20190423, Ch3 Angular Momentum

Saturday, April 20, 2019 7:42 PM

Home work

① $A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$, find eigenvalue λ, eigenstates $|\lambda\rangle$ of A. with $\sigma_z = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

show $\sigma_z \otimes A = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & -1 & -1 \\ 0 & 0 & -1 & -1 \end{pmatrix}$, find eigenvalue and eigenstates of $\sigma_z \otimes A$, show the relation with $\pm\lambda$, and $|0\rangle|\lambda\rangle$, $|1\rangle|\lambda\rangle$.

① "W-state", for N-spin-$\frac{1}{2}$ particles, one can construct

$$|W_N\rangle \equiv \frac{1}{\sqrt{N}}\left(|100\cdots0\rangle + |010\cdots0\rangle + \cdots + |00\cdots10\rangle + |00\cdots01\rangle\right)$$

with all the permutation of one of the particles at state $|1\rangle$ and the other particles at $|0\rangle$ state.

what is the probability of measuring "$|1\rangle$" state for the first particle?

if we measure "$|0\rangle$" for the first particle, find the relation of the remaining state and $|W_{N-1}\rangle$.

② evolution of coupled spin-$\frac{1}{2}$ system.

$H = \Omega(\sigma_z \otimes \mathbb{1} + \mathbb{1} \otimes \sigma_z)$ $|\Psi(t=0)\rangle = \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle)$, find $|\Psi(t)\rangle$

hint: find the eigenstates and eigenvalues of H first.

Recap

§1 orbital angular momentum

$\vec{L}^2$, L_x, L_y, L_z, $[\vec{L}^2, L_i] = 0$, $[L_k, L_l] = i\varepsilon_{klm}L_m$

eigenfunctions for $\vec{L}^2$ and L_i

spherical harmonics Y_l^m, $l = 0, 1, 2, \cdots$, $m = -l, -l+1, \cdots, l-1, l$

§2 general properties of angular momentum

$\vec{J}^2$, J_x, J_y, J_z, $[\vec{J}^2, J_i] = 0$, $[J_k, J_l] = i\varepsilon_{klm}J_m$, eigenstate $|j,m\rangle$

J_+, J_-, $[J_z, J_+] = \hbar J_+$, $[J_z, J_-] = -\hbar J_-$

$\vec{J}^2|j,m\rangle = \hbar^2 j(j+1)|j,m\rangle$, $J_z|j,m\rangle = m\hbar|j,m\rangle$, $\begin{cases} j: \text{integer or } \frac{\text{integer}}{2} \\ m: -j, -j+1, \cdots, j-1, j \end{cases}$

$J_+|j,m\rangle = \hbar\sqrt{j(j+1) - m(m+1)}\,|j,m+1\rangle$, $J_+|j,j\rangle = 0$

$J_-|j,m\rangle = \hbar\sqrt{j(j+1) - m(m-1)}\,|j,m-1\rangle$, $J_-|j,-j\rangle = 0$

proof:

$J_+|j,m\rangle = C|j,m+1\rangle$

$\langle j,m|J_-J_+|j,m\rangle = |C|^2$, $J_-J_+ = (J_x - iJ_y)(J_x + iJ_y) = J_x^2 + J_y^2 + i[J_x, J_y]$

$= J^2 - J_z^2 - \hbar J_z$

$\Rightarrow |C|^2 = \hbar^2 j(j+1) - \hbar^2 m^2 - m\hbar^2 = \hbar^2(j(j+1) - m(m+1))$

$\Rightarrow J_+|j,m\rangle = \hbar\sqrt{j(j+1) - m(m+1)}\,|j,m+1\rangle$. check $J_+|j,j\rangle = 0$ ✓

similarly $J_-|j,m\rangle = \hbar\sqrt{j(j+1) - m(m-1)}\,|j,m-1\rangle$ (apply $J_+J_- = J^2 - J_z^2 + \hbar J_z$)

$\Rightarrow J_+|j,m\rangle = \hbar\sqrt{j(j+1)-m(m+1)}\,|j,m+1\rangle$, and $J_+|j,j\rangle = 0$ ✓

similarly $J_-|j,m\rangle = \hbar\sqrt{j(j+1)-m(m-1)}\,|j,m-1\rangle$ (apply $J_+J_- = J^2 - J_z^2 + \hbar J_z$)

check $J_-|j,-j\rangle = 0$ ✓

$Y_0^0(\theta,\varphi) = \frac{1}{\sqrt{4\pi}}$, $Y_l^m(\theta,\varphi) = \langle\theta,\varphi|l,m\rangle \propto e^{im\varphi}$

