively. Then $T_q^{(k)} = \sum_{q_1}\sum_{q_2} \langle k_1k_2;q_1q_2|k_1k_2;kq\rangle X_{q_1}^{(k_1)} Z_{q_2}^{(k_2)}$ is a spherical (irreducible) tensor of rank k. For our problem $k_1 = k_2 = k = 1$, hence

$$T_q^{(1)} = \sum_{q_1}\sum_{q_2} \langle 11;q_1q_2|11;1q\rangle U_{q_1}^{(1)} V_{q_2}^{(1)} \tag{1}$$

From (1), we have $T_{-1}^{(1)} = \frac{1}{2^{1/2}}(-U_{-1}^{(1)}V_o^{(1)} + U_o^{(1)}V_{-1}^{(1)})$, $T_o^{(1)} = \frac{1}{2^{1/2}}(U_1^{(1)}V_{-1}^{(1)} - U_{-1}^{(1)}V_1^{(1)})$ and $T_1^{(1)} = \frac{1}{2^{1/2}}(-U_o^{(1)}V_1^{(1)} + U_1^{(1)}V_o^{(1)})$. In terms of $U_{x,y,z}$ and $V_{x,y,z}$, we have

$$\begin{aligned}
T_{-1}^{(1)} &= \tfrac{1}{2}[-(U_x-iU_y)V_z + (V_x-iV_y)U_z] \\
T_o^{(1)} &= \frac{i}{2^{1/2}}[U_xV_y - U_yV_x] \\
T_1^{(1)} &= \tfrac{1}{2}[-(U_x+iU_y)V_z + (V_x+iV_y)U_z]
\end{aligned} \tag{2}$$

(b) For $k_1 = k_2 = 1$, $k = 2$, we have

$$T_q^{(2)} = \sum_{q_1,q_2} \langle 11;q_1q_2|11;2q\rangle U_{q_1}^{(1)} V_{q_2}^{(1)} \ . \tag{3}$$

From (3), we find $T_{-2}^{(2)} = U_{-1}^{(1)}V_{-1}^{(1)}$, $T_{-1}^{(2)} = \frac{1}{2^{1/2}}(U_{-1}^{(1)}V_o^{(1)} + U_o^{(1)}V_{-1}^{(1)})$, $T_o^{(2)} = \frac{1}{6^{1/2}}(U_{-1}^{(1)}V_{+1}^{(1)} + 2U_o^{(1)}V_o^{(1)} + U_{+1}^{(1)}V_{-1}^{(1)})$, $T_1^{(2)} = \frac{1}{2^{1/2}}(U_{+1}^{(1)}V_o^{(1)} + U_o^{(1)}V_{+1}^{(1)})$, and $T_2^{(2)} = U_{+1}^{(1)}V_{+1}^{(1)}$. In terms of $U_{x,y,z}$ and $V_{x,y,z}$ we have

$$\begin{aligned}
&T_{-2}^{(2)} = \tfrac{1}{2}(U_x-iU_y)(V_x-iV_y),\quad T_{-1}^{(2)} = \tfrac{1}{2}[(U_x-iU_y)V_z + U_z(V_x-iV_y)], \\
&T_o^{(2)} = \frac{1}{2\sqrt{6}}[-(U_x-iU_y)(V_x+iV_y) + 4U_zV_z - (U_x+iU_y)(V_x-iV_y)], \\
&T_1^{(2)} = -\tfrac{1}{2}[(U_x+iU_y)V_z + U_z(V_x+iV_y)],\quad T_2^{(2)} = \tfrac{1}{2}(U_x+iU_y)(V_x+iV_y)
\end{aligned} \tag{4}$$

(Remark: (3) is similar to $Y_2^m = \sum_{m_1m_2} \langle 11;m_1m_2|11;2m\rangle Y_1^{m_1}Y_2^{m_2}$ for spherical harmonics)

31. *This solution reprinted from the solutions manual for the revised edition.*

(a) According to (3.10.31), the Wigner-Eckart theorem for our problem where $R^{(1)}_{\pm1} = \mp \frac{1}{2^{1/2}}(x \pm iy)$ and $R^{(1)}_{o} = z$ form three components of a spherical tensor of rank 1, reads

$$\langle n',\ell',m'|R^{(1)}_{q}|n,\ell,m\rangle = \frac{\langle \ell 1;mq|\ell 1;\ell' m'\rangle\langle n'\ell'||R^{(1)}||n\ell\rangle}{\sqrt{2\ell+1}} \tag{1}$$

where the "double bar" matrix element is independent of m and m'. Since $\langle \ell 1;mq|\ell 1;\ell' m'\rangle = 0$ unless $m' = m+q$ and $\ell' = |\ell\pm1|,\ell$, therefore $\langle n',\ell',m'|R^{(1)}_{q}|n,\ell,m\rangle = 0$ unless $m' = m+q$ and $\ell' = |\ell\pm1|,\ell$.

Furthermore, since we are dealing with a central force potential, the $|n,\ell,m\rangle$ are eigenstates of U_P (parity operator). Hence $U_P|n,\ell,m\rangle = (-1)^{\ell}|n,\ell,m\rangle$ and $U_P^{-1}R^{(1)}U_P = -R^{(1)}$ and we have $-\langle n',\ell',m'|R^{(1)}|n,\ell,m\rangle = (-1)^{\ell}(-1)^{\ell'}\langle n',\ell',m'|R^{(1)}|n,\ell,m\rangle$ or $\ell+\ell' =$ odd. Combine with Clebsch-Gordan selection rule from (1), we have

$$\langle n',\ell',m'|R^{(1)}_{q}|n,\ell,m\rangle = 0, \text{ unless } m'=m+q,\ \ell' = |\ell\pm1|. \tag{2}$$

Again from (1), we have

$$\frac{\langle n',\ell',m_1'|R^{(1)}_{\pm1}|n,\ell,m_1\rangle}{\langle n',\ell',m_2'|R^{(1)}_{o}|n,\ell,m_2\rangle} = \frac{\langle \ell 1;m_1,\pm1|\ell 1;\ell' m_1'\rangle}{\langle \ell 1;m_2,0|\ell 1;\ell' m_2'\rangle} \tag{3}$$

where ℓ',m' satisfy selection rule (2).

(b) Use now wave function $\psi(\vec{x}) = R_{n\ell}(r)Y^{m}_{\ell}(\theta,\phi)$. We have

$$\langle n',\ell',m'|R^{(1)}_{\pm,o}|n,\ell,m\rangle = \int R^{*}_{n'\ell'}(r)Y^{m'*}_{\ell'}(\theta,\phi)[R^{(1)}_{\pm1,o}]R_{n\ell}(r)Y^{m}_{\ell}(\theta,\phi)d^3\vec{x}$$

$$= \sqrt{4\pi/3}\int_0^\infty R^{*}_{n'\ell'}(r)r^3R_{n\ell}(r)dr\int d\Omega Y^{m'*}_{\ell'}(\theta,\phi)Y^{\pm,o}_{1}(\theta,\phi)Y^{m}_{\ell}(\theta,\phi). \tag{4}$$

Let $\overline{r^3_{n'n,\ell'\ell}} = \int_0^\infty r^3R^{*}_{n'\ell'}(r)R_{n\ell}(r)dr$, than (4) reads (using (3.7.73))

$$\langle n',\ell',m'|R^{(1)}_{q}|n,\ell,m\rangle = (4\pi/3)^{1/2}\overline{r^3_{n'n,\ell'\ell}}\sqrt{(2\ell+1)3/4\pi(2\ell'+1)}\langle \ell 1;00|\ell 1;\ell' 0\rangle \times$$
$$\langle \ell 1;mq|\ell' m'\rangle = \overline{r^3_{n'n,\ell'\ell}}\frac{(2\ell+1)^{1/2}}{(2\ell'+1)^{1/2}}\langle \ell 1;00|\ell 1;\ell' 0\rangle\langle \ell 1;mq|\ell' m'\rangle,\ \ell\neq\ell' \tag{5}$$
$$= 0, \text{ if } \ell = \ell'$$

where $q = \pm1,0$. We have thus the selection rule

$$\langle n',\ell',m'|R^{(1)}_{q}|n,\ell,m\rangle = 0 \text{ unless } m' = m+q,\ \ell' = |\ell\pm1| \tag{6}$$

which is identical to part (a). Also note from (5) we have at once the ratio equality (3) where $\ell' = |\ell\pm1|$, $m_1' = m_1\pm1$, $m_2' = m_2$.

32. *This solution reprinted from the solutions manual for the revised edition.*

(a) From (3.10.17), $Y_2^{\pm 2} = (\frac{15}{32\pi})^{\frac{1}{2}}\frac{(x^2-y^2\pm 2ixy)}{r^2}$, thence $xy = i(\frac{2\pi}{15})^{\frac{1}{2}}(Y_2^{-2}-Y_2^{+2})r^2$.

Similarly, by $Y_2^{\pm 1} = \mp(\frac{15}{8\pi})^{\frac{1}{2}}\frac{(x\pm iy)z}{r^2}$, we have $xz = (\frac{2\pi}{15})^{\frac{1}{2}}(Y_2^{-1}- Y_2^{+1})r^2$, and again by $Y_2^{\pm 2}$ we have $x^2-y^2 = (8\pi/15)^{\frac{1}{2}}(Y_2^{2}+Y_2^{-2})r^2$. Note $Y_2^m(m=0,\pm 1,\pm 2)$ are components of a spherical (irreducible) tensor of rank 2.

(b) $Q \equiv e\langle\alpha,j,m=j|(3z^2-r^2)|\alpha,j,m=j\rangle$. First note that $Y_2^0 = (\frac{5}{16\pi})^{\frac{1}{2}}\frac{(3z^2-r^2)}{r^2}$, hence

$Q = e\langle\alpha,j,j|\sqrt{16\pi/5}\; r^2 Y_2^0|\alpha,j,j\rangle$. Now apply the Wigner-Eckart theorem (3.10.31), we have

$$Q = e(\frac{16\pi}{5})^{\frac{1}{2}}\frac{\langle j2;j0|j2;jj\rangle\langle\alpha j||r^2Y_2||\alpha j\rangle}{\sqrt{2j+1}} \tag{1}$$

By the same token use of W-E theorem on $e\langle\alpha,j,m'|(x^2-y^2)|\alpha,j,m=j\rangle = e(\frac{8\pi}{15})^{\frac{1}{2}}\times$ $\langle\alpha,j,m'|r^2(Y_2^2 + Y_2^{-2})|\alpha,j,m=j\rangle$ leads to

$$\begin{aligned} &e(\frac{8\pi}{15(2j+1)})^{\frac{1}{2}}[\langle j2;j2|j2;jm'\rangle\langle\alpha j||r^2Y_2||\alpha j\rangle+\langle j2;j-2|j2;jm'\rangle\langle\alpha j||r^2Y_2||\alpha j\rangle] \\ &= e\sqrt{8\pi/15(2j+1)}\langle j2;j-2|j2;jm'\rangle\langle\alpha j||r^2Y_2||\alpha j\rangle. \end{aligned} \tag{2}$$

Substitute $\langle\alpha j||r^2Y_2||\alpha j\rangle$ of (1) into (2), we have

$$e\langle\alpha,j,m'|(x^2-y^2)|\alpha,j,m=j\rangle = (1/\sqrt{2})[\frac{\langle j2;j-2|j2;jm'\rangle}{\langle j2;j0|j2;jj\rangle}]Q. \tag{3}$$

33. *This solution reprinted from the solutions manual for the revised edition.*

In expression for $H_{int.}$, we recognize that $S_x^2 = \frac{1}{4}(S_+^2 + S_-^2 + \{S_+,S_-\})$ and $S_y^2 = -\frac{1}{4}(S_+^2 + S_-^2 - \{S_+,S_-\})$ with $S_\pm = S_x \pm iS_y$ and $\{S_+,S_-\} = 2(\vec{S}^2 - S_z^2)$. Thus

$$H_{int.} = \frac{eQ}{2s(s-1)\hbar^2}[(\frac{\partial^2\phi}{\partial x^2})_o \frac{1}{4}\{S_+^2 + S_-^2 + 2(\vec{S}^2 - S_z^2)\} + (\frac{\partial^2\phi}{\partial y^2})_o \frac{[2(\vec{S}^2 - S_z^2) - S_+^2 - S_-^2]}{4}$$

$$+(\frac{\partial^2\phi}{\partial z^2})_o S_z^2]$$

$$= \frac{eQ}{2s(s-1)\hbar^2}[\frac{1}{4}\{(\frac{\partial^2\phi}{\partial x^2})_o - (\frac{\partial^2\phi}{\partial y^2})_o\}(S_+^2+S_-^2) + \frac{1}{2}\{(\frac{\partial^2\phi}{\partial x^2})_o + (\frac{\partial^2\phi}{\partial y^2})_o\}(\vec{S}^2 - S_z^2) + (\frac{\partial^2\phi}{\partial z^2})_o S_z^2].$$

Using $\nabla^2\phi = 0$, we can write

$$H_{int.} = A(3S_z^2 - \vec{S}^2) + B(S_+^2 + S_-^2) \tag{1}$$

where $A = \frac{eQ}{4s(s-1)\hbar^2}(\partial^2\phi/\partial z^2)_o$ and $B = \frac{eQ}{8s(s-1)\hbar^2}\{(\partial^2\phi/\partial x^2)_o - (\partial^2\phi/\partial y^2)_o\}$.

From (1) we note that $H_{int.}$ acts on states of definite $|s,m\rangle$ where $s=3/2$ as follows:-

$$H_{int.}|sm\rangle = A(3S_z^2 - \vec{S}^2)|sm\rangle + B(S_+^2 + S_-^2)|sm\rangle$$

$$= 3Am^2\hbar^2|sm\rangle - \frac{15A\hbar^2}{4}|sm\rangle + B\sqrt{(s-m)(s+m+1)(s-m-1)(s+m+2)}\,\hbar^2|s,m+2\rangle$$

$$+ B\sqrt{(s+m)(s-(m-1))(s+m-1)(s-(m-2))}\,\hbar^2|s,m-2\rangle. \tag{2}$$

In the $m,m' = 3/2,-1/2$ and $m,m' = 1/2, -3/2$ basis, the matrix $H_{int.}$ using (2) can be written in block form as

$$H_{int.}^{mm'} = \begin{pmatrix} 3A & 2\sqrt{3}B & 0 & 0 \\ 2\sqrt{3}B & -3A & 0 & 0 \\ 0 & 0 & -3A & 2\sqrt{3}B \\ 0 & 0 & 2\sqrt{3}B & 3A \end{pmatrix}\hbar^2 . \tag{3}$$

Diagonalizing each block of (3), we see that $\lambda_{\pm}\hbar^2 = \pm(12B^2 + 9A^2)^{1/2}\hbar^2$ are the energy eigenvalues for both m,m' = 3/2, -1/2 <u>and</u> m,m' = 1/2, -3/2 basis. The eigenstates $\begin{pmatrix}\alpha_1\\ \alpha_2\end{pmatrix}$ can be determined for each 2×2 matrix block as

$$\begin{pmatrix} 3A & 2\sqrt{3}B \\ 2\sqrt{3}B & -3A \end{pmatrix}\begin{pmatrix}\alpha_1\\ \alpha_2\end{pmatrix} = \lambda_{\pm}\begin{pmatrix}\alpha_1\\ \alpha_2\end{pmatrix}, \quad \begin{pmatrix} -3A & 2\sqrt{3}B \\ 2\sqrt{3}B & 3A \end{pmatrix}\begin{pmatrix}\alpha_1\\ \alpha_2\end{pmatrix} = \lambda_{\pm}\begin{pmatrix}\alpha_1\\ \alpha_2\end{pmatrix} \;. \tag{4}$$

Hence for m,m' = 3/2, -1/2 we have $\alpha_2/\alpha_1 = (\lambda_{\pm}-3A)/2\sqrt{3}B$, while for m,m' = -3/2, 1/2 we have $\alpha_2/\alpha_1 = (\lambda_{\pm}+3A)/2\sqrt{3}B$. The energy eigenstates are

$$|\lambda_{\pm}\rangle = 2\sqrt{3}B\,|3/2,3/2\rangle + (\lambda_{\pm} - 3A)\,|3/2,-1/2\rangle \tag{5a}$$

$$|\lambda_{\pm}\rangle = 2\sqrt{3}B\,|3/2,-3/2\rangle + (\lambda_{\pm} + 3A)\,|3/2,+1/2\rangle. \tag{5b}$$

Note from (5a) and (5b), there exists a two-fold degeneracy, namely there exists two states corresponding to each value of $\lambda(\lambda_+$ and $\lambda_-)$.

Chapter Four

1. *This solution reprinted from the solutions manual for the revised edition.*

(a) Assume these particles can be distinguished, in other words they are non-identical particles. Since the three particles do not interact, so the Hamiltonian operator $H = -\frac{\hbar^2}{2m}\nabla_1^2 - \frac{\hbar^2}{2m}\nabla_2^2 - \frac{\hbar^2}{2m}\nabla_3^2 + V(1,2,3)$ can be separated, thus the energy for particle i is $E^{(i)} = (\hbar^2\pi^2/2mL^2)\sum_{j=1}^{3} n_{ij}^2$ where n_{ij} are non-zero integers, and the total energy for the system is $E = E^{(1)}+E^{(2)}+E^{(3)} = \frac{\hbar^2\pi^2}{2mL^2}\sum_{i,j=1}^{3} n_{ij}^2$.

Obviously the lowest energy state is the state with all indices $n_{ij} = 1$, and $E_1 = \frac{9\hbar^2\pi^2}{2mL^2}$. The second lowest energy will be $E_2 = \frac{\hbar^2\pi^2}{2mL^2}(2^2+\underbrace{1+\ldots\ldots+1}_{8\text{ times}}) = \frac{12\hbar^2\pi^2}{2mL^2}$ $= 6\hbar^2\pi^2/mL^2$. The third lowest energy will be $E_3 = \frac{\hbar^2\pi^2}{2mL^2}(2^2+2^2+\underbrace{1+\ldots\ldots+1}_{7\text{ times}}) =$ $15\hbar^2\pi^2/2mL^2$.

<u>Degeneracy</u>. For energy E_1, we have only <u>one</u> spatial wave function, because all indices are 1. For energy E_2, we have 9 spatial wave functions. The reason is that the nine indices (n_{ij}) with i,j = 1,2,3 are such that each of them has an equal chance to be 2, while others equal to 1. So the number of distinct possibilities is $\frac{9!}{(9-1)!1!} = 9$. Evidently for E_3 we have $\frac{9!}{(9-2)!2!} = 36$ distinct spatial wave functions. In addition we have $2^3 = 8$ spin wave functions, they are $|+++\rangle, |++-\rangle, |+-+\rangle, |+--\rangle, |-++\rangle, |-+-\rangle, |--+\rangle$, and $|---\rangle$. So in short E_1 has degeneracy 1x8 = 8, E_2 has degeneracy 9x8 = 72, while E_3 has degeneracy 36x8 = 288.

(b) For four non-identical spin-½ particles system, we have total energy $E = \frac{\hbar^2\pi^2}{2mL^2}\sum_{i=1}^{4}\sum_{j=1}^{3} n_{ij}^2$ where i refers to the i^{th} particle while j refers to the three dimensional space index. Therefore $E_1 = 12\hbar^2\pi^2/2mL^2 = 6\pi^2\hbar^2/mL^2$ and again the degeneracy for spatial wave function is 1. $E_2 = 15\hbar^2\pi^2/2mL^2$ and the number of distinct spatial wave function is $\frac{12!}{(12-1)!1!} = 12$. $E_3 = 9\hbar^2\pi^2/mL^2$, and the number of distinct spatial wave function is $\frac{12!}{(12-2)!2!} = 66$. At the same time we have $2^4 = 16$ spin wave functions $|{+}{+}{+}{+}\rangle, |{+}{+}{+}{-}\rangle, |{+}{+}{-}{+}\rangle, |{+}{+}{-}{-}\rangle$, etc. Hence the three lowest energy levels have degeneracies $16\text{x}1 = 16$ for E_1, $16\text{x}12 = 192$ for E_2, and $16\text{x}66 = 1056$ for E_3.

2. *This solution reprinted from the solutions manual for the revised edition.*

(a) $\mathcal{T}_{\vec{d}}\psi(\vec{x}) = \psi(\vec{x}+\vec{d})$, $\mathcal{T}_{\vec{d}'}\mathcal{T}_{\vec{d}}\psi(\vec{x}) = \psi(\vec{x}+\vec{d}'+\vec{d})$ and $\mathcal{T}_{\vec{d}}\mathcal{T}_{\vec{d}'}\psi(\vec{x}) = \mathcal{T}_{\vec{d}}\psi(\vec{x}+\vec{d}') = \psi(\vec{x}+\vec{d}+\vec{d}')$, so $[\mathcal{T}_{\vec{d}},\mathcal{T}_{\vec{d}'}]\psi(\vec{x}) = 0$. Since $\psi(\vec{x})$ is arbitrary, we have $[\mathcal{T}_{\vec{d}},\mathcal{T}_{\vec{d}'}] = 0$. They commute.

(b) $\mathcal{D}(\hat{n},\phi)$ does not commute with $\mathcal{D}(\hat{n}',\phi')$. This is easily seen by taking the case $\hat{n} = \hat{x}$, $\hat{n}' = \hat{y}$ where we know the rotation around x-axis does not commute with the rotation around y-axis.

(c) $\mathcal{T}_{\vec{d}}$ and Π do not commute. $\Pi\psi(\vec{x}) = \psi(-\vec{x})$ while $\mathcal{T}_{\vec{d}}\Pi\psi(\vec{x}) = \psi(-\vec{x}+\vec{d})$. On the other hand, $\mathcal{T}_{\vec{d}}\psi(\vec{x}) = \psi(\vec{x}+\vec{d})$ while $\Pi\mathcal{T}_{\vec{d}}\psi(\vec{x}) = \psi(-\vec{x}-\vec{d}) \neq \mathcal{T}_{\vec{d}}\,\Pi\psi(\vec{x})$. Hence $[\Pi,\mathcal{T}_{\vec{d}}] \neq 0$.

(d) $\Pi\mathcal{D}(\hat{n},\phi)\psi(\vec{x}) = \Pi\psi(\vec{x}') = \psi(-\vec{x}')$ where $\vec{x}' = \mathcal{D}(\hat{n},\phi)\vec{x}$. On the other hand, $\mathcal{D}(\hat{n},\phi)\Pi\psi(\vec{x}) = \mathcal{D}(\hat{n},\phi)\psi(-\vec{x}) = \psi(-\vec{x}')$. So $\Pi\mathcal{D}(\hat{n},\phi)\psi(\vec{x}) = \mathcal{D}(\hat{n},\phi)\Pi\psi(\vec{x})$ and since $\psi(\vec{x})$ is arbitrary, we have $[\Pi,\mathcal{D}(\hat{n},\phi)] = 0$. They commute.

3. *This solution reprinted from the solutions manual for the revised edition.*

$\{A,B\} = AB+BA = 0$. Suppose it is possible, than there exists $|a',b'\rangle$ such that $AB|a',b'\rangle = -BA|a',b'\rangle$ or $a'b' = -b'a'$, thus $a' = 0$ or $b' = 0$. If $A = \vec{p}$ and $B = \Pi$, than $\{\vec{p},\Pi\} = 0$ [because $\Pi^{-1}\vec{p}\Pi = -\vec{p}$], hence momentum eigenstate is usually not parity eigenstate, except for $\vec{p}' = 0$ state.

4. *This solution reprinted from the solutions manual for the revised edition.*

From (3.7.64) we know that

$$\mathcal{Y}_{\ell}^{j=\ell\pm\frac{1}{2},m} = \frac{1}{(2\ell+1)^{\frac{1}{2}}}\begin{pmatrix} \pm\sqrt{\ell\pm m+\frac{1}{2}}\; Y_{\ell}^{m-\frac{1}{2}}(\theta,\phi) \\ \sqrt{\ell\mp m+\frac{1}{2}}\; Y_{\ell}^{m+\frac{1}{2}}(\theta,\phi) \end{pmatrix} \tag{1}$$

(a) For $\ell=0$, only $j=\frac{1}{2}$ (upper sign) is possible, so from (1) we have

$$\mathcal{Y}_{\ell=0}^{j=\frac{1}{2},m=\frac{1}{2}} = \frac{1}{(4\pi)^{\frac{1}{2}}}\begin{pmatrix}1\\0\end{pmatrix} \tag{2}$$

(b)

$$\vec{\sigma}\cdot\vec{x}\,\frac{1}{(4\pi)^{\frac{1}{2}}}\begin{pmatrix}1\\0\end{pmatrix} = \frac{1}{(4\pi)^{\frac{1}{2}}}\begin{pmatrix} z & x-iy \\ x+iy & -z\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix} = \frac{r}{(4\pi)^{\frac{1}{2}}}\begin{pmatrix}\cos\theta \\ \sin\theta e^{i\phi}\end{pmatrix}$$

$$= -r\begin{pmatrix} -Y_1^0(\theta,\phi)/\sqrt{3} \\ (2/3)^{\frac{1}{2}} Y_1^1(\theta,\phi)\end{pmatrix}, \tag{3}$$

where we recall $Y_1^0 = (3/4\pi)^{\frac{1}{2}}\cos\theta$ and $Y_1^1 = -(3/8\pi)^{\frac{1}{2}}\sin\theta e^{i\phi}$. Compare with $\mathcal{Y}_{\ell}^{j,m}$ in (1), we see that m must be $\frac{1}{2}$, ℓ must be 1. Take lower sign in (1) hence $j = \ell-\frac{1}{2} = 1-\frac{1}{2} = \frac{1}{2}$. So (3) becomes

$$\vec{\sigma}\cdot\vec{x}\,\frac{1}{(4\pi)^{\frac{1}{2}}}\begin{pmatrix}1\\0\end{pmatrix} = (-r/\sqrt{3})\begin{pmatrix} -\sqrt{1-\frac{1}{2}+\frac{1}{2}}\; Y_1^0 \\ \sqrt{1+\frac{1}{2}+\frac{1}{2}}\; Y_1^1\end{pmatrix} = -r\mathcal{Y}_{\ell=1}^{j=\frac{1}{2},m=\frac{1}{2}}.$$

Conclusion: Apart from $-r$, we get $\mathcal{Y}_{\ell}^{j,m}$ with ℓ changed ($\ell=0 \to \ell=1$) and j,m both unchanged from Eq.(2).

(c) The result obtained in (b) is not surprising: $\vec{S}\cdot\vec{x}$ is scalar (spherical tensor of rank 0) under rotation, hence by Wigner-Eckart theorem it cannot change j and m. But under space inversion $\vec{S}\cdot\vec{x}$ is odd. So $\vec{S}\cdot\vec{x}$ connects even parity with odd parity, and we note $\ell=0$ and $\ell=1$ have opposite parity.

5. *This solution reprinted from the solutions manual for the revised edition.*

$\vec{S}\cdot\vec{p}$ is invariant under rotations but changes sign under parity. So it is pseudoscalar. Now since $\delta^3(\vec{x})$ is scalar, so the entire V is pseudoscalar. This means V must connect ℓ odd with ℓ even but cannot change j,m. From elementary first order perturbation theory we have

$$C_{n'\ell'j'm'} = \frac{\langle n',\ell',j',m'|V|n,\ell,j,m\rangle}{E_{n\ell j} - E_{n'\ell'j'}} \tag{1}$$

where $\ell' = \ell\pm1$ (note however $|\Delta\ell| \geq 2$ is impossible because j must remain the same) and $m'=m$, $j'=j$. It is more difficult to evaluate $\langle n',\ell',j',m'|V|n,\ell,j,m\rangle$. The wave function for $|n,\ell,j,m\rangle$ can be written as $R_{n\ell j}\mathcal{Y}_\ell^{j=\ell\pm\frac{1}{2},m}$ where $\mathcal{Y}_\ell^{j,m}$ is the spin angular function and for low Z, $R_{n\ell j}$ has no dependence on j. So $\langle n',\ell',j',m'|V|n,\ell,j,m\rangle$ becomes

$$= \lambda\int d^3x\, R_{n'\ell'j'}(r)\mathcal{Y}_{\ell'}^{j'=\ell'\pm\frac{1}{2},m}\,[\delta^{(3)}(\vec{x})\vec{S}\cdot(-i\hbar\vec{\nabla})+(-i\hbar\vec{\nabla})\cdot\vec{S}\delta^{(3)}(\vec{x})]$$
$$\cdot\, R_{n\ell j}(r)\mathcal{Y}_\ell^{j=\ell\pm\frac{1}{2},m} \tag{2}$$

where $(-i\hbar\vec{\nabla})$ in the second term of (2) operates on the wave function to the left. Because of $\delta^{(3)}(\vec{x})$ function, the matrix element vanishes unless $R_{n'\ell'j'}(r)$ or $R_{n\ell j}(r)$ is finite at the origin. This implies that we must have $S_{\frac{1}{2}}$ or $P_{\frac{1}{2}}$ for $|n,\ell,j,m\rangle$ to obtain non-vanishing contributions to $C_{n'\ell'j'm'}$.

6. *This solution reprinted from the solutions manual for the revised edition.*

Consider our symmetric rectangular double-well potential, as divided into three regions: (I) $-a-b<x<-a$; (II) $-a\leq x\leq +a$; and (III) $a<x<a+b$. We have the symmetric states $u_I(x) = A\sin(k_s(x+a+b))$, $u_{II}(x) = B\cosh\kappa_s x$, $u_{III}(x) = -A\sin(k_s(x-a-b))$, and antisymmetric states $v_I(x) = C\sin(k_a(x+a+b))$, $v_{II}(x) = D\sinh\kappa_a x$, $v_{III}(x) = +C\sin(k_a(x-a-b))$. All of which satisfy Schrödinger's equations and the appropriate boundary conditions,

$$k_s = \sqrt{2mE_s/\hbar^2},\ k_a = \sqrt{2mE_a/\hbar^2},\ \kappa_s = \sqrt{2m(V_o-E_s)/\hbar^2},\ \kappa_a = \sqrt{2m(V_o-E_a)/\hbar^2}$$

where because we assume $V_o \gg E_a, E_s$, $\kappa = \kappa_s \cong \kappa_a$. Matching solutions and derivatives at each boundary we have $A\sin k_s b = B\cosh\kappa a$, $C\sin k_a b = -D\sinh\kappa a$ and

$Ak_s\cos k_s b = -B\kappa\sinh\kappa a$, $Ck_a\cos k_a b = +D\kappa\cosh\kappa a$. Therefore we have the eigenvalue conditions

$$\tan k_s b/k_s = -\coth\kappa a/\kappa\ ,\ \tan k_a b/k_a = -\tanh\kappa a/\kappa. \tag{1}$$

Since $V_o \gg E_{a,s}$, we expect the energy levels to be approximately those of a particle in a box (one dimensional, with infinite walls) in regions (I) and (III). Hence $\tan k_{a,s}b = \tan(\pi+\epsilon_{a,s}) = \tan\epsilon_{a,s} \cong \epsilon_{a,s} = k_{a,s}b-\pi$, and (1) can be rewritten as

$$(k_s b-\pi)/k_s = -\coth\kappa a/\kappa,\ (k_a b-\pi)/k_a = -\tanh\kappa a/\kappa. \tag{2}$$

From (2) we have $k_s = \frac{\pi}{b+\coth\kappa a/\kappa}$, $k_a = \frac{\pi}{b+\tanh\kappa a/\kappa}$, and the lowest lying states are $E_s = \hbar^2k_s^2/2m = \frac{\hbar^2\pi^2}{2m}(b+\coth\kappa a/\kappa)^{-2}$ and $E_a = \hbar^2k_a^2/2m = \hbar^2\pi^2(b+\tanh\kappa a/\kappa)^{-2}/2m$.
So $\Delta E = E_a - E_s = \frac{\hbar^2\pi^2}{2mb^2}\{(1+\tanh\kappa a/\kappa b)^{-2} - (1+\coth\kappa a/\kappa b)^{-2}\}$.

7. *This solution reprinted from the solutions manual for the revised edition.*

(a) The plane wave is $\psi(\vec{x},t) = e^{i(\vec{p}.\vec{x}/\hbar-\omega t)}$, hence $\psi^*(\vec{x},-t) = e^{-i(\vec{p}.\vec{x}/\hbar+\omega t)} = e^{i(-\vec{p}.\vec{x}/\hbar-\omega t)}$ and is a plane wave with momentum direction reversed $(-\vec{p})$.

(b) From (3.2.52) with $\alpha=\gamma$, we have $\chi_+(\hat{n}) = \cos\beta/2\ e^{-i\gamma/2}\begin{pmatrix}1\\0\end{pmatrix} + \sin\beta/2\ e^{+i\gamma/2}\begin{pmatrix}0\\1\end{pmatrix}$.

$\chi_+^*(\hat{n}) = \cos\beta/2\ e^{i\gamma/2}\begin{pmatrix}1\\0\end{pmatrix} + \sin\beta/2\ e^{-i\gamma/2}\begin{pmatrix}0\\1\end{pmatrix}$, thus $-i\sigma_2\chi_+^*(n) = \cos\frac{\beta}{2}\ e^{\frac{i\gamma}{2}}\begin{pmatrix}0&-1\\1&0\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix}$ $+\sin\frac{\beta}{2}e^{\frac{-i\gamma}{2}}\begin{pmatrix}0&-1\\1&0\end{pmatrix}\begin{pmatrix}0\\1\end{pmatrix} = \cos\frac{\beta}{2}e^{\frac{i\gamma}{2}}\begin{pmatrix}0\\1\end{pmatrix} + \sin\frac{\beta}{2}e^{\frac{-i\gamma}{2}}\begin{pmatrix}-1\\0\end{pmatrix}$. But by explicit calculation $(\vec{S}\cdot\hat{n})(-i\sigma_2\chi_+^*(\hat{n})) = (-\hbar/2)(-i\sigma_2\chi_+^*(\hat{n}))$ where $\vec{S} = (\hbar/2)\vec{\sigma}$. Hence $\chi_-(\hat{n}) = -i\sigma_2\chi_+^*(\hat{n})$ is the two component eigenspinor with the spin direction reversed.

8. *This solution reprinted from the solutions manual for the revised edition.*

(a) is proved in (4.4.59) and (4.4.60) of text. (b) The wave function of a plane wave $e^{i\vec{p}.\vec{x}/\hbar}$ can be complex without violating time reversal invariance, because it is <u>degenerate</u> with $e^{-i\vec{p}.\vec{x}/\hbar}$.

9. *This solution reprinted from the solutions manual for the revised edition.*

In momentum space $|\alpha\rangle = \int d^3p'|\vec{p}'\rangle\langle\vec{p}'|\alpha\rangle$ where $\langle\vec{p}'|\alpha\rangle = \phi(\vec{p}')$ is the momentum space wave function for $|\alpha\rangle$. Apply Θ to $|\alpha\rangle$ (using $\Theta|\vec{p}'\rangle = |-\vec{p}'\rangle$) we have

$$\Theta|\alpha\rangle = \int d^3p'|-\vec{p}'\rangle\langle\vec{p}'|\alpha\rangle^* = \int d^3p'|\vec{p}'\rangle\langle-\vec{p}'|\alpha\rangle^*$$

where $\langle-\vec{p}'|\alpha\rangle^*$ is the momentum space wave function for $\Theta|\alpha\rangle$. So $\phi^*(-\vec{p}')$ is the momentum space wave function for the time reversed state.

<u>Alternative method</u>: The momentum space wave function $\phi(\vec{p}') = [\frac{1}{(2\pi\hbar)}]^{\frac{3}{2}}\int d^3x' e^{\frac{-i\vec{p}'.\vec{x}'}{\hbar}}\psi$, when complex conjugated, becomes $\phi^*(\vec{p}') = [\frac{1}{(2\pi\hbar)}]^{\frac{3}{2}}\int d^3x' e^{i\vec{p}'.\vec{x}'/\hbar}\psi^*(\vec{x}')$. Thus

momentum space wave function for time reversed state $\phi^*(-\vec{p}')$ is

$$\phi^*(-\vec{p}') = \frac{1}{(2\pi\hbar)}3/2\ [\int d^3x'\ e^{-i\vec{p}'.\vec{x}'/\hbar}\psi^*(\vec{x}')]$$

where $\psi^*(x')$ is the position space wave function for time reversed state.

10. *This solution reprinted from the solutions manual for the revised edition.*

(a) Let Θ be the time reversal operator than $|\alpha\rangle = \mathcal{D}(R)|j,m\rangle$ behaves under time reversal as follows: $\Theta|\alpha\rangle = \Theta\mathcal{D}(R)|j,m\rangle = \Theta e^{-i\vec{J}\cdot\hat{n}\theta/\hbar}|j,m\rangle = \Theta e^{-i\vec{J}\cdot\hat{n}\theta/\hbar}\Theta^{-1}\Theta|j,m\rangle$. But $\Theta\vec{J}\Theta^{-1} = -\vec{J}$ and Θ changes $i \to -i$, therefore $[\Theta,\mathcal{D}(R)] = 0$ and we have $\Theta|\alpha\rangle = \Theta\mathcal{D}(R)|j,m\rangle = \mathcal{D}(R)\Theta|j,m\rangle = (-1)^m\mathcal{D}(R)|j,-m\rangle$, where we have used (4.4.78).

