HW: ① use $[a, a^+] = \mathbb{1}$, arrange operator $A = aa^+a^+aa^+$ into a form so that $\hat{a}^+$ are all on the left, and $\hat{a}$ are all on the right, for example $a^+a^+a^+aa + 2a^+a$, etc. before arranging, A has three a^+ and two a, so one more a^+ than a. Does this still hold after arranging?

② evaluate $[a, A]$, $A = aa^+a^+a$.

③ for a Fock state $|n\rangle$, calculate $\langle \Delta x^2\rangle$, $\langle \Delta p^2\rangle$ and does it obey uncertainty principle?

④ $\xi(\lambda) = e^{i\lambda a^+a} a e^{-i\lambda a^+a}$, λ is a number

(4.1) evaluate $\xi'(\lambda)$

(4.2) get a differential equation in the form $\xi'(\lambda) = c\,\xi(\lambda)$, and obtain $\xi(\lambda)$.

recap:

1D harmonic oscillator $\hat{H} = \frac{\hat{p}^2}{2m} + \frac{1}{2}m\omega^2\hat{x}^2$,

define $\hat{a} = \frac{1}{\sqrt{\hbar\omega}}\left(i\frac{\hat{p}}{\sqrt{2m}} + \sqrt{\frac{1}{2}m\omega^2}\,\hat{x}\right)$, annihilation op.

$\hat{a}^+ = \frac{1}{\sqrt{\hbar\omega}}\left(-i\frac{\hat{p}}{\sqrt{2m}} + \sqrt{\frac{1}{2}m\omega^2}\,\hat{x}\right)$, creation op.

get
$$\begin{cases} \hat{H} = \hbar\omega\left(\hat{a}^+\hat{a} + \frac{1}{2}\right) \\ [\hat{a}, \hat{a}^+] = \hat{\mathbb{1}} \\ [\hat{H}, \hat{a}] = -\hbar\omega\hat{a}, \quad [\hat{H}, \hat{a}^+] = \hbar\omega\hat{a}^+ \end{cases}$$

so far haven't specified basis or representation.

$\underline{H|\psi\rangle = E|\psi\rangle}$

$\underline{aH}|\psi\rangle = Ea|\psi\rangle$, $\begin{cases} [\hat{H}, \hat{a}] = -\hbar\omega\hat{a} \\ Ha - aH = -\hbar\omega\hat{a} \Rightarrow aH = Ha + \hbar\omega\hat{a} \end{cases}$

$(Ha + \hbar\omega a)|\psi\rangle = Ea|\psi\rangle$

$\rightarrow H\underbrace{a|\psi\rangle} = (E - \hbar\omega)\underbrace{a|\psi\rangle}$

$\rightarrow H \underbrace{a|\psi\rangle}_{\text{eigenstate}} = (\underset{\uparrow\ \text{eigen energy}}{E-\hbar\omega}) \underbrace{a|\psi\rangle}$

applying $\hat{a}$ on eigenstate $|\psi\rangle$ of energy E
gets another eigenstate with energy $E-\hbar\omega$
$\hat{a}$ annihilation operator

we restrict a bound of the energy
$|u_0\rangle$ so that $\underline{a|u_0\rangle = 0}$
$|u_0\rangle$ is the ground state

$H|u_0\rangle = E_0|u_0\rangle \Rightarrow H = \hbar\omega(a^\dagger a + \frac{1}{2})$

$H|u_0\rangle = \hbar\omega(a^\dagger a + \frac{1}{2})|u_0\rangle = \hbar\omega a^\dagger \underbrace{\cancel{(a|u_0\rangle)}}_{0}{}^{0} + \frac{1}{2}\hbar\omega|u_0\rangle$

$= \frac{1}{2}\hbar\omega|u_0\rangle$

☆ for the ground state, energy is $\frac{1}{2}\hbar\omega$.
"zero-point energy"

- creation operator

$H|\psi\rangle = E|\psi\rangle$

apply $a^\dagger$ on left $\Rightarrow a^\dagger H|\psi\rangle = E a^\dagger|\psi\rangle$.

$[H, a^\dagger] = \hbar\omega a^\dagger \Rightarrow Ha^\dagger - a^\dagger H = \hbar\omega a^\dagger$

$(Ha^\dagger - \hbar\omega a^\dagger)|\psi\rangle = E a^\dagger|\psi\rangle$

$H \underbrace{a^\dagger|\psi\rangle} = (E + \underset{\uparrow\ \text{energy up by } \hbar\omega.}{\hbar\omega}) \underbrace{a^\dagger|\psi\rangle}$

$|u_0\rangle$ ground state.
$|u_1\rangle$ 1st excited state by $\underline{a^\dagger|u_0\rangle}$.

