20190416, Ch3 Angular Momentum

Tuesday, April 9, 2019 6:10 PM

HW ① with $\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$, $\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$,

and $\vec{n} = (\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta)$, $\vec{\sigma} = (\sigma_x, \sigma_y, \sigma_z)$

find eigenvalue and eigenstates of $\vec{\sigma}\cdot\vec{n}$

② with $\psi(\theta,\varphi) = \frac{1}{\sqrt{3}}\left(\sqrt{2}\, Y_1^0(\theta,\varphi) + Y_1^1(\theta,\varphi)\right)$, without integrations,
find $\langle \vec{L}^2 \rangle$, $\langle L_z \rangle$

③ with $\phi(t=0) = \psi(\theta,\varphi)$ as above, $H = \frac{\vec{L}^2}{2mR^2}$,
find $\phi(t=T)$

④ uncertainty principle.
with $J_z|j,m\rangle = m\hbar|j,m\rangle$, $\vec{J}^2|j,m\rangle = \hbar^2 j(j+1)|j,m\rangle$

$$\langle \Delta A \rangle = \sqrt{\langle l,m|A^2|l,m\rangle - (\langle l,m|A|l,m\rangle)^2}.$$

find $\langle \Delta J_x \rangle \langle \Delta J_y \rangle$ and $\langle [J_x, J_y] \rangle$, check $\langle \Delta J_x \rangle \langle \Delta J_y \rangle \geq \frac{1}{2}|\langle [J_x, J_y] \rangle|$
when $\langle \Delta J_x \rangle \langle \Delta J_y \rangle = \frac{1}{2}|\langle [J_x, J_y] \rangle|$, what's the requirments of m?

⑤ write out $J_- J_+$ as a matrix in the basis of J_z, with $j=1$, and
find the eigenvalue and eigenstates of $J_- J_+$.

Homework Q & A Sunday 3PM with Mr. Zhang
Quiz II 4/30 in class.

Sakurai P157-172, 178-185, 191-203.

motivation.

position, momentum. ⟵⟶ Linear.
rotation? ⟶ angular momentum
angular position → angle.

rotation ⟵⟶ spin
coupled spin system.

- Some prerequisites (预修)

3D - system.

→

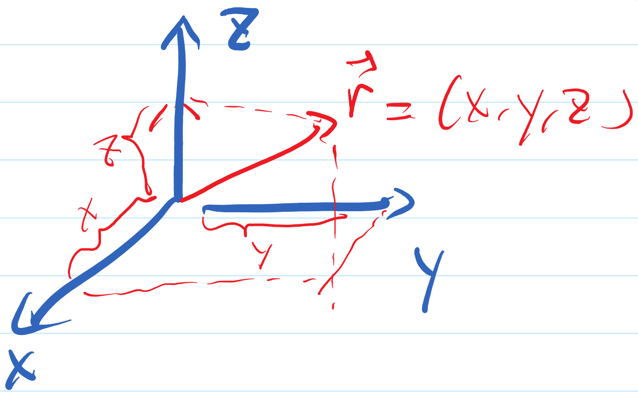
position in 3D $r = (x, y, z) \to$ coordinates.

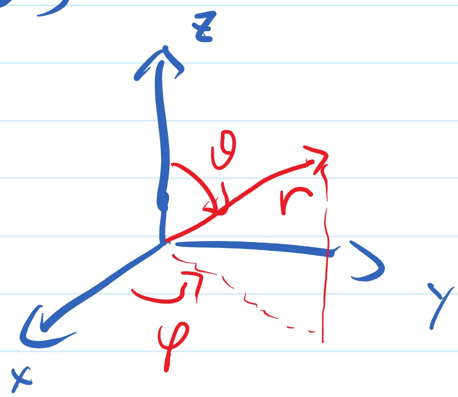
momentum $\vec{p} = (p_x, p_y, p_z)$

$p_x \leftrightarrow -i\hbar \frac{\partial}{\partial x} \implies \vec{p} = (-i\hbar\frac{\partial}{\partial x}, -i\hbar\frac{\partial}{\partial y}, -i\hbar\frac{\partial}{\partial z})$

$\vec{\nabla} \equiv (\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}) \Rightarrow \vec{p} \equiv -i\hbar\vec{\nabla}$

- spherical coordinate. (r, θ, φ)





comments.

