Chap 5 Jacobians: Velocities and Static Forces

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Time-varying Position and Orientation -1

- Differentiation of a position vector P_Q

Q向量在B坐标系下的速度

$$ {}^{B}V_Q = \frac{d}{dt}\,{}^{B}P_Q = \lim_{\Delta t \to 0} \frac{{}^{B}P_Q(t+\Delta t) - {}^{B}P_Q(t)}{\Delta t} $$

Derivative of position vector ${}^{B}P_Q$ relative to frame $\{B\}$

如果要求Q向量在A参考系下的速度，从定义上来解决就是这个式子

$$ {}^{A}({}^{B}V_Q) = {}^{A}(\frac{d}{dt}\,{}^{B}P_Q) $$

Expressed in frame $\{A\}$

Q对时间的微分，在{B}坐标下看，其实就是Q在B坐标下面的速度

$$ = {}^{A}_{B}R\,{}^{B}({}^{B}V_Q) = {}^{A}_{B}R\,{}^{B}V_Q $$

此处就是假设在B坐标下微分并且在B坐标下看，然后再通过旋转矩阵转移到由A坐标看

When both frames are the same

$$ \upsilon_C = {}^{U}V_{C\,ORG} $$

vc默认是对地的速度

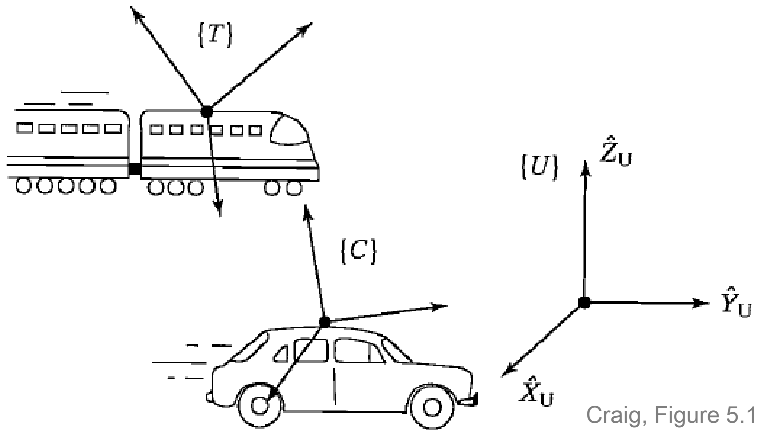
Velocity of the origin of frame $\{C\}$ relative to the universe reference frame $\{U\}$

Time-varying Position and Orientation -2

- Example

$^{U}V_T = 100\hat{\imath}$

$^{U}V_C = 30\hat{\imath}$



Craig, Figure 5.1

$$^{U}\left(\frac{d}{dt}\,^{U}P_{C\,ORG}\right) = \,^{U}V_{C\,ORG} = v_C = 30\hat{\imath}$$

$$^{C}(\,^{U}V_{T\,ORG}) = \,^{C}v_T = \,^{C}_{U}R(v_T) = \,^{C}_{U}R(100\hat{\imath}) = \,^{U}_{C}R^{-1}100\hat{\imath}$$

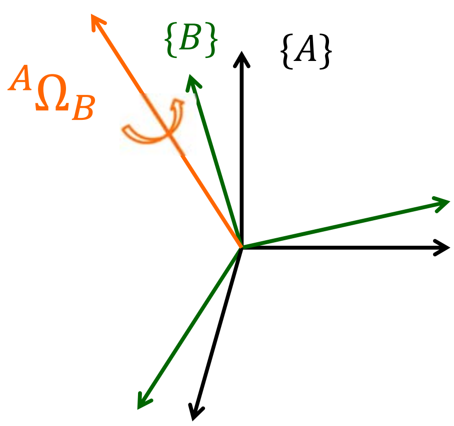
$$^{C}(\,^{T}V_{C\,ORG}) = \,^{C}_{T}R(\,^{T}(\,^{T}V_{C\,ORG})) = \,^{C}_{T}R(\,^{T}V_{C\,ORG})$$
$$= \,^{C}_{U}R\,^{U}_{T}R(-70\hat{\imath}) = -\,^{U}_{C}R^{-1}\,^{U}_{T}R70\hat{\imath}$$

Time-varying Position and Orientation -3

- Angular velocity vector $^{A}\Omega_B$

B坐标对着A坐标以Ω方式旋转，Ω是一个向量，因为是在空间中转动

  - The rotation of frame $\{B\}$ relative to frame $\{A\}$
  - Direction of $^{A}\Omega_B$: The instantaneous axis of rotation
  - Magnitude of $^{A}\Omega_B$: The speed of rotation



$^{C}(\,^{A}\Omega_B)$

Expressed in frame $\{C\}$

$\omega_c = \,^{U}\Omega_C$

Angular velocity of frame $\{C\}$ relative to the universe reference frame $\{U\}$

Rigid Body Motion -1

- Freshman Dynamics

$$\overrightarrow{r_A}=x_A\hat{I}+y_A\hat{J}$$
$$=\overrightarrow{r_B}+\overrightarrow{r_{A/B}}$$
$$=\left(x_B\hat{I}+y_B\hat{J}\right)+\left(x_{A/B}\hat{I}+y_{A/B}\hat{J}\right)$$
$$=\overrightarrow{r_B}+\overrightarrow{r_{A/B}}$$

