HW

① prove $e^{-i\omega t\hat{\sigma}_x} = \cos\omega t\,\mathbb{1} - i\sin\omega t\,\hat{\sigma}_x$

② detuned Rabi flopping

for a spin-$\frac{1}{2}$ particle with energy spacing ω_0 apply an oscillating magnetic with frequency $\omega_0+\delta$, and Rabi rate Ω, so we have

$$H(t) = \hbar\frac{\omega_0}{2}\hat{\sigma}_z + \hbar\frac{\Omega}{2}\hat{\sigma}_x\cos((\omega+\delta)t)$$

(2.1) choose a proper transformation and apply rotating wave approximation to make $H(t)$ time-independent, so that

$$H_{int} = -\hbar\frac{\delta}{2}\sigma_z + \hbar\frac{\Omega}{2}\sigma_x$$

(2.2) solve for eigenvalue λ_+, λ_- and eigenstate $|\psi_+\rangle, |\psi_-\rangle$ for H_{int} in the basis of σ_z $\{|0\rangle, |1\rangle\}$, $\sigma_z = \begin{pmatrix}1&0\\0&-1\end{pmatrix}$, $\sigma_x = \begin{pmatrix}0&1\\1&0\end{pmatrix}$.

(2.3) with $|\psi(t=0)\rangle = |\psi_0\rangle = \begin{pmatrix}1\\0\end{pmatrix}$ solve for the overlap between $|\psi_0\rangle$ and $|\psi(t)\rangle$, defined as $|\langle\psi_0|\psi(t)\rangle|^2$.

hint: use $|\psi(t)\rangle = e^{-iHt/\hbar}|\psi(t=0)\rangle$, and $H = \lambda_+|\psi_+\rangle\langle\psi_+| + \lambda_-|\psi_-\rangle\langle\psi_-|$. can assume Ω is real for simplicity.

③ proof $Tr(\rho^2)=1$ correspond to $\rho = |\psi\rangle\langle\psi|$, a pure state

recap

$H = \sum_i E_i|a_i\rangle\langle a_i|$, $|\psi(t=0)\rangle = \sum_i c_i|a_i\rangle \Rightarrow |\psi(t)\rangle = \sum_i c_i e^{-iE_it/\hbar}|a_i\rangle$

$H = \hbar\frac{\omega_0}{2}\sigma_z$, $i\hbar\frac{\partial}{\partial t}|\psi\rangle = H|\psi\rangle$, in σ_z basis $|\psi(t=0)\rangle = \begin{pmatrix}c_0\\c_1\end{pmatrix}$

$$\Rightarrow |\psi(t)\rangle = \begin{pmatrix}c_0 e^{-i\omega_0 t/2}\\ c_1 e^{i\omega_0 t/2}\end{pmatrix}$$

o spin resonance

we have a time-dependent $H(t)$, try to solve dynamics

spin: a degree of freedom for microscopic particles
behaves like classical spinning charges with angular momentum. S-G exp.

spin-$\frac{1}{2}$ particle has two distinct state to span Hilbert space "spin-up" "spin-down"
"north pole" "south pole"

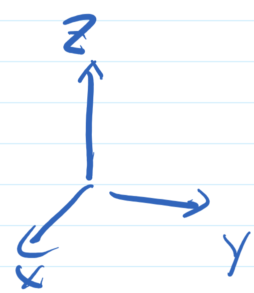
spin-m particle, orthogonal state labeled by quantum number $\underbrace{-m, -m+1, \cdots, m-1, m}_{\text{in total } 2m+1 \text{ diff states}}$

σ_z eigenvalue $+1, -1$.

in this lecture, m can be integer, of half integer.