(b) Consider the matrix element $\langle j,-m'|\Theta\mathcal{D}(R)|j,m\rangle = \langle j,-m'|(-1)^m\mathcal{D}(R)|j,-m\rangle = (-1)^m\mathcal{D}^{(j)}_{-m',-m}(R)$. But $\langle j,-m'|\,\Theta\mathcal{D}(R)|j,m\rangle = \sum_{m''}\langle j,-m'|\Theta|j,m''\rangle\langle j,m''|\mathcal{D}(R)|j,m\rangle^* = \sum_{m''}(-1)^{m''}\delta_{-m',-m''}\mathcal{D}^{*(j)}_{m'',m}(R) = (-1)^{m'}\mathcal{D}^{*(j)}_{m',m}(R)$ also (remember Θ contains complex conjugation). Comparing the two expressions for $\langle j,-m'|\Theta\mathcal{D}(R)|j,m\rangle$, we have

$$(-1)^m\mathcal{D}^{(j)}_{-m',-m}(R) = (-1)^{m'}\mathcal{D}^{*(j)}_{m',m}(R) \text{ or } (-1)^{m-m'}\mathcal{D}^{(j)}_{-m',-m}(R) = \mathcal{D}^{*(j)}_{m',m}(R).$$

(c) From part (a) we have $\Theta|\alpha\rangle = (-1)^m\mathcal{D}(R)|j,-m\rangle = \mathcal{D}(R)\Theta|j,m\rangle$, but $i^2 = (-1)$, hence $\mathcal{D}(R)\Theta|j,m\rangle = \mathcal{D}(R)(i^{2m})|j,-m\rangle$ or $\Theta|j,m\rangle = i^{2m}|j,-m\rangle$.

Remarks: The above discussion is for j integer. For $j=\frac{1}{2}$ integer we need to proceed with (4.4.73) with $\eta = +i$ to obtain consistency with (4.4.72a).

11. *This solution reprinted from the solutions manual for the revised edition.*

Under time reversal $\vec{p} \to -\vec{p}$, $\vec{r} \to \vec{r}$, then $[H,\Theta] = 0$ implies invariance under time reversal. Let $|\alpha\rangle$ be an energy eigenket, than $H\Theta|\alpha\rangle = \Theta H|\alpha\rangle = E\Theta|\alpha\rangle$. Hence $\Theta|\alpha\rangle$ is also an eigenket of H with same energy as $|\alpha\rangle$. By the non degenerate assumption we have $\Theta|\alpha\rangle = |\tilde{\alpha}\rangle = e^{i\delta}|\alpha\rangle$ where δ is real. Consider $\langle\alpha|\vec{L}|\alpha\rangle = \langle\tilde{\alpha}|\Theta\vec{L}\Theta^{-1}|\tilde{\alpha}\rangle = -\langle\tilde{\alpha}|\vec{L}|\tilde{\alpha}\rangle = -e^{-i\delta}\langle\alpha|\vec{L}|\alpha\rangle e^{i\delta} = -\langle\alpha|\vec{L}|\alpha\rangle$. Hence $\langle\alpha|\vec{L}|\alpha\rangle = 0$.

If $\psi_\alpha(\vec{x}) = \langle\vec{x}|\alpha\rangle = \sum_{\ell,m} \langle\vec{x}|\ell,m\rangle\langle\ell,m|\alpha\rangle = \sum_{\ell,m} \langle\hat{n}|\ell,m\rangle\, F_{\ell m}(r) = \sum_{\ell,m} F_{\ell m}(r) Y_\ell^m(\theta,\phi)$, where we have used (3.6.22) and (3.6.23), than $\langle\vec{x}|\Theta|\alpha\rangle = e^{i\delta}\langle\vec{x}|\alpha\rangle$ and thus $\psi_\alpha(\vec{x}) = e^{-i\delta}\langle\vec{x}|\tilde{\alpha}\rangle = e^{-i\delta}\psi^*_\alpha(\vec{x}) = e^{-i\delta}\sum_{\ell,m} F^*_{\ell m}(r)[Y_\ell^m(\theta,\phi)]^* = e^{-i\delta}[\sum_{\ell,m} F^*_{\ell m}(r)(-1)^m Y_\ell^{-m}(\theta,\phi)]$

$= e^{-i\delta}\ [\sum_{\ell,m} F^*_{\ell,-m}(r)(-1)^{-m} Y_\ell^m(\theta,\phi)]$, where we have used (3.6.38). Compare the coefficient of $Y_\ell^m(\theta,\phi)$ for the two forms of $\psi_\alpha(\vec{x})$ we have

$$F_{\ell,m}(r) = (-1)^m e^{-i\delta} F^*_{\ell,-m}(r)$$

12. *This solution reprinted from the solutions manual for the revised edition.*

Hamiltonian for a spin-one system is $H = AS_z^2 + B(S_x^2 - S_y^2)$. This problem is similar to problem 29 in Chapter 3. Here

$$S_x = (\hbar/\sqrt{2})\begin{pmatrix}0&1&0\\1&0&1\\0&1&0\end{pmatrix},\ S_y = (\hbar/\sqrt{2})\begin{pmatrix}0&-i&0\\i&0&-i\\0&i&0\end{pmatrix},\ S_z = \hbar\begin{pmatrix}1&0&0\\0&0&0\\0&0&-1\end{pmatrix}.$$

$$H = \hbar^2\begin{pmatrix}A&0&B\\0&0&0\\B&0&A\end{pmatrix}$$

The 'block' matrix that needs to be diagonalized is of form $\begin{pmatrix}A&B\\B&A\end{pmatrix}$. Hence eigenvalues of H are $E = \hbar^2(A\pm B)$, 0 and the eigenvectors are (in terms of $|s,s_z\rangle$) $\frac{1}{2^{1/2}}(|1,1\rangle + |1,-1\rangle)$, $\frac{1}{2^{1/2}}(|1,1\rangle - |1,-1\rangle)$, and $|1,0\rangle$.

Assume that H is Hermitian than A,B are real, and $\Theta H\Theta^{-1} = A\Theta S_z\Theta^{-1}\Theta S_z\Theta^{-1} + B[\Theta S_x\Theta^{-1}\Theta S_x\Theta^{-1} - \Theta S_y\Theta^{-1}\Theta S_y\Theta^{-1}] = A(-S_z)^2 + B[(-S_x)^2 - (-S_y)^2] = H$. Hence Hamiltonian is invariant under time reversal. Since from (4.4.78) $\Theta|j,m\rangle = (-1)^m|j,-m\rangle$, we have $\Theta[\frac{1}{2^{1/2}}(|1,1\rangle + |1,-1\rangle)] = -\frac{1}{2^{1/2}}(|1,1\rangle + |1,-1\rangle)$, $\Theta[\frac{1}{2^{1/2}}(|1,1\rangle - |1,-1\rangle)] = +\frac{1}{2^{1/2}}(|1,1\rangle - |1,-1\rangle)$, $\Theta|1,0\rangle = |1,0\rangle$.

Chapter Five

1. *This solution reprinted from the solutions manual for the revised edition.*

(a) The first order correction is via (5.1.37) just $\langle 0|bx|0\rangle = 0$. The second order correction for the energy is (c.f. (5.1.42) and (5.1.43))

$$\Delta E = -\sum_n \frac{|\langle n|bx|0\rangle|^2}{E_n - E_o} = -b^2 \sum_n \frac{|\langle n|x|0\rangle|^2}{E_n - E_o},$$

where $E_n = (n+\frac{1}{2})\hbar\omega$. Now $\langle n|x|0\rangle = \sqrt{\hbar/2m\omega}\,\delta_{n1}$, so $\Delta E = -b^2(\sqrt{\hbar/2m\omega})^2/(E_1 - E_o) = -b^2/2m\omega^2$ is the energy shift, and the energy of the ground state becomes $E^{(o)} = \frac{1}{2}\hbar\omega + \Delta E = \frac{1}{2}\hbar\omega - b^2/2m\omega^2$.

(b) The Schrödinger equation for this problem is

$$\frac{-\hbar^2}{2m}\frac{d^2\psi}{dx^2} + (\tfrac{1}{2}m\omega^2 x^2 + bx)\psi = E^{(o)}\psi.$$

Let $x' = x + b/m\omega^2$, than above equation can be reduced to

$$\frac{-\hbar^2}{2m}\frac{d^2\psi}{dx'^2} + \tfrac{1}{2}m\omega^2[x'^2 - (b/m\omega^2)^2]\psi = E^{(o)}\psi$$

that is

$$\frac{-\hbar^2}{2m}\frac{d^2\psi}{dx'^2} + \tfrac{1}{2}m\omega^2 x'^2\psi = (E^{(o)} + b^2/2m\omega^2)\psi.$$

This is again a SHO equation with $E' = E^{(o)} + b^2/2m\omega^2$. For lowest energy value $E' = \frac{1}{2}\hbar\omega$, hence $E^{(o)} = \frac{1}{2}\hbar\omega - b^2/2m\omega^2$ which is exactly the same as the perturbation result in (a).

2. *This solution reprinted from the solutions manual for the revised edition.*

From (5.1.44) with $k \leftrightarrow n$ and $\lambda \to g$, we have

$$|k\rangle = |k^{(o)}\rangle + g\sum_{n\neq k}\frac{|n^{(o)}\rangle V_{nk}}{E_k^{(o)} - E_n^{(o)}} + \cdots\cdots\cdots$$

Using orthonormality of $|k^{(o)}\rangle$ and $|n^{(o)}\rangle$ we have

$$\langle k|k\rangle = 1 + g^2\sum_{n\neq k}\frac{|V_{nk}|^2}{(E_k^{(o)} - E_n^{(o)})^2} + \cdots\cdots\cdots$$

and

$$\frac{|\langle k|k^{(o)}\rangle|^2}{|\langle k|k\rangle|^2} = 1 - g^2\sum_{n\neq k}\frac{|V_{nk}|^2}{(E_k^{(o)} - E_n^{(o)})^2} + O(g^3)$$

3. *This solution reprinted from the solutions manual for the revised edition.*

Solving the Schrödinger equation for the unperturbed system, we can easily find the energy eigenfunctions. They are $\psi_G = \sqrt{2/L}\sqrt{2/L}\ \sin\pi x/L\ \sin\pi y/L = \frac{2}{L}\sin\frac{\pi x}{L}\sin\frac{\pi y}{L}$ for ground state, and $\psi_{e1}^{(1)} = \frac{2}{L}\sin\frac{\pi x}{L}\sin\frac{2\pi y}{L}$ or $\psi_{e1}^{(2)} = \frac{2}{L}\sin\frac{2\pi x}{L}\ \sin\frac{\pi y}{L}$ for the first excited state. So obviously the <u>zeroth</u> order eigenfunction for the <u>ground</u> state is just $\psi_G = \frac{2}{L}\sin\frac{\pi x}{L}\sin\frac{\pi y}{L}$, with the first order energy shift of $\langle 1|\lambda xy|1\rangle =$ $\int_0^L\int_0^L \frac{4}{L^2}\ \lambda xy\sin^2\pi x/L\ \sin^2\pi y/L\ dxdy = \frac{1}{4}\lambda L^2$, i.e. $\Delta E^{(o)} = \lambda L^2/4$. For the first excited state, there is degeneracy and the perturbation in general lift the degeneracy. We need to construct the perturbation matrix by evaluating

$$\langle\psi_{e1}^{(1)}|V_1|\psi_{e1}^{(1)}\rangle = \frac{4\lambda}{L^2}\int_0^L\int_0^L xy\sin^2\pi x/L\ \sin^2 2\pi y/L\ dxdy = \frac{1}{4}\lambda L^2$$

$$\langle\psi_{e1}^{(1)}|V_1|\psi_{e1}^{(2)}\rangle = \frac{4\lambda}{L^2}\int_0^L\int_0^L xy\sin\frac{\pi x}{L}\sin\frac{2\pi x}{L}\sin\frac{2\pi y}{L}\sin\frac{\pi y}{L}dxdy = \frac{4^4}{81}\lambda L^2/\pi^4$$

while by symmetry $\langle\psi_{e1}^{(2)}|V_1|\psi_{e1}^{(2)}\rangle = \langle\psi_{e1}^{(1)}|V_1|\psi_{e1}^{(1)}\rangle$ and $\langle\psi_{e1}^{(2)}|V_1|\psi_{e1}^{(1)}\rangle = \langle\psi_{e1}^{(1)}|V_1|\psi_{e1}^{(2)}\rangle$. So the perturbation matrix is

$$\Delta = \frac{\lambda L^2}{4\pi^4}\begin{pmatrix}\pi^4 & 4^5/81\\ 4^5/81 & \pi^4\end{pmatrix}.$$

Diagonalizing Δ with $\det(\Delta-\lambda'I) = 0$ and

$$(\Delta-\lambda'I)\begin{pmatrix}a\psi_{e1}^{(1)}\\ b\psi_{e1}^{(2)}\end{pmatrix} = 0$$

where $a^2 + b^2 = 1$ (normalization), we get $a = 1/\sqrt{2}$, $b = \pm 1/\sqrt{2}$ and

$$\Delta' = \frac{\lambda L^2}{4\pi^4}\begin{pmatrix}\pi^4+4^5/81 & 0\\ 0 & \pi^4-4^5/81\end{pmatrix}.$$

Hence energy shifts for the first excited state are

$$\frac{(\pi^4+4^5/81)\lambda L^2}{4\pi^4} = 0.28\lambda L^2 \text{ and } \frac{(\pi^4-4^5/81)\lambda L^2}{4\pi^4} = 0.22\lambda L^2$$

with corresponding zeroth order energy eigenfunctions

$$\frac{1}{\sqrt{2}}\ \frac{2}{L}[\sin\frac{\pi x}{L}\sin\frac{2\pi y}{L} + \sin\frac{2\pi x}{L}\sin\frac{\pi y}{L}] \text{ and } \frac{1}{\sqrt{2}}\ \frac{2}{L}[\sin\frac{\pi x}{L}\sin\frac{2\pi y}{L} - \sin\frac{2\pi x}{L}\sin\frac{\pi y}{L}]$$

respectively.

4. *This solution reprinted from the solutions manual for the revised edition.*

(a) State vector for energy eigenstate is characterized by $|n_x,n_y\rangle$, and wave function is given by $\psi_{n_x}(x)\psi_{n_y}(y)$ where $\psi_{n_x}(x)$ and $\psi_{n_y}(y)$ are individually wave functions for one dimensional SHO. The energy for the isotropic two dimensional oscillator is just the sum of the energies for one dimensional oscillators, i.e. $E_{n_x n_y} = \hbar\omega(n_x + \frac{1}{2} + n_y + \frac{1}{2})$. The three lowest-lying states are $(n_x,n_y) = (0,0)$, $(1,0)$, $(0,1)$ with energies $\hbar\omega$, $2\hbar\omega$, $2\hbar\omega$, respectively. Evidently the first excited states are doubly degenerate.

(b) The first order energy shift is clearly zero for the ground state (0,0), since $\langle 0,0|xy|0,0\rangle = 0$ because in $\langle 0|x|0\rangle$ (and $\langle 0|y|0\rangle$) $n_x(n_y)$ must change by one unit. For the first excited states we use the formalism of degenerate perturbation theory by diagonalizing $V = \delta m\omega^2 xy$. In the (1,0) and (0,1) basis

$$V = \delta m\omega^2 \begin{pmatrix} 0 & x_{10}y_{01} \\ x_{01}y_{10} & 0 \end{pmatrix} = \tfrac{1}{2}\delta\hbar\omega \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

and hence behaves like σ_x. By same method as problem 3 above, we get zeroth order energy eigenkets $\frac{1}{2^{1/2}}(|10\rangle+|01\rangle)$ with $\Delta^{(1)} = \frac{1}{2}\delta\hbar\omega$ and $\frac{1}{2^{1/2}}(|10\rangle-|01\rangle)$ with $\Delta^{(1)} = -\frac{1}{2}\delta\hbar\omega$. So to summarize we have ground state $|0,0\rangle$ with energy $E = \hbar\omega$ (no first order shift) and first excited states $\frac{1}{2^{1/2}}(|10\rangle+|01\rangle)$ with $E = (2+\delta/2)\hbar\omega$ and $\frac{1}{2^{1/2}}(|10\rangle-|01\rangle)$ with $E = (2-\delta/2)\hbar\omega$.

(c) Now $m\omega^2(x^2+y^2)/2 + \delta m\omega^2 xy = \frac{m\omega^2}{2}[(1+\delta)(x+y)^2/2 + (1-\delta)(x-y)^2/2]$. Let us rotate coordinates by 45°, than $X \equiv (x+y)/\sqrt{2}$, $Y \equiv (x-y)/\sqrt{2}$. So

$$H = p_X^2/2m + p_Y^2/2m + m[\omega^2(1+\delta)]X^2/2 + m[\omega^2(1-\delta)]Y^2/2$$

and is effectively again a two dimensional SHO with ω replaced by $\sqrt{1\pm\delta}\,\omega$ in the

(X,Y) system. The exact energy for the ground state is $\frac{1}{2}\hbar\omega\sqrt{1+\delta} + \frac{1}{2}\hbar\omega\sqrt{1-\delta} = \hbar\omega + O(\delta^2)$. There is therefore no change in energy if only terms linear in δ are kept. The exact energy for $(n_x,n_y) = (1,0)$ is $\hbar\omega\sqrt{1+\delta}(1+\frac{1}{2}) + \hbar\omega\sqrt{1-\delta}\,\frac{1}{2} = \hbar\omega(2+\delta/2) + O(\delta^2)$; similarly for $(n_x,n_y) = (0,1)$, by letting $\delta \to -\delta$, we have exact energy $\hbar\omega(2-\delta/2) + O(\delta^2)$. Ignoring $O(\delta^2)$ contributions, the results are the same as in (b).

5. *This solution reprinted from the solutions manual for the revised edition.*
The Hamiltonian for the system is $H = H_o + \frac{1}{2}\epsilon m\omega^2 x^2 = p_x^2/2m + \frac{1}{2}(1+\epsilon)m\omega^2x^2$, hence $V_{ko} = \langle k|V|0\rangle = \langle k|\frac{1}{2}\epsilon m\omega^2x^2|0\rangle \propto \langle k|x^2|0\rangle$. So our task is to evaluate $\langle k|x^2|0\rangle$ or x^2_{ko}. Since from (2.3.24) $x = \sqrt{\hbar/2m\omega}(a + a^\dagger)$ where a and $a^\dagger$ satisfy $a|n\rangle = c_-|n-1\rangle$ and $a^\dagger|n\rangle = c_+|n+1\rangle$, then $x|0\rangle = \sqrt{\hbar/2m\omega}\,(a|0\rangle + a^\dagger|0\rangle) = \sqrt{\hbar/2m\omega}|1\rangle$ while $x^2|0\rangle = (\sqrt{\hbar/2m\omega})^2(a + a^\dagger)|1\rangle = c_1|0\rangle + c_2|2\rangle$. So $V_{ko} \propto \langle k|x^2|0\rangle = c_1\delta_{ko} + c_2\delta_{k2}$, and only V_{oo} and V_{2o} are relevant to our discussion. Explicit evaluation of c_1 and c_2 (remembering that $(a^\dagger/\sqrt{2})|1\rangle = |2\rangle$ from (2.3.21)), we have $c_1 = \hbar/2m\omega$, $c_2 = \hbar\sqrt{2}/2m\omega$. Thus $V_{oo} = \frac{1}{2}\epsilon m\omega^2\langle 0|x^2|0\rangle = c_1\epsilon m\omega^2/2 = \frac{\hbar}{2m\omega}\frac{\epsilon m\omega^2}{2} = \epsilon\hbar\omega/4$, and $V_{2o} = \frac{1}{2}\epsilon m\omega^2\langle 2|x^2|0\rangle = c_2\epsilon m\omega^2/2 = \frac{\hbar\sqrt{2}}{2m\omega}\frac{\epsilon m\omega^2}{2} = \epsilon\hbar\omega/2\sqrt{2}$.

6. *This solution reprinted from the solutions manual for the revised edition.*
Let $\omega = \omega_x = \omega_y$, $\omega_z = (1+\epsilon)\omega$ where $\epsilon \ll 1$. Full Hamiltonian is

$$H = \frac{1}{2\mu}(\vec{p} - q\vec{A}/c)^2 + \tfrac{1}{2}\mu\omega^2(x^2+y^2) + \tfrac{1}{2}\mu(1+\epsilon)^2\omega^2z^2. \tag{1}$$

Choosing $\vec{A} = \frac{1}{2}\vec{B}_o\times\vec{r}$ with $\vec{B}_o$ along x-axis, $\vec{r} = x\vec{i}+y\vec{j}+z\vec{k}$, we have

$$H = \frac{1}{2\mu}p^2 + \tfrac{1}{2}\mu\omega^2r^2 - \frac{qB_o}{2\mu c}L_x + \epsilon\mu\omega^2z^2 + \tfrac{1}{2}\epsilon^2\mu\omega^2z^2 + \frac{q^2B_o^2}{8\mu c^2}(y^2+z^2). \tag{2}$$

We are told third and fourth terms on r.h.s. of (2) are comparably small, hence to lowest (first) order in perturbation theory we can drop fifth and sixth terms on r.h.s. of (2) [of order ϵ^2 and B_o^2]. In this approximation

$$H = \frac{1}{2\mu}p^2 + \tfrac{1}{2}\mu\omega^2r^2 - \frac{qB_o}{2\mu c}L_x + \epsilon\mu\omega^2z^2. \tag{3}$$

To simplify, rotate through 90^o in $x - z$ plane s.t. $x \to z'$, $-z \to x'$, $y \to y'$ and then drop primes, Eq. (3) becomes

$$H = \frac{p^2}{2\mu} + \tfrac{1}{2}\mu\omega^2 r^2 - \frac{qB_0}{2\mu c}L_z + \mu\epsilon\omega^2 x^2 = H_o + \mathcal{H} \tag{4}$$

where $H_o = p^2/2\mu + \tfrac{1}{2}\mu\omega^2 r^2$, $\mathcal{H} = -\frac{qB_0}{2\mu c}L_z + \mu\epsilon\omega^2 x^2$.

H_o has eigenvalues $\epsilon_n = (n+\tfrac{1}{2})\hbar\omega$ $[n = 0,1,2,\ldots.]$, with $(n+1)(n+2)/2$ fold degeneracy and eigenfunctions of form $\phi_{n\ell m} = Ne^{-r^2/2}r^{\ell}L_k^{\ell+\frac{1}{2}}(r^2)Y_{\ell m}(\theta,\phi)$. First excited state $n = 1$ is three fold degenerate and the three wave functions are $u_1 = \phi_{110} = N(3/4\pi)^{\frac{1}{2}}\cos\theta re^{-r^2/2}$, $u_2 = \phi_{111} = N(3/8\pi)^{\frac{1}{2}}\sin\theta e^{i\phi}re^{-r^2/2}$, $u_3 = \phi_{11-1} = -N(3/8\pi)^{\frac{1}{2}}\sin\theta e^{-i\phi}re^{-r^2/2}$, where $N^2 = 8/3\pi^{\frac{1}{2}}$ and r is measured in units of $(\hbar/\mu\omega)^{\frac{1}{2}}$. The unperturbed energy $\epsilon_1 = (5/2)\hbar\omega$ and the 3×3 secular determinantal equation for computing E to first order in perturbation is

$$\det[\epsilon_1 - E + \mathcal{H}_{ij}] = 0 \qquad (i,j = 1,2,3). \tag{5}$$

Since $[L_z, H_o] = 0$, our eigenstates of H_o are also eigenstates of L_z, so that $(-qB_0/2\mu c)[L_z]_{ij} = 0$ if $i \neq j$ and the diagonal entries are 0, $-qB_0\hbar/2\mu c$, $qB_0\hbar/2\mu c$ for 11, 22, 33 respectively. The other term in $\mathcal{H}$ is $\epsilon\mu\omega^2 x^2$ which in units of $\hbar/\mu\omega$ becomes $\epsilon\mu\omega^2(\hbar/\mu\omega)x^2 = \epsilon\hbar\omega x^2$, and so we have to evaluate its matrix elements say $\epsilon\hbar\omega(x^2)_{ij}$ where $(x^2)_{ij} = \langle u_i|x^2|u_j\rangle$. Now since $x^2 = r^2\sin^2\theta\cos^2\phi$, the matrix elements $(x^2)_{12}$, $(x^2)_{21}$, $(x^2)_{13}$, $(x^2)_{31}$ involve an integral of form $\int_0^{2\pi}\cos^2\phi e^{\pm i\phi}d\phi = 0$, and hence $(x^2)_{12} = (x^2)_{21} = (x^2)_{13} = (x^2)_{31} = 0$. Since L_z piece in $\mathcal{H}$ is already diagonal, we have $\mathcal{H}_{12} = \mathcal{H}_{21} = \mathcal{H}_{31} = \mathcal{H}_{13} = 0$. Evaluation of $(x^2)_{ii}$ $[i = 1,2,3]$ is straightforward using the explicit form for u_i above, we have

$$\begin{aligned}(x^2)_{11} &= (8/3\pi^{\frac{1}{2}})(3/4\pi)\int_0^{\infty} r^6e^{-r^2}dr \int_0^{\pi}\sin^3\theta\cos^2\theta d\theta \int_0^{2\pi}\cos^2\phi d\phi \\ &= (8/3\pi^{\frac{1}{2}})(3/4\pi)[15/16\pi^{\frac{1}{2}}](4/15)\pi = \tfrac{1}{2} = \tfrac{1}{2}(x^2)_{22} = \tfrac{1}{2}(x^2)_{33}.\end{aligned} \tag{6}$$

Computation of $(x^2)_{23}$ and $(x^2)_{32}$ is also straightforward and yields

$$\begin{aligned}(x^2)_{23} = (x^2)_{32} &= (8/3\pi^{\frac{1}{2}})(3/8\pi)\int_0^{\infty} r^6e^{-r^2}dr\int_0^{\pi}\sin^5\theta d\theta\int_0^{2\pi}\cos^2\phi e^{-2i\phi}d\phi \\ &= \tfrac{1}{2}\ .\end{aligned} \tag{7}$$

So $\mathcal{H}_{32} = \mathcal{H}_{23} = \tfrac{1}{2}\epsilon\hbar\omega$, and $\mathcal{H}_{11} = \tfrac{1}{2}\epsilon\hbar\omega$, $\mathcal{H}_{22} = \epsilon\hbar\omega - \frac{qB_0}{2\mu c}\hbar$, $\mathcal{H}_{33} = \epsilon\hbar\omega + \frac{qB_0}{2\mu c}\hbar$. Hence secular equation for energy becomes

$$\begin{vmatrix} \frac{5}{2}\hbar\omega - E + \frac{\varepsilon\hbar\omega}{2} & 0 & 0 \\ 0 & \frac{5}{2}\hbar\omega - E + \varepsilon\hbar\omega - \frac{qB_0\hbar}{2\mu c} & \frac{1}{2}\varepsilon\hbar\omega \\ 0 & \frac{1}{2}\varepsilon\hbar\omega & \frac{5}{2}\hbar\omega - E + \varepsilon\hbar\omega + \frac{qB_0\hbar}{2\mu c} \end{vmatrix} = 0 \qquad (8)$$

So one solution is $E_1 = (5/2)\hbar\omega + (\varepsilon/2)\hbar\omega$, and the other two solutions are $E_2 = (5/2 + \varepsilon)\hbar\omega - \frac{1}{2}[(\varepsilon\hbar\omega)^2 + (qB_0\hbar/\mu c)^2]^{\frac{1}{2}}$, $E_3 = (5/2 + \varepsilon)\hbar\omega + \frac{1}{2}[(\varepsilon\hbar\omega)^2 + (qB_0\hbar/\mu c)^2]^{\frac{1}{2}}$.

Various limiting cases. In limit that $\varepsilon \to 0$, we just get the Zeeman splittings

$$E_1 = \varepsilon_1 = \frac{5}{2}\hbar\omega,\ E_2 = \frac{5}{2}\hbar\omega - \frac{qB_0\hbar}{2\mu c},\ E_3 = \frac{5}{2}\hbar\omega + \frac{qB_0\hbar}{2\mu c}\ . \qquad (9)$$

In the limit that $B_0 \to 0$,

$$E_1 \to \tfrac{1}{2}(5 + \varepsilon)\hbar\omega,\ E_2 \to \frac{5}{2}\hbar\omega + \ \hbar\omega/2 = E_1,\ E_3 \to (5/2 + 3\varepsilon/2)\hbar\omega. \qquad (10)$$

Thus we have degeneracy between E_1 and E_2.

7. *This solution reprinted from the solutions manual for the revised edition.*

Here $V = -ez|\vec{E}|$, and the perturbed ground state ket $|1,0,0\rangle'$ and unperturbed ground state ket $|1,0,0\rangle$ in the $|n,\ell,m\rangle$ notation are related by

$$|1,0,0\rangle' = |1,0,0\rangle + \sum_{n\ell m} \frac{|\vec{E}|(-e)\langle n,\ell,m|z|1,0,0\rangle|n,\ell,m\rangle}{E_{100} - E_{n\ell m}}$$

where E_{100} and $E_{n\ell m}$ are unperturbed energies (actually independent of m). Take expectation value of ez

$$(\langle 1,0,0| + \sum_{n'\ell'm'} \frac{(-e)|\vec{E}|\langle 1,0,0|z|n',\ell',m'\rangle\langle n',\ell',m'|}{E_{100} - E_{n'\ell'm'}})ez(|1,0,0\rangle + \sum_{n\ell m}$$

$$\frac{-e|\vec{E}|\langle n\ell m|z|100\rangle|n\ell m\rangle}{E_{100} - E_{n\ell m}}) = -2e^2 \sum_{n\ell m} \frac{|\langle 100|z|n\ell m\rangle|^2}{E_{100} - E_{n\ell m}}|\vec{E}|,\ (\ell=1, m=0 \text{ in our case}) \quad (1)$$

where we have used the fact that $\langle 100|z|100\rangle = 0$. Also from (5.1.63), (5.1.67), and (5.1.68) we have for the energy shift of the ground state computed to second order

$$\Delta = -\tfrac{1}{2}\alpha|\vec{E}|^2,\ \alpha = -2e^2 \sum_{n\ell m} \frac{|\langle 100|z|n\ell m\rangle|^2}{E_{100} - E_{n\ell m}}\ . \qquad (2)$$

Hence from (1), we have induced dipole moment $\alpha|\vec{E}|$, where α is the same α which appears in $\Delta = -\frac{1}{2}\alpha|\vec{E}|^2$ of (2).

8. *This solution reprinted from the solutions manual for the revised edition.*

(a) $\langle n=2, \ell=1, m=0|x|n=2, \ell=0, m=0\rangle = 0$, because x is rank 1 tensor $(k=1, q=\pm 1)$ and behaves like $Y_1^1 - Y_1^{-1}$, so m value must change.

(b) $\langle n=2, \ell=1, m=0|p_z|n=2, \ell=0, m=0\rangle = 0$, since $p_z = \frac{m}{i\hbar}[z, H]$ we get $\langle p_z\rangle = im/\hbar \times (E_{210} - E_{200})\langle n=2, \ell=1, m=0|z|n=2, \ell=0, m=0\rangle$, but $E_{210} - E_{200} = 0$ by "accidental degeneracy" (2s − 2p degeneracy).

(c) From (3.7.64), we note that $|j=9/2, m=7/2, \ell=4\rangle$ is represented by

$$\mathcal{Y}_{\ell=4}^{j=4+\frac{1}{2}, 7/2} = (1/\sqrt{9})\begin{pmatrix}\sqrt{4+7/2+1/2}\; Y_{\ell=4}^{7/2-1/2}\\ \sqrt{4-7/2+1/2}\; Y_{\ell=4}^{7/2+1/2}\end{pmatrix},$$

hence $\langle L_z\rangle = (\sqrt{8/9})^2 3\hbar + (\sqrt{1/9})^2 4\hbar = (28/9)\hbar$.

(Alternative method: Use $\langle L_z\rangle = m\hbar - \langle S_z\rangle$ with $S_z = \pm m\hbar/(2\ell+1)$ (c.f. (5.3.31)) for $j = \ell \pm \frac{1}{2}$.)

(d) To evaluate $\langle \text{singlet}, m=0|(S_z^{(e^-)} - S_z^{(e^+)})|\text{triplet}, m=0\rangle$, first note

$$(S_z^{(e^-)} - S_z^{(e^+)})|\text{triplet}, m=0\rangle = (S_z^{(e^-)} - S_z^{(e^+)})\frac{1}{2^{1/2}}(|\uparrow\rangle_{e^-}|\downarrow\rangle_{e^+} + |\downarrow\rangle_{e^-}|\uparrow\rangle_{e^+})$$

$$= (\tfrac{1}{2}\hbar - (-\tfrac{1}{2})\hbar)\frac{1}{2^{1/2}}(|\uparrow\rangle_{e^-}|\downarrow\rangle_{e^+}) + ((-\tfrac{1}{2}\hbar) - (\tfrac{1}{2}\hbar))\frac{1}{2^{1/2}}(|\downarrow\rangle_{e^-}|\uparrow\rangle_{e^+})$$

$$= \frac{\hbar}{2^{1/2}}[|\uparrow\rangle_{e^-}|\downarrow\rangle_{e^+} - |\downarrow\rangle_{e^-}|\uparrow\rangle_{e^+}] = \hbar|\text{singlet}, m=0\rangle.$$

So $\langle \text{singlet}, m=0|(S_z^{(e^-)} - S_z^{(e^+)})|\text{triplet}, m=0\rangle = \hbar$.

(e) Ground state of H_2 molecule: For "homopolar" binding, the space part is symmetric, hence spin part is in singlet state. Thus

$$\langle \vec{S}_1 \cdot \vec{S}_2\rangle = \tfrac{1}{2}(\vec{S}_{\text{tot.}}^2 - \vec{S}_1^2 - \vec{S}_2^2) = -\tfrac{1}{2}\cdot 2\cdot (3/4)\hbar^2 = -\tfrac{3}{4}\hbar^2$$

where expectation value of $\langle \vec{S}_{\text{tot.}}^2\rangle$ gives zero for a spin singlet state.

9. *This solution reprinted from the solutions manual for the revised edition.*

(a) $\langle n,\ell=1,m=\pm1,0|V|n,\ell=1,m=\pm1,0\rangle$, $\frac{V}{\lambda} = x^2-y^2 = r^2\sin^2\theta(\cos^2\phi - \sin^2\phi) =$ $r^2\sin^2\theta\cos2\phi = r^2\sin^2\theta(e^{2i\phi}+ e^{-2i\phi})/2$. So the perturbation connects $m = \pm1$ with $m = \mp1$. The type of non vanishing V-matrix elements are of form

$$I = \lambda\int\frac{\sin^2\theta}{2}e^{\pm i\phi}e^{\mp2i\phi}\sin^2\theta e^{\pm i\phi}d\Omega \int r^2R_{nl}^2\, r^2dr$$

between $m = +1$ to $m = -1$ and $m = -1$ to $m = +1$ respectively. Hence perturbation matrix

$$V = \begin{pmatrix} 0 & 0 & I \\ 0 & 0 & 0 \\ I & 0 & 0 \end{pmatrix}$$

and evidently the "correct" zeroth order energy eigenstates that diagonalize the perturbation is

$$\frac{1}{2^{1/2}}[\,|n,\ell=1,m=+1\rangle \pm |n,\ell=1,m=-1\rangle] \tag{1}$$

(b) We are dealing with states whose angular dependence are spherical harmonics. Under time reversal: $Y_\ell^m \rightarrow Y_\ell^{m*} = (-1)^mY_\ell^{-m}$, hence $\Theta|n,\ell=1,m=\pm1\rangle = -|n,\ell=1,m=\mp1\rangle$. Therefore (1) evidently go into itself (up to a phase factor or sign) under time reversal.

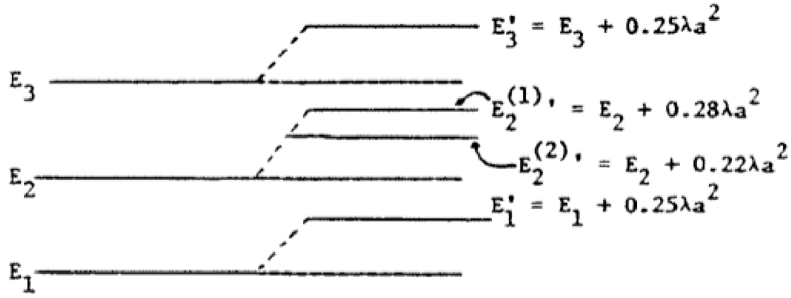
10. *This solution reprinted from the solutions manual for the revised edition.*

This problem is rather similar to problem 3 above with L replaced by a. For (a) the Hamiltonian of the unperturbed system is H_o, where $H_o = -\frac{\hbar^2}{2m}\nabla^2 + V$, and by using the method of separation of variables, we can easily find the energy eigenvalues and eigenfunctions

$$E_n = \frac{\hbar^2\pi^2}{2ma^2}(n_x^2 + n_y^2),\ \psi_n(x,y) = \sin(n_x\pi x/a)\sin(n_y\pi y/a) \tag{1}$$

where n_x, n_y are non-zero integers. Thus the three lowest states correspond to $n_x=n_y=1$; $n_x=2$, $n_y=1$ and $n_x=1$, $n_y=2$; and $n_x=2$, $n_y=2$ respectively, and from (1) we have $E_1 = \hbar^2\pi^2/ma^2$ with $\psi_1(x,y) = (2/a)\sin(\frac{\pi x}{a})\sin(\frac{\pi y}{a})$ and nondegenerate, $E_2 =$ $5\hbar^2\pi^2/2ma^2$ with $\psi_2(x,y) = (2/a)\sin(\frac{2\pi x}{a})\sin(\frac{\pi y}{a})$ or $(2/a)\sin(\frac{\pi x}{a})\sin(\frac{2\pi y}{a})$ and hence two fold degenerate, $E_3 = 4\hbar^2\pi^2/ma^2$ with $\psi_3(x,y) = (2/a)\sin(\frac{2\pi x}{a})\sin(\frac{2\pi y}{a})$ and non-degenerate.

