

機械手臂 軌跡規劃

Manipulator Trajectory Planning

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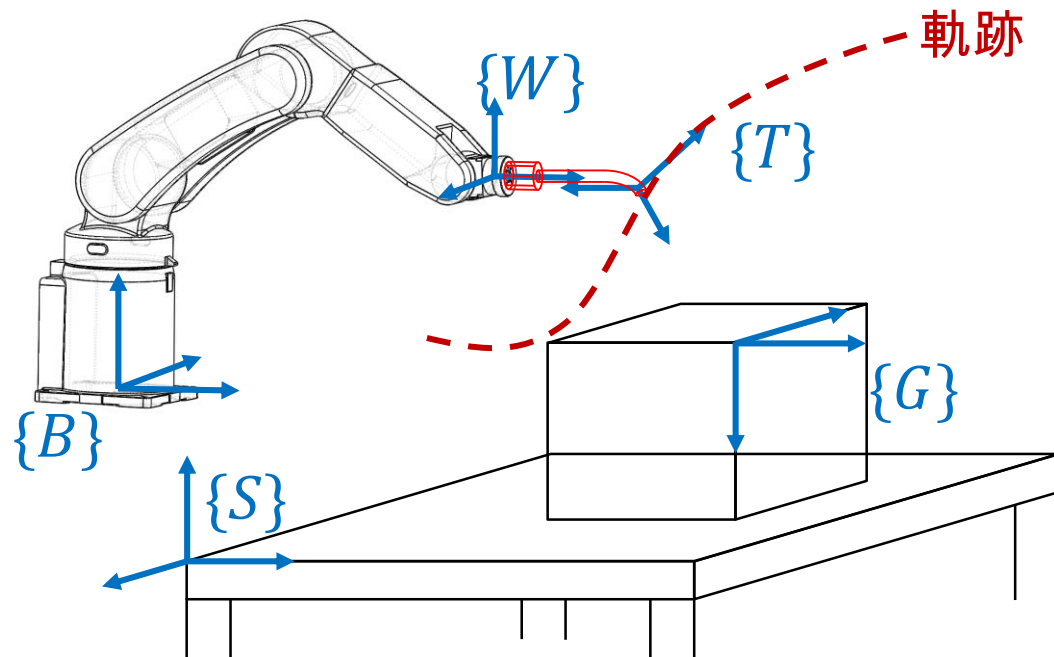