$(x, y, z) \longrightarrow$ all components in length unit

$(r, \theta, \varphi) \longrightarrow$ "r" in length unit

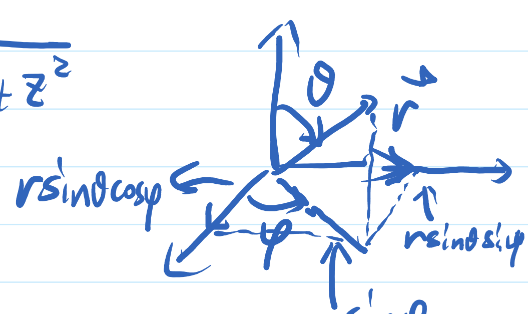
"θ", "φ" unit less or in radient in angle unit.

for a point in space, we use a ket $|\psi\rangle$ to represent it.

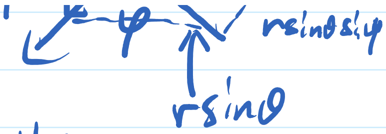
in spherical coordinate, we have r, θ, φ for this point

but we can also use x, y, z for the same point

$$\begin{cases} x = r\sin\theta\cos\varphi \\ y = r\sin\theta\sin\varphi \\ z = r\cos\theta \end{cases} \Rightarrow r = \sqrt{x^2+y^2+z^2}$$



- rotation along $\hat{z}$ direction



- rotation along $\hat{z}$ direction

$\longrightarrow \varphi \to \varphi + d\varphi$, and r, θ remain the same.

$|x, y, z\rangle \xrightarrow{\text{rotation}} |x', y', z'\rangle$

$(r, \theta, \varphi) \qquad\qquad |r, \theta, \varphi + d\varphi\rangle$

what are x', y', z' in terms of $r, \theta, \varphi, d\varphi$.

$$\begin{cases} x' = r\sin\theta\cos(\varphi + d\varphi) \simeq \underline{r\sin\theta\cos\varphi} - \underline{r\sin\theta\sin\varphi}\, d\varphi \\ y' = r\sin\theta\sin(\varphi + d\varphi) \simeq r\sin\theta\sin\varphi + r\sin\theta\cos\varphi\, d\varphi \\ z' = r\cos\theta = z \end{cases}$$

$d\varphi \ll 1$

$\cos(\varphi + d\varphi) = \cos\varphi\cos d\varphi - \sin\varphi\sin d\varphi \simeq \cos\varphi - (\sin\varphi)d\varphi$

$\sin(\varphi + d\varphi) = \sin\varphi\cos d\varphi + \cos\varphi\sin d\varphi \simeq \sin\varphi + \cos\varphi\, d\varphi$

$$\Rightarrow \begin{cases} x' \simeq x - y\,d\varphi \\ y' \simeq y + x\,d\varphi \\ z' = z \end{cases}$$

- prerequisites on spin-$\frac{1}{2}$ system ☆

Pauli operators $\sigma_x, \sigma_y, \sigma_z$

in the σ_z basis, $\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$, $\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

$[\sigma_k, \sigma_l] = 2i\,\varepsilon_{klm}\sigma_m$, $\quad \varepsilon_{klm} = \begin{cases} 1, & \text{if } k, l, m \text{ in order} \\ -1, & \text{if out of order} \end{cases}$

define spin operator $\begin{cases} \vec{S} = \frac{\hbar}{2}\vec{\sigma}, \quad S_z = \frac{\hbar}{2}\sigma_z \\ \vec{\sigma} = (\sigma_x, \sigma_y, \sigma_z) \end{cases}$

$\Rightarrow \boxed{[S_k, S_l] = i\hbar\,\varepsilon_{klm} S_m}$ $\longleftarrow$ generally true for angular momentum operator including spin-$\frac{1}{2}$ operators.