(b) For (i) the first order energy shift is $\Delta E_n = \langle n|V_1|n\rangle = \lambda\langle n|xy|n\rangle \propto \lambda$, hence the energy shift is <u>linear</u> in λ, in otherwords proportional to λ. For (ii) $\Delta E_3 = \langle 3|\lambda xy|3\rangle = (\frac{2}{a})^2\lambda \int_0^a\int_0^a x\sin^2(\frac{2\pi x}{a})y\sin^2(\frac{2\pi y}{a})dxdy = \frac{1}{4}\lambda a^2$. The energy shifts for degenerate state E_2 are given from problem 3 as $\Delta E_2^{(1)} = 0.28\lambda a^2$ and $\Delta E_2^{(2)} = 0.22\lambda a^2$, while that for nondegenerate E_1 is $\Delta E_1 = \frac{1}{4}\lambda a^2 = 0.25\lambda a^2$. (iii) The energy level diagrams for unperturbed levels (E_n) and perturbed levels $E_n + \Delta E_n = E_n'$ look as follows:



unperturbed levels perturbed levels

11. *This solution reprinted from the solutions manual for the revised edition.*

(a) The energy eigenvalues E_1 and E_2 are found from secular equation

$$\begin{vmatrix} E_1^o - E & \lambda\Delta \\ \lambda\Delta & E_2^o - E \end{vmatrix} = 0$$

therefore $E_{1,2} = (E_1^o + E_2^o)/2 \pm \sqrt{(E_1^o - E_2^o)^2/4 + \lambda^2\Delta^2}$. To find the eigenfunctions, we write $\psi_{1,2} = \begin{pmatrix} a_{1,2} \\ 1 \end{pmatrix}$, then $H\psi = E\psi$ gives $E_1^o a_{1,2} + \lambda\Delta = E_{1,2}a_{1,2}$ and thus up to normalization

$$\psi_{1,2} = \begin{pmatrix} \lambda\Delta/(E_{1,2} - E_1^o) \\ 1 \end{pmatrix}$$

with $E_{1,2}$ as given above. Note also that this problem is completely analogous to problem 11 of Chapter 1, if we make the substitution $E_1^o \leftrightarrow H_{11}$, $E_2^o \leftrightarrow H_{22}$, and $\lambda\Delta \leftrightarrow H_{12}$. Hence an alternative way to parametrize $\psi_{1,2}$ in normalized form is

$$\psi_1 = \begin{pmatrix} \cos\frac{\beta}{2} \\ \sin\frac{\beta}{2} \end{pmatrix} \quad , \psi_2 = \begin{pmatrix} -\sin\frac{\beta}{2} \\ \cos\frac{\beta}{2} \end{pmatrix} \text{ where } \beta = \tan^{-1}[\frac{2\lambda\Delta}{E_1^o - E_2^o}]$$

(b) For H as given,

$$H_o = \begin{pmatrix} E_1^o & 0 \\ 0 & E_2^o \end{pmatrix}, \; V = \begin{pmatrix} 0 & \lambda\Delta \\ \lambda\Delta & 0 \end{pmatrix},$$

hence $V_{11} = V_{22} = 0$, so first order energy shifts vanish in time-independent perturbation theory, and we must go to second-order. Here second order shifts are

$$\Delta_1^{(2)} = \frac{|V_{12}|^2}{E_1^o - E_2^o} = \frac{\lambda^2\Delta^2}{E_1^o - E_2^o} \;, \quad \Delta_2^{(2)} = \frac{|V_{21}|^2}{E_2^o - E_1^o} = \frac{\lambda^2\Delta^2}{E_2^o - E_1^o}$$

But the exact energy solution for $\lambda|\Delta| \ll |E_1^o - E_2^o|$ is

$$E_{1,2} = \frac{(E_1^o + E_2^o)}{2} \pm \frac{(E_1^o - E_2^o)}{2}[1 + \frac{4\lambda^2\Delta^2}{(E_1^o - E_2^o)^2}]^{\frac{1}{2}} \cong \begin{cases} E_1^o + \lambda^2\Delta^2/(E_1^o - E_2^o) \\ E_2^o - \lambda^2\Delta^2/(E_1^o - E_2^o) \end{cases}$$

in agreement with perturbation results $E_1^o + \Delta_1^{(2)}$, and $E_2^o + \Delta_2^{(2)}$.

(c) Now suppose $E_1^o \sim E_2^o \equiv E^o$. Then $H = E^o\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \lambda\Delta\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Since the perturbation term is proportional to σ_x, we know right away that the eigenfunctions are those of σ_x,

$$\phi_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad \phi_2 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}; \text{ with } E_1 = E^o + \lambda\Delta,\; E_2 = E^o - \lambda\Delta.$$

Note $\phi_1 = \phi_1^o + \phi_2^o$, $\phi_2 = \phi_2^o - \phi_1^o$, i.e. linear combinations of degenerate states. From (a), we have if $E_1^o = E_2^o = E^o$, than $E_{1,2} = E^o \pm \lambda\Delta$ and $\phi_{1,2} = \begin{pmatrix} \pm 1 \\ 1 \end{pmatrix}$ which agrees with (c).

12. *This solution reprinted from the solutions manual for the revised edition.*
Using the secular equation method, we diagonalize the perturbed Hamiltonian matrix to obtain the exact energy eigenvalues. The secular equation reads

$$(E_1-\lambda)((E_1-\lambda)(E_2-\lambda) - |b|^2) + a((\lambda-E_1)a^*) = 0 .$$

Evidently $E_1 = \lambda$ is one solution, and the other two solutions are roots of $\lambda^2 - (E_1+E_2)\lambda + E_1E_2 - |a|^2 - |b|^2 = 0$, i.e. $\lambda_+ \equiv E_1 + \frac{(|a|^2+|b|^2)}{(E_1 - E_2)}$ and $\lambda_- \equiv E_2 - \frac{(|a|^2+|b|^2)}{(E_1 - E_2)}$, where we have assumed $|a|, |b| << |E_2-E_1|$.

Formally non-degenerate second order perturbation theory (5.1.42), translated into our notation, reads $\Delta_1 = |a|^2/(E_1-E_2)$, $\Delta_2 = |b|^2/(E_1-E_2)$ and $\Delta_3 = \frac{|a|^2+|b|^2}{E_2-E_1}$,

hence energy levels are $E_1+\Delta_1$, $E_1+\Delta_2$, and $E_2+\Delta_3$ respectively. The non-degenerate second order perturbation results are unjustified because degeneracy is not removed to first order.

Use degenerate perturbation theory à la Gottfried (1966) (see p. 397, for details). We have here a degenerate two level subspace (E_1 twice and E_2), hence to second order in degenerate perturbation theory the energy shifts are given by

$$(\Delta - \frac{|a|^2}{E_1-E_2})(\Delta - \frac{|b|^2}{E_1-E_2}) = |\frac{ab}{E_1-E_2}|^2$$

i.e. $\Delta_1 = 0$, $\Delta_2 = \frac{|a|^2+|b|^2}{E_1-E_2}$, which agrees with the exact solution above where we had $E_1 = \lambda$(with $\Delta_1 = 0$), and $\lambda_+ \equiv E_1 + (|a|^2+|b|^2)/(E_1-E_2) = E_1 + \Delta_2$, at least in the approximation $|a|, |b| << |E_2-E_1|$.

13. *This solution reprinted from the solutions manual for the revised edition.*
The Hamiltonian is $H = \vec{p}^2/2m - e^2/r + e\varepsilon z$, where $e\varepsilon z$ is the perturbation potential. In terms of the $2S_{1/2}$ and $2P_{1/2}$ levels of hydrogen, our Hamiltonian can be represented as

$$H \equiv \begin{pmatrix} E_2^S + \langle s|e\varepsilon z|s\rangle + \delta & \langle s|e\varepsilon z|p\rangle \\ \langle p|e\varepsilon z|s\rangle & E_2^P + \langle p|e\varepsilon z|p\rangle \end{pmatrix} \qquad (1)$$

where δ is the Lamb shift, and E_2^S, E_2^P are the unperturbed energies for $2S_{1/2}$ and

$2P_{1/2}$ respectively. It is evident (from parity selection rule) that $\langle s|e\varepsilon z|s\rangle =$ $\langle p|e\varepsilon z|p\rangle = 0$, while $\langle s|e\varepsilon z|p\rangle = \langle p|e\varepsilon z|s\rangle = e\varepsilon\langle s|r|p\rangle \int Y^{*\, j=1/2,m}_{\ell=0}\, Y^{j=1/2,m}_{\ell=1}\cos\theta d\Omega$. Using (3.7.64), we have $\langle s|e\varepsilon z|p\rangle = \langle p|e\varepsilon z|s\rangle = \mp\sqrt{3}e\varepsilon a_o$ for $m = \pm\frac{1}{2}$. Hence (1) becomes

$$H = \begin{pmatrix} E_2^S + \delta & \mp\sqrt{3}e\varepsilon a_o \\ \mp\sqrt{3}e\varepsilon a_o & E_2^P \end{pmatrix} \tag{2}$$

We diagonalize (2) to obtain eigenvalues Λ, where we recognize that $E_2^S = E_2^P =$ E_2, this gives

$$\Lambda = E_2 + \delta/2 \pm [(\delta/2)^2 + 3e^2\varepsilon^2a_o^2]^{1/2} \tag{3}$$

The energy shift from the mean $E_2+\delta/2$ is $\pm[(\delta/2)^2+3e^2\varepsilon^2a_o^2]^{1/2}$. Hence $\Delta E_S = -\Delta E_P$ $= [(\delta/2)^2+3e^2\varepsilon^2a_o^2]^{1/2} \cong \frac{\delta}{2}[1 + 6e^2\varepsilon^2a_o^2/\delta^2]$ for $e\varepsilon a_o \ll \delta$, and $[(\delta/2)^2+3e^2\varepsilon^2a_o^2]^{1/2} \cong$ $\sqrt{3}e\varepsilon a_o(1 + \frac{\delta^2}{24e^2\varepsilon^2a_o^2})$ for $e\varepsilon a_o \gg \delta$. Note for $e\varepsilon a_o \ll \delta$ the shift from $E_2^S + \delta$ is quadratic in ε, while for $e\varepsilon a_o \gg \delta$ the dominant shift is linear in ε.

Whereas parity restricts $\langle s|e\varepsilon z|s\rangle = \langle p|e\varepsilon z|p\rangle = 0$, time reversal invariance of our Hamiltonian places no similar restriction. Nevertheless (c.f. (4.4.84)) it imposes the restriction that expectation value $\langle\vec{x}\rangle$ (hence $\langle z\rangle$ as a special case) vanishes when taken with respect to eigenstates of j,m. For example $|j,m\rangle$ of our problem need not be parity eigenkets, and could be $c_s|s_{1/2}\rangle + c_p|p_{1/2}\rangle$, yet it remains true that $\langle j,m|\vec{x}|j,m\rangle = 0$ under time reversal invariance - i.e. no electric dipole moment.

14. *This solution reprinted from the solutions manual for the revised edition.*

Let the electric field be in z-direction, i.e. $\vec{\varepsilon} = \varepsilon\vec{k}$, so the potential is expressed as $V = +e\vec{\varepsilon}.\vec{r} = re\varepsilon\cos\theta$. Assuming ε is small, we can use perturbation theory. The wave functions for n=3 are $\psi_{n\ell m_\ell}$, $n = 3$, $\ell = 0,1,2$, obviously $\langle 3\ell m_\ell|V|3\ell'm_\ell'\rangle \propto \langle R_{3\ell}|r|R_{3\ell'}\rangle\langle Y_\ell^{m_\ell}|\cos\theta|Y_{\ell'}^{m_\ell'}\rangle$. Now $\langle Y_\ell^{m_\ell}|\cos\theta|Y_{\ell'}^{m_\ell'}\rangle \propto \delta_{m_\ell m_\ell'}\langle P_\ell^{m_\ell}|\cos\theta$ $|P_{\ell'}^{m_\ell'}\rangle$, while $\cos\theta P_{\ell'}^{m_\ell'} \propto [(\ell'-|m_\ell'|+1)P_{\ell'+1}^{m_\ell'} + (\ell'+|m_\ell'|)P_{\ell'-1}^{m_\ell'}]$, thus $\langle Y_\ell^{m_\ell}|\cos\theta|Y_{\ell'}^{m_\ell'}\rangle$

$\propto \delta_{m_\ell m'_\ell}\{[\ell'-|m'_\ell|+1]\delta_{\ell,\ell'+1} + [\ell'+|m'_\ell|]\delta_{\ell,\ell'-1}\}$. So matrix elements vanish unless $m_\ell=m'_\ell$, $\ell=\ell'+1$ or $\ell=\ell'-1$, and we have non-vanishing matrix elements $\langle 321|V|311\rangle$, $\langle 32-1|V|31-1\rangle$, $\langle 320|V|310\rangle$, $\langle 310|V|320\rangle$, $\langle 311|V|321\rangle$, $\langle 31-1|V|32-1\rangle$, $\langle 310|V|300\rangle$, $\langle 300|V|310\rangle$. As a first step let us calculate these non-vanishing matrix elements, remembering that $\psi_{n\ell m}=R_{n\ell}Y_\ell^{m_\ell}$ where Y_ℓ^m is given by (A.5.6) and $R_{n\ell}(r)$ by (A.6.3). Straightforward evaluation leads to

$$\langle 321|V|311\rangle=\langle 311|V|321\rangle=\langle 32-1|V|31-1\rangle=\langle 31-1|V|32-1\rangle= -27\varepsilon a_o e/2$$

$$\langle 320|V|310\rangle=\langle 310|V|320\rangle= -9\sqrt{3}\varepsilon a_o e$$

$$\langle 310|V|300\rangle=\langle 300|V|310\rangle= -9\sqrt{6}\varepsilon a_o e.$$

Diagonalizing the (9×9) V-matrix ($\sum_{m_\ell=0}^{2}(2m_\ell+1) = 9$), we have the matrix equation (with eigenvalues $\lambda= e\varepsilon a_o r$)

$$\begin{pmatrix} r&0&0&0&0&0&0&0&0\\ 0&r&0&0&0&a&0&0&0\\ 0&0&r&0&0&0&b&0&0\\ 0&0&0&r&0&0&0&a&0\\ 0&0&0&0&r&0&0&0&0\\ 0&a&0&0&0&r&0&0&0\\ 0&0&b&0&0&0&r&0&c\\ 0&0&0&a&0&0&0&r&0\\ 0&0&0&0&0&0&c&0&r \end{pmatrix} \begin{pmatrix} A_1\\A_2\\A_3\\A_4\\A_5\\A_6\\A_7\\A_8\\A_9 \end{pmatrix} = 0 \qquad (1)$$

where $a = 27/2$, $b = 9\sqrt{3}$, $c = 9\sqrt{6}$. and secular equation is $r^3[r^2-a^2]^2[r^2-b^2-c^2] = 0$ i.e. $r=0$. $r=\pm a$, $r=\pm(b^2+c^2)^{1/2}$. Substitute $r=0$ into Eq. (1) gives $A_2=A_4=A_6=A_7=A_8=0$, $A_3b+cA_9 = 0$, no information on A_1 and A_5. So for $r=0$, we can choose three combinations for A_1, A_3, A_5, A_9. They are $A_1=1$, $A_i=0$ $(i\neq 1)$, i.e. ψ_{322}; $A_5=1$, $A_i = 0$ $(i\neq 5)$, i.e. ψ_{32-2}; $A_i = 0$ $(i\neq 3,9)$, $A_3 = \sqrt{2/3}$, $A_9 = -\sqrt{1/3}$, i.e. $(\sqrt{2/3}\psi_{320} - \sqrt{1/3}\psi_{300})$. Here our notation is $A_1 \ldots\ldots A_5$ correspond to $\ell=2$, $m_\ell=2,1,0,-1,-2$; A_6, A_7, A_8 correspond to $\ell=1$, $m_\ell=1,0,-1$; A_9 corresponds to $\ell=0$, $m_\ell=0$. To summarize:

For $r=0$, $\Delta E=0$, wave functions are $\psi_{322}, \psi_{32-2}, \sqrt{2/3}\psi_{320} - \sqrt{1/3}\psi_{300}$. (2)

For $r=a=27/2$, i.e. $\lambda= 27e\varepsilon a_o/2$, we have from Eq.(1), $A_1=0$, $A_5=0$, $A_3=A_7=A_9=0$ and either $A_2 = -A_6 = 1/\sqrt{2}$, $A_4=A_8=0$ or $A_2=A_6=0$, $A_4= -A_8 = 1/\sqrt{2}$ i.e.

$$r=a,\ \Delta E = 27e\varepsilon a_o/2;\ \text{wave functions}\ \begin{cases} \frac{1}{2^{1/2}}(\psi_{321}-\psi_{311}) \\ \frac{1}{2^{1/2}}(\psi_{32-1}-\psi_{31-1}) \end{cases} . \tag{3}$$

For $r= -a = -27/2$, i.e. $\lambda = -27e\varepsilon a_o/2$, we find from Eq.(1) $A_1=A_5=0$, $A_3=A_7=A_9 = 0$ and either $A_2=A_6=1/\sqrt{2}$, $A_4=A_8=0$ or $A_2=A_6=0$, $A_4=A_8=1/\sqrt{2}$, i.e.

$$r=-a,\ \Delta E = -27e\varepsilon a_o/2;\ \text{wave functions}\ \begin{cases} \frac{1}{2^{1/2}}(\psi_{321}+\psi_{311}) \\ \frac{1}{2^{1/2}}(\psi_{32-1}+\psi_{31-1}) \end{cases} . \tag{4}$$

Finally, for $r=\pm(b^2+c^2)^{1/2} = \pm 9\sqrt{9}$, i.e. $\lambda = \pm 9\sqrt{9}e\varepsilon a_o$, we find $A_1=A_5=A_2=A_4=A_6=A_8=0$. For $r=+9\sqrt{9}$, $A_3=1/\sqrt{6}$, $A_7= -1/\sqrt{2}$, $A_9=1/\sqrt{3}$, i.e.

$$r=+(b^2+c^2)^{1/2},\ \Delta E=+9\sqrt{9}e\varepsilon a_o;\ \text{wave function is}\ [\tfrac{1}{6^{1/2}}\psi_{320}-\tfrac{1}{2^{1/2}}\psi_{310}+\tfrac{1}{3^{1/2}}\psi_{300}] \tag{5}$$

and for $r=-9\sqrt{9}$, $A_3=1/\sqrt{6}$, $A_7=1/\sqrt{2}$, $A_9=1/\sqrt{3}$, i.e.

$$r=-(b^2+c^2)^{1/2},\ \Delta E=-9\sqrt{9}e\varepsilon a_o;\ \text{wave function is}\ [\tfrac{1}{6^{1/2}}\psi_{320}+\tfrac{1}{2^{1/2}}\psi_{310}+\tfrac{1}{3^{1/2}}\psi_{300}] \tag{6}$$

15. *This solution reprinted from the solutions manual for the revised edition.*

For electric dipole $V = -\vec{\mu}_e\cdot\vec{E}$ where $\vec{\mu}_e = \mu_e\vec{\sigma}$. The Coulomb field of the nucleus may be written as: $\vec{E} = -\frac{1}{e}\hat{r}dV_c/dr$ (V_c = Coulomb potential). Now $\vec{\sigma}\cdot\hat{r}$ may be written as: $\vec{\sigma}\cdot\hat{r} = [\sigma_+(x-iy)+\sigma_-(x+iy)+\sigma_z z]/r = (4\pi/3)^{1/2}[\sqrt{2}(\sigma_+Y_1^{-1}-\sigma_-Y_1^{1})+\sigma_zY_1^0]$. Hence there are selection rules governing which matrix elements of V are non-zero. For $\Delta m_\ell=0$ the matrix elements of Y_1^0 are needed. These vanish unless $\Delta\ell=\pm1$. For $\Delta m_\ell=\pm1$, $\Delta\ell$ is also ±1. This is expected since $\hat{r}$ is a vector operator and connects states of different parity. The radial contribution is proportional to: $\int_0^\infty R_{n\ell}\frac{dV}{dr}cR_{n'\ell'}r^2dr = -\int_0^\infty R_{n\ell}R_{n'\ell'}dr$. One may verify that for $\ell-\ell' =\pm1$, this integral vanishes for $n=n'$.

The ground state of Na has n=3 (degeneracy n^2=9). But from the above, we know that $\Delta n \neq 0$ therefore the effects of this perturbation V on the energy levels are seen in second order. Mixings will occur between 3s and 4p states and similarly between 4s and 3p, 3d and 4p etc. Using eigenstates of L^2, L_z, S^2, S_z, the following expression for $\langle 3s|V|4p\rangle$ is true for $\Delta \ell_z = 0$.

$$\langle 3s|V|4p\rangle_{\Delta\ell_z=0} = \frac{Z}{(-e)} \int_0^\infty R_{30}(r)R_{41}(r)\,dr\left(\frac{4\pi}{3}\right)^{\frac{1}{2}}\langle 00\tfrac{1}{2}\tfrac{1}{2}|Y_1^0\sigma_z|10\tfrac{1}{2}\tfrac{1}{2}\rangle$$

$$= \frac{Z}{-e}\int_0^\infty R_{30}R_{41}\,dr\left(\frac{4\pi}{3}\right)^{\frac{1}{2}}\int_{-1}^{+1}\int_0^{2\pi}\left(\frac{1}{4\pi}\right)^{\frac{1}{2}}\left(\frac{3}{4\pi}\right)^{\frac{1}{2}}\cos\theta\times\left(\frac{3}{4\pi}\right)^{\frac{1}{2}}\cos\theta\, d(\cos\theta)\,d\phi$$

$$= \left(\frac{Z}{-e}\right)\left(\frac{1}{3}\right)^{\frac{1}{2}}\int_0^\infty R_{30}R_{41}\,dr = \left(\frac{Z}{-e}\right)\left(\frac{1}{3}\right)^{\frac{1}{2}} I_R \ .$$

So the second (lowest) order shift in the 3s state of Na would be (using (5.2.18))

$$\Delta_{3s} = \left(\frac{-\mu_e Z I_R}{e\sqrt{3}}\right)^2 \Big/ (E_{n=3} - E_{n=4})$$

where $E_n = -Z^2 me^4/2\hbar^2 n^2$.

16. *This solution reprinted from the solutions manual for the revised edition.*

(a) This is the central force problem with spherically symmetric potential V(r). As usual let $\psi(r) = cu(r)/r$ where $\psi(r)$ satisfies the usual radial Schrödinger equation and u(r) satisfies (c.f. (A.5.8))

$$-\frac{\hbar^2}{2m}\frac{d^2u}{dr^2} + V(r)u = Eu \quad (\text{for } \ell=0 \text{ S-states}). \tag{1}$$

Multiply (1) by $u' \equiv du/dr$, we have

$$-\frac{\hbar^2}{2m}\frac{1}{2}\frac{d(u')^2}{dr} + \frac{1}{2}\frac{d(uVu)}{dr} - \frac{1}{2}\frac{dV}{dr}u^2 = \frac{E}{2}\frac{d(u^2)}{dr}. \tag{2}$$

Integrate (2) from 0 to ∞ on both sides, we have

$$-\frac{\hbar^2}{4m}(u')^2\Big|_0^\infty + \tfrac{1}{2}(uVu)\Big|_0^\infty - \tfrac{1}{2}\int_0^\infty \frac{dV}{dr}u^2\,dr = \frac{E}{2}u^2\Big|_0^\infty \ . \tag{3}$$

But $\lim_{r\to 0} u(r) = 0$, and $\lim_{r\to\infty} u(r) = 0$, therefore (3) gives

$$-\frac{\hbar^2}{4m}(u')^2\Big|_0^\infty - \tfrac{1}{2}\int_0^\infty \frac{dV}{dr}u^2\,dr = 0. \tag{4}$$

From $u(r) = r\psi(r)/c$, we get $u'(r) = \psi(r)/c + r\psi'(r)/c$, where at ∞ the right hand side functions are well behaved and must vanish as $r\to\infty$. Thus (4) gives

$$\frac{\hbar^2}{4mc^2}|\psi(0)|^2 = \frac{1}{2c^2}\int_0^\infty r^2(dV/dr)\psi^2(r)dr = \frac{1}{4\pi(2c^2)}\langle\frac{dV}{dr}\rangle ,$$

and therefore

$$|\psi(0)|^2 = (\frac{m}{2\pi\hbar^2})\langle dV/dr\rangle \tag{5}$$

(b) For the hydrogen atom $V(r) = -e^2/r$ and for the ground state (from (A.6.7)) we have $R_{10}(\rho) = (2/a_o^{3/2})e^{-\rho/2}$, where $\rho = 2r/a_o$ and $a_o = \hbar^2/m_e e^2$ is the Bohr radius (c.f. (A.6.3)). $R_{10}(0) = 2/a_o^{3/2} = 2/(\hbar^2/me^2)^{3/2} = 2e^3m^{3/2}/\hbar^3$, and

$$\langle dV/dr\rangle = e^2 4\pi\int_0^\infty \frac{1}{r^2}\, r^2 R_{10}^2(r)dr = \frac{4\pi e^2.4}{a_o^3}\int_0^\infty e^{-\rho}dr$$

and since $dr = a_o d\rho/2$, we have $\langle dV/dr\rangle = 8\pi e^2/a_o^2$. Therefore $\frac{m}{2\pi\hbar^2}\langle dV/dr\rangle =$ $\frac{m}{2\pi\hbar^2}\cdot\frac{8\pi e^2}{a_o^2} = 4me^2/\hbar^2 a_o^2 = 4/a_o^3 = |R_{10}(0)|^2$. Hence relation (5) is verified.

For the three dimensional harmonic oscillator $V(r) = \frac{1}{2}kr^2$, the ground state is $n_x=n_y=n_z=0$, and wave function $\psi = X_o(x)Y_o(y)Z_o(z)$ is such that

$$X_o(x)=N_oH_o(\alpha x)e^{-\frac{1}{2}\alpha^2x^2},\ Y_o(y)=N_oH_o(\alpha y)e^{-\frac{1}{2}\alpha^2y^2},\ Z_o(z)=N_oH_o(\alpha z)e^{-\frac{1}{2}\alpha^2z^2}$$

where $N_o=(\alpha/\pi^{\frac{1}{2}})^{\frac{1}{2}}$ and $\alpha = (mk/\hbar^2)^{\frac{1}{4}}$. So $|\psi(0)|^2 = N_o^6H_o^6(0)$, while

$$\langle dV/dr\rangle = N_o^6\int_{-\infty}^{+\infty}H_o^2(\alpha x)H_o^2(\alpha y)H_o^2(\alpha z)[kr]e^{-\alpha^2r^2}dxdydz.$$

From (A.4.5) we see that $H_o(\xi) = 1$, hence $|\psi(0)|^2 = N_o^6$, while $\langle dV/dr\rangle = N_o^6\int_0^\infty kr\times$ $e^{-\alpha^2r^2}r^2dr(4\pi) = N_o^6(4\pi)k\int_0^\infty r^3e^{-\alpha^2r^2}dr = N_o^6(4\pi)k\frac{1}{2\alpha^4} = N_o^6(2\pi)\hbar^2/m$. Thus $(\frac{m}{2\pi\hbar^2})\times$ $\langle dV/dr\rangle = N_o^6 = |\psi(0)|^2$ for the three dimensional isotropic harmonic oscillator also.

17. *This solution reprinted from the solutions manual for the revised edition.*

(a) Rotate the system in such a way that the z'-axis is along the magnetic field $\vec{B}$, we then have $H = A\vec{L}^2+(B^2+C^2)^{\frac{1}{2}}L'_z$ where in the y-z plane the angle θ between Oz and Oz' is given by $\tan\theta = C/B$. We then have eigenkets $|\ell,m'\rangle$ with eigenvalues

$$E = A\ell(\ell+1)\hbar^2 + (B^2+C^2)^{\frac{1}{2}}m'\hbar \tag{1}$$

where $|\ell,m'\rangle = \mathcal{D}(\pi/2,\beta,0)|\ell,m\rangle = \sum_{m=-\ell}^{\ell}|\ell,m\rangle\mathcal{D}^{(\ell)}_{mm'}(\pi/2,\beta,0)$. When $B \gg C$, we treat $H_o = A\vec{L}^2 + BL_z$ as the unperturbed Hamiltonian, and CL_y as the perturbation, than unperturbed eigenvalues $E^{(o)}_{\ell,m}$ and eigenkets are $A\hbar^2\ell(\ell+1)+Bm\hbar$ and $|\ell,m\rangle$ respectively. Hence to second order in perturbation

$$E^{(2)}=A\hbar^2\ell(\ell+1)+Bm\hbar + \langle\ell,m|CL_y|\ell,m\rangle + C^2\sum_{\substack{\ell',m'\\ \neq \ell,m}}\frac{|\langle\ell',m'|L_y|\ell,m\rangle|^2}{E^{(o)}_{\ell,m} - E^{(o)}_{\ell',m'}}\ . \tag{2}$$

Use next $L_y = \frac{1}{2i}(L_+-L_-)$ and (3.5.41), (2) becomes

$$E^{(2)} = A\hbar^2\ell(\ell+1) + Bm\hbar + C^2\hbar m/2B. \tag{3}$$

From the exact solution (1), we may expand for $B \gg C$ to get

$$E = A\ell(\ell+1)\hbar^2 + Bm'\hbar + \frac{C^2}{2B}\hbar m' + \ldots\ldots\ldots \tag{4}$$

Hence in this approximation ($B \gg C$), the second-order perturbed energy (3) reproduces the exact solution for $m'\to m$.

(b) We consider $\langle n'\ell'm'_\ell m'_s|O|n\ell m_\ell m_s\rangle$ where $O = 3z^2-r^2$, xy. Note that the operator O is <u>spin-independent</u>, hence $\Delta m_s = m'_s-m_s=0$. Now $3z^2-r^2 \sim (3\cos^2\theta-1) \sim Y_2^0$, hence $\langle\ell'm'_\ell|Y_2^0|\ell,m_\ell\rangle$ must satisfy $\Delta m_\ell = m_{\ell'}-m_\ell = 0$, and $-2\leq\Delta\ell=\ell'-\ell\leq +2$. However $|\Delta\ell| \neq 1$ because of parity conservation. <u>Summary</u>: $\Delta m_s = m'_s - m_s = 0$, $\Delta m_\ell = m_{\ell'}-m_\ell=0$, $\Delta\ell= 0,\pm2$. (Actually we have also the constraint $\ell+\ell'\geq 2$.)

Consider next $O = xy$, now $Y_2^2 \propto (x+iy)^2$, $Y_2^{-2} \propto (x-iy)^2$, hence $Y_2^2 - Y_2^{-2} \propto xy$. So $\langle\ell'm'_\ell|(Y_2^2-Y_2^{-2})|\ell m_\ell\rangle$ satisfies $\Delta m_\ell=2,-2$; $\Delta\ell= \pm1$ remains forbidden by parity conservation, hence $\Delta\ell=0,\pm2$. <u>Summary</u>: $\Delta m_s = m'_s-m_s=0, \Delta m_\ell = \pm2, \Delta\ell=0,\pm2$ $(\ell+\ell'\geq 2)$.

<u>Remarks</u>: The above selection rules are different from those for dipole radiations which require $\Delta m_s = 0, \Delta m_\ell=0,\pm1$, $\Delta\ell=0,\pm1$, which is not surprising since for instance $3z^2-r^2$ relates to quadrupole radiation.

18. *This solution reprinted from the solutions manual for the revised edition.*

The perturbation Hamiltonian (see (5.3.25)) is $e^2A^2/2m_ec^2 = e^2B^2(x^2+y^2)/8m_ec^2$, where we have used $A_x = -\frac{1}{2}By$, $A_y = \frac{1}{2}Bx$, $A_z = 0$ and noted that the perturbation is spin independent (hence okay to ignore spin). So we must evaluate $\langle x^2+y^2\rangle$ for the ground state. Now by symmetry $\langle x^2+y^2\rangle = \frac{2}{3}\langle r^2\rangle$ because $\langle x^2\rangle = \langle y^2\rangle = \langle z^2\rangle$ and $\langle x^2+y^2+z^2\rangle = \langle r^2\rangle$. So the integral to be evaluated relative to the ground-state of hydrogen atom is $4\pi\int(1/\pi a_o^3)e^{-2r/a_o}\, r^2r^2dr = \frac{4}{a_o^3}(a_o/2)^5 4!$. Hence

$$\Delta = \frac{e^2B^2a_o^2}{m_ec^2}\,\frac{4.4.3.2}{8.2.2.2.2.2}(2/3) = \frac{e^2B^2a_o^2}{4m_ec^2}$$

and $\chi = -\,e^2a_o^2/2m_ec^2$, the negative sign is because the induced dipole moment has opposite sign for diamagnetism.

19. *This solution reprinted from the solutions manual for the revised edition.*

In this problem, we work out the quadratic Zeeman effect with the help of vector potential $\vec{A} = \frac{1}{2}\vec{B}\times\vec{r}$ for uniform magnetic field $\vec{B} = B_o\hat{e}_z$ (we notice $\vec{B} = \vec{\nabla}\times\vec{A}$). Using the Lorentz gauge $\vec{\nabla}.\vec{A} = 0$, then $[\vec{p},\vec{A}] = -i\hbar\vec{\nabla}.\vec{A} = 0$ or $\vec{A}.\vec{p} = \vec{p}.\vec{A}$ for particle momentum $\vec{p}$. Then $\vec{A}.\vec{p} = \frac{1}{2}(\vec{B}\times\vec{r}).\vec{p} = \frac{1}{2}\vec{B}.\vec{r}\times\vec{p} = \frac{1}{2}\vec{B}.\vec{L} = \frac{1}{2}B_oL_z$, $\vec{A}^2 = \frac{1}{4}(\vec{B}\times\vec{r}).(\vec{B}\times\vec{r}) = \frac{1}{4}[\vec{B}^2\vec{r}^2 - (\vec{B}.\vec{r})^2] = \frac{1}{4}B_o^2(x^2+y^2)$, and the total Hamiltonian will be

$$H = \frac{1}{2m_e}(\vec{p} - \frac{e}{c}\vec{A})^2 = \frac{\vec{p}^2}{2m_e} - \frac{eB_o}{2m_ec}L_z + e^2B_o^2(x^2+y^2)/8m_ec^2 - \frac{Ze^2}{r}$$

with the perturbation term $V = \frac{-eB_o}{2m_ec}L_z + \frac{e^2B_o^2}{8m_ec^2}(r^2\sin^2\theta)$. For zero angular momentum $\ell=0$ (S-state), we have $\vec{L} = 0$, $L_z = 0$, and in this simple case, for an atomic electron in the n=1 ground state of an atom with atomic number Z, the energy change will be

$$\Delta E^{(1)}_{Z,m_\ell=0} = \int_0^{2\pi}d\phi\int_{-1}^{+1}d(\cos\theta)\int_0^\infty r^2dr\frac{Z^3}{\pi a_o^3}\, e^{-2Zr/a_o}(e^2B_o^2r^2\sin^2\theta/8m_ec^2)$$
$$= [1/(2Z)^2].\frac{e^2B_o^2a_o^2}{m_ec^2} = [1/(2Z)]^2.\lambda_e^3B_o^2/\alpha \qquad (1)$$

in which $\lambda\!\!\!^{-}_e = \hbar/m_e c$ is the electron Compton wavelength and $a_o = \hbar^2/m_e e^2$ is the Bohr atomic radius and $\alpha = e^2/\hbar c \simeq 1/137$ is the fine structure constant. The above integral was carried out by using $\int_0^\infty d\xi \xi^N e^{-\rho\xi} = N!/\rho^{N+1}$.

Now for the helium atom the result would be twice that we obtained for an atomic electron in (1) with effective atomic number $Z = 2-5/16 \simeq 1.7$:

$$\Delta E^{(1)}_{He,m_\ell=0} = 2\times \frac{1}{(2Z)^2\alpha} \lambda\!\!\!^{-3}_e B_o^2 \Big|_{Z\simeq1.7} \simeq 23.7\lambda\!\!\!^{-3}_e B_o^2 \ . \tag{2}$$

For one mole of helium the energy change is $N_o \Delta E^{(1)}_{He,m_\ell=0}$ where $N_o \simeq 6.022\times10^{23}$/mole (the Avogadro's number). Thus the magnetic susceptibility per mole of helium, χ_{He}, is going to be

$$N_o \Delta E^{(1)}_{He,m_\ell=0} = -\tfrac{1}{2}\chi_{He} B_o^2 \ \rightarrow \ \chi_{He} = -2N_o\times 23.7\lambda\!\!\!^{-3}_e. \tag{3}$$

Expressed in terms of a_o(atomic unit), we have $\lambda\!\!\!^{-}_e = 7.2973\times10^{-3}a_o$, then

$$\chi_{He} = -1.109\times10^{19} a_o^3 \ /\text{mole} \equiv -1.643\times10^{-6}\text{cm}^3/\text{mole}. \tag{4}$$

The experimental result is -1.88×10^{-6}cm^3/mole which is in fairly good agreement with our perturbation calculation.

20. *This solution reprinted from the solutions manual for the revised edition.*

$$\overline{H} = \frac{(-\hbar^2/2m)\int_{-\infty}^{+\infty} e^{-\beta|x|}\frac{d^2}{dx^2} e^{-\beta|x|}dx + \int_{-\infty}^{+\infty} e^{-2\beta|x|}(m\omega^2x^2/2)dx}{\int_{-\infty}^{+\infty} e^{-2\beta|x|}dx}$$

$$= \frac{-\frac{2\hbar^2}{2m}\int_0^\infty \beta^2 e^{-2\beta x}dx - \frac{\hbar^2}{2m}(-2\beta) + \frac{m\omega^2}{2}(2)\int_0^\infty e^{-2\beta x}x^2dx}{2\int_0^\infty e^{-2\beta x}dx}$$

where the term $-\frac{\hbar^2}{2m}(-2\beta)$ in numerator is the contribution from the first derivative at x=0. So $\overline{H} = \hbar^2\beta^2/2m + m\omega^2/4\beta^2$, and $\partial\overline{H}/\partial\beta = 0$ implies $2\hbar^2\beta/2m - m\omega^2/2\beta^3 = 0$ or $\beta^2 = m\omega/\sqrt{2}\hbar$. Hence $(\overline{H})_{min} = \frac{\hbar^2}{2\sqrt{2}}\frac{m\omega}{m\hbar} + \frac{m\omega^2\sqrt{2}\hbar}{4m\omega} = \hbar\omega(\frac{1}{2\sqrt{2}} + \frac{\sqrt{2}}{4}) = \frac{\hbar\omega\sqrt{2}}{2}$,

where $(\hbar\omega/2)$ is the true energy.