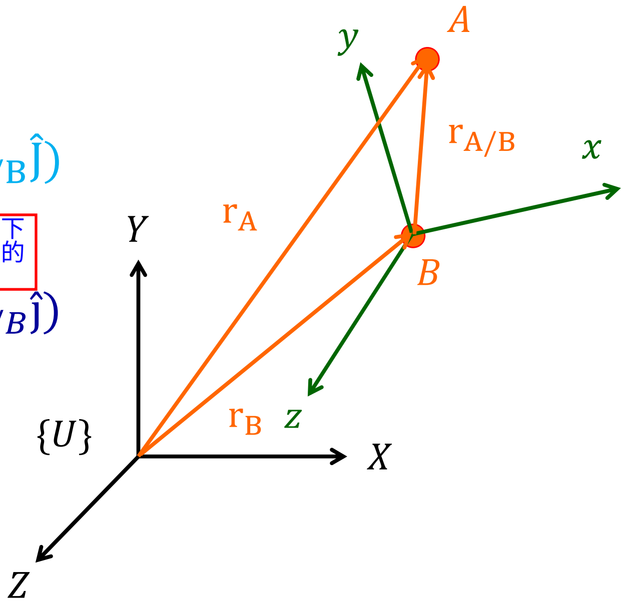
不仅可以看成在U坐标下，也可以看成在B下面的向量

$$=\left(x_B\hat{I}+y_B\hat{J}\right)+\left(x_{A/B}\hat{\imath}+y_{A/B}\hat{\jmath}\right)$$

diff.

此处其实I向量和J向量都做了微分，但是由于这两个向量在空间中固定不动，所以微分后还是本身

$$\overrightarrow{v_A}=\dot{\overrightarrow{r_A}}=\dot{x}_A\hat{I}+\dot{y}_A\hat{J}$$
$$=\dot{\overrightarrow{r_B}}+\dot{\overrightarrow{r_{A/B}}}$$
$$=\left(\dot{x}_B\hat{I}+\dot{y}_B\hat{J}\right)+\left(\dot{x}_{A/B}\hat{I}+\dot{y}_{A/B}\hat{J}\right)$$



Rigid Body Motion -2

- $$\overrightarrow{v_A}=\dot{\overrightarrow{r_B}}+\dot{\overrightarrow{r_{A/B}}}$$

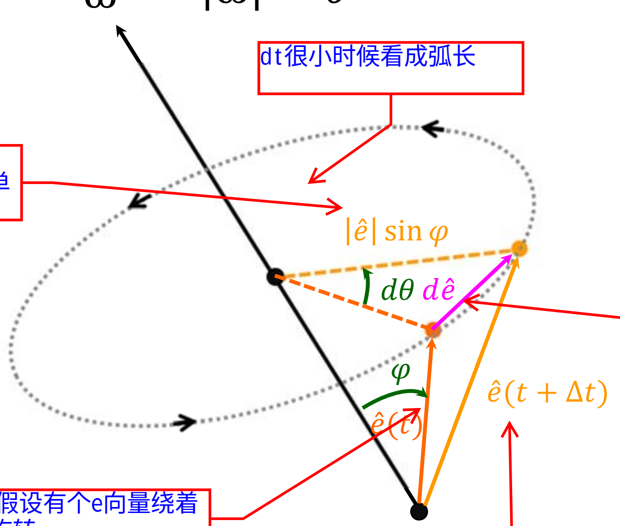
$$=\left(\dot{x}_B\hat{I}+\dot{y}_B\hat{J}\right)+\left(\dot{x}_{A/B}\hat{\imath}+\dot{y}_{A/B}\hat{\jmath}\right)+\underline{\left(x_{A/B}\,\dot{\hat{\imath}}+y_{A/B}\dot{\hat{\jmath}}\right)}$$

因为i，j会变动，I，J不会变动

$$=x_{A/B}\left(\vec{\omega}\times\hat{\imath}\right)+y_{A/B}\left(\vec{\omega}\times\hat{\jmath}\right)$$

这里这个图解释为什么对i向量求微分是w向量和i向量的叉乘

$$\vec{\omega}\qquad|\vec{\omega}|=\dot{\theta}$$



dt很小时候看成弧长

好好理解这个esinφ，放在空间中想象，e为单位向量，绕着转轴旋转

这里假设有个e向量绕着Ω轴旋转

经过代尔塔t之后转动到这里

Magnitude:

两边同时除以dt得到下面的式子

$$|d\hat{e}|=|\hat{e}|\sin\varphi d\theta$$
$$|\dot{\hat{e}}|=|\hat{e}|\sin\varphi\dot{\theta}=|\hat{e}||\vec{\omega}|\sin\varphi$$

θ求导就是w

Direction:

$$d\hat{e}\perp\hat{e}$$
$$d\hat{e}\perp\vec{\omega}$$

所以这个才是经过微分后的值

$$\Rightarrow\quad\dot{\hat{e}}=\vec{\omega}\times\hat{e}$$

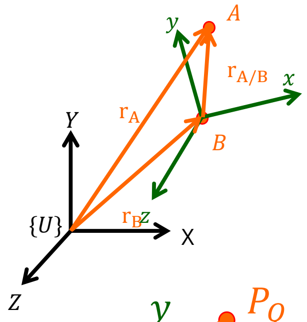
e向量转动的微分是自己和依赖转动的向量的叉乘

Rigid Body Motion -3

$$\overrightarrow{v_A} = (\dot{x}_B\hat{I} + \dot{y}_B\hat{J}) + (\dot{x}_{A/B}\hat{\imath} + \dot{y}_{A/B}\hat{\jmath}) + \vec{\omega} \times (x_{A/B}\hat{\imath} + y_{A/B}\hat{\jmath})$$

