

HW ① with  $\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ ,  $\sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$ ,  $\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ ,

and  $\vec{n} = (\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta)$ ,  $\vec{\sigma} = (\sigma_x, \sigma_y, \sigma_z)$

find eigenvalue and eigenstates of  $\vec{\sigma} \cdot \vec{n}$

② with  $\psi(\theta, \varphi) = \frac{1}{\sqrt{3}} (Y_1^0(\theta, \varphi) + Y_1^1(\theta, \varphi))$ , without integrations,  
find  $\langle \vec{L}^2 \rangle$ ,  $\langle L_z \rangle$

③ with  $\psi(t=0) = \psi(\theta, \varphi)$  as above,  $H = \frac{\vec{L}^2}{2mR^2}$ ,  
find  $\psi(t=T)$

④ uncertainty principle.

With  $J_z |j, m\rangle = m \hbar |j, m\rangle$ ,  $\vec{J}^2 |j, m\rangle = \hbar(j(j+1)) |j, m\rangle$

$$\langle \Delta A \rangle = \sqrt{\langle l, m | A^2 | l, m \rangle - (\langle l, m | A | l, m \rangle)^2}.$$

find  $\langle AJ_x \rangle \langle \Delta J_y \rangle$  and  $\langle [J_x, J_y] \rangle$ . check  $\langle AJ_x \rangle \langle AJ_y \rangle \geq \frac{1}{2} |\langle [J_x, J_y] \rangle|$   
when  $\langle AJ_x \rangle \langle \Delta J_y \rangle = \frac{1}{2} |\langle [J_x, J_y] \rangle|$ , what's the requirements of  $m$ ?

⑤ write out  $J \cdot J_t$  as a matrix in the basis of  $J_z$ , with  $j=1$ , and  
find the eigenvalue and eigenstates of  $J \cdot J_t$ .

Homework Q&A Sunday 3PM with Mr.Zhang

Quiz II 4/30 in class.

Sakurai: P(57-172, 178-185) 191-203.

motivation.

position, momentum.  $\leftarrow$  linear.

rotation?  $\rightarrow$  angular momentum

angular position  $\rightarrow$  angle.

rotation  $\longleftrightarrow$  spin

coupled spin system.

• Some prerequisites (预备知识)

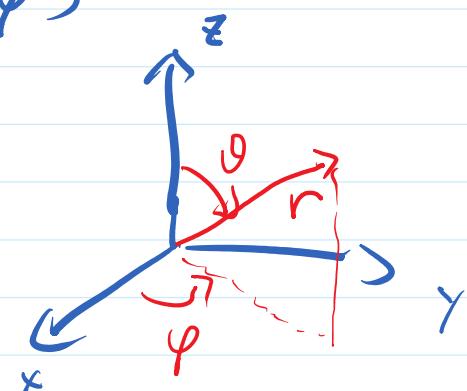
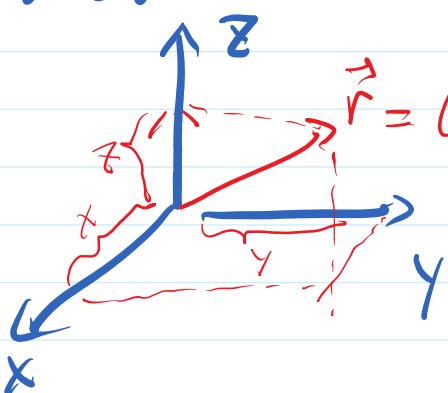
3D - system.  $\rightarrow \dots$

position in 3D  $\vec{r} = (x, y, z) \rightarrow$  coordinates.  
 momentum  $\vec{P} = (P_x, P_y, P_z)$

$$P_x \leftarrow -i\hbar \frac{\partial}{\partial x} \Rightarrow \vec{p} = \left( -i\hbar \frac{\partial}{\partial x}, -i\hbar \frac{\partial}{\partial y}, -i\hbar \frac{\partial}{\partial z} \right)$$

$$\vec{\nabla} = \left( \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right) \Rightarrow \vec{p} = -i\hbar \vec{\nabla}$$

• spherical coordinate.  $(r, \theta, \phi)$



Comments.