21. *This solution reprinted from the solutions manual for the revised edition.*

The equation $d^2\psi/dx^2 + (\lambda-|x|)\psi = 0$ can be written as $-d^2\psi/dx^2 + |x|\psi = \lambda\psi$ and hence is like Schrödinger equation $H\psi = \lambda\psi$ with $\hbar^2/2m = 1$. Let us set c=1 and worry about normalization later, than

$$d\psi/dx = \begin{cases} -1 \text{ for } 0<x<\alpha \\ +1 \text{ for } -\alpha<x<0 \end{cases}, \quad \int_{-\varepsilon}^{\varepsilon} d^2\psi/dx^2\, dx = (d\psi/dx)_{x=\varepsilon} - (d\psi/dx)_{x=-\varepsilon},$$

hence $d^2\psi/dx^2 = -2\delta(x)$, and $\langle\psi|H|\psi\rangle = 2\psi(0) + 2\int_0^\alpha x(\alpha-x)^2 dx = 2\alpha + \alpha^4/12$. Also $\langle\psi|\psi\rangle = 2\int_0^\alpha (\alpha-x)^2 dx = 2\alpha^3/3$. Therefore from (5.4.2), we have

$$\lambda \leqslant \frac{\langle\psi|H|\psi\rangle}{\langle\psi|\psi\rangle} = \frac{2\alpha+2\alpha^4/12}{2\alpha^3/3} = \frac{3}{\alpha^2} + \frac{\alpha}{4}. \tag{1}$$

Hence $d\lambda/d\alpha = 0$ implies $(3/\alpha^3)(-2) + \frac{1}{4} = 0$ or $\alpha = 24^{1/3} = 2\times 3^{1/3}$, and $\lambda <$ $3/4\times 3^{2/3} + 2\times 3^{1/3}/4 = 1.081$. So the true λ must be lower than 1.081 which is not bad compared to exact value 1.019 for such a crude trial function. Note normalization of ψ is taken care of via $\langle\psi|\psi\rangle$ in denominator of (1).

22. *This solution reprinted from the solutions manual for the revised edition.*

Here $V(t) = F_o x\cos\omega t$, and we set $\omega_{1o} \equiv (E_1-E_0)/\hbar = \omega_o$. From (5.6.17) we see that $c_o(t) \equiv 1$ up to first order, while

$$\begin{aligned} c_1(t) &\equiv (-i/\hbar)\frac{F_o}{2} \int_0^t \langle 1|x|0\rangle e^{i\omega_{1o}t'}[e^{i\omega t'} + e^{-i\omega t'}]dt' \\ &= -(F_o/2\hbar)\langle 1|x|0\rangle\left[\frac{e^{i(\omega_o+\omega)t}-1}{(\omega_o+\omega)} + \frac{e^{i(\omega_o-\omega)t}-1}{(\omega_o-\omega)}\right]. \end{aligned} \tag{1}$$

Let us compute x in the Schrödinger picture, than

$$\begin{aligned} \langle x\rangle_S &= (\langle 0|e^{i\omega_o t/2} + c_1^*\langle 1|e^{3i\omega_o t/2})x(|0\rangle e^{-i\omega_o t/2} + c_1|1\rangle e^{-3i\omega_o t/2}) \\ &= c_1^*(t)\langle 1|x|0\rangle e^{i\omega_o t} + c_1(t)\langle 0|x|1\rangle e^{-i\omega_o t} \\ &= -\left(\frac{F_o}{2\hbar}\right)|\langle 1|x|0\rangle|^2\left[\frac{e^{-i(\omega_o+\omega)t}e^{i\omega_o t} - e^{i\omega_o t}}{(\omega_o+\omega)} + \frac{e^{i(\omega_o-\omega)t}e^{-i\omega_o t} - e^{-i\omega_o t}}{(\omega_o-\omega)}\right] \\ &\quad + \text{c.c. (complex conjugate)} \end{aligned} \tag{2}$$

where we have used (1) and the constancy of F_o in arriving at (2). Since $\langle 1|x|0\rangle = (\hbar/2m\omega_o)^{\frac{1}{2}}$, (2) becomes

$$\langle x\rangle_S = -\frac{F_o}{m}\left(\frac{\cos\omega t - \cos\omega_o t}{\omega_o^2 - \omega^2}\right) . \tag{3}$$

This is more or less what you expect classically. As $\omega \cong \omega_o$, $\cos\omega t - \cos\omega_o t \cong -\frac{1}{2}\omega^2 t^2 + \frac{1}{2}\omega_o^2 t^2 = \frac{1}{2}t^2(\omega_o^2-\omega^2)$, thus $\langle x\rangle_S \cong \frac{-F_o}{m}\frac{1}{2}t^2$. Treating $-F_o/m$ as a classical uniform acceleration a, $\langle x\rangle_S = \frac{1}{2}at^2$ is the classical rectilinear motion starting from rest, however procedure breaks down for $\omega \cong \omega_o$.

23. *This solution reprinted from the solutions manual for the revised edition.*

(a) For a force $F(t) = F_o e^{-t/\tau}$, we have $-dV/dx = F_o e^{-t/\tau}$, hence $V = -F_o x e^{-t/\tau}$. Again from (5.6.17), $c_o^{(o)}(t) = 1$, and $\omega_{1o} \equiv (E_1-E_0)/\hbar = \omega$, while

$$\begin{aligned} c_1^{(1)}(t) &= (-i/\hbar)\int_o^t e^{i\omega t'}e^{-t'/\tau}dt'\langle 1|x|0\rangle F_o \\ &= (-i/\hbar)\left[\frac{e^{i\omega t-t/\tau}-1}{(i\omega - 1/\tau)}\right]\langle 1|x|0\rangle F_o . \end{aligned} \tag{1}$$

Hence

$$|c_1^{(1)}(t)|^2 = \frac{1}{\hbar^2}\left[\frac{1+e^{-2t/\tau}-(2\cos\omega t)e^{-t/\tau}}{\omega^2 + (1/\tau)^2}\right]|F_o|^2(\hbar/2m\omega). \tag{2}$$

Note that as $t\to\infty$, $|c_1^{(1)}(t)|^2$ is independent of t. This is reasonable since for sufficiently large t, the perturbation is no longer on.

(b) Take (5.6.17) again, we see to first order the nth excited state is

$$c_n^{(1)}(t) = (-i/\hbar)\int_o^t e^{i\omega_{no}t'}V_{no}(t')dt' \tag{3}$$

where $\omega_{no} \equiv (E_n-E_o)/\hbar$, and $n\geq 2$. However $V_{no}(t')$ would contain multiplicative factor $\langle n|x|0\rangle$ which vanishes for $n\geq 2$. Nevertheless for $\langle n'|x|n\rangle = \sqrt{\hbar/2m\omega}(\sqrt{n}\delta_{n',n-1} + \sqrt{n+1}\delta_{n',n+1})$, we know that $\langle 2|x|1\rangle = \sqrt{2}(\hbar/2m\omega)^{\frac{1}{2}}$ while $\langle 1|x|0\rangle = \sqrt{\hbar/2m\omega}$. Thus to <u>second order</u>

$$c_2^{(2)}(t) = (-i/\hbar)^2\int_o^t dt'\int_o^{t'}dt'' e^{i\omega_{21}t'}V_{21}(t')e^{i\omega_{1o}t''}V_{10}(t'') \tag{4}$$

gives a non-vanishing contribution, since V_{21} and V_{10} are non-vanishing ($\omega_{21} = (E_2-E_1)/\hbar$). Thus there is a finite probability to find the oscillator in its <u>second</u> excited state E_2, and the argument can be pursued to even higher order terms and corresponding higher order excited states.

24. *This solution reprinted from the solutions manual for the revised edition.*
The initial state is $|0\rangle$, so from (5.6.17), we have

$$c_n^{(o)}(t) = \delta_{no},\ c_n^{(1)}(t) = (-i/\hbar)\int_o^t e^{-i(E_o-E_n)t'/\hbar}\langle n|H'(x,t')|0\rangle dt'. \quad (1)$$

Next we note that

$$\langle n|H'(x,t)|0\rangle = Ae^{-t/\tau}\langle n|x^2|0\rangle \quad (2)$$

and from (2.3.24), we have $x^2|0\rangle = \frac{1}{2}(\hbar/m\omega)(a+a^\dagger)(a+a^\dagger)|0\rangle$. Since $a|0\rangle = 0$, $a^\dagger|0\rangle = |1\rangle$, $a|1\rangle = |0\rangle$, $a^\dagger|1\rangle = \sqrt{2}|2\rangle$, thus $x^2|0\rangle = (\hbar/2m\omega)[|0\rangle + \sqrt{2}|2\rangle]$, and therefore $\langle n|x^2|0\rangle = (\hbar/2m\omega)[\delta_{no}+\sqrt{2}\delta_{n2}]$. We see that if $n \neq 0$ or $n \neq 2$, $c_n^{(1)}(t)$ of (1) vanishes because $\langle n|x^2|0\rangle$ vanishes in (2). Only the following coefficients are relevant to our discussion: $c_o^{(o)}=1$, $c_2^{(o)}=0$, $c_o^{(1)} = (-i/\hbar)\int_0^t(\hbar/2m\omega)\times Ae^{-t'/\tau}dt' = \frac{iA}{2m\omega}(e^{-t/\tau}-1)\tau$ (which for $t/\tau >>1$, gives $c_o^{(1)} \simeq -iA\tau/2m\omega$), $c_2^{(1)} = (-i/\hbar)\frac{\hbar}{2m\omega}\sqrt{2}\int_0^t \exp[-i(E_o-E_2)t'/\hbar]Ae^{-t'/\tau}dt' = -i\sqrt{2}A/2m\omega(1/\tau - 2\omega i)$.

After a long time duration of perturbation, the state becomes [see (5.5.4) and (5.6.1)]

$$|\psi\rangle = [1 - iA\tau/2m\omega]e^{-i\omega t/2}|0\rangle - \frac{i\sqrt{2}A}{2m\omega(1/\tau-2i\omega)}e^{-i5\omega t/2}|2\rangle \quad (3)$$

(Remark: higher order terms like A^2, A^3,...... are ignored.) So the probability for the system to be transmitted to the second excited state is

$$P_2 = \frac{|A|^2}{2m^2\omega^2(1/\tau^2+4\omega^2)} \Big/ [1+\frac{|A|^2\tau^2}{4m^2\omega^2} + \frac{1}{2}\frac{|A|^2}{m^2\omega^2(4\omega^2+1/\tau^2)}]. \quad (4)$$

There is no probability for transition to other states such as $|1\rangle, |3\rangle$,.......

25. *This solution reprinted from the solutions manual for the revised edition.*

$$H = \begin{pmatrix} E_1^{(o)} & \lambda\cos\omega t \\ \lambda\cos\omega t & E_2^{(o)} \end{pmatrix} = H_o + V(t)$$

(a) Let us write $|1\rangle = \begin{pmatrix}1\\0\end{pmatrix}$ and $|2\rangle = \begin{pmatrix}0\\1\end{pmatrix}$. A general state is

$$|\alpha,t\rangle = c_1(t)\exp[-iE_1^{(o)}t/\hbar]\ |1\rangle + c_2(t)\exp[-iE_2^{(o)}t/\hbar]\ |2\rangle$$

with $c_1(0) = 1$, and $c_2(0) = 0$. Now this problem can be solved exactly, but we are told to proceed via time-dependent perturbation theory. Take (5.6.17) - (5.6.19) of text, we have (for n=2)

$$c_2^{(1)}(t) = -\frac{i}{\hbar}\lambda \int_0^t \exp[i\omega_{21}t']\cos\omega t'\, dt'$$

$$= (-i/\hbar)\lambda\int_0^t \tfrac{1}{2}[\exp(i[\omega_{21}+\omega]t' + \exp(i[\omega_{21}-\omega]t')]dt'$$

$$= \left(\frac{-\lambda i}{\hbar}\right)\left[\frac{e^{i(\omega_{21}+\omega)t/2}\sin(\omega_{21}+\omega)t/2}{(\omega_{21}+\omega)} + \frac{e^{i(\omega_{21}-\omega)t/2}\sin(\omega_{21}-\omega)t/2}{(\omega_{21}-\omega)}\right].$$

Now $|c_2^{(1)}(t)|^2$ is the transition probability which becomes

$$|c_2^{(1)}(t)|^2 = \frac{\lambda^2}{\hbar^2}\left[\frac{\sin^2(\omega_{21}+\omega)t/2}{(\omega_{21}+\omega)^2} + \frac{\sin^2(\omega_{21}-\omega)t/2}{(\omega_{21}-\omega)^2} + \frac{\cos\omega t(\cos\omega t-\cos\omega_{21}t)}{(\omega_{21}^2-\omega^2)}\right]$$

(b) Since $\omega_{21} = (E_2^{(o)} - E_1^{(o)})/\hbar$, we see that $\omega_{21}\pm\omega = 0$ would correspond to vanishing denominators in our perturbation expression for $|c_2^{(1)}(t)|^2$ above, and hence a breakdown of the approximation scheme.

26. *This solution reprinted from the solutions manual for the revised edition.*
Perturbation potential added is $-F(t)x$. The ground state energy $E_o = \frac{1}{2}\hbar\omega$ and the first excited state has energy $E_1 = \hbar\omega(1+\frac{1}{2})$ where $\omega_{1o} = \frac{1}{\hbar}(E_1-E_0) = \omega$. From (5.6.17), we have

$$c_1^{(1)}(\infty) = +(-i/\hbar)\frac{F_o\tau}{\omega}\langle 1|x|0\rangle \int_{-\infty}^{+\infty}\frac{e^{i\omega t}}{\tau^2+t^2}dt = (i/\hbar)\frac{F_o\tau}{\omega}\langle 1|x|0\rangle I\ . \qquad (1)$$

The integral I may be evaluated using complex variable theory. Since $\omega>0$, we close contour in upper half t-plane ($\mathrm{Im}(t)>0$) with no contribution from semicircle as $|t| \to \infty$. The pole at $t=+i\tau$ gives through the method of residues, contribution $I = (\pi/\tau)e^{-\omega\tau}$. Since $\langle 1|x|0\rangle = \sqrt{\hbar/2m\omega}\,(\sqrt{n}\delta_{n',n-1} + \sqrt{n+1}\delta_{n',n+1}) = \sqrt{\hbar/2m\omega}$ for $n=0, n'=1$, we have putting everything together

$$c_1^{(1)}(\infty) = \frac{i}{\hbar}\,\frac{F_o\tau}{\omega}\sqrt{\hbar/2m\omega}\left(\frac{\pi}{\tau}\right)e^{-\omega\tau}\ . \qquad (2)$$

Probability for being found in the first excited state is $|c_1^{(1)}(\infty)|^2 = \frac{\pi^2F_o^2}{2m\hbar\omega^3}e^{-2\omega\tau}$.

"Challenge for experts". Yes, it is reasonable. If the perturbation is turned on very slowly, and then turned off very slowly (as in the $\tau >> \frac{1}{\omega}$ case), the oscillator can be visualized to be in the ground state all the time. This is because the only effect of the applied force (uniform in space) is just a very slow change in the equilibrium point of the oscillator; at each instant of time, you can solve the time-<u>independent</u> Schrödinger equation for the ground state.

This problem can also be attacked semiclassically. The action integral $\oint pdq$ (related to $(n+\frac{1}{2})\hbar$) is "adiabatically invariant". This means that there is no sudden quantum jump as long as the external parameters change very slowly.

27. *This solution reprinted from the solutions manual for the revised edition.*

(a) Again from (5.6.17), $c^{(1)}(t) = (-i/\hbar)\int_{t_o}^{t} \langle f|V(t')|i\rangle e^{i\omega_{fi}(t'-t_o)}dt'$, and using fact that $\delta(x-ct') = \frac{1}{c}\delta(x/c-t')$, we have

$$c^{(1)}(t) = (-i/\hbar)\int_{t_o}^{t} dt' \int dx \langle f|x\rangle \frac{A}{c}\delta(x/c-t')\langle x|i\rangle e^{i\omega_{fi}(t'-t_o)}$$

$$= (-iA/\hbar c)\int_{-\infty}^{+\infty} dx \langle f|x\rangle\langle x|i\rangle e^{i\omega_{fi}x/c} \underbrace{e^{-i\omega_{fi}t_o}}_{\text{uninteresting phase factor}}$$

as $t_o \to -\infty$, and $t \to \infty$. So probability for finding system in state $|f\rangle$ is given by $|c^{(1)}(t)|^2 = \frac{|A|^2}{\hbar^2 c^2}\left|\int_{-\infty}^{\infty} u_f^*(x)u_i(x)e^{i\omega_{fi}x/c}dx\right|^2$ with $\omega_{fi} \equiv (E_f-E_i)/\hbar$.

(b) $\delta(x-ct)$ pulse can be regarded as superposition of harmonic perturbation of form $e^{i\omega x/c}e^{-i\omega t}$ with $\omega>0$ (absorption) as well as $\omega<0$ (emission). Our result in (a) shows that the travelling pulse can give up energy $\hbar\omega = E_f-E_i$ so that the particle gets excited to state $|f\rangle$. The form of $|c^{(1)}|^2$ shows that only that part of the harmonic perturbation with the "right" frequency is relevant, just as expected from energy conservation. Note that the space integral $\int u_f^* u_i dx \times e^{i\omega_{fi}}$ is identical to the case where only one frequency component ("monochromatic wave") is present.

28. *This solution reprinted from the solutions manual for the revised edition.*

To first order 1s → 2s transition is forbidden since the matrix element of perturbation is $\langle 200|z|100\rangle = 0$ by parity. Likewise, since z is proportional to a spherical tensor of rank 1, the only 1s → 2p transition which is allowed, to this (first) order, is when $\Delta m = 0$.

With potential energy $V = -eE_o z e^{-t/\tau}$ for t>0, we have for the only non vanishing transition amplitude is (see (5.6.17))

$$c^{(1)}(t) = -(-i/\hbar)eE_o \int_0^t dt' \langle 210|z|100\rangle e^{(i\omega - 1/\tau)t'} \quad . \tag{1}$$

Therefore to this first order we have selection rule $\Delta \ell = 1$, $\Delta m = 0$. By simple integration, (1) can be rewritten as

$$c^{(1)}(t) = \frac{-(-ieE_o/\hbar)\langle 210|z|100\rangle(e^{[i\omega - 1/\tau]t} - 1)(-i\omega - 1/\tau)}{(\omega^2 + 1/\tau^2)} \quad . \tag{2}$$

From (2) we have probability

$$|c^{(1)}(t)|^2 = \frac{e^2E_o^2}{\hbar^2}\frac{|\langle 210|z|100\rangle|^2}{(\omega^2+1/\tau^2)}[1+e^{-2t/\tau} - 2e^{-t/\tau}(\cos\omega t)]. \tag{3}$$

After a long time $t \gg \tau$(essentially set $t \to \infty$), we have

$$|c^{(1)}(\infty)|^2 = \frac{e^2E_o^2}{\hbar^2}\frac{|\langle 210|z|100\rangle|^2}{(\omega^2 + 1/\tau^2)} \tag{4}$$

where $\langle 210|z|100\rangle = 2\pi\int_{-1}^{+1} d(\cos\theta) \int_0^\infty r^2 dr R_{21} Y_1^o r\cos\theta R_{10} Y_o^o = \frac{2^{15/2}}{3^5} a_o$. and $\omega = (E_{2p} - E_{1s})/\hbar = 3e^2/8a_o\hbar$ (with $a_o = \hbar^2/me^2$).

29. *This solution reprinted from the solutions manual for the revised edition.*

First we observe that

$$\frac{1}{\hbar^2}\vec{S}_1\cdot\vec{S}_2 = \frac{-(\vec{S}_1^2 + \vec{S}_2^2) + (\vec{S}_1 + \vec{S}_2)^2}{2\hbar^2} = \begin{cases} 1/4 \text{ for triplet} \\ -3/4 \text{ for singlet} \end{cases} .$$

Therefore eigenkets of H are triplet and singlet, and eigenvalues are

$$E = \begin{cases} \Delta \text{ for triplet} \\ -3\Delta \text{ for singlet} \end{cases}$$

(a) At $t=0$, $|+-\rangle = \frac{1}{2^{1/2}}(|1,0\rangle + |0,0\rangle)$ where $|1,0\rangle$ is a triplet $m=0$ state and $|0,0\rangle$ is a singlet state. For a later time

$$|\alpha;t\rangle = \frac{1}{2^{1/2}}(|1,0\rangle e^{-i\Delta t/\hbar} + |0,0\rangle e^{+3i\Delta t/\hbar})$$

where $|1,0\rangle = \frac{1}{2^{1/2}}(|+-\rangle+|-+\rangle)$ and $|0,0\rangle = \frac{1}{2^{1/2}}(|+-\rangle-|-+\rangle)$. So

$$\begin{aligned} |\langle+-|\alpha;t\rangle|^2 &= \tfrac{1}{4}|e^{-i\Delta t/\hbar} + e^{3i\Delta t/\hbar}|^2 = \tfrac{1}{2} + \tfrac{1}{2}\cos(4\Delta t/\hbar) \\ |\langle-+|\alpha;t\rangle|^2 &= \tfrac{1}{4}|e^{-i\Delta t/\hbar} - e^{3i\Delta t/\hbar}|^2 = \tfrac{1}{2} - \tfrac{1}{2}\cos(4\Delta t/\hbar) \end{aligned} \qquad (1)$$

and obviously $|\langle++|\alpha;t\rangle|^2 = |\langle--|\alpha;t\rangle|^2 = 0$.

(b) Use first order perturbation theory

$$c_{+-}^{(1)}(t) = (-i/\hbar)\int_0^t \langle+-|\frac{4\Delta}{\hbar^2}\,\vec{S}_1\cdot\vec{S}_2|+-\rangle dt', \quad c_{-+}^{(1)}(t) = (\frac{-i}{\hbar})\int_0^t \langle-+|\frac{4\Delta}{\hbar^2}\,\vec{S}_1\cdot\vec{S}_2|+-\rangle dt'$$

where we note that $\langle+-| = \frac{1}{2^{1/2}}\langle 1,0| + \frac{1}{2^{1/2}}\langle 0,0|$, $\langle-+| = \frac{1}{2^{1/2}}\langle 1,0| - \frac{1}{2^{1/2}}\langle 0,0|$ and similarly for the dual corresponding (DC) kets. Hence $c_{+-}^{(1)}(t) = -\frac{i\Delta t}{\hbar}(1-3)/2 = \frac{i\Delta t}{\hbar}$; $c_{-+}^{(1)}(t) = -\frac{i\Delta t}{\hbar}(1+3)/2 = -i2\Delta t/\hbar$. Note that $c_{--}^{(1)}(t) = c_{++}^{(1)}(t) = 0$ because $\vec{S}_1\cdot\vec{S}_2$ connects only states of the same $m_{tot.}$ values.

Probability for $|+-\rangle$ is $|1+i\Delta t/\hbar|^2 = 1 + \Delta^2t^2/\hbar^2$, this does not quite agree with exact treatment because $c_{+-}^{(2)}$ interfering with $c_{+-}^{(0)}$ also gives Δ^2t^2 term. Probability for $|-+\rangle$ is $4\Delta^2t^2/\hbar^2$ which agrees with exact treatment up to $O(\frac{\Delta^2t^2}{\hbar^2})$. Note expansion of exact results from (1) gives

$$|\langle+-|\alpha;t\rangle|^2 \simeq 1 - \frac{1}{2}\frac{16\Delta^2t^2}{\hbar^2}, \quad |\langle-+|\alpha;t\rangle|^2 \simeq \tfrac{1}{2} - \tfrac{1}{2}(1 - \frac{16\Delta^2t^2}{2\hbar^2}). \qquad (2)$$

Hence validity of first order perturbation theory for $|+-\rangle$ is never satisfied, for $|-+\rangle$ validity is questionable when $t \gg \hbar/\Delta$ since lowest order expansion in (2) gives a poor approximation to the exact answer.

30. *This solution reprinted from the solutions manual for the revised edition.*

(a) From (5.5.17) for a two channel problem we have

$$i\hbar\begin{pmatrix}\dot{c}_1\\ \dot{c}_2\end{pmatrix} = \begin{pmatrix}0 & \gamma e^{i\omega t}e^{i\omega_{12}t}\\ \gamma e^{-i\omega t}e^{i\omega_{21}t} & 0\end{pmatrix}\begin{pmatrix}c_1\\ c_2\end{pmatrix} \qquad (1)$$

where $\omega_{21} = -\omega_{12} = (E_2-E_1)/\hbar$. Try for a solution of form

$$\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} e^{i(\omega-\omega_{21})t/2} a_1 \\ e^{-i(\omega-\omega_{21})t/2} a_2 \end{pmatrix} \tag{2}$$

into (1), we have upon simplification

$$\frac{i(\omega-\omega_{21})}{2} a_1 + \dot{a}_1 = \frac{\gamma}{i\hbar} a_2; \quad \frac{-i(\omega-\omega_{21})}{2} a_2 + \dot{a}_2 = \frac{\gamma}{i\hbar} a_1 \tag{3}$$

It is straightforward to see from (3) that

$$\ddot{a}_1 = -[\gamma^2/\hbar^2 + \frac{(\omega-\omega_{21})^2}{4}]a_1; \quad \ddot{a}_2 = -[\gamma^2/\hbar^2 + \frac{(\omega-\omega_{21})^2}{4}]a_2. \tag{4}$$

Hence for instance

$$a_2 \sim \{{\sin \atop \cos}\} [(\gamma^2/\hbar^2 + (\omega-\omega_{21})^2/4)^{\frac{1}{2}}t] . \tag{5}$$

Since $c_2(0) = 0$, we must have from (2)

$$c_2(t) \propto e^{-i(\omega-\omega_{21})t/2}\sin\{[\gamma^2/\hbar^2 + (\omega-\omega_{21})^2/4]^{\frac{1}{2}}t\} \tag{6}$$

Again from (1), we have since $c_1(0) = 1$, that $i\hbar\dot{c}_2|_{t=0} = \gamma$. Hence

$$c_2(t) = \frac{\gamma}{i\hbar[\gamma^2/\hbar^2+(\omega-\omega_{21})^2/4]^{\frac{1}{2}}} e^{-i(\omega-\omega_{21})t/2}\sin\{[\frac{\gamma^2}{\hbar^2} + \frac{(\omega-\omega_{21})^2}{4}]^{\frac{1}{2}}t\} \tag{7}$$

and

$$|c_2(t)|^2 = \frac{(\gamma^2/\hbar^2)}{(\gamma^2/\hbar^2)+(\omega-\omega_{21})^2/4}\sin^2\{[\gamma^2/\hbar^2+(\omega-\omega_{21})^2/4]^{\frac{1}{2}}t\}. \tag{8}$$

Now from (1), we have $i\hbar\dot{c}_2 = \gamma e^{-i\omega t}e^{i\omega_{21}t} c_1$, hence

$$c_1 = (i\hbar/\gamma)e^{i(\omega-\omega_{21})t/2} \dot{c}_2 \tag{9}$$

and using (7) it is easy to verify that

$$|c_1(t)|^2 = 1 - |c_2(t)|^2 \tag{10}$$

with $|c_2(t)|^2$ given by (8).

(b) Perturbation approach, let us use (5.6.17), than

$$c_2^{(1)}(t) = \left(\frac{-i}{\hbar}\right)\gamma\int_0^t e^{-i(\omega-\omega_{21})t'}dt' = \frac{(\gamma/\hbar)[e^{-i(\omega-\omega_{21})t} - 1]}{(\omega-\omega_{21})} \tag{11}$$

and

$$|c_2^{(1)}(t)|^2 = \frac{4(\gamma/\hbar)^2}{(\omega-\omega_{21})^2}\sin^2\left[\frac{(\omega-\omega_{21})t}{2}\right] . \tag{12}$$

Compare (12) with exact result (8), we see that γ^2 in denominator (as well as the γ^2 in the radical sign) is missing in the perturbation expression. However, as long as $|\omega-\omega_{21}| >> 2|\gamma|/\hbar$, the perturbation result is justifiable. When $\omega \cong \omega_{21}$, $|c_2^{(1)}|^2$ can exceed unity even with small γ. As for c_1, we have $c_1^{(1)} = 0$ (since $i\hbar\dot{c}_1^{(1)} = 0$), so $|c_1|^2 \cong |c_1^{(o)}|^2 = 1$.

31. *This solution reprinted from the solutions manual for the revised edition.*
If the perturbation potential V is constant in time, then the second term in Eq. (5.6.36) will be rapidly oscillating and gives no contribution to the transition probability.

However, if the perturbation is assumed to be slowly time-dependent, i.e. $V \to Ve^{\eta t}$, where η is small, the rapid oscillating term does give some non-vanishing contribution, which grows linearly in time: With $V \to Ve^{\eta t}$, (5.6.36) becomes

$$c_n^{(2)}(t) = \left(\frac{-i}{\hbar}\right)^2 \sum_m V_{nm}V_{mi} \int_{-\infty}^{t} dt' e^{i\omega_{nm}t'+\eta t'} \int_{-\infty}^{t'} dt'' e^{i\omega_{mi}t''+\eta t''}$$

$$= (i/\hbar)\sum_m \frac{V_{nm}V_{mi}}{E_m-E_i-i\eta\hbar}\int_{-\infty}^{t}dt' e^{i\omega_{ni}t'+2\eta t'} = \frac{e^{i\omega_{ni}t+2\eta t}}{E_n-E_i-2i\eta\hbar}\cdot\sum_m \frac{V_{nm}V_{mi}}{E_m-E_i-i\eta\hbar}$$

$$= \sum_m \frac{V_{nm}V_{mi}}{(E_m-E_i-i\eta\hbar)(E_n-E_i-2i\eta\hbar)} + \frac{e^{i\omega_{ni}t+2\eta t}-1}{E_n-E_i-2i\eta\hbar}\sum_m\frac{V_{nm}V_{mi}}{E_m-E_i-i\eta\hbar} . \tag{1}$$

When $\eta \to 0$, the first term above (in (1)) is exactly the first term in (5.6.36). On the other hand, the second term has a coefficient

$$\lim_{\omega_{ni}\to 0} \frac{e^{i\omega_{ni}t}-1}{E_n-E_i} \to (i/\hbar)t \tag{2}$$

which is linear in time when $\omega_{ni} \to 0$. That $|c_n^{(2)}(t)|^2$ has a quadratic dependence on time is not disturbing (c.f. (5.6.26) and subsequent discussion). Hence a non vanishing contribution to the transition probability from the second term in (5.6.36) is realizable since the total transition rate $\Gamma_{i\to n}(t)$ is defined to be

$$\Gamma_{i\to n}(t) = \frac{d}{dt}(\sum_{\alpha}|c_n^{(\alpha)}|^2). \tag{3}$$

32. *This solution reprinted from the solutions manual for the revised edition.*

Our Hamiltonian is

(a) $$H = H_o + V = A\vec{S}_1\cdot\vec{S}_2 + (eB/m_ec)(S_{1z} - S_{2z}). \tag{1}$$

The four unperturbed states of positronium are

$$\psi_1^{+1} = |++\rangle,\ \psi_1^o = \frac{1}{2^{1/2}}[|+-\rangle+|-+\rangle],\ \psi_1^{-1} = |--\rangle \quad \text{(triplet)}$$
$$\psi_o^o = \frac{1}{2^{1/2}}[|+-\rangle-|-+\rangle] \quad \text{(singlet)}. \tag{2}$$

The unperturbed energy levels must be determined, with $H_o = A\vec{S}_1\cdot\vec{S}_2 = \frac{A}{2}[(\vec{S}_1+\vec{S}_2)^2 -\vec{S}_1^2 - \vec{S}_2^2]$, hence $H_o\psi_1^{\pm1,0} = \frac{A\hbar^2}{2}[2-3/4-3/4]\psi_1^{\pm1,0} = \frac{A\hbar^2}{4}\psi_1^{\pm1,0}$, while $H_o\psi_o^o = (A\hbar^2/2)\times[0-3/4-3/4]\psi_o^o = \frac{-3A\hbar^2}{4}\psi_o^o$. So unperturbed energy levels are

$$E_1^{(o)} = A\hbar^2/4 \text{ (triplet state)},\ E_o^{(o)} = -3A\hbar^2/4 \text{ (singlet state)}. \tag{3}$$

Now it is evident from (2) that $(S_{1z} - S_{2z})\psi_1^{\pm1} = 0$ and $(S_{1z}-S_{2z})\psi^0_{0,1} = \hbar\psi^0_{1,0}$ therefore the first order energy level shifts are zero. Because the matrix elements of V between degenerate states are all vanishing, there is no problem abou using non-degenerate perturbation theory in this case. We next compute the firs order corrections to the unperturbed states. From $\langle\psi_1^{\pm1}|S_{1z}-S_{2z}|\psi^0_{0,1}\rangle$, we see that there is no mixing of the $S_z = \pm1$ states with the $S_z = 0$ states. Using (5.1.53a), the mixing between the two $S_z = 0$ states are given by

$$\delta\psi_o^o = \frac{\psi_1^o\langle\psi_1^o|V|\psi_o^o\rangle}{E_o^{(o)} - E_1^{(o)}},\ \delta\psi_1^o = \frac{\psi_o^o\langle\psi_o^o|V|\psi_1^o\rangle}{E_1^{(o)} - E_o^{(o)}} \tag{4}$$

where $\langle\psi_1^o|V|\psi_o^o\rangle = \frac{eB}{m_ec}\langle\psi_1^o|S_{1z} - S_{2z}|\psi_o^o\rangle = \frac{eB}{m_ec}\hbar$. Hence using (3), we have

$$\delta\psi_o^o = \psi_1^o \frac{eB\hbar}{m_e c}(-1/A\hbar^2) \text{ and } \delta\psi_1^o = \psi_o^o \frac{eB\hbar}{m_e c}(1/A\hbar^2). \quad (5)$$

Also from (5.1.53b), we have

$$\Delta E_o = (\frac{eB\hbar}{m_e c})^2(-1/A\hbar^2), \ \Delta E_1 = (\frac{eB\hbar}{m_e c})^2(1/A\hbar^2). \quad (6)$$

Therefore to second order in perturbation theory

$$E_1(m=\pm 1) = A\hbar^2/4, \ E_1(m=0) = \frac{A\hbar^2}{4}[1+4(eB/m_e cA\hbar)^2],$$
$$E_o = -\frac{A\hbar^2}{4}[3+4(eB/m_e cA\hbar)^2]. \quad (7)$$

Assuming the field B to be weak, the term $[1+4(eB/m_e cA\hbar)^2]^{\frac{1}{2}}$ may be approximated by $1+2(eB/m_e cA\hbar)^2$ in the exact expression for energy, than we see that exact expression for the m = 0 energy levels yields the second order results found above.

(b) We may write this new time dependent perturbation as

$$V'(t) = \frac{eB'e^{i\omega t}}{m_e c}(S_{1\hat{B}'} - S_{2\hat{B}'}) \quad (8)$$

where ω is the angular frequency of the energy difference. To determine which direction to orient $\hat{B}'$ the matrix elements of $(S_{1j} - S_{2j})$ with j=x,y,z between χ_1 and χ_o will be examined, where $\chi_1 = Z_1^{\frac{1}{2}}(\psi_1^o + a_1\psi_o^o)$ and $\chi_o = Z_o^{\frac{1}{2}}(\psi_o^o + a_o\psi_1^o)$ are the general forms of mixture between the two m=0 states. Let us use $S_x = \frac{\hbar}{2} \times [|+\rangle\langle -| + |-\rangle\langle +|]$ representation, than from (2)

$$S_{1x}\psi_o^o = \frac{\hbar}{2} \times \frac{1}{2^{\frac{1}{2}}}[|--\rangle - |++\rangle], \ S_{2x}\psi_o^o = \frac{\hbar}{2}\times\frac{1}{2^{\frac{1}{2}}}[|++\rangle - |--\rangle], \quad (9)$$

hence

$$(S_{1x} - S_{2x})\psi_o^o = \frac{\hbar}{2^{\frac{1}{2}}}(\psi_1^{-1} - \psi_1^{+1}), \quad (10)$$

also

$$S_{1x}\psi_1^o = \frac{\hbar}{2}\times\frac{1}{2^{\frac{1}{2}}}[|--\rangle + |++\rangle], \quad S_{2x}\psi_1^o = \frac{\hbar}{2}\times\frac{1}{2^{\frac{1}{2}}}[|++\rangle + |--\rangle] \quad (11)$$

and thus

$$(S_{1x} - S_{2x})\psi_1^o = 0. \quad (12)$$

From (10) and (12), we see that by orthonormality of ψ_ℓ^m states, $\langle\chi_o|(S_{1x}-S_{2x})\times$

$|\chi_1\rangle = 0$; similarly it can be shown that $\langle\chi_o|(S_{1y}-S_{2y})|\chi_1\rangle = 0$. However we have $(S_{1z}-S_{2z})\psi^o_{o,1} = \hbar\psi^o_{1,o}$, thence $(S_{1z}-S_{2z})|\chi_1\rangle = Z_1^{\frac{1}{2}}\hbar(\psi^o_o + a_1\psi^o_1)$ and $\langle\chi_o|(S_{1z}-S_{2z})\times|\chi_1\rangle = Z_1^{\frac{1}{2}}Z_o^{\frac{1}{2}}\hbar(1+a_o^*a_1)$. From orthogonality of χ_1 and χ_o we know that $a_o^* = -a_1$, hence in general $\langle\chi_o|(S_{1z}-S_{2z})|\chi_1\rangle$ does not vanish. Therefore the $\hat{B}'$ field shou be in the z-direction.