$|u_1\rangle \xrightarrow{\text{normalized}} \frac{1}{\sqrt{C}} a^\dagger|u_0\rangle$

$\langle u_1|u_1\rangle = 1 \Rightarrow \frac{1}{C}\langle u_0|aa^\dagger|u_0\rangle = 1$.

$a|u_0\rangle = 0 \quad [a, a^\dagger] = 1 \Rightarrow aa^\dagger = 1 + a^\dagger a$.

$a|u_0\rangle = 0$, $[a, a^\dagger] = \mathbb{1} \Rightarrow aa^\dagger = \mathbb{1} + a^\dagger a$.

left $= \frac{1}{c}\langle u_0|(\mathbb{1} + a^\dagger a)|u_0\rangle = \frac{1}{c}\left(\langle u_0|u_0\rangle + \cancel{\langle u_0|a^\dagger \underline{a|u_0\rangle}}\right)$, with $\underline{a|u_0\rangle} \to 0$

$= 1 =$ right

$\frac{1}{c}\langle u_0|u_0\rangle = 1 \quad \Rightarrow c = 1$.

$$\boxed{|u_1\rangle = a^\dagger|u_0\rangle}$$

keep applying $a^\dagger$ to get $|u_n\rangle$

$$|u_{n+1}\rangle = \underline{\frac{1}{\sqrt{c_{n+1}}} a^\dagger |u_n\rangle} = \frac{1}{\sqrt{n+1}} a^\dagger |u_n\rangle.$$

$$1 = \langle u_{n+1}|u_{n+1}\rangle = \underline{\frac{1}{c_{n+1}}\langle u_n|aa^\dagger|u_n\rangle}$$

$H|u_n\rangle = E_n|u_n\rangle$, n^{th} state $E_n = \frac{1}{2}\hbar\omega + n\hbar\omega$

$H = \hbar\omega a^\dagger a + \frac{1}{2}\hbar\omega$

left $= \hbar\omega a^\dagger a|u_n\rangle + \frac{1}{2}\hbar\omega|u_n\rangle$

right $= (\frac{1}{2}\hbar\omega + n\hbar\omega)|u_n\rangle$

$$\boxed{a^\dagger a|u_n\rangle = n|u_n\rangle}, \quad \hat{N} \equiv \hat{a}^\dagger \hat{a} \text{ number op.}$$

$$\Rightarrow 1 = \frac{1}{c_{n+1}}\langle u_n|aa^\dagger|u_n\rangle \overset{[a,a^\dagger]=\mathbb{1}}{=\!=} \frac{1}{c_{n+1}}\langle u_n|a^\dagger a + \mathbb{1}|u_n\rangle$$

$$= \frac{1}{c_{n+1}}(n+1) \quad \Rightarrow c_{n+1} = n+1.$$

$$\Rightarrow \boxed{|u_{n+1}\rangle = \frac{1}{\sqrt{n+1}} a^\dagger |u_n\rangle} = \frac{1}{\sqrt{n+1}} a^\dagger\left(\frac{1}{\sqrt{n}} a^\dagger |u_{n-1}\rangle\right)$$

$$\boxed{|u_{n+1}\rangle = \frac{1}{\sqrt{(n+1)!}}(a^\dagger)^n|u_0\rangle} \quad \text{"Fock state"}$$

$$a^\dagger|u_n\rangle = \sqrt{n+1}\,|u_{n+1}\rangle.$$

$$a|u_n\rangle = \sqrt{n}\,|u_{n-1}\rangle$$

- wavefunction in position representation.