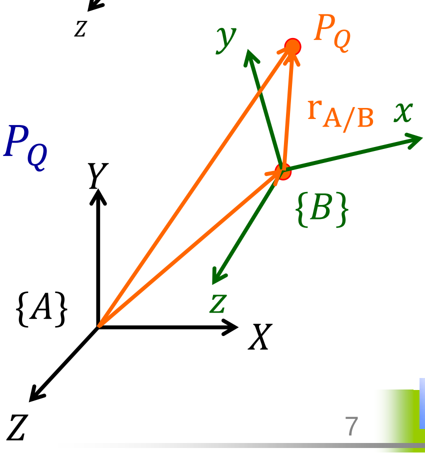
$$= (\dot{x}_B\hat{I} + \dot{y}_B\hat{J}) + (\dot{x}_{A/B}\hat{\imath} + \dot{y}_{A/B}\hat{\jmath}) + \vec{\omega} \times (x_{A/B}\hat{I} + y_{A/B}\hat{J})$$

$$\Rightarrow \quad \overrightarrow{v_A} = \overrightarrow{v_B} + \underbrace{\overrightarrow{v_{rel}}}_{\text{“relative” velocity}} + \vec{\omega} \times \overrightarrow{r_{A/B}}$$



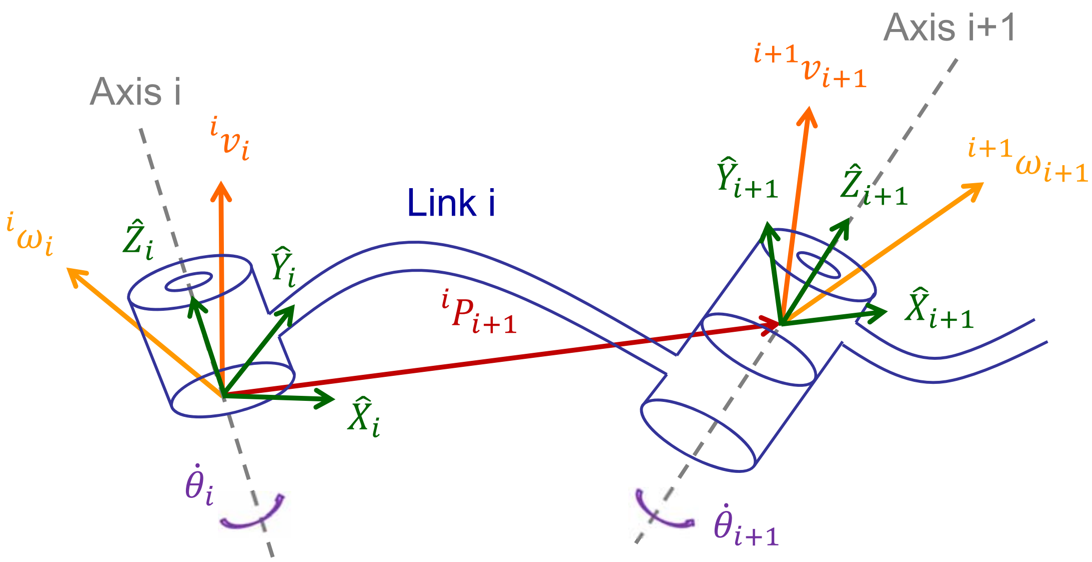
- Thus,

$$^{A}V_Q = {}^{A}V_{B\,ORG} + {}^{A}_{B}R\,\underbrace{^{B}V_Q}_{\text{“relative” velocity}} + {}^{A}\Omega_B \times {}^{A}_{B}R\,{}^{B}P_Q$$



Velocity “Propagation” from Link to Link -1

- Strategy: Represent linear and angular velocities of link i in frame $\{i\}$, and find their relationship to those of neighboring links



Velocity “Propagation” from Link to Link -2

- Rotational Joint (Link i+1)

  - Angular velocity propagation

i+1的角速度从i坐标看，i+1的坐标轴比i轴多了一个旋转，就是说i+1轴的转动速度(从i坐标系看)就是坐标系的转动速度加上i+1轴相对于i轴的转动速度

$$ {}^{i}\omega_{i+1} = {}^{i}\omega_i + {}_{i+1}^{\;\;i}R\,\underline{\dot{\theta}_{i+1}\,{}^{i+1}\hat{Z}_{i+1}} $$

$$ \dot{\theta}_{i+1}\,{}^{i+1}\hat{Z}_{i+1} = {}^{i+1}\begin{bmatrix} 0 \\ 0 \\ \dot{\theta}_{i+1} \end{bmatrix} $$

$$ \Big\downarrow\ {}_{i}^{i+1}R $$

$$ {}^{i+1}\omega_{i+1} = {}_{i}^{i+1}R\ {}^{i}\omega_i + \dot{\theta}_{i+1}\,{}^{i+1}\hat{Z}_{i+1} $$