$Y_1^0(\theta,\varphi) = \sqrt{\frac{3}{4\pi}}\cos\theta$ $\quad Y_1^{\pm1}(\theta,\varphi) = \sqrt{\frac{3}{8\pi}}\sin\theta\, e^{\pm i\varphi}$

$Y_2^0(\theta,\varphi) = \sqrt{\frac{5}{16\pi}}(3\cos^2\theta - 1)$, $Y_2^{\pm1}(\theta,\varphi) = \sqrt{\frac{15}{8\pi}}\sin\theta\cos\theta\, e^{\pm i\varphi}$, $Y_2^{\pm2}(\theta,\varphi) = \sqrt{\frac{15}{32\pi}}\sin^2\theta\, e^{\pm2i\varphi}$

$Y_l^m(\theta,\varphi) \propto e^{im\varphi}$

- Quiz 4/30, two pieces of A4-size double sided material, problems covering ch2.

{ basic calculus + linear algebra tutoring
  Mr. Zhang Sat. 3PM this week.

if we have $|\psi(t=0)\rangle = \sum_i c_i|\alpha_i\rangle \longrightarrow |\psi(t)\rangle = \sum_i c_i e^{-iE_it/\hbar}|\alpha_i\rangle$

$H|\alpha_i\rangle = E_i|\alpha_i\rangle$

$\langle\theta,\varphi|\psi(t=0)\rangle = \sum_{l,m} c_{l,m} Y_l^m(\theta,\varphi) \Rightarrow \langle\theta,\varphi|\psi(t)\rangle = \sum_{l,m} c_{lm} Y_l^m e^{-i\frac{E_l t}{\hbar}}$

§3. Coupled spin-$\frac{1}{2}$ system.

① we have two particles, each with spin-$\frac{1}{2}$

i.e. each has two distinct state. $\{|0\rangle, |1\rangle\}$

combination.

both at $|0\rangle$, first at $|0\rangle$, second at $|1\rangle$

both at $|1\rangle$ first at $|1\rangle$, second at $|0\rangle$.

"$\otimes$" direct product.

if we have $|\psi_1\rangle$ for the 1st particle

$|\psi_2\rangle$ for the 2nd

then the overall system, the state is written as $|\psi_1\rangle\otimes|\psi_2\rangle$

in matrix representation, particularly spin-$\frac{1}{2}$

$|\psi_1\rangle = \begin{pmatrix} a_1 \\ b_1 \end{pmatrix}$, $|\psi_2\rangle = \begin{pmatrix} a_2 \\ b_2 \end{pmatrix}$

then $|\psi_1\rangle\otimes|\psi_2\rangle = \begin{pmatrix} a_1 \\ \cdot \end{pmatrix}\otimes\begin{pmatrix} a_2 \\ \cdot \end{pmatrix} = \begin{pmatrix} a_1a_2 \\ a_1b_2 \\ \cdots \end{pmatrix} = \begin{pmatrix} a_1\begin{pmatrix} a_2 \\ b_2 \end{pmatrix} \\ \cdots \end{pmatrix}$

then $|\psi_1\rangle \otimes |\psi_2\rangle = \begin{pmatrix} a_1 \\ b_1 \end{pmatrix} \otimes \begin{pmatrix} a_2 \\ b_2 \end{pmatrix} = \begin{pmatrix} a_1 a_2 \\ a_1 b_2 \\ b_1 a_2 \\ b_1 b_2 \end{pmatrix} = \begin{pmatrix} a_1 \begin{pmatrix} a_2 \\ b_2 \end{pmatrix} \\ b_1 \begin{pmatrix} a_2 \\ b_2 \end{pmatrix} \end{pmatrix}$

this expands two dimension Hilbert space into 4 dimension.

examples. , $|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $|1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$|0\rangle \otimes |0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \times \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ 0 \times \begin{pmatrix} 1 \\ 0 \end{pmatrix} \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$

$|0\rangle \otimes |1\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$

$|1\rangle \otimes |0\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$, $|1\rangle \otimes |1\rangle = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$

some notes: $|\psi_1\rangle \otimes |\psi_2\rangle$, we shorthanded it as $|\psi_1\rangle|\psi_2\rangle$
$\hookrightarrow |\psi_1\rangle|\psi_2\rangle$, also $|\psi_1, \psi_2\rangle$.