ground state $\hat{a}|u_0\rangle = 0$

$$\hat{a} = \frac{1}{\sqrt{\hbar\omega}}\left(\frac{\hbar}{\sqrt{2m}}\frac{\partial}{\partial x} + \sqrt{\tfrac{1}{2}m\omega}\, x\right), \quad \langle x|u_0\rangle = u_0(x)$$

$$\frac{\hbar}{\sqrt{2m}}\frac{\partial}{\partial x}u_0(x) + \sqrt{\tfrac{1}{2}m\omega}\, x\, u_0(x) = 0.$$

$$\Rightarrow \quad u_0(x) = C\, e^{-\frac{1}{2}m\omega x^2/\hbar}$$

$$\frac{\partial}{\partial x}e^{f(x)} = f'(x)\, e^{f(x)}.$$

normalization $\int_{-\infty}^{+\infty} dx\, u_0^*(x)\, u_0(x) = 1$

$$\Rightarrow 1 = |C|^2 \int_{-\infty}^{+\infty} dx\, e^{-m\omega x^2/\hbar} \xlongequal{\int_{-\infty}^{+\infty} dx\, e^{-x^2} = \sqrt{\pi}} |C|^2 \sqrt{\frac{\hbar}{m\omega}}\sqrt{\pi}$$

define $x' = \sqrt{\frac{m\omega}{\hbar}}\, x$

$$\int_{-\infty}^{+\infty} dx\, e^{-m\omega x^2/\hbar} = \int_{-\infty}^{+\infty} \frac{1}{\sqrt{\frac{m\omega}{\hbar}}}\, dx'\, e^{-x'^2}$$

$$\Rightarrow \quad C = \left(\frac{m\omega}{\pi\hbar}\right)^{\frac{1}{4}}$$

ground state wavefunction

$$u_0(x) = \left(\frac{m\omega}{\pi\hbar}\right)^{\frac{1}{4}} e^{-\frac{1}{2}m\omega x^2/\hbar}.$$

$$u_n(x) = \langle x|u_n\rangle \xlongequal{|u_n\rangle = \frac{1}{\sqrt{n!}}(a^\dagger)^n|u_0\rangle} \frac{1}{\sqrt{n!}}\langle x|(a^\dagger)^n|u_0\rangle$$

$$= \frac{1}{\sqrt{n!}}\left(-\frac{\hbar}{\sqrt{2m}}\frac{\partial}{\partial x} + \sqrt{\tfrac{1}{2}m\omega}\, x\right)^n u_0(x)$$

sketch:

∘ matrix representation of $\hat{a}$

$\{|u_n\rangle\}$ basis.

$\{|u_n\rangle\}$ basis.

$$\begin{cases} \hat{a}|u_n\rangle = \sqrt{n}\,|u_{n-1}\rangle \\ \hat{a}|u_0\rangle = 0 \end{cases} \Rightarrow \begin{array}{l} \langle u_{n-1}|a|u_n\rangle = \sqrt{n} \\ \langle u_n|\underbrace{a|u_n\rangle}_{\sqrt{n}|u_{n-1}\rangle} = 0 \end{array}$$

$$\Rightarrow \langle u_m|a|u_n\rangle = \begin{cases} 0 \\ \sqrt{n}, \text{ if } m=n-1. \end{cases}$$

a in basis of $\{|u_n\rangle\}$

$$\hat{a} \longrightarrow \begin{pmatrix} 0 & \sqrt{1} & & \\ & 0 & \sqrt{2} & \\ & & & \ddots \end{pmatrix}$$

$$|u_0\rangle \rightarrow \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \end{pmatrix}, \quad |u_1\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \end{pmatrix} \Rightarrow a|u_0\rangle \rightarrow \begin{pmatrix} 0 & \sqrt{1} & & \\ & 0 & \sqrt{2} & \\ & & & \ddots \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

$$a|u_1\rangle = \begin{pmatrix} 0 & \sqrt{1} & & \\ & 0 & \sqrt{2} & \\ & & & \ddots \end{pmatrix}\begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \end{pmatrix}$$

- What if we solved Schrödinger's equation directly. (not in quiz or exam)

$$H = \frac{P^2}{2m} + \frac{1}{2}m\omega^2 x^2 \xrightarrow{P=-i\hbar\frac{\partial}{\partial x}} \left(-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + \frac{1}{2}m\omega^2 x^2\right)\psi(x) = E\psi(x)$$

$$\epsilon = \frac{2mE}{\hbar^2}, \quad y = \left(\frac{m\omega}{\hbar}\right)^{\frac{1}{2}} x \Leftarrow y \text{ unitless 无量纲化}$$

$$\Rightarrow \frac{\partial^2}{\partial y^2}u(y) + (\epsilon - y^2)u(y) = 0.$$

Step 1: we need $u(y) \to 0$ at $y = \pm\infty$

at $y \gg \epsilon \Rightarrow$ throw away ϵ in above equation

$$\underline{\frac{\partial^2}{\partial y^2}u(y) - y^2 u(y) = 0} \Rightarrow u(y) \overset{y\gg\epsilon}{\simeq} e^{-\frac{1}{2}y^2}$$