引言 -1

□ 軌跡：機械手臂（的末端點或操作點）的位置、速度、加速對對時間的歷程

□ 可進一步定義成 $\{T\}$ 相對 $\{G\}$ 的狀態歷程

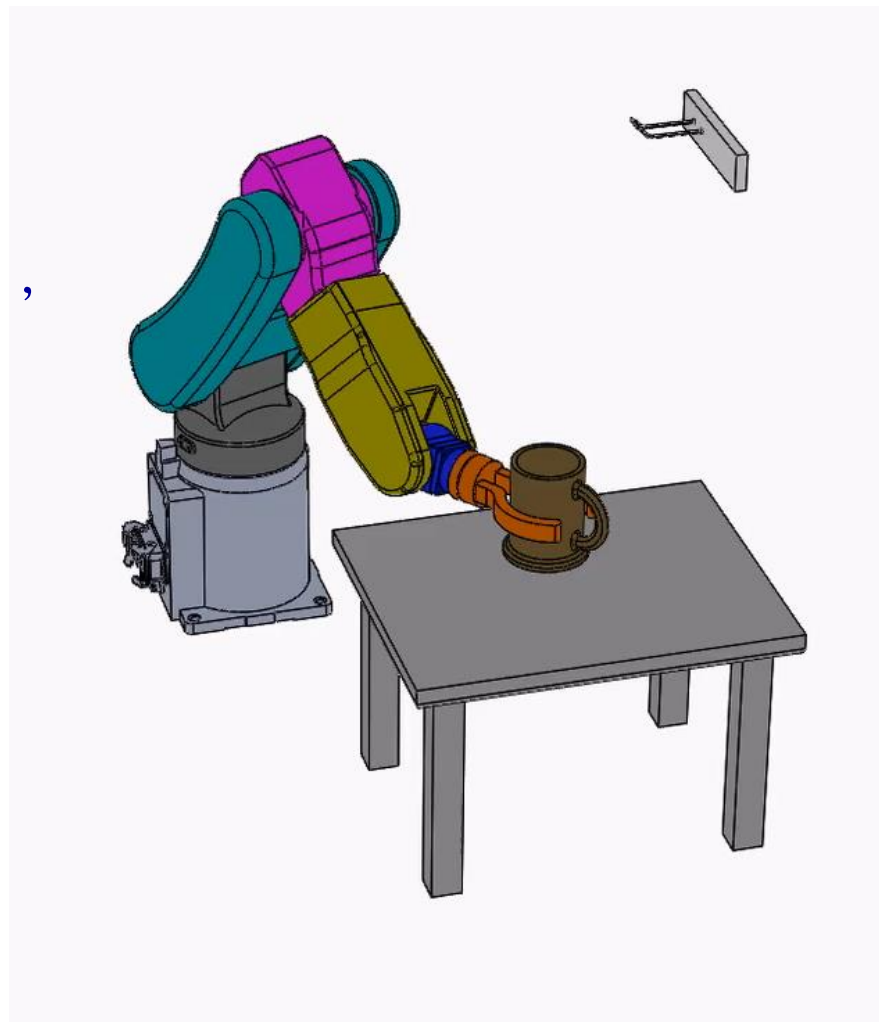
和手臂種類無關， $\{G\}$ 也可隨時間變動（如輸送帶）



引言 -2

□ 物件取放任務中手臂末端點（或物件）的運動歷程，即是軌跡規劃所進行的內容

- ◆ Example情境：
- ◆ 機械手臂夾住放在桌上的杯子，
移動手臂將杯子掛到
牆上的杯架

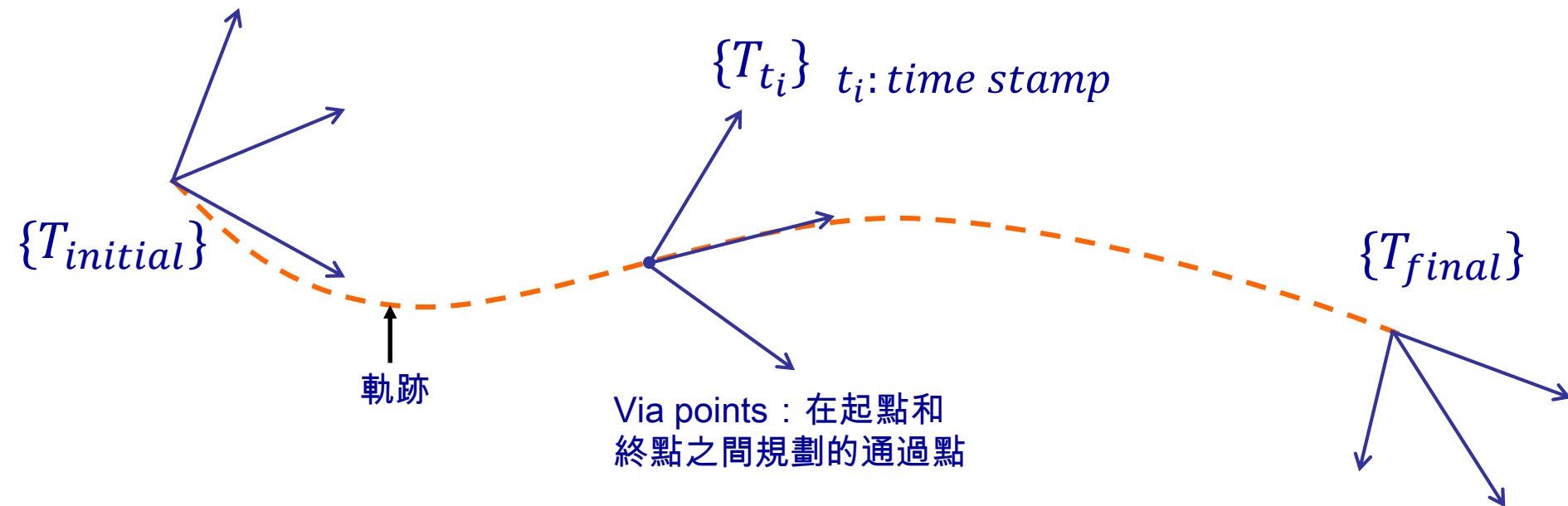


引言 -3

- 理想軌跡：Smooth path (i.e., continuous with continuous first derivative)

Smooth表示速度一定要Smooth，不可以突變，
这样轨迹会十分僵硬

常常為 G_T



Joint-space下的軌跡規劃 -1

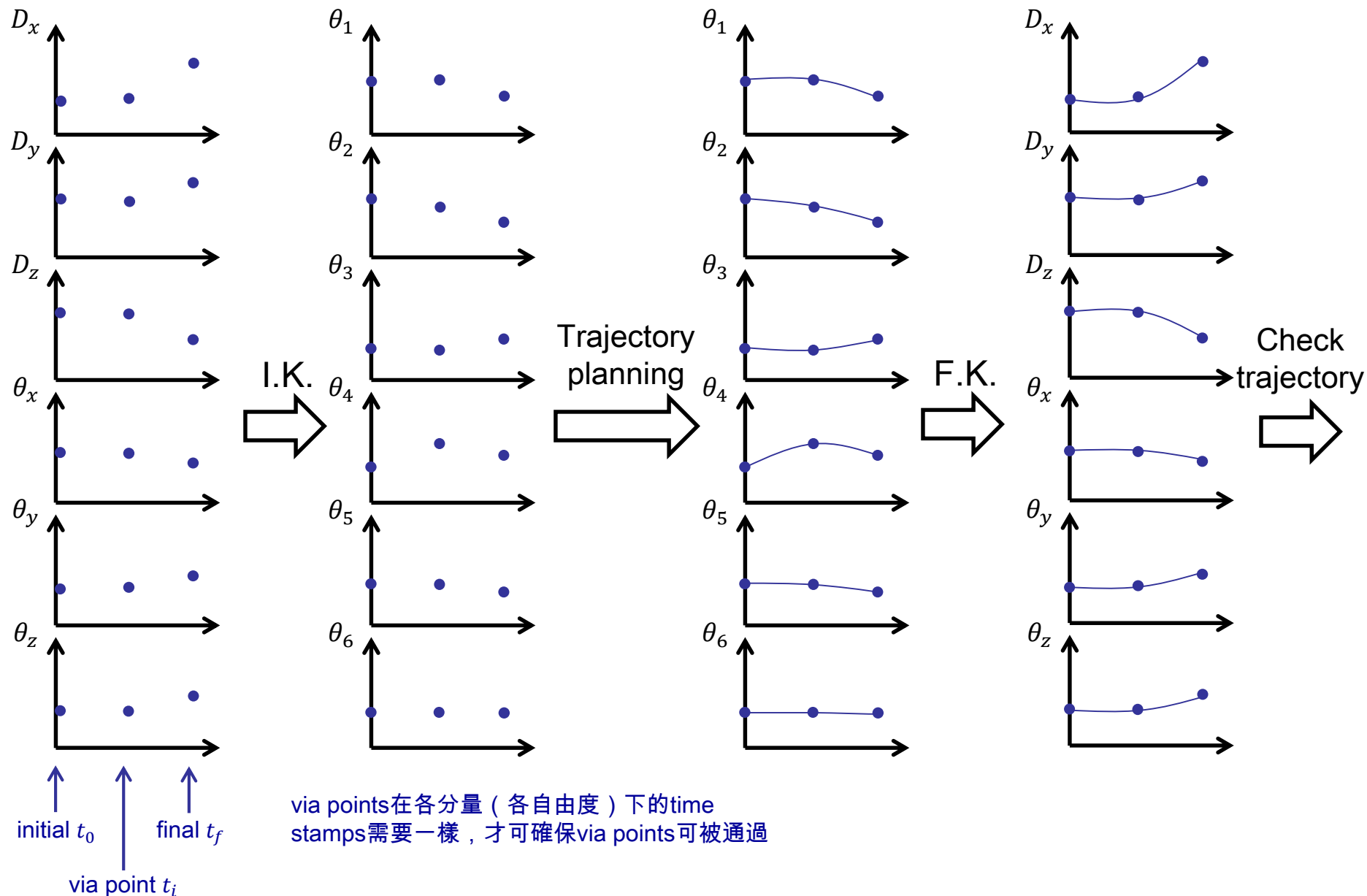
□ 步驟

- ◆ 定義{T}相對於{G}的initial, via, & final points , ${}^G_T T_i$
(包含移動和轉動自由度)
 - i=1 initial
 - i=2~N-1 via points
 - i=N+1 final

將 ${}^G_T T_i$ 以6個參數方式來表達 ${}^G X_T = \begin{bmatrix} {}^G P_{T \text{ org}} \\ \text{ROT}({}^G \hat{K}_T, \theta) \end{bmatrix}$
非rotation matrix的方式表達

- ◆ Inverse kinematics: 將手臂末端點狀態轉換到joint狀態 : ${}^G X_T \rightarrow \Theta_i$
- ◆ 對所有joints規劃smooth trajectories
- ◆ Forward Kinematics: 將joint狀態轉換到手臂末端點狀態，檢查莫短點在Cartesian-space下軌跡的可行性

Joint-space下的軌跡規劃 -2



Cartesian-space下的軌跡規劃 -1

□ 步驟

- ◆ 定義{T}相對於{G}的initial, via, & final points , ${}^G T_i$
(包含移動和轉動自由度)

$i=1$ initial
 $i=2 \sim N-1$ via points
 $i=N+1$ final

將 ${}^G T_i$ 以6個參數方式來表達 ${}^G X_T = \begin{bmatrix} {}^G P_{T \text{ org}} \\ ROT({}^G \hat{K}_T, \theta) \end{bmatrix}$

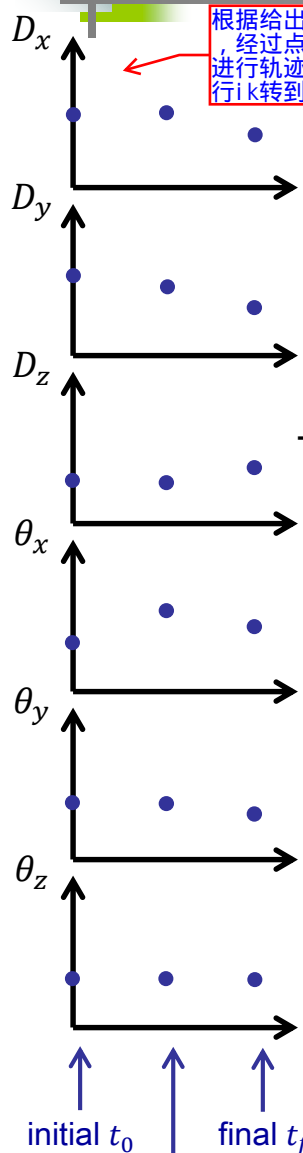
- ◆ 對所有手臂末端點狀態規劃smooth trajectories
- ◆ 將規劃好手臂末端點狀態的軌跡點轉換到joint狀態： ${}^G X_T \rightarrow \Theta_i$
- ◆ 檢查joint狀態在Joint-space下軌跡的可行性