$(x, y, z) \rightarrow$  all components in length unit

$(r, \theta, \phi) \rightarrow$  "r" in length unit

" $\theta$ ", " $\phi$ " unit less or in radient  
in angle unit.

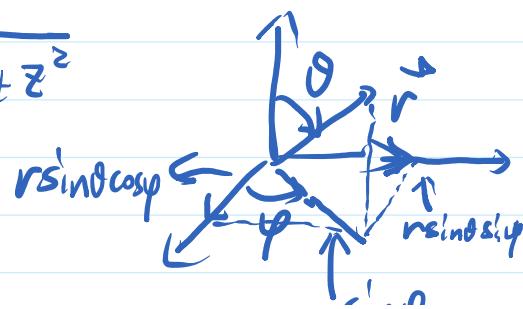
for a point in space, we use a bracket to represent it.

in spherical coordinate, we have  
 $r, \theta, \phi$  to represent this point

but we can also use  $x, y, z$  for the same point

$$\begin{cases} x = r \sin \theta \cos \phi \\ y = r \sin \theta \sin \phi \\ z = r \cos \theta \end{cases} \Rightarrow$$

$$r = \sqrt{x^2 + y^2 + z^2}$$



• rotation along  $\hat{z}$  direction

1. - - - -

◦ rotation along  $\hat{z}$  direction

$\rightarrow \varphi \rightarrow \varphi + d\varphi$ , and  $r, \theta$  remain the same.

$(x, y, z) \xrightarrow{\text{rotation}} (x', y', z')$

$(r, \theta, \varphi)$

$(r, \theta, \varphi + d\varphi)$



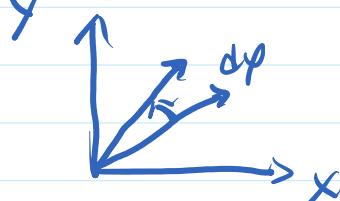
what are  $x', y', z'$  in terms of  $r, \theta, \varphi, d\varphi$ .

$$\begin{cases} x' = r \sin \theta \cos(\varphi + d\varphi) \cong r \sin \theta \cos \varphi - r \sin \theta \sin \varphi d\varphi \\ y' = r \sin \theta \sin(\varphi + d\varphi) \cong r \sin \theta \sin \varphi + r \sin \theta \cos \varphi d\varphi \\ z' = r \cos \theta = z \end{cases}$$

$$\cos(\varphi + d\varphi) = \cos \varphi \cos d\varphi - \sin \varphi \sin d\varphi \cong \cos \varphi - (\sin \varphi) d\varphi$$

$$\sin(\varphi + d\varphi) = \sin \varphi \cos d\varphi + \cos \varphi \sin d\varphi \cong \sin \varphi + (\cos \varphi) d\varphi$$

$$\Rightarrow \begin{cases} x' \cong x - y d\varphi \\ y' \cong y + x d\varphi \\ z' = z \end{cases}$$



◦ prerequisites on spin- $\frac{1}{2}$  system

Pauli operators  $\sigma_x, \sigma_y, \sigma_z$



in the  $\sigma_z$  basis,  $\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

$$[\sigma_k, \sigma_l] = 2i \epsilon_{klm} \sigma_m, \quad \epsilon_{klm} = \begin{cases} 1 & \text{if } k, l, m \text{ in order} \\ -1 & \text{if out of order} \end{cases}$$

define spin operator  $\vec{s} = \frac{\hbar}{2} \vec{\sigma}, s_z = \frac{\hbar}{2} \sigma_z$

$$\vec{\sigma} = (\sigma_x, \sigma_y, \sigma_z)$$

$$[\sigma_k, \sigma_l] = i\hbar \epsilon_{klm} \sigma_m$$

← generally true for angular momentum operator including spin- $\frac{1}{2}$  operators.