(c) For the singlet state to first order "$|\psi^o_o\rangle$" $= |\psi^o_o\rangle + \dfrac{\langle\psi^o_1|V|\psi^o_o\rangle}{E_o^{(o)}-E_1^{(o)}}|\psi^o_1\rangle$ $= |\psi^o_o\rangle + [eB\hbar/m_ec\ /\ -A\hbar^2]|\psi^o_1\rangle$, where we have used the fact that $(S_{1z}-S_{2z})|\psi^o_o\rangle = \hbar|\psi^o_1\rangle$ and hence $\langle\psi^o_1|V|\psi^o_o\rangle = eB\hbar/m_ec$. Analogously for the triplet state to first order we have "$|\psi^o_1\rangle$" $= |\psi^o_1\rangle + [eB\hbar/m_ec/A\hbar^2]|\psi^o_o\rangle$. The $|m=\pm1,\text{triplet}\rangle$ is an eigenstate of $H_o + V$ and so to first order in perturbation theory, the perturbed "$|m=0,\text{triplet}\rangle$" eigenvector has to be orthogonal to the $|m=\pm1,\text{triplet}\rangle$ ones, and hence cannot contain components in the $|m=\pm1,\text{triplet}\rangle$ directions.

33. *This solution reprinted from the solutions manual for the revised edition.*

We need to digress here on time independent degenerate perturbation theory. Let $(H_0+V)\psi_n = E_n\psi_n \equiv (E_n^{(o)} + E_n^{(1)} + E_n^{(2)} + \ldots\ldots)\psi_n$. Write $\psi_n = \psi_n^{(o)} + \psi_n^{(1)} + \psi_n^{(2)}\ldots$ We have 1. $H_o\psi_n^{(1)}+V\psi_n^{(o)} = E_n^{(o)}\psi_n^{(1)}+E_n^{(1)}\psi_n^{(o)}$; 2. $H_o\psi_n^{(2)}+V\psi_n^{(1)} = E_n^{(o)}\psi_n^{(2)}+E_n^{(1)}\psi_n^{(1)}+E_n^{(2)}\psi_n^{(o)}$; 3. $H_o\psi_n^{(3)}+V\psi_n^{(2)} = E_n^{(o)}\psi_n^{(3)}+E_n^{(1)}\psi_n^{(2)}+E_n^{(2)}\psi_n^{(1)}+ E_n^{(3)}\psi_n^{(o)}$ for the first three orders of perturbation. Let $\psi_n^{(o)}$, $\psi_{n'}^{(o)}\ldots$ be degenerate eigenfunctions of H_o with eigenvalue $E_n^{(o)}$. We saw earlier that we need to choose these to be eigenfunctions of V so these would not be mixed by the perturbation. We want to solve $\psi_n^{(1)}= \sum_k C_k\psi_k^{(o)}$. Take scalar product of first order relation 1. above with $\psi_n^{(o)}$ we have $E_n^{(1)} = V_{nn}$. Take scalar product of 1. with $\psi_p^{(o)}$ where $E_p^{(o)} \neq E_n^{(o)}$, we have $C_p = V_{pn}/[E_n^{(o)}-E_p^{(o)}]$ which is correct to 1st order. Note if we had taken scalar product with $\psi_{n'}^{(o)}$ in 1. we would have got $C_{n'}E_n^{(o)} = E_n^{(o)}C_{n'}$ because $V_{n'n}=0$ and hence $C_{n'}$ is <u>not</u> fixed by the 1st order equation. Multiply second order eqn. 2. by $\psi_{n'}^{(o)}$ we have $C_{n'}(V_{nn} - V_{n'n'}) = \sum_{k\neq n,n'\ldots} C_kV_{n'k}$. Although this is formally a 2nd equation, note however that if the perturbation <u>removes</u> the degeneracy, i.e. $V_{nn}\neq V_{n'n'}$ then $C_{n'} = \sum_{k\neq n,n'\ldots} C_kV_{n'k}/[V_{nn}-V_{n'n'}] = \sum_{k\neq n,n'\ldots}\frac{V_{kn}V_{n'k}}{(E_n^{(o)}-E_k^{(o)})(V_{nn}-V_{n'n'})}$ which is formally 1st order! The reason is that 1. above does <u>not</u> determine $C_{n'}$ hence we need to go to second order equation 2. and obtain the curious $V_{nn}-V_{n'n'}$ denominator for $C_{n'}$. But for our case, n & n' were nondegenerate only in 1st order, i.e. $V_{nn}-V_{n'n'}$ is 1st order so that we must need a 2nd order numerator (and hence use 2nd order eqn.) to get first order answer for the wavefunction/eigenvector. This was the reason why an arbitrarily <u>small</u> perturbation makes a big effect <u>unless</u> we work in the correct basis.

(a) Write $\vec{S}_1\cdot\vec{S}_2 = \frac{1}{2}[\vec{S}^2-\vec{S}_1^2-\vec{S}_2^2]$. Again eigenstates are triplets and singlets. Treat $V \equiv eBS_{1z}/m_ec \equiv \alpha S_{1z}$ as a perturbation and denote state as $|e,p\rangle$. Then $S_{1z}[|+-\rangle - |-+\rangle]/2^{\frac{1}{2}} = \frac{1}{2}\hbar[|+-\rangle + |-+\rangle]/2^{\frac{1}{2}}$ and thus $S_{1z}|\text{singlet}\rangle = \frac{1}{2}\hbar|\text{triplet}\rangle_{m=0}$ $S_{1z}|\text{triplet},m=0\rangle = \frac{1}{2}\hbar|\text{singlet}\rangle$, $S_{1z}|++\rangle = \frac{1}{2}\hbar|++\rangle$, $S_{1z}|--\rangle = -\frac{1}{2}\hbar|--\rangle$. The singlet state clearly obtains no shift to first order in perturbation theory. For the triplet state, we must again use degenerate perturbation theory and choose a bas where V is diagonal with respect to the triplets. Fortunately, our basis alread has this property. The $m = \pm 1$ components of the triplet suffer a first order energy shift $E^{(1)}_{m=\pm 1} = \langle\pm|V|\pm\rangle = \frac{1}{2}\hbar\alpha$.

Second order perturbation theory. For the singlet state, usual perturbation the formula is okay. We have

$$\Delta E^{(2)}_{\text{singlet}} = \sum_k V_{nk}V_{kn}/(E^{(o)}_n - E^{(o)}_k), \quad |n\rangle = \text{singlet}$$

This has only a non-vanishing matrix element with the $m = 0$ triplet, i.e. $|k\rangle$ m be triplet $m = 0$ state. We have

$$\langle m=0,\text{triplet}|V|\text{singlet}\rangle = \tfrac{1}{2}\hbar\alpha$$

and since $E^{(o)}_{\text{singlet}} = -3A\hbar^2/4$, $E^{(o)}_{\text{triplet}} = A\hbar^2/4$, we have

$$\Delta E^{(2)}_{\text{singlet}} = (\tfrac{1}{2}\hbar\alpha)^2/-A\hbar^2$$

Shift of triplet wavefunction in 1st order. The $|m=\pm,\text{triplet}\rangle$ states are eigen functions of V and so they don't mix. The $|m=0,\text{triplet}\rangle$ does mix with the sing and $C_{\text{singlet}} = \frac{1}{2}\hbar\alpha/[A\hbar^2/4 - (-3A\hbar^2/4)] = \frac{1}{2}\hbar\ /A\hbar^2$. There is no mixing with the other triplet components as can be seen unambiguously from our expression for C equation above. Writing "$|m=0,\text{triplet}\rangle$" for the first order shift, we have

$$"|m=0,\text{triplet}\rangle" = |m=0,\text{triplet}\rangle^{(o)} + \frac{\frac{1}{2}\hbar}{A\hbar^2}|\text{singlet}\rangle$$

Finally, multiplying Eq. 2. by $\psi^{(o)}_n$, we have

$$(\psi^{(o)}_n,\ V\psi^{(1)}_n) = E^{(2)}_n \quad \text{(where we have used } E^{(1)}_n = 0\text{)}$$

and thus $E^{(2)}_{\text{triplet}}(m=0) = C_{\text{singlet}}\langle m=0,\text{triplet}|V|\text{singlet}\rangle = \frac{1}{2}\hbar/A\hbar^2 \times \frac{1}{2}\hbar\alpha =$ $(\hbar\alpha)^2/4A\hbar^2$.

(b) The new time dependent perturbative term is for this problem

$$V'(t) = (eB'/m_e c)e^{i\omega t}\vec{S}_1\cdot\hat{B}'$$

Using the expressions for χ_1 and χ_o in terms of ψ^o_1 and ψ^o_o and Eq.(9) and Eq.(1 of solution to Problem 32(b), we find readily that $\langle\chi_1|S_{1x}|\chi_o\rangle = 0$, and simila

$\langle\chi_1|S_{1y}|\chi_0\rangle = 0$. However again like Problem 32(b) $\langle\chi_1|S_{1z}|\chi_0\rangle$ does not vanish. Therefore the $\hat{B}'$ field should again be in the z-direction to cause m=0 transitions.

(c) The first order eigenvector "$|m=0,\text{triplet}\rangle$" was already given in part (a), for "$|m=0,\text{singlet}\rangle$" $= |\psi_0^0\rangle + [\langle\psi_1^0|V|\psi_0^0\rangle]/(E_0^{(0)} - E_1^{(0)})]|\psi_1^0\rangle = |\psi_0^0\rangle + (\tfrac{1}{2}\hbar\alpha/-A\hbar^2)|\psi_1^0\rangle$.

Note for this problem n = triplet, m=0; n' = triplet, m=±1; k = singlet; hence $V_{n'k} = 0$ and thus $C_{n'} = 0$ and does not actually contribute here in first order.

34. *This solution reprinted from the solutions manual for the revised edition.*

From (5.7.1) the photon-electron interaction is written as

$$V = \frac{-e}{m_e c}\vec{A}\cdot\vec{p} \tag{1}$$

and for emission (see (5.7.6)), we have $\vec{A} = A_0\vec{\varepsilon}e^{-i\omega\hat{n}\cdot\vec{x}/c+i\omega t}$ where $\vec{\varepsilon}$ is the polarization. Matrix element

$$V_{ni} = \frac{-eA_0}{m_e c}\langle n|e^{-i\omega\hat{n}\cdot\vec{x}/c}\vec{\varepsilon}\cdot\vec{p}|i\rangle \tag{2}$$

and the transition rate from $i\to n$ is (c.f. analogous case for absorption in (5.7.8))

$$w_{i\to n} = (2\pi/\hbar)\frac{e^2}{m_e^2c^2}|A_0|^2|\langle n|e^{-i\omega\hat{n}\cdot\vec{x}/c}\,\vec{\varepsilon}\cdot\vec{p}|i\rangle|^2\delta(E_n-E_i+\hbar\omega). \tag{3}$$

In the dipole approximation $e^{-i(\omega/c)\hat{n}\cdot\vec{x}} \to 1$, so

$$w_{i\to n} = (2\pi/\hbar)\frac{e^2|A_0|^2}{m_e^2c^2}|\vec{\varepsilon}\cdot\langle n|\vec{p}|i\rangle|^2\delta(E_n-E_i+\hbar\omega). \tag{4}$$

From $[x,H_0] = i\hbar p/m_e$ we write as in (5.7.21) $\langle n|p_x|i\rangle = im_e\omega_{ni}\langle n|x|i\rangle$, etc. Henc

$$w_{i\to n} = (2\pi/\hbar)\frac{e^2|A_0|^2}{m_e^2c^2}m_e^2\omega_{ni}^2|\vec{\varepsilon}\cdot\langle n|\vec{x}|i\rangle|^2\delta(E_n-E_i+\hbar\omega). \tag{5}$$

Now $m_n - m_i = -1$, and remember that $\vec{x}$ is a spherical tensor of rank 1, we have from Sakurai (1967) [see equations (2.127) and (2.128)] that $\vec{d}_{ni} = \langle n|\vec{x}|i\rangle \sim \hat{x}-i$ and

$$|\vec{\varepsilon}\cdot\vec{d}_{ni}|^2 \sim |\varepsilon_x - i\varepsilon_y|^2 = \varepsilon_x^2 + \varepsilon_y^2\ . \tag{6}$$

Since $\hat{n}$ is perpendicular to $\vec{\varepsilon}$, if we have a rotating polarization vector $\vec{\varepsilon}$, its projection on to the x-y plane will be $(\varepsilon_x^2+\varepsilon_y^2)^{1/2} = |\vec{\varepsilon}|\cos\theta$ where θ is the angle between $\hat{n}$ and the z-axis. Therefore the angular distribution is proportional to $\cos^2\theta$.

Since the atom loses one unit of angular momentum in the z-direction, this must be carried off by the photon. Therefore if the photon is emitted in the positive z direction it must be right polarized (i.e. spin parallel to momentum) and if the photon is emitted in the negative z direction then its spin point in the +z direction and its polarization must be left-handed.

35. *This solution reprinted from the solutions manual for the revised edition.*
Since the electron's wave function does not change discontinuously, it remains in the ground state of ^{3}H for a short while, before it leaks into a definite eigenstate of ^{3}He. Thus all we need is the overlap of the initial wave function with the ground state of ^{3}He.

$$^3\text{H}: \; \psi_{o,o}(r) = \frac{1}{\pi^{1/2}}\frac{1}{a_o^{3/2}}e^{-r/a_o}\,, \quad a_o = \hbar/m_e c\alpha = 0.53\text{Å}$$

$$^3\text{He}: \; \psi_{o,o}(r) = \frac{1}{\pi^{1/2}}(2/a_o)^{3/2}e^{-2r/a_o}\,.$$

The probability amplitude $C_o = \int d^3x\psi_{oo}^{*^3\text{He}}\psi_{oo}^{^3\text{H}}$ (where $\int d\Omega = 4\pi$) $= \int_0^\infty drr^2 4(\frac{2}{a_o^2})^{3/2} \times e^{-3r/a_o} = 4(2/a_o^2)^{3/2}(a_o/3)^3\int_0^\infty dxx^2e^{-x} = 16\sqrt{2}/27 = 0.838$. Hence probability $= |C_o|^2 = 0.702$ (or a 70% chance).

36. This problem shows that Berry's Phase is a real number, and it is not hard, but the notation is a little tricky. Remember that a differential operator acts to the right, and that you can differentiate a ket (or a bra) with respect to the parameters on which it depends, and get a different ket (or bra). Being very explicit to make this clear, we have

$$0 = \boldsymbol{\nabla}_{\mathbf{R}}\left[1\right] = \boldsymbol{\nabla}_{\mathbf{R}}\left[\langle n;t|n;t\rangle\right] = \boldsymbol{\nabla}_{\mathbf{R}}\left[\left(\langle n;t|\right)\left(|n;t\rangle\right)\right] = \left(\boldsymbol{\nabla}_{\mathbf{R}}\langle n;t|\right)|n;t\rangle + \langle n;t|\left(\boldsymbol{\nabla}_{\mathbf{R}}|n;t\rangle\right)$$

However $\left(\boldsymbol{\nabla}_{\mathbf{R}}\langle n;t|\right)|n;t\rangle = \left(\langle n;t|\left[\boldsymbol{\nabla}_{\mathbf{R}}|n;t\rangle\right]\right)^*$. So $\langle n;t|\left[\boldsymbol{\nabla}_{\mathbf{R}}|n;t\rangle\right] = -\left(\langle n;t|\left[\boldsymbol{\nabla}_{\mathbf{R}}|n;t\rangle\right]\right)^*$, in which case it is a purely imaginary quantity. Therefor $\mathbf{A}_n(\mathbf{R})$ in (5.6.23) must be real.

37. The state vector is well known, in Problem 1.11 and (3.2.52). In spherical coordinates,

$$|n;t\rangle = \cos\left(\frac{\theta}{2}\right)|+\rangle + e^{i\phi}\sin\left(\frac{\theta}{2}\right)|-\rangle$$

To be sure, we want the state $|n;t\rangle$ which depends on the vector-of-parameters $\mathbf{R}(t)$ that is the magnetic field. However, the state does not depend on the magnitude of the field, only on the field's coordinates, in the usual three spatial coordinates. Consequently, a gradient with respect to B_θ or B_ϕ is the same as the usual three dimensional spatial gradient. Therefore

$$\begin{aligned}
\boldsymbol{\nabla}_{\mathbf{R}}|n;t\rangle &= \left[\hat{\boldsymbol{\theta}}\frac{\partial}{\partial\theta} + \hat{\boldsymbol{\phi}}\frac{1}{\sin\theta}\frac{\partial}{\partial\phi}\right]|n;t\rangle \\
&= -\frac{1}{2}\sin\left(\frac{\theta}{2}\right)\hat{\boldsymbol{\theta}}|+\rangle + e^{i\phi}\frac{1}{2}\cos\left(\frac{\theta}{2}\right)\hat{\boldsymbol{\theta}}|-\rangle + \frac{i}{\sin\theta}e^{i\phi}\sin\left(\frac{\theta}{2}\right)\hat{\boldsymbol{\phi}}|-\rangle \\
\langle n;t|\left[\boldsymbol{\nabla}_{\mathbf{R}}|n;t\rangle\right] &= \frac{i}{\sin\theta}\sin^2\left(\frac{\theta}{2}\right)\hat{\boldsymbol{\phi}} \\
\mathbf{A}_n(\mathbf{R}) &= i\langle n;t|\left[\boldsymbol{\nabla}_{\mathbf{R}}|n;t\rangle\right] = -\frac{1}{\sin\theta}\sin^2\left(\frac{\theta}{2}\right)\hat{\boldsymbol{\phi}} \equiv A_\phi(\theta)\hat{\boldsymbol{\phi}} \\
\boldsymbol{\nabla}_{\mathbf{R}}\times\mathbf{A}_n(\mathbf{R}) &= \hat{\mathbf{r}}\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta A_\phi) = -\hat{\mathbf{r}}\frac{1}{\sin\theta}2\frac{1}{2}\sin\left(\frac{\theta}{2}\right)\cos\left(\frac{\theta}{2}\right) = -\hat{\mathbf{r}}\frac{1}{2}
\end{aligned}$$

At this point, you will notice that we have essentially derived the first part of (5.6.42) for this particular state vector. The rest follows from the definition of the solid angle, but for the sake of completeness, we can carry it through. So

$$\gamma_n(C) = \int\left[\boldsymbol{\nabla}_{\mathbf{R}}\times\mathbf{A}_n(\mathbf{R})\right]\cdot d\mathbf{a} = -\frac{1}{2}\int\hat{\mathbf{r}}\cdot d\mathbf{a} = -\frac{1}{2}\Omega$$

If you really want, you could carry out the integral, at which point you will simply derive the expression for the solid angle subtended by a cone of half-angle θ, namely $2\pi(1-\cos\theta)$.

38. *This solution reprinted from the solutions manual for the revised edition.*

Write $V(\vec{x},t) = \frac{1}{2}[V_o\exp(i\omega z/c - i\omega t) + V_o\exp(-i\omega z/c + i\omega t)]$ where $V_o e^{i\omega z/c - i\omega t}$ is responsible for absorption of energy $\hbar\omega$ while $V_o e^{-i\omega z/c + i\omega t}$ is responsible for emission of energy $\hbar\omega$. Since $E_i < E_f$, only absorption part contributes and absorption rate is

$$|c^{(1)}|^2/t = (2\pi/\hbar)|V_o/2|^2|\langle\vec{k}_f|e^{i\omega z/c}|S\rangle|^2\delta(E_{\vec{k}_f} - E_S - \hbar\omega) \qquad (1)$$

where $\langle\vec{k}_f|$ is a plane wave bra state and $|S\rangle$ is atomic ket state.

The basic differences with photoelectric case are (i) $|V_o/2|^2$ in place of $e^2|A_o|^2/m_e^2c^2$ (c.f. (5.7.1) and (5.7.3)) and (ii) $|\langle\vec{k}_f|e^{i\omega z/c}|S\rangle|^2$ in place of $|\langle\vec{k}_f|\vec{\varepsilon}.\vec{p}e^{i\omega z/c}|S\rangle|^2$, where note $\vec{\varepsilon}.\vec{p}$ is absent in our case.

The integral to be evaluated is

$$\int \frac{e^{-i\vec{k}_f.\vec{x}}\, e^{i\omega z/c}}{L^{3/2}}\psi_S(\vec{x})d^3x \tag{2}$$

where $\psi_S(x)$ is the atomic wave function. Compare (2) with the space integral in the photoelectric case (see (5.7.33))

$$\int \frac{e^{-i\vec{k}_f.\vec{x}}}{L^{3/2}}\, \vec{\varepsilon}.(-i\hbar\vec{\nabla})e^{i\omega z/c}\psi_S(\vec{x})d^3x \tag{3}$$

where we let $-i\hbar\vec{\nabla}$ operate on plane wave (i.e. integrate by part, or use Hermiticity of $-i\hbar\vec{\nabla}$). This picks up $\vec{k}_f.\vec{\varepsilon}$ which can be taken outside integral in (3). Thus

$$|\langle\vec{k}_f|e^{i\omega z/c}|S\rangle|^2 = \frac{1}{\hbar^2(\vec{k}_f.\vec{\varepsilon})^2}|\langle\vec{k}_f|\vec{\varepsilon}.\vec{p}e^{i\omega z/c}|S\rangle|^2 \tag{4}$$

and in terms of angles shown in Fig.5.10, we have

$$(\vec{k}_f.\vec{\varepsilon})^2 = k_f^2\sin^2\theta\cos^2\phi. \tag{5}$$

So the angular distribution differs by the absence (our case) or presence (photo electric) of $\sin^2\theta\cos^2\phi$.

The energy dependence is such that energy conservation $E_{\vec{k}_f} - E_S = \hbar\omega$ must be be satisfied in both cases. This means that k_f is determined by $\hbar^2k_f^2/2m - E_S = \hbar\omega$. But suppose we now vary ω . Then the transition rate integrated with the density of final states (linear in k_f) goes as k_f (our case) but k_f^3 (in photoelectric case).

39. *This solution reprinted from the solutions manual for the revised edition.*

A particle of mass m, constrained by an infinite-wall potential within the interval $0 \leq x \leq L$, satisfies the boundary condition $\sin kL = \sin(n\pi)$, or

$$k = n\pi/L, \quad n=0, \pm 1, \pm 2, \ldots\ldots \tag{1}$$

The (one dimensional) energy is $E = \hbar^2 k^2/2m = \hbar^2 n^2 \pi^2/2mL^2$. What we need to calculate is ndn expressed as the density of states (i.e. number of states per unit energy interval) viz: $\rho(E)dE$. Here $dE = \hbar^2\pi^2 ndn/mL^2$, so $ndn = (mL^2/\hbar^2\pi^2)dE$. The assumption of high energy is needed in order to work in a n-continuum space rather than the discrete set given by (1). Dimension is consistent with that found in problem 36 above for two dimensions, namely dimensionless as required.

40. *This solution reprinted from the solutions manual for the revised edition.*

Use (5.7.32) and (5.7.33) where in (5.7.33) we replace the hydrogen atom wave function by ψ_S, the ground state wave function of a three-dimensional isotropic harmonic oscillator of angular frequency ω_o. The differential cross section reads therefore as

$$\frac{d\sigma}{d\Omega} = \frac{4\pi^2\alpha\hbar}{m^2\omega}\left|\int \frac{e^{-i\vec{k}_f\cdot\vec{x}}}{L^{3/2}} e^{i\omega\hat{n}\cdot\vec{x}/c}(-i\hbar\vec{\nabla})\cdot\hat{\epsilon}\psi_S d^3x\right|^2 \frac{mL^3 k_f}{\hbar^2(2\pi)^3} \tag{1}$$

where $(-i\hbar\vec{\nabla})$ operates on the final state wave function using Hermiticity. Equation (1) simplifies to

$$\frac{d\sigma}{d\Omega} = \frac{4\pi^2\alpha m}{m^2\omega}\frac{k_f\hbar^2}{\hbar(2\pi)^3}(\vec{k}_f\cdot\hat{\epsilon})^2\left|\int e^{-i\vec{q}\cdot\vec{x}}\psi_S d^3x\right|^2 \tag{2}$$

where $\vec{q} \equiv \vec{k}_f - \frac{\omega}{c}\hat{n}$, and $\psi_S = (m\omega_o/\pi\hbar)^{3/4}\, e^{-m\omega_o r^2/2\hbar}$ (note: $\omega \neq \omega_o$, the oscillator frequency). Energy conservation requires that

$$\hbar\omega + \frac{1}{2}\hbar\omega_o = \hbar^2 k_f^2/2m. \tag{3}$$

Let us evaluate the integral in Eq.(2), i.e.

$$I = \int e^{-i\vec{q}\cdot\vec{x}}\psi_S d^3x = \left(\frac{m\omega_o}{\pi\hbar}\right)^{3/4} I_x I_y I_z \qquad (4)$$

where $I_x \equiv \int_{-\infty}^{+\infty} e^{-iq_x x} e^{-m\omega_o x^2/2\hbar} dx$, I_y, I_z are similar expressions with $x \to y,z$ and $q_x \to q_y, q_z$. By method of quadrature, we have $I_x = \int_{-\infty}^{+\infty} e^{-m\omega_o(x+iq_x\hbar/m\omega_o)^2/2\hbar}$ $e^{-\frac{1}{2}(\hbar q_x^2/m\omega_o)} dx = \sqrt{2\pi\hbar/m\omega_o}\, e^{-\hbar q_x^2/2m\omega_o}$. So

$$I^2 = \left(\frac{m\omega_o}{\pi\hbar}\right)^{3/2}\left(\frac{2\pi\hbar}{m\omega_o}\right)^3 e^{-\hbar(q_x^2+q_y^2+q_z^2)/m\omega_o} \qquad (5)$$

and in terms of the angles (θ,ϕ) shown in Fig. 5.10, we have $(\vec{k}_f\cdot\hat{\varepsilon})^2 = k_f^2\sin^2\theta \times \cos^2\phi$, and

$$I^2 = \left(\frac{4\pi\hbar}{m\omega_o}\right)^{3/2} e^{-\frac{\hbar}{m\omega_o}[k_f^2-2k_f(\omega/c)\cos\theta+(\omega/c)^2]}. \qquad (6)$$

Thus from (2) we have putting everything together

$$\frac{d\sigma}{d\Omega} = \frac{4\alpha\hbar^2 k_f^3}{m^2\omega\omega_o}\left(\frac{\pi\hbar}{m\omega_o}\right)^{\frac{1}{2}} e^{\frac{-\hbar}{m\omega_o}[k_f^2+(\omega/c)^2]}\sin^2\theta\cos^2\phi\, e^{\left(\frac{2\hbar k_f}{m\omega_o}\frac{\omega}{c}\right)\cos\theta} \qquad (7)$$

Let's check the dimension of (7), α: dimensionless, $\hbar$: ML^2/T, k_f: $1/L$, ω,ω_o: $1/T$ hence dimension of (7) is

$$\frac{M^2L^4}{T^2}\ \frac{1}{L^3}\ \frac{T^2}{M^2}\ [ML^2/T/M/T]^{\frac{1}{2}} = L^2 \text{ (dimension of area)} \qquad (8)$$

41. *This solution reprinted from the solutions manual for the revised edition.*

Via Fourier transform, we know

$$\phi(\vec{p}) = \frac{1}{(2\pi\hbar)^{3/2}}\int d^3\vec{x}\, e^{-i\vec{p}\cdot\vec{x}/\hbar}\psi_{100}(\vec{r}). \qquad (1)$$

Now $d^3\vec{x} = r^2 dr\sin\theta d\theta d\phi$, and for ground-state of hydrogen atom we have $\psi_{100}(\vec{r}) = \psi_{100}(r) = (1/\pi a_o^3)^{\frac{1}{2}} e^{-r/a_o} = \gamma e^{-r/a_o}$. Then

$$\phi(\vec{p}) = \frac{-\gamma}{(2\pi\hbar)^{3/2}}\int_0^\infty\int_0^\pi\int_0^{2\pi} r^2 dr\, d(\cos\theta)\, d\phi\, e^{-(r/a_o+i\vec{p}\cdot\vec{r}/\hbar)}. \qquad (2)$$

We choose the z-axis in the direction of $\vec{p}$, therefore $\vec{p}\cdot\vec{r} = p_z z = pz = pr\cos\theta$, and we integrate out θ,ϕ variables to get

$$\phi(\vec{p}) = \frac{4\pi\gamma\hbar}{p(2\pi\hbar)^{3/2}} \int_0^\infty r dr e^{-r/a_o} \sin(pr/\hbar). \tag{3}$$

The r-integration is also straightforward, we have

$$\phi(\vec{p}) = \frac{4\pi\gamma\hbar}{p(2\pi\hbar)^{3/2}} \left\{ \frac{re^{-r/a_o}[-\frac{1}{a_o}\sin(pr/\hbar) - \frac{p}{\hbar}\cos(pr/\hbar)]}{(\frac{1}{a_o^2} + p^2/\hbar^2)} - \frac{e^{-r/a_o}\{[\frac{1}{a_o^2} - \frac{p^2}{\hbar^2}]\sin(pr/\hbar) + \frac{2p}{a_o\hbar}\cos(pr/\hbar)\}}{(\frac{1}{a_o^2} + p^2/\hbar^2)^2} \right\}\Bigg|_0^\infty , \tag{4}$$

at $r\to\infty$ contribution vanishes because of dominance of e^{-r/a_o}, the r=0 contribution gives

$$\phi(\vec{p}) = \frac{4\pi\gamma\hbar}{p(2\pi\hbar)^{3/2}} \frac{(2p/a_o\hbar)}{[\frac{1}{a_o^2} + p^2/\hbar^2]^2} . \tag{5}$$

Since $\gamma = (1/\pi a_o^3)^{1/2}$, $|\phi(\vec{p})|^2$ assumes form

$$|\phi(\vec{p})|^2 = \left(\frac{2^3}{\pi^2}\right) \frac{a_o^3\hbar^5}{[\hbar^2 + a_o^2 p^2]^4} . \tag{6}$$

42. *This solution reprinted from the solutions manual for the revised edition.*
This problem is spontaneous emission in the dipole approximation (E1) for a hydrogen atom (or a hydrogen-like atom with only one valence electron). The complete treatment leading to $\tau(2p\to 1s) = 1.6\times 10^{-9}$ sec. is well described on p.41 – 44 of J. J. Sakurai, Advanced Quantum Mechanics (1967).

Chapter Six

1. *This solution reprinted from the solutions manual for the revised edition.*

(a) From (7.1.6) and (7.1.7), the Lippmann-Schwinger equation reads (in one dimension) $|\psi^{(\pm)}\rangle = |\phi\rangle + \frac{1}{E - H_o \pm i\varepsilon} V|\psi^{(\pm)}\rangle$ or $\langle x|\psi^{(\pm)}\rangle = \langle x|\phi\rangle + \int dx'\langle x| \frac{1}{E - H_o \pm i\varepsilon}|x'\rangle\langle x'|V|\psi^{(\pm)}\rangle$ in position basis. The singular operator $\frac{1}{E-H_o}$ is handled by the $E \to E+i\varepsilon$ prescription if we are to have a transmitted wave for $x>a$; for reflected wave in $x<-a$ we need prescription $E \to E-i\varepsilon$. Hence for transmitted-reflected Green's function we have $G_\pm(x,x') = \frac{\hbar^2}{2m}\langle x|\frac{1}{E-H_o \pm i\varepsilon}|x'\rangle$. So

$$
\begin{aligned}
G_\pm(x,x') &= \left(\frac{\hbar^2}{2m}\right)\int\int_{-\infty}^{+\infty} dp''\langle x|p'\rangle\langle p'|\frac{1}{E-H_o \pm i\varepsilon}|p''\rangle\langle p''|x'\rangle dp' \\
&= \left(\frac{\hbar^2}{2m}\right)\int_{-\infty}^{+\infty} dp'\left(e^{ip'x/\hbar}/\sqrt{2\pi}\right)\frac{1}{E-p'^2/2m \pm i\varepsilon}\left(e^{-ip'x'/\hbar}/\sqrt{2\pi}\right) \qquad (1)\\
&= (1/2\pi)\int_{-\infty}^{+\infty} dq\left[e^{iq(x-x')}/(k^2-q^2 \pm i\varepsilon)\right]
\end{aligned}
$$

where we have used the one dimensional version of (7.1.14) and (7.1.15). The poles of (1) are at $q = \pm(k^2 \pm i\varepsilon)^{1/2} \cong \pm k \pm i\varepsilon'$. By straightforward method of residue contour integration in q-plane, we have

$$G_+(x,x') = -\frac{i}{2k}e^{ik|x-x'|}, G_-(x,x') = -\frac{i}{2k}e^{-ik|x-x'|}. \qquad (2)$$

Hence integral equation for $\langle x|\psi^{(+)}\rangle$ is

$$\langle x|\psi^{(+)}\rangle = \langle x|\phi\rangle - (i/2k)(2m/\hbar^2)\int_{-a}^{+a} dx' e^{ik|x-x'|}V(x')\langle x'|\psi^{(+)}\rangle \qquad (3)$$

For transmitted wave $x>a$ (hence $|x-x'| = x-x'$), we have

$$\langle x|\psi^{(+)}\rangle = e^{ikx}/\sqrt{2\pi} - \frac{im}{k\hbar^2}\int_{-a}^{+a} e^{ik(x-x')}V(x')\langle x'|\psi^{(+)}\rangle dx'. \qquad (3')$$

Similarly for a reflected wave $x<-a$, we have from (2)

$$\langle x|\psi^{(-)}\rangle = e^{ikx}/\sqrt{2\pi} - \frac{im}{k\hbar^2}\int_{-a}^{+a} e^{-ik(x-x')}V(x')\langle x'|\psi^{(-)}\rangle dx', \qquad (4)$$

where the first term on r.h.s. of (4) is really the original wave for $x<-a$.

(b) Take now $V = -(\gamma\hbar^2/2m)\delta(x)$ where $\gamma>0$, and substitute into (3) we have

$$\langle x|\psi^{(+)}\rangle = \langle x|\phi\rangle + \frac{i\gamma}{2k}e^{ik|x|}\langle 0|\psi^{(+)}\rangle. \qquad (5)$$

Set x=0 (center of range $-a<x<a$ where $V(x) \neq 0$), (5) becomes

$$<0|\psi^{(+)}> = \frac{1}{(2\pi)^{1/2}}\frac{1}{[1 - i\gamma/2k]} . \tag{6}$$

Substitute (6) into (5), we have

$$<x|\psi^{(+)}> = e^{ikx}/\sqrt{2\pi} + \frac{1}{(2\pi)^{1/2}} e^{ik|x|}[i\gamma/(2k-i\gamma)]. \tag{7}$$

Hence for $x>a$, transmission coefficient is $T = 1 + i\gamma/(2k-i\gamma)$. Similarly from (4) for $x<-a$, we have for reflection coefficient $R = i\gamma/(2k-i\gamma)$. This checks with Gottfried (1966), p.52.

(c) It is seen explicitly from our expressions for T and R, that they have poles at $k = i\gamma/2$, and $<x|\phi> \sim e^{-\gamma|x|/2}$. From problem 22 (Chapter 2), we see that the Schrödinger equation

$$-\frac{\hbar^2}{2m} d^2\psi/dx^2 - \frac{\gamma\hbar^2}{2m}\delta(x)\phi = -|E|\phi \tag{8}$$

has solutions of form $\phi = Ae^{-\kappa x}$ ($x>0$) and $\phi = Ae^{+\kappa x}$ ($x<0$) with $k = i\kappa = i(\frac{2m|E|}{\hbar^2})^{1/2}$ and satisfy

$$d\phi/dx|_{0^+} - d\phi/dx|_{0^-} = -\gamma\phi(0). \tag{9}$$

Eq.(9) implies that $\kappa = \gamma/2$ (or $k = i\gamma/2$) thus $\phi \propto e^{-\gamma|x|/2}$ in agreement with the discussion of T and R and bound state poles when k is treated as a complex variable.

2. *This solution reprinted from the solutions manual for the revised edition.*

(a) From (7.1.33) - (7.1.36), we have $<\vec{x}|\psi^{(+)}> = (1/2\pi)^{3/2}[e^{i\vec{k}.\vec{x}}+f(\vec{k}',\vec{k})e^{ikr}/r]$ and the differential cross-section $d\sigma/d\Omega = |f(\vec{k},\vec{k}')|^2$ where $f(\vec{k},\vec{k}')$ in the first Born approximation is given by (7.2.2)

$$f^{(1)}(\vec{k}',\vec{k}) = -\frac{1}{4\pi}(2m/\hbar^2)\int d^3x' e^{i(\vec{k}-\vec{k}').\vec{x}'}V(\vec{x}'). \tag{1}$$

Hence

$$d\sigma/d\Omega = (m^2/4\pi^2\hbar^4)\int d\vec{x}' d\vec{x}'' \, e^{i(\vec{k}-\vec{k}').\vec{x}'}V(\vec{x}')e^{-i(\vec{k}-\vec{k}').\vec{x}''}V(\vec{x}'') \tag{2}$$

and

$$\sigma = \int(m^2/4\pi^2\hbar^4)\int d\Omega_{k'} e^{-i\vec{k}'.(\vec{x}'-\vec{x}'')} d\vec{x}' d\vec{x}'' e^{i\vec{k}.(\vec{x}'-\vec{x}'')}V(\vec{x}')V(\vec{x}''). \tag{3}$$

Now $\int d\Omega_{k'} e^{-i\vec{k}'\cdot(\vec{x}'-\vec{x}'')} = 2\pi\int_{-1}^{+1} e^{ik|\vec{x}'-\vec{x}''|\cos\theta}d(\cos\theta) = 4\pi\sin k|\vec{x}'-\vec{x}''|/k|\vec{x}'-\vec{x}''|$,

where θ is angle between $\vec{k}'$ and $\vec{x}''-\vec{x}'$, and $|\vec{k}| = |\vec{k}'| = k$.