$\vec{S}^2 = S_x^2 + S_y^2 + S_z^2 = \frac{\hbar^2}{4}(\sigma_x^2 + \sigma_y^2 + \sigma_z^2) = \hbar^2 \cdot \frac{3}{4}\mathbb{1}$

$S^2 = S_x^2 + S_y^2 + S_z^2 = \frac{\hbar^2}{4}(\sigma_x^2 + \sigma_y^2 + \sigma_z^2) = \hbar^2 \frac{3}{4} \mathbb{1}$

for each k in x, y, z, $\boxed{\sigma_k^2 = \mathbb{1}}$

$\vec{S}^2 \propto \mathbb{1}$

$$\boxed{[\vec{S}^2, S_k] = 0}$$

if we define unit length vector

$\vec{n} = (\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta)$

we have

$$\vec{\sigma}\cdot\vec{n} = \sigma_x \sin\theta\cos\varphi + \sigma_y \sin\theta\sin\varphi + \sigma_z\cos\theta.$$

$$e^{i\phi\hat{\sigma}_x} = \mathbb{1} + i\phi\sigma_x - \frac{\phi^2\sigma_x^2}{2} + i\frac{\phi^3\sigma_x^3}{3!} + \cdots$$

$$\begin{cases} e^{\hat{A}} = \mathbb{1} + \hat{A} + \frac{\hat{A}^2}{2!} + \cdots + \frac{\hat{A}^n}{n!} + \cdots \\ \sigma_x^2 = \mathbb{1} \end{cases}$$

cube

$$\Rightarrow e^{i\phi\hat{\sigma}_x} = \left(\mathbb{1} - \frac{\phi^2}{2!} + \frac{\phi^4}{4!} + \cdots\right) + i\sigma_x\left(\phi - \frac{\phi^3}{3!} + \frac{\phi^5}{5!} + \cdots\right)$$

$$= \cos\phi + i\sin\phi\,\sigma_x$$

factorial

§1. orbital angular momentum.

for a small rotation $\varphi \to \varphi + d\varphi$: translation 位移 of angle φ

$$\begin{cases} x' \simeq x - y\,d\varphi \\ y' \simeq y + x\,d\varphi \\ z' = z \end{cases}$$

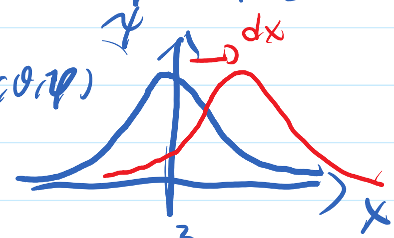
if we have a state $|\psi\rangle$ expressed in position coordinate $\langle x, y, z|\psi\rangle = \psi(x, y, z)$

apply a rotation to the state. $\psi(x, y, z)$

which means in spherical basis

we get from $\psi(r,\theta,\varphi) \longrightarrow \psi(r,\theta,\varphi - d\varphi)$

$$\psi(r,\theta,\underline{\varphi - d\varphi}) \simeq \psi(r,\theta,\varphi) - d\varphi \cdot \frac{\partial}{\partial\varphi}\psi(r,\theta,\varphi)$$



we can express this small rotation as $\mathbb{1} - \frac{\partial}{\partial\varphi} = \mathbb{1} + \frac{1}{i\hbar}\hat{L}_z$

e^{-x^2}

$e^{-(x-dx)^2}$

orbital angular momentum $\hat{L}_z \xrightarrow{\varphi \text{ coordinat}} -i\hbar\frac{\partial}{\partial\varphi} \longleftrightarrow P_x \leftrightarrow -i\hbar\frac{\partial}{\partial x}$