for two spin-$\frac{1}{2}$ particles, a general state $|\psi\rangle$ is expressed as $|\psi\rangle = C_0 |0\rangle \otimes |0\rangle + C_1 |0\rangle \otimes |1\rangle + C_2 |1\rangle \otimes |0\rangle + C_3 |1\rangle \otimes |1\rangle$

we have a basis formed by $\{|0\rangle \otimes |0\rangle, |0\rangle \otimes |1\rangle, |1\rangle \otimes |0\rangle, |1\rangle \otimes |1\rangle\}$

② operators $\hat{A}$ for 1st particle, $\hat{B}$ for 2nd particle
we have operator $\hat{A} \otimes \hat{B}$

$$\boxed{(\hat{A} \otimes \hat{B}) \cdot (|\psi_1\rangle \otimes |\psi_2\rangle) = (\hat{A}|\psi_1\rangle) \otimes (\hat{B}|\psi_2\rangle)}$$

clarity with $\hat{A}\hat{B}|\psi_1\rangle$

example: suppose $\underline{\hat{B} = \mathbb{1}}$, so we have $\underline{\hat{A} \otimes \hat{B}} = \underline{\hat{A} \otimes \hat{\mathbb{1}}}$.

$$(\hat{A} \otimes \hat{\mathbb{1}}) \cdot (|\psi_1\rangle \otimes |\psi_2\rangle) = (\hat{A}|\psi_1\rangle) \otimes (\underline{\hat{\mathbb{1}}|\psi_2\rangle})$$
$$= (\hat{A}|\psi_1\rangle) \otimes |\psi_2\rangle$$

back to single particle case.

in matrix form,

$$\hat{A} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}, \quad \hat{B} = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}$$

$$\hat{A} \otimes \hat{B} = \begin{pmatrix} a_{11}\begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} & a_{12}\begin{pmatrix} & \\ & \end{pmatrix} \\ a_{21}\begin{pmatrix} & \\ & \end{pmatrix} & a_{22}\begin{pmatrix} & \\ & \end{pmatrix} \end{pmatrix}$$

quick example $\hat{A} \otimes \hat{\mathbb{1}} = \begin{pmatrix} a_{11}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} & a_{12}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\ a_{21}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} & a_{22}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \end{pmatrix} = \begin{pmatrix} a_{11} & 0 & a_{12} & 0 \\ 0 & a_{11} & 0 & a_{12} \\ a_{21} & 0 & a_{22} & 0 \\ 0 & a_{21} & 0 & a_{22} \end{pmatrix}$

$$|\psi_1\rangle = \begin{pmatrix} c_1 \\ d_1 \end{pmatrix}, \quad |\psi_2\rangle = \begin{pmatrix} c_2 \\ d_2 \end{pmatrix}$$

$$|\psi_1\rangle \otimes |\psi_2\rangle = \begin{pmatrix} c_1\begin{pmatrix} c_2 \\ d_2 \end{pmatrix} \\ d_1\begin{pmatrix} c_2 \\ d_2 \end{pmatrix} \end{pmatrix} = \begin{pmatrix} c_1 c_2 \\ c_1 d_2 \\ d_1 c_2 \\ d_1 d_2 \end{pmatrix}$$

$$(\hat{A} \otimes \hat{\mathbb{1}}) \cdot (|\psi_1\rangle \otimes |\psi_2\rangle) = \begin{pmatrix} a_{11} & 0 & a_{12} & 0 \\ 0 & a_{11} & 0 & a_{12} \\ a_{21} & 0 & a_{22} & 0 \\ 0 & a_{21} & 0 & a_{22} \end{pmatrix} \cdot \begin{pmatrix} c_1 c_2 \\ c_1 d_2 \\ d_1 c_2 \\ d_1 d_2 \end{pmatrix} = \begin{pmatrix} a_{11} c_1 \underline{c_2} + a_{12} d_1 c_2 \\ a_{11} c_1 \underline{d_2} + a_{12} d_1 \underline{d_2} \\ a_{21} c_1 \underline{c_2} + a_{22} d_1 \underline{c_2} \\ a_{21} c_1 \underline{d_2} + a_{22} d_1 \underline{d_2} \end{pmatrix}$$