□ Comments

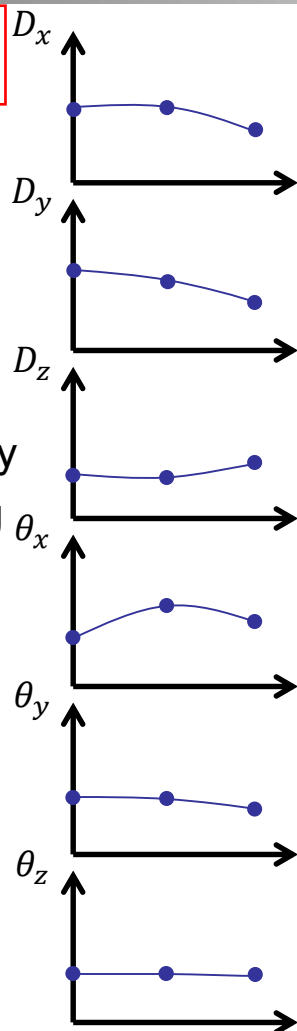
- ◆ 較具物理直觀的軌跡
- ◆ 較高的運算負載 (i.e., IK)

Cartesian-space下的軌跡規劃 -2

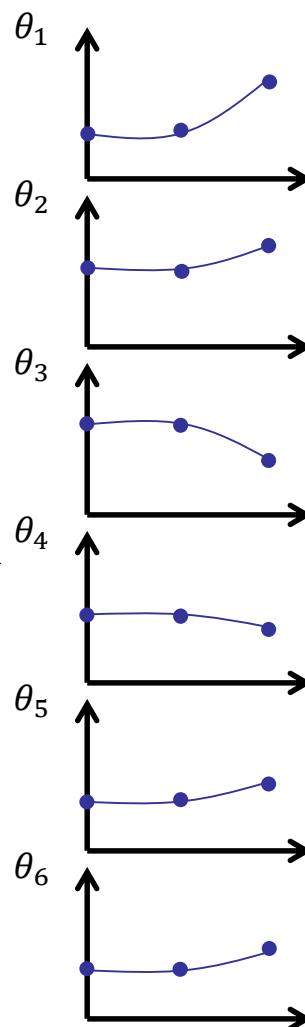
根据给出的点：初始点，经过点，末端点，来进行轨迹选择，之后进行ik转到关节空间下面



Trajectory
planning



I.K.



F.K.

Check
trajectory

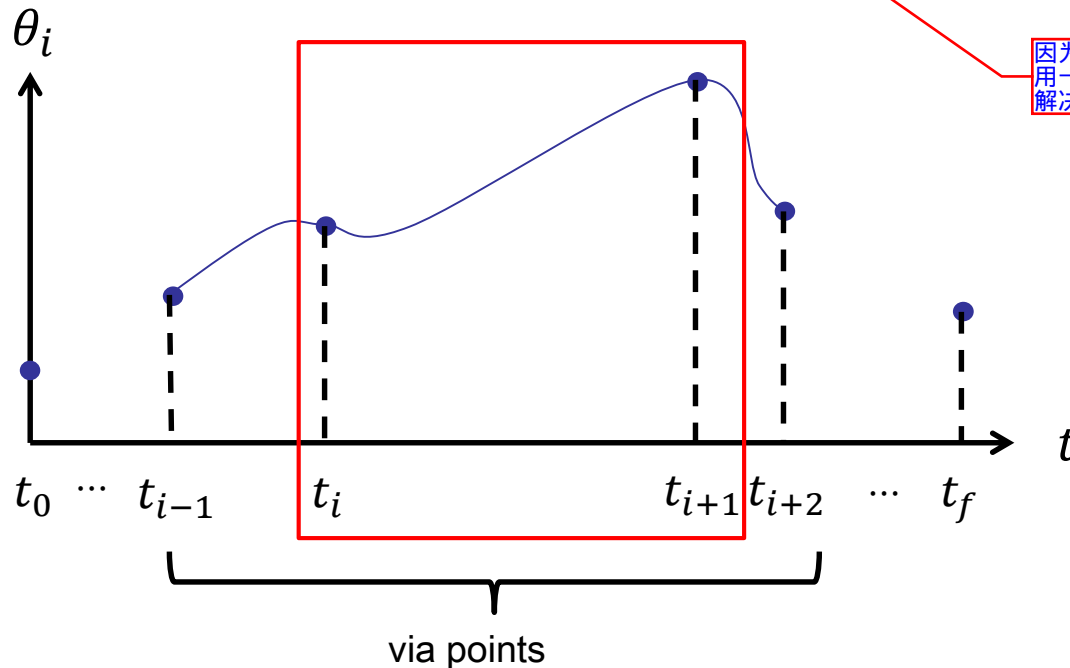
Cubic Polynomials說明 -1

□ 原則

- ◆ 軌跡：不同軌跡區段 $[t_i \ t_{i+1}]$ 以不同參數的函數來規劃
- ◆ Smooth: 需定義各函數的邊界條件（包含位置和速度
 $\theta(t_i), \theta(t_{i+1}), \dot{\theta}(t_i), \dot{\theta}(t_{i+1})$ ，有4個條件）
- ◆ 以cubic polynomial（三次多項式）來規劃軌跡

如果要保证轨迹是smooth的，那么就要保证头尾两个地方的位置条件和速度条件

因为是4个条件，所以要用一个三次的多项式来解决



- 1 linear
- 2 quadratic
- 3 cubic
- 4 quartic
- 5 quintic
- 6 hexic (sextic)
- 7 heptic (septic)
- 8 octic
- 9 nonic
- 10 decic

Cubic Polynomials說明-2

□ 解cubic polynomial

◆ 通式

$$\theta(\tilde{t}) = a_0 + a_1\tilde{t} + a_2\tilde{t}^2 + a_3\tilde{t}^3 \quad 4 \text{ unknowns: } a_j \quad j=0\sim 3$$

◆ 對每一個區段： $t \in [t_i, t_{i+1}]$

$$\tilde{t} = t - t_i \quad \text{so } \tilde{t}|_{t=t_i} = 0 \text{ and } \tilde{t}|_{t=t_{i+1}} \equiv \Delta t = t_{i+1} - t_i > 0$$

每一個區段 $[t_i, t_{i+1}]$ 的 Δt 可以不同，取決於via points的設定

邊界條件

$$\theta(\tilde{t}|_{t=t_i}) = \theta_i = a_0 \quad \text{①}$$

$$\theta(\tilde{t}|_{t=t_{i+1}}) = \theta_{i+1} = a_0 + a_1\Delta t + a_2\Delta t^2 + a_3\Delta t^3 \quad \text{②}$$

$$\dot{\theta}(\tilde{t}|_{t=t_i}) = \dot{\theta}_i = a_1 \quad \text{③}$$

$$\dot{\theta}(\tilde{t}|_{t=t_{i+1}}) = \dot{\theta}_{i+1} = a_1 + 2a_2\Delta t + 3a_3\Delta t^2 \quad \text{④}$$