$$\frac{\partial}{\partial y} e^{-\frac{1}{2}y^2} = (-y) e^{-\frac{1}{2}y^2}$$

$$\frac{\partial^2}{\partial y^2} e^{-\frac{1}{2}y^2} = \frac{\partial}{\partial y}\left(-y e^{-\frac{1}{2}y^2}\right) = y^2 e^{-\frac{1}{2}y^2} - e^{-\frac{1}{2}y^2}$$

$$\simeq y^2 e^{-\frac{1}{2}y^2} \qquad h(y) = \sum_{m=-\infty}^{+\infty} a_m y^m.$$

asymptoptic behavior 渐近解.

step 2 $\quad u(y) = h(y) e^{-\frac{1}{2}y^2}$

know something about $h(y)$

is that at $y \to \pm\infty$ $h(y)$ behave nicely.

plug $u(y) = h(y) e^{-\frac{1}{2}y^2}$ into $\frac{\partial^2}{\partial y^2} u(y) + (\epsilon - y^2) u(y) = 0$

$$\Rightarrow \quad h''(y) - 2y h'(y) + (\epsilon - 1) h(y) = 0$$

$$h(y) = y^s \sum_{m=0}^{\infty} a_m y^m \Leftarrow \text{polynomial with lowest order } y^s$$

also we have $a_M = 0$ so that this polynomial is bounded.

at $y \to 0$ we don't have $u(y) \to \infty$.

so this is bounded.

to obtain iterative condition of a_m.

$$0 = \sum_{m=0}^{\infty} a_m \left\{ (m+s)(m+s-1) y^{m+s-2} + [-2(m+s) + (\epsilon - 1)] y^{m+s} \right\}$$

needs to be true for all y

we have to compare the orders

$$\Rightarrow \sum^{\infty} a_{m'} \quad (m'+s+2)(m'+s+1) y^{m'+s}$$

we have to compare the terms

$$\Rightarrow \sum_{m'=-2}^{\infty} a_{m'+2}(m'+s+2)(m'+s+1)y^{m'+s}$$
$$+\sum_{m=0}^{\infty} a_m[-2(m+s)+\epsilon-1]y^{m+s} = 0$$

$\Rightarrow m' = -2: \quad a_0 s(s-1) = 0 \Rightarrow \begin{cases} s=0 \\ s=1 \end{cases}$

for other m' we have

$$a_{m+2}(m+s+2)(m+s+1) + a_m[-2(m+s)+\epsilon-1] = 0$$

$$a_{m+2} = a_m \frac{2(m+s)+1-\epsilon}{(m+s+2)(m+s+1)}.$$ even/odd series

$m\to\infty$ we have $m \gg s \Rightarrow \frac{a_{m+2}}{a_m} \xrightarrow{m\to\infty} \frac{2}{m}$.

if we compare $e^{y^2} = \sum_{k=0}^{\infty} \frac{(y^2)^k}{k!} = \sum_{k=0}^{\infty} b_k y^{2k}$

between y^{2k+2} and y^{2k} we have

$$\frac{b_{k+1}}{b_k} = \frac{1}{k}$$

we can conclude the series is not going to converge, and behave like e^{y^2} at $m\to\infty$.

$$u(y) = h(y)e^{-\frac{1}{2}y^2} \xrightarrow[y\to\infty]{h(y)\simeq e^{y^2}} e^{y^2} \text{ at } y\to\infty.$$

the way out is that we require

$$\boxed{2(M+s)+1-\epsilon = 0 \quad \text{at some } m=M.}$$

then the entire series is bounded being a finite polynomial.

$\epsilon = \frac{2E}{\ldots}$, s is a number, and M is

$\epsilon = \frac{2E}{\hbar\omega}$, s is a number, and M is also a number.

as we have before $S = 0$ or 1, integer 整数.

M is also an integer.

$$\epsilon = 2(M+s)+1 \overset{N = M+s}{=\!=} 2N+1$$

N is integer $N \geq 0$

$$\Rightarrow \quad E = \hbar\omega\left(N+\frac{1}{2}\right)$$

§2 spin-$\frac{1}{2}$ system.

for S-G experiment

$$\hat{H} = -\frac{q}{m}\vec{S}\cdot\vec{B}$$

make analogy to classical magnet with moment $\vec{\mu}$. $H = -\vec{\mu}\cdot\vec{B}$, $\vec{B}$ magnetic field.