- express $\hat{L}_z$ in x, y, z coordinates.

$$\psi(x,y,z) \xrightarrow{\mathbb{1} - \frac{\partial}{\partial\varphi}} \psi(x',y',z')$$

rotate axis along z by $d\varphi$

$$\begin{cases} x' = x - y\,d\varphi \\ y' = y + x\,d\varphi \\ z' = z \end{cases} \xrightarrow{\text{want } -d\varphi} \begin{cases} x' = x + y\,d\varphi \\ y' = y - x\,d\varphi \\ z' = z \end{cases}$$

$$\psi(x',y',z') \simeq \psi(x + y\,d\varphi,\ y \underline{- x\,d\varphi},\ z)$$

$$= \psi(x,y,z) + y\,d\varphi\frac{\partial}{\partial x}\psi(x,y,z) - x\,d\varphi\frac{\partial}{\partial y}\psi(x,y,z)$$

$$\left(\begin{array}{l} \text{for 1D} \quad \psi(x+dx) \simeq \psi(x) + dx\frac{\partial}{\partial x}\psi(x) \\ \quad \psi(x+dx, y) \simeq \psi(x,y) + dx\frac{\partial}{\partial x}\psi(x,y) \longleftarrow \text{calculus 高数/微积分} \end{array} \right)$$

$P_x \leftrightarrow -i\hbar\frac{\partial}{\partial x}$, $P_y \leftrightarrow -i\hbar\frac{\partial}{\partial y}$

$$\psi(x',y',z') = \psi(x,y,z) + d\varphi\, y\left(\frac{\hat{P}_x}{-i\hbar}\right)\psi(x,y,z) - d\varphi\, x\left(\frac{\hat{P}_y}{-i\hbar}\right)\psi(x,y,z)$$

$$= \left(\hat{\mathbb{1}} + \frac{1}{(-i\hbar)}d\varphi\,(yP_x - xP_y)\right)\psi(x,y,z)$$

$$= \left(\mathbb{1} + \frac{1}{i\hbar}\hat{L}_z\right)\psi(x,y,z)$$

$\Rightarrow$ $\boxed{\hat{L}_z = \hat{x}\hat{P}_y - \hat{y}\hat{P}_x}$ $\longleftarrow$ $\hat{L}_z$ expressed with x, y, z [illegible]

$$\Rightarrow \hat{L}_z \quad \boxed{\hat{L}_z = \hat{x}\hat{P}_y - \hat{y}\hat{P}_x} \longleftarrow$$ $\hat{L}_z$ expressed with x, y, z coordinate with respected operators.

☆

$$\hat{L}_x = \hat{y}\hat{P}_z - \hat{z}\hat{P}_y, \quad \hat{L}_y = \hat{z}\hat{P}_x - \hat{x}\hat{P}_z$$

$$\vec{L} = \underline{\vec{r}\times\vec{p}} \longrightarrow \text{same as classical}$$

$$= \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ x & y & z \\ P_x & P_y & P_z \end{vmatrix} = (yP_z - zP_y)\hat{x} + \cdots$$

we get an orbital angular momentum operator in analogy to classical

$L_k = r_l P_m - r_m P_l$, k, l, m in order

$$\begin{cases} L_z = -i\hbar \dfrac{\partial}{\partial\varphi} \\ L_y = zP_x - xP_z = i\hbar\left(-\cos\varphi \dfrac{\partial}{\partial\theta} + \cot\theta \sin\varphi \dfrac{\partial}{\partial\varphi}\right) \\ L_x = i\hbar\left(\sin\varphi \dfrac{\partial}{\partial\varphi} + \cot\theta\cos\varphi \dfrac{\partial}{\partial\varphi}\right) \end{cases}$$

not in the exam/quiz

can be derived using $P_y \leftrightarrow -i\hbar\frac{\partial}{\partial y}$

$y = r\sin\theta\sin\varphi$ etc.