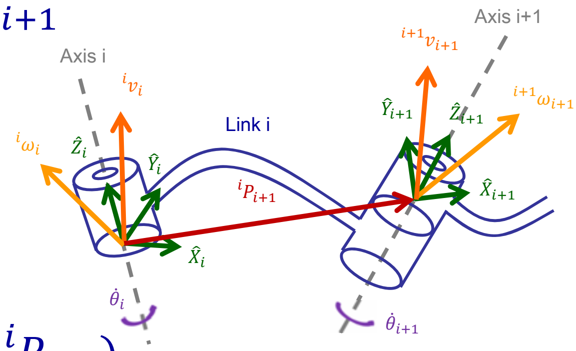
  - Linear velocity propagation

参考前面的推导ra rb ra/b那块

$$ {}^{i}v_{i+1} = {}^{i}v_i + {}^{i}\omega_i \times {}^{i}P_{i+1} $$

$$ \Big\downarrow\ {}_{i}^{i+1}R $$

$$ {}^{i+1}v_{i+1} = {}_{i}^{i+1}R\left({}^{i}v_i + {}^{i}\omega_i \times {}^{i}P_{i+1}\right) $$



Velocity “Propagation” from Link to Link -3

- Prismatic joint (Link i+1)

假设i+1轴沿着axis i+1移动

  - Angular velocity propagation

$$ {}^{i}\omega_{i+1} = {}^{i}\omega_i $$

因为只有移动没有转动的情况下

$$ \Big\downarrow\ {}_{i}^{i+1}R $$

$$ {}^{i+1}\omega_{i+1} = {}_{i}^{i+1}R\ {}^{i}\omega_i $$

  - Linear velocity propagation

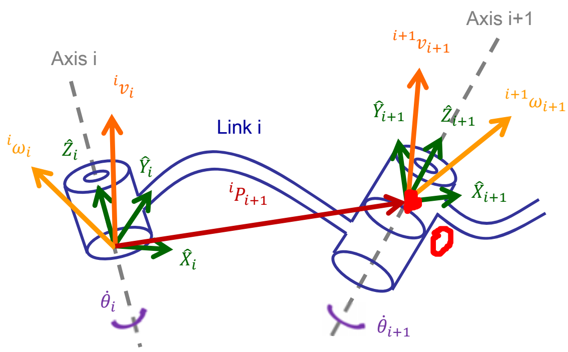
$$ {}^{i}v_{i+1} = \left({}^{i}v_i + {}^{i}\omega_i \times {}^{i}P_{i+1}\right) + {}_{i+1}^{\;\;i}R\,\underline{\dot{d}_{i+1}\,{}^{i+1}\hat{Z}_{i+1}} $$

$$ \dot{d}_{i+1}\,{}^{i+1}\hat{Z}_{i+1} = {}^{i+1}\begin{bmatrix} 0 \\ 0 \\ \dot{d}_{i+1} \end{bmatrix} $$

i+1相对于i轴的速度就是o点相对于i的速度加上o在z轴上面的移动

$$ \Big\downarrow\ {}_{i}^{i+1}R $$

$$ {}^{i+1}v_{i+1} = {}_{i}^{i+1}R\left({}^{i}v_i + {}^{i}\omega_i \times {}^{i}P_{i+1}\right) + \dot{d}_{i+1}\,{}^{i+1}\hat{Z}_{i+1} $$



Jacobians -1

- A multidimensional form of the derivative

$$y_1 = f_1\,(x_1, x_2, x_3, x_4, x_5, x_6)$$

六个输入对应六个输出

$$y_2 = f_2\,(x_1, x_2, x_3, x_4, x_5, x_6)$$

$$\vdots$$

$$y_6 = f_6\,(x_1, x_2, x_3, x_4, x_5, x_6)$$

$\Rightarrow$ $Y = F(X)$

Jacobians -2

- Calculating the differentials of y_i as a function of differentials of x_i

$$\delta y_1 = \frac{\partial f_1}{\partial x_1}\delta x_1 + \frac{\partial f_1}{\partial x_2}\delta x_2 + \cdots + \frac{\partial f_1}{\partial x_6}\delta x_6$$

$$\delta y_2 = \frac{\partial f_2}{\partial x_1}\delta x_1 + \frac{\partial f_2}{\partial x_2}\delta x_2 + \cdots + \frac{\partial f_2}{\partial x_6}\delta x_6$$

$$\vdots$$

$$\delta y_6 = \frac{\partial f_6}{\partial x_1}\delta x_1 + \frac{\partial f_6}{\partial x_2}\delta x_2 + \cdots + \frac{\partial f_6}{\partial x_6}\delta x_6$$

Jacobian, “linear transformation”

$\Rightarrow$ $$\delta Y = \frac{\partial F}{\partial X}\delta X = J(\underline{X})\delta X$$

Function of X, if f_i is nonlinear

$\Rightarrow$ $\dot{Y} = J(X)\dot{X}$

Jacobians -3

- In robotics
  - Relating joint velocities to Cartesian velocities of the tip of the arm

末端点对地的速度和角速度

$$ {}^{0}\boldsymbol{\nu} = \begin{bmatrix} {}^{0}v \\ {}^{0}\omega \end{bmatrix} = {}^{0}J(\Theta)\dot{\Theta} $$

input端的θ

3x1 : plane motion

6x1 : spatial motion

平面运动是一个3*1的向量，空间运动是一个6*1的向量

- Changing a Jacobian's frame of reference (spatial motion)

$$ {}^{B}\boldsymbol{\nu} = \begin{bmatrix} {}^{B}v \\ {}^{B}\omega \end{bmatrix} = {}^{B}J(\Theta)\dot{\Theta} $$