$\frac{q\vec{S}}{m}$ is the magnetic moment for a fundamental particle. — spin.

in analogy to classical current loop.

$$\vec{S} = \frac{1}{2}\hbar(\sigma_x, \sigma_y, \sigma_z) = \frac{1}{2}\hbar\vec{\sigma}$$

$(\ , \ , \ )$ is vector in 3-dimensional.

so $\vec{S}$ is a vector-like operator.

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

this is in σ_z basis.

this is in σ_z basis.

$$\boxed{\sigma_x\sigma_y = i\sigma_z, \quad \sigma_y\sigma_z = i\sigma_x, \quad \sigma_z\sigma_x = i\sigma_y}$$

important. in exams.

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\cdot\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} = i\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\sigma_k\sigma_l = i\,\varepsilon_{klm}\sigma_m, \quad \varepsilon_{klm} = \begin{cases} 1, & \text{if } k,l,m \text{ in order } x,y,z \\ -1, & \text{other cases.} \end{cases}$$

$$\sigma_y\sigma_x = -i\sigma_z, \quad \sigma_z\sigma_y = -i\sigma_x, \quad \sigma_x\sigma_z = -i\sigma_y$$

$$\hat{H} = -\frac{q}{m}\vec{S}\cdot\vec{B}, \quad \begin{cases} q \text{ is the charge} \\ m \text{ is mass.} \end{cases}$$

let $\vec{B} = B\cdot\hat{z}$ ← direction. not operator.

$$\hat{H} = -\underbrace{\frac{q\hbar}{2m}B}_{\text{number}}\sigma_z \equiv \frac{1}{2}\hbar\omega_0\,\sigma_z$$

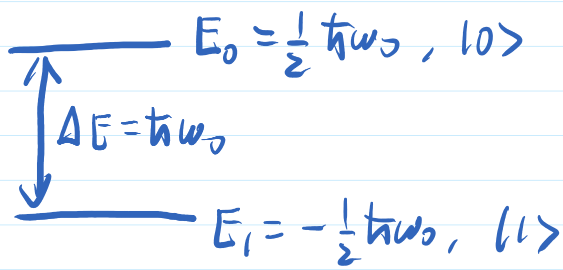
eigenvalue and eigenstate of σ_z

$+1, \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $-1, \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

for $\hat{H}$ again we have eigenenergy $\pm\frac{1}{2}\hbar\omega_0$

eigenstate $|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \end{pmatrix} = |1\rangle$

energy diagram



since we have $\vec{S}\cdot\vec{B}$, so for a general vector of $\vec{B} = B\vec{n} = B(\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta)$

in x, y, z coordinate.

$$\vec{\sigma}\cdot\vec{n} = \sigma_x \sin\theta\cos\varphi + \sigma_y \sin\theta\sin\varphi + \sigma_z \cos\varphi.$$

- Schrödinger's equation

$$\hat{H} = \frac{1}{2}\hbar\omega_0 \sigma_z$$

$$H|\psi\rangle = i\hbar\frac{\partial}{\partial t}|\psi\rangle$$

we ask for a $|\psi(t=0)\rangle$, what is $|\psi(t)\rangle$. $\Longleftarrow$ dynamics.

① express $|\psi(t)\rangle$ in the energy/eigen basis of H, in general, we have for discrete case $\{|u_n\rangle\}$

$$|\psi(t)\rangle = \sum_n C_n(t)\,|u_n\rangle.$$

If we have $H \neq H(t)$ time independent so that the basis remains the same.

$t=0$, $|\psi(0)\rangle = \sum_n C_n |u_n\rangle$

$$|\psi(t)\rangle = \sum_n C_n e^{-iHt/\hbar}|u_n\rangle$$

$$= \sum_n C_n e^{-iE_nt/\hbar}|u_n\rangle.$$

$$\Rightarrow \boxed{C_n(t) = C_n e^{-iE_nt/\hbar}}$$

② second way, we express H, $|\psi\rangle$ in some arbitrary basis.
then we solve differential equation

directly.

example. $\hat{H} = \frac{1}{2}\hbar\omega_0 \sigma_z$.