- time dependent spin-$\frac{1}{2}$ system

$$i\hbar \frac{\partial}{\partial t} |\psi(t)\rangle = H(t) |\psi(t)\rangle$$



$$H(t) = -\vec{\mu} \cdot \vec{B}(t)$$

$$\vec{B}(t) = B_z \hat{z} + B_x \sin(\omega t) \hat{x}$$

$B_z \gg B_x$ for simplicity

for example in NMR, B_z on the order of a few Tesla

$\vec{\mu} = \frac{gq}{2m}\vec{S}$, $\begin{cases} g\text{-factor unit less} \\ m \quad \text{mass} \\ q \quad \text{charge} \to -e \text{ for an electron} \\ \vec{S} = \frac{\hbar}{2}\vec{\sigma} = \frac{\hbar}{2}(\sigma_x \hat{x} + \sigma_y \hat{y} + \sigma_z \hat{z}) \end{cases}$

not an operator, unit of direction

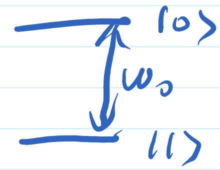
$$\vec{\mu} = -\frac{e\hbar}{2m}\vec{\sigma}$$

$$H(t) = \frac{e\hbar}{2m}\left(\sigma_z B_z + \sigma_x B_x \cos(\omega t)\right)$$

$$\frac{e\hbar}{2m} B_z = \frac{\hbar\omega_0}{2}, \quad \frac{e\hbar}{2m} B_x = \hbar\Omega$$

$$H(t) = \hbar\left(\frac{\omega_0}{2}\sigma_z + \Omega\sigma_x \cos(\omega t)\right)$$

① physically, the first part give a spin-$\frac{1}{2}$ system with energy spacing of ω_0



② spin resonance: when $\omega \simeq \omega_0$, transition between $|0\rangle \longleftrightarrow |1\rangle$ states
跃迁

interaction picture — a basis change technique

$H = \underline{H_0} + \underline{H_1}$, $\quad H_0 = \hbar\frac{\omega_0}{2}\sigma_z$, $\quad H_1 = \hbar\Omega\sigma_x\cos(\omega t)$

$$i\hbar\frac{\partial}{\partial t}|\psi(t)\rangle = H|\psi(t)\rangle$$

$$|\psi(t)\rangle = e^{-iH_0t/\hbar}|\phi(t)\rangle$$ — H_0 to be time independent.

$$\text{left} = i\hbar\left(-i\frac{H_0}{\hbar}e^{-iH_0t/\hbar}|\phi(t)\rangle + e^{-iH_0t/\hbar}\frac{\partial}{\partial t}|\phi(t)\rangle\right.$$

$$\text{right} = (H_0 + H_1)e^{-iH_0t/\hbar}|\phi(t)\rangle$$

$$\text{left} = H_0 e^{-iH_0t/\hbar}|\phi(t)\rangle + i\hbar e^{-iH_0t/\hbar}\frac{\partial}{\partial t}|\phi(t)\rangle$$

$$\Rightarrow H_1 e^{-iH_0t/\hbar}|\phi(t)\rangle = i\hbar e^{-iH_0t/\hbar}\frac{\partial}{\partial t}|\phi(t)\rangle$$

apply $e^{iH_0t/\hbar}$ on both sides

$$i\hbar\frac{\partial}{\partial t}|\phi(t)\rangle = \underbrace{e^{iH_0t/\hbar}H_1e^{-iH_0t/\hbar}}_{H_{1,int}}|\phi(t)\rangle$$

$$i\hbar\frac{\partial}{\partial t}|\phi(t)\rangle = H_{1,int}|\phi(t)\rangle$$

$H = H_0 + H_1, \quad H_0 = \hbar \frac{\omega_0}{2}\sigma_z, \quad H_1 = \hbar\Omega \sin(\omega t)\,\sigma_x$

$$H_{1,int} = \hbar\Omega\, e^{i\frac{\omega_0}{2}\sigma_z t}\,\sigma_x \cos(\omega t)\, e^{-i\frac{\omega_0}{2}\sigma_z t}$$

$$\cos\omega t = \frac{1}{2}\left(e^{i\omega t} + e^{-i\omega t}\right)$$

$$e^{i\frac{\omega_0}{2}\sigma_z t} \xrightarrow[\quad]{\text{basis of } \sigma_z} e^{i\frac{\omega_0}{2}t\left(|0\rangle\langle 0| - |1\rangle\langle 1|\right)} = e^{i\frac{\omega_0}{2}t}|0\rangle\langle 0| + e^{-i\frac{\omega_0}{2}t}|1\rangle\langle 1|$$