已知在B坐标下的速度和角速度，求得在A坐标下的速度和角速度

$$ {}^{A}\boldsymbol{\nu} = \begin{bmatrix} {}^{A}v \\ {}^{A}\omega \end{bmatrix} = {}^{A}J(\Theta)\dot{\Theta} = \begin{bmatrix} {}^{A}_{B}R & 0 \\ 0 & {}^{A}_{B}R \end{bmatrix} \begin{bmatrix} {}^{B}v \\ {}^{B}\omega \end{bmatrix} $$

$$ \Rightarrow \quad {}^{A}J(\Theta) = \begin{bmatrix} {}^{A}_{B}R & 0 \\ 0 & {}^{A}_{B}R \end{bmatrix} {}^{B}J(\Theta) $$

Jacobians -4

- Invertibility

已知速度反推输入

$$ \dot{\Theta} = J^{-1}(\Theta)\boldsymbol{v} $$

如果这个J的逆矩阵不存在称这个矩阵singular

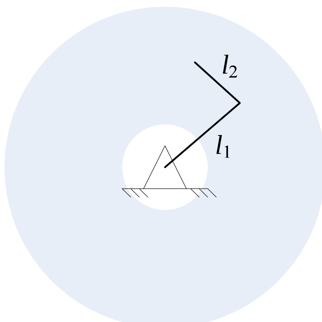
  - Singular: When the Jacobian J is NOT invertible
    - Workspace-boundary singularities

      Ex: When the manipulator is fully stretch out or folded back on itself
    - Workspace-interior singularities

奇异解的情况两种

  - When a manipulator is in a singular configuration
    - Lost one or more DOF

当处于singular状态时候，假设处于完全伸直状态，那么手臂就会失去沿着手臂向外方向的速度，速度方向会受限，如果在这个浅蓝色以内某点则可以有各种速度方向，不受限



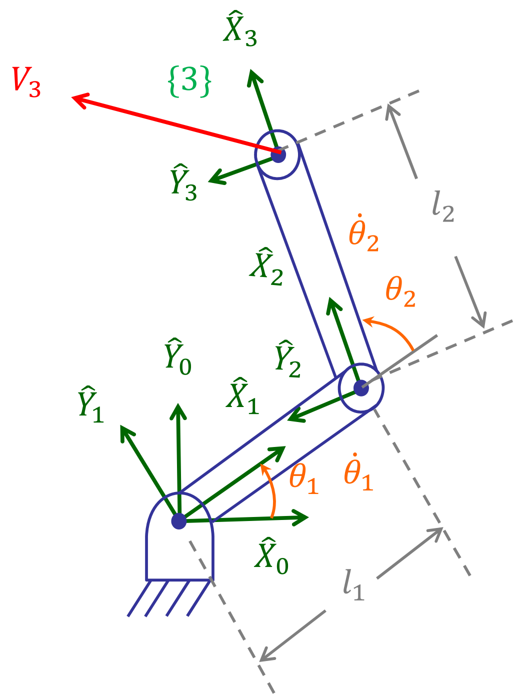
Example: A RR Manipulator -1

- Method 1: Velocity "propagation" from link to link

$$ {}^0_1T = \begin{bmatrix} c_1 & -s_1 & 0 & 0 \\ s_1 & c_1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

$$ {}^1_2T = \begin{bmatrix} c_2 & -s_2 & 0 & l_1 \\ s_2 & c_2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

$$ {}^2_3T = \begin{bmatrix} 1 & 0 & 0 & l_2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} $$



Example: A RR Manipulator -2

- Link "propagation"

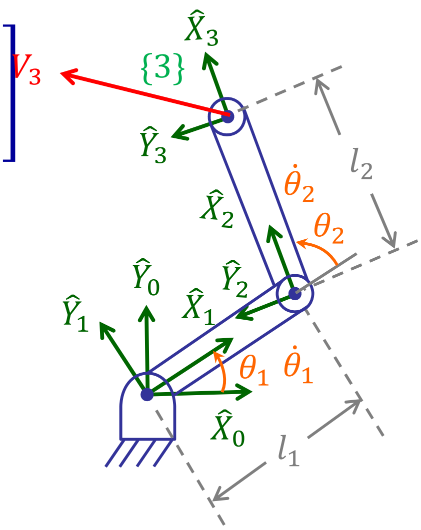
地杆不动，这个项为0

$$ {}^1\omega_1 = {}^1_0R\ {}^0\cancel{\omega_0} + \dot{\theta}_1\ {}^1\hat{Z}_1 = \dot{\theta}_1\ {}^1\hat{Z}_1 = \begin{bmatrix} 0 \\ 0 \\ \dot{\theta}_1 \end{bmatrix} $$

$$ {}^1v_1 = {}^1_0R(\ {}^0\cancel{v_0} + \ {}^0\cancel{\omega_0} \times \ {}^0\cancel{P_1}) = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} $$

$$ {}^2\omega_2 = {}^2_1R\ {}^1\omega_1 + \dot{\theta}_2\ {}^2\hat{Z}_2 = \begin{bmatrix} 0 \\ 0 \\ \dot{\theta}_1 + \dot{\theta}_2 \end{bmatrix} $$

$$ {}^2v_2 = {}^2_1R(\ {}^1\cancel{v_1} + \ {}^1\omega_1 \times \ {}^1P_2) = \begin{bmatrix} c_2 & s_2 & 0 \\ -s_2 & c_2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ l_1\dot{\theta}_1 \\ 0 \end{bmatrix} = \begin{bmatrix} l_1 s_2 \dot{\theta}_1 \\ l_1 c_2 \dot{\theta}_1 \\ 0 \end{bmatrix} $$