$$\vec{s}^2 = s_x^2 + s_y^2 + s_z^2 = \frac{\hbar^2}{4} (\sigma_x^2 + \sigma_y^2 + \sigma_z^2) = \frac{\hbar^2}{4} \cdot \frac{3}{4} \mathbb{1}$$

$$\delta = \delta_x + \delta_y + \delta_z = \frac{\hbar}{4}(\delta_x + \delta_y + \delta_z) = \hbar \cdot \frac{\delta}{4} \mathbf{1}$$

for each  $k$  in  $x, y, z$ ,  $\boxed{\delta_k^2 = 1}$   $\star$

$$\vec{\delta}^2 \propto 1$$

$$\boxed{[\vec{\delta}, \delta_k] = 0}$$

if we define unit length vector

$$n = (\sin\theta \cos\phi, \sin\theta \sin\phi, \cos\theta)$$

we have

$$\vec{\delta} \cdot \vec{n} = \delta_x \sin\theta \cos\phi + \delta_y \sin\theta \sin\phi + \delta_z \cos\theta.$$

$$\star e^{i\phi \hat{\delta}_x} = 1 + i\phi \delta_x - \frac{\phi^2 \delta_x^2}{2} + i\phi \frac{\delta_x^3}{3!} + \dots$$

$$\left\{ \begin{array}{l} e^{\hat{A}} = 1 + \hat{A} + \frac{\hat{A}^2}{2!} + \dots + \frac{\hat{A}^n}{n!} + \dots \\ \delta_x^2 = 1 \end{array} \right.$$

cube

$$\Rightarrow e^{i\phi \hat{\delta}_x} = \left( 1 - \frac{\phi^2}{2!} + \frac{\phi^4}{4!} + \dots \right) + i\delta_x (\phi - \frac{\phi^3}{3!} + \frac{\phi^5}{5!} + \dots)$$

$$= \cos\phi + i\sin\phi$$

$\uparrow$

$\uparrow$   
factorial

§1. orbital angular momentum.

for a small rotation  $\phi \rightarrow \phi + d\phi$  : translation  
位移

of angle  $\phi$

$$\left\{ \begin{array}{l} x' \approx x - y d\phi \\ y' \approx y + x d\phi \\ z' = z \end{array} \right.$$

if we have a state  $|\psi\rangle$  expressed in position coordinate  $\langle x, y, z | \psi \rangle = \psi(x, y, z)$   
apply a rotation to the state.  $\psi'(x, y, z)$   
which means in spherical basis

we get from  $\psi(r, \theta, \varphi) \rightarrow \psi(r, \theta, \varphi - d\varphi)$

$$\underline{\psi(r, \theta, \varphi - d\varphi)} \simeq \underline{\psi(r, \theta, \varphi)} - d\varphi \cdot \frac{\partial}{\partial \varphi} \psi(r, \theta, \varphi)$$

We can express this small rotation

$$as \quad 1 - \frac{\partial}{\partial \varphi} = 1 + \frac{1}{i\hbar} \hat{L}_z$$

$$\begin{aligned} \text{orbital} \\ \text{angular} \\ \text{momentum} \end{aligned} \quad \hat{L}_z \xrightarrow{\varphi \text{ coordinate}} -i\hbar \frac{\partial}{\partial \varphi} \leftrightarrow P_x \xrightarrow{-i\hbar \frac{\partial}{\partial x}} e^{-(x-dx)^2}$$

express  $\hat{L}_z$  in  $x, y, z$  coordinates.