Cubic Polynomials 說明-3

□ 解聯立方程式 ① ② ③ ④

$$a_2 = \frac{3}{\Delta t^2} (\theta_{i+1} - \theta_i) - \frac{2}{\Delta t} \dot{\theta}_i - \frac{1}{\Delta t} \dot{\theta}_{i+1}$$

$$a_3 = -\frac{2}{\Delta t^3} (\theta_{i+1} - \theta_i) + \frac{1}{\Delta t^2} (\dot{\theta}_{i+1} + \dot{\theta}_i)$$

Cubic Polynomials說明-4

- 以Matrix operation方式運算

$$\begin{bmatrix} \theta_i \\ \theta_{i+1} \\ \dot{\theta}_i \\ \dot{\theta}_{i+1} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & \Delta t & \Delta t^2 & \Delta t^3 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 2\Delta t & 3\Delta t^2 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix}$$

$$\Theta = T_{4 \times 4}(\Delta t) \cdot A$$

$$\det(T_{4 \times 4}) = -\Delta t^4 \neq 0 \text{ as long as } \Delta t \neq 0$$

所以

$$A = T_{4 \times 4}^{-1} \cdot \Theta$$

$$\begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\frac{3}{\Delta t^2} & \frac{3}{\Delta t^2} & -\frac{2}{\Delta t} & -\frac{1}{\Delta t} \\ \frac{2}{\Delta t^3} & -\frac{2}{\Delta t^2} & \frac{1}{\Delta t^2} & \frac{1}{\Delta t^2} \end{bmatrix} \begin{bmatrix} \theta_i \\ \theta_{i+1} \\ \dot{\theta}_i \\ \dot{\theta}_{i+1} \end{bmatrix}$$

多段Cubic Polynomials -1

□ 如何選擇速度條件 $\dot{\theta}_i$ 和 $\dot{\theta}_{i+1}$?

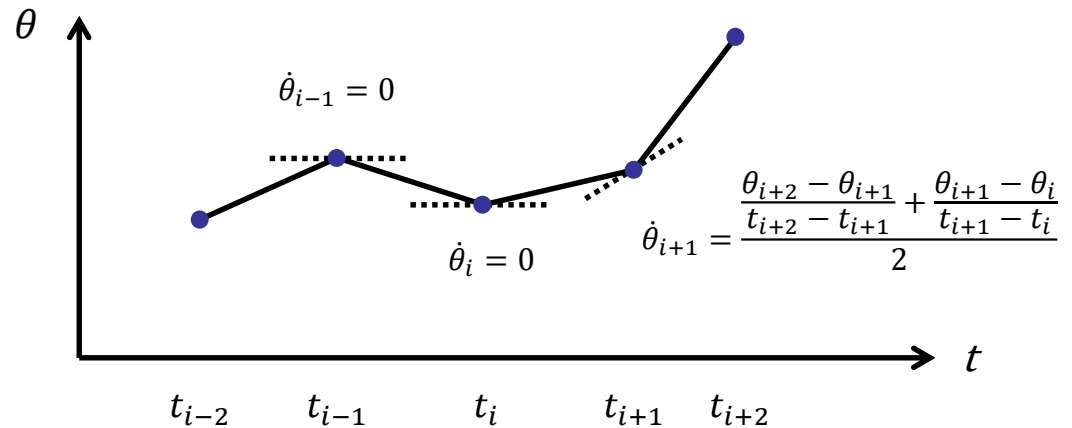
◆ 直接定義，不論在 Cartesian space 或 joint space

不建議，過於複雜，尤其軌跡落在singular points附近時

◆ 自動生成

Ex: 如果 $\dot{\theta}_i$ 在 t_i 前後變號，選擇 $\dot{\theta}_i = 0$

如果 $\dot{\theta}_i$ 在 t_i 前後同號，選擇平均



以此兩個方式，不同區段的Cubic polynomials可以分開求解

多段Cubic Polynomials -2

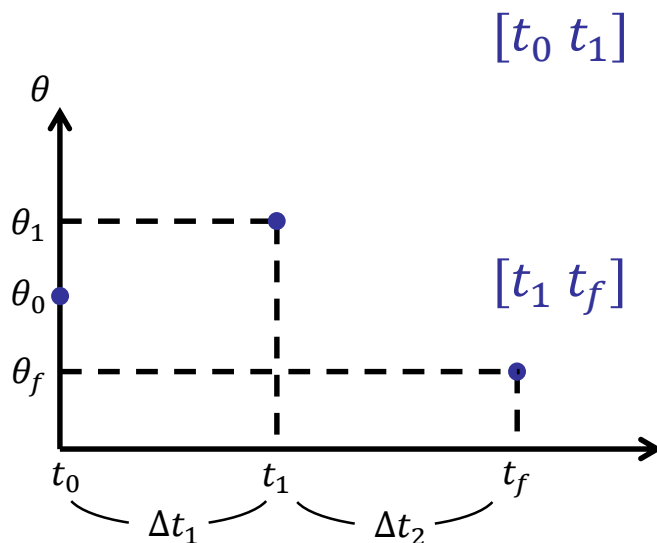
- ◆ 規劃速度使過程中加速度也CONTINUOUS

(有效使用此可調控的變數)

不同區段的Cubic polynomials需要整合一起，一併求解

- ◆ Example: A trajectory with one via point

各段在運算時將時間平移到0開始



$[t_0 t_1]$

$$\tilde{t} = t_1 - t_0 \quad [0 \Delta t_1]$$

$$\theta_I(\tilde{t}) = a_{10} + a_{11}\tilde{t} + a_{12}\tilde{t}^2 + a_{13}\tilde{t}^3$$

$[t_1 t_f]$

$$\tilde{t} = t_f - t_1 \quad [0 \Delta t_2]$$

$$\theta_{II}(\tilde{t}) = a_{20} + a_{21}\tilde{t} + a_{22}\tilde{t}^2 + a_{23}\tilde{t}^3$$

$\Rightarrow 8 \text{ unknowns}$

多段Cubic Polynomials -3

- ◆ Example: A trajectory with one via point (cont.)

4 position B.C.s

2 for each $\theta_j(t)$ $j = I, II$

$$\begin{cases} \theta_0 = a_{10} \\ \theta_1 = a_{10} + a_{11}\Delta t_1 + a_{12}\Delta t_1^2 + a_{13}\Delta t_1^3 \\ \theta_1 = a_{20} \\ \theta_f = a_{20} + a_{21}\Delta t_2 + a_{22}\Delta t_2^2 + a_{23}\Delta t_2^3 \end{cases}$$

2 velocity B.C.s

$$\begin{cases} \dot{\theta}_0 = 0 = a_{11} \\ \dot{\theta}_f = 0 = a_{21} + 2a_{22}\Delta t_2 + 3a_{23}\Delta t_2^2 \end{cases}$$

not necessary "0"

等号左边是第一段的式子，此时对于第一段来说是右边界，右边是第二段的式子，对于第二段来说是左边界，代入0

Via point

velocity continuity

acceleration continuity

$$\begin{cases} \dot{\theta}_1 = a_{11} + 2a_{12}\Delta t_1 + 3a_{13}\Delta t_1^2 = a_{21} \\ \ddot{\theta}_1 = 2a_{12} + 6a_{13}\Delta t_1 = 2a_{22} \end{cases}$$

$\Rightarrow 8 \text{ equations}$

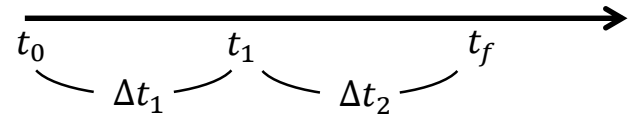
多段Cubic Polynomials -4

- ◆ Example: A trajectory with one via point (cont.)