$$= \begin{pmatrix} (\underline{a_{11} c_1 + a_{12} d_1}) c_2 \\ (\underline{a_{11} c_1 + a_{12} d_1}) d_2 \\ (\underline{a_{21} c_1 + a_{22}} d_1) c_2 \\ (\underline{a_{21} c_1} + a_{22} d_1) d_2 \end{pmatrix} = \begin{pmatrix} a_{11} c_1 + a_{12} d_1 \\ a_{21} c_1 + a_{21} d_1 \end{pmatrix} \otimes \begin{pmatrix} c_2 \\ d_2 \end{pmatrix}$$

$$= (A|\psi_1\rangle) \otimes |\psi_2\rangle$$

- properties of outer products of operators.

$$(\hat{A} \otimes \hat{B}) \cdot (\hat{C} \otimes \hat{D}) \stackrel{?}{=} (\hat{A}\hat{C}) \otimes (\hat{B}\hat{D})$$

proof. $(\hat{C} \otimes \hat{D}) \cdot (|\psi_1\rangle \otimes |\psi_2\rangle) = (\hat{C}|\psi_1\rangle) \otimes (\hat{D}|\psi_2\rangle)$

proof. $(\hat{C}\otimes\hat{D})\cdot(|\psi_1\rangle\otimes|\psi_2\rangle) = (\hat{C}|\psi_1\rangle)\otimes(\hat{D}|\psi_2\rangle)$

$$(\hat{A}\otimes\hat{B})\cdot(\hat{C}\otimes\hat{D})\cdot(|\psi_1\rangle\otimes|\psi_2\rangle) = (\hat{A}\otimes\hat{B})\cdot(\hat{C}|\psi_1\rangle\otimes\hat{D}|\psi_2\rangle)$$

$$= (\hat{A}\hat{C}|\psi_1\rangle)\otimes(\hat{B}\hat{D}|\psi_2\rangle)$$

$$\underline{\hat{A}\otimes\hat{B} + \hat{C}\otimes\hat{D}} \overset{?}{=} (\hat{A}\hat{C}) + (\hat{B}\hat{D})$$

back to K.G. $\hat{A}+\hat{B}$

$$(\hat{A}+\hat{B})|\psi\rangle = \hat{A}|\psi\rangle + \hat{B}|\psi\rangle$$

$$(\hat{A}\otimes\hat{B} + \hat{C}\otimes\hat{D})\cdot(|\psi_1\rangle\otimes|\psi_2\rangle) = \hat{A}|\psi_1\rangle\otimes\hat{B}|\psi_2\rangle + \hat{C}|\psi_1\rangle\otimes\hat{D}|\psi_2\rangle$$

$$\hat{A} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}_{2\times2}, \quad \hat{B} = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}_{2\times2}$$

$$\hat{A}\otimes\hat{B} = \begin{pmatrix} a_{11}b_{11} & a_{11}b_{12} & a_{12}b_{11} & a_{12}b_{12} \\ a_{11}b_{21} & a_{11}b_{22} & a_{12}b_{21} & a_{12}b_{22} \\ a_{21}b_{11} & a_{21}b_{12} & a_{22}b_{11} & a_{22}b_{12} \\ a_{21}b_{21} & a_{21}b_{22} & a_{22}b_{21} & a_{22}b_{22} \end{pmatrix}_{4\times4 = (2\times2)\times(2\times2)}$$

$A_{n\times n}$, $B_{m\times m}$, $(A\otimes B)_{nm\times nm}$.

③ entanglement, quantum logic gates

(not in quiz / exam).

$|\psi\rangle = |a\rangle\otimes|b\rangle$ — a separable state.
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entangled state $|\psi\rangle \neq |a\rangle\otimes|b\rangle$. non-separable.

example: $|\psi\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$ $\qquad |00\rangle = |0\rangle\otimes|0\rangle$

try to make $|\psi\rangle = |a\rangle\otimes|b\rangle$.