$$\psi(x, y, z) \xrightarrow{1 - \frac{\partial}{\partial \varphi}} \psi(x', y', z')$$

rotate axis along  $z$  by  $d\varphi$

$$\begin{cases} x' = x - y d\varphi \\ y' = y + x d\varphi \\ z' = z \end{cases}$$

$$\xrightarrow{\text{want } -d\varphi} \begin{cases} x' = x + y d\varphi \\ y' = y - x d\varphi \\ z' = z \end{cases}$$

$$\psi(x', y', z') \simeq \psi(x + y d\varphi, y - x d\varphi, z)$$

$$= \psi(x, y, z) + y d\varphi \frac{\partial}{\partial x} \psi(x, y, z) - x d\varphi \frac{\partial}{\partial y} \psi(x, y, z)$$

$$\left( \text{for 1D} \quad \psi(x + dx) \simeq \psi(x) + dx \frac{\partial}{\partial x} \psi(x) \right)$$

$$\psi(x + dx, y) \simeq \psi(x, y) + dx \frac{\partial}{\partial x} \psi(x, y) \quad \xleftarrow{\text{calculus}}$$

高数/微积

$$P_x \leftrightarrow -i\hbar \frac{\partial}{\partial x}, \quad P_y \leftrightarrow -i\hbar \frac{\partial}{\partial y}$$

$$\begin{aligned} \psi(x', y', z') &= \psi(x, y, z) + d\varphi Y \left( \frac{P_x}{-i\hbar} \right) \psi(x, y, z) - d\varphi X \left( \frac{P_y}{-i\hbar} \right) \psi(x, y, z) \\ &= \left( \mathbb{1} + \frac{1}{(-i\hbar)} d\varphi (Y P_x - X P_y) \right) \psi(x, y, z) \end{aligned}$$

$$= \left( \mathbb{1} + \frac{1}{i\hbar} \hat{L}_z \right) \psi(x, y, z)$$

$$\boxed{\hat{L}_z = \hat{x} \hat{P}_y - \hat{y} \hat{P}_x} \quad \xleftarrow{\hat{L}_z \text{ expressed with } x, y, z}$$

$$\hat{L}_z = \hat{x}\hat{P}_y - \hat{y}\hat{P}_x$$

$\hat{L}_z$  expressed with  $x, y, z$  coordinate with respected operators.

$$\hat{L}_x = \hat{y}\hat{P}_z - \hat{z}\hat{P}_y, \quad \hat{L}_y = \hat{z}\hat{P}_x - \hat{x}\hat{P}_z$$

$$\hat{\vec{L}} = \frac{\vec{r} \times \vec{p}}{i\hbar} \rightarrow \text{same as classical}$$

$$= \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ x & y & z \\ P_x & P_y & P_z \end{vmatrix} = (yP_z - zP_y)\hat{x} + \dots$$

we get an orbital angular momentum operator in analogy to classical

$$L_k = r_l P_m - r_m P_l, \quad k, l, m \text{ in order}$$

$$\left\{ \begin{array}{l} L_z = i\hbar \frac{\partial}{\partial \varphi} \\ L_y = zP_x - xP_z = i\hbar \left( -\cos\varphi \frac{\partial}{\partial \theta} + \sin\theta \sin\varphi \frac{\partial}{\partial \varphi} \right) \\ L_x = i\hbar \left( \sin\varphi \frac{\partial}{\partial \varphi} + \cot\theta \cos\varphi \frac{\partial}{\partial \theta} \right) \end{array} \right.$$

not in the exam/  
quiz

can be derived using  $P_y \leftrightarrow -i\hbar \frac{\partial}{\partial x}$   
 $y = r \sin\theta \sin\varphi$  etc.

some properties of orbital angular momentum

$$\textcircled{1} \quad [L_x, L_y] = i\hbar L_z \quad \left[ S_x, S_y \right] = i\hbar S_z$$

proof:  $[yP_z - zP_y, zP_x - xP_z]$

$$= [yP_z, zP_x] + [yP_z, -xP_z] + [-zP_y, zP_x]$$

$$+ [-zP_y, -xP_z]$$

$[x, y] = 0, [P_x, P_y] = 0$  etc.