$$\sigma_x = |0\rangle\langle 1| + |1\rangle\langle 0|$$

$$e^{-i\frac{\omega_0}{2}\sigma_z t} = e^{-i\frac{\omega_0}{2}t}|0\rangle\langle 0| + e^{i\frac{\omega_0}{2}t}|1\rangle\langle 1|$$

$$e^{i\frac{\omega_0}{2}\sigma_z t}\,\sigma_x\, e^{-i\frac{\omega_0}{2}\sigma_z t} = e^{i\omega_0 t}|0\rangle\langle 1| + e^{-i\omega_0 t}|1\rangle\langle 0|$$

$$H_{1,int} = \hbar\Omega\left(e^{i\omega_0 t}|0\rangle\langle 1| + e^{-i\omega_0 t}|1\rangle\langle 0|\right)\frac{e^{i\omega t} + e^{-i\omega t}}{2}$$

$$= \frac{\hbar\Omega}{2}\Big(e^{i(\omega_0+\omega)t}|0\rangle\langle 1| + e^{i(\omega_0-\omega)t}|0\rangle\langle 1| + e^{i(\omega-\omega_0)t}|1\rangle\langle 0| + e^{-i(\omega+\omega_0)t}|1\rangle\langle 0| \Big)$$

$\omega \simeq \omega_0$, let $\omega = \omega_0$, in practice $\omega_0 \simeq$ few hundred MHz $= 10^6$ Hz

$t \simeq \mu s \quad 10^{-6}$ s

$(\omega_0 + \omega)t \rightarrow$ large

Rotating wave approximation (RWA) $\quad e^{i(\omega_0+\omega)t} \rightarrow 0$

$e^{i(\omega-\omega_0)t}$ survives

let $\omega = \omega_0$, and apply RWA

$$\Rightarrow H_{1,int} \simeq \frac{\hbar\Omega}{2}\left(|0\rangle\langle 1| + |1\rangle\langle 0|\right) = \frac{\hbar\Omega}{2}\sigma_x$$

a recap $\quad H = \hbar\frac{\omega_0}{2}\sigma_z + \hbar\Omega\,\sigma_x \cos\omega t$

$|0\rangle$, ω_0, $|1\rangle$

$\downarrow$ apply a time-dependent basis change

$$H_{1,int} = \hbar\Omega\, e^{i\frac{\omega_0}{2}t\sigma_z}\,\sigma_x\, e^{-i\frac{\omega_0}{2}t\sigma_z}\cos\omega t$$

$\updownarrow \omega_0$ $|1\rangle$

$$H_{1,int} = \hbar\Omega e^{i\frac{\omega_0}{2}t\sigma_z}\,\sigma_x\, e^{-i\frac{\omega_0 t}{2}\sigma_z}\cos\omega t$$

$\Big\downarrow$ RWA

$$H_{1,int} \simeq \frac{\hbar\Omega}{2}\sigma_x \leftarrow \text{a result of } \omega = \omega_0$$

a hint: if $\omega_0 \neq \omega$, it may be better to choose

$$H_0 = \hbar\frac{\omega}{2}\sigma_z$$

$$H_{1,int} = \frac{\hbar\Omega}{2}\sigma_x$$

time-evolution operator $U = e^{-iH_{1,int}t/\hbar} = e^{-i\frac{\Omega t}{2}\sigma_x}$

$$U = \cos\frac{\Omega t}{2}\hat{\mathbb{1}} - i\sin\frac{\Omega t}{2}\hat{\sigma}_x \leftarrow \text{hint of proving this}$$

$$\begin{cases} e^{\hat{A}} = \mathbb{1} + \hat{A} + \frac{\hat{A}^2}{2!} + \cdots + \frac{\hat{A}^n}{n!} + \cdots \\ \sigma_x^2 = \mathbb{1} \end{cases}$$