We now average over all incident beam direction $\vec{k}$ (assuming that V is spherically symmetric), than

$$\bar{\sigma} = \frac{m^2}{\pi\hbar^4}\int d\vec{x}' d\vec{x}'' \frac{\sin k|\vec{x}'-\vec{x}''|}{k|\vec{x}'-\vec{x}''|} V(|\vec{x}'|)V(|\vec{x}''|)\frac{1}{4\pi}\int d\Omega_k e^{i\vec{k}\cdot(\vec{x}'-\vec{x}'')}$$

$$= \frac{m^2}{\pi\hbar^4}\int d\vec{x}' d\vec{x}'' \, V(r')V(r'')\sin^2 k|\vec{x}'-\vec{x}''|/k^2|\vec{x}'-\vec{x}''|^2 \qquad (4)$$

(b) Let us now apply the optical theorem for $\vec{k} = \vec{k}'$ (forward scattering) given by (7.3.9) in the second-order Born approximation (7.2.23)

$$\sigma_{tot.} = \frac{4\pi}{k}\,\mathrm{Im}(f_B^{(2)}(\vec{k}=\vec{k}'))$$

$$= \mathrm{Im}\{\frac{4\pi}{k}(-\frac{1}{4\pi}\frac{2m}{\hbar^2})^2\int d^3x'\int d^3x'' e^{-i\vec{k}\cdot(\vec{x}'-\vec{x}'')}e^{ik|\vec{x}'-\vec{x}''|}V(\vec{x}')V(\vec{x}'')/|\vec{x}'-\vec{x}''|\}. \qquad (5)$$

Since (5) is dependent on the angle between $\vec{k}$ and $(\vec{x}'-\vec{x}'')$, we need to take an average over all $\hat{k}$ direction, i.e. $\frac{1}{4\pi}\int d\Omega_k e^{-i\vec{k}\cdot(\vec{x}'-\vec{x}'')}$ must be computed. Hence

$$\bar{\sigma}_{tot.} = \mathrm{Im}\,\{\frac{m^2}{\pi k\hbar^4}\int d^3x'\int d^3x'' \frac{\sin k|\vec{x}'-\vec{x}''|}{k|\vec{x}'-\vec{x}''|}\frac{e^{ik|\vec{x}'-\vec{x}''|}}{|\vec{x}'-\vec{x}''|}V(r')V(r'')\}$$

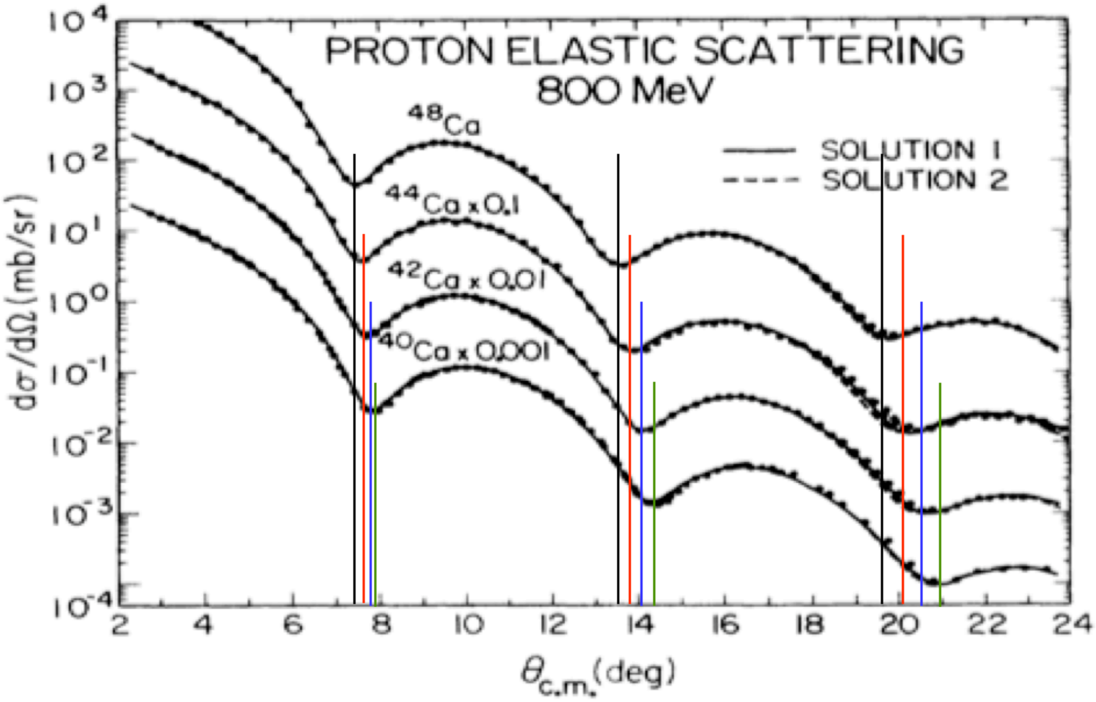
$$= \frac{m^2}{\pi\hbar^4}\int d^3x'\int d^3x'' \, V(r')V(r'')\sin^2 k|\vec{x}'-\vec{x}''|/k^2|\vec{x}'-\vec{x}''|^2. \qquad (6)$$

This is the same as (4) and again V is assumed to be spherically symmetric.

3. The figure is reprinted here from the original paper, along with vertical lines to help read off the positions of the minima. We determine momentum transfer from $q = 2k\sin(\theta/2)$ where $\hbar^2k^2/2m = (\hbar c)^2k^2/mc^2 = 800$ MeV, so $k = \sqrt{800\cdot 934}/200 = 4.32$/fm. For a square well of radius a, the first three minima are at $qa = 4.49$, 7.73, and 10.9. (See page 401.) So,

		Minimum #1			Minimum #2			Minimum #3		
Isotope	$1.4A^{1/3}$	θ	q	a	θ	q	a	θ	q	a
^{40}Ca	4.79	7.95°	1.20	3.76	14.2°	2.12	3.65	20.9°	3.08	3.54
^{42}Ca	4.87	7.85°	1.18	3.80	14.0°	2.09	3.70	20.5°	3.03	3.60
^{44}Ca	4.94	7.6°	1.14	3.93	13.8°	2.06	3.75	20.1°	2.97	3.67
^{48}Ca	5.09	7.3°	1.10	4.09	13.4°	2.00	3.86	19.5°	2.88	3.78

Several remarks are in order. Firstly, as mentioned in the text, the minimum shifts to lower



angles as the number of neutrons is increased. In other words, the radius increases with neutron number, just as expected. The quantitative agreement with the liquid drop formula $a = 1.4A^{1/3}$ is marginal, but you can only expect so much when comparing one crude approximation (liquid drop) with another (square well). The position of the minima, though, are reasonably consistent with each other, each leading to a radius that is within $\sim 5\%$ of the others for a given isotope.

4. *This solution reprinted from the solutions manual for the revised edition.*

From (7.6.35) and (7.6.34), and the method of partial waves (7.6.50), we have $\beta_\ell = \left(\frac{r\,dA_\ell}{A_\ell\,dr}\right)_{r=R}$, $\tan\delta_\ell = (kRj'_\ell(kR) - \beta_\ell j_\ell(kR))/(kRn'_\ell(kR) - \beta_\ell n_\ell(kR))$ and $\sigma_{tot.} = \frac{4\pi}{k^2}\sum_{\ell=0}^{\ell\simeq kR}(2\ell+1)\sin^2\delta_\ell$.

Again from (7.6.36) – (7.6.38), we have for the radial wave function $A_\ell = u_\ell/r$, $u''_\ell + [k'^2 - \ell(\ell+1)/r^2]u_\ell = 0$ where $k'^2 = 2m(E-V_0)/\hbar^2$, and $u_\ell = 0$ at $r=0$. So our solution is $u_\ell(k'r) = rj_\ell(k'r)$ or $A_\ell(k'r) = j_\ell(k'r)$, hence $\beta_\ell = k'Rj'_\ell(k'R)/j_\ell(k'R)$. Since $kR \ll 1$, and $|V_0| \ll E = \hbar^2k^2/2m$, therefore $k'R \ll 1$. But from the general recursion for $j_\ell(kR)$, we have $j'_\ell(k'R) = \ell j_\ell(k'R)/k'R -$

$j_{\ell+1}(k'R)$. Hence

$$\beta_\ell = \ell - (k'R)j_{\ell+1}(k'R)/j_\ell(k'R) = \ell - (k'R)^2/(2\ell+3). \tag{1}$$

Our expression for $\tan\delta_\ell$ becomes

$$\tan\delta_\ell = \frac{(k'R)^2 j_\ell(kR) - (2\ell+3)(kR)j_{\ell+1}(kR)}{(k'R)^2 n_\ell(kR) - (2\ell+3)(kR)n_{\ell+1}(kR)}$$

$$\simeq \frac{(k'^2-k^2)R^2 2^\ell \ell!(kR)^\ell 2^{\ell+1}(\ell+1)!}{(2\ell+1)!(2\ell+3)!\ (kR)^{-(\ell+1)}}$$

$$\simeq \frac{-2mV_oR^2}{\hbar^2(2\ell+3)}[2^\ell\ell!/(2\ell+1)!]^2(kR)^{2\ell+1} \simeq \sin\delta_\ell \simeq \delta_\ell. \tag{2}$$

Clearly only S-wave ($\ell=0$) will have significant contribution to total scattering, hence

$$\sigma_{tot.} \simeq \frac{4\pi}{k^2}\sin^2\delta_o \simeq \frac{16\pi}{9\hbar^4}m^2V_o^2R^6. \tag{3}$$

If the energy is raised slightly, we must take into account δ_1 contribution. From (2) above we have $\delta_1 \simeq -2mV_oR^2(kR)^3/45\hbar^2$. Now from (7.6.17) we have

$$f(\theta) = \frac{1}{k}(e^{i\delta_o}\sin\delta_o + 3e^{i\delta_1}\sin\delta_1 P_1(\cos\theta) + \ldots\ldots), \tag{4}$$

hence $|f(\theta)|^2 = (1/k^2)\sin^2\delta_o + \frac{3}{k^2}(e^{i(\delta_1-\delta_o)} + e^{+i(\delta_o-\delta_1)})\sin\delta_1\sin\delta_o\cos\theta + \ldots\ldots$ or

$$\frac{d\sigma}{d\Omega} = |f(\theta)|^2 = A+B\cos\theta = \sin^2\delta_o/k^2 + 6\cos(\delta_1-\delta_o)\sin\delta_1\sin\delta_o\cos\theta/k^2.$$

We see therefore that $B/A = 6\sin\delta_1\sin\delta_o\cos(\delta_1-\delta_o)/\sin^2\delta_o \simeq 6\sin\delta_1/\sin\delta_o$ and using (2) $B/A \simeq \frac{2}{5}(kR)^2$ where we have set $\cos(\delta_1-\delta_o) \simeq 1$.

5. *This solution reprinted from the solutions manual for the revised edition.*

(a) Let's take

$$f_k^{(1)}(\theta) = \sum_{\ell=o}^{\infty}\frac{(2\ell+1)}{k}e^{i\delta_\ell}\sin\delta_\ell P_\ell(\cos\theta) = -\frac{2mV_o}{\hbar^2\mu}\frac{1}{2k^2[(1+\mu^2/2k^2)-\cos\theta]}$$

and denote $\xi = \cos\theta$, $\zeta = 1+\mu^2/2k^2 > 1$ (for $\mu>0$). Then we know how to expand $1/(\zeta-\xi) = \sum_{\ell=o}^{\infty}a_\ell P_\ell(\xi)$ in the domain $-1\leq\xi\leq+1$ where $a_\ell = \frac{(2\ell+1)}{2}\int_{-1}^{+1}\frac{P_\ell(\xi)d\xi}{\zeta-\xi} =$

$(2\ell+1)Q_\ell(\zeta)$ (and we have in mind the orthogonality of the Legendre Polynomials $\int_{-1}^{+1} d\xi P_\ell(\xi)P_{\ell'}(\xi) = \frac{2}{(2\ell+1)}\delta_{\ell\ell'}$).

Again let us rewrite $f_k^{(1)}(\theta)$ as

$$f_k^{(1)}(\theta) = \sum_{\ell=0}^{\infty} \frac{(2\ell+1)}{2ik}(e^{2i\delta_\ell}-1)P_\ell(\xi) \equiv \sum_{\ell=0}^{\infty} \frac{(2\ell+1)}{k}\left[\frac{-mV_o Q_\ell(\zeta)}{\hbar^2\mu k}\right]P_\ell(\xi).$$

Hence compare coefficient of $P_\ell(\xi)$ on both sides, we have

$$(e^{2i\delta_\ell}-1)/2i = \frac{-mV_o}{\hbar^2\mu k} Q_\ell(\zeta).$$

Assume $|\delta_\ell| << 1$ than $e^{2i\delta_\ell}-1 \cong 2i\delta_\ell$, and we find

$$\delta_\ell \cong -\frac{mV_o}{\hbar^2\mu k} Q_\ell(\zeta) = -\frac{mV_o}{\hbar^2\mu^2}\sqrt{2(\zeta-1)}\, Q_\ell(\zeta).$$

(b) (i) Obviously from above we can write $\delta_\ell \cong -V_o K_\ell(\zeta)$ with

$$K_\ell(\zeta) = \frac{m}{\hbar^2\mu^2}\sqrt{2(\zeta-1)}\, Q_\ell(\zeta) > 0, \quad (\zeta>1).$$

This is evident from the explicit expansion form for $Q_\ell(\zeta)$ (or at least $K_\ell(\zeta) \to 0$ for $k \to 0$). Hence we see for repulsive potential $V_o>0$: $\delta_\ell = -V_o K_\ell(\zeta)<0$, and for attractive potential $V_o<0$: $\delta_\ell = -V_o K_\ell(\zeta)>0$.

(ii) Now the deBroglie wave length $\lambda = h/p = 2\pi\hbar/p = 2\pi/k$ (or $\lambdabar = 1/k$), while the range of the potential $R = 1/\mu$. Hence $\lambdabar/R >> 1$ implies $\eta = \mu/k >> 1$. Hence $1/\zeta = 1/[1+\frac{1}{2}(\mu/k)^2] \cong 2(k/\mu)^2 = 2\eta^{-2}$ and therefore the polynomial $K_\ell(\zeta)$ will be reduced to $K_\ell(\eta)$ as follows

$$K_\ell(\zeta) \cong \frac{m}{\hbar^2\mu^2}\eta \frac{\ell!}{(2\ell+1)!!}\left[\ (2\eta^{-2})^{\ell+1} + \ldots\ldots\right].$$

This gives the approximate form for δ_ℓ in terms of η as follows:

$$\delta_\ell \cong -\frac{mV_o}{\hbar^2\mu^2}\cdot\frac{2^{\ell+1}\ell!}{(2\ell+1)!!}\eta^{-2\ell-1} = -\frac{2mV_o}{\hbar^2\mu^{2\ell+3}}\cdot\frac{(2\ell)!!}{(2\ell+1)!!}k^{2\ell+1}. \qquad (1)$$

According to Gottfried (1966) [p.124, Eq. (17)] for small δ_ℓ

$$\delta_\ell(k)_{\ell>>\ell_{max}} \cong \tan\delta_\ell = -\frac{k^{2\ell+1}}{[(2\ell+1)!!]^2}\int_0^\infty r^{2\ell+2}U(r)dr \qquad (2)$$

where $U(r) = (2m/\hbar^2)V(r) = (2mV_o/\hbar^2)e^{-\mu r}/\mu r$. Since $\int_0^\infty \rho^{2\ell+1}e^{-\rho}d\rho = (2\ell+1)!$ (remember $\int_0^\infty x^n e^{-x}dx = n!$), we have

$$\delta_\ell \cong -\frac{k^{2\ell+1}.2^{2\ell}.(\ell!)^2}{[(2\ell+1)!]^2}\left(\frac{2mV_o}{\hbar^2\mu^{2\ell+3}}\int_0^\infty e^{-\rho}\rho^{2\ell+1}d\rho\right)_{\rho=\mu r} .$$

We notice that $1/(2\ell+1)!! = 2^\ell \ell!/(2\ell+1)!$, hence

$$\delta_\ell \cong -\frac{2mV_o}{\hbar^2\mu^{2\ell+3}}\cdot\frac{(2^\ell\ell!)^2}{(2\ell+1)!}k^{2\ell+1} = -\frac{2mV_o}{\hbar^2\mu^{2\ell+3}}\cdot\frac{(2\ell)!!}{(2\ell+1)!!}k^{2\ell+1}, \qquad (3)$$

and (3) is the same as (1) above obtained using the $Q_\ell(\zeta)$ expansion formula.

6. *This solution reprinted from the solutions manual for the revised edition.*

The ground state wave function for the hard sphere can be written as $\psi(r,\theta,\phi) = Y_{oo}(\theta,\phi)R(r) \equiv (1/4\pi)^{\frac{1}{2}}\chi(r)/r$, where $\chi(r)$ obeys the equation $(-\hbar^2/2m)d^2\chi/dr^2 = E_o\chi$ $(r<a)$ and $\chi(r) = 0$ for $r>a$. Thus $\chi(r) = A\sin\alpha r + B\cos\alpha r$ for $r<a$, with $\alpha = [2mE_o/\hbar^2]^{\frac{1}{2}}$. The requirement that $R(r)$ be finite at $r=0$ demands that $B = 0$. At the boundary $r=a$, $\chi(a)=0$. Thus we impose $\alpha a = \pi$ or $E_o = \frac{\hbar^2}{2m}(\pi/a)^2$. The normalization constant A is fixed by $\int\psi^*\psi r^2 d\phi d\cos\theta dr = \frac{1}{4\pi}\int_0^\infty 4\pi\chi^2(r)dr = 1$, or $A^2\int_0^a \sin^2\alpha r dr = 1$. This in turn implies $A = \sqrt{2\alpha/\pi} = [8mE_o/\pi^2\hbar^2]^{\frac{1}{2}} = \sqrt{2/a}$.

Now we check explicitly the uncertainty relation in $x - p_x$, $(\Delta x)^2 = \langle x^2\rangle - \langle x\rangle^2$ but $\langle x^2\rangle = \int r^2\sin^2\theta\cos^2\phi|\psi|^2 r^2 d\phi d\cos\theta dr = \frac{1}{2a\pi}\cdot\frac{4\pi}{3}\int_0^a r^2\sin^2\alpha r dr = \frac{1}{3\alpha^2}[\pi^2/3 - \frac{1}{2}]$ and $\langle x\rangle = 0$. On the other hand $\langle p_x^2\rangle = -\hbar^2\int\psi^*\frac{d^2}{dx^2}\psi d^3x$, now

$$\frac{d^2\psi}{dx^2} = \frac{1}{(4\pi)^{\frac{1}{2}}}\sqrt{2/a}\,\frac{d^2}{dx^2}(\sin\alpha r/r) = \sqrt{1/2a\pi}\left\{\frac{d}{dx}\left[-\frac{\sin\alpha r}{r^2} + \frac{\alpha\cos\alpha r}{r}\right]\frac{\partial r}{\partial x}\right\}$$

$$= \left(\frac{1}{2a\pi}\right)^{\frac{1}{2}}\left[\frac{2\sin\alpha r}{r^3} - \frac{2\alpha\cos\alpha r}{r^2} - \frac{\alpha^2\sin\alpha r}{r}\right]\left(\frac{\partial r}{\partial x}\right)^2 + \left(\frac{1}{2a\pi}\right)^{\frac{1}{2}}\left[-\frac{\sin\alpha r}{r^2} + \frac{\alpha\cos\alpha r}{r}\right]\frac{\partial^2 r}{\partial x^2}$$

fore

$$\langle p_x^2\rangle = -\frac{\hbar^2}{2a\pi}\int\left[\frac{\sin^2\alpha r}{r^4}(2-\alpha^2r^2) - \frac{2\alpha\sin\alpha r\cos\alpha r}{r^3}\right](\sin^2\theta\cos^2\phi)r^2 d\phi d\cos\theta dr$$

$$-\frac{\hbar^2}{2a\pi}\int\left[-\frac{\sin^2\alpha r}{r^3} + \frac{\alpha\sin\alpha r\cos\alpha r}{r^2}\right]\frac{1}{r}(1-\sin^2\theta\cos^2\phi)r^2 d\phi d\cos\theta dr$$

$$= -\frac{\hbar^2}{2a\pi}\cdot\frac{4\pi}{3}\int_0^a\left[\frac{\sin^2\alpha r}{r^2}(2-\alpha^2r^2) - \frac{2\alpha\sin\alpha r\cos\alpha r}{r}\right]dr$$

$$-\frac{\hbar^2}{2a\pi}\cdot\frac{8\pi}{3}\int_0^a\left[-\frac{\sin^2\alpha r}{r^2} + \frac{\alpha\sin\alpha r\cos\alpha r}{r}\right]dr = +\frac{2\hbar^2\alpha^2}{3a}\int_0^a \sin^2\alpha r dr = \frac{\alpha^2}{3}\hbar^2.$$

It can be readily seen that $\langle p_x\rangle = 0$, thus we have

$$(\Delta x)^2(\Delta p_x)^2 = \frac{1}{9}[\pi^2/3 - \tfrac{1}{2}]\hbar^2 \geq \hbar^2/4,$$

which is consistent with the Heisenberg uncertainty relation.

7. *This solution reprinted from the solutions manual for the revised edition.*

(a) The full wave function for r>a can be written in partial wave analysis as

$$\langle\vec{x}|\psi^{(+)}\rangle = \frac{1}{(2\pi)^{3/2}}\sum_\ell i^\ell(2\ell+1)A_\ell(r)P_\ell(\cos\theta)$$

with $A_\ell = c_\ell^{(1)}h_\ell^{(1)}(kr) + c_\ell^{(2)}h_\ell^{(2)}(kr)$ where $h_\ell^{(1)}$ and $h_\ell^{(2)}$ are the Hankel functions of the first and second kind, respectively. When we consider large r behavior, we have (c.f. (7.6.33)):

$$A_\ell(r) = e^{i\delta_\ell}[j_\ell(kr)\cos\delta_\ell - n_\ell(kr)\sin\delta_\ell].$$

Asymptotically $h_\ell^{(1)} \to e^{i(kr-\ell\pi/2)}/ikr$, $h_\ell^{(2)} \to e^{-i(kr-\ell\pi/2)}/ikr$, while the large r behavior of $\langle\vec{x}|\psi^{(+)}\rangle$ is (from (7.6.8) and (7.6.16))

$$\langle\vec{x}|\psi^{(+)}\rangle \to \frac{1}{(2\pi)^{3/2}}\sum_\ell(2\ell+1)[e^{2i\delta_\ell}e^{ikr}/2ikr - e^{-i(kr-\ell\pi)}/2ikr]P_\ell(\cos\theta).$$

So clearly $c_\ell^{(1)} = \tfrac{1}{2}e^{2i\delta_\ell}$ and $c^{(2)} = \tfrac{1}{2}$. Thus

$$A_\ell(r) = e^{i\delta_\ell}[\cos\delta_\ell j_\ell(kr) - \sin\delta_\ell n_\ell(kr)].$$

For hard sphere, the boundary condition at r=a is $A_\ell(r)|_{r=a} = 0$ because the sphere is impenetrable. This means $j_\ell(ka)\cos\delta_\ell - n_\ell(ka)\sin\delta_\ell = 0$ or $\tan\delta_\ell = j_\ell(ka)/n_\ell(ka)$. For $\ell=0$ $\tan\delta_0 = \frac{\sin(ka)/ka}{-\cos(ka)/ka} = -\tan(ka)$ or $\delta_0 = -ka$.

(b) We have $f(\theta) = \frac{1}{k}\sum_{\ell=0}^{\infty}(2\ell+1)e^{i\delta_\ell}\sin\delta_\ell\, P_\ell(\cos\theta)$ and in the limit when $k\to 0$, the $\ell=0$ partial wave dominates the scattering. Thus $f(\theta) \approx \frac{1}{k}e^{-ika}\sin(ka)$, and knowing that $d\sigma/d\Omega = |f(\theta)|^2$ we have for the total cross section $\sigma = \int(\frac{d\sigma}{d\Omega})\cdot d\Omega = \int|f(\theta)|^2 d\Omega = \int\frac{1}{k^2}\sin^2(ka)d\Omega = 4\pi\sin^2(ka)/k^2 \approx 4\pi a^2$.

Classically, the "geometric cross section" is πa^2. By "geometric cross section" we mean the area of the disc of radius a that blocks the propagation of the plane wave (and has the same cross section area as that of a hard sphere). Low energy scattering of course means a very large wave length scattering and we do not necessarily expect a classically reasonable result.

8. *This solution reprinted from the solutions manual for the revised edition.*

(a) For the Gaussian potential (c.f. (7.4.14)), we have $\Delta_G(b) \equiv \frac{-m}{2k\hbar^2}\int_{-\infty}^{\infty} V(\sqrt{b^2+z^2})dz$ where $V(r) = V_o e^{-r^2/a^2}$. This implies that

$$\Delta_G(b) = \frac{-mV_o}{2k\hbar^2}\int_{-\infty}^{\infty} e^{-(b^2+z^2)/a^2}dz$$

$$= \frac{-mV_o}{2k\hbar^2}\, e^{-(b/a)^2}\int_{-\infty}^{\infty} e^{-(z/a)^2}dz = \frac{-\sqrt{\pi}}{2}\,\frac{mV_o a}{k\hbar^2}\, e^{-(b/a)^2}.$$

Since we are given that $\delta_\ell^G = \Delta(b)\big|_{b=\ell/k}$, hence

$$\delta_\ell^G = -\frac{\sqrt{\pi}}{2}\,\frac{mV_o a}{k\hbar^2}\, e^{-(\ell/ka)^2}$$

(b) For the Yukawa potential $V(r) = V_o e^{-\mu r}/\mu r$, we have

$$\Delta_Y(b) = -\frac{mV_o}{2k\hbar^2}\int_{-\infty}^{\infty}\frac{1}{\mu r}\, e^{-\mu r}\Big|_{r=\sqrt{b^2+z^2}}dz = -\frac{mV_o}{2k\hbar^2}\int_{-\infty}^{\infty}\frac{e^{-\mu\sqrt{b^2+z^2}}}{\mu(b^2+z^2)^{1/2}}dz.$$

The integral (remembering $r^2 = b^2+z^2$)

$$I = \int_{-\infty}^{\infty}\frac{e^{-\mu(b^2+z^2)^{1/2}}}{(b^2+z^2)^{1/2}}dz = 2\int_0^{\infty}\frac{e^{-\mu(b^2+z^2)^{1/2}}}{(b^2+z^2)^{1/2}}dz = 2\int_0^{\infty}\frac{e^{-\mu r}}{r}\frac{rdr}{(r^2-b^2)^{1/2}}$$

$$= 2\int_0^{\infty}\frac{e^{-\mu r}dr}{(r^2-b^2)^{1/2}} = 2K_o(\mu b)$$

where K_o is the modified Bessel function. Thus

$$\Delta_Y(b) = -\frac{mV_o}{2k\hbar^2}\,\frac{2K_o(\mu b)}{\mu} = -\frac{mV_o}{\mu k\hbar^2}\,K_o(\mu b)$$

hence $\delta_\ell^Y = \Delta_Y(b)\big|_{b=\ell/k}$ assumes value

$$\delta_\ell^Y = -\frac{mV_o}{\mu k\hbar^2}\,K_o(\mu\ell/k).$$

In case of Gaussian potential $\delta_\ell^G \propto e^{-(\ell/ka)^2}$, and as ℓ increases $\delta_\ell^G \to 0$ very rapidly as $e^{-\ell^2/k^2a^2}$. In the case of the Yukawa potential $\delta_\ell^Y \propto K_o(\mu\ell/k)$, for $\ell >> k/\mu$ ($R\sim 1/\mu$) we have $K_o(\mu\ell/k) \sim \sqrt{\pi/2}(k/\mu\ell)^{\frac{1}{2}}e^{-\mu\ell/k}$ thus δ_ℓ^Y also goes to zero very rapidly as ℓ increases.

9. *This solution reprinted from the solutions manual for the revised edition.*

(a) From (7.1.11) and (7.1.12), we have

$$\frac{\hbar^2}{2m}\langle\vec{x}|\frac{1}{E-H_o+i\varepsilon}|\vec{x}'\rangle = G_+(\vec{x},\vec{x}') = -\frac{1}{4\pi}\frac{e^{+ik|\vec{x}-\vec{x}'|}}{|\vec{x}-\vec{x}'|}. \quad (1)$$

This Green's function turns out to be the out-going wave solution to the Helmholtz equation:

$$(\vec{\nabla}^2 + k^2)G_+(\vec{x},\vec{x}') = \delta^{(3)}(\vec{x}-\vec{x}'). \quad (2)$$

To solve this equation, first notice that the δ-function in spherical coordinates can be represented as

$$\delta(\vec{x}-\vec{x}') = \frac{1}{r^2}\delta(r-r')\sum_{\ell=o}^{\infty}\sum_{m=-\ell}^{\ell}Y_{\ell m}^*(\theta',\phi')Y_{\ell m}(\theta,\phi). \quad (3)$$

Expanding the Green's function in spherical harmonics, we have

$$G_+(\vec{x},\vec{x}') = -\sum_{\ell,m} g_\ell(r,r')Y_{\ell m}^*(\theta',\phi')Y_{\ell m}(\theta,\phi). \quad (4)$$

Substitute (3) and (4) into (2), we are led to an equation for $g_\ell(r,r')$

$$\left[\frac{d^2}{dr^2} + \frac{2}{r}\frac{d}{dr} + k^2 - \frac{\ell(\ell+1)}{r^2}\right]g_\ell = -\frac{1}{r^2}\delta(r-r'). \quad (5)$$

The boundary conditions are that g_ℓ be finite at the origin and infinity. This in turn requires that

$$g_\ell(r,r') = Aj_\ell(kr_<)h_\ell^{(1)}(kr_>). \quad (6)$$

When we match the discontinuity in slope (at $r=r'$), we find

$$A = +ik. \quad (7)$$

Thus the expansion of the Green's function is

$$\frac{\hbar^2}{2m}\langle\vec{x}|\frac{1}{E-H_o+i\varepsilon}|\vec{x}'\rangle = -ik\sum_{\ell=o}^{\infty}j_\ell(kr_<)h_\ell^{(1)}(kr_>)\sum_{m=-\ell}^{\ell}Y_{\ell m}^*(\theta',\phi')Y_{\ell m}(\theta,\phi) \quad (8)$$

(b) In $\vec{x}$ - representation

$$\langle\vec{x}|E\ell m(+)\rangle = \langle\vec{x}|E\ell m\rangle + \int d^3x'd^3x''\langle\vec{x}|\frac{1}{E-H_o+i\epsilon}|\vec{x}'\rangle\langle\vec{x}'|V|\vec{x}''\rangle\langle\vec{x}''|E\ell m(+)\rangle \quad (9)$$

where $\langle\vec{x}|\frac{1}{E-H_o+i\epsilon}|\vec{x}'\rangle$ can be evaluated from (8), and (c.f. (7.5.21b)) $\langle\vec{x}|E\ell m\rangle =$ $\frac{i^\ell}{\hbar}\sqrt{2mk/\pi}\; j_\ell(kr)Y_\ell^m(\hat{r})$ while we write $\langle\vec{x}|E\ell m(+)\rangle \equiv \frac{i^\ell}{\hbar}\sqrt{2mk/\pi}A_\ell(k;r)Y_\ell^m(\hat{r})$. Assume that the potential is local, i.e. $\langle\vec{x}'|V|\vec{x}''\rangle = V(\vec{x}')\delta^{(3)}(\vec{x}'-\vec{x}'')$, then (9) can be rewritten as

$$A_\ell(k;r)Y_\ell^m(\hat{r}) = j_\ell(kr)Y_\ell^m(\hat{r}) - \frac{2mik}{\hbar^2}\int d^3x'd^3x''\sum_{\ell'}\sum_{m'}Y_{\ell'}^{m'}(\hat{r})Y_{\ell'}^{m'*}(\hat{r}')Y_\ell^m(\hat{r}'') \tag{10}$$
$$\times j_{\ell'}(kr_<)h_{\ell'}^{(1)}(kr_>)V(\vec{x}')\delta^{(3)}(\vec{x}'-\vec{x}'')A_\ell(k;r'').$$

The second term on r.h.s. of (10) becomes

$$\frac{2mik}{\hbar^2}\int d^3x'\sum_{\ell'}\sum_{m'}Y_{\ell'}^{m'}(\hat{r})Y_{\ell'}^{m'*}(\hat{r}')Y_\ell^m(\hat{r}')j_{\ell'}(kr_<)h_{\ell'}^{(1)}(kr_>)V(\vec{x}')A_\ell(k;r')$$
$$= \frac{2mik}{\hbar^2}\int_0^\infty r'^2dr'\sum_{\ell'}\sum_{m'}Y_{\ell'}^{m'}(\hat{r})\delta_{\ell\ell'}\delta_{mm'}j_{\ell'}(kr_<)h_{\ell'}^{(1)}(kr_>)V(\vec{x}')A_\ell(k;r')$$
$$= \frac{2mik}{\hbar^2}\,Y_\ell^m(\hat{r})\int_0^\infty r'^2dr'j_\ell(kr_<)h_\ell^{(1)}(kr_>)V(r')A_\ell(k;r'). \tag{11}$$

Thus

$$A_\ell(k;r) = j_\ell(kr) - \frac{2mik}{\hbar^2}\int_0^\infty j_\ell(kr_<)h_\ell^{(1)}(kr_>)V(r')A_\ell(k;r')r'^2dr'. \tag{12}$$

As $r \to \infty$, it is clear that we should identify $r_>$ as r, and $r_< = r'$. But from (7.6.7) and (A.5.19), we have

$$j_\ell(kr)\xrightarrow[r\to\infty]{}\frac{e^{i(kr-\ell\pi/2)}-e^{-i(kr-\ell\pi/2)}}{2ikr},\quad h_\ell^{(1)}(kr)\xrightarrow[r\to\infty]{}\frac{e^{i(kr-(\ell+1)\pi/2)}}{kr}. \tag{13}$$

So from (12)

$$A_\ell(k;r)\xrightarrow[r\to\infty]{}\frac{i^{-\ell}}{2ik}\{[1-\frac{4mik}{\hbar^2}\int_0^\infty j_\ell(kr')A_\ell(k;r')V(r')r'^2dr']\frac{e^{ikr}}{r} - \frac{e^{-i(kr-\ell\pi)}}{r}\}. \tag{14}$$

On the other hand, for sufficiently large r, there are only the plane incoming wave and the spherical outgoing wave, with scattering amplitude $f(\theta) = \sum_{\ell=0}^{\infty}(2\ell+1)f_\ell(k)P_\ell(\cos\theta)$. The ℓ^{th} partial wave $f_\ell(k)$ contributes to $A_\ell(k;r)$

[c.f. (7.6.8)] as

$$A_\ell(k;r) \xrightarrow[r\to\infty]{} \frac{i^{-\ell}}{2ik}\{[1+2ikf_\ell(k)]\frac{e^{ikr}}{r} - \frac{e^{-i(kr-\ell\pi)}}{r}\}. \tag{15}$$

Comparing (14) and (15) and noting (c.f. (7.6.14) and (7.6.15)) that $S_\ell \equiv e^{2i\delta_\ell} = 1 + 2ikf_\ell(k)$, we have

$$f_\ell(k) = \frac{e^{i\delta_\ell}\sin\delta_\ell}{k} = -(2m/\hbar^2)\int_0^\infty j_\ell(kr)A_\ell(k;r)V(r)r^2dr. \tag{16}$$

10. *This solution reprinted from the solutions manual for the revised edition.*

(a) From (7.6.29), the scattering wave is $\langle\vec{x}|\psi^{(+)}\rangle = \frac{1}{(2\pi)^{3/2}}\sum_\ell i^\ell(2\ell+1)A_\ell(r) \times P_\ell(\cos\theta)$ and $A_\ell(r)$ satisfies (c.f. (7.6.36)) $u_\ell'' + (k^2 - \frac{2mV(r)}{\hbar^2} - \frac{\ell(\ell+1)}{r^2})u_\ell(r) = 0$ with $u_\ell(r) = rA_\ell$. For S-wave and $(2m/\hbar^2)V(r) = \gamma\delta(r-R)$, we consider $\ell=0$ only. Hence $u_0'' + (k^2-\gamma\delta(r-R))u_0 = 0$ and for $r<R$ the solution can be written as $u_0(r) = B_0 r\sin kr/kr$ while for $r>R$ (using (7.6.33) and (7.6.45)) we have $u_0(r) = re^{i\delta_0}\sin(kr+\delta_0)/kr$. These two solutions must match at $r=R$, i.e. $u_0|_{r=R+} = u_0|_{r=R-} = u_0(R)$ while $u_0'|_{r=R+} - u_0'|_{r=R-} = \gamma u_0(R)$. Therefore

$$\begin{aligned} \frac{Re^{i\delta_0}\sin(kR+\delta_0)}{kR} &= \frac{B_0 R\sin(kR)}{kR} \\ e^{i\delta_0}\cos(kR+\delta_0) - B_0\cos(kR) &= \frac{\gamma B_0}{k}\sin(kR). \end{aligned} \tag{1}$$

Solving (1) for $\tan\delta_0$, we have

$$\tan\delta_0 = \frac{(-\gamma/k)\sin^2(kR)}{1 + (\gamma/k)\sin(kR)\cos(kR)}. \tag{2}$$

(b) Assume $\gamma \gg 1/R,\ k$, from (1) we obtain

$$\tan(kR+\delta_0) = \frac{\sin(kR)}{\cos(kR)+(\gamma/k)\sin(kR)} \approx \frac{\tan(kR)}{(\gamma/k)\tan(kR)} = \frac{k}{\gamma} \ll 1, \tag{3}$$

thus $-kR\approx\delta_0$, and this resembles the hard sphere scattering (7.6.44). Again from (2) above we have

$$\cot\delta_0 = 0 \text{ when } 1 + (\gamma/k)\sin(kR)\cos(kR) = 0 \tag{4}$$

i.e. $\sin(2kR) = -2k/\gamma \cong 0$. Ostensibly we have solutions $(kR)_r \cong n\pi$, $(n+\frac{1}{2})\pi$, but $(n+\frac{1}{2})\pi$ is eliminated since $\cot\delta_o$ then goes through zero from below (negative side). So we write $k_rR \cong n\pi$ (where $d\cot\delta_o/dk < 0$ as k increases) and $k_rR = n\pi - \varepsilon$, $\varepsilon \ll 1$. Hence $\sin(2k_rR) = -\sin(2\varepsilon) = -2k/\gamma$, and $\varepsilon \cong k/\gamma$ to first order, an $k_rR = n\pi - k/\gamma$ as the resonance condition. The resonance energy is

$$E_r = \hbar^2k_r^2/2m \cong \frac{\hbar^2}{2m}\frac{n^2\pi^2}{R^2}(1 - 2/R\gamma). \tag{5}$$

For a particle confined inside potential $V = 0$, $r<R$, and $V = \infty$ for $r>R$ and in S-wave, we have $u'' + k^2u = 0$ where $u(0) = 0$ and $u(R) = 0$. Solution is $u(r) = A\sin(kr)$ $(0\leq r\leq R)$ and from boundary condition $kR = n\pi$, bound state energies are $E_b = \hbar^2k^2/2m = \hbar^2n^2\pi^2/2mR^2$. Hence from (5), we have

$$E_r \cong E_b(1-2/R\gamma). \tag{6}$$

Finally from (2), we have

$$\begin{aligned} d(\cot\delta_o)/dE &= (d(\cot\delta_o)/dk)(dk/dE) \\ &= \frac{1}{\gamma\sin^4(kR)}[R\cos(kR)(k+\gamma\sin(2kR)/2)\sin(kR) - (1+\gamma R\cos(kR))\sin^2(kR)]\frac{m}{\hbar^2k} \end{aligned} \tag{7}$$

At $E=E_r$, since $k_rR \cong n\pi(1-\frac{1}{\gamma R})$, $\sin^2(k_rR) \cong (n\pi/\gamma R)^2$ and $\cos(2k_rR) \cong 1$, we have from (7)

$$\Gamma = -2/[d(\cot\delta_o)/dE]\big|_{E=E_r} \cong \frac{2\hbar^2}{m}\frac{(n\pi)^3}{\gamma^2R^4}. \tag{8}$$

Notice that because of the $1/\gamma^2$ dependence in (8), $\Gamma\to 0$ as γ becomes large, thus the resonances become extremely sharp.