- some properties of orbital angular momentum

① $\underline{[L_x, L_y] = i\hbar L_z} \longleftarrow [S_x, S_y] = i\hbar S_z$

proof: $[yP_z - zP_y, \; zP_x - xP_z]$

$$= [yP_z, zP_x] + [yP_z, \cancel{-xP_z}] + [-zP_y, zP_x] + [-zP_y, -xP_z]$$

$[x, y] = 0$, $[P_x, P_y] = 0$ etc.

$$= [yP_z, zP_x] + [-zP_y, -xP_z]$$

$$= [yP_z, zP_x] + [-zP_y, -xP_z]$$

$$= yP_x[P_z, z] + P_y x [z, P_z]$$

$$= i\hbar (xP_y - yP_x) = i\hbar L_z$$

$$[yP_z, zP_x] = yP_z zP_x - zP_x yP_z = yP_x P_z z - P_x y z P_z$$

$$= yP_x P_z z - yP_x zP_z$$

$$= yP_x [P_z, z]$$

$$\boxed{[L_k, L_l] = i\hbar \varepsilon_{klm} L_m}$$

$$[L_y, L_x] = -i\hbar L_z$$

② $\boxed{[\vec{L}^2, L_i] = 0}$ ← similar to $[\vec{S}^2, S_i] = 0$

proof: $\vec{L}^2 = L_x^2 + L_y^2 + L_z^2$

$$[A^2, B] = A[A, B] + [A, B]A \checkmark$$

$$A^2B - BA^2 = A^2B - ABA + ABA - BA^2$$

$$= A[A, B] + [A, B]A.$$

$$[\vec{L}^2, L_x] = [L_x^2 + L_y^2 + L_z^2, L_x] = [L_y^2, L_x] + [L_z^2, L_x]$$

$$= L_y[L_y, L_x] + [L_y, L_x]L_y + L_z[L_z, L_x] + [L_z, L_x]L_z$$

$[L_y, L_x] = -i\hbar L_z$

$$= -i\hbar \underline{L_y L_z} - i\hbar \underline{L_z L_y} + i\hbar \underline{L_z L_y} + i\hbar \underline{L_y L_z}$$

$[L_z, L_x] = i\hbar L_y$

$$= 0$$

comment.
$$\begin{cases} [L_k, L_l] = i\hbar \varepsilon_{klm} L_m \\ [\vec{L}^2, L_i] = 0 \end{cases}$$
← independent of representation.

representation.

- eigen equation with $\vec{L}^2$ and L_z. ⟸ useful in solving S-eq with azimuthal symmetry.

because $[\vec{L}^2, L_z] = 0$.

if $\hat{A}, \hat{B}$ are Hermitian, and $[\hat{A}, \hat{B}] = 0$
then there exists $\{|\psi\rangle\}$ giving $\begin{cases} A|\psi\rangle = a|\psi\rangle \\ B|\psi\rangle = b|\psi\rangle \end{cases}$
or $\{|\psi\rangle\}$ is the mutual eigenbasis / eigenstate.
共同的

example $[\mathbb{1}, \sigma_x] = 0$, then we can find
$|+\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$, $|-\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ to be eigenstate of both $\hat{\mathbb{1}}, \hat{\sigma}_x$

$$\begin{cases} \mathbb{1}|+\rangle = |+\rangle, & \mathbb{1}|-\rangle = |-\rangle \\ \sigma_x|+\rangle = |+\rangle, & \sigma_x|-\rangle = -|-\rangle \end{cases}$$

let $|y\rangle$ to be one these states

so $\begin{cases} L_z|y\rangle = m\hbar\,|y\rangle \\ \vec{L}^2|y\rangle = \beta\hbar^2|y\rangle \end{cases}$, $\begin{cases} m \text{ is a number} \\ \beta \end{cases}$