example of spin-resonance

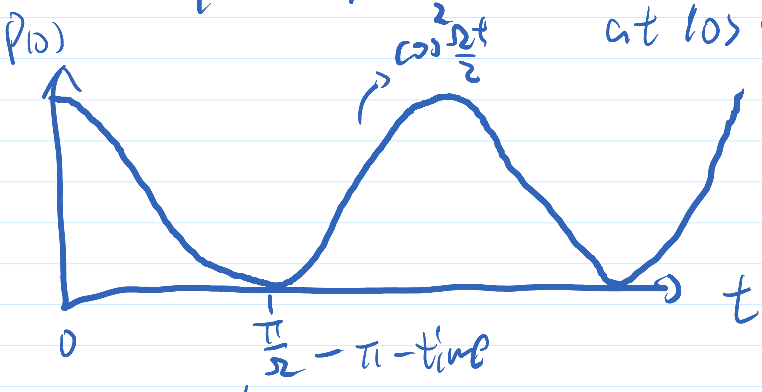
if $|\psi(t=0)\rangle = |0\rangle$, $|\phi(t=0)\rangle = |0\rangle \leftarrow$ in the rotating frame

$$|\phi(t)\rangle = U(t)|\phi(t=0)\rangle = \left(\cos\frac{\Omega t}{2}\hat{\mathbb{1}} - i\sin\frac{\Omega t}{2}\hat{\sigma}_x\right)|0\rangle$$

$\hat{\sigma}_x|0\rangle = |1\rangle$, $\hat{\mathbb{1}}|0\rangle = |0\rangle$

$$\Rightarrow \underline{|\phi(t)\rangle} = \cos\frac{\Omega t}{2}|0\rangle - i\sin\frac{\Omega t}{2}|1\rangle \leftarrow \begin{cases} t=0,\ |\phi\rangle = |0\rangle \\ t = \frac{\pi}{\Omega},\ |\phi(t=\frac{\pi}{\Omega})\rangle = -i|1\rangle \\ \uparrow \text{"}\pi\text{-pulse"} \\ t = \frac{\pi}{2\Omega},\ |\phi(t=\frac{\pi}{2\Omega})\rangle = \frac{1}{\sqrt{2}}(|0\rangle - i|1\rangle) \end{cases}$$

$P(0) = |\langle 0|\phi(t)\rangle|^2 \leftarrow$ population at $|0\rangle$ state



form a quantum logic

formulate quantum logic gate.

"Rabi flopping"

Ω: Rabi frequency

"coherent evolution" $|0\rangle \leftrightarrow |1\rangle$

$|0\rangle \rightarrow |1\rangle$ stuck ✗.

- density matrix. Sakurai P178-P186

$|\psi\rangle, \langle\psi| \rightarrow$ pure state

for example, in a typical quantum experiment, we need to accumulate statistics.

if the state is expressed as a pure state, it means at each shot of exp. the states are all the same.

① if we have shot to shot variation, it's variation in time domain

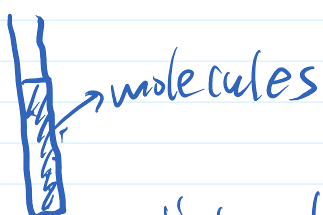
for one shot of exp. starts with $|\psi\rangle$

in another shot, starts with $|\psi'\rangle \neq |\psi\rangle$

$\longrightarrow$ non pure state, "mixed state"

② working with an ensemble of particles, each with different states to begin with.

NMR



非均匀性

mixed in the particle domain, inhomogeniety

③ during evolution, there exists noise that can drive the particle to different states to get different outcome.

∘ mixed state $\longrightarrow$ density matrix

$$\boxed{\rho = \sum_i p_i |\psi_i\rangle\langle\psi_i|}$$ in the discrete case as an example.

"rho"

∘ $|\psi_i\rangle$ is normalized state, and $|\psi_i\rangle, |\psi_j\rangle$ are not necesaily orthogonal.