8 equations, 8 unknowns

代數解法

(when $\Delta t_1 = \Delta t_2 = \Delta t$)



$$a_{10} = \theta_0$$

$$a_{11} = 0$$

$$a_{12} = \frac{12\theta_1 - 3\theta_f - 9\theta_0}{4\Delta t^2}$$

$$a_{13} = \frac{-8\theta_1 + 3\theta_f + 5\theta_0}{4\Delta t^3}$$

$$a_{20} = \theta_1$$

$$a_{21} = \frac{3\theta_f - 3\theta_0}{4\Delta t}$$

$$a_{22} = \frac{-12\theta_1 + 6\theta_f + 6\theta_0}{4\Delta t^2}$$

$$a_{23} = \frac{8\theta_1 - 5\theta_f - 3\theta_0}{4\Delta t^3}$$

多段Cubic Polynomials -5

- ◆ Example: A trajectory with one via point (cont.)

矩陣解法

$$\Theta_{8 \times 1} = T_{8 \times 8} A_{8 \times 1}$$

$$[0 \ \Delta t_1] \quad \theta_I(\tilde{t}) = a_{10} + a_{11}\tilde{t} + a_{12}\tilde{t}^2 + a_{13}\tilde{t}^3$$

$$[0 \ \Delta t_2] \quad \theta_{II}(\tilde{t}) = a_{20} + a_{21}\tilde{t} + a_{22}\tilde{t}^2 + a_{23}\tilde{t}^3$$

$$\begin{bmatrix} \theta_0 \\ \theta_1 \\ \theta_1 \\ \theta_f \\ \dot{\theta}_0 \\ \dot{\theta}_f \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & \Delta t_1 & \Delta t_1^2 & \Delta t_1^3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & \Delta t_2 & \Delta t_2^2 & \Delta t_2^3 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 2\Delta t_2 & 3\Delta t_2^2 \\ 0 & 1 & 2\Delta t_1 & 3\Delta t_1^2 & 0 & -1 & 0 & 0 \\ 0 & 0 & 2 & 6\Delta t_1 & 0 & 0 & -2 & 0 \end{bmatrix} \begin{bmatrix} a_{10} \\ a_{11} \\ a_{12} \\ a_{13} \\ a_{20} \\ a_{21} \\ a_{22} \\ a_{23} \end{bmatrix}$$

$$A_{8 \times 1} = T_{8 \times 8}^{-1} \Theta_{8 \times 1}$$

中间矩阵已知，如果有解那么中间矩阵不可以为0

$$\det(T_{8 \times 8}) = 4\Delta t_1^4 \Delta t_2^3 + 4\Delta t_1^3 \Delta t_2^4$$

$$\neq 0 \text{ as long as } \Delta t_1 \neq 0, \Delta t_2 \neq 0, \Delta t_1 \neq -\Delta t_2$$

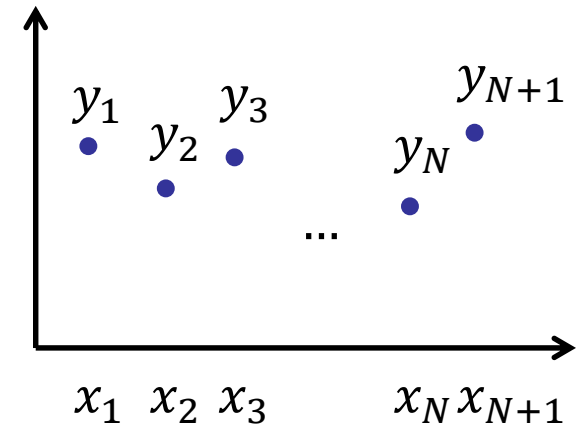
把之前的八个等式转化成矩阵形式

General Cubic Polynomials -1

之前讨论的是中间点为1个的情况下，现在开始讨论中间点有N-1个并且保证速度连续的情况

□ General cubic spline function

N+1 set points (x_i, y_i) $\left\{ \begin{array}{l} 1 \\ N-1 \\ 1 \end{array} \right.$ $\begin{array}{l} \text{initial} \\ \text{via} \\ \text{final} \end{array}$



N cubic functions

$$s_j(x) = a_j + b_j x + c_j x^2 + d_j x^3$$

$$\begin{array}{l} x_j \leq x \leq x_{j+1} \\ j = 1 \dots N \end{array}$$

⇒ total 4N unknown coefficients

General Cubic Polynomials -2

Position conditions at both ends of each $s_j(x)$

⇒ $2N$ conditions

头尾的位置条件，一段就有两个，总共有N段，就有 $2 \cdot N$ 个

Velocity & acceleration continuity conditions at via points

⇒ $2(N-1)$ conditions

via point的速度条件

還需要 **2 CONDITIONS** 以求解

举例子，假设有3个点2段，那么下面列出了6个等式，还差两个条件，复合上面的推论

Revisit example: A trajectory with one via point

$$\begin{bmatrix} y_1 = s_1(x_1) \\ y_2 = s_1(x_2) \end{bmatrix} \quad \begin{bmatrix} y_2 = s_2(x_2) \\ y_3 = s_2(x_3) \end{bmatrix}$$

$$\dot{y}_2 = \dot{s}_1(x_2) = \dot{s}_2(x_2)$$

$$\ddot{y}_2 = \ddot{s}_1(x_2) = \ddot{s}_2(x_2)$$

共 6 conditions

General Cubic Polynomials -3

定义头尾点的条件

最後 2 conditions 的選擇方法

(1) $\ddot{s}_1(x_1) = \ddot{s}_N(x_{N+1}) = 0$

定義加速度，Natural cubic spline

3选1

(2) $\dot{s}_1(x_1) = u \quad \dot{s}_N(x_{N+1}) = v$

定義速度，Clamped cubic spline

(3) $if \quad s_1(x_1) = s_N(x_{N+1})$
 $use \quad \dot{s}_1(x_1) = \dot{s}_N(x_{N+1})$
 $\quad \ddot{s}_1(x_1) = \ddot{s}_N(x_{N+1})$