- $|y\rangle$ needs to be normalized. $\langle y|y\rangle = 1$

$$\beta\hbar^2 = \langle y|\vec{L}^2|y\rangle = \langle y|L_x^2 + L_y^2 + L_z^2|y\rangle$$
$$\geq \langle y|L_z^2|y\rangle = m^2\hbar^2$$

$$\Rightarrow \boxed{\beta \geq m^2}$$

in spherical coordinate, as above we can express $\vec{L}^2$, L_z all in just θ, φ without r
$\vec{L}^2$, L_z only relates to angular coordinates ☆

$\vec{L}^2$, L_z only relates to angular coordinates ☆

$\{|\theta,\varphi\rangle\}$ as the angular representation.

$\langle\theta,\varphi|Y\rangle = Y(\theta,\varphi) \longleftrightarrow \langle x|\psi\rangle = \psi(x)$

as a wavefunction.

$\langle\theta,\varphi|L_z|Y\rangle = m\hbar\langle\theta,\varphi|Y\rangle$

$$= -i\hbar\frac{\partial}{\partial\varphi}\langle\theta,\varphi|Y\rangle$$

$\Rightarrow -i\hbar\frac{\partial}{\partial\varphi}Y(\theta,\varphi) = m\hbar Y(\theta,\varphi) \Rightarrow Y(\theta,\varphi) \propto e^{im\varphi}$

proportional 正比例

$\varphi \to \varphi + 2\pi$ should give the same wavefunction

so we have $e^{im\varphi} = e^{im(\varphi+2\pi)} \Rightarrow e^{i2\pi m} = 1$

$\Rightarrow m$ to be an integer $m = 0, \pm 1, \pm 2, \cdots$

complex analysis (intro)

$\alpha = |\alpha|e^{i\phi}$

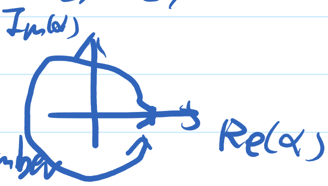
$= |\alpha|\cos\phi + i|\alpha|\sin\phi$

$\alpha' = |\alpha|e^{i(\phi+2\pi)} = |\alpha|e^{i\phi}$

a rotation of 2π is back to itself.

α is a complex number

复数



if $\alpha = 1$, $\alpha = 1\cdot e^{i\cdot 0}$

here we have $1 = e^{i(2\pi m)} = \cos 2\pi m + i\sin 2\pi m$

$1 = e^{i2\pi m} = e^{i2\pi(M+\delta)} = e^{i2\pi\delta} \Rightarrow \delta = 0$

M: integer, $\delta \in [0,1)$; $\cos 2\pi m$: 1, $\sin 2\pi m$: 0

$$\boxed{L_z|Y\rangle = m\hbar|Y\rangle \to m \text{ to be integer.}}$$

for the continuous of angular coordinates.

further, we have

$$\langle\theta,\varphi|\vec{L}^2|Y\rangle = \beta\hbar^2\langle\theta,\varphi|Y\rangle$$

$$\vec{L}^2 = -\hbar^2\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}\right]$$

by $\vec{L}^2 = L_x^2 + L_y^2 + L_z^2$

not in the exams.

$$\Rightarrow -\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\frac{\partial}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}\right]Y(\theta,\varphi) = \beta Y(\theta,\varphi)$$

$L_z Y(\theta,\varphi) = m\hbar Y(\theta,\varphi)$, $-i\hbar\frac{\partial}{\partial\varphi}Y(\theta,\varphi) = m\hbar Y(\theta,\varphi)$

$\frac{\partial^2}{\partial\varphi^2}Y(\theta,\varphi) = -m^2 Y(\theta,\varphi)$ consistent with $Y(\theta,\varphi) \propto e^{im\varphi}$

$$\Rightarrow \left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{m^2}{\sin^2\theta} - \beta\right]Y(\theta,\varphi) = 0$$