∘ p_i is the statistical probability that the state happens to be at $|\psi_i\rangle$, $\sum_i p_i = 1$, $1 \geq p_i \geq 0$

example 50% $|\psi\rangle$, 50% $|\psi'\rangle$ mixed

$\Rightarrow \rho = \frac{1}{2}|\psi\rangle\langle\psi| + \frac{1}{2}|\psi'\rangle\langle\psi'|$.

Example 100% $|\psi\rangle \Rightarrow \rho = |\psi\rangle\langle\psi|$.

properties.

① $Tr(\rho) = 1$, proof: $|\psi_i\rangle = \sum_j a_{ij}|\alpha_j\rangle$, in $\{|\alpha_j\rangle\}$ basis

$\Rightarrow Tr(\rho) = \sum_k \langle\alpha_k|\rho|\alpha_k\rangle = \sum_{k,i} p_i \langle\alpha_k|\psi_i\rangle\langle\psi_i|\alpha_k\rangle$

$= \sum_{k,i} p_i \langle\psi_i|\alpha_k\rangle\langle\alpha_k|\psi_i\rangle = \sum_i p_i \langle\psi_i|\big(\underbrace{\sum_k |\alpha_k\rangle\langle\alpha_k|}_{\mathbb{1}}\big)|\psi_i\rangle$

$= \sum_i p_i \langle\psi_i|\psi_i\rangle = \sum_i p_i = 1$.

② time evolution. — we are talking about coherent case

$\rho(t) = U\rho(t=0)U^\dagger$, proof: for each $|\psi_i(t)\rangle = U|\psi_i(0)\rangle$

then we have $\rho(t) = \sum_i p_i |\psi_i(t)\rangle\langle\psi_i(t)|$

$\rho(t) = U\rho(t=0)U^\dagger$, proof: for each $|\psi_i(t)\rangle = U|\psi_i(0)\rangle$

then we have $\rho(t) = \sum_i P_i |\psi_i(t)\rangle\langle\psi_i(t)|$

$$= \sum_i P_i U |\psi_i(0)\rangle\langle\psi_i(0)| U^\dagger$$

$$= U\Big(\sum_i P_i |\psi_i(0)\rangle\langle\psi_i(0)|\Big)U^\dagger$$

$\uparrow$ $\rho(t=0)$

$$= U\rho(t=0)U^\dagger$$

③ $Tr(\hat{A}\rho) = Tr(\rho\hat{A})$, $\hat{A}$ is an operator.

proof: because $\rho = \sum_i P_i \underbrace{|\psi_i\rangle\langle\psi_i|}_{\text{by itself an operator}}$

$Tr(\hat{A}\hat{B}) \overset{?}{=} Tr(\hat{B}\hat{A})$, $\hat{B}$ is also operator.

by working in a general basis

$$Tr(\hat{A}\hat{B}) = \sum_{ij} a_{ij} b_{ji} = \sum_{ji} b_{ji} a_{ij} = Tr(\hat{B}\hat{A})$$

$$\Rightarrow \quad Tr(U\rho U^\dagger) = Tr(\underbrace{U^\dagger U}_{1}\rho) = Tr(\rho)$$

④ measurement. $\hat{A}$ measure result is $Tr(\rho\hat{A})$

proof: recap. $|\psi_i\rangle$, measurement of $\hat{A}$ give $\langle\psi_i|\hat{A}|\psi_i\rangle$

we expect for statistical P_i of each $|\psi_i\rangle$

result should be $\sum_i P_i \langle\psi_i|\hat{A}|\psi_i\rangle \overset{?}{=} Tr(\rho\hat{A})$

$\uparrow$ true.

$$Tr(\rho\hat{A}) = Tr\Big(\sum_i P_i |\psi_i\rangle\langle\psi_i|\hat{A}\Big)$$

$$= \sum_{ij} \langle\alpha_j|\psi_i\rangle\langle\psi_i|\hat{A}|\alpha_j\rangle P_i$$

$$= \sum_{ij} P_i \langle\psi_i|\hat{A}\underbrace{|\alpha_j\rangle\langle\alpha_j|}_{1}\psi_i\rangle = \sum_i P_i \langle\psi_i|\hat{A}|\psi_i\rangle$$