$|a\rangle = a_0|0\rangle + a_1|1\rangle$, $|b\rangle = b_0|0\rangle + b_1|1\rangle$

right $= |a\rangle\otimes|b\rangle = (a_0|0\rangle + a_1|1\rangle)\otimes(b_0|0\rangle + b_1|1\rangle)$

$= a_0|0\rangle \otimes b_0|0\rangle + a_0|0\rangle \otimes b_1|1\rangle$
$+ a_1|1\rangle \otimes b_0|0\rangle + a_1|1\rangle \otimes b_1|1\rangle$

$= a_0 b_0|00\rangle + a_0 b_1|01\rangle + a_1 b_0|10\rangle + a_1 b_1|11\rangle$

left $\quad |\psi\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$

$$\begin{cases} a_0 b_0 = \frac{1}{\sqrt{2}}, \quad \underbrace{a_0 b_1 = 0}_{\text{either } a_0 \text{ or } b_1 \text{ is } 0}, \quad a_1 b_0 = 0, \quad a_1 b_1 = \frac{1}{\sqrt{2}}. \\ \text{are in contradiction!} \end{cases}$$

there is no way you can write $\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$ into $|a\rangle|b\rangle$.

special correlation.

if we define $|\psi_1\rangle = |00\rangle$
$|\psi_2\rangle = |11\rangle$

$|\psi\rangle = \frac{1}{\sqrt{2}}(|\psi_1\rangle + |\psi_2\rangle)$ — superposition

entanglement is one special form of superposition.

example 2. $\quad$ if $|\psi_2\rangle = |01\rangle$

$|\psi\rangle = \frac{1}{\sqrt{2}}(|\underline{0}0\rangle + |\underline{0}1\rangle) = |0\rangle \otimes \left(\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)\right)$

example 3 $\quad$ if $|\psi_2\rangle = |10\rangle$ $\quad$ separable.

$|\psi\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |10\rangle) = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \otimes |0\rangle$.

example 4 $\quad$ if $|\psi_2\rangle = |+1\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \otimes |1\rangle$

$|\psi\rangle = c\left(|00\rangle + \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \otimes |1\rangle\right)$

$= c\left(|00\rangle + \frac{1}{\sqrt{2}}|01\rangle + \frac{1}{\sqrt{2}}|11\rangle\right) \neq |a\rangle \otimes |b\rangle$

not so much entangled.

Some other important entangled state

① Bell state $\quad$ C J.S. Bell $\quad$ physics $\quad$ physique $\quad$ Fizika

1, 195, (1964)

$\frac{1}{\sqrt{2}}(|00\rangle \pm |11\rangle)$, $\frac{1}{\sqrt{2}}(|01\rangle \pm |10\rangle)$ ← Bell basis.

③ W-state

n spin-$\frac{1}{2}$ particle, but we have only one of them in $|1\rangle$ state, the rest are in $|0\rangle$ state.

$$|W\rangle = \frac{1}{\sqrt{n}}\left(|100\cdots0\rangle + |010\cdots0\rangle + \cdots + |00\cdots10\rangle + |00\cdots01\rangle\right)$$

permutations of the required state.
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∘ properties of entangled states

① measurement.

$$\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$$

measure on the 1st particle, along σ_z

"$|0\rangle$" → original $|00\rangle$ → "$|0\rangle$" on 2nd

"$|1\rangle$" → $|11\rangle$ → "$|1\rangle$" on 2nd particle.

⇒ if we measured "$|0\rangle$" on 1st particle, we immediately know 2nd one is also on "$|0\rangle$" state.

so as if we measured "$|1\rangle$" on 1st, we get "$|1\rangle$" on 2nd.

∘ distinguishing $\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$ with classical case that half of the times you get $|00\rangle$, another half at $|11\rangle$.

② quantum correlation vs. classical correlation.

if we have $\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$

we can do a "half flip". $\begin{cases} |0\rangle \to |+\rangle = \frac{1}{\sqrt{2}}(|0\rangle+|1\rangle) \\ |1\rangle \to |-\rangle = \frac{1}{\sqrt{2}}(|0\rangle-|1\rangle) \end{cases}$

$$|\psi\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) \longrightarrow \frac{1}{\sqrt{2}}\left(|++\rangle + |--\rangle\right) = \frac{1}{2\sqrt{2}}\left(\begin{array}{l}|00\rangle+|01\rangle+|10\rangle+|11\rangle \\ +|00\rangle+|11\rangle-|10\rangle-|01\rangle\end{array}\right)$$

but if we have only $|00\rangle \to |++\rangle$

if we have only $|11\rangle \to |--\rangle$

$$\to |\psi'\rangle = \frac{1}{2\sqrt{2}}\left(2|00\rangle + 2|11\rangle\right) = \frac{1}{\sqrt{2}}\left(|00\rangle + |11\rangle\right)$$

for $|00\rangle \longrightarrow |++\rangle = \frac{1}{2}\left(|00\rangle + |01\rangle + |10\rangle + |11\rangle\right)$