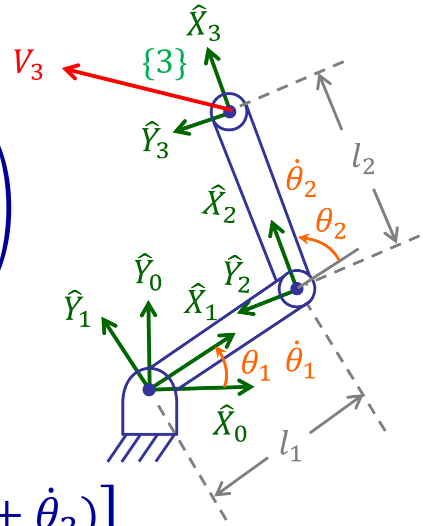
Example: A RR Manipulator -3

$$ {}^3\omega_3 = {}^2\omega_2 $$

$$ {}^3v_3 = {}^3_2R\left({}^2v_2 + {}^2\omega_2 \times {}^2P_3\right) $$

$$ = I\left(\begin{bmatrix} l_1 s_2\dot{\theta}_1 \\ l_1 c_2\dot{\theta}_1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \dot{\theta}_1 + \dot{\theta}_2 \end{bmatrix} \times \begin{bmatrix} l_1 \\ 0 \\ 0 \end{bmatrix}\right) $$

$$ = \begin{bmatrix} l_1 s_2\dot{\theta}_1 \\ l_1 c_2\dot{\theta}_1 + l_2(\dot{\theta}_1 + \dot{\theta}_2) \\ 0 \end{bmatrix} $$



$$ {}^0v_3 = \underline{{}^0_3R}\ {}^3v_3 = \begin{bmatrix} -l_1 s_1\dot{\theta}_1 - l_2 s_{12}(\dot{\theta}_1 + \dot{\theta}_2) \\ l_1 c_1\dot{\theta}_1 + l_2 s_{12}(\dot{\theta}_1 + \dot{\theta}_2) \\ 0 \end{bmatrix} $$

$$ = {}^0_1R\,{}^1_2R\,{}^2_3R $$

$$ = \begin{bmatrix} c_{12} & -s_{12} & 0 \\ s_{12} & c_{12} & 0 \\ 0 & 0 & 1 \end{bmatrix} $$

Example: A RR Manipulator -4

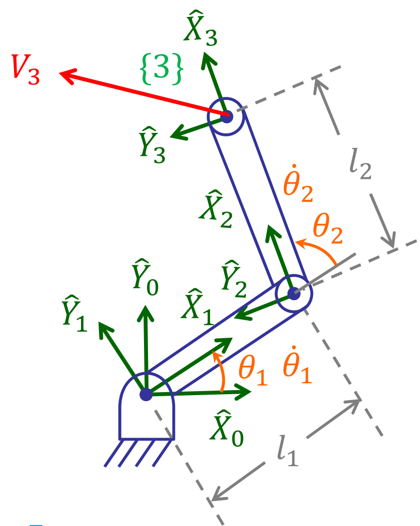
- Therefore

$$ {}^3\boldsymbol{v} = \begin{bmatrix} v_x \\ v_y \\ \omega \end{bmatrix} = \begin{bmatrix} l_1 s_2 & 0 \\ l_1 c_2 + l_2 & l_2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix} $$

$$ = {}^3J(\Theta)\dot{\Theta} $$

$$ det \begin{vmatrix} l_1 s_2 & 0 \\ l_1 c_2 + l_2 & l_2 \end{vmatrix} = l_1 l_2 s_2 = 0 $$

奇异解 $\Rightarrow \theta_2 = 0 \ or \ 180$



$$ {}^0\boldsymbol{v} = \begin{bmatrix} v_x \\ v_y \\ \omega \end{bmatrix} = \begin{bmatrix} -l_1 s_1 - l_2 s_{12} & -l_2 s_{12} \\ l_1 c_1 + l_2 c_{12} & l_2 c_{12} \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix} $$

$$ = {}^0J(\Theta)\dot{\Theta} $$

Example: A RR Manipulator -5

- Method 2: Direct differentiation

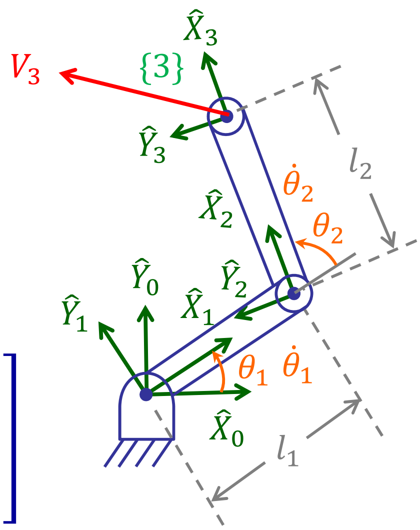
直接微分法来求

$$ {}^{0}\begin{bmatrix} p_x \\ p_y \\ \theta \end{bmatrix} = {}^{0}\begin{bmatrix} l_1 c_1 + l_2 c_{12} \\ l_1 s_1 + l_2 s_{12} \\ \theta_1 + \theta_2 \end{bmatrix} $$

$\downarrow$ diff.