⑤ $Tr(\rho^2) \leq 1$

special case if $\rho = |\psi_i\rangle\langle\psi_i|$, $\rho^2 = |\psi_i\rangle\underbrace{\langle\psi_i|\cdot|\psi_i\rangle}\langle\psi_i|$

if ρ is pure, $\rho^2 = \rho$, $Tr(\rho^2) = Tr(\rho) = 1$ $= |\psi_i\rangle\langle\psi_i| = \rho$

proof: $Tr(\rho^2) = Tr\left(\sum_{ij} P_i P_j |\psi_i\rangle\langle\psi_i|\cdot|\psi_j\rangle\langle\psi_j|\right)$

$$= \sum_{ijk} P_i P_j \langle\alpha_k|\psi_i\rangle\langle\psi_i|\psi_j\rangle\langle\psi_j|\alpha_k\rangle$$

$$= \sum_{ij} P_i P_j \underbrace{\langle\psi_j|\psi_i\rangle\langle\psi_i|\psi_j\rangle}_{CC^* \leq 1} \leq \sum_{ij} P_i P_j = \sum_i P_i \sum_j P_j = 1$$

⑥ dynamics of ρ

we have $i\hbar \frac{\partial}{\partial t}|\psi_i(t)\rangle_{n\times 1} = H|\psi_i(t)\rangle$

$$\rho(t) = \sum_i P_i |\psi_i(t)\rangle\langle\psi_i(t)|$$

$$\frac{\partial}{\partial t}\rho(t) = \sum_i P_i \left[\frac{\partial}{\partial t}|\psi_i(t)\rangle\right)\langle\psi_i(t)| + \sum_i P_i |\psi_i(t)\rangle\left(\frac{\partial}{\partial t}\langle\psi_i(t)|\right)$$

$$= \sum_i P_i \frac{H}{i\hbar}|\psi_i(t)\rangle\langle\psi_i(t)| + \sum_i P_i |\psi_i(t)\rangle\langle\psi_i(t)|\frac{H}{(-i\hbar)}$$

$\Rightarrow \frac{\partial}{\partial t}\rho(t)_{n\times n} = \frac{1}{i\hbar}[H, \rho(t)] \longrightarrow$ refer to Lindblad equation
master equation

⑦ spin-$\frac{1}{2}$ system, $\rho \to 2\times 2$ matrix.

for any matrix $\hat{A}$ express

$\hat{A} = C_0 \mathbb{1} + C_1\sigma_x + C_2\sigma_y + C_3\sigma_z$, C complex number.

$$\hat{A} = \begin{pmatrix} a_{00} & a_{01} \\ a_{10} & a_{11} \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \frac{1}{2}(\mathbb{1} + \sigma_z)$$

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + \frac{i}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \frac{1}{2}(\sigma_x + i\sigma_y)$$

$$\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = \frac{1}{2}(\sigma_x - i\sigma_y)$$

$$\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = \frac{1}{2}(\sigma_x - i\sigma_y)$$

$$\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \frac{1}{2}(\mathbb{1} - \sigma_z)$$

$\rho = \sum_i p_i |\psi_i\rangle\langle\psi_i| \Longleftarrow \underline{\rho^+ = \rho}$, p_i real

$\rho = \underbrace{\frac{1}{2}(\mathbb{1}}_{} + \vec{r}\cdot\vec{\sigma})$,

$Tr(\rho) = 1$, $Tr(\sigma_i) = 0$, $\sigma_i^+ = \sigma_i$

define $\vec{r}$: Bloch vector. $|\vec{r}| \in [0, 1]$.

- Evolution of expectation values.

a particle with $|\psi(t)\rangle$

$\langle\hat{x}\rangle = \langle\psi(t)|\hat{x}|\psi(t)\rangle$.