$$= [yP_z, zP_x] + [-zP_y, -xP_z]$$

$$= [yP_z, zP_x] + [-zP_y, -xP_z]$$

$$= yP_x [P_z, z] + P_y x [z, P_z]$$

$$= i\hbar (xP_y - yP_x) = i\hbar L_z$$

$$[yP_z, zP_x] = yP_z zP_x - zP_x yP_z = yP_x P_z z - P_x y zP_z$$

$$= yP_x P_z z - yP_x zP_z$$

$$= yP_x [P_z, z]$$

$$\boxed{[L_k, L_l] = i\hbar \sum_{klm} L_m}$$



$$\stackrel{\wedge}{[L_y, L_x]} = -i\hbar L_z$$



$$\textcircled{2} \quad \boxed{[\vec{L}, L_i] = 0} \quad \leftarrow \text{similar to } [\vec{S}, S_i] = 0$$

proof:  $\vec{L}^2 = L_x^2 + L_y^2 + L_z^2$

$$[A^2, B] = A[A, B] + [A, B]A \quad \checkmark$$

$$\hookrightarrow A^2 B - B A^2 = \underbrace{A^2 B - A B A}_{=} + \underbrace{A B A - B A^2}_{=}$$

$$= A[A, B] + [A, B]A.$$

$$[\vec{L}, L_x] = [L_x^2 + L_y^2 + L_z^2, L_x] = [L_y^2, L_x] + [L_z^2, L_x]$$

$$= L_y [L_y, L_x] + [L_y, L_x] L_y + L_z [L_z, L_x] + [L_z, L_x] L_z$$

$$[L_y, L_x] = -i\hbar L_z$$

$$= -i\hbar \underline{L_y L_z} - i\hbar \underline{L_z L_y} + i\hbar \underline{L_z L_y} + i\hbar \underline{L_y L_z}$$

$$[L_z, L_x] = i\hbar L_y$$

$$= 0$$

Comment.  $\begin{cases} [L_k, L_l] = i\hbar \sum_{klm} L_m \\ [\vec{L}, L_i] = 0 \end{cases}$

$\curvearrowleft$  independent of representation.

representation.

- o Eigen equation with  $\vec{L}^2$  and  $L_z$ .  $\leftarrow$  useful in because  $[\vec{L}^2, L_z] = 0$ . solving 5-eq with azimuthal symmetry.

if  $\hat{A}, \hat{B}$  are Hermitian, and  $[\hat{A}, \hat{B}] = 0$   
then there exists  $\{|y\rangle\}$  giving  $\langle A|y\rangle = a|y\rangle$   
 $\langle B|y\rangle = b|y\rangle$

or  $\{|y\rangle\}$  is the mutual eigenbasis/eigenstate.  
共同的

example  $[\hat{I}, \hat{\sigma}_x] = 0$ , then we can find  $|+\rangle = \frac{1}{\sqrt{2}}(|1\rangle + |-\rangle)$ ,  $|-\rangle = \frac{1}{\sqrt{2}}(|1\rangle - |-\rangle)$  to be eigenstate of both  $\hat{I}, \hat{\sigma}_x$

$$\begin{cases} \hat{I}|+\rangle = |+\rangle, \hat{I}|-\rangle = |-\rangle \\ \hat{\sigma}_x|+\rangle = |+\rangle, \hat{\sigma}_x|-\rangle = -|-\rangle \end{cases}$$

let  $|y\rangle$  to be one these states

$$\Rightarrow \begin{cases} L_z|y\rangle = m\hbar|y\rangle, \{m \text{ is a number}\} \\ \vec{L}^2|y\rangle = \beta\hbar^2|y\rangle \end{cases}$$

$|y\rangle$  needs to be normalized.  $\langle y|y\rangle = 1$

$$\beta\hbar^2 = \langle y|\vec{L}^2|y\rangle = \langle y|L_x^2 + L_y^2 + L_z^2|y\rangle$$

$$\geq \langle y|L_z^2|y\rangle = m^2\hbar^2$$

$$\Rightarrow \boxed{\beta \geq m^2}$$

in spherical coordinate, as above we can express  $\vec{L}^2, L_z$  all in just  $\theta, \phi$ . without  $r$   
 $\vec{L}^2, L_z$  only relates to angular coordinates 

$L_x^2, L_z$  only relates to angular coordinates  $\{\theta, \varphi\}$  as the angular representation.