$\Rightarrow$ has a solution proportional to a special function $P_l^m(\cos\theta)$ associated Legendre polynomial

we need
$$\begin{cases} \beta = l(l+1), & l = 0,1,2,\cdots \text{ positive integer} \\ m = -l, -l+1, \cdots, l-1, l \end{cases}$$

(a hint to solve this equation, is to let $\xi = \cos\theta$ get a new equation $(1-\xi^2)\frac{\partial^2 Y}{\partial\xi^2} - 2\xi\frac{\partial Y}{\partial\xi} + \left(\beta - \frac{m^2}{1-\xi^2}\right)Y = 0$

full solution: spherical Harmonics 球谐

function $Y_{lm} = Y(\theta, \varphi) = \sqrt{\frac{(2l+1)}{4\pi} \frac{(l-m)!}{(l+m)!}} P_l^m(\cos\theta) e^{im\varphi}$

as normalized solution, with $\begin{cases} l = 0, 1, 2, \cdots \\ m = \underbrace{-l, -l+1, \cdots l-1, l}_{2l+1 \text{ choices of } m} \end{cases}$

Quantum number 量子数

solution Y_{lm} needs two numbers to specify.

like ID 身份证.

- some properties of Y_{lm}

normalized $\underbrace{\int_0^{2\pi} d\varphi \int_0^{\pi} d\theta \sin\theta}_{\text{solid angle}} \left(Y_{l'm'}(\theta,\varphi)\right)^* Y_{lm}(\theta,\varphi)$

$= \delta_{ll'} \delta_{mm'} = \begin{cases} 1, & \text{if } l=l', m=m' \\ 0, & \text{otherwise} \end{cases}$

Y_{lm} we know $|Y\rangle = |l, m\rangle$

$\{|l, m\rangle\}$ form a basis, $2l+1$ dimension

§2 General properties of angular momentum

section 3.5 of Sakurai

similar to $\vec{S}$, $\vec{L}$, we use $\vec{J}$ for general case

have eigenstate as $|j, m\rangle$

with ① $[J_k, J_l] = i\hbar \varepsilon_{klm} J_m$.

② $[\vec{J}^2, J_i] = 0$

$J_z|j,m\rangle = m\hbar|j,m\rangle$

$J^2|j,m\rangle = \beta\hbar^2|j,m\rangle$. $\beta = \beta(j,m) \overset{?}{=} j(j+1)$

will be prooved soon.

∘ ladder operator

J_+, J_-

$J_+ = J_x + iJ_y$, $J_- = J_x - iJ_y$. $J_- = J_+^\dagger$ (recall definition of $a, a^\dagger$)

$\Rightarrow$ $[J_z, J_+] = \hbar J_+$, $[J_z, J_-] = -\hbar J_-$

$[J_+, J_-] = 2\hbar J_z$

$J_-J_+ = J_x^2 + J_y^2 + i[J_x, J_y] = J_x^2 + J_y^2 - \hbar J_z$

$J_-J_+ \xlongequal{J_x^2+J_y^2+J_z^2=J^2} J^2 - J_z^2 - \hbar J_z$

$J_+J_- \xlongequal{\quad} J^2 - J_z^2 + \hbar J_z$

with $J_z|j,m\rangle = m\hbar|j,m\rangle$ try to find properties

apply J_+

$$J_z \underbrace{J_+|j,m\rangle}_{|\xi\rangle} \xlongequal{[J_z,J_+]} ([J_z, J_+] + J_+J_z)|j,m\rangle$$

$$= (\hbar J_+ + J_+ \overset{m\hbar}{J_z})|j,m\rangle$$

$$= (m+1)\hbar J_+|j,m\rangle$$

$J_z|\xi\rangle = \underline{(m+1)\hbar}\,\underline{\underline{|\xi\rangle}}$, $|\xi\rangle = J_+|j,m\rangle$

we know $|\xi\rangle$ has something to do with $|j,m+1\rangle$

$J_+|j,m\rangle \longrightarrow c|j,m+1\rangle$ $\longrightarrow$ increases quantum # by one unit.