週期運動的連續性，Periodic cubic spline

Note: Matlab® command *spline*

$$[YY] = spline(x, y, XX)$$

Example: A RRR Manipulator -1

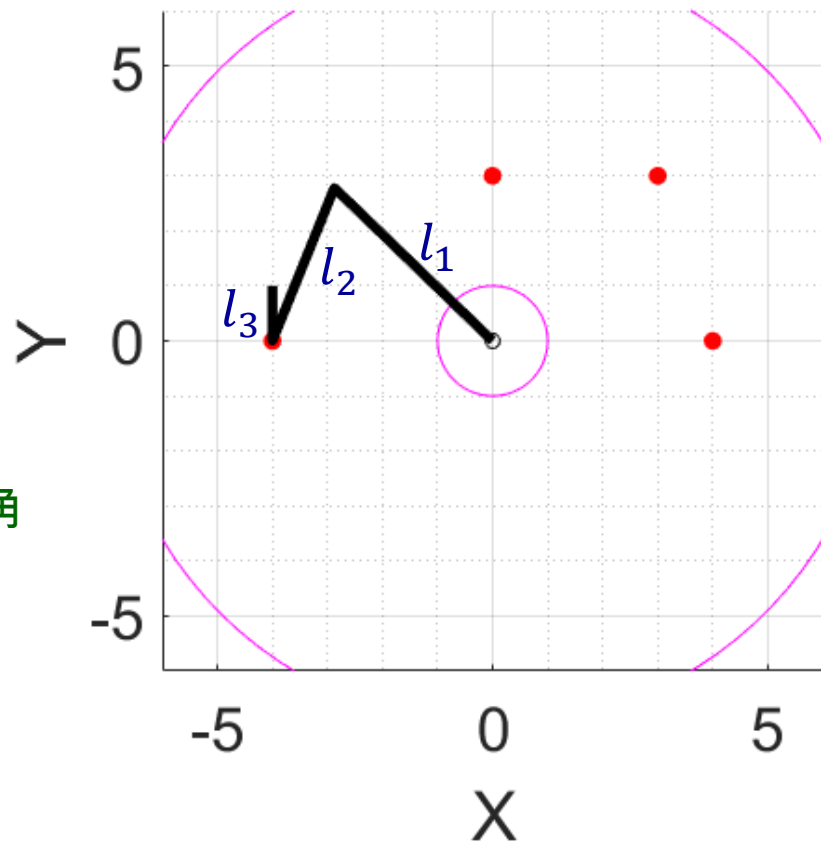
- 平面RRR手臂長度： $l_1 = 4$, $l_2 = 3$, and $l_3 = 1$

下表定義 initial, via, via, 和 final points的位置

t	x	y	theta
0	-4	0	90
2	0	3	45
4	3	3	30
7	4	0	0

(X,Y) 定義在第二桿件的末端

Theta為第三桿件對X座標軸的夾角



Example: A RRR Manipulator -2

□ 方法一：以cubic polynomials在Cartesian-space下規劃軌跡

1. 求出3個DOF (X, Y, θ) 各自cubic polynomials的coefficients

需通過4個點：每個DOF有3個cubic polynomials，共12個未知數

$$\mathbf{Q}_{12 \times 1} = \mathbf{T}_{12 \times 12} \mathbf{A}_{12 \times 1}$$

為 $(\Delta t_1, \Delta t_2, \Delta t_3)$ 函數

X/Y/ θ

$$\begin{array}{l}
 \text{6 position} \\
 \text{B.C.s} \\
 \text{2 init., final} \\
 \text{vel.} \\
 \text{4 vel. \& acl.} \\
 \text{continuity}
 \end{array}
 \begin{bmatrix}
 \theta_0 \\
 \theta_1 \\
 \theta_1 \\
 \theta_2 \\
 \theta_2 \\
 \theta_f \\
 \dot{\theta}_0 \\
 \dot{\theta}_f \\
 0 \\
 0 \\
 0 \\
 0
 \end{bmatrix}
 =
 \begin{bmatrix}
 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 1 & \Delta t_1 & \Delta t_1^2 & \Delta t_1^3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 0 & 0 & 0 & 0 & 1 & \Delta t_2 & \Delta t_2^2 & \Delta t_2^3 & 0 & 0 & 0 & 0 \\
 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & \Delta t_3 & \Delta t_3^2 & \Delta t_3^3 \\
 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 2\Delta t_3 & 3\Delta t_3^2 \\
 0 & 1 & 2\Delta t_1 & 3\Delta t_1^2 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
 0 & 0 & 2 & 6\Delta t_1 & 0 & 0 & -2 & 0 & 0 & 0 & 0 & 0 \\
 0 & 0 & 0 & 0 & 0 & 1 & 2\Delta t_2 & 3\Delta t_2^2 & 0 & -1 & 0 & 0 \\
 0 & 0 & 0 & 0 & 0 & 0 & 2 & 6\Delta t_2 & 0 & 0 & -2 & 0
 \end{bmatrix}
 \begin{bmatrix}
 a_{10} \\
 a_{11} \\
 a_{12} \\
 \vdots \\
 \vdots \\
 \vdots \\
 \vdots \\
 \vdots \\
 \vdots \\
 a_{31} \\
 a_{32} \\
 a_{33}
 \end{bmatrix}$$

Example: A RRR Manipulator -3

- 方法一：以cubic polynomials在Cartesian-space下規劃軌跡

將三個DOF在各點的參數代入matlab後將會是：

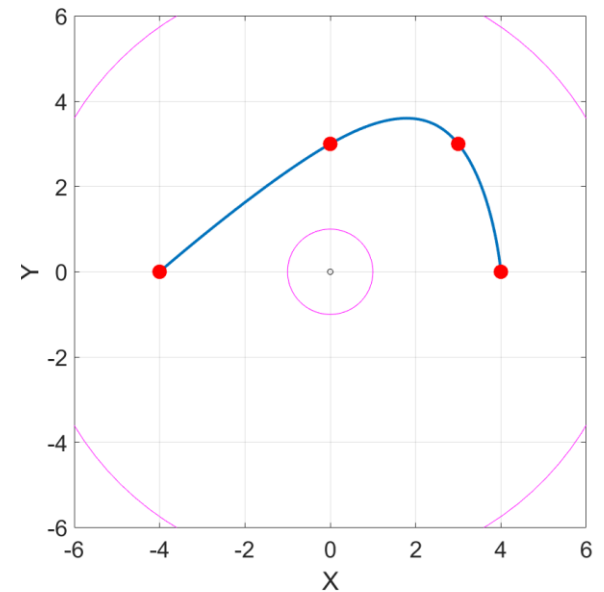
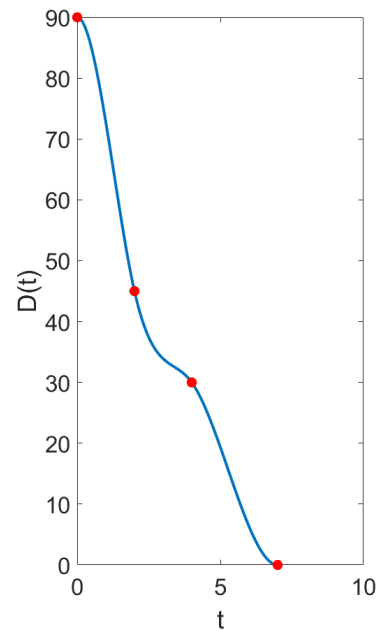
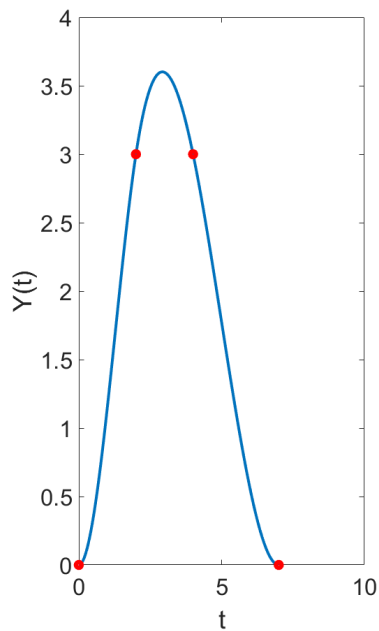
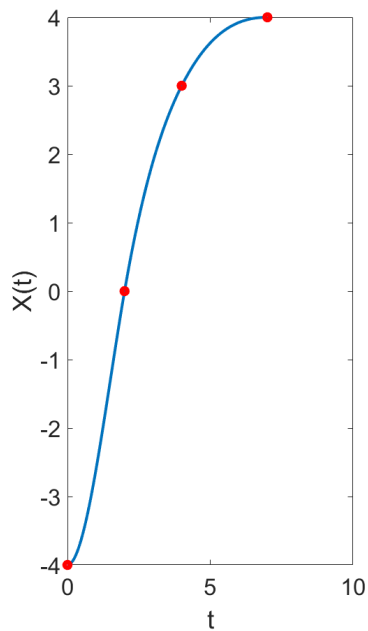
$$\theta_{12 \times 3} = T_{12 \times 12} A_{12 \times 3}$$