$$\langle \theta, \varphi | Y \rangle = Y(\theta, \varphi) \leftarrow \langle x | \psi \rangle = \psi(x)$$

as a wavefunction.

$$\langle \theta, \varphi | L_z | Y \rangle = m \hbar \langle \theta, \varphi | Y \rangle$$

$$= -i\hbar \frac{\partial}{\partial \varphi} \langle \theta, \varphi | Y \rangle$$

$$\Rightarrow -i\hbar \frac{\partial}{\partial \varphi} Y(\theta, \varphi) = m \hbar Y(\theta, \varphi) \Rightarrow Y(\theta, \varphi) \propto e^{im\varphi}$$

$\varphi \rightarrow \varphi + 2\pi$  should give the same proportional wavefunction

$$\text{so we have } e^{im\varphi} = e^{i(m\varphi + 2\pi)} \Rightarrow e^{i2\pi m} = 1$$

$\Rightarrow m$  to be an integer  $m = 0, \pm 1, \pm 2, \dots$

{ complex analysis (intro)

$$\alpha = |\alpha| e^{i\phi}$$

$$= |\alpha| \cos \phi + i |\alpha| \sin \phi$$

$$\alpha' = |\alpha| e^{i(\phi + 2\pi)} = |\alpha| e^{i\phi}$$

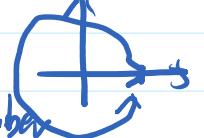
a rotation of  $2\pi$  is back to itself.

$$\text{here we have } 1 = e^{i(2\pi m)} = \cos 2\pi m + i \sin 2\pi m$$

$$1 = e^{i2\pi m} = e^{i2\pi(M+\delta)} = e^{i2\pi\delta} \Rightarrow \delta = 0$$

integer  $\delta \in [0, 1)$

$\alpha$  is a complex number



$\text{Re}(\alpha)$

$\overline{\alpha}$  is also  
if  $\alpha = 1$ ,  $\alpha = 1 \cdot e^{i \cdot 0}$

$$[L_z|Y\rangle = m\hbar|Y\rangle \rightarrow m \text{ to be integer.}]$$

for the continuous of angular coordinates.

further, we have

$$\langle \theta, \varphi | \vec{L}^2 | Y \rangle = \beta \hbar^2 \langle \theta, \varphi | Y \rangle$$

$$\vec{L}^2 = -\hbar^2 \left[ \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial}{\partial \theta} \right) + \frac{1}{\sin^2 \theta} \frac{\partial^2}{\partial \varphi^2} \right]$$

$$\text{by } \vec{L}^2 = L_x^2 + L_y^2 + L_z^2$$

not in the exams.

$$\Rightarrow - \left[ \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial}{\partial \theta} \right) + \frac{1}{\sin^2 \theta} \frac{\partial^2}{\partial \varphi^2} \right] Y(\theta, \varphi) = \beta Y(\theta, \varphi)$$

$$L_z Y(\theta, \varphi) = m\hbar Y(\theta, \varphi), \quad -i\hbar \frac{\partial}{\partial \varphi} Y(\theta, \varphi) = m\hbar Y(\theta, \varphi)$$

$$\frac{\partial^2}{\partial \varphi^2} Y(\theta, \varphi) = -m^2 Y(\theta, \varphi) \quad \text{consistent with } Y(\theta, \varphi) \propto e^{im\varphi}$$

$$\Rightarrow \left[ \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial}{\partial \theta} \right) + \frac{m^2}{\sin^2 \theta} - \beta \right] Y(\theta, \varphi) = 0$$

$\Rightarrow$  has a solution proportional to a special function  $P_l^m(\cos \theta)$  associated Legendre polynomial

$$\text{we need } \begin{cases} \beta = l(l+1), \quad l=0, 1, 2, \dots \text{ positive integer} \\ m = -l, -l+1, \dots, l-1, l \end{cases}$$