$|11\rangle \longrightarrow |--\rangle = \frac{1}{2}\left(|00\rangle + |11\rangle - |01\rangle - |10\rangle\right)$

if I repeat measurement along σ_z after "half flip"

- entangling operations

the Controlled-NOT Gate (CNOT)

$\vec{A} \otimes \vec{B} \to 4\times4$ matrix

CNOT $-$ 4×4

$$\text{CNOT} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

truth table.

$$\text{CNOT}\,|00\rangle = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$\text{CNOT}\,|01\rangle = |01\rangle$

$\text{CNOT}\,|10\rangle = |11\rangle$

$CNOT|11\rangle = |10\rangle$.

truth table (真值表)

before		after	
control	target	control	target
$|0\rangle \otimes$	$|0\rangle$	$|0\rangle \otimes$	$|0\rangle$
$|0\rangle \otimes$	$|1\rangle$	$|0\rangle \otimes$	$|1\rangle$
$|1\rangle \otimes$	$|0\rangle$	$|1\rangle \otimes$	$|1\rangle$
$|1\rangle \otimes$	$|1\rangle$	$|1\rangle \otimes$	$|0\rangle$

NOT Gate

$NOT \cdot |0\rangle = |1\rangle$, $NOT \cdot |1\rangle = |0\rangle$.

suppose $|\psi_{input}\rangle = |+\rangle \otimes |0\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \otimes |0\rangle$

$$= \begin{pmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ \frac{1}{\sqrt{2}} \\ 0 \end{pmatrix}$$

$CNOT |\psi_{in}\rangle = |\psi_{out}\rangle$

$$= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ \frac{1}{\sqrt{2}} \\ 0 \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ 0 \\ \frac{1}{\sqrt{2}} \end{pmatrix} = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$$ entangled!

example. if we have 99.9% rate making CNOT to be successful, then we can only apply CNOT for roughly $O(1000)$ times.

- how to construct a CNOT operation.

suppose we have $H_1 = \hbar\omega\, \sigma_z^A \otimes \sigma_x^B$

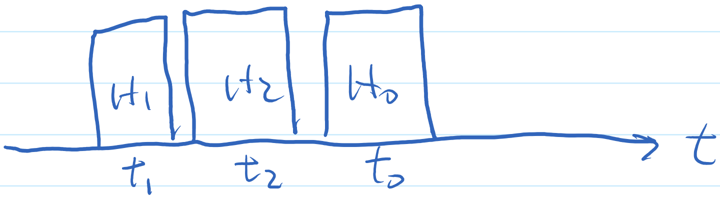
we have evolution $U = e^{-iH_1 t/\hbar} = e^{-i\omega t\, \sigma_z^A \otimes \sigma_x^B}$

$$\Rightarrow U = \cos\omega t - i\sin\omega t\, (\sigma_z^A \otimes \sigma_x^B)$$

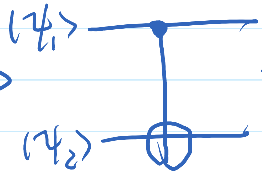
we have $\begin{cases} H_0 = \hbar\omega_0 \sigma_z^{\text{on }1^{st}} = \hbar\omega_0\, \sigma_z \otimes \mathbb{1} \\ H_2 = \hbar\omega_2 \sigma_x^{\text{on }2^{nd}\text{ particle}} = \hbar\omega_2\, \mathbb{1} \otimes \sigma_x \end{cases}$

$$\text{CNOT} = e^{-i\frac{\pi}{4}}\, e^{-iH_0 t_0}\, e^{-iH_2 t_2}\, e^{-iH_1 t_1}$$

$$\begin{cases} CNOT = e^{-i\frac{\pi}{4}} \, e^{-iH_0 t_0} \, e^{-iH_2 t_2} \, e^{-iH_1 t_1} \\ t_0 = \frac{7\pi}{4\omega_0}, \quad t_2 = \frac{7\pi}{4\omega_2}, \quad t_1 = \frac{\pi}{4\omega_1} \end{cases}$$



$\Longleftrightarrow$



$\Rightarrow CNOT|\psi_1\rangle|\psi_2\rangle$

"pulse sequence"

$\hookrightarrow$ a gate, when we apply at our request.