$$\begin{matrix} X & Y & \theta \end{matrix} \begin{bmatrix} -4 & 0 & 90 \\ 0 & 3 & 45 \\ 0 & 3 & 45 \\ 3 & 3 & 30 \\ 3 & 3 & 30 \\ 4 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 2 & 4 & 8 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 2 & 4 & 8 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 3 & 9 & 27 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 6 & 27 & 0 \\ 0 & 1 & 4 & 12 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 12 & 0 & 0 & -2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 4 & 12 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2 & 12 & 0 & 0 & -2 & 0 & 0 \end{bmatrix} \begin{bmatrix} A_{12 \times 3} \end{bmatrix}$$

Example: A RRR Manipulator -4

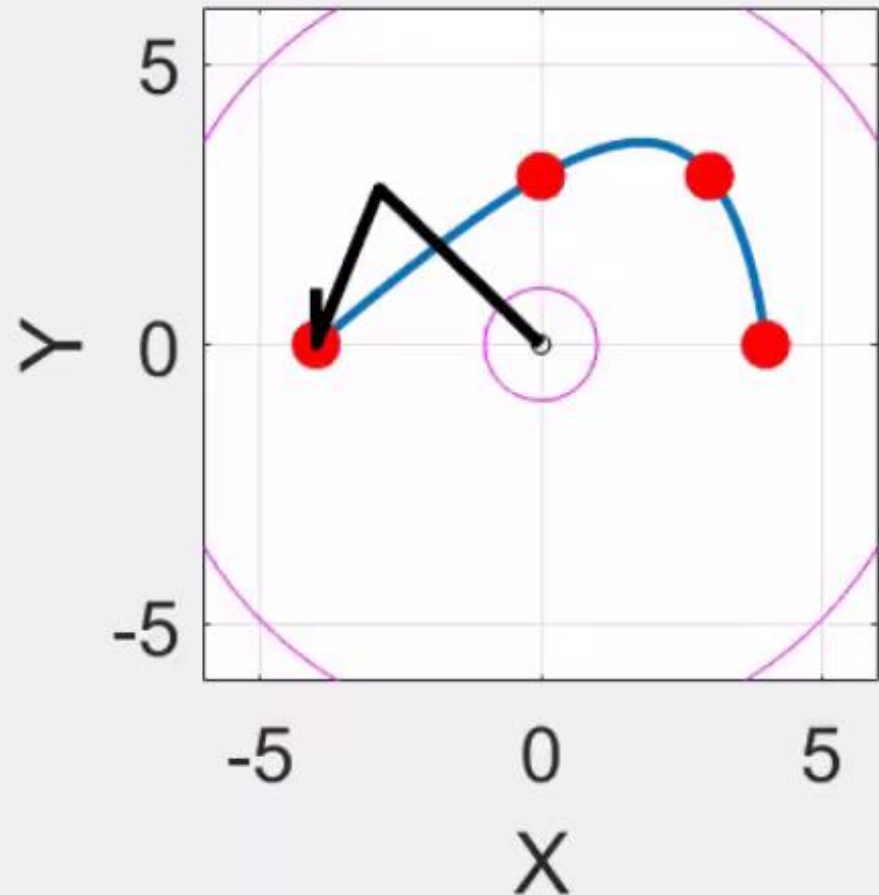
2. 對所有DOF規劃smooth trajectories

t	x	y	theta
0	-4	0	90
2	0	3	45
4	3	3	30
7	4	0	0



Example: A RRR Manipulator -5

3. I.K.，找出各Joints的對應軌跡
4. 將Joints帶入手臂模擬動作，確認手臂末端點軌跡如規劃運作



Example: A RRR Manipulator -6

□ 方法二：以cubic polynomials在Joint-space下規劃軌跡

1. I.K.，求出initial、via、final points的Joint angles ($\theta_1, \theta_2, \theta_3$)

2. 求出各($\theta_1, \theta_2, \theta_3$) cubic polynomials的coefficients

需通過4個點：每個Joint angle有3個cubic polynomials，共12個未知數

$$\Theta_{12 \times 3} = T_{12 \times 12} A_{12 \times 3}$$

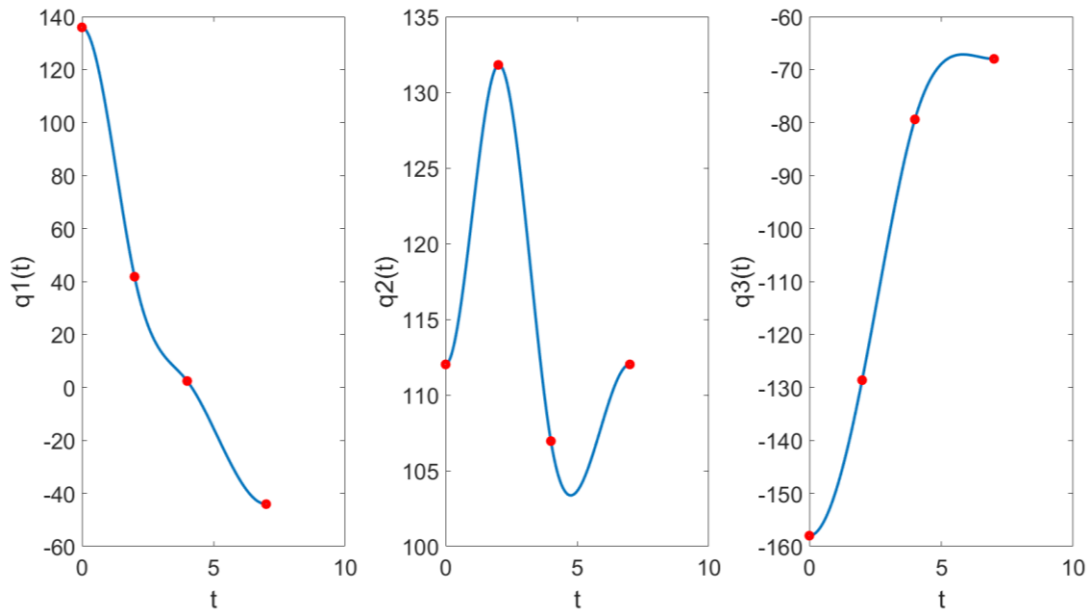
4 vel. & acl. continuity 6 position B.C.s

$$\begin{bmatrix} \theta_1 & \theta_2 & \theta_3 \end{bmatrix} = \begin{bmatrix} 2.3728 & 1.9552 & -2.7572 \\ 0.7297 & 2.3005 & -2.2449 \\ 0.7297 & 2.3005 & -2.2449 \\ 0.0426 & 1.8668 & -1.3858 \\ 0.0426 & 1.8668 & -1.3858 \\ -0.7688 & 1.9552 & -1.1864 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 2 & 4 & 8 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 2 & 4 & 8 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 3 & 9 & 27 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 6 & 27 \\ 0 & 1 & 4 & 12 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 12 & 0 & 0 & -2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 4 & 12 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2 & 12 & 0 & 0 & -2 & 0 \end{bmatrix} \begin{bmatrix} A_{12 \times 3} \end{bmatrix}$$