11. *This solution reprinted from the solutions manual for the revised edition.*
Assume that initially (t=0) the particle is in an eigenstate $|i\rangle$. The potential $V(\vec{r},t) = V(\vec{r})\cos\omega t$ is turned on at t=0. Take the perturbation expansion of the state amplitudes $c_n(t)$ up to first order $c_n(t) = c_n^{(o)}(t) + c_n^{(1)}(t) + \ldots\ldots$ Then obviously $c_n^{(o)}(t) = \delta_{ni}$. Let the final state be $|f\rangle$, then

$$c_f^{(1)}(t) = (-i/\hbar)\int_o^t V_{fi}(\vec{r})\cos\omega t' e^{i\omega_{fi}t'}dt', \tag{1}$$

where $V_{fi}(\vec{r}) \equiv \langle f|V(\vec{r})|i\rangle$ and $\omega_{fi} = (E_f - E_i)/\hbar$. Integrate (1) gives

$$c_f^{(1)}(t) = \frac{V_{fi}}{2\hbar}\left[\frac{1-e^{i(\omega+\omega_{fi})t}}{(\omega+\omega_{fi})} + \frac{1-e^{i(\omega_{fi}-\omega)t}}{(-\omega+\omega_{fi})}\right]. \tag{2}$$

Obviously, as $t\to\infty$, $|c_f^{(1)}|^2$ is appreciable only if

$$\begin{aligned} &(i)\ \omega_{fi} + \omega \cong 0 \quad \text{or } E_f \cong E_i - \hbar\omega \\ &(ii)\ \omega_{fi} - \omega \cong 0 \text{ or } E_f \cong E_i + \hbar\omega\ . \end{aligned} \tag{3}$$

The transition rate is then

$$w_{i\to f} = \frac{2\pi}{\hbar}|V_{fi}|^2\{\rho(E_f)|_{E_f=E_i-\hbar\omega} + \rho(E_f)|_{E_f=E_i+\hbar\omega}\}d\Omega. \tag{4}$$

Using box normalization (c.f. (7.11.23)), we have

$$\rho(E_f) = n^2 dn/dE_f = (L/2\pi)^3 k_f m/\hbar^2, \tag{5}$$

where k_f is the momentum of the final state. On the other hand, the incident flux $\vec{j}$ (c.f. (7.11.26)) is $|\vec{j}| = \hbar k_i/mL^3$. From (7.11.25) we know that the transition rate $w_{i\to f}$ = (incident flux)$\times(d\sigma/d\Omega)d\Omega$, we obtain

$$\begin{aligned} \frac{d\sigma}{d\Omega} &= \left(\frac{mL^3}{\hbar k_i}\right)\left(\frac{2\pi}{\hbar}\right)\left(\frac{L}{2\pi}\right)^3 \frac{m}{\hbar^2}|\langle f|V(\vec{r})|i\rangle|^2\{k_f|_{E_f=E_i-\hbar\omega} + k_f|_{E_f=E_i+\hbar\omega}\} \\ &= \frac{m^2}{4\pi^2\hbar^4}\left|\int d^3\vec{r}\,V(\vec{r})e^{i(\vec{k}_f-\vec{k}_i)\cdot\vec{r}}\right|^2 \times \frac{1}{k_i}\{(k_i^2-2m\omega/\hbar)^{\frac{1}{2}} + (k_i^2+2m\omega/\hbar)^{\frac{1}{2}}\} \end{aligned} \tag{6}$$

where initial and final states are assumed to be plane waves and momentum of the two final states are related by $E_f = \hbar^2 k_f^2/2m = E_i \pm \hbar\omega = \hbar^2 k_i^2/2m \pm \hbar\omega$.

Since

$$c_f^{(2)} = (-i/\hbar)^2 \sum_m \int_0^t dt' e^{i\omega_{fm}t'} V_{fm}(t') \int_0^{t'} dt'' e^{i\omega_{mi}t''} V_{mi}(t''), \tag{7}$$

similar to (2) we have

$$\begin{aligned} c_f^{(2)}(t) &= \frac{i}{2}\left(\frac{-i}{\hbar}\right)^2 \sum_m \int_0^t dt' e^{i\omega_{fm}t'} V_{fm}\cos\omega t' \cdot V_{mi}\left[\frac{1-e^{i(\omega+\omega_{mi})t'}}{\omega+\omega_{mi}} + \frac{1-e^{i(\omega_{mi}-\omega)t'}}{-\omega+\omega_{mi}}\right] \\ &= \left(\frac{i}{2}\right)^2\left(\frac{-i}{\hbar}\right)^2 \sum_m V_{fm}V_{mi}\left\{\left(\frac{1}{\omega+\omega_{mi}} + \frac{1}{-\omega+\omega_{mi}}\right)\cdot\left[\frac{1-e^{i(\omega+\omega_{fm})t}}{\omega+\omega_{fm}}\right.\right. \end{aligned}$$

$$+ \frac{1-e^{i(\omega_{fm}-\omega)t}}{-\omega+\omega_{fm}}] + \frac{(-1)}{\omega+\omega_{mi}} \cdot [\frac{1-e^{i(2\omega+\omega_{mi}+\omega_{fm})t}}{2\omega+\omega_{mi}+\omega_{fm}} + \frac{1-e^{i(\omega_{mi}+\omega_{fm})t}}{\omega_{mi}+\omega_{fm}}]$$

$$+ \frac{-1}{-\omega+\omega_{mi}} \cdot [\frac{1-e^{i(\omega_{fm}+\omega_{mi}+\omega)t}}{\omega+\omega_{fm}+\omega_{mi}} + \frac{1-e^{i(\omega_{fm}+\omega_{mi}-\omega)t}}{-\omega+\omega_{fm}+\omega_{mi}}]\} \, . \tag{8}$$

Looking at the square brackets of (8) we see that these terms contribute only i the denominators are close to zero, which means

$$\omega_f \cong \omega_m \pm \omega, \; \omega_f \cong \omega_i - 2\omega, \; \omega_f \cong \omega_i, \; \omega_f \cong \omega_i \pm \omega \, . \tag{9}$$

The $\omega_f \cong \omega_i \pm \omega$ last condition of (9) is the same as in the first order transitic (3) whereas the other three conditions are new. In particular, there can be a "second harmonic" generation where $\omega_i - \omega_f \cong 2\omega$.

Furthermore, the first condition ($\omega_f \cong \omega_m \pm \omega$) of (9) implies that the interme diate states that are $\pm\omega\hbar$ away from the final state $|f>$ will contribute most among all intermediate states.

These observations can be generalized to even higher order perturbations. example, in 3^{rd} order perturbation we expect to see a "third harmonic" transit with $\omega_f \cong \omega_i \pm 3\omega$.

12. *This solution reprinted from the solutions manual for the revised edition.*

The potential for the elastic scattering of a fast electron by the ground stat of the hydrogen atom is

$$V = -e^2/r + e^2/|\vec{x} - \vec{x}'| \; , \; r = |\vec{x}| . \tag{1}$$

So the matrix element for elastic scattering is

$$<\vec{k}'0|V|\vec{k}0> = \frac{1}{L^3}\int d^3x e^{i\vec{q}\cdot\vec{x}} <0|\frac{-e^2}{r} + \frac{e^2}{|\vec{x}-\vec{x}'|}|0>$$

$$= \frac{1}{L^3}\int d^3x e^{i\vec{q}\cdot\vec{x}} \int d^3x' \psi_o^*(\vec{x}')[\frac{-e^2}{r} + \frac{e^2}{|\vec{x}-\vec{x}'|}]\psi_o(\vec{x}') \tag{2}$$

where $\vec{q} = \vec{k}' - \vec{k}$.

As explained in section 7.12, the first term in V does not contribute to th $\int d^3x'$ integration, and from Eq. (7.12.10)

$$\int d^3\vec{x}\; e^{i\vec{q}\cdot\vec{x}}/r = 4\pi/q^2. \tag{3}$$

Furthermore, after shifting the coordinate variable $\vec{x} \to \vec{x} + \vec{x}'$ we have

$$\int d^3\vec{x}\; e^{i\vec{q}\cdot\vec{x}}/|\vec{x}-\vec{x}'| = \frac{4\pi}{q^2}\, e^{i\vec{q}\cdot\vec{x}'} \tag{4}$$

so

$$\langle\vec{k}'0|V|\vec{k}0\rangle = \frac{4\pi e^2}{q^2}[-\langle 0|0\rangle + \langle 0|e^{i\vec{q}\cdot\vec{x}}|0\rangle]L^{-3}. \tag{5}$$

Notice that $\langle 0|0\rangle = 1$, and $\langle\vec{x}|0\rangle = (1/a_o)^{3/2}2e^{-r/a_o}\, Y_{oo} = \frac{2}{(4\pi)^{1/2}}(\frac{1}{a_o})^{3/2}e^{-r/a_o}$, so

$$\langle 0|e^{i\vec{q}\cdot\vec{x}'}|0\rangle = (\frac{1}{a_o})^3 \int_{-1}^{+1}\int_0^\infty 2r'^2 d\cos\theta\, dr' e^{-2r'/a_o}\, e^{iqr'\cos\theta}$$

$$= (\frac{1}{a_o})^3\, \frac{4}{q} \int_0^\infty r' e^{-2r'/a_o} \sin qr' dr' ,$$

therefore

$$\langle 0|e^{i\vec{q}\cdot\vec{x}}|0\rangle = (\frac{1}{a_o})^3\, \frac{4}{q}\cdot \frac{4q/a_o}{(4/a_o^2+q^2)^2} = \frac{16}{[4+(qa_o)^2]}2 \, . \tag{6}$$

Thus

$$\langle\vec{k}'0|V|\vec{k}0\rangle = -\frac{4\pi e^2}{q^2}\{1 - \frac{16}{[4 + (qa_o)^2]^2}\}L^{-3} \, . \tag{7}$$

For the differential cross section (c.f. (7.12.6)) with $k' = k$

$$\frac{d\sigma}{d\Omega} = L^6|\frac{1}{4\pi}\frac{2m}{\hbar^2}\langle\vec{k}'0|V|\vec{k}0\rangle|^2 = \frac{4m^2e^4}{\hbar^4q^4}\{1 - \frac{16}{[4+(qa_o)^2]^2}2\}^2 \tag{8}$$

13. *This solution reprinted from the solutions manual for the revised edition.*
(See Finkelstein: Non Relativistic Mechanics (1973), p.292 for background material). Energy E is

$$E = E(J_1, J_2, J_3) \tag{1}$$

In the case of a central potential, it turns out that

$$E = E(J_r, J_\phi + J_\theta), \tag{2}$$

where

$$J_\phi = \oint p_\phi d\phi = \oint \frac{\partial W}{\partial\phi}\, d\phi = 2\pi\alpha_\phi$$
$$J_\theta = \oint p_\theta d\theta = \oint \frac{\partial W}{\partial\theta}\, d\theta = \oint [\alpha_\theta^2 - \alpha_\theta^2 /\sin^2\theta]^{1/2} d\theta = 2\pi(\alpha_\theta - \alpha_\phi)$$

$$J_r = \oint p_r dr = \oint \frac{\partial W}{\partial r} dr = \oint [2\mu\alpha_1 - 2\mu V + \alpha_\theta^2/r^2]^{\frac{1}{2}} dr \tag{3}$$

and the function W and constants α_ϕ, α_θ, and α_1 are defined by the Hamilton-Jacobi equation:

$$H(\partial W/\partial q_i,\ q_i) = \alpha_1 = E. \tag{4}$$

Equation (2) arises because (3) gives

$$J_r = \oint [2\mu E - 2\mu V(r) - (J_\theta + J_\phi)^2/4\pi^2 r^2]^{\frac{1}{2}} dr. \tag{5}$$

When $V(r)$ is the Coulomb potential, $V(r) = -e^2/r$, we have

$$J_r = \oint [2\mu E + 2\mu e^2/r - (J_\theta + J_\phi)^2/4\pi^2 r^2]^{\frac{1}{2}} dr$$

and with some algebra, this integration gives (c.f. Goldstein, Classical Mechanics (1980), p.475)

$$E = -\frac{2\pi^2 \mu e^4}{(J_r + J_\theta + J_\phi)^2} = E(J_r + J_\theta + J_\phi). \tag{6}$$

Compare (2) and (3), we see that for a central potential in general, J_ϕ and J_θ always appear in the combination $(J_\theta + J_\phi)$, hence there is at least 'singly' degeneracy. On the otherhand, in the Coulomb case J_r, J_ϕ, J_θ appear always in combination $(J_r + J_\theta + J_\phi)$, hence there is at least a double degeneracy.

In the case of Coulomb potential, there is, in addition to the angular momentum $\vec{L}$, yet another invariance of the action $\vec{A}$:

$$\vec{A} = \vec{L} \times \vec{p} + e^2 \mu \hat{r} \tag{7}$$

which determines the direction of the major axis and the eccentricity of the conic.

If one writes for the general central potential

$$V(r) = -e^2/r + \phi(r) \tag{8}$$

then

$$d\vec{A}/dt = \left(\frac{-d\phi}{r\,dr}\right)(\vec{L} \times \vec{r}). \tag{9}$$

Therefore, $\vec{A}$ precesses in general according to this equation. Consequently, the general case of motion in a central potential may be pictured in terms of a precessing conic, that also has a changing eccentricity. In terms of action and angle variables, this means that the Coulombian motion is distinguished by a single period whereas the motion of a central field problem is generally characterized by two periods which are not commensurable.

The explicit expression of $\vec{A}^2$ from (7) (for our classical system) is

$$\vec{A}^2 = \ell^2\vec{p}^2 - \frac{2e^2\mu\ell^2}{r} + \mu^2e^4 = 2\mu\ell^2(\vec{p}^2/2\mu - \frac{e^2}{r}) + \mu^2e^4. \tag{10}$$

Other than the last term μ^2e^4 in $\vec{A}^2$ of (10), the first two terms are proportional to the kinetic and potential energies of the Coulomb problem. It is thus clear that the Hamiltonian $H = \vec{p}^2/2\mu + V(r) + F(\vec{A}^2)$ is a polynomial in $(\frac{\vec{p}^2}{2\mu} - \frac{e^2}{r})$ plus the extra term $\phi(r)$ in (8). It follows that all the algebra of the Poisson brackets remain the same (as that without $F(\vec{A}^2)$), and the previous statements are still valid.

To describe quantum systems, we modify the Poisson brackets into commutators:

$$\begin{aligned}
[L_i,L_j] &= i\hbar\varepsilon_{ijk}\, L_k \\
[L_i,M_j] &= i\hbar\varepsilon_{ijk}\, M_k \\
[M_i,M_j] &= i\hbar\varepsilon_{ijk}\, L_k
\end{aligned} \tag{11}$$

where $M_i \equiv \sqrt{1/2\mu H}\, A_i$ and $L_i \equiv \frac{1}{2}\varepsilon_{ijk}(L_jp_k+p_kL_j)$ such that L_i is Hermitian. It follows that (for Hamiltonian H)

$$\vec{A}^2 = (\mu e^2)^2+(2\mu H)(\vec{L}^2+\hbar^2),\ \vec{M}^2 = \frac{\mu^2e^4}{-2\mu H} - \vec{L}^2 - \hbar^2. \tag{12}$$

These lead to

$$H = -\frac{1}{2}\left(\frac{\mu e^2}{\vec{L}^2 + \vec{M}^2 + \hbar^2}\right) . \tag{13}$$

Let

$$\vec{J} = \tfrac{1}{2}(\vec{M} + \vec{L}),\ \vec{K} = \tfrac{1}{2}(\vec{M} - \vec{L}) \tag{14}$$

then one has

$$[J_i, J_j] = i\hbar\epsilon_{ijk}J_k,\ [K_i, K_j] = -i\hbar\epsilon_{ijk}K_k,\ [K_i, J_j] = 0 \tag{15}$$

with the constraint $\vec{J}^2 = \vec{K}^2$ (c.f. also discussion in Schiff, <u>Quantum Mechanics</u> (1968), p.236-239). Thus $H = -\frac{1}{2}\frac{\mu e^4}{(4\vec{J}^2+\hbar^2)}$ and the possible values of $\vec{J}^2$ are $j(j+1)\hbar^2$. The complete set of commuting observables can thus be chosen as $\vec{J}^2$, J_z and K_z, with

$$\begin{aligned}
\vec{J}^2\mathcal{D}^k_{mn}(\alpha\beta\gamma) &= k(k+1)\hbar^2\mathcal{D}^k_{mn}(\alpha\beta\gamma) \\
J_z\mathcal{D}^k_{mn}(\alpha\beta\gamma) &= m\hbar\mathcal{D}^k_{mn}(\alpha\beta\gamma) \\
K_z\mathcal{D}^k_{mn}(\alpha\beta\gamma) &= n\hbar\mathcal{D}^k_{mn}(\alpha\beta\gamma) \\
H\ \mathcal{D}^k_{mn}(\alpha\beta\gamma) &= E(k)\mathcal{D}^k_{mn}(\alpha\beta\gamma)
\end{aligned} \tag{16}$$

where $E(k) = -\frac{\mu e^4}{2\hbar^2}\cdot\frac{1}{(2k+1)^2}$.

In terms of the usual quantum numbers (n, ℓ, m) we have

$$H = -\frac{\mu e^4}{2\hbar^2}\cdot\frac{1}{(2\ell+1)^2} \equiv -\frac{\mu e^4}{2\hbar^2}\cdot\frac{1}{n^2}\ . \tag{17}$$

Thus the eigenvalues depend only on the principal quantum number n, and the number of degeneracy is

$$n^2 = (2\ell+1)^2 \tag{18}$$

for the Coulomb problem. The degeneracy for the central potential problem, on the other hand, is most easily seen by the (k,m,n) representation defined by Eqs. (14) - (16). While in the Coulomb problem the two commuting conserved vectors $\vec{J}$ and $\vec{K}$ are derived from the conserved vectors $\vec{A}$ and $\vec{L}$, in the central potential problem $\vec{A}$ is no longer a conserved vector (in general). Thus the two commuting sets of observables reduce to one, and the degeneracy reduces to

$$k = (2m+1) \tag{19}$$

Since the Schrödinger equation in x-space governing the wave function $\mathcal{D}^k_{mn}$ separates the same way as the Hamilton-Jacobi equation, namely, in spherical (and parabolic) coordinates, we get Laguerre functions for the spherical case.

Chapter Seven

1. With $E \approx 3kT/2$, have $\lambda = h/p = hc/\sqrt{2(mc^2)E} = 2\pi(\hbar c)/\sqrt{3(mc^2)kT}$
$= 2\pi(2\times 10^{-7})/\sqrt{3\cdot 4\times 10^9\cdot 8.6\times 10^{-5}\cdot 2.17}$ m $= 8.4$Å, for helium. As the size of a helium atom is around 1Å, at this temperature, the DeBroglie wavelength spans many atoms, and that is the key. For heavier elements, the temperature needs to be proportionally smaller to get the same wavelength, and the atoms are larger, suggesting that even longer wavelengths are required. However, higher Z noble gases have interactions that prevent their remaining liquid at very low temperatures. Neon, for example, freezes at 27K.

2. *This solution reprinted from the solutions manual for the revised edition.*

(a) Assume each particle's motion is only due to the SHO potential, than the energy states for any one particle are $\frac{1}{2}\hbar\omega$, $3\hbar\omega/2, \ldots, (n+\frac{1}{2})\hbar\omega, \ldots$ From Fermi-Dirac statistical distribution, we have the probability for state with energy E being occupied is $p = 2/(1 + e^{(E-E_F)/kT})$ where E_F is the Fermi energy and the constant 2 is due to spin multiplicity (2s+1). So

$$N = \Sigma p_i = \sum_{E_i} \frac{2}{1+\exp[(E_i-E_F)/kT]} \; ; \; E_i = (n_i+\tfrac{1}{2})\hbar\omega. \qquad (1)$$

In principle, if we know ω and temperature T, solving (1) for E_F would yield the Fermi energy E_F. In practice this is far from being elementary. The ground state $E^{(0)}$ is

$$E^{(0)} = 2\{\tfrac{1}{2}\hbar\omega + 3\hbar\omega/2 + \ldots + ([N/2]-\tfrac{1}{2})\hbar\omega\} + ([N/2]+\tfrac{1}{2})\hbar\omega\delta \qquad (2)$$

where $\delta = 0$ if N is even, $\delta = 1$ for N odd, and [N/2] is the integer part of $\frac{N}{2}$. $E^{(0)}$ can be rewritten as

$$E^{(0)} = [N/2]\hbar\omega + [N/2]([N/2]-1)\hbar\omega + ([N/2]+\tfrac{1}{2})\hbar\omega\delta. \qquad (3)$$

Thence for N even and N odd, we have

$$E^{(0)} = \frac{N^2}{4}\hbar\omega \text{ (N even)}, \quad E^{(0)} = [(N-1)^2/4 + N/2]\hbar\omega \text{ (N odd)}. \qquad (4)$$

Note for N even, we have N/2 energy states while for N odd we have [N/2]+1 states. Also for the ground state, we have from the definition of Fermi energy that

$$E_F = \begin{cases} (N-1)\hbar\omega/2 \text{ for } N \text{ even} \\ (N/2)\hbar\omega \text{ for } N \text{ odd} \end{cases} \tag{5}$$

(b) If we assume $N >> 1$, and no mutual interaction as in part (a), than from (4) and (5) ground state energy $E^{(0)} \cong \frac{N^2}{4}\hbar\omega$ while Fermi energy $E_F \cong (N/2)\hbar\omega$.

3. *This solution reprinted from the solutions manual for the revised edition.*
From the Clebsch-Gordan Coefficients table, the combination of two spin-1 particles lead to 9 states. These are in the $|m_1, m_2\rangle$ basis representation,

$$j=2 \ : |11\rangle, |-1-1\rangle, \frac{1}{6^{1/2}}[|1-1\rangle+2|00\rangle+|-11\rangle], \frac{1}{2^{1/2}}[|10\rangle+|01\rangle], \frac{1}{2^{1/2}}[|0-1\rangle+|-10\rangle] \tag{1a}$$

$$j=1: \qquad \frac{1}{2^{1/2}}[|10\rangle-|01\rangle], \frac{1}{2^{1/2}}[|1-1\rangle-|-11\rangle], \frac{1}{2^{1/2}}[|0-1\rangle-|-10\rangle] \tag{1b}$$

$$j=0: \qquad \frac{1}{3^{1/2}}[|1-1\rangle-|00\rangle+|-11\rangle]. \tag{1c}$$

For two identical particles which are bosons (with no orbital angular momentum) and both of spin 1, Bose statistics require symmetry for the states. Evidently from (1b) the $j=1$ states are anti-symmetric under $m_1 \leftrightarrow m_2$ while (1a) and (1c) are acceptable, forming six symmetric states with $j=2$ and $j=0$ respectively.

4. *This solution reprinted from the solutions manual for the revised edition.*
If the electron were a spinless boson, then the total wave function (with now no spin part) must be symmetric, viz:-

$$\psi(x_1,x_2) = \frac{1}{2^{1/2}}(\psi_\alpha(x_1)\psi_\beta(x_2) + \psi_\beta(x_1)\psi_\alpha(x_2)) \quad \text{if } \alpha \neq \beta$$

$$= \psi_\alpha(x_1)\psi_\alpha(x_2). \qquad \text{if } \alpha = \beta$$

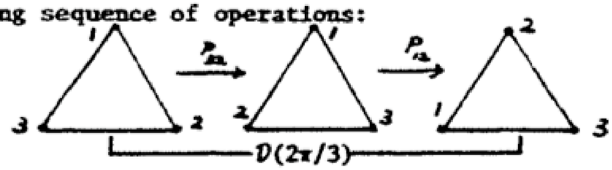
We have only "singlet" parahelium. If we assume that the interaction due to spin is small, then there is no "triplet" orthohelium and the levels of parahelium remains the same.

5. *This solution reprinted from the solutions manual for the revised edition.*

Consider rotation operator around z-axis:

$$\mathcal{D}(\phi) = e^{-iJ_z\phi/\hbar}. \tag{1}$$

Let $\phi = 2\pi/3$, $(4\pi/3, \ldots\ldots)$. Note however that rotation by $2\pi/3$, is equivalent to the following sequence of operations:



The system must return to its original configuration, and from (1) we have

$$\mathcal{D}(2\pi/3)|\alpha\rangle = \text{const.}|\alpha\rangle \tag{2}$$

where const. in (2) must be +1 because $P_{12}P_{32}$ gives +1. Therefore $e^{-im\phi}|_{\phi=2\pi/3}$ = 1, which in turn implies

$$m = 0, \pm 3, \pm 6, \pm 9, \ldots\ldots\ldots\ldots\ldots\ldots \tag{3}$$

6. *This solution reprinted from the solutions manual for the revised edition.*

(a) The spin states must be totally symmetric. (i) $|+\rangle|+\rangle|+\rangle$ is obviously of $S=3$ where S is total spin. (ii) $\frac{1}{3^{1/2}}(|+\rangle|+\rangle|0\rangle + |+\rangle|0\rangle|+\rangle + |0\rangle|+\rangle|+\rangle)$. This construction can be obtained by applying $S_- = S_{1-} + S_{2-} + S_{3-}$ to $|+\rangle|+\rangle|+\rangle$. Since $S^2_{tot.}$ commutes with S_-, we have $S = 3$ in this case also. (iii) Simply write down all possible states with $+,-,0$ and of equal amplitude, we have $\frac{1}{6^{1/2}}[|+\rangle|0\rangle|-\rangle + |+\rangle|-\rangle|0\rangle + |0\rangle|-\rangle|+\rangle + |0\rangle|+\rangle|-\rangle + |-\rangle|+\rangle|0\rangle + |-\rangle|0\rangle|+\rangle]$. This <u>cannot</u> be a pure $S=3$ state since $S=3$, $m_S=0$ would contain $|0\rangle|0\rangle|0\rangle$. Proof: Imagine applying $(S_-)^3$ to $|+\rangle|+\rangle|+\rangle$, there would be a contribution from $S_{3-}S_{2-}S_{1-}|+\rangle|+\rangle|+\rangle \propto |0\rangle|0\rangle|0\rangle$. It cannot be a pure $S=1$ state either. Proof: The symmetric $S=1$ state would look like $\vec{a}(\vec{b}\cdot\vec{c}) + \vec{b}(\vec{c}\cdot\vec{a}) + \vec{c}(\vec{a}\cdot\vec{b})$ which necessarily contains $|0\rangle|0\rangle|0\rangle$. Now $S=2$ states cannot be totally symmetric. Proof: $S=2$ states with $m_S=1$ must be orthogonal to the symmetric $S=3$ state. We can construct only one totally symmetric state out of $++0$, $+0+$, and $0++$. Answer for part (iii) is some linear combination of $S=3$ and $S=1$.

(b) This time the spin part must be totally antisymmetric by Bose statistics. Cases (i) and (ii) above are clearly impossible for total antisymmetry, e.g. for $++0$, $+0+$, $0++$ no matter how we distribute the signs we cannot arrange for an antisymmetric state of spin. For case (iii) it is possible, we have:- $\frac{1}{6^{1/2}}[|+\rangle|0\rangle|-\rangle - |+\rangle|-\rangle|0\rangle + |0\rangle|-\rangle|+\rangle - |0\rangle|+\rangle|-\rangle + |-\rangle|+\rangle|0\rangle - |-\rangle|0\rangle|+\rangle]$. There is in fact only one totally antisymmetric spin state possible. It goes like $\vec{a}\cdot(\vec{b}\times\vec{c})$ and is necessarily a singlet $S=0$.

7. Obviously N is Hermitian, so $N|\eta\rangle = \eta|\eta\rangle$ where η is a real number. We know that the eigenvalues η of N cannot be negative since

$$\eta = \langle\eta|N|\eta\rangle = \left[\langle\eta|a^\dagger\right]\left[a|\eta\rangle\right] = \langle\alpha|\alpha\rangle \geq 0$$

based on the "positivity postulate" of quantum mechanics. This was our starting point for working out the algebra of the simple harmonic oscillator, establishing a minimum value for η.

Now, consider the state $a^\dagger|\eta\rangle$. We have

$$\begin{aligned} N\left[a^\dagger|\eta\rangle\right] &= a^\dagger a a^\dagger|\eta\rangle \\ &= a^\dagger\left[1-N\right]|\eta\rangle \\ &= (1-\eta)\left[a^\dagger|\eta\rangle\right] \end{aligned}$$

Therefore $a^\dagger|\eta\rangle$ is also an eigenstate of N but with eigenvalue $(1-\eta)$ which must also be positive. Therefore, there is a *maximum* value of η as well.

8. *This solution reprinted from the solutions manual for the revised edition.*

Possible spin states for spin 3/2 particles are $2\cdot\frac{3}{2}+1=4$. So the configuration is $(1s)^4(2s)^4(2p)^{12}$. High degeneracy is because the 2p orbitals can accomodate up to $4(2\ell+1) = 12$ electrons, typically $\binom{12}{2} = \frac{12!}{2!10!} = \frac{12\times 11}{2} = 66$, i.e. 66-fold degeneracy and hence a very large number.

The ground state (lowest term) should have spin states as symmetric as possible, and space states as antisymmetric as possible [c.f. discussion of C-atom]. The only antisymmetric space states are P-wave, i.e. $\ell_1=\ell_2=1$, $\ell_{tot.}=1$. With $S_{tot.} = 3/2+3/2 = 3$, we have a spin 7-plet. For the total angular momentum, $\vec{L}$ and $\vec{S}$ should be as "antiparallel" as possible, this implies that $J_{tot.} = 2$. Hence ground state is $^7P_2$.

9. *This solution reprinted from the solutions manual for the revised edition.*

(a) The wave function for a single particle is $\psi_n^{(1)}(x_i) = \sqrt{2/L}\,\sin(n\pi x_i/L)$ with energy $E_n^{(1)} = n^2\pi^2\hbar^2/2mL^2$. For a two particle system the wave function is

$$\psi^{(2)}(x_1,x_2) = \sum_{i,j} c_{ij}\psi_i^{(1)}(x_1)\psi_j^{(1)}(x_2) \qquad (1)$$

where c_{ij} is determined by symmetry and the filling of energy levels according to Pauli principle. For ground state of two spin ½ fermions, the total wave function must be antisymmetric and the spin part is triplet (hence symmetric), therefore the space wave function (1) must be antisymmetric. Thus $\psi^{(2)} = \frac{1}{2^{1/2}}(\psi_1^{(1)}(x_1)\psi_2^{(1)}(x_2) - \psi_2^{(1)}(x_1)\psi_1^{(1)}(x_2))$ and to obtain a <u>non-vanishing</u> ground state space wave function, we must choose

$$\psi^{(2)}(x_1,x_2) = (2/\sqrt{2}L)[\sin\pi x_1/L\ \sin 2\pi x_2/L - \sin 2\pi x_1/L\ \sin\pi x_2/L] \qquad (2)$$

and $E_{tot.} = \frac{\pi^2\hbar^2}{2mL^2}(1^2+2^2) = 5\pi^2\hbar^2/2mL^2$.

(b) If spin part is a singlet state (which is antisymmetric), than space wave function must be symmetric. Hence (1) must assume for the ground state the form

$$\psi^{(2)}(x_1,x_2) = (2/L)\sin\pi x_1/L\ \sin\pi x_2/L \qquad (3)$$

and $E_{tot.} = \frac{\pi^2\hbar^2}{2mL^2}(1^2+1^2) = \pi^2\hbar^2/mL^2$.

(c) First for triplet and singlet state the first order energy shift is

$$\Delta E = -\lambda\int dx_1 dx_2 \psi^{(2)*}(x_1,x_2)\delta(x_1-x_2)\psi^{(2)}(x_1,x_2). \qquad (4)$$

Use the explicit form (2) and integrate over δ-function, we find for triplet state $\Delta E = (2/L)^2(-\lambda/2)\int dx_1(\sin\pi x_1/L\ \sin 2\pi x_1/L - \sin 2\pi x_1/L\ \sin\pi x_1/L)^2 = 0$ and for singlet state $\Delta E = -\lambda(2/L)^2\int_0^L \sin^4(\pi x_1/L)dx_1 = -3\lambda/2L$.