$$ {}^{0}\begin{bmatrix} v_x \\ v_y \\ \omega \end{bmatrix} = {}^{0}\begin{bmatrix} -l_1 s_1 \dot{\theta}_1 - l_2 s_{12}(\dot{\theta}_1 + \dot{\theta}_2) \\ l_1 c_1 \dot{\theta}_1 + l_2 s_{12}(\dot{\theta}_1 + \dot{\theta}_2) \\ \dot{\theta}_1 + \dot{\theta}_2 \end{bmatrix} $$

$$ = {}^{0}\begin{bmatrix} -l_1 s_1 - l_2 s_{12} & -l_2 s_{12} \\ l_1 c_1 + l_2 c_{12} & l_2 c_{12} \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix} $$



$$ \dot{X} = {}^{0}J(\Theta)\dot{\Theta} $$

Note: NO 3x1 orientation vector whose derivative is ω

Static Forces in Manipulators -1

本节是假设系统定格在某个瞬间下的力的输入和输出的关系

- When considering static forces
  - Lock all the joints
  - Write force-moment relationship
  - Compute static torque (ignore gravity)

只考虑关节之间的，不考虑重力作用下关节重量

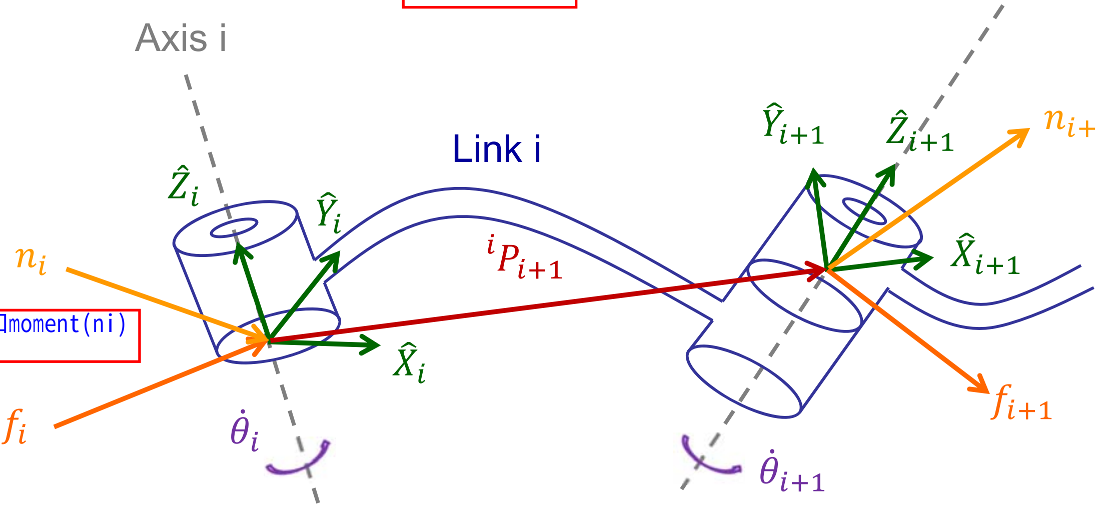
Static Forces in Manipulators -2

- $${}^{i}f_i = {}^{i}f_{i+1} \qquad {}^{i}n_i = {}^{i}n_{i+1} + {}^{i}P_{i+1} \times {}^{i}f_{i+1}$$

$$\downarrow {}^{i}_{i+1}R \qquad \downarrow {}^{i}_{i+1}R$$

$${}^{i}f_i = {}^{i}_{i+1}R\,{}^{i+1}f_{i+1} \qquad {}^{i}n_i = {}^{i}_{i+1}R\,{}^{i+1}n_{i+1} + {}^{i}P_{i+1} \times {}^{i}f_i$$

由末端点往回算



force(fi)和moment(ni)

Static Forces in Manipulators -3

- The joint toque required to maintain the static equilibrium
  - Revolute joint

做一个投影才是某个关节方向的

$$\tau_i = {}^{i}n_i^T\ {}^{i}\hat{Z}_i$$

  - Prismatic joint

$$\tau_i = {}^{i}f_i^T\ {}^{i}\hat{Z}_i$$

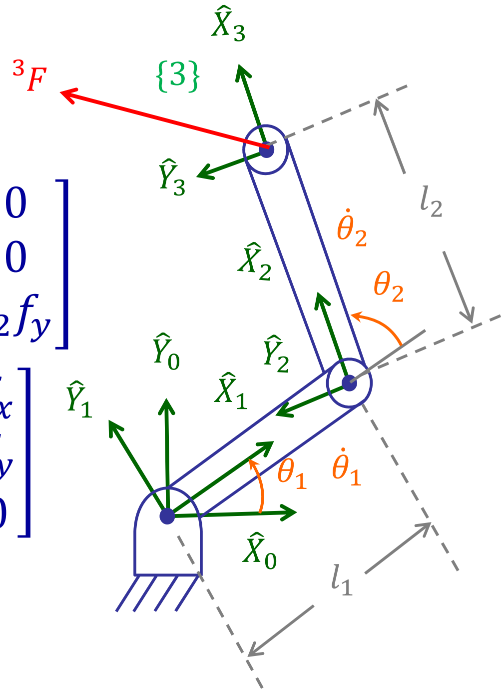
Example: A RR Manipulator -1

- Force “propagation” from link to link

$$ {}^2f_2 = {}^2_3R\ {}^3f_3 = I\ {}^3F = \begin{bmatrix} f_x \\ f_y \\ 0 \end{bmatrix} $$

$$ {}^2n_2 = {}^2_3R\ {}^3n_3 + {}^2P_3 \times {}^2f_2 = \begin{bmatrix} 0 \\ 0 \\ l_2 f_y \end{bmatrix} $$

$$ {}^1f_1 = {}^1_2R\ {}^2f_2 = \begin{bmatrix} c_2 & -s_2 & 0 \\ s_2 & c_2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} f_x \\ f_y \\ 0 \end{bmatrix} = \begin{bmatrix} c_2 f_x - s_2 f_y \\ s_2 f_x + c_2 f_y \\ 0 \end{bmatrix} $$