Example: A RRR Manipulator -7

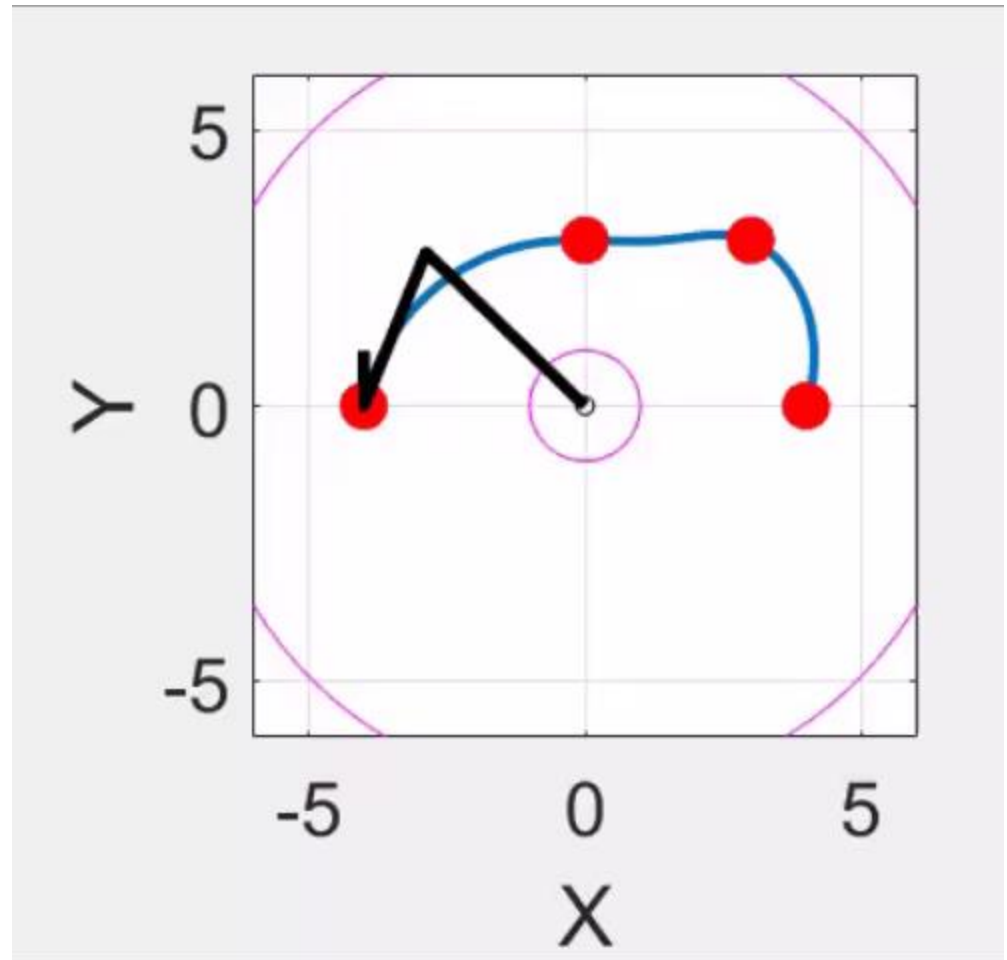
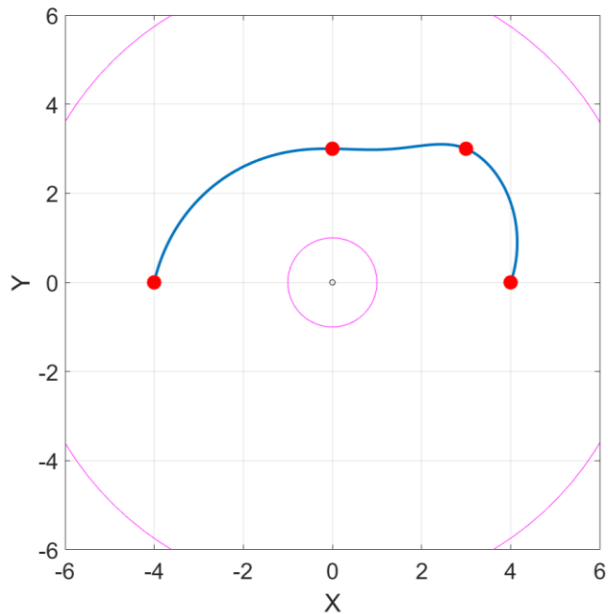
3. 對所有DOF規劃smooth trajectories

t	x	y	theta
0	-4	0	90
2	0	3	45
4	3	3	30
7	4	0	0



Example: A RRR Manipulator -8

4. 將Joints帶入手臂模擬動作，確認手臂末端點軌跡如規劃運作



High-order Polynomials

□ 如果位置、速度、和加速度都必須要規劃

⇒ Quintic polynomial $\theta(t) = a_0 + a_1t + a_2t^2 + a_3t^3 + a_4t^4 + a_5t^5 = \sum_{i=0}^5 a_i t^i$

有6個條件，quintic polynomial有6個未知數

$$\left\{ \begin{array}{l} \theta_0 = a_0 \\ \theta_f = a_0 + a_1\Delta t + a_2\Delta t^2 + a_3\Delta t^3 + a_4\Delta t^4 + a_5\Delta t^5 \\ \dot{\theta}_0 = a_1 \\ \dot{\theta}_f = a_1 + 2a_2\Delta t + 3a_3\Delta t^2 + 4a_4\Delta t^3 + 5a_5\Delta t^4 \\ \ddot{\theta}_0 = 2a_2 \\ \ddot{\theta}_f = 2a_2 + 6a_3\Delta t + 12a_4\Delta t^2 + 20a_5\Delta t^3 \end{array} \right.$$

$$a_0 = \theta_0 \quad a_3 = \frac{20(\theta_f - \theta_0) - (8\dot{\theta}_f + 12\dot{\theta}_0)\Delta t - (3\ddot{\theta}_0 - \ddot{\theta}_f)\Delta t^2}{2\Delta t^3}$$

$$a_1 = \dot{\theta}_0 \quad a_4 = \frac{30(\theta_0 - \theta_f) + (14\dot{\theta}_f + 16\dot{\theta}_0)\Delta t - (3\ddot{\theta}_0 - 2\ddot{\theta}_f)\Delta t^2}{2\Delta t^4}$$

$$a_2 = \frac{1}{2}\ddot{\theta}_0 \quad a_5 = \frac{12(\theta_f - \theta_0) - (6\dot{\theta}_f + 6\dot{\theta}_0)\Delta t - (\ddot{\theta}_0 - \ddot{\theta}_f)\Delta t^2}{2\Delta t^5}$$

1 linear

2 quadratic

3 cubic

4 quartic

5 quintic

6 hexic (sextic)

7 heptic (septic)

8 octic

9 nonic

10 decic