(a hint to solve this equation, is to let  $\xi = \cos \theta$  get a new equation  $(1-\xi^2) \frac{\partial^2 Y}{\partial \xi^2} - 2\xi \frac{\partial Y}{\partial \xi} + (\beta - \frac{m^2}{1-\xi^2}) Y = 0$ )

full solution: spherical harmonics definition

$$\text{function } Y_{lm} = Y(l\theta, \varphi) = \sqrt{\frac{(2l+1)(l-m)!}{4\pi(l+m)!}} P_l^m(\cos\theta) e^{im\varphi}$$

as normalized solution, with  $l=0, 1, 2, \dots$   
 $m = -l, -l+1, \dots, l$   
 $\underbrace{2l+1 \text{ choices of } m}$

Quantum number 量子数  
solution  $Y_{lm}$  needs two numbers to specify.  
like 1D 身份证.

• Some properties of  $Y_{lm}$

normalized  $\int_0^{2\pi} d\varphi \int_0^\pi d\theta \sin\theta (Y_{l'm'}(\theta, \varphi))^* Y_{lm}(\theta, \varphi)$

$\underbrace{\quad}_{\text{solid angle}}$

$$= \delta_{ll'} \delta_{mm'} = \begin{cases} 1, & \text{if } l=l', m=m' \\ 0, & \text{otherwise} \end{cases}$$

$Y_{lm}$  we know  $|Y\rangle = |l, m\rangle$

$\{|l, m\rangle\}$  form a basis,  $2l+1$  dimension

§2 General properties of angular momentum

section 3.5 of Sakurai

similar to  $\vec{s}, \vec{l}$ , we use  $\vec{J}$  for general case

have eigenstate as  $|\underline{j, m}\rangle$

with  $\Omega [J_k, J_L] = i\hbar \sum_{klm} J_m$ .

$$\textcircled{2} \quad [\hat{J}^2, J_z] = 0$$

$$J_z |j, m\rangle = m\hbar |j, m\rangle$$

$$J^2 |j, m\rangle = \beta \hbar^2 |j, m\rangle, \quad \beta = \beta(j, m) \neq j(j+1)$$

will be proved soon.

### o Ladder operator

$$J_+, J_-$$

$$J_+ = J_x + i J_y, \quad J_- = J_x - i J_y, \quad J_z = J_+^\dagger \quad (\text{recall definition of } a, a^\dagger)$$

$$\Rightarrow [J_z, J_+] = \hbar J_+, \quad [J_z, J_-] = -\hbar J_-$$

$$[J_+, J_-] = 2\hbar J_z$$

$$J_z J_+ = J_x^2 + J_y^2 + i[J_x, J_y] = J_x^2 + J_y^2 - \hbar J_z$$

$$J_- J_+ \underbrace{= J_x^2 + J_y^2 + J_z^2 = J^2}_{=} \quad J^2 - J_z^2 - \hbar J_z$$

$$J_+ J_- \underbrace{=} J^2 - J_z^2 + \hbar J_z$$

with  $J_z |j, m\rangle = m\hbar |j, m\rangle$  try to find properties  
apply  $J_+$

$$\begin{aligned} \underline{\underline{J_z J_+ |j, m\rangle}} &= ([J_z, J_+] + J_+ J_z) |j, m\rangle \\ &= (\hbar J_+ + J_+ J_z) |j, m\rangle \\ &= (m+1) \hbar J_+ |j, m\rangle \end{aligned}$$

$$J_z |\underline{\underline{j}}\rangle = (m+1) \hbar |\underline{\underline{j}}\rangle, \quad |\underline{\underline{j}}\rangle = J_+ |j, m\rangle$$

We know  $|\underline{\underline{j}}\rangle$  has something to do with  $(j, m+1)$   
 $J_+ |j, m\rangle \rightarrow |j, m+1\rangle \rightarrow$  increases quantum # by one unit.