10. To prove the orthogonality relations (7.6.11), start with the definitions

$$\begin{aligned}
\hat{\mathbf{e}}_{\mathbf{k}\pm} &= \mp\frac{1}{\sqrt{2}}\left(\hat{\mathbf{e}}_{\mathbf{k}}^{(1)} \pm i\hat{\mathbf{e}}_{\mathbf{k}}^{(2)}\right) \qquad \text{so we have} \\
\hat{\mathbf{e}}_{\mathbf{k}\lambda}^{*}\cdot\hat{\mathbf{e}}_{\pm\mathbf{k}\lambda'} &= \left[-\lambda\frac{1}{\sqrt{2}}\left(\hat{\mathbf{e}}_{\mathbf{k}}^{(1)} - \lambda i\hat{\mathbf{e}}_{\mathbf{k}}^{(2)}\right)\right]\cdot\left[-\lambda'\frac{1}{\sqrt{2}}\left(\hat{\mathbf{e}}_{\pm\mathbf{k}}^{(1)} + \lambda' i\hat{\mathbf{e}}_{\pm\mathbf{k}}^{(2)}\right)\right] \\
&= \frac{\lambda\lambda'}{2}\left[\hat{\mathbf{e}}_{\mathbf{k}}^{(1)}\cdot\hat{\mathbf{e}}_{\pm\mathbf{k}}^{(1)} + i\lambda'\hat{\mathbf{e}}_{\mathbf{k}}^{(1)}\cdot\hat{\mathbf{e}}_{\pm\mathbf{k}}^{(2)} - i\lambda\hat{\mathbf{e}}_{\mathbf{k}}^{(2)}\cdot\hat{\mathbf{e}}_{\pm\mathbf{k}}^{(1)} + \lambda\lambda'\hat{\mathbf{e}}_{\mathbf{k}}^{(2)}\cdot\hat{\mathbf{e}}_{\pm\mathbf{k}}^{(2)}\right] \\
&= \frac{\lambda\lambda'}{2}\left[\pm 1 + 0 - 0 \pm \lambda\lambda'\right] \\
&= \pm 1 \qquad \text{if}\lambda = \lambda' \\
&= 0 \qquad \text{if}\lambda \neq \lambda' \\
\hat{\mathbf{e}}_{\mathbf{k}\lambda}^{*}\times\hat{\mathbf{e}}_{\pm\mathbf{k}\lambda'} &= \left[-\lambda\frac{1}{\sqrt{2}}\left(\hat{\mathbf{e}}_{\mathbf{k}}^{(1)} - \lambda i\hat{\mathbf{e}}_{\mathbf{k}}^{(2)}\right)\right]\times\left[-\lambda'\frac{1}{\sqrt{2}}\left(\hat{\mathbf{e}}_{\pm\mathbf{k}}^{(1)} + \lambda' i\hat{\mathbf{e}}_{\pm\mathbf{k}}^{(2)}\right)\right] \\
&= \frac{\lambda\lambda'}{2}\left[\hat{\mathbf{e}}_{\mathbf{k}}^{(1)}\times\hat{\mathbf{e}}_{\pm\mathbf{k}}^{(1)} + i\lambda'\hat{\mathbf{e}}_{\mathbf{k}}^{(1)}\times\hat{\mathbf{e}}_{\pm\mathbf{k}}^{(2)} - i\lambda\hat{\mathbf{e}}_{\mathbf{k}}^{(2)}\times\hat{\mathbf{e}}_{\pm\mathbf{k}}^{(1)} + \lambda\lambda'\hat{\mathbf{e}}_{\mathbf{k}}^{(2)}\times\hat{\mathbf{e}}_{\pm\mathbf{k}}^{(2)}\right] \\
&= \frac{\lambda\lambda'}{2}\left[0 \pm i\lambda'\hat{\mathbf{k}} \pm i\lambda\hat{\mathbf{k}} + 0\right] \\
&= \pm i\hat{\mathbf{k}} \qquad \text{if}\lambda = \lambda' \\
&= 0 \qquad \text{if}\lambda \neq \lambda'
\end{aligned}$$

The first result (7.6.11a) serves to collapse the two sums over λ and λ' into one, when calculating $|\mathbf{E}|^2 = \mathbf{E}^*\cdot\mathbf{E}$ from (7.6.14), and the integral (7.6.15) collapses the two sums over $\mathbf{k}$ and $\mathbf{k}'$ into one, leading to (7.6.16). The expression for the magnetic field is

$$\mathbf{B}(\mathbf{x},t) = \boldsymbol{\nabla}\times\mathbf{A}(\mathbf{x},t) = \frac{i}{c}\sum_{\mathbf{k},\lambda}\omega_k\left[\mathbf{A}_{\mathbf{k},\lambda}e^{-i(\omega_k t-\mathbf{k}\cdot\mathbf{x})} - \mathbf{A}_{\mathbf{k},\lambda}^{*}e^{i(\omega_k t-\mathbf{k}\cdot\mathbf{x})}\right]\hat{\mathbf{k}}\times\hat{\mathbf{e}}_{\mathbf{k}}^{(\lambda)}$$

which is very similar to (7.6.14), differing by the presence of $\hat{\mathbf{k}}\times\hat{\mathbf{e}}_{\mathbf{k},\lambda}$ instead of $\hat{\mathbf{e}}_{\mathbf{k},\lambda}$. But

$$\hat{\mathbf{k}}\times\hat{\mathbf{e}}_{\mathbf{k},\pm} = -\mp\frac{1}{\sqrt{2}}\left[\hat{\mathbf{k}}\times\hat{\mathbf{e}}_{\mathbf{k}}^{(1)} \pm i\hat{\mathbf{k}}\times\hat{\mathbf{e}}_{\mathbf{k}}^{(2)}\right] = -\mp\frac{1}{\sqrt{2}}\left[\hat{\mathbf{e}}_{\mathbf{k}}^{(2)} \mp i\hat{\mathbf{e}}_{\mathbf{k}}^{(1)}\right] = i\hat{\mathbf{e}}_{\mathbf{k},\pm}$$

so that the calculation of $|\mathbf{B}|^2 = \mathbf{B}^*\cdot\mathbf{B}$ carries through directly as for the electric field. The cross terms, however, have opposite sign, and therefore cancel when adding the contributions to the energy from electric and magnetic fields, leading to (7.6.17).

Chapter Eight

1. (a) (Note the typo in the exponent.) $m_p c^2 = (1.67\times10^{-27}\text{ kg})(3.00\times10^8\text{ m/s})^2 = 1.5\times10^{-10}$ joule (1 eV/1.60×10^{-19} joule $= 9.39\times10^8$ eV $= 0.939$ GeV.
(b) $E = pc \sim (\hbar/1\text{ fm})c = 200\text{ MeV}\cdot\text{fm}/1\text{ fm} = 200$ MeV, which is about the same as the pion mass. The point is that, at these distances, mass and energy are not easily distinguished. That is, in this regime (called "particle physics") the formalism has to be relativistic.
(c) Using "$[a]$" to mean "dimensions of a", i.e. M, L, or T for mass, length or time, we note that $[G]=L^3M^{-1}T^{-2}$, $[\hbar]=ML^2T^{-1}$, and $[c]=LT^{-1}$. Writing $M_P = G^x\hbar^y c^z$ we must have $1=-x+y$, $0=3x+2y+z$, and $0=-2x-y-z$, so $x=-1/2$, $y=1/2$, and $z=1/2$. So $M_Pc^2=\sqrt{\hbar c^5/G}=\sqrt{(1.05\times10^{-34})(3\times10^8)^5/(6.67\times10^{-11})}=1.96{\times}10^9$ j$=1.2{\times}10^{19}$ GeV.

2. This problem is trivial, but the implications are important. Since

$$\begin{bmatrix}1&0&0&0\\0&-1&0&0\\0&0&-1&0\\0&0&0&-1\end{bmatrix}\begin{bmatrix}1&0&0&0\\0&-1&0&0\\0&0&-1&0\\0&0&0&-1\end{bmatrix}=\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix}$$

the metric tensor is its own inverse, i.e. $\eta^{\mu\lambda}\eta_{\lambda\nu}=\delta^\mu_\nu$. It is therefore simple to show that the contravariant form of the metric follows appropriately from the covariant form, that is $\eta^{\mu\lambda}\eta^{\nu\sigma}\eta_{\lambda\sigma}=\delta^\mu_\sigma\eta^{\nu\sigma}=\eta^{\nu\mu}=\eta^{\mu\nu}$ since it is also symmetric. Also $a^\mu b_\mu = a_\nu\eta^{\mu\nu}b^\lambda\eta_{\lambda\mu}=a_\nu b^\lambda\delta^\nu_\lambda=a_\nu b^\nu=a_\mu b^\mu$.

3. For (8.1.11) to be a conserved current, we must show that $\partial_\mu j^\mu=0$:

$$\begin{aligned}\partial_\mu j^\mu &= \frac{i}{2m}\left[\partial_\mu(\Psi^*\partial^\mu\Psi)-\partial_\mu\left((\partial^\mu\Psi)^*\Psi\right)\right]\\ &= \frac{i}{2m}\left[(\partial_\mu\Psi^*)(\partial^\mu\Psi)+\Psi^*(\partial^2\Psi)-(\partial^2\Psi^*)\Psi-(\partial^\mu\Psi)^*(\partial_\mu\Psi)\right]\\ &= \frac{i}{2m}\left[\Psi^*(-m^2\Psi)-(-m^2\Psi^*)\Psi\right]=0\end{aligned}$$

4. This is a silly, trivial problem. The Klein-Gordon equation basically comes from writing $E^2=p^2+m^2$ with E replaced by $i\partial^0$ and p replaced by $-i\boldsymbol{\nabla}$. In other words

$$\left[E^2-p^2\right]\Psi=\left[-(\partial^0)^2+\boldsymbol{\nabla}^2\right]\Psi=-\partial^\mu\partial_\mu\Psi=m^2\Psi \qquad\text{or}\qquad (\partial^\mu\partial_\mu+m^2)\Psi=0$$

which is (8.1.8). This can be read as replacing $p^\mu p_\mu=E^2-p^2$ with the operator $-\partial^\mu\partial_\mu$, i.e. p^μ with $-i\partial^\mu$. So, the minimal electromagnetic substitution $p^\mu\to p^\mu-eA^\mu$ becomes $-i\partial^\mu\to-i\partial^\mu-eA^\mu=-i(\partial^\mu-ieA^\mu)=-iD^\mu$ where $D^\mu\equiv\partial^\mu-ieA^\mu$. (Did I make a sign error in the definition of D^μ in the textbook? I guess so.)

5. Rewrite (8.1.14) using $D_\mu D^\mu = D_t^2 - \mathbf{D}^2$ as $D_t^2\Psi = \mathbf{D}^2\Psi - m^2\Psi$. Then, using (8.1.15),

$$\begin{aligned}
iD_t\phi &= \frac{i}{2}D_t\Psi - \frac{1}{2m}D_t^2\Psi = \frac{i}{2}D_t\Psi - \frac{1}{2m}\mathbf{D}^2\Psi + \frac{m}{2}\Psi \\
&= -\frac{1}{2m}\mathbf{D}^2\Psi + \frac{m}{2}\left[\Psi + \frac{i}{m}D_t\Psi\right] = -\frac{1}{2m}\mathbf{D}^2\Psi + m\phi \\
\text{and}\qquad iD_t\chi &= \frac{i}{2}D_t\Psi + \frac{1}{2m}D_t^2\Psi = \frac{i}{2}D_t\Psi + \frac{1}{2m}\mathbf{D}^2\Psi - \frac{m}{2}\Psi \\
&= +\frac{1}{2m}\mathbf{D}^2\Psi - \frac{m}{2}\left[\Psi - \frac{i}{m}D_t\Psi\right] = +\frac{1}{2m}\mathbf{D}^2\Psi - m\chi
\end{aligned}$$

which are (8.1.16) since $\Psi = \phi + \chi$. These two equations obviously become (8.1.18) since

$$\tau_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \qquad \text{and} \qquad \tau_3 + i\tau 2 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} + i\begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}$$

6. Write the solutions as

$$\Upsilon(\boldsymbol{x},t) = \begin{pmatrix} a \\ b \end{pmatrix} e^{-iEt+i\boldsymbol{p}\cdot\boldsymbol{x}}$$

which results in the matrix equation

$$E\begin{pmatrix} a \\ b \end{pmatrix} = \left[\frac{p^2}{2m}\begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix} + m\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\right]\begin{pmatrix} a \\ b \end{pmatrix}$$

or

$$\begin{pmatrix} \frac{p^2}{2m} + m - E & \frac{p^2}{2m} \\ -\frac{p^2}{2m} & -\frac{p^2}{2m} - m - E \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Take the determinant to get the characteristic equation

$$-\left(\frac{p^2}{2m} + m\right)^2 + E^2 + \left(\frac{p^2}{2m}\right)^2 = 0$$

so that

$$E^2 = 2\frac{p^2}{2m}m + m^2 = p^2 + m^2$$

in which case the energy eigenvalues are

$$E = \pm E_p \qquad \text{where} \qquad E_p = \sqrt{p^2 + m^2}$$

In order to find the eigenfunctions, first rewrite the characteristic equation by multiplying through by $2m$ and also writing $p^2 = E_p^2 - m^2$, so

$$\begin{pmatrix} E_p^2 + m^2 - 2mE & E_p^2 - m^2 \\ m^2 - E_p^2 & -E_p^2 - m^2 - 2mE \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

In the case of *positive energy eigenvalues* $E = +E_p$, we have

$$\begin{bmatrix} (E_p-m)^2 & (E_p+m)(E_p-m) \\ (m-E_p)(m+E_p) & -(E_p+m)^2 \end{bmatrix} \begin{bmatrix} a^+ \\ b^+ \end{bmatrix} = 0$$

which implies that $(E_p-m)a^+ + (E_p+m)b^+ = 0$. Normalizing using the relation $\Upsilon^\dagger \tau_3 \Upsilon = +1$ we have

$$\left(a^+\right)^2 - \left(b^+\right)^2 = \left(a^+\right)^2 \left[1 - \frac{(E_p-m)^2}{(E_p+m)^2}\right] = \left(a^+\right)^2 \left[\frac{4mE_p}{(E_p+m)^2}\right] = +1$$

in which case

$$a^+ = \frac{E_p+m}{2\sqrt{mE_p}} \qquad \text{and} \qquad b^+ = \frac{m-E_p}{2\sqrt{mE_p}}$$

Similarly, for *negative energy eigenvalues* $E = -E_p$, we have

$$\begin{bmatrix} (E_p+m)^2 & (E_p+m)(E_p-m) \\ (m-E_p)(m+E_p) & -(E_p-m)^2 \end{bmatrix} \begin{bmatrix} a^- \\ b^- \end{bmatrix} = 0$$

which implies that $(E_p+m)a^- + (E_p-m)b^- = 0$. This time we normalize using the relation $\Upsilon^\dagger \tau_3 \Upsilon = -1$ and therefore

$$\left(a^-\right)^2 - \left(b^-\right)^2 = \left(a^-\right)^2 \left[1 - \frac{(E_p+m)^2}{(E_p-m)^2}\right] = \left(a^-\right)^2 \left[\frac{-4mE_p}{(E_p-m)^2}\right] = -1$$

in which case

$$a^- = \frac{m-E_p}{2\sqrt{mE_p}} \qquad \text{and} \qquad b^- = \frac{E_p+m}{2\sqrt{mE_p}}$$

7. First, a mea culpa. I wrote this problem (from Landau's book) years before the manuscript was completed and I came to work out this solution. There are a few mistakes, I realize, in the problem and a little in the text. One has to do with the sign of e, which in this book (unlike any others I know) is negative. Therefore $D_\mu \equiv \partial_\mu - ieA_\mu$ (see problem 4) and $A_0 = \Phi = -Z|e|/r = +Ze/r$. In the problem statement, I misused k in the argument of $u(r)$ and also in the definition of ρ. Also, I wrote "Work the upper component", but I don't recall why. It may be the default when using the positive energy solution.

Anyway, we start with the Klein-Gordon Equation (8.1.14), namely

$$\left[D_\mu D^\mu + m^2\right] \Psi(\mathbf{x}, t) = 0$$

where $D_\mu \equiv \partial_\mu - ieA_\mu$. With $\mathbf{A} = 0$ and $eA_0 = Ze^2/r = Z\alpha/r$, this becomes

$$\left[\left(\partial_t - i\frac{Z\alpha}{r}\right)^2 - \boldsymbol{\nabla}^2 + m^2\right] \Psi(\mathbf{x}, t) = 0$$

Next, as suggested, put $\Psi(\mathbf{x},t) = Ne^{-iEt}[u_l(r)/r]Y_{lm}(\theta,\phi)$. Then

$$\left[\left(E+\frac{Z\alpha}{r}\right)^2+\boldsymbol{\nabla}^2-m^2\right]\frac{u_l(r)}{r}Y_{lm}(\theta,\phi)=0$$

From (3.6.21) the Laplacian $\boldsymbol{\nabla}^2$ can be written as

$$\boldsymbol{\nabla}^2=\frac{\partial^2}{\partial r^2}+\frac{2}{r}\frac{\partial}{\partial r}-\frac{1}{r^2}\mathbf{L}^2$$

where, in this context, $\mathbf{L}^2$ is a differential operator in θ and ϕ. Therefore

$$\left[\left(E+\frac{Z\alpha}{r}\right)^2-m^2-\frac{l(l+1)}{r^2}\right]\frac{u}{r}+\frac{1}{r}\frac{d^2u}{dr^2}=0$$

Finally, with $\gamma^2\equiv 4(m^2-E^2)$ and $\rho\equiv\gamma r$, this becomes

$$\frac{d^2u}{d\rho^2}+\left[\frac{2EZ\alpha}{\gamma\rho}-\frac{1}{4}-\frac{l(l+1)-(Z\alpha)^2}{\rho^2}\right]u=0$$

For $\rho\to\infty$, this becomes $d^2u/d\rho^2=u/4$ or $u(\rho)=\exp(\pm\rho/2)$. Only the negative sign gives a normalizable solution, so we write $u(\rho)=w(\rho)\exp(-\rho/2)$ in which case

$$\frac{d^2w}{d\rho^2}-\frac{dw}{d\rho}+\left[\frac{2EZ\alpha}{\gamma\rho}-\frac{l(l+1)-(Z\alpha)^2}{\rho^2}\right]w=0$$

Now substitute $w(\rho)=\sum_{q=0}^{\infty}C_n\rho^{k+q}$ and collect terms into the same power of ρ by redefining the index of summation. This gives

$$\begin{aligned}&[k(k-1)-l(l+1)+(Z\alpha)^2]C_0\rho^{k-2}\\ +\textstyle\sum_{q=0}^{\infty}\quad&\Big\{[(k+q+1)(k+q)-l(l+1)+(Z\alpha)^2]C_{q+1}\\ &-\left[(k+q)-\frac{2EZ\alpha}{\gamma}\right]C_q\Big\}\,\rho^{k+q-1}=0\end{aligned}$$

This series is set to zero term by term. Solving $k(k-1)-l(l+1)+(Z\alpha)^2=0$ for k gives

$$k=\frac{1}{2}\pm\frac{1}{2}\left[1+4l(l+1)-4(Z\alpha)^2\right]^{1/2}=\frac{1}{2}\pm\left[\left(l+\frac{1}{2}\right)^2-(Z\alpha)^2\right]^{1/2}$$

Near the origin, the wave function goes like $u(r)/r\propto r^{k-1}$, so the expectation value of kinetic energy goes like $r^{k-1}r^{k-3}r^2=r^{2k-2}$. Consider the negative sign solution for k. For $l=0$, k is close to zero, and negative for nonzero l, so the kinetic energy diverges too rapidly. For the positive sign solution, with $Z\alpha\ll 1$, $k\approx l+1$ and the wave function goes like r^l, which is the nonrelativistic result from the Schrödinger equation. All this points to taking the positive sign in the solution for k.

Now set the higher power terms of ρ to zero. We have

$$C_{q+1} = \frac{k + q - 2EZ\alpha/\gamma}{(k + q + 1)(k + q) - l(l + 1) + (Z\alpha)^2} C_q \sim \frac{1}{q} C_n \qquad \text{as } q \to \infty$$

so that the series approaches e^ρ. In other words, the wave function is proportional to $e^{+\rho/2}$ for large ρ which is unacceptably divergent. Let the series terminate at $q = N$, then

$$k + N - 2EZ\alpha/\gamma = k + N - EZ\alpha/\sqrt{m^2 - E^2} = 0$$

Note that $N = 0$ is possible. As in (3.7.51), then, define the principle quantum number $n = N = l + 1$, and so $(m^2 - E^2)(k + n - l - 1)^2 = E^2(Z\alpha)^2$. Solving for E we find

$$E = \frac{m}{\left(1 + (Z\alpha)^2 \left[n - l - \frac{1}{2} + \sqrt{\left(l + \frac{1}{2}\right)^2 - (Z\alpha)^2}\right]^{-2}\right)^{1/2}}$$

A Taylor expansion of this expression is straightforward, but I used MAPLE instead:

$$\begin{aligned} E &= m - \frac{m(Z\alpha)^2}{2n^2} \\ &\quad + \frac{m(Z\alpha)^2}{2n^2} \left[\frac{3}{4n^2} - \frac{1}{n(l + 1/2)}\right] (Z\alpha)^2 \\ &\quad + \frac{m(Z\alpha)^2}{2n^2} \left[\frac{3}{2n^3(l + 1/2)} - \frac{n + 3(l + 1/2)}{4n^2(l + 1/2)^3} - \frac{5}{8n^4}\right] (Z\alpha)^4 + \cdots \end{aligned}$$

The first term is just the rest energy, the second is the Balmer formula, and the third is the relativistic correction to the kinetic energy; the spin-orbit term is, of course, missing. (See the solution to Problem 16.)

Jenkins and Kunselman give a large number of transitions, both experimental values and Klein-Gordon solutions, for π^- atoms. For example, the $3D \to 2P$ transition in ^{59}Co is 384.6 ± 1.0 keV, while the "Klein-Gordon energy" is listed as 378.6 keV. With $m = m_\pi = 139.57$ MeV and $\alpha = 1/137.036$ (from PDG 2010) and $Z = 27$, the Balmer transition energy is $-2709.1 \times (1/9 - 1/4) = 376.3$ keV, and the relativistic correction to the kinetic energy adds to this an amount

$$376.3 \text{ keV} \times \left[\frac{3}{36} - \frac{3}{16} - \frac{1}{15/2} + \frac{1}{3}\right] \times (27\alpha)^2 = 1.4 \text{ keV}$$

for a total (to first order) transition energy 377.7 keV. The next order correction will be smaller by $\sim (Z\alpha)^2 = 4/100$ and will not account for the difference between this and the number in the paper; theirs is likely due to older values for the pion mass.

8. All of these follow from equations (8.2.4), and $\mathsf{Tr}(AB) = \mathsf{Tr}(BA)$. For example

$$\mathsf{Tr}(\beta) = \mathsf{Tr}(\gamma^0) = -\mathsf{Tr}(\gamma^i\gamma^i\gamma^0) = -\mathsf{Tr}(\gamma^i\gamma^0\gamma^i) = +\mathsf{Tr}(\gamma^0\gamma^i\gamma^i) = -\mathsf{Tr}(\gamma^0)$$

and so $\mathsf{Tr}(\gamma^0) = 0$. In other words, insert an appropriate γ^μ twice to get a factor ± 1, then split the pair using the commutativity property above, then reverse the order using (8.2.4c) to pick up a minus sign, then contract the two γ^μ that you inserted in the first place. You always get that the trace equals its own negative, so must be zero.

9. We are to construct the γ matrices from

$$\boldsymbol{\alpha} = \begin{bmatrix} 0 & \boldsymbol{\sigma} \\ \boldsymbol{\sigma} & 0 \end{bmatrix} \qquad \text{and} \qquad \beta = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

where

$$\alpha_i \equiv \gamma^0\gamma^i \qquad \text{and} \qquad \beta \equiv \gamma^0$$

so that

$$\gamma^0 = \beta \qquad \text{and} \qquad \gamma^i = \gamma^0\alpha^i$$

and we can write the γ matrices explicitly as

$$\begin{aligned}
\gamma^0 &= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \\
\gamma^1 &= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & \sigma^1 \\ -\sigma^1 & 0 \end{bmatrix} \\
\gamma^2 &= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & \sigma^2 \\ -\sigma^2 & 0 \end{bmatrix} \\
\gamma^3 &= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & \sigma^3 \\ -\sigma^3 & 0 \end{bmatrix}
\end{aligned}$$

It is simple enough to multiply out the 4×4 matrices, or even to use the compact 2×2 form we derive here, and show that the Clifford Algebra is satisfied.

10. The Schrödinger equation with the Dirac Hamiltonian is

$$\begin{aligned}
i\partial_t\Psi &= H\Psi = (\boldsymbol{\alpha}\cdot\mathbf{p} + \beta m)\Psi \\
&= -i\boldsymbol{\alpha}\cdot\boldsymbol{\nabla}\Psi + \beta m\Psi \\
\text{so that} \qquad -i\partial_t\Psi^\dagger &= +i(\boldsymbol{\nabla}\Psi^\dagger)\cdot\boldsymbol{\alpha} + \beta m\Psi^\dagger
\end{aligned}$$

Note that $\boldsymbol{\alpha}$ and β are Hermitian. Now take the time derivative of the probability density

$$\frac{\partial \rho}{\partial t} = \partial_t(\Psi^\dagger \Psi) = (\partial_t \Psi^\dagger)\Psi + \Psi^\dagger \partial_t \Psi = \left[-(\boldsymbol{\nabla}\Psi^\dagger)\cdot\boldsymbol{\alpha}\Psi - \Psi^\dagger\boldsymbol{\alpha}\cdot\boldsymbol{\nabla}\Psi\right] = -\boldsymbol{\nabla}\cdot\left(\Psi^\dagger\boldsymbol{\alpha}\Psi\right)$$

Therefore $\rho \equiv \Psi^\dagger\Psi$ satisfies the continuity equation for the current $\mathbf{j} \equiv \Psi^\dagger\boldsymbol{\alpha}\Psi$.

11. As indicated in the text, this decouples into two eigenvalue problems, one for u_1 and u_3, and the other for u_2 and u_4. That is, we have

$$\begin{bmatrix} m & p \\ p & -m \end{bmatrix}\begin{bmatrix} u_1 \\ u_3 \end{bmatrix} = E\begin{bmatrix} u_1 \\ u_3 \end{bmatrix} \qquad \text{and} \qquad \begin{bmatrix} m & -p \\ -p & -m \end{bmatrix}\begin{bmatrix} u_2 \\ u_4 \end{bmatrix} = E\begin{bmatrix} u_2 \\ u_4 \end{bmatrix}$$

The first equation implies that $(m-E)(-m-E)-p^2 = -(m^2-E^2)-p^2 = 0$ which implies that $E = \pm E_p$ where $E_p \equiv \sqrt{p^2+m^2}$. The second gives the same characteristic equation, so the eigenvalues are once again $E = \pm E_p$.

12. Since $j^\mu \equiv \bar{\Psi}\gamma^\mu\Psi$ with $\bar{\Psi} \equiv \Psi^\dagger\gamma^0$, $j^0 = \Psi^\dagger\gamma^0\gamma^0\Psi = \Psi^\dagger\Psi$ and $\mathbf{j} = \Psi^\dagger\gamma^0\boldsymbol{\gamma}\Psi = \Psi^\dagger\boldsymbol{\alpha}\Psi$. So, for if Ψ has four (real) components a, b, c, and d, then

$$\begin{aligned}
j^0 &= a^2+b^2+c^2+d^2 \\
j^1 &= \begin{bmatrix} a & b & c & d \end{bmatrix}\begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = 2(ad+bc) \\
j^2 &= \begin{bmatrix} a & b & c & d \end{bmatrix}\begin{bmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = 0 \\
j^3 &= \begin{bmatrix} a & b & c & d \end{bmatrix}\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix}\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = 2(ac-bd)
\end{aligned}$$

For each of the spinors (8.2.22), the (normalized) probability density is

$$j^0 = \frac{E_p+m}{2E_p}\left[1+\frac{p^2}{(E_p+m)^2}\right] = 1$$

that is, a constant, independent of momentum. (Note that the exponential plane wave factors multiply to one in any combination of $\Psi^\dagger\Psi$.) Also for each of the spinors, $j^1 = 0$. For both positive energy solutions $u_R^{(+)}(p)$ and $u_L^{(+)}(p)$, we find

$$j^3 = \frac{E_p+m}{2E_p}\frac{2p}{E_p+m} = \frac{p}{E_p}$$

that is, the velocity of the particle. For both negative energy solutions $u_R^{(-)}(p)$ and $u_L^{(-)}(p)$, we find

$$j^3 = \frac{E_p + m}{2E_p} \frac{-2p}{E_p + m} = -\frac{p}{E_p}$$

that is, the velocity of the particle moving in the direction opposite from the momentum.

13. Work out $U_T = \gamma^1\gamma^3$ as follows:

$$U_T = \begin{pmatrix} 0 & \sigma^1 \\ -\sigma^1 & 0 \end{pmatrix} \begin{pmatrix} 0 & \sigma^3 \\ -\sigma^3 & 0 \end{pmatrix} = \begin{pmatrix} -\sigma^1\sigma^3 & 0 \\ 0 & -\sigma^1\sigma^3 \end{pmatrix} = i \begin{pmatrix} \sigma^2 & 0 \\ 0 & \sigma^2 \end{pmatrix} = i\sigma^2 \otimes I$$

where (3.2.34) and (3.2.35) imply that $\sigma^1\sigma^3 = i\varepsilon_{13k}\sigma^k = -i\sigma^2$.

14. Refer to **(9)** for the γ matrices in explicit form. The positive helicity, positive energy electron free particle Dirac wave function is

$$\Psi(\mathbf{x}, t) = u_R^{(+)}(p) e^{-ip_\mu x^\mu} = \begin{bmatrix} 1 \\ 0 \\ p/(E_p + m) \\ 0 \end{bmatrix} e^{-i(E_p t - \mathbf{p}\cdot\mathbf{x})}$$

We can therefore construct the following

$$
\begin{aligned}
\mathcal{P}\Psi(\mathbf{x},t) = \gamma^0\Psi(-\mathbf{x},t) \quad &= \begin{bmatrix} 1 \\ 0 \\ -p/(E_p+m) \\ 0 \end{bmatrix} e^{-i(E_pt+\mathbf{p}\cdot\mathbf{x})} \\
\mathcal{C}\Psi(\mathbf{x},t) = i\gamma^2\Psi^*(\mathbf{x},t) \quad &= \begin{bmatrix} 0 \\ -p/(E_p+m) \\ 0 \\ 1 \end{bmatrix} e^{+i(E_pt-\mathbf{p}\cdot\mathbf{x})} \\
\mathcal{CP}\Psi(\mathbf{x},t) = i\gamma^2\gamma^0\Psi^*(-\mathbf{x},t) \quad &= \begin{bmatrix} 0 \\ p/(E_p+m) \\ 0 \\ 1 \end{bmatrix} e^{+i(E_pt+\mathbf{p}\cdot\mathbf{x})} \\
\mathcal{T}\Psi(\mathbf{x},t) = \gamma^1\gamma^3\Psi^*(\mathbf{x},t) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix}\Psi^*(\mathbf{x},t) \quad &= \begin{bmatrix} 0 \\ -1 \\ 0 \\ -p/(E_p+m) \end{bmatrix} e^{+i(E_pt-\mathbf{p}\cdot\mathbf{x})} \\
\mathcal{PT}\Psi(\mathbf{x},t) \quad &= \begin{bmatrix} 0 \\ -1 \\ 0 \\ p/(E_p+m) \end{bmatrix} e^{+i(E_pt+\mathbf{p}\cdot\mathbf{x})} \\
\mathcal{CPT}\Psi(\mathbf{x},t) \quad &= \begin{bmatrix} p/(E_p+m) \\ 0 \\ 1 \\ 0 \end{bmatrix} e^{-i(E_pt+\mathbf{p}\cdot\mathbf{x})}
\end{aligned}
$$

We see that $\mathcal{CPT}\Psi(\mathbf{x},t)$ is the wave function for a negative energy, right-handed electron with momentum $-\mathbf{p}$. This is the "hole" that we call a positron.

15. This is also a silly problem, with the solution pretty much outlined in the text. For large i, (8.4.39) shows that b_i is proportional to a_i. Furthermore, for large i, equations (8.4.38) show that a_i is proportional to $+1/i$. Therefore, each of the series (8.4.32) or (8.4.33) look like $x^i/i!$, that is e^x for large x. As they are multiplied by factors $\exp[-(1-\varepsilon^2)^{1/2}x]$ but $\varepsilon < 1$, the functions $u(x)$ and $v(x)$ will grow without bound for large x unless the series terminates.

16. First, expand the second term in the denominator of (8.4.43) to second order in $(Z\alpha)^2$:

$$
\begin{aligned}
\frac{(Z\alpha)^2}{\left[\sqrt{(j+1/2)^2-(Z\alpha)^2}+n'\right]^2} &= (Z\alpha)^2\left[\left(j+\frac{1}{2}\right)\left(1-\frac{1}{2}\frac{(Z\alpha)^2}{(j+1/2)^2}\right)+n'\right]^{-2} \\
&= (Z\alpha)^2\left[n-\frac{1}{2}\frac{(Z\alpha)^2}{j+1/2}\right]^{-2} = \frac{(Z\alpha)^2}{n^2}\left[1+\frac{(Z\alpha)^2}{n(j+1/2)}\right]
\end{aligned}
$$

where $n \equiv j+1/2+n'$ as defined in the text. Now on to (8.4.43). Recall that

$$
\begin{aligned}
(1+x)^{-1/2} &= 1-\frac{1}{2}x+\frac{3}{8}x^2+\cdots \qquad \text{and, so} \\
E &= mc^2\left\{1-\frac{1}{2}\frac{(Z\alpha)^2}{n^2}\left[1+\frac{(Z\alpha)^2}{n(j+1/2)}\right]+\frac{3}{8}\frac{(Z\alpha)^4}{n^4}+\cdots\right\} \\
&= mc^2-\frac{mc^2(Z\alpha)^2}{2n^2}-\frac{mc^2(Z\alpha)^4}{2n^2}\left[\frac{1}{n(j+1/2)}-\frac{3}{4n^2}\right]
\end{aligned}
$$

In other words, to this order, the energy is shifted by an amount

$$
\Delta = E_0(Z\alpha)^2\left[\frac{1}{n(j+1/2)}-\frac{3}{4n^2}\right]
$$

where $E_0=-mc^2(Z\alpha)^2/2n^2$ is the energy level to lowest order. Perturbatively, the energy shift is given by the sum of the relativistic correction to the kinetic energy (5.3.10) and the spin-orbit energy (5.3.31). That is, we expect $\Delta=\Delta_{\rm rel}+\Delta_{\rm so}$ where

$$
\begin{aligned}
\Delta_{\rm rel} &= E_0(Z\alpha)^2\left[-\frac{3}{4n^2}+\frac{1}{n(l+1/2)}\right] \\
\text{and} \qquad \Delta_{\rm so} &= -E_0(Z\alpha)^2\frac{1}{2nl(l+1)(l+1/2)}\left\{\begin{array}{cl} l & \text{for} \quad j=l+1/2 \\ -(l+1) & \text{for} \quad j=l-1/2 \end{array}\right.
\end{aligned}
$$

Adding l-dependent terms for $j=l+1/2$ gives

$$
\frac{1}{n(l+1/2)}-\frac{1}{2n(l+1)(l+1/2)}=\frac{1}{2nj}\left[2-\frac{1}{j+1/2}\right]=\frac{1}{n(j+1/2)}
$$

Adding l-dependent terms for $j=l-1/2$ gives

$$
\frac{1}{n(l+1/2)}+\frac{1}{2nl(l+1/2)}=\frac{1}{2n(j+1)}\left[2+\frac{1}{j+1/2}\right]=\frac{1}{n(j+1/2)}
$$

Therefore, for both $j=l\pm 1/2$ we find

$$
\Delta_{\rm rel}+\Delta_{\rm so}=E_0(Z\alpha)^2\left[-\frac{3}{4n^2}+\frac{1}{n(j+1/2)}\right]
$$

in agreement with our second order expansion of the Dirac energy level.

17. For (a) and (b) we can make the comparison using the second order approximation derived in Problem 16. However, we need to take into account that the energies of the two states with the same j but different l are not the same. See the discussion of Lamb Shift, part (d) of this problem. If we just average those two levels in each case, we find (in eV),

n	j_1	j_2	Experiment	Theory
2	1/2	3/2	4.318×10^{-5}	4.528×10^{-5}
4	5/2	7/2	9.4338985×10^{-7}	9.4338739×10^{-7}

Clearly, Dirac's theory does a much better job for the higher lying energy levels. As for the $1S \to 2S$ transition energy, the question is leading. If we tabulate the answer for the Balmer formula and for the Dirac formula using the next order approximation from Problem 16, and then also the exact Dirac formula, we find (in cm^{-1})

Balmer	82302.98684444
Dirac (approx)	82303.99122362
Dirac (exact)	82303.99125026
Experiment	82258.95439928

The Dirac formula makes a small correction to the Balmer formula, but even the exact form for Dirac is (relatively) far from the precise value. Now the Lamb Shift is a direct violation of the Dirac formula, resulting from quantum field effects. It shows up six times in the table, for any two states with the same n and j but different l values. We have (in (in cm^{-1})

n	Splitting	Value
2	SP	0.0353
3	SP	0.0105
3	PD	1.78×10^{-4}
4	SP	0.00444
4	PD	7.631×10^{-5}
4	DF	2.700×10^{-5}

The Lamb Shift gets smaller with increasing n, and apparently very much smaller for higher angular momenta. The moral of the story is that quantum field theory is important for understanding the energy levels of the hydrogen atom, especially for the lower lying ones.

Following is the MATLAB code used to calculate the numbers in this solution:

```
clear all
%Fundamental constants from 2010 PDG
hc=2*pi*197.3269631*1.0E6*1.0E-13; %From h-bar c
alpha=1/137.035999679;
mc2=0.510998910E6;
%
E0n1=-mc2*alpha^2/2;
%
%Analyzes precision hydrogen atomic energy differences
%
load EnljHAtom.dat
n=EnljHAtom(:,1);
l=EnljHAtom(:,2);
j2=2*EnljHAtom(:,3);
EDel=hc*EnljHAtom(:,4);
clear EnljHAtom
%
% Fine structure split in n=2, j=1/2 and 3/2, using 1st order expression
nset=2
DEexpt=EDel(find(n==nset & j2==3))-mean(EDel(find(n==nset & j2==1)))
DEcalc=(E0n1/nset^2)*alpha^2*(1/2-1)/nset
%
% Fine structure split in n=4, j=5/2 and 7/2, using 1st order expression
nset=4
DEexpt=EDel(find(n==nset & j2==7))-mean(EDel(find(n==nset & j2==5)))
DEcalc=(E0n1/nset^2)*alpha^2*(1/4-1/3)/nset
%
% 1S-2S energy difference using balmer, approximate, and exact formulas
format('long')
DE12balmr=E0n1*(1/4-1)/hc
DE12apprx=(E0n1/4)*(1+alpha^2*(1/2-3/16))-E0n1*(1+alpha^2*(1-3/4));
DE12apprx=DE12apprx/hc
DE12exact=mc2*(1/sqrt(1+alpha^2/(sqrt(1-alpha^2)+1)^2)-1/sqrt(1+alpha^2/(1-alpha^2)));
DE12exact=DE12exact/hc
%
% Lamb Shift data
LS2SP=EDel(find(n==2 & j2==1 & l==0))-EDel(find(n==2 & j2==1 & l==1));
LS2SP=LS2SP/hc
%
LS3SP=EDel(find(n==3 & j2==1 & l==0))-EDel(find(n==3 & j2==1 & l==1));
LS3SP=LS3SP/hc
LS3PD=EDel(find(n==3 & j2==3 & l==1))-EDel(find(n==3 & j2==3 & l==2));
LS3PD=LS3PD/hc
%
LS4SP=EDel(find(n==4 & j2==1 & l==0))-EDel(find(n==4 & j2==1 & l==1));
LS4SP=LS4SP/hc
LS4PD=EDel(find(n==4 & j2==3 & l==1))-EDel(find(n==4 & j2==3 & l==2));
LS4PD=LS4PD/hc
LS4DF=EDel(find(n==4 & j2==5 & l==2))-EDel(find(n==4 & j2==5 & l==3));
LS4DF=LS4DF/hc
```