work in the eigenbasis of σ_z $|0\rangle = \begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix} = |1\rangle$

$$|\psi(t)\rangle = c_0(t)|0\rangle + c_1(t)|1\rangle$$
$$= \begin{pmatrix} c_0(t) \\ c_1(t) \end{pmatrix}$$

$$i\hbar \frac{\partial}{\partial t}|\psi(t)\rangle = H|\psi(t)\rangle, \quad H = \begin{pmatrix} \frac{1}{2}\hbar\omega_0 & 0 \\ 0 & -\frac{1}{2}\hbar\omega_0 \end{pmatrix}$$

$$i\hbar \begin{pmatrix} \dot{c}_0(t) \\ \dot{c}_1(t) \end{pmatrix} = \begin{pmatrix} \frac{1}{2}\hbar\omega_0 & 0 \\ 0 & -\frac{1}{2}\hbar\omega_0 \end{pmatrix} \begin{pmatrix} c_0(t) \\ c_1(t) \end{pmatrix}$$

$\dot{g}(t) = \frac{\partial}{\partial t} g(t)$

$$= \begin{pmatrix} \frac{1}{2}\hbar\omega_0 c_0(t) \\ -\frac{1}{2}\hbar\omega_0 c_1(t) \end{pmatrix}$$

$$\Rightarrow \begin{cases} i\hbar \dot{c}_0(t) = \frac{1}{2}\hbar\omega_0 c_0(t) \\ i\hbar \dot{c}_1(t) = -\frac{1}{2}\hbar\omega_0 c_1(t). \end{cases} \Rightarrow \begin{cases} c_0(t) = e^{-i\frac{1}{2}\hbar\omega_0 t/\hbar} c_0 \\ c_1(t) = e^{i\frac{1}{2}\hbar\omega_0 t/\hbar} c_1 \end{cases}$$

$$c_0(t) = c_0 e^{-i\frac{1}{2}\omega_0 t}, \quad c_1(t) = c_1 e^{i\frac{1}{2}\omega_0 t}$$

consistent with above $c_n(t) = c_n(t=0) e^{-iE_n t/\hbar}$

$$|\psi(t)\rangle = \begin{pmatrix} c_0 e^{-iE_0 t/\hbar} \\ c_1 e^{-iE_1 t/\hbar} \end{pmatrix} \text{ in } \sigma_z \text{ basis.}$$

③ time evolution operator

$$U(t) = e^{-iHt/\hbar} = \mathbb{1} - \frac{i\hat{H}t}{\hbar} + \frac{1}{2!}\left(-\frac{i\hat{H}t}{\hbar}\right)^2 + \cdots$$

example $\hat{H} = \frac{1}{2}\hbar\omega_0 \sigma_z$

example $\hat{H} = \frac{1}{2}\hbar\omega_0 \sigma_z$

$$U(t)\,|\psi(t=0)\rangle = |\psi(t)\rangle$$

what is $U(t)$?

$$U(t) = \mathbb{1} - i\frac{\frac{1}{2}\hbar\omega_0\sigma_z t}{\hbar} + \frac{1}{2!}\left(-i\frac{\frac{1}{2}\hbar\omega_0\hat{\sigma}_z t}{\hbar}\right)^2 + \cdots$$

$$\sigma_z^2 = \mathbb{1}, \quad \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \mathbb{1}.$$

$$\boxed{\sigma_x^2 = \sigma_y^2 = \sigma_z^2 = \mathbb{1}}$$ ☆ important.

$$U(t) = \mathbb{1} - i\left(\frac{1}{2}\omega_0 t\right)\sigma_z + \frac{1}{2!}\left(\frac{i\omega_0 t}{2}\right)^2 \cancel{\hat{\sigma}_z^2}^{\,\mathbb{1}} + \cdots$$

$$= \mathbb{1}\cdot\cos\frac{\omega_0 t}{2} - i\hat{\sigma}_z \sin\frac{\omega_0 t}{2}$$

$$= \begin{pmatrix} e^{-i\frac{\omega_0 t}{2}} & 0 \\ 0 & e^{+i\frac{\omega_0 t}{2}} \end{pmatrix}$$

$$|\psi(t)\rangle = U(t)\,|\psi(t=0)\rangle = \begin{pmatrix} e^{-i\frac{\omega_0 t}{2}} & 0 \\ 0 & e^{i\frac{\omega_0 t}{2}} \end{pmatrix}\begin{pmatrix} c_0 \\ c_1 \end{pmatrix}$$

$$= \begin{pmatrix} c_0 e^{-i\frac{\omega_0 t}{2}} \\ c_1 e^{i\frac{\omega_0 t}{2}} \end{pmatrix} \leftarrow \text{consistent as above.}$$