Similarly, we have  $J_- |j, m\rangle \rightarrow c' |j, m-1\rangle$   
 here we try to find  $c$ , and  $c'$

① upper bound for  $|j, m\rangle = |j, m=M_{up}\rangle$

$$J_+ |j, M_{up}\rangle = 0$$

$$\begin{aligned} 0 &= J - J_+ |j, M_{up}\rangle = (J^2 - J_z^2 - \hbar J_z) |j, M_{up}\rangle \\ &= (\beta b^2 - M_{up}^2 \hbar^2 - M_{up} \hbar^2) |j, M_{up}\rangle \\ \Rightarrow \beta &= M_{up}(M_{up} + 1) \end{aligned}$$

② lower bound of  $|j, m\rangle = |j, m=M_{low}\rangle$

$$J_- |j, M_{low}\rangle = 0$$

$$0 = J_+ J_- |j, M_{low}\rangle \Rightarrow \beta = M_{low}(M_{low} - 1)$$

if we have

$$\beta = \frac{(-M_{low})(-M_{low} - 1)}{(-M_{low})(-M_{low} + 1)}$$

$$M_{up} = -M_{low} = j$$

$$\boxed{\beta = j(j+1)}$$

$$\boxed{J^2 |j, m\rangle = j(j+1) \hbar^2 |j, m\rangle}$$

$$\boxed{\begin{aligned} J_z |j, m\rangle &= m\hbar |j, m\rangle \\ m &= -j, -j+1, \dots, j-1, j \end{aligned}}$$

from  $[J^2, J_+] = 0$ ,  $[J^2, J_-] = 0$

$$\underline{J^2 J_+ |j, m\rangle = J_+ J^2 |j, m\rangle = j(j+1) \hbar^2 J_+ |j, m\rangle}$$

$$\boxed{\begin{aligned} J_+ |j, m\rangle &= \hbar \sqrt{j(j+1) - m(m+1)} |j, m+1\rangle \\ J_- |j, m\rangle &= \hbar \sqrt{j(j+1) - m(m-1)} |j, m-1\rangle \end{aligned}} \quad \star$$

some other properties

$$\Gamma J^2 \cdot J_+ J_- = 0 \quad \Gamma J^2 \cdot J_- J_+ = 0$$

o some other properties

$$[J^2, J+] = 0 \quad [J^2, J-] = 0$$

$\{j, m\}$  form a basis,  $\langle j', m' | j, m \rangle = \delta_{jj'} \delta_{mm'}$

example for  $|j, m\rangle$

find eigenstates of  $J_x$  in the basis of  $\{|j, m\rangle\}$

$$J_x = \frac{J+ + J-}{2} \Rightarrow \text{matrix element}$$

$$\begin{aligned} \langle j', m' | J_x | j, m \rangle &= \langle j', m' | \frac{J+ + J-}{2} | j, m \rangle \\ &= \frac{1}{2} \delta_{jj'} \left( \delta_{m', m+1} \sqrt{j(j+1) - m(m+1)} + \delta_{m', m-1} \right. \\ &\quad \left. \sqrt{j(j+1) + m(m-1)} \right) \end{aligned}$$

for example  $j=1$

$$J_x \text{ in this form } \cancel{j, m} \quad \begin{array}{ccccc} l, -1 & l, 0 & l, 1 & l, -1 & l, 0 \\ \cancel{j, m} & l, -1 & l, 0 & l, 1 & l, -1 \\ J_x = & \begin{pmatrix} 0 & \sqrt{2}/2 & 0 & 0 & \sqrt{2}/2 \\ \sqrt{2}/2 & 0 & 0 & \sqrt{2}/2 & 0 \end{pmatrix} & & & \end{array}$$

$$\Rightarrow \lambda=0 : |\psi_{\lambda=0}\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$$

$$\lambda=+1 \quad |\psi_{\lambda=1}\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} \frac{1}{\sqrt{2}} \\ 1 \\ \frac{1}{\sqrt{2}} \end{pmatrix}$$

$$\lambda=-1 \quad |\psi_{\lambda=-1}\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} \frac{1}{\sqrt{2}} \\ -1 \\ \frac{1}{\sqrt{2}} \end{pmatrix}$$