$i\hbar\frac{\partial}{\partial t}|\psi(t)\rangle = H|\psi(t)\rangle$.

$$\Rightarrow \frac{d}{dt}\langle\hat{x}\rangle = \left(\frac{\partial}{\partial t}\langle\psi(t)|\right)\hat{x}|\psi(t)\rangle + \langle\psi(t)|\hat{x}\left(\frac{\partial}{\partial t}|\psi(t)\rangle\right)$$

$$= \frac{i}{\hbar}\langle\psi(t)|[H, x]|\psi(t)\rangle$$

$$\frac{d}{dt}\langle\hat{x}\rangle = \frac{i}{\hbar}\langle[H, x]\rangle \longrightarrow \frac{d}{dt}\langle A\rangle = \frac{i}{\hbar}\langle[H, A]\rangle$$

$H = \frac{p^2}{2m} + \underline{V(x)}$, $\Rightarrow [H, \underline{x}] = [\frac{p^2}{2m}, x] = \frac{1}{2m}(p^2x - xp^2)$

$[x, p] = i\hbar$, $px = xp - i\hbar$

$p^2x = p(px) = p(xp - i\hbar) = \underline{px}p - i\hbar p$

$= (xp - i\hbar)p - i\hbar p = -2i\hbar p + xp^2$

$\Rightarrow p^2x - xp^2 = -2i\hbar p$.

notice in momentum basis. $x \leftrightarrow i\hbar\frac{\partial}{\partial p}$.

$p^2x - xp^2 = p^2(i\hbar\frac{\partial}{\partial p}) - (i\hbar\frac{\partial}{\partial p})p^2$

Write in momentum basis. $x \leftrightarrow i\hbar\frac{\partial}{\partial p}$.

$$p^2x - xp^2 = p^2\left(i\hbar\frac{\partial}{\partial p}\right) - \underline{\left(i\hbar\frac{\partial}{\partial p}\right)p^2}$$

$$i\hbar\frac{\partial}{\partial p}p^2\psi(p) = (i\hbar)\left(2p\psi(p) + p^2\frac{\partial}{\partial p}\psi(p)\right) = i\hbar\left(2p + p^2\frac{\partial}{\partial p}\right)\psi(p)$$

$$\Rightarrow \quad i\hbar\frac{\partial}{\partial p}p^2 = i\hbar 2p + i\hbar p^2\frac{\partial}{\partial p}$$

$$\Rightarrow p^2x - xp^2 = \cancel{p^2\left(i\hbar\frac{\partial}{\partial p}\right)} - i\hbar(2p) + \cancel{p^2\left(i\hbar\frac{\partial}{\partial p}\right)} = -2i\hbar p.$$

$$p^nx - xp^n = -in\hbar p.$$

$$\Rightarrow [H, x] = -\frac{2i\hbar p}{2m} = -\frac{i\hbar}{m}p.$$

$$\frac{d}{dt}\langle x\rangle = \frac{i}{\hbar}\langle[H,x]\rangle = \frac{i}{\cancel{\hbar}}\left\langle -\frac{i\hbar}{m}p\right\rangle = \frac{1}{m}\langle p\rangle$$

$$\underline{m\langle\dot{x}\rangle = \langle p\rangle}$$

$$\frac{d}{dt}\langle p\rangle = \frac{i}{\hbar}\langle[H,p]\rangle, \quad H = \frac{p^2}{2m} + V(x)$$

$$[H,p] = [V(x), p] \xrightarrow{x \text{ repres.}} V(x)\left(-i\hbar\frac{\partial}{\partial x}\right) - \left(-i\hbar\frac{\partial}{\partial x}\right)V(x)$$

$$= i\hbar\left(\frac{\partial}{\partial x}V(x)\right) = i\hbar V'(x)$$

$$\frac{d}{dt}\langle p\rangle = \frac{i}{\hbar}\langle i\hbar V'(x)\rangle = \underbrace{-\langle V'(x)\rangle}_{\text{force}}$$

$$m\frac{d^2}{dt^2}\langle x\rangle = -\langle V'(x)\rangle$$

$F = ma$ in classical.