similarly, we have $J_-|j,m\rangle \to c'|j,m-1\rangle$

here we try to find c, and c'

(1) upper bound for $|j,m\rangle = |j, m=M_{up}\rangle$

$$J_+|j,M_{up}\rangle = 0$$

$$0 = J_-J_+|j,M_{up}\rangle = (J^2 - J_z^2 - \hbar J_z)|j,M_{up}\rangle$$

$$= (\beta\hbar^2 - M_{up}^2\hbar^2 - M_{up}\hbar^2)|j,M_{up}\rangle$$

$$\Rightarrow \quad \underline{\beta = M_{up}(M_{up}+1)}$$

(2) lower bound of $|j,m\rangle = |j, m=M_{low}\rangle$

$$J_-|j,M_{low}\rangle = 0$$

$$0 = J_+J_-|j,M_{low}\rangle \Rightarrow \beta = \underline{M_{low}(M_{low}-1)}$$

if we have $\quad \beta = (-M_{low})((-M_{low})+1)$

$$M_{up} = -M_{low} = j$$

$$\boxed{\beta = j(j+1)} \qquad \boxed{J^2|j,m\rangle = j(j+1)\hbar^2|j,m\rangle}$$

$$\boxed{\begin{array}{l} J_z|j,m\rangle = m\hbar|j,m\rangle \\ m = -j, -j+1, \cdots, j-1, j \end{array}}$$

- from $[J^2, J_+] = 0$, $[J^2, J_-] = 0$

$$J^2 \underline{J_+|j,m\rangle} = J_+J^2|j,m\rangle = j(j+1)\hbar^2 J_+|j,m\rangle$$

$$\boxed{\begin{array}{l} J_+|j,m\rangle = \hbar\sqrt{j(j+1) - m(m+1)}\;|j,m+1\rangle \\ J_-|j,m\rangle = \hbar\sqrt{j(j+1) - m(m-1)}\;|j,m-1\rangle \end{array}}$$

- some other properties

$[J^2, J_+] = 0 \qquad [J^2, J_-] = 0$

o some other properties

$[J^2, J_+] = 0 \qquad [J^2, J_-] = 0$

$\{|j,m\rangle\}$ form a basis. $\langle j',m'|j,m\rangle = \delta_{j'j}\delta_{m'm}$

example for $|j,m\rangle$

find eigen states of J_x in the basis of $\{|j,m\rangle\}$

$$J_x = \frac{J_+ + J_-}{2} \Rightarrow \text{matrix element}$$

$$\langle j',m'|J_x|j,m\rangle = \langle j',m'|\frac{J_+ + J_-}{2}|j,m\rangle$$

$$= \frac{1}{2}\underline{\underline{\delta_{j'j}}}\left(\delta_{m',m+1}\sqrt{j(j+1)-m(m+1)} + \delta_{m',m-1}\sqrt{j(j+1)-m(m-1)}\right)$$

for example $j=1$

J_x in this form

$$J_x = \begin{array}{c|ccc} j'm' \backslash jm & 1,-1 & 1,0 & 1,1 \\ \hline 1,-1 & 0 & \sqrt{2}/2 & 0 \\ 1,0 & \sqrt{2}/2 & 0 & \sqrt{2}/2 \\ 1,1 & 0 & \sqrt{2}/2 & 0 \end{array}$$

$\Rightarrow \quad \lambda = 0 : \quad |\psi_{\lambda=0}\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$

$\lambda = +1 \quad |\psi_{\lambda=1}\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} \frac{1}{\sqrt{2}} \\ 1 \\ \frac{1}{\sqrt{2}} \end{pmatrix}$

$\lambda = -1 \quad |\psi_{\lambda=-1}\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} \frac{1}{\sqrt{2}} \\ -1 \\ \frac{1}{\sqrt{2}} \end{pmatrix}$