Example: A RR Manipulator -2

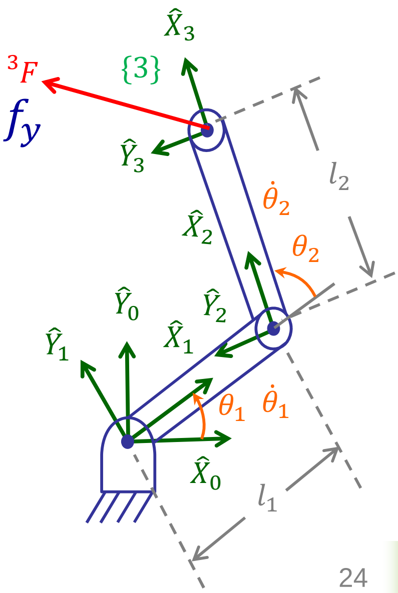
$$ {}^1n_1 = {}^1_2R\ {}^2n_2 + {}^1P_2 \times {}^1f_1 = \begin{bmatrix} 0 \\ 0 \\ l_1 s_2 f_x + l_1 c_2 f_y + l_2 f_y \end{bmatrix} $$

- Therefore,

$$ \tau_1 = {}^1n_1^T\ {}^1\widehat{Z_1} = l_1 s_2 f_x + (l_1 c_2 + l_2) f_y $$

$$ \tau_2 = {}^2n_2^T\ {}^2\widehat{Z_2} = l_2 f_y $$

$$ \Rightarrow \tau = \begin{bmatrix} l_1 s_2 & l_1 c_2 + l_2 \\ 0 & l_2 \end{bmatrix} \begin{bmatrix} f_x \\ f_y \end{bmatrix} $$



Jacobian in the Force Domain

- The principal of virtual work

$$F \cdot \delta\mathcal{X} = \Gamma \cdot \delta\Theta$$

$$F^T \delta\mathcal{X} = F^T J \delta\Theta = \Gamma^T \delta\Theta$$

$$\Gamma = J^T F$$

Respect to frame $\{0\}$

$$\Rightarrow \quad \Gamma = {}^0J^T \, {}^0F$$

"inverse" Cartesian torque to joint torque without using IK technique

Cartesian Transformation -1

- General velocity and force representations

$$\boldsymbol{\nu} = \begin{bmatrix} v \\ \omega \end{bmatrix} \qquad \boldsymbol{\mathcal{F}} = \begin{bmatrix} F \\ N \end{bmatrix}$$

- Frame transformation

$${}^{i+1}\omega_{i+1} = {}^{i+1}_{i}R \; {}^{i}\omega_i + \dot{\theta}_{i+1} \, {}^{i+1}\hat{Z}_{i+1}$$

$${}^{i+1}v_{i+1} = {}^{i+1}_{i}R(\, {}^{i}v_i + \, {}^{i}\omega_i \times \, {}^{i}P_{i+1})$$

$$\Big\downarrow \; i = A, i+1 = B, \; \dot{\theta} = 0$$

$$\begin{bmatrix} {}^{A}v_A \\ {}^{A}\omega_A \end{bmatrix} = \begin{bmatrix} {}^{A}_{B}R & {}^{A}P_{B\,ORG} \times {}^{A}_{B}R \\ 0 & {}^{A}_{B}R \end{bmatrix} \begin{bmatrix} {}^{B}v_B \\ {}^{B}\omega_B \end{bmatrix}$$

$${}^{A}\boldsymbol{\nu}_A = {}^{A}_{B}T_v \, {}^{B}\boldsymbol{\nu}_B \qquad\qquad P \times = \begin{bmatrix} 0 & -p_z & p_y \\ p_z & 0 & -p_x \\ -p_y & p_x & 0 \end{bmatrix}$$

Cartesian Transformation -2

$$\Big\downarrow \text{“}inverse\text{"}$$

$$\begin{bmatrix} {}^{B}v_B \\ {}^{B}\omega_B \end{bmatrix} = \begin{bmatrix} {}^{B}_{A}R & -{}^{B}_{A}R\,{}^{A}P_{BORG}\times \\ 0 & {}^{B}_{A}R \end{bmatrix} \begin{bmatrix} {}^{A}v_A \\ {}^{A}\omega_A \end{bmatrix}$$

$${}^{B}\boldsymbol{\nu}_B = {}^{B}_{A}T_v\,{}^{A}\boldsymbol{\nu}_A$$

- Similarly,

$$\begin{bmatrix} {}^{A}F_A \\ {}^{A}N_A \end{bmatrix} = \begin{bmatrix} {}^{A}_{B}R & 0 \\ {}^{A}P_{BORG}\times {}^{A}_{B}R & {}^{A}_{B}R \end{bmatrix} \begin{bmatrix} {}^{B}F_B \\ {}^{B}N_B \end{bmatrix}$$

$${}^{A}\boldsymbol{\mathcal{F}}_A = {}^{A}_{B}T_f\,{}^{B}\boldsymbol{\mathcal{F}}_B$$

$$\Rightarrow \quad {}^{A}_{B}T_f = {}^{A}_{B}T_v{}^{T}$$

The End

- Questions?